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On Rich Modal Logics

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If you want the hits, be prepared for the misses

Carl Yastrzemski

On Rich Modal Logics

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Resumo

Esta dissertação trata do enriquecimento de lógicas modais. O termo enriquecimento é usado em dois sentidos distintos. No primeiro deles, de fundo semântico, propomos uma semântica difusa para diversas lógicas modais normais e demonstramos um resultado de completude para uma extensa classe dessas lógicas enriquecidas com múltiplas instâncias do axioma da confluência. Um fato curioso a respeito dessa semântica é que ela se comporta como as semânticas de Kripke usuais. O outro enriquecimento diz respeito à expressividade da lógica e se dá por meio da adição de novos conectivos, especialmente de negações modais. Neste sentido, estudamos inicialmente o fragmento da lógica clássica positiva estendido com uma negação modal paraconsistente e mostramos que essa linguagem é forte o suficiente para expressar as linguagens modais normais. Vemos que também é possível definir uma negação modal paracompleta e conectivos de restauração que internalizam as noções de consistência e determinação a nível da linguagem-objeto. Esta lógica constitui-se em uma Lógica da Inconsistência Formal e em uma Lógica da Indeterminação Formal. Em tais lógicas, com o objetivo de recuperar inferências clássicas perdidas, demonstram-se Teoremas de Ajuste de Derivabilidade. No caso da lógica estendida com uma negação paraconsistente, se removermos a implicação ainda lidaremos com uma linguagem bastante rica, com ambas negações paranormais e seus respectivos conectivos de restauração. Sobre esta linguagem estudamos a lógica modal normal minimal definida por meio de um cálculo de Gentzen apropriado, à diferença dos demais sistemas estudados até então, que são apresentados via cálculo de Hilbert. Em seguida após demonstrarmos a completude do sistema dedutivo associado a este cálculo, introduzimos algumas extensões desse sistema e buscamos Teoremas de Ajuste de Derivabilidade adequados.

Palavras-chave: Lógica Modal, Lógica Paranormal, Lógica Difusa.

On Rich Modal Logics

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Abstract

This thesis is about the enrichment of modal logics. We use the term enrichment in two distinct ways. In the first of them, it is a semantical enrichment. We propose a fuzzy semantics to different normal modal logics and we prove a completeness result for a generous class of this logics enriched with multiple instances of the axiom of confluence. A curious fact about this semantics is that it behaves just like the usual boolean-based Kripke semantics for modal logics. The other enrichment is about the expressibility of the logic and it occurs by means of the addition of new connectives, essentially modal negations. In this sense, firstly we study the positive fragment of classical logic extended with a paraconsistent modal negation and we show that this language is sufficiently strong to express the normal modal logics. It is also possible to define a paracomplete modal negation and restoration connectives that internalize at the level object-language the notions of consistency and determinedness. This logic constitutes a Logic of Formal Inconsistency and a Logic of Formal Undeterminedness. In such logics, with the objective of recovering lost inferences of classical logic, Derivability Adjustment Theorems are proved. In the case of the logic with one paraconsistent negation, if we remove the implication we still have a rich language, with both paranormal negations and its respective connectives of restoration. In this logic we study the minimal normal modal logic defined by means of a Gentzen calculus, differently of the others modal systems studied, which are presented by means of Hilbert calculus. Next, after we prove a completeness result of the deductive system associated to this calculus, we present some extensions of this system and we look for appropriate Derivability Adjustment Theorems.

Keywords: Modal Logics, Paranormal Logics, Fuzzy Logics.

Contents

1 Introduction

In this Thesis we are interested in modal logics and we investigate some ways of enriching certain modal logics. These enrichments are done in two distinct ways. The first of them is a semantical enrichment, in the sense that we adopt a fuzzy Kripke semantics for normal modal logics, instead of the usual Boolean-based Kripke semantics. The other type of enrichment is linguistic, in the sense that we add new connectives to the signature of certain logics.

The study of modal logic in the context of fuzzy logics is made in Chapter 2. We propose a fuzzy semantics in which the operations of T-norm, S-norm, fuzzy negation and fuzzy implication, that are used to interpret the connectives of the language, take values on the real interval $[0, 1]$. Based on the binary partition $\{[0, i), [i, 1]\}$, we impose some restrictions over these fuzzy operations with the aim of obtaining a semantics that behaves like Boolean-based Kripke semantics. Then, we use this semantics to characterize a minimal normal modal logic and we show how to extend this characterization to a class of modal logics of confluence, of which many other usual normal modal logics are particular cases. This investigation resulted in the paper (Dodó; Marcos; Bergamaschi, 2013), where Flaulles Bergamaschi was our co-author, published in the Proceedings of the 2013 Joint IFSA World Congress NAFIPS Annual Meeting (IFSA/NAFIPS), which took place in Edmonton, Canada. This paper was based on the early paper (Dodó; Marcos, 2012) published in the Proceedings of the CBSF 2012.

With respect to the linguistic enrichment we essentially study logics that have modal negations among their connectives. Traditionally, normal modal logics are presented in terms of the modalities $\Box$ and $\Diamond$, that are instances of positive modalities. Is it possible to present modal normal logics in terms of negative modalities? One affirmative answer for this question is given by Marcos in (Marcos, 2005a). In this work the usual normal modal logics are obtained from a paranormal logic (a paracomplete and paraconsistent logic) with the help of connectives of restoration that internalize at the object-language level the notions of consistency and determinedness. Indeed this logic, under specific conditions, constitutes a Logic of Formal Inconsistency (LFI) and a Logic of Formal Undeterminedness (LFU) too. In such logics the paranormal negations fail some classical inferences, that can be recovered by means of so-called Derivability Adjustment Theorems (DAT). These theorems say basically that adding adequate assumptions of both consistency and determinedness the logic can recover the lost inferences from classical logic.

In (Marcos, 2005a), Marcos extends the positive fragment of classical logic (where the binary connectives $\wedge$, $\vee$ and $\supset$ represent conjunction, disjunction and implication respectively) with two connectives, one of them being a paraconsistent negation and the other a consistency connective. This language is sufficiently expressive to characterize every normal modal logic in the standard signature containing the connectives $\wedge, \vee, \supset$, $\Box$ and $\Diamond$. In what follows the author proposes an extension with a determinedness connective and a paracomplete negation instead of a consistency connective and a paraconsistent negation. However, in this case, the logic is able to express the same normal modal logics in the signature $\langle \wedge, \vee, \supset, \Box, \Diamond \rangle$ only for extensions of the logic *KT*. Another result says that the usual normal modal logics are definable already from a logic that extends the classical positive logic only with a paraconsistent negation. A similar result can be obtained, if it is used a paracomplete negation instead a paraconsistent negation, for extensions of the logic *KT*. In the latter case, however, if one uses a coimplication instead an implication the result holds for every normal modal logic.

In the same work, Marcos sketches an axiomatization, in terms of a Hilbert Calculus, for a normal minimal logic using only the positive fragment of classical logic and a paraconsistent negation. The author suggests, although he does not show the proofs of completeness, how to axiomatize other modal systems. Based on these ideas, in the Chapter 3 of the present thesis, we study logics under this axiomatization: the *LAB* logics. We prove completeness results for the minimal *LAB* logic and some extension of it. As we will show, it is possible to define the positive modalities $\Box$ and $\Diamond$ in these logics. This confirms the fact that normal modal logics can be alternatively presented by means of negative modalities.

In the case of the classical positive logic extended with a paraconsistent negation, what does happen if we remove the implication from the language? In Chapter 4 we will answer that question: we still have a rich language, with two paranormal negations and its respective connectives of restoration. And more, this logic is both an LFI and an LFU. In Chapter 4 we also present definitions of the positive and negative modalities, and of corresponding connectives of restoration in terms of Universal Logic. We define a deductive system for this minimal system by means of a Gentzen calculus, called $\mathcal{K}^n$, instead of a Hilbert style system. In $\mathcal{K}^n$, besides the usual structural rules and the rules for classical conjunction and classical disjunction, we have rules for the connectives of restoration and rules for the interaction between our modal negations. A Kripke semantics is proposed for our modal logic and we prove that our deductive system is sound and complete with respect to this semantics. After this, two extensions of $\mathcal{K}^n$ are studied. One of them is characterized by the class of reflexive frames and the other by the class of symmetric frames. Finally we study how the inferences of more standard logic systems may be recovered with the use of our rich modal language, by way of appropriate DATs. The ideas developed in this chapter resulted in the paper (Dodó; Marcos, 2014), that was published in Electronic Notes in Theoretical Computer Science.

Chapters 2 and 4 has structure similar to the papers published in the above mentioned vehicles. We preserve the same division of sections. However, particularly in chapter 2 we add full proofs of some results, that were omitted in the original papers by reasons of space limitations. We also add some definitions and more details of the subjects thereby explained.

2 Confluence in Fuzzy Modal Logic

In this chapter, we investigate classic-like aspects of Kripke models endowed with a fuzzy accessibility relation and a fuzzy notion of satisfaction. We prove general completeness result concerning the fuzzy semantics of a generous class of normal modal systems enriched with multiple instances of the axiom of confluence.

2.1 Introduction

With different goals, several papers in the literature have proposed to 'modalize' fuzzy logics or to 'fuzzify' modal logics. In (Mironov, 2005), for instance, the author aims at constructing logical calculi with languages appropriate for specifying dynamical systems whose behavior and structure is only modeled approximately. Other authors are also interested in providing adequate axiomatizations for such logics. For example, in (Caicedo; Rodriguez, 2010) the authors provide an axiomatization for the $\Box$-fragment and the $\Diamond$-fragment of the so-called Gödel modal logics, based on the many-valued Gödel logic and some well-known logics from the literature on modal logics. In (Bou et al., 2011) the authors characterize minimal many-valued modal logics for a $\Box$ operator defined over finite residuated lattices. All papers cited above have one thing in common: the semantical framework used to characterize the modal systems is based on Kripke-style structures.

The semantics that we utilize here is also a many-valued Kripke-style semantics. Our particular aim, though, is to characterize a generous class of many-valued modal systems with locally bivalent semantics that behave just like the usual boolean-based Kripke semantics for modal logics. In (Bedregal et al., 2011), the authors study models for a certain kind of fuzzy modal logics and prove weak completeness results for a couple of modal extensions of classic-like fuzzy models of some traditional normal modal systems, viz. K, T, D, B, $S4$, and $S5$. In (Dodó; Marcos, 2012) we followed a similar thread to prove completeness results for a much more inclusive class of fuzzy normal modal systems which contain instances of the *axiom of confluence* ($G^{k,l,m,n}$) $\Diamond^k\Box^l\varphi \supset \Box^m\Diamond^n\varphi$, where k, l, m and n are natural numbers. It should be clear that the systems $K + G^{k,l,m,n}$ encompass the above traditional systems, and a lot else. Indeed, one may observe that the characteristic modal axioms (T) $\Box\varphi \supset \varphi$, ($D$) $\Box\varphi \supset \Diamond\varphi$, ($B$) $\varphi \supset \Box\Diamond\varphi$, (4) $\Box\varphi \supset \Box\Box\varphi$ and (5) $\Diamond\varphi \supset \Box\Diamond\varphi$ are particular instances of ($G^{k,l,m,n}$) where $\langle k, l, m, n\rangle$ are $\langle 0, 1, 0, 0\rangle$, $\langle 0, 1, 0, 1\rangle$, $\langle 0, 0, 1, 1\rangle$, $\langle 0, 1, 2, 0\rangle$ and $\langle 1, 0, 1, 1\rangle$, respectively.

In our preliminary study, (Dodó; Marcos, 2012), we have followed (Bedregal et al., 2011) in producing for the real-valued unit interval $[0, 1]$ the 'canonical' binary partition $\{[0, 1), [1, 1]\}$ and in putting certain restrictions on the fuzzy operators which we have used to interpret the connectives of our language. Notions of satisfaction and validity of a formula are straighforwardly defined based on this partition. A weak completeness result was then established for a large class of modal systems. In the present Chapter, our 'crisp semantics' is more general: instead $\{[0, 1), [1, 1]\}$ we use a partition $\{[0, i), [i, 1]\}$, with $i \neq 0$. We have shown, then, how to extend the completeness result for a much larger class of classic-like fuzzy modal logics.

The so-called *Geach axiom* ($G^{1,1,1,1}$) is well-known to characterize, in terms of the associated notion of accessibility $\rightsquigarrow$ (and its inverse $\leftsquigarrow$) in the corresponding Kripke frames, the *diamond property*, namely: if $y \leftsquigarrow x \rightsquigarrow z$, then there is some w such that $y \rightsquigarrow w \leftsquigarrow z$. As noted in (Lemmon; Scott, 1977), where $\overset{i}{\rightsquigarrow}$ denotes an i-long sequence of $\rightsquigarrow$ transitions (and similarly for $\overset{i}{\leftsquigarrow}$ and $\leftsquigarrow$ transitions), the natural generalization of the diamond property is the following $\langle k, l, m, n\rangle$-*confluence property*: if $y \overset{k}{\leftsquigarrow} x \overset{l}{\rightsquigarrow} z$, then there is some w such that $y \overset{m}{\rightsquigarrow} w \overset{n}{\leftsquigarrow} z$. From the logical viewpoint, a general completeness proof based directly on the axiom of confluence, thus, is attractive in having the potential to subsume a denumerable number of particular instances of ($G^{k,l,m,n}$). At any rate, it should be noted that the confluence property has importance on its own. In abstract rewriting systems and type theory, for instance, one deals with frames in which accessibility characterizes some appropriate notion of reduction. There, confluence is used together with termination to guarantee convergence of reductions, which on its turn guarantees the existence of normal forms and has applications on the design of decision procedures. Strong normalization, in particular, is a much desirable property of lambda calculi, and is a property guaranteed by theorems of confluence à la Church-Rosser, with applications to programming language theory. The availability of modal logics of confluence, and in fact of fuzzy versions of such logics, allows one to expect to have a local perspective on rewrite systems and on program evaluation, and this time imbued with varying degrees of uncertainty, customized to the user's discretion.

The plan of the Chapter is as follows: in Section 2.2 we introduce the usual fuzzy operators; in section 2.3 we present the concept of classic-like fuzzy semantics and show that there exist fuzzy logics with the same set of tautologies of classical propositional logic; in Section 2.4 we present a particular kind of fuzzy Kripke semantics for modal logics; in Section 2.5 we prove completeness results for the modal system K extended with instances of the axiom of confluence.

2.2 Fuzzy Operators

We first review some useful terminology and easy results:

Definition 2.2.1. *Throughout the Chapter we shall use* $\mathcal{O}$ *to denote the* ***boolean*** *domain* $\{0, 1\}$ *of classical logic, and* $\mathcal{U}$ *to denote the* ***unit*** *interval* $[0, 1]$*, typical of fuzzy logics. By* $\leq$ *we will always denote the natural* ***total order*** *on* $\mathcal{U}$*. Given a k-ary operator* $©_b$ *on* $\mathcal{O}$ *and a k-ary operator* $©_u$ *on* $\mathcal{U}$*, we shall say that* $©_u$ ***agrees with*** $©_b$ *if* $©_u|_{\mathcal{O}} = ©_b$*. Given some* $i \in \mathcal{U} \setminus \{0\}$*, we will use* Π *to denote the* ***partition*** $\{\Pi_0, \Pi_1\}$ *of* $\mathcal{U}$*, where* $\Pi_0 = [0, i)$ *and* $\Pi_1 = [i, 1]$.

We list in what follows the defining properties of the most standard fuzzy operators used to interpret their homonymous classical counterparts, some references for these definitions are ((Hájek, 1998), (Klement; Mesiar; Pap, 2000)).

Definition 2.2.2. *A* ***fuzzy conjunction****, or* ***t-norm****, is a binary operation* T *on* $\mathcal{U}$ *such that:* (T0) *T agrees with classical conjunction,* (T1) *T is commutative,* (T2) *T is associative,* (T3) *T is monotone, that is, order-preserving, on both arguments, and* (T4) *T has* 1 *as neutral element. We call* $x \in \mathcal{U}$ *a* Π_0*-**divisor** of a t-norm* T *if there exists some* $y \in \mathcal{U}$ *such that* $T(x, y) \in \Pi_0$*; such* Π_0*-divisor is called* ***non-trivial*** *if both* $x, y \in \Pi_1$*. We say that* T *is* ***left-continuous*** *if it preserves limits of non-decreasing sequences, that is, if* $\lim_{n\to\infty} T(x_n, y) = T(\lim_{n\to\infty} x_n, y)$*, for every non-decreasing sequence* $\{x_n\}_{n\in\mathbb{N}}$.

For every left-continuous t-norm T there is a unique binary operation R_T on $\mathcal{U}$, called the **residuum** of T, such that $T(x, z) \leq y$ iff $z \leq R_T(x, y)$ for all x, y and $z \in \mathcal{U}$.

The following are examples of t-norms.

- Godel t-norm: $T_G(x, y) = \min(x, y)$
- Drastic t-norm: $T_D(x, y) = \begin{cases} \min(x, y), & \text{if } \max(x, y) = 1 \\ 0, & \text{otherwise} \end{cases}$

Definition 2.2.3. *A* ***fuzzy disjunction****, or* ***s-norm****, is a binary operation S on $\mathcal{U}$ such that:* (S0) *S agrees with classical disjunction,* (S1) *S is commutative,* (S2) *S is associative,* (S3) *S is monotone on both arguments, and* (S4) *S has 0 as neutral element. We call $x \in \mathcal{U}$ a Π_1-**divisor** of a s-norm S if there exists some $y \in \mathcal{U}$ such that $S(x, y) \in \Pi_1$; such Π_1-divisor is called* ***non-trivial*** *if both $x, y \in \Pi_0$.*

Some examples of s-norms are:

- Godel s-norm: $S_G(x, y) = \max(x, y)$
- Drastic s-norm: $S_D(x, y) = \begin{cases} \max(x, y), & \text{if } \min(x, y) = 0 \\ 1, & \text{otherwise} \end{cases}$

Some easily checkable important derived properties of the above operators include:

Proposition 2.2.1. *For any t-norm T, s-norm S, and every $x, y \in \mathcal{U}$:*
(i) *If $T(x, y) \in \Pi_1$, then $x \in \Pi_1$ and $y \in \Pi_1$.*
(ii) *If $S(x, y) \in \Pi_0$, then $x \in \Pi_0$ and $y \in \Pi_0$.*

Proof. For part (i), as $y \leq 1$, by (T3) it follows that $T(x, y) \leq T(x, 1)$. But by (T4) we have $T(x, 1) = x$, so we conclude that (a) $T(x, y) \leq x$. Similarly, we see that (b) $T(x, y) \leq y$. From (a) and (b) it follows that $T(x, y) \leq \min(x, y)$. Since $T(x, y) \in \Pi_1$, then $i \leq T(x, y)$. So, $i \leq \min(x, y)$. From this we conclude that $x \geq i$ and $y \geq i$, that is, $x \in \Pi_1$ and $y \in \Pi_1$. For part (ii) we have that $0 \leq x$ and $0 \leq y$. Then, by $(S3)$, it follows that $S(x, 0) \leq S(x, y)$ and $S(y, 0) \leq S(x, y)$. From this and $(S4)$, $\max(x, y) \leq S(x, y)$. Since $S(x, y) \in \Pi_0$, then $x, y \in \Pi_0$. *QED*

Note that small[1] t-norms such as the 'drastic t-norm' fail the converse of Prop. 2.2.1(i). Dually, large[2] s-norms such as the 'drastic s-norm' fail the converse of Prop. 2.2.1(ii). It is interesting to observe the following:

Proposition 2.2.2. *Let T be a t-norm, and S be an s-norm. Then:*
(i) *If T lacks non-trivial Π_0-divisors, then $x \in \Pi_1$ and $y \in \Pi_1$ imply $T(x, y) \in \Pi_1$, for every $x, y \in \mathcal{U}$.*
(ii) *If S lacks non-trivial Π_1-divisors, then $x \in \Pi_0$ and $y \in \Pi_0$ imply $S(x, y) \in \Pi_0$, for every $x, y \in \mathcal{U}$.*

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[1] In the sense that is the pointwise smallest t-norm. That is $T_D(x, y) \leq T(x, y)$ for every t-norm T.
[2] In the sense that is the pointwise largest s-norm. That is $S(x, y) \leq S_D(x, y)$ for every s-norm S.

