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Truncated Quantile Regression

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Truncated Quantile Regression

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Abstract

This paper deals with the estimation of conditional quantiles of linear truncated regression models with known truncation point. The truncated quantile model is shown to posses an important property related to $\Theta_{or} \in (0,1)$, a set of quantiles of the original latent model: **truncation-invariance**. Truncation-invariance means that there is a one-to-one function $F : \Theta_{or} \to \Theta_{tr}$, $\Theta_{tr} \in (0,1)$ and where Θ_{tr} is a set of quantiles of the truncated variable. That property turns out to be fundamental to identification of the model, as well as estimation and inference. In fact, simplicity is a major appeal of our semi-parametric approach compared to alternatives estimator of truncated models, as it does not require any modification of available software.

1 Introduction

Since the seminal paper of Koenker and Bassett (1978), the literature on quantile regression has experienced a steady growth. This semi-parametric approach to estimation has influenced many aspects of econometrics both at theoretical and applied grounds. While Koenker and Bassett (1978) set the theoretical foundations of quantile regression, an assessment about the type of possible applications of that technique can be found on Buchinsky (1994), Buchinsky (1998) and the special edition about quantile regression on *Empirical Economics*, volume 26, 2001. See, specially, Ribeiro (2001) for a study of asymmetric labor supply in Brazil, Koenker and Bilias (2001)

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for an application of quantile regression to duration data and Bierens and Ginther (2001) for a consistent test of the linearity of the quantile model. A quick overview of the entire spectrum of quantile regression is Buchinsky (1998). Notwithstanding that, there are still open problems in econometrics that could be fruitfully approached by the use of quantile regression. We believe that truncated regression models is one of those.

2 The Truncated Quantile Regression Model

The quantile regression with fractional variable model, henceforth **QRF**, is defined by the following equations:

$$
\begin{aligned}
L(y) &= \ln[y \cdot (1-y)^{-1}] \\
&= X'\beta + u \qquad \text{for } y \in (0,1)
\end{aligned} \tag{1}
$$

$$
l(y) = \begin{cases} p_0 & \text{if } y = 0 \\ f(y) & \text{if } y \in (0,1) \\ p_1 & \text{if } y = 1 \end{cases} \tag{2}
$$

$$
\mathcal{Q}_\theta\big[L(y)|X\big] = X'\beta_\theta \qquad \text{for } \theta \in (p_0, 1-p_1) \tag{3}
$$

Where equation (1) just repeats the log-odds model, leaving unspecified $L(0)$ and $L(1)$. The function $l(.)$ is the probability density function of Y. Thus, Y is a mixed discrete-continuous random variable. Finally, $\mathcal{Q}_\theta[.|X]$ represents the θ'th quantile operator. The distribution of the error term u_θ is left unspecified and it is assumed that:

Assumption 1 (Zero Conditional Quantile) $\mathcal{Q}_\theta[u_\theta|X] = 0$.

The model partially outlined will try to overcome the drawbacks of the conditional expectation model. We approach the **recovery** issue by adopting a conditional quantile specification instead of a conditional expectation. Since the function $g(y) = \ln\left[y \cdot (1-y)^{-1}\right]$ is monotonic increasing, the operator conditional quantile is easy to deal with. Accordingly to Powell (1991), if $h(y)$ is monotonic increasing, $\mathcal{Q}_\theta\left[h(y)|X\right] = h\big(\mathcal{Q}_\theta[y|X]\big)$. Thus, it follows that:

$$\mathcal{Q}_\theta\left[\ln[y \cdot (1-y)^{-1}]|X\right] = \ln\left[\mathcal{Q}_\theta(y|X) \cdot (1 - \mathcal{Q}_\theta(y|X))^{-1}\right] \tag{4}$$

Hence, from a model specified by Equations (1), (2), (3) and from inspection of Equation (4) the quantity of interest, $\mathcal{Q}_\theta(y|X)$, has the following expression:

$$\mathcal{Q}_\theta(y|X, y \in (0,1)) = \big(1 + \exp(X'\beta_\theta)\big)^{-1} \tag{5}$$

To deal with the **boundary** problem we make use of the fact that truncation of a random variable is analogous to a monotonic increasing transformation of the original (non truncated) cumulative distribution function . To see this let the cumulative distribution function of a random variable Y be $F(.)$, for all $y \in \mathcal{Y}$. Also, let A be a subset of $\mathcal{Y}$ and $\mathcal{Y} \setminus A$ the complement of the set A with respect to $\mathcal{Y}$, A_0 the left interval of truncation and A_1 the right interval of truncation. $A_0 \cup A_1 \cup A = \mathcal{Y}$ and $A_i \cap A = \emptyset$ for $i = 0, 1$. As long as $P(\mathcal{Y} \setminus A) > 0$, the following conditional cumulative distribution function is well-defined:

$$\widetilde{G}(y|y \in \mathcal{Y} \setminus A) = \frac{F(y) - P(A_0)}{P[y \in \mathcal{Y} \setminus A]} \qquad \text{for all } y \in \mathcal{Y} \setminus A \tag{6}$$

Thus, the monotonic feature of the truncation has the potential to overcome the **boundary** problem by noting the following. Suppose that one is interested in estimating a fractional regression model and that the model appearing in Equations (1), (2) and (3) is correctly specified. Also, suppose that the econometrician does not want to use a subset of available observations. The first consequence of that type trimming is that Equation (6) should the right one to describe the new sample instead of $F(z)$. However, $\widetilde{G}(.)$ is again just a monotonic increasing transformation of $F(.)$. The quantile set up once more makes its point as it is well known that the same truncation would demand a much more complicated analysis had the model been set in terms of a conditional mean. Nonetheless the simplicity of the preceding argument, there are three issues related to the specific process of trimming: first, even for a fixed initial sample size, the available sample size is no longer a fixed number but, instead, a random variable; second, not all quantiles from the original model might be identified; third, the truncation changes the location of quantiles[1].

Before elaborating on our sample strategy, it is worth to compare our trimmed strategy with other uses of trimming. The use of trimmed samples has its origins in the statistical literature of robust estimation. A classical example, with roots on an old French custom, is the trimming of extreme observations to calculate the mean. The α-trimmed mean consists in ordering the n observations of a sample from an *i.i.d.* random variable X, removing a proportion of α from both extremes of the sample and, finally, taking the average of the trimmed sample. Taking robustness[2] as synonym to insensitiveness to outliers in the sample, the trimmed mean estimator is a very simple robust estimator. Its computation is easy:

$$T_\alpha = (n - [2 \cdot \alpha \cdot n])^{-1} \cdot \sum_{j=[\alpha \cdot n+1]}^{n-[2\cdot\alpha\cdot n]} X_j. \tag{7}$$

Where [.] is the *greatest integer* function. The objective of trimming the sample is to remove the influence of outliers on the estimated mean. The α-trimmed sample has its asymptotics developed in Stigler (1973). The econometrics literature has also made use of trimmed samples. The approach of heteroscedasticity of the Tobit model by Powell (1986) is an interesting exam-

[1]Indeed, if the variable is truncated both in the left and right side this change is non-trivial.

[2]See Huber (1972) for a non-technical discussion about the different concepts of robustness in the statistical literature. For a thorough exposition of robust inference, see, Maddala and Rao (1997).

ple[3]. Now, the "trimming" is used to turn a non-symmetric distribution into a symmetric one. With that, ordinary least squares can deliver a consistent estimate for the true parameter, regardless the presence of heteroscedasticity. Notwithstanding the differences between these two examples, they share a common feature: the available sample size continues to be predetermined. This is in contrast to our proposed sample scheme. Also, our initial hope was to use a trimmed sample not to protect against outliers nor to symmetrize our distribution but simply to make the specification of our model at $y = 0$ and $y = 1$ completely dispensable.

The fundamental question that arises now is under what conditions can β_θ be consistently estimated if the trimmed sample is to be used in the place of the complete sample? It turns out to be that as long as the trimmed model satisfies some very mild conditions, β_θ can be consistently estimated, at least for $\theta \in \Delta \subset [0, 1]$. Hence, in order to consistently estimate quantiles of the log-odds model, any attempt to specify a transformation at the boundaries is an unnecessary complication. All that is needed is an specification for the interval $(0, 1)$. Nevertheless, the implications of the trimmed sample on the efficiency of the quantile estimator is not considered here. In fact we believe this is an interesting topic for future research. In order to define precisely the model to be estimated, we make the following additional assumptions[4]:

Assumption 2 *The cumulative distribution function of Y, $F(.)$ is continuous in the set $y \in \mathcal{Y} \setminus A$.*

Assumption 3 $F(.) > 0$ *for all* $z \in [0, 1]$.