Proof. For part (i) suppose by contraposition that $T(x, y) \in \Pi_0$ for some $x, y \in \mathcal{U}$. Since T has only trivial Π_0-divisors, then $x \in \Pi_0$ or $y \in \Pi_0$.
For part (ii) suppose also by contraposition that $S(x, y) \in \Pi_1$. Since S has only trivial Π_1-divisors, then $x \in \Pi_1$ or $x \in \Pi_1$. *QED*

Definition 2.2.4. *A* ***fuzzy negation*** *is a unary operation N on $\mathcal{U}$ such that:* (N0) *N agrees with classical negation,* (N1) *N is antitone, that is, order-reversing.*

Definition 2.2.5. *A* ***fuzzy implication*** *is a binary operation I on $\mathcal{U}$ such that:* (I0) *I agrees with classical implication,* (I1) *I is antitone on the first argument, and* (I2) *I is monotone on the second argument.*

Given that the unit interval $\mathcal{U} = [0, 1]$ is closed and bounded, the Bolzano-Weierstrass theorem guarantees that:

Proposition 2.2.3. *The image of a left-continuous t-norm is complete (in the sense that its subsets contain their own suprema).*

Proof. Let T be a t-norm, S be a closed subset of $\mathcal{U} \times \mathcal{U}$ and M be the supremum of $T(S)$, $T(S)$ is a subset of $\mathcal{U}$ that collects the images of all elements of S. This supremum must exist because S is bounded. We will show that there is a $z \in S$ such that $T(z) = M$. Pick an arbitrary sequence $\{y_n\}_{n \in \mathbb{N}}$ of elements in the range of T that converges to M. For each n, let x_n be an element of S such that $y_n = T(x_n)$. Then, the sequence $\{T(x_n)\}_{n \in \mathbb{N}}$ converges to M. Let now $\{x_{n_k}\}_{k \in \mathbb{N}}$ be a convergent non-decreasing subsequence of $\{x_n\}_{n \in \mathbb{N}}$, and let $z = \lim_{k \to \infty} x_{n_k}$. This is guaranteed to exist by the Bolzano-Weierstrass theorem. We conclude that $z \in S$, given than S is closed. Given that T is left-continuous, we have that $T(z) = T(\lim_{k \to \infty} x_{n_k}) = \lim_{k \to \infty} T(x_{n_k}) = \lim_{n \to \infty} T(x_n) = M$. *QED*

Residuation allows us to define a particularly interesting kind of fuzzy implication:

Proposition 2.2.4. *The residuum I of a left-continuous t-norm is a fuzzy implication. Moreover, $I(x, y) = 1$ iff $x \leq y$.*

2.3 Fuzzy Semantics

Let P be a denumerable set of propositional variables, and let the set of formulas of classical propositional logic, L_P, be inductively defined by:

$$\varphi ::= p \mid (\neg\varphi) \mid (\varphi_1 \wedge \varphi_2) \mid (\varphi_1 \vee \varphi_2) \mid (\varphi_1 \supset \varphi_2)$$

where p ranges over elements of P.

The following definition employs the standard fuzzy operators in interpreting the above symbols for the classical connectives:

Definition 2.3.1. *A **fuzzy evaluation** of the propositional variables is any total function $e : P \longrightarrow \Pi_0 \cup \Pi_1$. The structure $\mathbb{S} = \langle N, T, S, I \rangle$ will be called a **fuzzy semantics** for the propositional connectives $\langle \neg, \wedge, \vee, \supset \rangle$. By way of a fuzzy semantics, an evaluation e may be recursively extended to a **fuzzy valuation** $e^{\mathbb{S}} : L_P \longrightarrow \Pi_0 \cup \Pi_1$ as follows:*

$$\begin{array}{lcl} e^{\mathbb{S}}(p) & = & e(p) \text{ for each } p \in P \\ e^{\mathbb{S}}(\neg\alpha) & = & N(e^{\mathbb{S}}(\alpha)) \\ e^{\mathbb{S}}(\alpha \wedge \beta) & = & T(e^{\mathbb{S}}(\alpha), e^{\mathbb{S}}(\beta)) \\ e^{\mathbb{S}}(\alpha \vee \beta) & = & S(e^{\mathbb{S}}(\alpha), e^{\mathbb{S}}(\beta)) \\ e^{\mathbb{S}}(\alpha \supset \beta) & = & I(e^{\mathbb{S}}(\alpha), e^{\mathbb{S}}(\beta)) \end{array}$$

*A formula $\alpha \in L_P$ is called an **$\mathbb{S}$-tautology**, denoted by $\vDash_{\mathbb{S}} \alpha$, if for every fuzzy evaluation e we have $e^{\mathbb{S}}(\alpha) \in \Pi_1$. We shall denote by $\mathbb{T}(L_P)$ the set of all classical tautologies in L_P and by $\mathbb{T}^{\mathbb{S}}(L_P)$ the set of all $\mathbb{S}$-tautologies in L_P.*

The fact that each fuzzy operator agrees with the corresponding classical operator immediately guarantees the following result:

Proposition 2.3.1. *All fuzzy tautologies are classical tautologies, that is, $\mathbb{T}^{\mathbb{S}}(L_P) \subseteq \mathbb{T}(L_P)$, for any fuzzy semantics $\mathbb{S}$.*

The following definitions, from (Bedregal; Cruz, 2008), and the subsequent result aim at capturing the core of classical semantics from within the context of fuzzy semantics:

Definition 2.3.2. *$\mathbb{S}$ is a **classic-like fuzzy semantics** if $\mathbb{T}(L_P) \subseteq \mathbb{T}^{\mathbb{S}}(L_P)$.*

Definition 2.3.3. *Let $\mathbb{S} = \langle N, T, S, I \rangle$ be a fuzzy semantics and Π be a partition for $\mathcal{U}$. We say that: (1) N is **crisp with respect to Π** when $N(x) \in \Pi_0$ if and only if $x \in \Pi_1$; (2) T is **crisp with respect to Π** when $T(x, y) \in \Pi_1$ if and only if $x, y \in \Pi_1$; (3) S is **crisp with respect to Π** when $S(x, y) \in \Pi_0$ if and only if $x, y \in \Pi_0$; (4) I is **crisp with respect to Π** when $I(x, y) \in \Pi_0$ if and only if $x \in \Pi_1$ and $y \in \Pi_0$. When the above conditions are all satisfied we say that $\mathbb{S}$ is Π-**crisp**.*

Notice in particular that crisp t-norms and crisp s-norms are fully characterized by Prop. 2.2.1 and Prop. 2.2.2. Part of what it takes for a fuzzy implication to be crisp is also guaranteed by Prop. 2.2.4. To show now that a Π-crisp fuzzy semantics is a classic-like fuzzy semantics we prove first the following result.

Proposition 2.3.2. *Given a fuzzy valuation $e^{\mathbb{S}}$ of a Π-crisp fuzzy semantics $\mathbb{S}$, there is a classical valuation $v : L_P \longrightarrow \mathcal{O}$ that simulates it, that is, such that*

$$v(\varphi) = 1 \quad \text{iff} \quad e^{\mathbb{S}}(\varphi) \in \Pi_1$$

holds for every $\varphi \in L_P$.

Proof. Let $\flat : \mathcal{U} \longrightarrow \mathcal{O}$ be such that $\flat(x) = 1$ if $x \in \Pi_1$ and $\flat(x) = 0$ otherwise. We will show that $v = \flat \circ e^{\mathbb{S}}$ defines a standard boolean valuation. The base step is trivial. In the inductive step, for the case of a negated formula $\neg\psi$, note that $v(\neg\psi) = 1$ iff $\flat(e^{\mathbb{S}}(\neg\psi)) = 1$ iff $e^{\mathbb{S}}(\neg\psi) \in \Pi_1$ iff $N(e^{\mathbb{S}}(\psi)) \in \Pi_1$. As $\mathbb{S}$ is Π-crisp, $N(e^{\mathbb{S}}(\psi)) \in \Pi_1$ iff $e^{\mathbb{S}}(\psi) \in \Pi_0$. The induction hypothesis applies to ψ, thus we conclude that $e^{\mathbb{S}}(\psi) \in \Pi_0$ iff $v(\psi) = 0$. From all this we conclude that $v(\neg\psi) = 1$ iff $v(\psi) = 0$, exactly as one would expect of the standard classical semantics of negation. The cases of the remaining operators are analogous. *QED*

Corollary 2.3.1. *All classical tautologies are tautologies of a Π-crisp fuzzy semantics, that is, $\mathbb{S}$ is a classic-like fuzzy semantics whenever $\mathbb{S} = \langle N, T, S, I\rangle$ is Π-crisp.*

Proof. Consider a classical tautology φ, and pick an arbitrary fuzzy valuation $e^{\mathbb{S}}$. In view of Prop. 2.3.2, there is a classical valuation v that simulates $e^{\mathbb{S}}$. But the formula φ is a tautology, so v must satisfy it, hence $e^{\mathbb{S}}$ must equally satisfy this formula. *QED*

2.4 Fuzzy Kripke Semantics

The set of modal formulas, LM_P, is defined by adding $(\Diamond\phi)$ to the inductive clauses defining L_P. The connective $\Box$ may be introduced by definition, setting $\Box\alpha := \neg\Diamond\neg\alpha$.

Definition 2.4.1. *Generalizing the notion of a characteristic function to the domain of fuzzy logic, a **fuzzy n-ary relation** B over a universe A is characterized by a membership function $\mu_B : A^n \longrightarrow \mathcal{U}$ which associates to each tuple $\overrightarrow{x} \in A^n$ its degree of membership $\mu_B(\overrightarrow{x})$ in B. In this context, a **fuzzy subset** is characterized by a fuzzy unary relation, or the corresponding unary membership function. A **crisp** n-ary relation is any fuzzy n-ary relation B over a given A such that $\mu_B(A^n) \subseteq \mathcal{O}$, and crisp sets are defined analogously.*

In the following, definitions of standard Kripke models are fuzzified:

Definition 2.4.2. *A* ***fuzzy frame*** $\mathbb{F}$ *is a structure* $\langle W, \rightsquigarrow \rangle$*, where* W *is a non-empty crisp set (of 'objects', 'worlds', or 'states') and* $\rightsquigarrow$ *is a fuzzy binary ('reduction', 'accessibility', or 'transition') relation over* W*. As expected, to characterize m-step accessibility,* $\overset{m}{\rightsquigarrow}$*, we set:*

- $\mu_{\overset{0}{\rightsquigarrow}}(w_i, w_j) \in \Pi_1$ *means that* $w_i = w_j$
- $\mu_{\overset{n+1}{\rightsquigarrow}}(w_i, w_j) \in \Pi_1$ *means that there is some* w_k *such that* $\mu_{\overset{n}{\rightsquigarrow}}(w_i, w_k) \in \Pi_1$ *and* $\mu_{\rightsquigarrow}(w_k, w_j) \in \Pi_1$

Furthermore, $w_i \overset{m}{\leftsquigarrow} w_j$ *is used to denote* $w_j \overset{m}{\rightsquigarrow} w_i$*.*

Definition 2.4.3. *Given a fuzzy frame* $\mathbb{F}$*, an* $\mathbb{F}$*-**evaluation** is any total function* $\rho : P \times W \longrightarrow \mathcal{U}$*. A* ***fuzzy Kripke model*** *is a structure* $\mathcal{K} = \langle \mathbb{F}, \mathbb{S}, V \rangle$*, where* $\mathbb{F}$ *is a fuzzy frame,* $\mathbb{S}$ *is a classic-like fuzzy semantics where* T *is a left-continuous t-norm and* V *is an* $\mathbb{F}$*-valuation. Given a fuzzy Kripke model* $\mathcal{K}$*, the associated* ***degree of satisfiability*** *is a total function* $\Vdash_{\mathcal{K}} : W \times LM_P \longrightarrow \mathcal{U}$ *recursively defined as follows (in infix notation, we write* $w \Vdash_{\mathcal{K}} \varphi$ *where* $w \in W$ *and* $\varphi \in LM_P$*; when there is no risk of ambiguity, we use more simply* $w \Vdash \varphi$ *instead of* $w \Vdash_{\mathcal{K}} \varphi$*):*

$$
\begin{array}{lcl}
w \Vdash \alpha & = & V(\alpha, w), \text{ if } \alpha \in P \\
w \Vdash \neg\alpha & = & N(w \Vdash \alpha) \\
w \Vdash \alpha \wedge \beta & = & T(w \Vdash \alpha, w \Vdash \beta) \\
w \Vdash \alpha \vee \beta & = & S(w \Vdash \alpha, w \Vdash \beta) \\
w \Vdash \alpha \supset \beta & = & I(w \Vdash \alpha, w \Vdash \beta) \\
w \Vdash \Diamond\alpha & = & \sup\{T(\mu_{\rightsquigarrow}(w, w'), w' \Vdash \alpha)/w' \in W\} \\
w \Vdash \Box\alpha & = & N(w \Vdash \Diamond\neg\alpha)
\end{array}
$$

A formula $\varphi \in LM_P$ *is said to be* ***true*** *in a fuzzy Kripke model* $\mathcal{K}$*, denoted by* $\vDash_{\mathcal{K}} \alpha$*, if* $(w \Vdash \varphi) \in \Pi_1$ *for every* $w \in W$*. Given a collection* $\mathfrak{K}$ *of fuzzy Kripke models, a formula* $\varphi \in LM_P$ *is said to be a* $\mathfrak{K}$*-**tautology** (denoted by* $\vDash_{\mathfrak{K}} \varphi$*), if* φ *is true in every model in* $\mathfrak{K}$*.*

Note that the above notion of satisfaction coincides with the standard interpretation in modal logics based on the standard bivalent semantics, with the fuzzy operators collapsing into their counterparts in classical logic, and with the interpretations of $\Diamond$ and $\Box$ coinciding with their standard interpretations in Kripke semantics.

Many standard properties of binary relations like

reflexivity	xRx for all x
seriality	for every x there exist y such that xRy
symmetry	if xRy, then yRx, for all x, y
transitivity	if xRy and yRz, then xRz, for all x, y and z
euclideanity	if xRy and xRz, then yRz, for all x, y and z

have natural fuzzy counterparts, as shown in the next definition.

Definition 2.4.4. *We say the fuzzy accessibility relation $\rightsquigarrow$ is:*

- Π-***reflexive*** *if $\mu_{\rightsquigarrow}(x,x) \in \Pi_1$, for every $x \in W$*
- Π-***serial*** *for every $x \in W$ there exists $y \in W$ such that $\mu_{\rightsquigarrow}(x,y) \in \Pi_1$*
- Π-***symmetric*** *if $\mu_{\rightsquigarrow}(x,y) \in \Pi_1$ implies $\mu_{\rightsquigarrow}(y,x) \in \Pi_1$, for every $x, y \in W$*
- Π-***transitive*** *if $\mu_{\overset{2}{\rightsquigarrow}}(x,y) \in \Pi_1$ implies $\mu_{\rightsquigarrow}(x,y) \in \Pi_1$ for every $x, y \in W$*
- Π-***euclidean*** *if $\mu_{\rightsquigarrow}(x,y) \in \Pi_1$ and $\mu_{\rightsquigarrow}(x,z) \in \Pi_1$ imply $\mu_{\rightsquigarrow}(y,z) \in \Pi_1$ for every $x, y, z \in W$*

*In general, given natural numbers k, l, m, n, we say that $\rightsquigarrow$ is Π-**(k,l,m,n)-confluent** if for each $x, y, z \in W$ such that $\mu_{\overset{k}{\rightsquigarrow}}(x,y) \in \Pi_1$ and $\mu_{\overset{m}{\rightsquigarrow}}(x,z) \in \Pi_1$ there exists $w \in W$ such that $\mu_{\overset{l}{\rightsquigarrow}}(y,w) \in \Pi_1$ and $\mu_{\overset{n}{\rightsquigarrow}}(z,w) \in \Pi_1$.*

2.5 Modal Systems Based on Instances of $G^{k,l,m,n}$

We will show that normal modal systems based on instances of $G^{k,l,m,n}$ can be characterized by adequate fuzzy Kripke models. First of all, we will prove the completeness of the K-Modal System with respect to the class all fuzzy Kripke models. Next, we will enrich this system with one or more instances of $G^{k,l,m,n}$ and prove a general completeness result for the systems thereby obtained.

Given a fuzzy Kripke model $\mathcal{M} = \langle W, \rightsquigarrow, \$, V\rangle$, in what follows let $\mathcal{M}^C$ bet the crisp fuzzy model $\langle W, \rightsquigarrow^C, \$, V\rangle$, associated to $\mathcal{M}$ where $\mu_{\rightsquigarrow^C} : W \times W \longrightarrow \mathcal{O}$ is such that

$$\mu_{\rightsquigarrow^C}(w,w') = \begin{cases} 1, & \text{if } \mu_{\rightsquigarrow}(w,w') \in \Pi_1 \\ 0, & \text{if } \mu_{\rightsquigarrow}(w,w') \in \Pi_0 \end{cases}$$

and $V^C : P \times W \longrightarrow \mathcal{O}$ is such that $V^C(p,w) = 1$ if $V(p,w) \in \Pi_1$ and $V^C(p,w) = 0$ otherwise.

The following result shows that each fuzzy modal semantics may be assumed to be based on a convenient crisp accessibility relation.