Assumption 4 *The set $\mathcal{Y} \setminus A$ is convex.*

Assumption 5 $P(y \in \mathcal{Y} \setminus A) > 0$.

Thus, given a vector of random variables (Y, X), $X \in \mathbb{R}^n$, we define our model by Equations (1), (2) , (3) and assumptions 1 - 5. The trimmed sample becomes the focus of analysis. In order to understand the consequences of trimming the sample, define the sampling scheme to be employed by the following two steps:

[3]Strictly speaking, there is no trimming of any observation. Powell (1986) truncates the random variable. Hence, observation are transformed and not thrown away.

[4]For the brevity of notation, the conditioning on X is taken for granted, although it does not explicitly appears in the formulas.

1. Fix a total sample size n to be draw;
2. During sampling, keep an observation if and only if it is on the interval $\mathcal{Y} \setminus A$, otherwise discharge it;

Call this scheme sequential trimming, **ST**. Denote by $N(n)$ the number of sampled observation that was retained. $N(n) = 1, 2, \cdots n$. The best way to see the effect of that deliberate trimming on the available sample is to use a triangular array. Denote y_{ij} the $j'th$ sample observation if only $i \leq n$ observations are in the set $\mathcal{Y} \setminus A$. The array assumes the following shape, as outlined by Lehmann (1999):

$$
\begin{array}{llll}
y_{11} & & & \\
y_{21} & y_{22} & & \\
\cdot & & & \\
\cdot & & & \\
\cdot & & & \\
y_{n1} & y_{n2} & \cdots & y_{nn}
\end{array}
$$

Hence, for fixed n the available sample size is $N(n)$, a random variable. However, since an observation is retained if and only if $y \in \mathcal{Y} \setminus A$, $N(n)$ and Y_{ij} are not independent and it becomes tricky how to do the asymptotics of this model. A key fact to the development of our strategy of estimation is that **ST** can be represented by a random variable with random indices. The following notation is adopted throughout the paper. We represent the fact that a sequence of random variables Z_n converges to a random variable Z in probability by $Z_n \quad \xrightarrow{p} \quad Z$. Convergence in law to the cumulative distribution function $H_z(.)$ is represented by $Z_n \quad \xrightarrow{\mathcal{L}} \quad H_z(.)$. Also, $\mathcal{Y} \setminus A \equiv \overline{A}$.

The next three sequences of random variables will play a key role in the rest of the paper:

- Y_n
- $\widetilde{Y_n}$
- $\widetilde{Y_{N(n)}}$

Where $n \to \infty$, $N(n) \to \infty$ and $N(n) \leq n$. Y_n is an *i.i.d.* sequence of random variables whose properties are given by Assumptions 2 and 3. $\widetilde{Y_n}$ is an

$i.i.d.$ sequence of random variables whose cumulative distribution function is given by $G(y) = \frac{F(y)-P(A_0)}{P(\overline{A})}$, for all $y \in \overline{A}$. $\widetilde{Y_{N(n)}}$ is an $i.i.d.$ sequence of random variables whose cumulative distribution function is also $G(.)$, for every fixed $N(n) = k$. Finally, $N(n)$ is a discrete random variable not necessarily independent of $\widetilde{Y_{N(n)}}$. The central issue is the limiting distribution of the sequence of random variables indexed by a discrete random variable represented by $\widetilde{Y_{N(n)}}$ for fixed n. Hence, conditions to prove convergence in law of $\widetilde{Y_{N(n)}}$ when both $N(n)$ and n go to infinity need to be developed.

The characterization of the limit distribution of $\widetilde{Y_{N(n)}}$ consists of two parts. First, we need to show that **ST** is equivalent to the random sequence $\widetilde{Y_{N(n)}}$. Secondly, it should be proved that $\lim_{N(n)\to\infty} P(\widetilde{Y_{N(n)}} \leq y)$ exists and is equal to $\lim_{n\to\infty} P(\widetilde{Y_n} \leq y)$, at every continuity point of y. The next proposition characterizes the trimmed sampling in terms of a sequence of random variables with random indexes.

Proposition 1 *The* ***ST*** *sample scheme is equivalent to an i.i.d. random sequence with random indices,* $\widetilde{Y_{N(n)}}$*, with cumulative distribution function* $G_{N(n)}(y) = \frac{F(y)-P(A_0)}{P(\overline{A})}$*, for all* $N(n) \leq n$.

Proof

The sequence of variables that generates $\widetilde{Y_{N(n)}}$ is Y_n. For fixed n and for any $N(n) = k$, $\widetilde{Y_{N(n)}}$ is a sequence of $i.i.d$ random variables whose cumulative distribution function is obtained by conditioning on the event that $Y \in \overline{A}$:

$$P(\widetilde{Y_{N(n)}} \leq y) = \frac{F(y)-P(A_0)}{P(\overline{A})}$$

Finally, $N(n)$ is just the sum of the number of success in n independent binomial random variables, with probability of success $P(y \in \overline{A})$.

$\mathcal{Q}.\mathcal{E}.\mathcal{D}$

In order to prove the second part, we use the results concerning limit theorems of sequences of random variables with random indices. The methods of proving such limit theorems date backs to Anscombe (1949) and Renyi

(1970). More general results can be found in Guiasu (1971). Galambos (1992) generalizes the results to functionals of random sequences. The theorem to be used is the classical Anscombe theorem for random sequences of random indices when $N(n)$ is not independent of $\widetilde{Y_{N(n)}}$. As it appears in Theorem 2 of Guiasu (1971), we use the following to characterize the limiting distribution of $\widetilde{Y_{N(n)}}$ in terms of the limiting distribution of $\widetilde{Y_n}$:

Proposition 2 *If the sequence $\widetilde{Y_{N(n)}}$ meets the conditions below:*

- ***C 1*** *There exists an increasing sequence $d(n)$ of positive integer numbers which tends to infinity with n and such that $\frac{N(n)}{d(n)}$ converges in probability to a random variable M with $P(M > 0) = 1$.*
- ***C 2*** *At every continuity point of $G(.)$, $\lim_{n\to\infty} P(\widetilde{Y_n} \leqslant a) = G(a)$.*
- ***C 3*** *For every $\varepsilon > 0$ and $\eta > 0$ there exist a small real number $s_0 = s_0(\varepsilon, \eta)$ and a natural number $n_0 = n_0(\varepsilon, \eta)$ such that for every $n > n_0$, $P\big(\max_i |\widetilde{Y_i} - \widetilde{Y_n}| > \varepsilon : |i - n| < s_0 \cdot n\big) < \eta$.*

Then, in every continuity point of the function $G(.)$, $\lim_{n\to\infty} P(\widetilde{Y_{N(n)}} \leqslant a) = G(a)$.

Before we proceed with the proof, it is worth mentioning that the assumption of $i.i.d$ sampling will significantly reduces the difficulties in proving the desired result. This is specially true for conditions $C2$ and $C3$. The proof follows in three steps, one for each condition.

Proof

[C1]. Let $d(n) = n$. Thus, the sequence $\frac{N(n)}{n}$ is the proportion of successes in n binomial trials with probability of success $P(\overline{A})$. Now, by the Weak Law of Large Numbers:

$$\mathbf{E}\Big(\frac{N(n)}{n} - P(\overline{A})\Big)^2 \to 0.$$

Hence, $\frac{N(n)}{n} \xrightarrow{p} P(\overline{A})$. This follows from the fact that convergence in quadratic mean implies convergence in probability. Condition *C1* is satisfied.

[C2]. It is trivially satisfied given our assumption of an $i.i.d$ sequence of Y_n.

[C3]. Fix $s_0 = 1$. Let $Z_{i,j} = \widetilde{Y}_i - \widetilde{Y}_j$, for all $i, j = 1, 2, 3 \cdots$. Define the following new sequence of random variables:

$C_1 = 0;$
$C_2 = \max\left\{|Z_{1,2}|, |Z_{3,2}|\right\};$
$C_3 = \max\left\{|Z_{1,3}|, |Z_{2,3}|, \cdots, |Z_{5,3}|\right\};$
$C_4 = \max\left\{|Z_{1,4}|, \cdots, |Z_{3,4}|, |Z_{5,4}|, \cdots, |Z_{7,4}|\right\};$
$\vdots$
$C_n = \max\left\{|Z_{1,n}|, \cdots, |Z_{n-1,n}|, |Z_{n+1,n}|, \cdots, |Z_{2\cdot n-1,n}|\right\};$

Since $\widetilde{Y_n}$ is an $i.i.d$ sequence of random variables, C_n is a sequence of independent but not identical random variables. Define the cumulative distribution function of C_n by $H_n(.)$ and the cumulative distribution function of each individual random variable $|Z_{i,j}|$ by $L(.)$, for all $z \in \overline{A}$.