Proposition 2.5.1. *Let $\mathcal{M} = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$ be a fuzzy Kripke model. Given an arbitrary $w \in W$ and $\alpha \in LM_P$, then $(w \Vdash_{\mathcal{M}} \alpha) \in \Pi_1$ iff $(w \Vdash_{\mathcal{M}^C} \alpha) = 1$, where $\mathcal{M}^C$ is the crisp fuzzy model associated to $\mathcal{M}$.*

Proof. This is checked by induction on the structure of α.
[Base step] α is some p $\in P$
$(w \Vdash_{\mathcal{M}} p) \in \Pi_1$ iff, by Def. 2.4.3, $V(p, w) \in \Pi_1$ iff, by definition of V^C , $V^C(p, w) = 1$ iff $(w \Vdash_{\mathcal{M}^C} p) = 1$.
[Step] Suppose, by Induction Hypothesis, that $(w \Vdash_{\mathcal{M}} \beta) \in \Pi_1$ iff $(w \Vdash_{\mathcal{M}^C} \beta) = 1$. We will check in detail the case where $\alpha = \Diamond\beta$. Suppose first that $(w \Vdash_{\mathcal{M}} \Diamond\beta) \in \Pi_1$. Then, $\sup\{T(\mu_{\rightsquigarrow}(w, w'), w' \Vdash_{\mathcal{M}} \beta) : w' \in W\} \in \Pi_1$. From 2.2.3 there exists w^* such that $T(\mu_{\rightsquigarrow}(w, w^*), w^* \Vdash_{\mathcal{M}} \beta) \in \Pi_1$. By Prop. 2.2.1 $\mu_{\rightsquigarrow}(w, w^*) \in \Pi_1$ and $(w^* \Vdash_{\mathcal{M}} \beta) \in \Pi_1$. By definition of $\mathcal{M}^C$ it's the case that $\mu_{\rightsquigarrow^C}(w, w^*) = 1$ and by Induction Hypothesis $(w^* \Vdash_{\mathcal{M}^C} \beta) = 1$. By the standard interpretation of $\Diamond$, it follows that $(w \Vdash_{\mathcal{M}^C} \Diamond\beta) = 1$. Conversely, using the fact that T is crisp with respect to Π, we can prove that if $(w \Vdash_{\mathcal{M}^C} \Diamond\beta) = 1$, then $(w \Vdash_{\mathcal{M}} \Diamond\beta) \in \Pi_1$. *QED*

As a straightforward consequence, it follows that:

Corollary 2.5.1. *Given an arbitrary fuzzy Kripke model $\mathcal{M}$ and $\alpha \in LM_P$, then $\vDash_{\mathcal{M}} \alpha$ iff $\vDash_{\mathcal{M}^C} \alpha$.*

2.5.1 The *K*-Modal System

Definition 2.5.1. *The K-modal system is the triple $\langle LM_P, \Delta \cup \{(K)\}, \{(MP), (Nec)\}\rangle$, where Δ is an axiomatization of Classical Propositional Logic,* (K) *is the axiom*

$$\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$$

and (MP) *and* (Nec) *are respectively the rules of Modus Ponens and Necessitation, namely:*

$$(MP) : \frac{\alpha, \alpha \supset \beta}{\beta} \qquad \textit{and} \qquad (Nec) : \frac{\vdash \alpha}{\vdash \Box\alpha}$$

Proposition 2.5.2. *Let $\alpha \in LM_P$. Then, α is a theorem in the K-modal system iff $\vDash_{\mathcal{K}} \alpha$ for each fuzzy Kripke model $\mathcal{K} = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$.*

Proof. (⇒) We already know, by Corollary 2.3.1, that the theorems of classical logic are all valid in any classic-like fuzzy semantics. It remains to be proven that the axiom (K) is valid and that the inferences rules preserve validity. Suppose that there exists a $w \in W$ such that $(w \Vdash \Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)) \in \Pi_0$. So by Def. 2.3.3 it follows that

$$(w \Vdash \Box(\alpha \supset \beta)) \in \Pi_1 \tag{2.1}$$

and

$$(w \Vdash \Box\alpha \supset \Box\beta) \in \Pi_0 \tag{2.2}$$

By (2.2) and Def. 2.3.3, we have

$$(w \Vdash \Box\alpha) \in \Pi_1 \tag{2.3}$$

and

$$(w \Vdash \Box\beta) \in \Pi_0 \tag{2.4}$$

By (2.4) and Def. 2.4.3,

$$N(\sup\{T(\mu_{\rightsquigarrow}(w, w'), N(w' \Vdash \beta))/w' \in W\}) \in \Pi_0 \tag{2.5}$$

By (2.5) and Def. 2.3.3, we have

$$\sup\{T(\mu_{\rightsquigarrow}(w, w'), N(w' \Vdash \beta))/w' \in W\} \in \Pi_1 \tag{2.6}$$

By (2.6) and Prop. 2.2.3 there exists a $w^* \in W$ such that

$$T(\mu_{\rightsquigarrow}(w, w^*), N(w^* \Vdash \beta)) \in \Pi_1 \tag{2.7}$$

By (2.7) and the Prop. 2.2.1, we have

$$\mu_{\rightsquigarrow}(w, w^*) \in \Pi_1 \quad (2.8) \qquad \text{and} \qquad N(w^* \Vdash \beta) \in \Pi_1 \quad (2.9)$$

From (2.9), by Def. 2.3.3 we know that

$$(w^* \Vdash \beta) \in \Pi_0 \tag{2.10}$$

By (2.1) and Def. 2.4.3,

$$\sup\{T(\mu_{\rightsquigarrow}(w, w'), N(w' \Vdash \alpha \supset \beta))/w' \in W\} \in \Pi_0 \tag{2.11}$$

By (2.11) and (2.8) in particular when $w' = w^*$ we have $N(w^* \Vdash \alpha \supset \beta)) \in \Pi_0$, by Def. 2.3.3, that is,

$$(w^* \Vdash \alpha \supset \beta) \in \Pi_1 \tag{2.12}$$

Using (2.3), (2.8) and Def. 2.3.3, analogously we conclude that

$$(w^* \Vdash \alpha) \in \Pi_1 \tag{2.13}$$

By (2.12), (2.13) and the interpretation of classic-like fuzzy implication it follows that

$$(w^* \Vdash \beta) \in \Pi_1 \tag{2.14}$$

But (2.14) contradicts (2.10) given that $\{\Pi_0, \Pi_1\}$ is a partition.

For (Nec) Rule assume that $\vDash_{\mathcal{K}} \beta$,, that is, for all w, $(w \Vdash \beta) \in \Pi_1$. Suppose by contradiction that $\vDash_{\mathcal{K}} \Box\beta$ is not the case. So there exists a $w \in W$ such that $(w \Vdash \Box\beta) \in \Pi_0$, that is, $N(\sup\{T(\mu_{\rightsquigarrow}(w, w'), N(w' \Vdash \beta))/w' \in W\}) \in \Pi_0$. It follows by Def. 2.3.3 that $\sup\{T(\mu_{\rightsquigarrow}(w, w'), N(w' \Vdash \beta))/w' \in W\} \in \Pi_1$. For some $w^* \in W$ it is the case that $T(\mu_{\rightsquigarrow}(w, w^*), N(w^* \Vdash \beta)) \in \Pi_1$. From this we conclude that $(w' \Vdash \beta) \in \Pi_0$, contradicting the assumption.

For (MP) Rule assume for an arbitrary w that $(w \Vdash \varphi) \in \Pi_1$ and $(w \Vdash \varphi \supset \psi) \in \Pi_1$. Suppose again by contradiction that $(w \Vdash \psi) \in \Pi_0$. Since $(w \Vdash \varphi) \in \Pi_1$, by Def. 2.3.3 it follows that $I(w \Vdash \varphi, w \Vdash \psi) \in \Pi_0$, that is $(w \Vdash \varphi \supset \psi) \in \Pi_0$. This is an absurd.

($\Leftarrow$) The K system is known to be complete with respect the class of all Kripke models. So, by Corollary 2.5.1, if $\vDash_{\mathcal{K}} \alpha$ then $\vdash_K \alpha$. *QED*

2.5.2 Completeness of $KG^{k,l,m,n}$

In what follows, we shall prove a sequence of lemmas which are used to establish the soundness result in Theorem 2.5.1.

Lemma 2.5.1. *Let $\mathcal{M} = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$ be a fuzzy Kripke model. If $(w \Vdash \Diamond^z\varphi) \in \Pi_1$, then there exists a w_z such that both $\mu_{\overset{z}{\rightsquigarrow}}(w, w_z) \in \Pi_1$ and $(w_z \Vdash \varphi) \in \Pi_1$.*

Proof. The proof proceeds by induction on z.
[*Basis*] $z = 1$
If $(w \Vdash \Diamond\beta) \in \Pi_1$, then $\sup\{T(\mu_{\rightsquigarrow}(w, w'), w' \Vdash \beta)/w' \in W\} \in \Pi_1$, by Def. 2.4.3. So, by Prop. 2.2.3 there is a $w_1 \in W$ such that $T(\mu_{\rightsquigarrow}(w, w_1), w_1 \Vdash \beta) \in \Pi_1$. By Prop. 2.2.1 we have $\mu_{\rightsquigarrow}(w, w_1) \in \Pi_1$ and $(w_1 \Vdash \beta) \in \Pi_1$.

[*Step*] Suppose by Induction Hypothesis that for $z = k$ the property is valid. Note that if $(w \Vdash \Diamond^{k+1}\beta) \in \Pi_1$, then, by Def. 2.4.3,

$$\sup\{T(\mu_{\rightsquigarrow}(w, w'), w' \Vdash \Diamond^k\beta)/w' \in W\} \in \Pi_1 \tag{2.15}$$

From (2.15) and Prop. 2.2.3 there exists a w_1 such that

$$T(\mu_{\rightsquigarrow}(w, w_1), w_1 \Vdash \Diamond^k\beta) \in \Pi_1 \tag{2.16}$$

By (2.16) and Prop. 2.2.1 we have:

$$\mu_{\rightsquigarrow}(w, w_1) \in \Pi_1 \tag{2.17}$$

and

$$(w_1 \Vdash \Diamond^k\beta) \in \Pi_1 \tag{2.18}$$

By (2.18) and Induction Hypothesis it follows that there exists a $w_{k'}$ such that $\mu_{\overset{k}{\rightsquigarrow}}(w_1, w_{k'}) \in \Pi_1$ and $(w_{k'} \Vdash \beta) \in \Pi_1$. Using (2.17) and setting $w_{k+1} = w_{k'}$ we conclude that $\mu_{\overset{k+1}{\rightsquigarrow}}(w, w_{k+1}) \in \Pi_1$ and $(w_{k+1} \Vdash \beta) \in \Pi_1$. *QED*

Lemma 2.5.2. *Let $\mathcal{M} = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$ be a fuzzy Kripke model. If $\mu_{\overset{m}{\rightsquigarrow}}(w, v) \in \Pi_1$ and $(w \Vdash \Box^m\varphi) \in \Pi_1$, then $(v \Vdash \varphi) \in \Pi_1$.*

Proof. The proof is carried out by induction on m.
[*Basis*] $m = 1$. Assume that:

$$\mu_{\rightsquigarrow}(w, v) \in \Pi_1 \tag{2.19}$$

and

$$(w \Vdash \Box\beta) \in \Pi_1 \tag{2.20}$$

By (2.20), Def. 2.3.3 and Def. 2.4.3 we have that

$$T(\mu_{\rightsquigarrow}(w, v), N(v \Vdash \beta)) \in \Pi_0 \tag{2.21}$$

By (2.19), (2.21) and Def. 2.3.3 we have that $N(v \Vdash \beta) \in \Pi_0$. By Def. 2.3.3 it follows that $(v \Vdash \beta) \in \Pi_1$.
[*Step*] $m = k + 1$
The (IH) Induction Hypothesis states that for $m = k$, if $\mu_{\overset{k}{\rightsquigarrow}}(w, v) \in \Pi_1$ and $(w \Vdash \Box^k\beta) \in \Pi_1$ then $(v \Vdash \beta) \in \Pi_1$.
Assume that

$$\mu_{\overset{k+1}{\rightsquigarrow}}(w, v) \in \Pi_1 \tag{2.22}$$

and

$$(w \Vdash \Box^{k+1}\beta) \in \Pi_1 \tag{2.23}$$

It follows from (2.23), Def. 2.3.3 and Def. 2.4.3 that

$$\sup\{T(\mu_{\rightsquigarrow}(w, w'), N(w' \Vdash \Box^k\beta))/w' \in W\} \in \Pi_0 \tag{2.24}$$

On the other hand, for every $w' \in W$ we have

$$T(\mu_{\rightsquigarrow}(w, w'), N(w' \Vdash \Box^k\beta)) \in \Pi_0 \tag{2.25}$$

By (2.22) there is a v_1 such that $\mu_{\rightsquigarrow}(w, v_1) \in \Pi_1$ and

$$\mu_{\overset{k}{\rightsquigarrow}}(v_1, v) \in \Pi_1 \tag{2.26}$$

For such v_1 it is thus the case that $T(\mu_{\rightsquigarrow}(w, v_1), N(v_1 \Vdash \Box^k\beta)) \in \Pi_0$. Since $\mu_{\rightsquigarrow}(w, v_1) \in \Pi_1$ theni it follows by the latter and Def. 2.3.3 that

$$(v_1 \Vdash \Box^k\beta) \in \Pi_1 \tag{2.27}$$

We conclude by (2.26), (2.27) and from the Induction Hypothesis that $(v \Vdash \beta) \in \Pi_1$. *QED*

The following result concerns equivalences between formulas with nested modalities. It will be useful in the proofs of propositions where formulas that contains iterated $\Box$ and $\Diamond$.

Lemma 2.5.3. *If $\mathcal{M} = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$ is a fuzzy Kripke model, and w is a element of W, then $(w \Vdash \neg\Diamond^m\varphi) \in \Pi_1$ iff $(w \Vdash \Box^m\neg\varphi) \in \Pi_1$.*

Proof. This is checked by induction on m.
[*Basis*] $m = 1$

$w \Vdash \neg\Diamond\varphi \in \Pi_1$	iff	(by Definition 2.4.3)
$N(\sup\{T(\mu_{\rightsquigarrow}(w, w'), w' \Vdash \varphi)w' \in W\}) \in \Pi_1$	iff	(by Def. 2.3.3)
$\sup\{T(\mu_{\rightsquigarrow}(w, w'), w' \Vdash \varphi)/w' \in W\} \in \Pi_0$	iff	
$\mu_{\rightsquigarrow}(w, w') \in \Pi_0$ or $(w' \Vdash \varphi) \in \Pi_0$ for all w'	iff	(by Def. 2.4.3 and Def. 2.3.3)
$\mu_{\rightsquigarrow}(w, w') \in \Pi_0$ or $(w' \Vdash \neg\neg\varphi) \in \Pi_0$	iff	
$\sup\{T(\mu_{\rightsquigarrow}(w, w'), w' \Vdash \neg\neg\varphi)/w' \in W\} \in \Pi_0$	iff	
$(w \Vdash \Diamond\neg\neg\varphi) \in \Pi_0$	iff	(by Def. 2.3.3)
$N(w \Vdash \Diamond\neg\neg\varphi) \in \Pi_1$	iff	(by Def. 2.4.3)
$(w \Vdash \Box\neg\varphi) \in \Pi_1$		

[*Step*] $m = k + 1$

Assume by Induction Hypothesis that $(w \Vdash \neg\lozenge^k\varphi) \in \Pi_1$ iff $(w \Vdash \Box^k\neg\varphi) \in \Pi_1$

Note that $w \Vdash \neg\lozenge^{k+1}\varphi \in \Pi_1$	iff	(by Def. 2.4.3 and Def. 2.3.3)
$w \Vdash \lozenge^{k+1}\varphi \in \Pi_0$	iff	(by Def. 2.4.3)
$\sup\{T(\mu_{\rightsquigarrow}(w, w'), w' \Vdash \lozenge^k\varphi)/w' \in W\} \in \Pi_0$	iff	
$\mu_{\rightsquigarrow}(w, w') \in \Pi_0$ or $(w' \Vdash \lozenge^k\varphi) \in \Pi_0$ for all w'	iff	
$\mu_{\rightsquigarrow}(w, w') \in \Pi_0$ or $(w' \Vdash \neg\lozenge^k\varphi) \in \Pi_1$	iff	(by Induction Hypothesis)
$\mu_{\rightsquigarrow}(w, w') \in \Pi_0$ or $(w' \Vdash \Box^k\neg\varphi) \in \Pi_1$	iff	
$\mu_{\rightsquigarrow}(w, w') \in \Pi_0$ or $N(w' \Vdash \Box^k\neg\varphi) \in \Pi_0$	iff	
$\mu_{\rightsquigarrow}(w, w') \in \Pi_0$ or $(w' \Vdash \neg\Box^k\neg\varphi) \in \Pi_0$	iff	
$\sup\{T(\mu_{\rightsquigarrow}(w, w'), w' \Vdash \neg\Box^k\neg\varphi)/w' \in W\} \in \Pi_0$	iff	(by Def. 2.4.3)
$(w \Vdash \lozenge\neg\Box^k\neg\varphi) \in \Pi_0$	iff	(by Def. 2.3.3)
$N(w \Vdash \lozenge\neg\Box^k\neg\varphi) \in \Pi_1$	iff	(by Def. 2.4.3)
$(w \Vdash \Box\Box^k\neg\varphi) \in \Pi_1$	iff	
$(w \Vdash \Box^{k+1}\neg\varphi) \in \Pi_1$		

QED

Lemma 2.5.4. *Let $\mathcal{M} = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$ be a fuzzy Kripke model. If $(w \Vdash \neg\lozenge^n\varphi) \in \Pi_1$ and $\mu_{\overset{n}{\rightsquigarrow}}(w, v) \in \Pi_1$, then $(v \Vdash \varphi) \in \Pi_0$.*

Proof. This is a straightforward consequence of the previous results. Indeed, note first that by Lemma 2.5.3 we have $(v \Vdash \neg\lozenge^m\varphi) \in \Pi_1$ iff $(v \Vdash \Box^m\neg\varphi) \in \Pi_1$. So we know that $(w \Vdash \Box^n\neg\varphi) \in \Pi_1$ and $\mu_{\overset{n}{\rightsquigarrow}}(w, v) \in \Pi_1$, and by applying Lemma 2.5.2 it follows that $(v \Vdash \neg\varphi) \in \Pi_1$. By Def. 2.3.3 we conclude that $(v \Vdash \varphi) \in \Pi_0$. *QED*

Lemma 2.5.5. *Let $\mathcal{M} = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$ be a fuzzy Kripke model. If $(w \Vdash \Box^n\varphi) \in \Pi_0$, then there exists some w_n such that $\mu_{\overset{n}{\rightsquigarrow}}(w, w_n) \in \Pi_1$ and $(w_n \Vdash \neg\varphi) \in \Pi_1$.*

Proof. This is checked by induction on n. The basis is straightforward using Def. 2.4.3. Assume by Induction Hypothesis that if $(w \Vdash \Box^k\varphi) \in \Pi_0$, then there exists w_1 such that $\mu_{\overset{k}{\rightsquigarrow}}(w, w_1) \in \Pi_1$ and $(w_1 \Vdash \neg\varphi) \in \Pi_1$.

Suppose that $(w \Vdash \Box^{k+1}\varphi) \in \Pi_0$. From this, Def. 2.3 and Def. of $\Box$ It follows that

$$(w \Vdash \lozenge\neg\Box^k\varphi) \in \Pi_1 \tag{2.28}$$

By (2.28) and Prop. 2.2.3 there exists w_1 such that

$$\mu_{\overset{1}{\rightsquigarrow}}(w, w_1) \in \Pi_1 \tag{2.29}$$

and $(w_1 \Vdash \neg\Box^k\varphi) \in \Pi_1$. Since $\mathfrak{S}$ is a Π-crisp, then $(w_1 \Vdash \Box^k\varphi) \in \Pi_0$. From the latter and Induction Hypothesis there exists w_{k+1} such that $\mu_{\overset{k}{\rightsquigarrow}}(w_1, w_{k+1}) \in \Pi_1$ and $(w_{k+1} \Vdash \neg\varphi) \in \Pi_1$. With the help of(2.29) the latter is the same that $\mu_{\overset{k}{\rightsquigarrow}}(w, w_{k+1}) \in \Pi_1$ and $(w_{k+1} \Vdash \neg\varphi) \in \Pi_1$. *QED*

The following lemma shows that the axiom $G^{k,l,m,n}$ is sound with respect fuzzy Kripke models in which $\rightsquigarrow$ is Π-(k, l, m, n)-confluent:

Lemma 2.5.6 (Soundness Lemma)**.** *If α is a formula of form $G^{k,l,m,n}$ and $\mathcal{G}$ is a fuzzy Kripke model where $\rightsquigarrow$ is Π-(k, l, m, n)-confluent, then $\vDash_{\mathcal{G}} \alpha$.*

Proof. Let α be $\Diamond^k\Box^l\beta \supset \Box^m\Diamond^n\beta$. Suppose that $(w \Vdash_{\mathcal{G}} \Diamond^k\Box^l\beta \supset \Box^m\Diamond^n\beta) \in \Pi_0$ for some $w \in W$. Then by Def. 2.3.1

$$(w \Vdash_{\mathcal{G}} \Diamond^k\Box^l\beta) \in \Pi_1 \tag{2.30}$$

and

$$(w \Vdash_{\mathcal{G}} \Box^m\Diamond^n\beta) \in \Pi_0 \tag{2.31}$$

By (2.30) and Lemma 2.5.1 there exists a w_k such that

$$\mu_{\overset{k}{\rightsquigarrow}}(w, w_k) \in \Pi_1 \tag{2.32}$$

and

$$(w_k \Vdash_{\mathcal{G}} \Box^l\beta) \in \Pi_1 \tag{2.33}$$

By (2.31) and Lemma 2.5.5 there exists a w_m such that

$$\mu_{\overset{m}{\rightsquigarrow}}(w, w_m) \in \Pi_1 \tag{2.34}$$

and

$$(w_m \Vdash_{\mathcal{G}} \neg\Diamond^n\beta) \in \Pi_1 \tag{2.35}$$

By (2.32), (2.34) and the appropriate instance of the Π-confluence property of $\rightsquigarrow$ there exists a $x \in W$ such that

$$\mu_{\overset{l}{\rightsquigarrow}}(w_k, x) \in \Pi_1 \tag{2.36}$$

and

$$\mu_{\overset{n}{\rightsquigarrow}}(w_m, x) \in \Pi_1 \tag{2.37}$$

By (2.33), (2.36) and Lemma 2.5.2 we conclude that

$$(x \Vdash_{\mathcal{G}} \beta) \in \Pi_1 \tag{2.38}$$

By (2.35), (2.37) and Lemma 2.5.4, on the other hand, we conclude that

$$(x \Vdash_{\mathcal{G}} \beta) \in \Pi_0 \tag{2.39}$$

Note that (2.39) contradicts (2.38). *QED*

Theorem 2.5.1. *For any $\alpha \in LM_P$, we have that α is a theorem of $KG^{k,l,m,n}$ iff $\vDash_{\mathcal{KG}} \alpha$ for each fuzzy Kripke model $\mathcal{KG} = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$ such that $\rightsquigarrow$ is Π-(k,l,m,n)-confluent.*