Clearly, $H_n(z) = [L(z)]^{2\cdot n-1}$ for $n > 3$. Since for a given n, $H_n(.)$ is an extreme function of a sequence of $i.i.d$ random variables, its asymptotic limit can be characterized by the following[5].

Let $z^{sup} = \sup(\overline{A})$ and $\theta > 0$ a small constant. The limit, as $n \longrightarrow \infty$, of $P(|C_n - z^{sup}| < \theta)$ is:

$$\begin{aligned} \lim_{n\to\infty} P(z^{sup} - \theta < C_n < z^{sup} + \theta) &= \lim_{n\to\infty} \left(1 - \left[L(z^{sup})\right]^{2\cdot n-1}\right) \\ &= 1 \end{aligned} \tag{8}$$

[5]A nice introduction to the asymptotic theory of extrem statistics is Galambos (1995).

Hence, $H_n(.) \xrightarrow{p} z^{sup}$. Convergence in probability to a constant is the same as convergence in law to a degenerate random variable. Hence, C_n converges in law and this implies that C_n is bounded in probability. We say that a sequence of random variables is bounded in probability if it satisfies condition **BP**. A sequence of random variable satisfies condition **BP** if:

For every $\epsilon > 0$ there exist a constant K and a natural number n_0 such that for every $n > n_0$, $P\big(|C_n| > K\big) < \epsilon$.

Finally, for $s_0 = 1$ satisfaction of condition **BP** clearly implies satisfaction of condition C3.

$\mathcal{Q.E.D}$

Hence, by proposition 2, $\widetilde{Y_{N(n)}}$ converges in law to the same cumulative distribution function of $\widetilde{Y_n}$ and this last random variable is just a monotonically increasing transformation of the cumulative distribution function of the original sequence Y_n, as shown in (6).

At least asymptotically, using the random-trimmed sample is equivalent to sampling from $G(y)$ for all $y \in \overline{A}$. In addition to that, assumption 4 guarantees that the quantile regression is well-behaved. Since we are trying to solve a model built in a set up different than the model's set up being used to estimation, it is necessary to characterize the change occurred in the quantiles of the original distributions due to the truncation of $F(.)$. This question gives rise to two issues. First, what can be identified and how from the original model if the trimmed sample is used instead of the full sample. Second, what are the asymptotic properties of this new estimator? Proposition 3 addresses both issues. Before that, we add to the set of invariant properties of the quantile regressor estimator appearing in Koenker and Bassett (1978) a new property related to the truncation of the original random variable.

Definition 1 *Let $\beta^*(\theta, y, x)$ be the estimated parameter of a regression model, where $\theta \in (0,1)$ indexes some set of population parameters. We say that a regression model is truncation-invariant if there is a convex subset $\Delta \subset [0,1]$ and a linear function $\pi : \Delta \to \mathbb{R}$ such that for any $\theta \in \Delta$, $\beta^*(\theta, y, x) =$*

$\beta^*(\pi(\theta), y_{trunc}, x)$. Where y_{trunc} is a random variable that results from a truncation of the original random variable y.

Proposition 3 *The quantile regression with fractional dependent variable model is truncation-invariant.*

Proof

Let $\Delta = (P(A_0), 1 - P(A_1))$, where $P(A_0)$ and $P(A_1)$ are the probabilities of sampling zero or one, respectively. For all $\theta \in \Delta$ define the θ'th conditional quantile of the original model by $\mathcal{Q}_\theta$. By continuity of $F(.)$, for all $y \in \Delta$, $\mathcal{Q}_\theta$ is the unique solution of $\theta = F^{-1}(\mathcal{Q}_\theta)$.

Call the truncated θ'th conditional quantile by $\overline{\mathcal{Q}_\theta}$. Note first that the cumulative distribution function for the truncated sample is $G(y) = P(\overline{A})^{-1} \cdot (F(y) - P(A_0))$.

For any $\theta \in \Delta$, clearly $\exists \quad \theta^* \in (0,1)$, such that $\mathcal{Q}_\theta = \overline{\mathcal{Q}_{\theta^*}}$. Where $\theta = P(\overline{A}) \cdot \theta^* + P(A_0)$. This is so because if $G(\overline{\mathcal{Q}_{\theta^*}}) = \theta^*$, then $P(\overline{A})^{-1} \cdot \big(F(\overline{\mathcal{Q}_{\theta^*}}) - P(A_0)\big) = \theta^*$. And that implies that:

$$F(\overline{\mathcal{Q}_{\theta^*}}) = P(\overline{A}) \cdot \theta^* + P(A_0).$$

$\mathcal{Q.E.D}$

Hence, estimating the θ'th quantile regression model with the original sample is equivalent to estimate the θ^*'th quantile regression model with the trimmed sample, where:

$$\theta^* = \frac{\theta - P(A_0)}{P(\overline{A})}. \tag{9}$$

Truncation-invariance establishes a one-to-one relationship between the quantile regression model that uses the original sample and the model that uses the trimmed sample. Hence, for all quantiles in the original model that can be identified by the truncated sample, say, $\theta \in \Delta$, consistency follows from the properties of the quantile regression developed in Koenker and Bassett (1978). Define the θ^* 'th randomly-trimmed quantile estimator by $\widehat{\beta_{RTQ}^{\theta^*}}$, and the true parameter corresponding to the θ'th quantile model by β_θ. Thus, for the randomly-trimmed quantile regressor:

$$\widehat{\beta^{\theta^*}_{RTQ}} \quad \xrightarrow{p} \quad \beta_\theta \qquad \text{for all } \theta^* : \theta^* = \frac{\theta - P(A_0)}{P(\overline{A})}.$$

As mentioned before, a study of relative efficiency of the randomly-trimmed quantile estimator vis-a-vis other estimators of models with fractional dependent variable is beyond the scope of the present article. Nonetheless, the quantile solution is well-known to outperform conditional mean models in non-gaussian situations. Next section contains a simple application of the randomly trimmed regression to model to the problem of estimating the percentage of a given sentence that is actually served by an inmate.

3 Conclusion

We have shown a new approach to estimate regression models with fractional dependent variable, when values of zero and one occur with positive probability. The randomly-trimmed quantile regression estimator appears to be a reasonable alternative to solve both the recovery and boundary problems. Asymptotically, it delivers consistent estimates of the θ'th quantile model, as long as, *theta* belongs to a specified subset of the unit interval, and it does that requiring a mild set of conditions. Notwithstanding those nice features, we believe the proposed estimator could be further improved in some directions.

A very important first point to be addressed is the issue of small sample properties. Econometricians are well aware of that consistent estimators can deliver very poor small sample properties. For instances, the case of the GMM estimator is illustrative: in small samples, accordingly to Podivinsky (1999), it can be *"badly biased, and asymptotic tests based on these estimators may have true sizes substantially different from presumed nominal sizes"*. So, a Monte Carlo study of the small sample property of our estimator should be one priority.

A second topic that deserves further attention has to do with the efficiency of the randomly-trimmed estimator vis-a-vis other competitors, specially the quasi-maximum likelihood approach of Papke and Wooldridge (1996) when the model is not correctly specified. Another interesting topic is related to specification testing of the original quantile model. However, the mixed continuous-discrete feature of the original model is an obstacle to tests like the ICM test for quantile regression of Bierens and Ginther (2001).

Finally, the estimator should pass the "real data" test. Given the straightforward implementation, one just needs to drop all zero and/or one observations and run a simple quantile regression model, we hope to see applied papers using our estimator. As a matter of fact, a thorough econometric package that, besides having a set of complete routines for data manipulation, estimation and inference, estimates quantile regression model is **EasyReg**, written by Prof. Herman Bierens from PennState University. It also contains the ICM test as it appears in Bierens and Ginther (2001). This freeware program can be found at **http://econ.la.psu.edu/~hbierens/EASYREG.HTM**.

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