Proof. ($\Rightarrow$) Let α be a theorem of the $KG^{k,l,m,n}$ and $\mathcal{KG}^{k,l,m,n}$ be a fuzzy Kripke model where $\rightsquigarrow$ is Π-(k,l,m,n)-confluent. We will prove that $\vDash_{\mathcal{KG}} \alpha$. In view of Prop. 2.5.2, however, it is sufficient to check the case where α is an instance of the $G^{k,l,m,n}$-axiom, i.e., to check that $(w \Vdash_{\mathcal{KG}} \Diamond^k \Box^l \beta \supset \Box^m \Diamond^n \beta) \in \Pi_1$ for each $w \in W$ and $\beta \in LM_P$, but from the Lemma 2.5.6 it is immediate.
($\Leftarrow$) In (Lemmon; Scott, 1977) the completeness of system $KG^{k,l,m,n}$ with respect the class of models that satisfies the confluence accessibility relation is established. By Corollary 2.5.1 it follows that the system $\mathcal{KG}$ is complete with respect the $KG^{k,l,m,n}$ system. So, if $\vDash_{\mathcal{KG}} \beta$ then $\vdash_{KG} \beta$. *QED*

The completeness results proven in Prop. 2.5.1 can be shown to hold not only for singular instances of $G^{k,l,m,n}$, but also for several such instances combined. Indeed:

Proposition 2.5.3. *Let $G^{k_1,l_1,m_1,n_1}, \ldots, G^{k_p,l_p,m_p,n_p}$ be instances of the schema $G^{k,l,m,n}$. Let $K + G^{k_1,l_1,m_1,n_1} + \ldots + G^{k_p,l_p,m_p,n_p}$ be the system which results from extending K with G^{k_1,l_1,m_1,n_1}, $\ldots, G^{k_p,l_p,m_p,n_p}$. A formula α is a theorem of $K + G^{k_1,l_1,m_1,n_1} + \ldots + G^{k_p,l_p,m_p,n_p}$ iff $\vDash_{\mathcal{KG}^+} \alpha$ for each fuzzy Kripke model $\mathcal{KG}^+ = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$ such that $\rightsquigarrow$ is Π-(k_1,l_1,m_1,n_1)-confluent, ..., Π-(k_p,l_p,m_p,n_p)-confluent.*

Proof. ($\Rightarrow$) By Theorem 2.5.1 this result is valid for $K + G^{k_1,l_1,m_1,n_1}$. If we add G^{k_2,l_2,m_2,n_2} and use Lemma 2.5.6 we can conclude that $K + G^{k_1,l_1,m_2,n_2} + G^{k_2,l_2,m_2,n_2}$ is sound in all fuzzy Kripke models such that $\rightsquigarrow$ is (k_1,l_1,m_1,n_1)-confluent and (k_2,l_2,m_2,n_2)-confluent. Using the same reasoning we can extend the result for each system $K + G^{k_1,l_1,m_1,n_1} + \ldots + G^{k_p,l_p,m_p,n_p}$.
($\Leftarrow$) From Corollary 2.5.1 this proof is analogous to the proof of completeness for extensions of K with finitely many instances of $G^{k,l,m,n}$, as done, e.g., in (Pizzi; Carnielli, 2008). *QED*

Notice that the completeness of the modal systems KT, KB and KD, for instance, are direct consequences of Theorem 2.5.1, while the completeness of B, $S4$ and $S5$, for instance, follows from Prop. 2.5.3. To illustrate, we show how obtain completeness for $S5$ (we use below $\Rightarrow$ and & for the classical metalinguistic implication and conjunction).

Example 2.5.1. *S5 is complete with respect all* Π*-reflexive and* Π*-euclidean fuzzy Kripke models.* The modal system S5 is axiomatized by K, T and 5, i.e. $K + \langle 0,1,0,0\rangle + \langle 1,0,1,1\rangle$. But $\rightsquigarrow$ is Π-$\langle 1,0,1,1\rangle$-confluent iff (by Definition 2.4.4) $\forall x\forall y\forall z((\mu_{\rightsquigarrow}(x,y) \in \Pi_1$ & $\mu_{\rightsquigarrow}(x,z) \in \Pi_1) \Rightarrow \exists w(y = w$ & $\mu_{\rightsquigarrow}(z,w) \in \Pi_1))$ iff for arbitrary $x, y, z \in W$ we have that $(\mu_{\rightsquigarrow}(x,y) \in \Pi_1$ & $\mu_{\rightsquigarrow}(x,z) \in \Pi_1) \Rightarrow (\mu_{\rightsquigarrow}(z,y) \in \Pi_1)$ iff $\forall x\forall y\forall z(\mu_{\rightsquigarrow}(x,y) \in \Pi_1$ & $\mu_{\rightsquigarrow}(x,z) \in \Pi_1) \rightarrow (\mu_{\rightsquigarrow}(z,y) \in \Pi_1))$ iff (by Definition 2.4.4) $\rightsquigarrow$ is Π-euclidean. Furthermore, using a similar reasoning we note that $\rightsquigarrow$ is Π_1-$\langle 0,1,0,0\rangle$-confluent iff $\forall x\forall y\forall z((x = y$ & $x = z) \rightarrow \exists w(\mu_{\rightsquigarrow}(y,w) \in \Pi_1$ & $z = w))$ iff $\forall x(\mu_{\rightsquigarrow}(x,x) \in \Pi_1)$ iff $\rightsquigarrow$ is Π-reflexive. So, by Theorem 2.5.3 follows the completeness of $K + \langle 0,1,0,0\rangle + \langle 1,0,1,1\rangle$ with respect all fuzzy Kripke models that are Π-reflexive and Π-euclidean.

2.6 Final Remarks

In this Chapter we generalize scattered results from (Bedregal et al., 2011) to a much more inclusive collection of modal logics, and also greatly generalizes our previous approach in (Dodó; Marcos, 2012) by the consideration of other classic-like partitions of the interval $[0,1]$ as $[0,i) \cup [i,1]$. The partition presupposed by most fuzzy logics in the literature takes $i = 1$, a constraint which seems by all means unnecessary. Furthermore, adapting the previous results to partitions of the form $[0,i] \cup (i,1]$ requires straightforward modifications to the above.

We believe it is possible to study a multimodal (diamond) version of the axiom of confluence by adding appropriate indices to the modalities, at the linguistic level, and adding corresponding fuzzy accessibility relations, at the semantic level (in such case, the initial case with iterated modalities will accordingly be reduced to distinct one-step modalities). Completeness should in this case be attainable, as in the case of normal modal logics extending classical logic, by adding appropriate interaction axioms. We also conjecture that the above results on the axiom of confluence and its corresponding collection of frames may be extended to every Sahlqvist-definable frame class. This thread of investigation, however, shall be left as matter for future work.

3 *LAB* Logics

Here we present the *LAB* logics[1], a family of modal logics that has a primitive paraconsistent negation among its connectives. We propose an axiomatization for the minimal *LAB* logics. Next we prove completeness results for this logic and some extensions of it. The modalities $\Box$ and $\Diamond$ and restoration connectives can be defined in *LAB* logics.

3.1 The regular *LAB* logics

Let $S_{\vee\wedge\supset\smile}$ be the language inductively defined over a set of denumerable propositional variables $\mathcal{P}$ as follows:

$$\varphi ::= p \,|\, (\varphi \vee \varphi) \,|\, (\varphi \wedge \varphi) \,|\, (\varphi \supset \varphi) \,|\, (\smile\varphi)$$

where $p \in \mathcal{P}$ and φ is a formula.

In this language, with the help of classical implication, we may define the following connectives:

$$\bot \overset{def}{=} \smile(\varphi \supset \varphi)$$

$$\sim\varphi \overset{def}{=} \varphi \supset \bot$$

The regular *LAB* logic is obtained by adding to a complete set of axioms and rules of inference for positive classical propositional logic the axiom

(A1) $\vdash \smile(\alpha \wedge \beta) \supset (\smile\alpha \vee \smile\beta)$

and the inference rules:

(R1) If $\vdash \alpha \supset \beta$ then $\vdash \smile\beta \supset \smile\alpha$

(R2) If $\vdash \alpha$, then $\vdash \smile\alpha \supset \beta$

[1]The name LAB is a reference to the word Laboratory. I think that the work of defining a logic is similar to the one done in an Laboratory, in the sense that we experiment with different definitions and we adjust several times the logical structures to achieve some desired results.

This logic, from now on will be called $K_\smile$. Let $\Gamma \cup \{\alpha\}$ be a set of formulas. We say that α is **derivable** in a logic $\mathcal{L}$ over $S_{\vee\wedge\supset\smile}$, denoted by $\Gamma \vdash_{\mathcal{L}} \alpha$, if there is a finite sequence $\varphi_1, \varphi_2, \ldots, \varphi_n$, $n \in \mathbb{N}$, of formulas of $S_{\wedge\vee\supset\smile}$ such that φ_n is α and for every $1 \leq i \leq n$ one of the following conditions hold:

(i) φ_i is an instance of an axiom;

(ii) $\varphi_i \in \Gamma$ (φ_i is a hypothesis);

(iii) φ_i follows from an application of an inference rule on premises that appears before line i.

We say that a formula φ is a **theorem** of a logic $\mathcal{L}$, denoted by $\vdash_{\mathcal{L}} \varphi$, if α is derived from the empty set of premisses. We denote the set of theorems of $\mathcal{L}$ by Th($\mathcal{L}$). By $\Gamma \nvdash_{\mathcal{L}} \alpha$ we denote that α is not derivable from Γ in $\mathcal{L}$.

A form of the Deduction Metatheorem may be proven for *LAB* logics.

Proposition 3.1.1. *(MTD) Let α and β be formulae of $S_{\wedge\vee\supset\smile}$ and $\Gamma \subseteq S_{\wedge\vee\supset\smile}$. Given a derivation for $\Gamma, \alpha \vdash \beta$, it is possible to build a derivation for $\Gamma \vdash \alpha \supset \beta$.*

Proof. See Appendix A. *QED*

Based in the above axiomatization we can prove for every regular *LAB* logic $\mathcal{L}$ that:

Proposition 3.1.2.

1. $\vdash \bot \supset \beta$
2. $\vdash \alpha \supset (\sim\alpha \supset \beta)$ *(Principle of Explosion)*
3. $\vdash \alpha \vee \sim\alpha$ *(Principle of Implosion)*
4. $(\sim\alpha \supset \beta) \supset ((\sim\alpha \supset \sim\beta) \supset \alpha)$
5. $\vdash \smile(\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_n) \supset \smile\beta_1 \vee \smile\beta_2 \vee \ldots \vee \smile\beta_n$, *for* $n \geq 2$.
6. $\sim\sim\alpha \vdash \alpha$
7. *If* $\Gamma, \alpha \vdash \bot$, *then* $\Gamma \vdash \sim\alpha$
8. $\alpha, \sim\alpha \vdash \bot$

Proof. In the proofs below [PC] represents an instance of a theorem of classical propositional logic, while [MP i, j] represents an application of *Modus Ponens* on the formulas in the lines i and j. A formula justified by [Def. of $\ominus$ in i], where $\ominus \in \{\sim, \bot\}$, follows from Definition of $\ominus$ on the formula that appears in the previous line i.

For (1) consider the following derivation:

1.	$\varphi \supset \varphi$	[PC]
2.	$\smile(\varphi \supset \varphi) \supset \beta$	[(R2) on 1]
3.	$\bot \supset \beta$	[Def. of $\bot$ in 2]

For (2) consider the following derivation:

1.	$\alpha \supset ((\alpha \supset \bot) \supset \bot)$	[PC]
2.	$\bot \supset \beta$	[Prop. 3.1.2.1]
3.	$(\alpha \supset ((\alpha \supset \bot) \supset \bot)) \supset ((\bot \supset \beta) \supset (\alpha \supset ((\alpha \supset \bot) \supset \beta))$	[PC]
4.	$(\bot \supset \beta) \supset (\alpha \supset ((\alpha \supset \bot) \supset \beta))$	[MP 1, 3]
5.	$\alpha \supset ((\alpha \supset \bot) \supset \beta)$	[MP 2, 4]
6.	$\alpha \supset (\sim\alpha \supset \beta)$	[Def. $\sim$ in 5]

For (3) consider the below derivation:

1.	$\alpha \vee (\alpha \supset \bot)$	[PC]
2.	$\alpha \vee \sim\alpha$	[Def. $\sim$ in 1]

For (4) consider the following derivation:

1.	$\sim\alpha \supset \beta$	[Hyp]
2.	$\sim\alpha \supset \sim\beta$	[Hyp]
3.	$\sim\alpha$	[Hyp]
4.	β	[MP 1, 3]
5.	$\sim\beta$	[MP 2, 3]
6.	$\beta \supset (\sim\beta \supset \alpha)$	[Prop. 3.1.2.2]
7.	$\sim\beta \supset \alpha$	[MP 4, 6]
8.	α	[MP 5, 7]

Thus, $\sim\alpha \supset \beta, \sim\alpha \supset \sim\beta, \sim\alpha \vdash \alpha$. By MTD it follows that $\sim\alpha \supset \beta, \sim\alpha \supset \sim\beta \vdash \sim\alpha \supset \alpha$. From the later there is a derivation $\mathcal{D}$ with Hypothesis $\sim\alpha \supset \beta, \sim\alpha \supset \sim\beta$ and conclusion $\sim\alpha \supset \alpha$, that is:

1.	$\sim\alpha \supset \beta$	[Hyp]
2.	$\sim\alpha \supset \sim\beta$	[Hyp]
	$\vdots$	
j.	$\sim\alpha \supset \alpha$	[Justification]

Consider the following derivation obtained by appending some lines to $\mathcal{D}$:

1.	$\sim\alpha \supset \beta$	[Hyp]
2.	$\sim\alpha \supset \sim\beta$	[Hyp]
	$\vdots$	
j.	$\sim\alpha \supset \alpha$	[Justification]
j+1.	$\alpha \supset \alpha$	[PC]
j+2.	$(\alpha \supset \alpha) \supset ((\sim\alpha \supset \alpha) \supset ((\alpha \supset \alpha) \wedge (\sim\alpha \supset \alpha)))$	[PC]
j+3.	$(\sim\alpha \supset \alpha) \supset ((\alpha \supset \alpha) \wedge (\sim\alpha \supset \alpha))$	[MP j+1, j+2]
j+4.	$(\alpha \supset \alpha) \wedge (\sim\alpha \supset \alpha)$	[MP j, j+3]
j+5.	$((\alpha \supset \alpha) \wedge (\sim\alpha \supset \alpha)) \supset ((\sim\alpha \vee \alpha) \supset \alpha)$	[PC]
j+6.	$(\sim\alpha \vee \alpha) \supset \alpha$	[MP j+4, j+5]
j+7.	$\sim\alpha \vee \alpha$	[Prop. 3.1.2.3]
j+8.	α	[MP j+6, j+7]

We conclude that $\sim\alpha \supset \beta, \sim\alpha \supset \sim\beta \vdash \alpha$. By applying twice MTD, it follows that $\vdash (\sim\alpha \supset \beta) \supset ((\sim\alpha \supset \sim\beta) \supset \alpha)$.

The proof of (5) follows from a induction on the size of n. The basis step, $k = 2$, is justified by the axiom (A1). Assume by Induction Hypothesis that $\vdash \smallsmile(\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k) \supset \smallsmile\beta_1 \vee \smallsmile\beta_2 \vee \ldots \vee \smallsmile\beta_k$ for an arbitrary k. The following derivation concludes the proof:

1.	$\smallsmile((\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k) \wedge \beta_{k+1}) \supset \smallsmile(\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k) \vee \smallsmile\beta_{k+1}$	[(A1)]
2.	$\smallsmile(\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k) \supset \smallsmile\beta_1 \vee \smallsmile\beta_2 \vee \ldots \vee \smallsmile\beta_k$	[I.H.]
3.	$(\smallsmile(\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k) \supset (\smallsmile\beta_1 \vee \smallsmile\beta_2 \vee \ldots \vee \smallsmile\beta_k)) \supset$ $((\smallsmile(\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k) \vee \smallsmile\beta_{k+1}) \supset ((\smallsmile\beta_1 \vee \smallsmile\beta_2 \vee \ldots \vee \smallsmile\beta_k) \vee \smallsmile\beta_{k+1}))$	[PC]
4.	$(\smallsmile(\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k) \vee \smallsmile\beta_{k+1}) \supset (\smallsmile\beta_1 \vee \smallsmile\beta_2 \vee \ldots \vee \smallsmile\beta_k \vee \smallsmile\beta_{k+1})$	[MP 2, 3]
5.	$\smallsmile(\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \beta_{k+1}) \supset (\smallsmile\beta_1 \vee \smallsmile\beta_2 \vee \ldots \vee \smallsmile\beta_k \vee \smallsmile\beta_{k+1})$	[1, 4 and transitivity of $\supset$]

For (6) consider the following derivation:

1. $\sim\sim\alpha$ [Hyp]
2. $(\sim\alpha \supset \sim\alpha) \supset ((\sim\alpha \supset \sim\sim\alpha) \supset \alpha)$ [Prop. 3.1.2.4]
3. $\sim\alpha \supset \sim\alpha$ [PC]
4. $(\sim\alpha \supset \sim\sim\alpha) \supset \alpha$ [MP 2, 3]
5. $\sim\sim\alpha \supset (\sim\alpha \supset \sim\sim\alpha)$ [PC]
6. $\sim\alpha \supset \sim\sim\alpha$ [MP 1, 4]
7. α [MP 4, 6]

Thus, $\sim\sim\alpha \vdash \alpha$.

For (7) assume that $\Gamma, \alpha \vdash \bot$, applying *MTD* we obtain that $\Gamma \vdash \alpha \supset \bot$, by definition of $\sim$, that is $\Gamma \vdash \sim\alpha$.

The derivation below justifies the item (8):

1. α [Hyp]
2. $\sim\alpha$ [Hyp]
3. $\alpha \supset \bot$ [Def. of $\bot$ in 2]
4. $\bot$ [MP 1, 3]

Thus, $\alpha, \sim\alpha \vdash \bot$. *QED*

The connective $\sim$ is like a classical negation since, by Prop. 3.1.2, $\sim$ satisfies both the principle of explosion and principle of implosion with respect the consequence relation $\vdash$.

Let $\mathcal{L}$ be a modal logic and Δ be a set of formulas. We say that the set of formulas Δ is **$\mathcal{L}$-coherent** if $\Delta \nvdash_{\mathcal{L}} \bot$, otherwise Δ is called $\mathcal{L}$-incoherent.

Lemma 3.1.1. *If Δ is a $\mathcal{L}$-coherent set of formulas and φ a formula then:*

1. *$\Delta \cup \{\varphi\}$ or $\Delta \cup \{\sim\varphi\}$ is $\mathcal{L}$-coherent*

Proof. Assume that Δ is a $\mathcal{L}$-coherent and suppose, by contradiction, that both sets $\Delta \cup \{\varphi\}$ and $\Delta \cup \{\sim\varphi\}$ are $\mathcal{L}$-incoherent. It follows that $\Delta, \varphi \vdash \bot$ and $\Delta, \sim\varphi \vdash \bot$. From the latter and Prop. 3.1.2.7 we have that (i) $\Delta \vdash \sim\varphi$ and (ii) $\Delta \vdash \sim\sim\varphi$ respectively. Since, by Prop. 3.1.2.2, $\Delta \vdash \sim\varphi \supset (\sim\sim\varphi \supset \bot)$, using (i), (ii) and MP we conclude that $\Delta \vdash \bot$, which contradicts our initial assumption. *QED*

We say that Δ is **$\mathcal{L}$-maximal coherent** if Δ is $\mathcal{L}$-coherent and

1. $\alpha \in \Delta$ or $\sim\alpha \in \Delta$, for every $\alpha \in \mathcal{S}_{\wedge\vee\supset\smile}$

The following properties can be checked for any set of $\mathcal{L}$-maximal coherent sets of formulas.

Proposition 3.1.3. *If $\mathcal{L}$ is a LAB logic and Γ is $\mathcal{L}$-maximal coherent, then:*

1. *Th($\mathcal{L}$)$\subseteq \Gamma$*
2. *Γ is closed under Modus Ponens*
3. *$\alpha \vee \beta \in \Gamma$ iff $\alpha \in \Gamma$ or $\beta \in \Gamma$*

Proof. (1) Assume that there is $\varphi \in$ Th($\mathcal{L}$) such that $\varphi \notin \Gamma$. Since Γ is $\mathcal{L}$-maximal coherent, then $\sim\varphi \in \Gamma$. Now, consider the following derivation:

1.	φ	[Theorem of $\mathcal{L}$]
2.	$\sim\varphi$	[Hypothesis]
3.	$\varphi \supset (\sim\varphi \supset \bot)$	[Theorem of $\mathcal{L}$]
4.	$\sim\varphi \supset \bot$	[MP 1, 3]
5.	$\varphi \supset (\sim\varphi \supset \bot)$	[MP 2, 4]

We conclude from this that $\sim\varphi \vdash \bot$. Since $\sim\varphi \in \Gamma$, then $\Gamma \vdash \bot$. This contradicts the $\mathcal{L}$-maximal coherency of Γ.

(2) Assume that $\alpha, \alpha \supset \beta \in \Gamma$. Suppose, by contradiction, that $\beta \notin \Gamma$. Since Γ is $\mathcal{L}$-maximal coherent, then $\sim\beta \in \Gamma$. From the Prop. 3.1.3.1 the theorem of classical propositional logic $(\alpha \supset \beta) \supset (\sim\beta \supset \sim\alpha)$ is also in Γ. By MP and Prop. 3.1.2.8 the set $\{\alpha, \alpha \supset \beta, \sim\beta\} \subseteq \Gamma$ derives $\bot$ by MP. Thus, Γ is $\mathcal{L}$-incoherent, which is an absurd.

(3) Assume that $\alpha \vee \beta \in \Gamma$ and suppose, by contradiction, that both $\alpha, \beta \notin \Gamma$. From this it follows, by the $\mathcal{L}$-maximality of Γ, that $\sim\alpha, \sim\beta \in \Gamma$. From the Prop. 3.1.3.1 the classical theorem $\sim\alpha \supset (\sim\beta \supset \sim(\alpha \vee \beta)) \in \Gamma$. The set $\{\alpha \vee \beta, \sim\alpha, \sim\beta, \sim\alpha \supset (\sim\beta \supset \sim(\alpha \vee \beta))\} \subseteq \Gamma$, by MP and Prop. 3.1.2.8, derives $\bot$. This contradicts the fact that Γ is $\mathcal{L}$-coherent. *QED*

Lemma 3.1.2. [Extension Lemma] *If Γ' is an $\mathcal{L}$-coherent set of formulas, then there is a $\mathcal{L}$-maximal coherent set of formulas Γ such that $\Gamma' \subseteq \Gamma$.*

Proof. Let $\mathcal{E} = (\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots)$ be an enumeration of the formulas of $S_{\wedge\vee\supset\smile}$. Define the following sequence of sets:

$$\Gamma_0 = \Gamma'$$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\}, & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is } \mathcal{L}\text{-coherent} \\ \Gamma_n, & \text{otherwise} \end{cases}$$

The proof that each of the sets Γ_i is $\mathcal{L}$-coherent proceeds by induction. The basis step is immediate, since $\Gamma_0 = \Gamma'$. Suppose by Induction Hypothesis that Γ_n is $\mathcal{L}$-coherent. The set Γ_{n+1} is either $\Gamma_n \cup \{\varphi_{n+1}\}$ or Γ_n. In the first case Γ_{n+1} is $\mathcal{L}$-coherent by construction, while in the second case Γ_{n+1} is $\mathcal{L}$-coherent by Induction Hypothesis.

We claim that $\Gamma = \bigcup_{i \geq 0} \Gamma_i$ is $\mathcal{L}$-maximal coherent. Suppose that Γ is not $\mathcal{L}$-coherent. Then there is a finite set $\{\varphi_{m_0}, \ldots, \varphi_{m_t}\}$ such that $\varphi_{m_0}, \ldots, \varphi_{m_t} \vdash \bot$, that is $\{\varphi_{m_0}, \ldots, \varphi_{m_t}\}$ is $\mathcal{L}$-incoherent. We have that $\{\varphi_{m_0}, \ldots, \varphi_{m_t}\} \subseteq \Gamma_i$ for some i sufficiently large, thus Γ_i is $\mathcal{L}$-incoherent. This an absurd, since each Γ_i is $\mathcal{L}$-coherent.

Now we prove the $\mathcal{L}$-maximality of Γ. Suppose that $\varphi_i \notin \Gamma$ for some φ_i in $\mathcal{E}$. By construction the set $\Gamma_i \cup \{\varphi_i\}$ is not $\mathcal{L}$-coherent. By Lemma 3.1.1 $\Gamma_i \cup \{\sim\varphi_i\}$ is $\mathcal{L}$-coherent. According the enumeration $\varphi_j = \sim\varphi_i$, for some $j \neq i$. We have to consider two cases:
[$j < i$]
In this case we have that $\Gamma_{j+1} = \Gamma_j \cup \{\varphi_j\}$. The set Γ_{j+1} is $\mathcal{L}$-coherent. Otherwise $\Gamma_i \supseteq \Gamma_j$ is $\mathcal{L}$-incoherent. Thus, $\sim\varphi_i \in \Gamma$.
[$j > i$]
In this case $\Gamma_i \subseteq \Gamma_j$. From this we have that $\Gamma_j \cup \{\varphi_i\}$ is $\mathcal{L}$-incoherent. By Lemma 3.1.1 $\Gamma_j \cup \{\sim\varphi_i\}$ is $\mathcal{L}$-coherent. The latter set is the same as Γ_{j+1}. Since $\Gamma_{j+1} \subseteq \Gamma$, then $\sim\varphi_i \in \Gamma$.
We conclude from the above that φ or $\sim\varphi \in \Gamma$ for every $\varphi \in \mathcal{E}$. *QED*

In Section 3.2 we present some extensions of the minimal *LAB* logics.

3.2 Other *LAB* Systems

In Table 1 we list some axioms. In section 3.4 we shall show that each of them corresponds to some specific property on the class of frames where it is valid.

$(D_\smile)$	$\smile\bot$
$(T_\smile)$	$\smile\alpha \vee \alpha$
$(B_\smile)$	$\smile\smile\alpha \supset \alpha$
$(4_\smile)$	$\smile\beta \supset (\smile\alpha \vee \smile\smile\alpha)$
$(5_\smile)$	$\smile\alpha \supset (\smile\smile\alpha \supset \beta)$

Table 1: Some axioms

The system $K_\smile$ extended with some of the axioms in Table 1 originates other systems, some of which are illustrated in Table 2:

$KT_{\smile}$	$K_{\smile} + (T_{\smile})$
$KB_{\smile}$	$K_{\smile} + (B_{\smile})$
$S4_{\smile}$	$K_{\smile} + (T_{\smile}) + (4_{\smile})$
$S5_{\smile}$	$K_{\smile} + (T_{\smile}) + (5_{\smile})$

Table 2: Some systems

3.3 Semantics

A frame $\mathcal{F} = \langle W, R\rangle$ is a structure where W is a non-empty set, the set of worlds, and R is a binary relation on W. A model $\mathcal{M} = \langle \mathcal{F}, V\rangle$ is an interpretation structure defined over a frame where $V : W \times \mathcal{P} \longrightarrow \{0, 1\}$. The notion of satisfiability of a formula at a world in a model is defined as follows:

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash p & \text{iff} & V(w, p) = 1 \\
\mathcal{M}, w \Vdash \alpha \vee \beta & \text{iff} & V(w, \alpha) = 1 \text{ or } V(w, \beta) = 1 \\
\mathcal{M}, w \Vdash \alpha \wedge \beta & \text{iff} & V(w, \alpha) = 1 \text{ and } V(w, \beta) = 1 \\
\mathcal{M}, w \Vdash \alpha \supset \beta & \text{iff} & V(w, \alpha) = 0 \text{ or } V(w, \beta) = 1 \\
\mathcal{M}, w \Vdash {\smile}\alpha & \text{iff} & (\exists v \in W)(wRv \;\&\; \mathcal{M}, v \nVdash \alpha)
\end{array}
$$

We write $\mathcal{M}, w \nVdash \alpha$ to say that $V(w, \alpha) = 0$. This notion of satisfiability permit us to prove the following proposition.

Proposition 3.3.1. *For any model $\mathcal{M} = \langle W, R, V\rangle$ and $w \in W$:*

1. $\mathcal{M}, w \nVdash \bot$
2. $\mathcal{M}, w \Vdash {\sim}\alpha$ *iff* $V(w, \alpha) = 0$
3. $\mathcal{M}, w \Vdash {\sim}{\smile}\alpha$ *iff* $(\forall v \in W)($ *if* wRv , *then* $\mathcal{M}, v \Vdash \alpha)$
4. $\mathcal{M}, w \Vdash {\smile}{\sim}\alpha$ *iff* $(\exists v \in W)(wRv \;\&\; \mathcal{M}, v \Vdash \alpha)$

Proof. (1) Assume by contradiction that $\mathcal{M}, w \Vdash \bot$. By Def. of $\bot$ this is the same that $\mathcal{M}, w \Vdash {\smile}(p \supset p)$. From the latter and the interpretation of ${\smile}$ there is a world u such that $\mathcal{M}, u \nVdash p \supset p$. At u we have that $\mathcal{M}, u \nVdash p$ and $\mathcal{M}, u \Vdash p$. This is an absurd.

(2) By Def. of ${\sim}$ we have that $\mathcal{M}, w \Vdash {\sim}\alpha$ iff $\mathcal{M}, w \Vdash \alpha \supset \bot$. From the latter and the interpretation of implication it follows that $\mathcal{M}, w \nVdash \alpha$ or $\mathcal{M}, w \Vdash \bot$. Since, from 3.3.1.1, $\mathcal{M}, w \nVdash \bot$ we have that $\mathcal{M}, w \nVdash \alpha$, i.e., $V(w, \alpha) = 0$.

(3) $\mathcal{M}, w \Vdash {\sim}{\smile}\alpha$ iff, by the previous item 2, $\mathcal{M}, w \nVdash {\smile}\alpha$. This is, by interpretation of ${\smile}$, the same as $(\forall v \in W)($ if wRv , then $\mathcal{M}, v \Vdash \alpha)$.

(4) $\mathcal{M}, w \Vdash \smile\sim\alpha$ iff, by interpretation of $\smile$, $(\exists v \in W)(wRv \ \& \ \mathcal{M}, v \nVdash \sim\alpha)$ iff $(\exists v \in W)(wRv \ \& \ \mathcal{M}, v \Vdash \alpha)$.

QED

A formula φ is **valid** in a frame $\mathcal{F}$, denoted by $\mathcal{F} \Vdash \varphi$, iff $\mathcal{M}, x \Vdash \varphi$ for every world x in every model $\mathcal{M}$ based on $\mathcal{F}$. A formula φ is valid in a class $\mathbb{F}$ of frames, denoted by $\models_{\mathbb{F}} \varphi$, iff is valid in every frame of $\mathbb{F}$. We say that a set of formulas Γ **entails** a formula φ, denoted by $\Gamma \models \varphi$ when for every model $\mathcal{M}$ and for every world w in $\mathcal{M}$ either $\mathcal{M}$ falsifies some formula of Γ at w or $\mathcal{M}$ satisfies φ at w.

Let $\sharp$ be a unary connective. We say that $\sharp$ is a **paraconsistent** negation when there is a model that satisfies the formula $\varphi \wedge \sharp\varphi$ for some formula φ and falsifies some other formula. We say that $\sharp$ is a **paracomplete** negation when there is a model that falsifies the formula $\varphi \vee \sharp\varphi$ for some formula φ and satisfies some other formula. If a negation is both paraconsistent and paracomplete, then it is a **paranormal** negation. Based on this and on the semantics presented above we can prove the following result:

Lemma 3.3.1.

1. *$\smile$ is a paranormal negation;*

Proof. To show that $\smile$ is a paraconsistent negation consider the model $\mathcal{M} = \langle W, R, V\rangle$ such that $W = \{w, u\}$, $V(w, p) = 1$, $V(u, p) = 0$ and wRu. Since wRu and $\mathcal{M}, u \nVdash p$, then $\mathcal{M}, w \Vdash \smile p$. From the latter and the fact that $\mathcal{M}, w \Vdash p$, we conclude that $\mathcal{M}, w \Vdash p \wedge \smile p$.

The model $\mathcal{M} = \langle W, R, V\rangle$, where $W = \{w\}$, $V(w, p) = 0$ and $R = \varnothing$, can be used to prove that $\smile$ is a paracomplete negation. *QED*

According to the next proposition the axioms and rules of $K_{\smile}$ are valid with respect this semantics. We say that an inference rule preserves validity if whenever that its premisses are valid on a frame, then its consequence is also valid on this frame.

Proposition 3.3.2. [Soundness] *The logic $K_{\smile}$ is sound for frame validity.*

Proof. For (A1) suppose that $\not\models \smile(\alpha \wedge \beta) \supset (\smile\alpha \vee \smile\beta)$. Then, there is a model $\mathcal{M} = \langle W, R, V\rangle$ and a world w in $\mathcal{M}$ such that $\mathcal{M}, w \nVdash \smile(\alpha \vee \beta) \supset (\smile\alpha \vee \smile\beta)$. From this it follows that (i) $\mathcal{M}, w \Vdash \smile(\alpha \wedge \beta)$ and also that $\mathcal{M}, w \nVdash \smile\alpha \vee \smile\beta$, that is, (ii) $\mathcal{M}, w \nVdash \smile\alpha$ and (iii) $\mathcal{M}, w \nVdash \smile\beta$. By (i) there is a world u such that wRu and (iv) $\mathcal{M}, u \nVdash \alpha \wedge \beta$. In u, by (ii) and (iii), we have that $\mathcal{M}$ satisfies both α and β at u, that is, $\mathcal{M}, u \Vdash \alpha \wedge \beta$. This contradicts (iv).

For the rule (R1) assume that $\mathcal{F} \Vdash \alpha \supset \beta$ and suppose by contradiction that $\mathcal{F} \nVdash \smile\beta \supset \smile\alpha$. From the latter it follows that (i) $\mathcal{M}, w \Vdash \smile\beta$ and (ii) $\mathcal{M}, w \nVdash \smile\alpha$ for some world w at some model $\mathcal{M}$. By (i) there is a world u such that wRu and (iii) $\mathcal{M}, u \nVdash \beta$. Since wRu, by (ii), it follows that $\mathcal{M}, u \Vdash \alpha$. From this and (iii) we have that $\mathcal{M}, u \nVdash \alpha \supset \beta$. Thus, $\mathcal{F} \nVdash \alpha \supset \beta$. This contradicts our initial assumption.

For (R2) assume that $\mathcal{F} \Vdash \alpha$ and suppose by contradiction that $\mathcal{F} \nVdash \smile\alpha \supset \beta$. From the latter there must be a a model $\mathcal{M}$ and a world w such that $\mathcal{M}, w \Vdash \smile\alpha$ and $\mathcal{M}, w \nVdash \beta$. From the latter, there is a world u accessible from w such that $\mathcal{M}, u \nVdash \alpha$. This is an absurd since $\mathcal{F} \Vdash \alpha$. *QED*

In next section the canonical model construction is used to prove the desired completeness results.

3.4 Completeness

The *canonical model* is the structure $\mathcal{M}_C = \langle W_C, R_C, V_C \rangle$, where:

1. W_C is the set of $K_\smile$-maximal coherent extensions
2. wR_Cu iff $\ni[w] \subseteq u$ where $\ni[w] = \{\varphi : \smile\varphi \notin w\}$
3. For every propositional variable p, $V_C(w, p) = 1$ iff $p \in w$

Lemma 3.4.1. *In the canonical model if $\smile\alpha \in w$, then there is v such that wR_Cv and $\alpha \notin v$.*

Proof. Assume that $\smile\alpha \in w$. Let v' be a set such that $v' = \{\sim\alpha\} \cup \ni[w]$. We claim that v' is $K_\smile$-coherent. Otherwise there is (a) $\{\beta_1, \beta_2, \ldots, \beta_n\} \subseteq \ni[w]$ such that $\beta_1, \beta_2, \ldots, \beta_n, \sim\alpha \vdash \bot$. By Prop. 3.1.2.7 it follows that $\beta_1, \beta_2, \ldots, \beta_n \vdash \sim\sim\alpha$. Since $\sim\sim\alpha \vdash \alpha$ (Prop. 3.1.2.6) it follows that $\beta_1, \beta_2, \ldots, \beta_n \vdash \alpha$. From this we have that $\vdash (\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_n) \supset \alpha$. By (R1), this implies that (b) $\vdash \smile\alpha \supset \smile(\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_n)$. Now, by Prop. 3.1.2.5 we know that (c) $\vdash \smile(\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_n) \supset \smile\beta_1 \vee \smile\beta_2 \vee \ldots \vee \smile\beta_n$. From (b), (c) and transitivity of classical implication we have that $\vdash \smile\alpha \supset \smile\beta_1 \vee \smile\beta_2 \vee \ldots \vee \smile\beta_n$. It follows that $\smile\beta_1 \vee \smile\beta_2 \vee \ldots \vee \smile\beta_n \in w$. By Prop. 3.1.3 it follows that $\smile\beta_i \in w$ for some $1 \leq i \leq n$, that is, $\beta_i \notin \ni[w]$. This contradicts (a). So, since v' is $K_\smile$-coherent then, by the Extension Lemma, we can extend v' to a $K_\smile$-maximal coherent v such that $v' \subseteq v$. Since $\sim\alpha \in v$, by

Prop. 3.1.3 it follows that $\alpha \notin v$. It remains to prove that wR_Cv. For this take an arbitrary $\varphi \in \ni[w]$. Then, by construction of v, it follows that $\varphi \in v$. *QED*

Lemma 3.4.2. *For every $w \in W_C$ and every φ,*

$$\mathcal{M}_C, w \Vdash \varphi \text{ iff } \varphi \in w.$$

Proof. The proof is by induction on the structure of φ. The base step is immediate from the definition of canonical model. Suppose by Induction Hypothesis that for all w in W_C and for a certain ψ we have that $M_C, w \Vdash \psi$ iff $\psi \in w$. We shall check the case where φ is $\smile\psi$. From the left to right assume that $\mathcal{M}_C, w \Vdash \smile\psi$. Then there is v such that wR_Cv and $\mathcal{M}_C, v \nVdash \psi$. From the latter it follows by Induction Hypothesis that $\psi \notin v$, since wR_Cv, then $\smile\psi \in w$. Conversely, assume that $\smile\psi \in w$, by Lemma 3.4.1 there is v such that wR_Cv and $\psi \notin v$, and that means precisely that $\mathcal{M}_C, w \Vdash \smile\psi$. *QED*

Let $\mathbb{F}$ be the class of all frames. We will prove the following completeness result:

Proposition 3.4.1 (Completeness). If $\Gamma \models_{\mathbb{F}} \alpha$, then $\Gamma \vdash_{K_\smile} \alpha$.

Proof. Suppose, by contraposition that $\Gamma \nvdash_{K_\smile} \alpha$. Then, the set $\Gamma \cup \{\sim\alpha\}$ is coherent. By Proposition 3.1.3 we can extend $\Gamma \cup \{\sim\alpha\}$ to a maximal coherent set w. Since $\sim\alpha \in w$, then $\alpha \notin w$. By Lemma 3.4.2, in the canonical model, $\mathcal{M}_C, w \nVdash \alpha$. Thus, $\Gamma \not\models \alpha$. *QED*

The logic $K_\smile$ is the minimal *LAB* logics in the sense that it is complete with respect the class of all frames. In what follows we prove that the axioms listed in Table 1 correspond to certain properties over the class of frames.

Proposition 3.4.2.

1. $\mathcal{F} \Vdash (D_\smile)$ *iff* $\mathcal{F}$ *is serial.*

2. $\mathcal{F} \Vdash (T_\smile)$ *iff* $\mathcal{F}$ *is reflexive.*

3. $\mathcal{F} \Vdash (B_\smile)$ *iff* $\mathcal{F}$ *is symmetric.*

4. $\mathcal{F} \Vdash (4_\smile)$ *iff* $\mathcal{F}$ *is transitive.*

5. $\mathcal{F} \Vdash (5_\smile)$ *iff* $\mathcal{F}$ *is euclidean.*

Proof. In what follows let $\mathcal{F} = \langle W, R \rangle$ be an arbitrary frame.

(1) Assume that $\mathcal{F}$ is serial. Suppose by contradiction that $\mathcal{F} \nVdash {\smile}\bot$. Then, there is a model $\mathcal{M}$ and a world w such that $\mathcal{M}, w \nVdash {\smile}\bot$. Since R is serial, for some v, wRv. In v we have that $\mathcal{M}, v \Vdash \bot$. This is a contradiction. For the other direction, suppose that $\mathcal{F}$ is not serial. Then, there is a world w, such that w does not access another world. At w all formulas of the form ${\smile}\alpha$ are falsified at w. For every model $\mathcal{M}$, based on $\mathcal{F}$, that contains w we have that $\mathcal{M}, w \nVdash {\smile}\bot$. Thus, $\mathcal{F} \nVdash {\smile}\bot$.

(2) Assume that $\mathcal{F}$ is reflexive. Suppose by contradiction that $\mathcal{F} \nVdash {\smile}p \vee p$. Then, there exists a model $\mathcal{M}$ and a world w such that $\mathcal{M}, w \nVdash {\smile}p \vee p$. From the later it follows that (i) $\mathcal{M}, w \nVdash {\smile}p$ and (ii) $\mathcal{M}, w \nVdash p$. Since wRw, by (i), we have that $\mathcal{M}, w \Vdash p$. Which contradicts (ii). For the other direction assume that $\mathcal{F}$ is not reflexive, that is $\langle x, x \rangle \notin W$ for some x. Let $\mathcal{M}_1 = \langle W, R, V_1 \rangle$ be the model where $V_1(x, p) = 0$ and $V_1(u, p) = 1$ for every u such that xRu. Since $V_1(x, p) = 0$, then (iii) $\mathcal{M}_1, x \nVdash p$. By construction of V_1 and interpretation of ${\smile}$, it follows that (iv) $\mathcal{M}_1, x \nVdash {\smile}p$. So, by (iii) and (iv), $\mathcal{M}_1, x \nVdash p \vee {\smile}p$.

(3) Assume that $\mathcal{F}$ is symmetric. Suppose, by contradiction, that $\mathcal{F} \nVdash {\smile}{\smile}p \supset p$. Then, there exists a w in a model $\mathcal{M}$ such that $\mathcal{M}, w \nVdash {\smile}{\smile}p \supset p$. That is, (i) $\mathcal{M}, w \Vdash {\smile}{\smile}p$ and (ii) $\mathcal{M}, w \nVdash p$. By (i) it follows that wRu and (iii) $\mathcal{M}, u \nVdash {\smile}p$ for some u. Since $\mathcal{F}$ is symmetric, uRw. From this and (iii) we conclude that $\mathcal{M}, w \Vdash p$. Which contradicts (ii). For the right to left direction assume that $\mathcal{F}$ is not symmetric. Then, there are worlds x and y such that xRy and $\langle y, x \rangle \notin R$. Now let $\mathcal{M}_1 = \langle W, R, V_1 \rangle$ be the model where $V_1(x, p) = 0$ and $V_1(z, p) = 1$ for all z such that yRz. Since $V_1(x, p) = 0$, then (iv) $\mathcal{M}_1, x \nVdash p$. By definition of V_1 we have that $\mathcal{M}_1, y \nVdash {\smile}p$. From the latter and by xRy, it follows that (v) $\mathcal{M}_1, x \Vdash {\smile}{\smile}p$. Thus, by (iv) and (v), $\mathcal{M}_1, x \nVdash {\smile}{\smile}p \supset p$.

(4) Assume that $\mathcal{F}$ is transitive. Suppose by contradiction that $\mathcal{F} \nVdash {\smile}\beta \supset ({\smile}\alpha \vee {\smile}{\smile}\alpha)$. Then there is a model $\mathcal{M}$ and a world w such that (i) $\mathcal{M}, w \Vdash {\smile}\beta$ and $\mathcal{M}, w \nVdash {\smile}\alpha \vee {\smile}{\smile}\alpha$, that is, (ii) $\mathcal{M}, w \nVdash {\smile}\alpha$ and (iii) $\mathcal{M}, w \nVdash {\smile}{\smile}\alpha$. By (i), there is a v accessible from w. In v, by (ii) and (iii) we have respectively that (iv) $\mathcal{M}, v \Vdash \alpha$ and (v) $\mathcal{M}, v \Vdash {\smile}\alpha$. From the latter, there is a world u such that vRu and $\mathcal{M}, u \nVdash \alpha$. Given $wRvRu$ and (ii), it follows that $\mathcal{M}, u \Vdash \alpha$. This is an absurd.
For the left to right direction, suppose by contraposition that, $\mathcal{F}$ is not transitive, then there are worlds x, y and z such that xRy, yRz and x does not access z. Let $\mathcal{M} = \langle \mathcal{F}, V \rangle$ be a model and p, q propositional variables such that $V(z, p) = V(y, q) = 0$, and for all u accessible from x, let $V(u, p) = 1$. Moreover, let $V(n, p) = 0$ if xRm and mRn. Since xRy and $V(y, q) = 0$, then (a) $V(x, {\smile}q) = 1$. Every world of $\mathcal{M}$ accessible from x satisfies p,

then (b) $V(x, \smile p) = 0$. By the construction of $\mathcal{M}$, $V(t, \smile p) = 1$ for every t such that xRt. Thus, (c) $V(x, \smile\smile p) = 0$. By (b) and (c), it follows that $V(x, \smile p \vee \smile\smile p) = 0$. From this and (a) we conclude that $V(x, \smile q \supset (\smile p \vee \smile\smile p)) = 0$.

(5) Assume that $\mathcal{F}$ is euclidean. Suppose by contradiction that $\mathcal{F} \nVdash \smile\alpha \supset (\smile\smile\alpha \supset \beta)$. Then, there is a model $\mathcal{M}$ and a world w, such that (i) $\mathcal{M}, w \Vdash \smile\alpha$ and (ii) $\mathcal{M}, w \Vdash \smile\smile\alpha$. By (i) there is a world x such that wRx and $\mathcal{M}, x \nVdash \alpha$, by (ii) there is z such that wRz and (iii) $\mathcal{M}, z \nVdash \smile\alpha$. Since wRz, wRx and $\mathcal{F}$ is euclidean, then it follows that zRx. From this and (iii) we conclude that $\mathcal{M}, x \Vdash \alpha$. This is an absurd.
For the other direction assume that $\mathcal{F}$ is not euclidean. Then, there are worlds x, y and z such that xRy, xRz and $\langle y, z\rangle \notin R$. Let $\mathcal{M} = \langle \mathcal{F}, V\rangle$ be the model such that $V(x, q) = V(z, p) = 0$ and $V(m, p) = 1$ for every m such yRm. From the latter we conclude that (i) $V(y, \smile p) = 0$. By xRy and (i) it follows that (ii) $V(x, \smile\smile p) = 1$. From (ii) and $V(x, q) = 0$ it follows that (iii) $V(x, \smile\smile p \supset q) = 0$. Since xRz and $V(z, p) = 0$, then $V(x, \smile p) = 1$. From this and (iii) we conclude that $V(x, \smile p \supset (\smile\smile p \supset q)) = 0$. *QED*

The extension of the minimal *LAB* logic with some of these axioms originate other *LAB* logics. For these logics the following completeness results can be now established:

Proposition 3.4.3.

1. $\vdash_{KT_\smile} \alpha$ *iff* α *is valid in all reflexive frames.*
2. $\vdash_{KB_\smile} \alpha$ *iff* α *is valid in all symmetric frames.*
3. $\vdash_{S4_\smile} \alpha$ *iff* α *is valid in all reflexive and transitive frames.*
4. $\vdash_{S5_\smile} \alpha$ *iff* α *is valid in all reflexive and euclidean frames.*

Proof. In what follows let $\mathcal{F} = \langle W_C, R_C, W_C\rangle$ be the canonical frame. The proof consists in showing that the relation of the canonical frame has the adequate properties.
(1) We have to show that wR_Cw, that is, $\Supset[w] \subseteq w$ for every $w \in W$. Assume that for some formula φ, (i) $\varphi \in \Supset[w]$ and (ii) $\varphi \notin w$. By (i), it follows that $\smile\varphi \notin w$. Since $\smile\varphi \vee \varphi \in w$, by Prop. 3.1.3 we have that $\varphi \in w$, which contradicts (ii). Thus, wRw.

(2) Assume that (i) $\Supset[w] \subseteq v$. We shall show that $\Supset[v] \subseteq w$. Otherwise suppose that there is (ii) $\varphi \in \Supset[v]$ such that (iii) $\varphi \notin w$. By (ii) we have that $\smile\varphi \notin v$. From this, (i) and Prop. 3.1.3, it follows that $\smile\varphi \notin \Supset[w]$, that is $\smile\smile\varphi \in w$. Since $\smile\smile\varphi \supset \varphi \in w$, from the latter we conclude that $\varphi \in w$. This contradicts (ii).

(3) Assume that (i) xRy and (ii) yRz for some $x, y, z \in W$. By (i) and (ii) it follows that (iii) $\ni[x] \subseteq y$ and (iv) $\ni[y] \subseteq z$ respectively. Suppose that $\varphi \in \ni[x]$ for an arbitrary φ. From this it follows that (v) $\smile\varphi \notin x$. Since y is a $\mathcal{L}$-maximal coherent set of formulas then $\bot \notin y$. From the latter and (iii) we have that (vi) $\smile\bot \in x$. The formula $\smile\bot \supset (\smile\varphi \vee \smile\smile\varphi)$, a theorem of $S4_{\smile}$, is an element of x. From this, (v) and Prop. 3.1.3.1 we have that $(\smile\varphi \vee \smile\smile\varphi) \in x$. By Prop. 3.1.3.2 and (vi) it follows that $\smile\smile\varphi \in x$. By this we have that $\smile\varphi \notin \ni[x]$. Now using (i) we have that $\smile\varphi \notin y$, that is, (vii) $\varphi \in \ni[y]$. We conclude from (iv) and (vii) that $\varphi \in z$.

(4) Assume that there are $u, v, w \in W$ such that (i) wRv, (ii) wRu and (iii) $\langle u, v \rangle \notin R$, that is $\ni[u] \subseteq v$. By (i) and (ii) we have (iv) $\ni[w] \subseteq v$ and (v) $\ni[w] \subseteq u$ respectively. By (iii) there is a formula φ such that (vi) $\varphi \in \ni[u]$ and (vii) $\varphi \notin v$. By (vi) it follows that (viii) $\smile\varphi \notin u$. By (vii) and (iii), it follows that $\varphi \notin \ni[w]$, that is (ix) $\smile\varphi \in w$. Since $\smile\varphi \supset (\smile\smile\varphi \supset \bot) \in w$, then $\smile\smile\varphi \supset \bot \in w$. The latter, by definition of $\sim$, is the same that $\sim\smile\smile\varphi$. Since w is a $\mathcal{L}$-maximal coherent set of formulas, then (x) $\smile\smile\varphi \notin w$. From this it follows that $\smile\varphi \in \ni[w]$. Thus, by (v), $\smile\varphi \in u$. This contradicts (viii). *QED*

The results proved in this section will be used to make a comparison between our *LAB* logics and some modal logics.

3.5 *LAB* Logics and Modal Logics

The items 3 and 4 of the Prop. 3.3.1 are the usual way of semantically defining the modalities $\Box$ and $\Diamond$, see((Blackburn; de Rijke; Venema, 2001),(Cresswell; Hughes, 2003)). It is not surprising that the completeness results proved in Propositions 3.4.1 and 3.4.3 show us an alternative way of characterizing some normal modal logics. Both logics $K_{\smile}$ and the normal modal logic K are characterized by the class of all frames. A similar analogy may be established between: the *LAB* logic $KB_{\smile}$ and the logic KB, characterized by the class of symmetric frames; the logic $KT_{\smile}$ and the logic KT by the class of all reflexives frames; $S4_{\smile}$ and $S4$ by the class of all transitive and reflexive frames; $S5_{\smile}$ and $S5$ by the class of all frames based on equivalence relations.

Our paranormal negation $\smile$, as shown in (Marcos, 2005a), is not paracomplete in extensions of the logic KT. But the author, in the same work, studied another paranormal negation $\frown$. That in the *LAB* logics is defined as follows: $\frown\alpha \overset{def}{=} \sim\smile\sim\alpha$. Other connectives, strongly related with this negations, are also studied there: the restoration connectives. One of them is the connective of consistency $\circ$, defined as $\circ\alpha \overset{def}{=} \alpha \supset \sim\smile\alpha$. It is easy to check that there are no model $\mathcal{M}$ neither world w in $\mathcal{M}$ such that $\mathcal{M}$ satisfies the formula $\alpha \wedge \smile\alpha \wedge \circ\alpha$ at w.

The axiom $\smile(\alpha\wedge\beta) \supset (\smile\alpha\vee\smile\beta)$ of *LAB* logics and the theorem $\Box(\alpha\wedge\beta) \supset (\Box\alpha\wedge\Box\beta)$ of *K* represent important properties of this connectives. We will see in next chapter that, in a multiple-conclusion environment, these properties and the property above described to the connective $\circ$, are used to characterize certain types of positive modalities, negative modalities and restoration modalities.

Another way of seeing the next chapter is as the study of the fragment of *LAB* logics without implication, where we will show a completeness result and ways of connecting this new logic with classical logic and De Morgan logic.

4 A Rich Language For Negative Modalities

We study a modal language for negative operators —an intuitionistic-like negation and its paraconsistent dual—added to (bounded) distributive lattices. For each non-classical negation an extra operator is hereby adjoined in order to allow for standard logical inferences to be opportunely restored. We present abstract characterizations and exhibit the main properties of each kind of negative modality, as well as of the associated connectives that express consistency and determinedness at the object-language level. Appropriate sequent-style proof systems and adequate Kripke semantics are also introduced, characterizing the minimal normal logic and a few other basic logics containing such negative modalities and their companions.

4.1 Context

Negationless normal modal logics with box-like and diamond-like operators were studied by Dunn in (Dunn, 1995), where the author obtains completeness results for the systems characterized by the class of all Kripke frames and by a few specific subclasses thereof. In (Celani; Jansana, 1997), Celani & Jansana extend that study so as to cover many other logics, and to that effect they consider Kripke-style semantics based on frames containing two relations —one of them being a preorder, as in intuitionistic logic, allowing for the expression of appropriate heredity conditions. Systems containing analogous negative modalities were studied by Dunn & Zhou, who investigate in (Dunn; Zhou, 2005) modal logics with conjunction, disjunction, an impossibility operator intended to play the role of an intuitionistic-like negation and a non-necessity operator intended to play the role of a paraconsistent negation. Restall, in (Restall, 1997), proposed a combination of positive and negative diamond-like modal operators; one of his aims was to use the resulting system to exhibit examples of modal logics that turn out to be undecidable even in the absence of classical negation. In the present chapter we will study a logic that contains the already mentioned negative modal operators over a strictly positive propositional basis (on a fragment agreed upon by intuitionists and classical logicians) to which we add extra operators that express at the object-language level the very notions of consistency and determinedness that allows one to recover much of the standard logical reasoning even when neither classical negation nor classical implication are available. The mentioned extra modal 'restoration' connectives were first proposed in (Marcos, 2005a). The basic universal logic apparatus used here is based on (Humberstone, 2011) and (Marcos, 2005b), and the proof-theoretical approach to the consistency operator is inherited from (Avron; Konikowska; Zamansky, 2013).

The structure of the chapter is as follows: in Section 4.2 we present the Universal Logic background, including the formulation of properties characterizing negative modalities and the properties that characterize connectives intended to express consistency and determinedness at the object-language level; in Section 4.3 a sequent system is used to define our main and most basic modal system, in which we include rules for introducing the restoration connectives and rules for the interaction between the non-classical negations; in Sections 4.4 and 4.5 the intended Kripke semantics is presented for our full modal language and our deductive system is shown to be sound and complete with respect to this semantics; a few extensions of the basic system are then formulated in Section 4.6; in Section 4.7 we study how the inferences of more standard logic systems may be recovered with the use of our rich modal language, by way of appropriate Derivability Adjustment Theorems; last, in Section 4.8, we briefly comment upon some directions for future research.

4.2 Universal Logic Perspective

Let $\mathcal{L}$ be a standard propositional language. As customary, we shall use small Greek letters to denote arbitrary sentences, and capital Greek letters for sets of sentences of $\mathcal{L}$. A generalized consequence relation (gcr) will here be assumed to be a relation $\rhd \subseteq \mathbf{2}^{\mathcal{L}} \times \mathbf{2}^{\mathcal{L}}$ that enjoys the following universal properties:

(ovl) $\Gamma, \varphi \rhd \varphi, \Delta$

(mon) If $\Gamma_1 \rhd \Delta_1$, then $\Gamma_2, \Gamma_1 \rhd \Delta_1, \Delta_2$

(trn) If $\Gamma_1, \varphi \rhd \Delta_1$ and $\Gamma_2 \rhd \varphi, \Delta_2$, then $\Gamma_1, \Gamma_2 \rhd \Delta_1, \Delta_2$

In writing a statement such as $\Pi \cup \{\pi\} \rhd \varnothing$ in the simplified form $\Pi, \pi \rhd$ we are simply aligning with standard usage from the literature. Here we shall write $\Gamma \blacktriangleright \Delta$ to indicate that $\Gamma \rhd \Delta$ fails, that is, that $\langle\Gamma, \Delta\rangle \notin \rhd$. Furthermore, aiming at a structured outlook on the above properties and on proofs based on them, we shall employ the following graphical representation:

$$\frac{}{\Gamma, \varphi \rhd \varphi, \Delta}\text{(ovl)} \qquad \frac{\Gamma_1 \rhd \Delta_1}{\Gamma_2, \Gamma_1 \rhd \Delta_1, \Delta_2}\text{(mon)} \qquad \frac{\Gamma_1, \varphi \rhd \Delta_1 \qquad \Gamma_2 \rhd \varphi, \Delta_2}{\Gamma_1, \Gamma_2 \rhd \Delta_1, \Delta_2}\text{(trn)}$$

A set $\Sigma \subseteq \mathcal{L}$ will be called a **$\rhd$-theory** if $\varphi \in \Sigma$ whenever $\Sigma \rhd \varphi, \Delta$ for every $\Delta \subseteq \mathcal{L}$. Dually, the set of sentences Σ will be called a **$\rhd$-cotheory** if $\varphi \in \Sigma$ whenever $\Gamma, \varphi \rhd \Sigma$ for every $\Gamma \subseteq \mathcal{L}$. Taking such definitions into account, a **$\rhd$-theory pair** will be any pair $\langle\Sigma_1, \Sigma_0\rangle$ where Σ_1 is a $\rhd$-theory and Σ_0 is a $\rhd$-cotheory. Given two $\rhd$-theory pairs $\Sigma = \langle\Sigma_1, \Sigma_0\rangle$ and $\Pi = \langle\Pi_1, \Pi_0\rangle$, we say that **$\Pi$ extends Σ** if $\Sigma_1 \subseteq \Pi_1$ and $\Sigma_0 \subseteq \Pi_0$ — we denote this by $\Sigma \subseteq \Pi$. In addition, fixed a given gcr $\rhd$, a theory pair $\Sigma = \langle\Sigma_1, \Sigma_0\rangle$ is called **unconnected** if $\Sigma_1 \blacktriangleright \Sigma_0$, and is called **closed** if $\Sigma_1 \cup \Sigma_0 = \mathcal{L}$. A gcr is called **trivial** if it does not allow for any unconnected theory pair.

A gcr $\rhd$ is called **finitary** if it enjoys the following property:

(fin) If $\Gamma \rhd \Delta$, then there are finite sets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\Gamma' \rhd \Delta'$

For finitary gcrs, thus, a connected theory pair extends some *finite* connected theory pair.

We confirm next that finitary gcrs enjoy the following property — a version of the well-known *Lindenbaum-Asser Lemma* (cf. (Segerberg, 1982)):

Proposition 4.2.1. *Let $\rhd$ be a finitary gcr. Then every unconnected $\rhd$-theory pair can be extended into a closed unconnected $\rhd$-theory pair.*

Proof. Assume $\Gamma \blacktriangleright \Delta$. Let $\mathcal{E}$ be the collection of unconnected extensions of $\langle\Gamma, \Delta\rangle$, partially ordered by inclusion, and let $\mathfrak{C} = \{\langle\mathfrak{C}^i_1, \mathfrak{C}^i_0\rangle\}_{i\in I}$ be some chain (a totally ordered set) on $\mathcal{E}$. We claim that $\bigcup\mathfrak{C} = \langle\bigcup_{i\in I}\mathfrak{C}^i_1, \bigcup_{i\in I}\mathfrak{C}^i_0\rangle$ is an upper bound for $\mathcal{E}$, i.e., we claim that $\Pi \subseteq \bigcup\mathfrak{C}$ for every $\Pi \in \mathcal{E}$ (which is obvious) and also claim that $(*)$ $\bigcup\mathfrak{C} \in \mathcal{E}$.

We check $(*)$. Where $\Lambda = \langle\Lambda_1, \Lambda_0\rangle$ is a $\rhd$-theory pair, let $\mathrm{Fin}(\Lambda)$ denote the set of $\rhd$-theory pairs $\langle\widehat{\Lambda_1}, \widehat{\Lambda_0}\rangle$ where $\widehat{\Lambda_1} \cup \widehat{\Lambda_0}$ is a finite set and Λ extends $\langle\widehat{\Lambda_1}, \widehat{\Lambda_0}\rangle$. Consider an arbitrary $\Phi = \langle\Phi_1, \Phi_0\rangle$ such that $\Phi \in \mathrm{Fin}(\bigcup\mathfrak{C})$. Then there is some $\mathfrak{C}^k \in \mathfrak{C}$ such that $\Phi \subseteq \mathfrak{C}^k$. As $\mathfrak{C}^k_1 \blacktriangleright \mathfrak{C}^k_0$, by (mon) we conclude that $\Phi_1 \blacktriangleright \Phi_0$. By (fin) it follows that $\bigcup\mathfrak{C}$ is unconnected. By Zorn's Lemma, if every chain in a partially ordered set has an upper bound, then there is a maximal element in that set; so, we conclude that $\mathcal{E}$ must have a maximal unconnected element $\langle\Gamma^\star, \Delta^\star\rangle \supseteq \langle\Gamma, \Delta\rangle$. To see that $\langle\Gamma^\star, \Delta^\star\rangle$ is indeed closed, suppose there is some $\varphi \in \mathcal{L}$ such that neither $\langle\Gamma^\star \cup \{\varphi\}, \Delta^\star\rangle$ nor $\langle\Gamma^\star, \Delta^\star \cup \{\varphi\}\rangle$ are unconnected. Then, by (trn), it would follow that $\Gamma^\star \rhd \Delta^\star$. *QED*

From this point on we consider some language specifics, concerning connectives of $\mathcal{L}$. A binary connective $\wedge$ in $\mathcal{L}$ will be called a **$\rhd$-ordinary conjunction** when it satisfies

$$\text{(oC)} \quad \Gamma, \varphi \wedge \psi \rhd \Delta \quad \text{iff} \quad \Gamma, \varphi, \psi \rhd \Delta$$

for arbitrary sentences $\varphi, \psi \in \mathcal{L}$ and arbitrary contexts $\Gamma, \Delta \subseteq \mathcal{L}$. In other words, to have a classic-like behavior, a conjunction will be expected to internalize, at the object-level, the meta-level commas that appear in the left-hand side of $\rhd$. Dually, a binary connective $\vee$ is called a **$\rhd$-ordinary disjunction** when it satisfies

$$\text{(oD)} \quad \Gamma \rhd \varphi \vee \psi, \Delta \quad \text{iff} \quad \Gamma \rhd \varphi, \psi, \Delta$$

In addition, a **$\rhd$-ordinary top** and a **$\rhd$-ordinary bottom** are 0-ary connectives $\top$ and $\bot$ satisfying

$$\text{(oT)} \quad \Gamma, \top \rhd \Delta \quad \text{iff} \quad \Gamma \rhd \Delta \qquad\qquad \text{(oB)} \quad \Gamma \rhd \bot, \Delta \quad \text{iff} \quad \Gamma \rhd \Delta$$

From such definitions one may easily check for instance that:

Proposition 4.2.2. *For any $\rhd$-ordinary conjunction $\wedge$ and any $\rhd$-ordinary disjunction $\vee$, the following rule-statements hold:*

$$\frac{\Gamma_1 \rhd \alpha, \Delta_1 \qquad \Gamma_2 \rhd \beta, \Delta_2}{\Gamma_1, \Gamma_2 \rhd \alpha \wedge \beta, \Delta_1, \Delta_2}\,\text{(Cj1)} \qquad\qquad \frac{\Gamma_1, \alpha \rhd \Delta_1 \qquad \Gamma_2, \beta \rhd \Delta_2}{\Gamma_1, \Gamma_2, \alpha \vee \beta \rhd \Delta_1, \Delta_2}\,\text{(Dj1)}$$

Proof. The proofs proceed as follows. We starting with rule-statement (Cj1):

Proof using the properties of gcr	Graphical representation of the proof
Assume (1) $\Gamma_1 \rhd \alpha, \Delta_1$ and (2) $\Gamma_2 \rhd \beta, \Delta_2$. By (ovl) we know that (3) $\alpha \wedge \beta \rhd \alpha \wedge \beta$. By (3) and (oC) it follows that (4) $\alpha, \beta \rhd \alpha \wedge \beta$. Using (trn) on (1) and (4) we obtain (5) $\Gamma_1, \beta \rhd \alpha \wedge \beta, \Delta_1$. From (5), (2) and (trn) we conclude that $\Gamma_1, \Gamma_2 \rhd \alpha \wedge \beta, \Delta_1, \Delta_2$.	$\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha \wedge \beta \rhd \alpha \wedge \beta}\,{\scriptstyle (ovl)}}{\alpha, \beta \rhd \alpha \wedge \beta}\,{\scriptstyle (oC)} \quad \Gamma_1 \rhd \alpha, \Delta_1}{\Gamma_1, \beta \rhd \alpha \wedge \beta, \Delta_1}\,{\scriptstyle (trn)} \quad \Gamma_2 \rhd \beta, \Delta_2}{\Gamma_1, \Gamma_2 \rhd \alpha \wedge \beta, \Delta_1, \Delta_2}\,{\scriptstyle (trn)}$

For rule-statement (Dj1) we rely directly on the corresponding tree-like presentation:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha \vee \beta \rhd \alpha \vee \beta}\,{\scriptstyle (ovl)}}{\alpha \vee \beta \rhd \alpha, \beta}\,{\scriptstyle (oD)} \quad \Gamma_1, \alpha \rhd \Delta_1}{\Gamma_1, \alpha \vee \beta \rhd \beta, \Delta_1}\,{\scriptstyle (trn)} \quad \Gamma_2, \beta \rhd \Delta_2}{\Gamma_1, \Gamma_2, \alpha \vee \beta \rhd \Delta_1, \Delta_2}\,{\scriptstyle (trn)}$$

QED

An immediate offshoot of the above result is that theories are closed under ordinary conjunctions and cotheories are closed under ordinary disjunctions:

Corollary 4.2.1. *Let $\wedge$ be a $\rhd$-ordinary conjunction, $\vee$ be a $\rhd$-ordinary disjunction, $\top$ be a $\rhd$-ordinary top and $\bot$ be a $\rhd$-ordinary bottom. Consider a $\rhd$-theory pair $\langle \Sigma_1, \Sigma_0 \rangle$. Then:*

(i) *If $\varphi \in \Sigma_1$ and $\psi \in \Sigma_1$, then $\varphi \wedge \psi \in \Sigma_1$*

(ii) *If $\varphi \in \Sigma_0$ and $\psi \in \Sigma_0$, then $\varphi \vee \psi \in \Sigma_0$*

(iii) $\top \in \Sigma_1$

(iv) $\bot \in \Sigma_0$

This will be very useful later on, in particular in Section 4.5. For closed theories, a further important result may be proven:

Proposition 4.2.3. *Let $\wedge$ be a $\rhd$-ordinary conjunction and $\vee$ be a $\rhd$-ordinary disjunction, and let $\langle \Sigma_1, \Sigma_0 \rangle$ be a closed unconnected $\rhd$-theory pair. Then:*

1. *If $\varphi \vee \psi \in \Sigma_1$, then $\varphi \in \Sigma_1$ or $\psi \in \Sigma_1$*
2. *If $\varphi \wedge \psi \in \Sigma_0$, then $\varphi \in \Sigma_0$ or $\psi \in \Sigma_0$*

Proof. For item (1), suppose by contraposition that $\varphi \notin \Sigma_1$ and $\psi \notin \Sigma_1$. By closure it follows that $\varphi \in \Sigma_0$ and $\psi \in \Sigma_0$. By the definition of cotheory, this means that $\Gamma, \varphi \rhd \Sigma_0$

and $\Gamma, \psi \rhd \Sigma_0$ for every $\Gamma \subseteq \mathcal{L}$. For any arbitrary such Γ it follows by (Dj1) that $\Gamma, \varphi \vee \psi \rhd \Sigma_0$. So, given that Σ_0 is a cotheory, we have $\varphi \vee \psi \in \Sigma_0$, and by unconnectedness it follows that $\varphi \vee \psi \notin \Sigma_1$. The proof of item (2) uses (Cj1). *QED*

Fix now a unary connective # in $\mathcal{L}$. We say that # is **$\rhd$-preserving** if it satisfies

$$\text{(Pvs\#)} \quad \varphi \rhd \psi \text{ implies } \#\varphi \rhd \#\psi$$

and say that # is **$\rhd$-reversing** if

$$\text{(Rvs\#)} \quad \varphi \rhd \psi \text{ implies } \#\psi \rhd \#\varphi$$

Following (Marcos, 2005b), the **minimal conditions** we will demand for calling # a **negation** consist on the existence of sentences φ and ψ such that $\#\varphi \blacktriangleright \varphi$ and $\psi \blacktriangleright \#\psi$, that is, such that the theory pairs $\langle \#\varphi, \varphi \rangle$ and $\langle \psi, \#\psi \rangle$ are unconnected. The underlying intuition is that negation should bring about some 'inversion' with respect to the underlying notion of consequence. It should be noticed that, in principle, a negation abiding to such minimal conditions need not be $\rhd$-reversing —yet, (Rvs#) is a typical and desirable property of *modal* negations such as the ones we will be studying in the present chapter. The following result introduces some properties that will play an important role in what follows:

Proposition 4.2.4. *Assume $\wedge$ to be a $\rhd$-ordinary conjunction and $\vee$ to be a $\rhd$-ordinary disjunction. For any $\rhd$-preserving connective #, the following statements hold:*

$$\text{(PM1.1\#)} \quad \#(\varphi \wedge \psi) \rhd \#\varphi \wedge \#\psi \qquad\qquad \text{(PM2.1\#)} \quad \#\varphi \vee \#\psi \rhd \#(\varphi \vee \psi)$$

If # is $\rhd$-reversing, the following alternative statements hold instead:

$$\text{(DM1.1\#)} \quad \#(\varphi \vee \psi) \rhd \#\varphi \wedge \#\psi \qquad\qquad \text{(DM2.1\#)} \quad \#\varphi \vee \#\psi \rhd \#(\varphi \wedge \psi)$$

Proof. The proofs below use rule-statements (Cj1) and (Dj1) from Prop. 4.2.2.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \rhd \varphi}\,\text{(ovl)}}{\varphi, \psi \rhd \varphi}\,\text{(mon)}}{\varphi \wedge \psi \rhd \varphi}\,\text{(oC)}}{\#(\varphi \wedge \psi) \rhd \#\varphi}\,\text{(Pvs\#)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\psi \rhd \psi}\,\text{(ovl)}}{\varphi, \psi \rhd \psi}\,\text{(mon)}}{\varphi \wedge \psi \rhd \psi}\,\text{(oC)}}{\#(\varphi \wedge \psi) \rhd \#\psi}\,\text{(Pvs\#)}}{\#(\varphi \wedge \psi) \rhd \#\varphi \wedge \#\psi}\,\text{(Cj1)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \rhd \varphi}\,\text{(ovl)}}{\varphi \rhd \varphi, \psi}\,\text{(mon)}}{\varphi \rhd \varphi \vee \psi}\,\text{(oD)}}{\#(\varphi \vee \psi) \rhd \#\varphi}\,\text{(Rvs\#)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\psi \rhd \psi}\,\text{(ovl)}}{\psi \rhd \varphi, \psi}\,\text{(mon)}}{\psi \rhd \varphi \vee \psi}\,\text{(oD)}}{\#(\varphi \vee \psi) \rhd \#\psi}\,\text{(Rvs\#)}}{\#(\varphi \vee \psi) \rhd \#\varphi \wedge \#\psi}\,\text{(Cj1)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \rhd \varphi}\,\text{(ovl)}}{\varphi \rhd \varphi, \psi}\,\text{(mon)}}{\varphi \rhd \varphi \vee \psi}\,\text{(oD)}}{\#\varphi \rhd \#(\varphi \vee \psi)}\,\text{(Pvs\#)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\psi \rhd \psi}\,\text{(ovl)}}{\psi \rhd \varphi, \psi}\,\text{(mon)}}{\psi \rhd \varphi \vee \psi}\,\text{(oD)}}{\#\psi \rhd \#(\varphi \vee \psi)}\,\text{(Pvs\#)}}{\#\varphi \vee \#\psi \rhd \#(\varphi \vee \psi)}\,\text{(Dj1)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \rhd \varphi}\,\text{(ovl)}}{\varphi, \psi \rhd \varphi}\,\text{(mon)}}{\varphi \wedge \psi \rhd \varphi}\,\text{(oC)}}{\#\varphi \rhd \#(\varphi \wedge \psi)}\,\text{(Rvs\#)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\psi \rhd \psi}\,\text{(ovl)}}{\varphi, \psi \rhd \psi}\,\text{(mon)}}{\varphi \wedge \psi \rhd \psi}\,\text{(oC)}}{\#\psi \rhd \#(\varphi \wedge \psi)}\,\text{(Rvs\#)}}{\#\varphi \vee \#\psi \rhd \#(\varphi \wedge \psi)}\,\text{(Dj1)}$$

QED

In what follows we shall say that # has **type** $\boxplus$ (read 'box-plus') if it respects

$$\text{(PM1.2\#)} \quad \#\varphi \land \#\psi \rhd \#(\varphi \land \psi)$$

and say that # has **type** $\diamond\!+$ ('diamond-plus') if it respects

$$\text{(PM2.2\#)} \quad \#(\varphi \lor \psi) \rhd \#\varphi \lor \#\psi$$

When a $\rhd$-ordinary top $\top$ is available, we will expect a **full** type $\boxplus$ connective # to also respect

$$\text{(PT\#)} \quad \rhd \#\top$$

Given an $\rhd$-ordinary bottom $\bot$, a **full** type $\diamond\!+$ connective # is also to respect

$$\text{(PB\#)} \quad \#\bot \rhd$$

Dually, we shall say that # has **type** $\boxminus$ (read as 'box-minus') if it respects

$$\text{(DM1.2\#)} \quad \#\varphi \land \#\psi \rhd \#(\varphi \lor \psi)$$

and say that # has **type** $\diamond\!-$ ('diamond-minus') if it respects

$$\text{(DM2.2\#)} \quad \#(\varphi \land \psi) \rhd \#\varphi \lor \#\psi$$

When a $\rhd$-ordinary top $\top$ or a $\rhd$-ordinary bot $\bot$ are available, a **full** type $\boxminus$ connective # will be expected to respect

$$\text{(DB\#)} \quad \rhd \#\bot$$

and a **full** type $\diamond\!-$ connective # will be expected to respect

$$\text{(DT\#)} \quad \#\top \rhd$$

We now turn to properties induced by our main (non-classical) negations and use them to characterize the restoration connectives that will accompany them. Here, given some specific sentence φ, the gcr $\rhd$ will be called **#-consistent with respect to $\boldsymbol{\varphi}$** in case it satisfies

$$\text{(Cns}\#\varphi) \quad \Gamma, \varphi, \#\varphi \rhd \Delta$$

for any choice of contexts Γ and Δ. Dually, the gcr $\rhd$ will be called **#-determined with respect to φ** in case it satisfies

$$\text{(Dtm}\#\varphi) \quad \Gamma \rhd \#\varphi, \varphi, \Delta$$

for any choice of contexts Γ and Δ. A gcr will be called **#-inconsistent** if there is some sentence φ with respect to which (Cns#φ) fails, and will be called **#-undetermined** if there is some sentence φ with respect to which (Dtm#φ) fails. If some negation # is available such that $\rhd$ turns out to be both #-consistent and #-determined with respect to all sentences, such # will be called a **$\rhd$-ordinary negation**. For #-inconsistent and for #-undetermined gcrs it will often be useful to have a way of internalizing, at the object-level, the corresponding notions of consistency and determinedness. To that effect, a **$\rhd$-ordinary consistency** connective will be defined as a unary symbol $\textcircled{\#}$ satisfying

$$\text{(GC\#)} \quad \Gamma \rhd \textcircled{\#}\varphi, \Delta \text{ iff } \Gamma, \varphi, \#\varphi \rhd \Delta$$

for any choice of contexts Γ and Δ and any sentence φ. Analogously, a **$\rhd$-ordinary determinedness** connective will be defined as a unary symbol $\textcircled{\#}$ satisfying

$$\text{(GD\#)} \quad \Gamma, \textcircled{\#}\varphi \rhd \Delta \text{ iff } \Gamma \rhd \#\varphi, \varphi, \Delta$$

A useful alternative abstract characterization of such new connectives is exhibited in what follows:

Proposition 4.2.5. *Let $\rhd$ be a* gcr*. Then:*

[EQ1] *Clause* (GC#) *is equivalent to the following three clauses taken together:*

$$\text{(Cb\#)}\ \Gamma, \textcircled{\#}\varphi, \#\varphi, \varphi \rhd \Delta \qquad \text{(Ck1\#)}\ \Gamma \rhd \varphi, \textcircled{\#}\varphi, \Delta \qquad \text{(Ck2\#)}\ \Gamma \rhd \#\varphi, \textcircled{\#}\varphi, \Delta$$

[EQ2] *Clause* (GD#) *is equivalent to the following three clauses taken together:*

$$\text{(Db\#)}\ \Gamma \rhd \varphi, \#\varphi, \textcircled{\#}\varphi, \Delta \qquad \text{(Dk1\#)}\ \Gamma, \textcircled{\#}\varphi, \varphi \rhd \Delta \qquad \text{(Dk2\#)}\ \Gamma, \textcircled{\#}\varphi, \#\varphi \rhd \Delta$$

Proof. Rule-statement (GC#) may be split in two halves, namely:

$$\frac{\Gamma, \varphi, \#\varphi \rhd \Delta}{\Gamma \rhd \textcircled{\#}\varphi, \Delta}\,\text{(GC1\#)} \qquad\qquad \frac{\Gamma \rhd \textcircled{\#}\varphi, \Delta}{\Gamma, \varphi, \#\varphi \rhd \Delta}\,\text{(GC2\#)}$$

These will help in attaining our goals below. We start by verifying [EQ1].

[Part 1] Assume (GC1#) and (GC2#) to hold. Then notice that:

$$\dfrac{\dfrac{}{\Gamma, \textcircled{\#}\varphi \rhd \textcircled{\#}\varphi, \Delta}\,(\text{ovl})}{\Gamma, \textcircled{\#}\varphi, \#\varphi, \varphi \rhd \Delta}\,(\text{GC2\#}) \quad\Bigg|\quad \dfrac{\dfrac{\dfrac{}{\Gamma, \varphi \rhd \varphi, \Delta}\,(\text{ovl})}{\Gamma, \varphi, \#\varphi \rhd \varphi, \Delta}\,(\text{mon})}{\Gamma \rhd \varphi, \textcircled{\#}\varphi, \Delta}\,(\text{GC1\#}) \quad\Bigg|\quad \dfrac{\dfrac{\dfrac{}{\Gamma, \#\varphi \rhd \#\varphi, \Delta}\,(\text{ovl})}{\Gamma, \varphi, \#\varphi \rhd \#\varphi, \Delta}\,(\text{mon})}{\Gamma \rhd \#\varphi, \textcircled{\#}\varphi, \Delta}\,(\text{GC1\#})$$

[Part 2] Assume (Cb#), (Ck1#) and (Ck2#) to hold. Then notice that:

$$\dfrac{\dfrac{\Gamma, \varphi, \#\varphi \rhd \Delta \qquad \dfrac{}{\Gamma \rhd \varphi, \textcircled{\#}\varphi, \Delta}\,(\text{Ck1\#})}{\Gamma, \#\varphi \rhd \textcircled{\#}\varphi, \Delta}\,(\text{trn}) \qquad \dfrac{}{\Gamma \rhd \#\varphi, \textcircled{\#}\varphi, \Delta}\,(\text{Ck2\#})}{\Gamma \rhd \textcircled{\#}\varphi, \Delta}\,(\text{trn}) \quad\Bigg|\quad \dfrac{\Gamma \rhd \textcircled{\#}\varphi, \Delta \qquad \dfrac{}{\Gamma, \textcircled{\#}\varphi, \#\varphi, \varphi \rhd \Delta}\,(\text{Cb\#})}{\Gamma, \varphi, \#\varphi \rhd \Delta}\,(\text{trn})$$

Verifying equivalence [EQ2], now, is an entirely analogous exercise, which we shall leave to the interested reader. [Part 1] Assume (GD1#) and (GD2#)

$$\dfrac{\dfrac{}{\Gamma, \textcircled{\#}\varphi \rhd \textcircled{\#}\varphi, \Delta}\,(\text{ovl})}{\Gamma \rhd \textcircled{\#}\varphi, \#\varphi, \varphi, \Delta}\,(\text{GD2\#}) \quad\Bigg|\quad \dfrac{\dfrac{\dfrac{}{\Gamma, \varphi \rhd \varphi, \Delta}\,(\text{ovl})}{\Gamma, \varphi \rhd \varphi, \#\varphi, \Delta}\,(\text{mon})}{\Gamma, \varphi, \textcircled{\#}\varphi \rhd \Delta}\,(\text{GD1\#}) \quad\Bigg|\quad \dfrac{\dfrac{\dfrac{}{\Gamma, \#\varphi \rhd \#\varphi, \Delta}\,(\text{ovl})}{\Gamma, \#\varphi \rhd \varphi, \#\varphi, \Delta}\,(\text{mon})}{\Gamma, \#\varphi, \textcircled{\#}\varphi \rhd \Delta}\,(\text{GD1\#})$$

[Part 2] Assume (Cb), (Ck1) and (Ck2).

$$\dfrac{\dfrac{\Gamma \rhd \varphi, \#\varphi, \Delta \qquad \dfrac{}{\Gamma, \varphi, \textcircled{\#}\varphi \rhd \Delta}\,(\text{Dk1})}{\Gamma, \textcircled{\#}\varphi \rhd \#\varphi, \Delta}\,(\text{trn}) \qquad \dfrac{}{\Gamma, \#\varphi, \textcircled{\#}\varphi \rhd \Delta}\,(\text{Dk2})}{\Gamma, \textcircled{\#}\varphi \rhd \Delta}\,(\text{trn}) \quad\Bigg|\quad \dfrac{\Gamma, \textcircled{\#}\varphi \rhd \Delta \qquad \dfrac{}{\Gamma \rhd \textcircled{\#}\varphi, \#\varphi, \varphi, \Delta}\,(\text{Db})}{\Gamma \rhd \varphi, \#\varphi, \Delta}\,(\text{trn})$$

QED

From this point on we shall fix a set of sentences $\mathcal{L}$ inductively defined by:

$$\varphi ::= p \mid (\varphi_1 \wedge \varphi_2) \mid (\varphi_1 \vee \varphi_2) \mid (\frown\varphi) \mid (\smile\varphi) \mid (\textcircled{$\frown$}\varphi) \mid (\textcircled{$\smile$}\varphi)$$

where p ranges over a denumerable set $\mathcal{P}$ of propositional variables, both $\frown$ and $\smile$ are symbols intended to represent negations, and the symbols $\textcircled{$\smile$}$ and $\textcircled{$\frown$}$ are intended to represent the restoration connectives that will be associated to the latter negation symbols. Fixed an arbitrary sentence φ, we will define $\top$ as an abbreviation for $\varphi \vee \frown\varphi \vee \textcircled{$\frown$}\varphi$ and will define $\bot$ as short for $\varphi \wedge \smile\varphi \wedge \textcircled{$\smile$}\varphi$. In the following sections we shall introduce a convenient deductive system involving the above connectives, and provide subsequently a characteristic modal interpretation for them. Using such proof system and such interpretation we will be able to easily classify each connective of $\mathcal{L}$ with respect to the terminology introduced above.

It is worth adding a few words on the connection between inconsistency, undeterminedness and the perhaps more usual terms 'paraconsistency' and 'paracompleteness', now very common in the literature on non-classical negations. Suppose the language of a given gcr $\rhd$ contains a symbol # satisfying the minimal conditions to be called a negation. In that case, we say that $\rhd$ is **#-paraconsistent** if there are sentences φ and ψ

such that $\varphi, \#\varphi \blacktriangleright \psi$, and say that $\rhd$ is **#-paracomplete** if there are sentences φ and ψ such that $\varphi \blacktriangleright \#\psi, \psi$. Obviously, in case a #-ordinary bottom $\bot$ is available, #-paraconsistency simply coincides with #-inconsistency, and in case a #-ordinary top $\top$ is available, #-paracompleteness coincides with #-undeterminedness. Paraconsistent logics equipped with ordinary consistency connnectives constitute particularly interesting examples of the so-called **logics of formal inconsistency**, or more simply **LFI**s (check (Carnielli; Marcos, 2002; Carnielli; Coniglio; Marcos, 2007)). Their duals, paracomplete logics with ordinary determinedness connectives, are called **logics of formal undeterminedness**, or **LFU**s.

As an additional useful matter of notation for the next sections, given $T \subseteq \mathcal{L}$ and any unary connective $\circledcirc$ we shall by $\circledcirc[T]$ denote the set $\{\circledcirc\varphi : \varphi \in T\}$, and by $\circledcirc^{-1}[T]$ denote the set $\{\varphi : \circledcirc\varphi \in T\}$. By $\overline{T}$ we will denote the complement of T relative to $\mathcal{L}$.

4.3 Proof-theoretical Presentation

We will introduce in what follows our main sequent systems, namely, proof formalisms with each rule has the format $\dfrac{\{A_i \Rightarrow B_i : i \in I\}}{A \Rightarrow B}$ (rule) where each A_k and each B_k, $k \in I$, represents a finite sequence of sentences of $\mathcal{L}$ and where I is a finite set of indices. As usual, given a collection $\mathcal{R}$ of inference rules, a deductive system is associated to $\mathcal{R}$ by defining $\Gamma \vdash \Delta$ to hold if there are finite sets $A \subseteq \Gamma$ and $B \subseteq \Delta$ such that $A \Rightarrow B$ is derivable from the rules in $\mathcal{R}$. In what follows we impose the standard structural rules, defining the system $\mathcal{S}$:

$$\frac{}{\varphi \Rightarrow \varphi}\ (\text{id}) \qquad \frac{A_1, \varphi \Rightarrow B_1 \qquad A_2 \Rightarrow \varphi, B_2}{A_1, A_2 \Rightarrow B_1, B_2}\ (\text{cut})$$

$$\frac{A \Rightarrow B}{A, \varphi \Rightarrow B}\ (\text{W}/) \qquad \frac{A \Rightarrow B}{A \Rightarrow \varphi, B}\ (/\text{W})$$

Such rules and the very definition of $\vdash$ are obviously sufficient to guarantee that the corresponding deductive system is a finitary gcr. As it is well-known, the system $\mathcal{DL}$ for distributive lattices is obtained from $\mathcal{S}$ by adding the standard rules for (classical) conjunction and disjunction:

$$\frac{A, \varphi, \psi \Rightarrow B}{A, \varphi \wedge \psi \Rightarrow B}\ (\wedge/) \qquad \frac{A \Rightarrow \varphi, B \qquad A \Rightarrow \psi, B}{A \Rightarrow \varphi \wedge \psi, B}\ (/\wedge)$$

$$\frac{A, \varphi \Rightarrow B \qquad A, \psi \Rightarrow B}{A, \varphi \vee \psi \Rightarrow B}\ (\vee/) \qquad \frac{A \Rightarrow \varphi, \psi, B}{A \Rightarrow \varphi \vee \psi, B}\ (/\vee)$$

Where $\vdash_{\text{dl}}$ is the gcr associated to $\mathcal{DL}$, the interplay between the structural rules and the logical rules allows us to easily check that both $\wedge$ and $\vee$ are $\vdash_{\text{dl}}$-ordinary, as well as to derive the usual distributivity rules involving the connectives $\wedge$ and $\vee$ — namely, to derive both $\varphi \wedge (\psi \vee \chi) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$ and $(\varphi \wedge \psi) \vee (\varphi \wedge \chi) \Rightarrow \varphi \wedge (\psi \vee \chi)$, as well as their duals, exchanging the roles of $\wedge$ and $\vee$.

Our main system $\mathcal{K}^{\mathrm{n}}$ adds to $\mathcal{DL}$ the following logical rules involving the remaining connectives of the language $\mathcal{L}$:

$$\dfrac{A \Rightarrow \varphi, B \qquad A \Rightarrow \smile\varphi, B}{A, \circledsmile\varphi \Rightarrow B}\ (\circledsmile/) \qquad \dfrac{A, \varphi, \smile\varphi \Rightarrow B}{A \Rightarrow \circledsmile\varphi, B}\ (/\circledsmile)$$

$$\dfrac{A \Rightarrow \varphi, \frown\varphi, B}{A, \circledfrown\varphi \Rightarrow B}\ (\circledfrown/) \qquad \dfrac{A, \varphi \Rightarrow B \qquad A, \frown\varphi \Rightarrow B}{A \Rightarrow \circledfrown\varphi, B}\ (/\circledfrown)$$

$$\dfrac{A \Rightarrow \varphi, B}{\frown[B], \smile\varphi \Rightarrow \smile[A]}\ (\smile\frown) \qquad \dfrac{A, \varphi \Rightarrow B}{\frown[B] \Rightarrow \frown\varphi, \smile[A]}\ (\frown\smile)$$

Using the structural rules and the rules for $\circledsmile$ and $\circledfrown$, it easily follows that (as in Prop. 4.2.5):

Proposition 4.3.1. *The following sequents are derivable in* $\mathcal{K}^{\mathrm{n}}$*:*

(GCb$\smile$)	$\circledsmile\varphi, \smile\varphi, \varphi \Rightarrow$	(GDb$\frown$)	$\Rightarrow \varphi, \frown\varphi, \circledfrown\varphi$
(GCk1$\smile$)	$\Rightarrow \varphi, \circledsmile\varphi$	(GDk1$\frown$)	$\circledfrown\varphi, \varphi \Rightarrow$
(GCk2$\smile$)	$\Rightarrow \smile\varphi, \circledsmile\varphi$	(GDk2$\frown$)	$\circledfrown\varphi, \frown\varphi \Rightarrow$

Proof. Witnessing derivations may be built as follows:

(GCb$\smile$) $$\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\ (\mathrm{id})}{\smile\varphi, \varphi \Rightarrow \varphi}\ (\mathrm{W}/) \qquad \dfrac{\dfrac{}{\smile\varphi \Rightarrow \smile\varphi}\ (\mathrm{id})}{\smile\varphi, \varphi \Rightarrow \smile\varphi}\ (\mathrm{W}/)}{\circledsmile\varphi, \smile\varphi, \varphi \Rightarrow}\ (\circledsmile/)$$

(GDb$\frown$) $$\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\ (\mathrm{id})}{\varphi \Rightarrow \varphi, \frown\varphi}\ (/\mathrm{W}) \qquad \dfrac{\dfrac{}{\frown\varphi \Rightarrow \frown\varphi}\ (\mathrm{id})}{\frown\varphi \Rightarrow \varphi, \frown\varphi}\ (/\mathrm{W})}{\Rightarrow \circledfrown\varphi, \frown\varphi, \varphi}\ (\circledsmile/)$$

(GCk1$\smile$) $$\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\ (\mathrm{id})}{\varphi, \smile\varphi \Rightarrow \varphi}\ (\mathrm{W}/)}{\Rightarrow \varphi, \circledsmile\varphi}\ (/\circledsmile)$$

(GDk1$\frown$) $$\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\ (\mathrm{id})}{\varphi \Rightarrow \varphi, \frown\varphi}\ (/\mathrm{W})}{\varphi, \circledfrown\varphi \Rightarrow}\ (\circledfrown/)$$

(GCk2$\smile$) $$\dfrac{\dfrac{\dfrac{}{\smile\varphi \Rightarrow \smile\varphi}\ (\mathrm{id})}{\smile\varphi, \varphi \Rightarrow \smile\varphi}\ (\mathrm{W}/)}{\Rightarrow \smile\varphi, \circledsmile\varphi}\ (/\circledsmile)$$

(GDk2$\frown$) $$\dfrac{\dfrac{\dfrac{}{\frown\varphi \Rightarrow \frown\varphi}\ (\mathrm{id})}{\frown\varphi \Rightarrow \varphi, \frown\varphi}\ (/\mathrm{W})}{\frown\varphi, \circledfrown\varphi \Rightarrow}\ (\circledfrown/)$$

QED

Recalling the appropriate definitions from Section 4.2 and substituting $\vdash_{\mathrm{n}}$ for $\rhd$, one may easily check in $\mathcal{K}^{\mathrm{n}}$ the following assertions as derived rules:

Proposition 4.3.2.

1. *$\perp$ is a $\vdash_{\mathrm{n}}$-ordinary bottom and $\top$ is a $\vdash_{\mathrm{n}}$-ordinary top*

2. $\frown$ *is a full type* [-] $\vdash_n$*-reversing connective, and*
   $\smile$ *is a full type* <-> $\vdash_n$*-reversing connective*

3. $\circleddash$ *is a* $\vdash_n$*-ordinary determinedness connective, and*
   $\ominus$ *is a* $\vdash_n$*-ordinary consistency connective*

Proof. [$\bot$ is a $\vdash_n$-ordinary bottom]

$$\dfrac{A \Rightarrow B}{A \Rightarrow \bot, B}\,(/W) \qquad \dfrac{\dfrac{A \Rightarrow \bot, B}{A \Rightarrow \varphi \wedge \smile\varphi \wedge \ominus\varphi, B}\,(\text{def. }\bot) \quad \dfrac{\dfrac{\dfrac{}{\varphi, \smile\varphi, \ominus\varphi \Rightarrow}\,(\text{GCb}\smile)}{\varphi \wedge \smile\varphi, \ominus\varphi \Rightarrow}\,(\wedge/)}{\varphi \wedge \smile\varphi \wedge \ominus\varphi \Rightarrow}\,(\wedge/)}{A \Rightarrow B}\,(\text{cut})$$

[$\top$ is a $\vdash_n$-ordinary top]

$$\dfrac{A \Rightarrow B}{A, \top \Rightarrow B}\,(W/) \qquad \dfrac{\dfrac{A, \top \Rightarrow B}{A, \varphi \vee \frown\varphi \vee \circleddash\varphi \Rightarrow B}\,(\text{def. }\top) \quad \dfrac{\dfrac{\dfrac{}{\Rightarrow \varphi, \frown\varphi, \circleddash\varphi}\,(\text{GDb}\frown)}{\Rightarrow \varphi \vee \frown\varphi, \circleddash\varphi}\,(/\vee)}{\Rightarrow \varphi \vee \frown\varphi \vee \circleddash\varphi}\,(/\vee)}{A \Rightarrow B}\,(\text{cut})$$

[$\frown$ is a full type [-] $\vdash_n$-reversing connective]

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi, \smile\varphi, \ominus\varphi \Rightarrow}\,(\text{GCb}\smile)}{\varphi \wedge \smile\varphi \wedge \ominus\varphi \Rightarrow}\,(\wedge/)\times 2}{\bot \Rightarrow}\,(\text{def. }\bot)}{\Rightarrow \frown\bot}\,(\frown\smile) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(\text{id}) \quad \dfrac{}{\psi \Rightarrow \psi}\,(\text{id})}{\varphi \vee \psi \Rightarrow \varphi, \psi}\,(\vee/)}{\frown\varphi, \frown\psi \Rightarrow \frown(\varphi \vee \psi)}\,(\frown\smile)}{\frown\varphi \wedge \frown\psi \Rightarrow \frown(\varphi \vee \psi)}\,(\wedge/) \qquad \dfrac{\varphi \Rightarrow \psi}{\frown\psi \Rightarrow \frown\varphi}\,(\frown\smile)$$

[$\smile$ is a full type <-> $\vdash_n$-reversing connective]

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Rightarrow \varphi, \frown\varphi, \circleddash\varphi}\,(\text{GDb}\frown)}{\Rightarrow \varphi \vee \frown\varphi \vee \circleddash\varphi}\,(/\vee)\times 2}{\Rightarrow \top}\,(\text{def. }\top)}{\smile\top \Rightarrow}\,(\smile\frown) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(\text{id}) \quad \dfrac{}{\psi \Rightarrow \psi}\,(\text{id})}{\varphi, \psi \Rightarrow \varphi \wedge \psi}\,(/\wedge)}{\smile(\varphi \wedge \psi) \Rightarrow \smile\varphi, \smile\psi}\,(\smile\frown)}{\smile(\varphi \wedge \psi) \Rightarrow \smile\varphi \vee \smile\psi}\,(/\vee) \qquad \dfrac{\varphi \Rightarrow \psi}{\smile\psi \Rightarrow \smile\varphi}\,(\smile\frown)$$

[$\circleddash$ is a $\vdash_n$-ordinary determinedness connective and $\ominus$ is a $\vdash_n$-ordinary consistency connective] To check this, note first that rules ($\circleddash$/) and (/$\ominus$) already do half of the job. As for the other half:

$$\dfrac{A, \circleddash\varphi \Rightarrow B \quad \dfrac{}{\Rightarrow \varphi, \frown\varphi, \circleddash\varphi}\,(\text{GDb}\frown)}{A \Rightarrow \varphi, \frown\varphi, B}\,(\text{cut}) \qquad \dfrac{A \Rightarrow \ominus\varphi, B \quad \dfrac{}{\varphi, \smile\varphi, \ominus\varphi \Rightarrow}\,(\text{GCb}\smile)}{A, \varphi, \smile\varphi \Rightarrow B}\,(\text{cut})$$

QED

The following simple observation follows from Prop. 4.3.2(2) and rules ($\ominus$/) and (/$\circleddash$), as may be easily checked:

Proposition 4.3.3. *The sequents* $\Rightarrow \bigcirc\!\!\!\!\!\smile\top$ *and* $\bigcirc\!\!\!\!\!\frown\bot \Rightarrow$ *are derivable in* $\mathcal{K}^{\mathrm{n}}$.

Proof.

$$\dfrac{\dfrac{\dfrac{}{\smile\top \Rightarrow}}{\smile\top, \top \Rightarrow}\,(\mathrm{W}/)}{\Rightarrow \bigcirc\!\!\!\!\!\smile\top}\,(/\bigcirc\!\!\!\!\!\smile) \qquad\qquad \dfrac{\dfrac{\dfrac{}{\Rightarrow \frown\bot}}{\Rightarrow \bot, \frown\bot}\,(/\mathrm{W})}{\bigcirc\!\!\!\!\!\frown\bot \Rightarrow}\,(\bigcirc\!\!\!\!\!\frown/)$$

QED

The following result concerns $\vdash$-theory pairs, and will play an important role in Section 4.5:

Proposition 4.3.4. *Let* $\langle\Sigma_1, \Sigma_0\rangle$ *be an unconnected* $\vdash$*-theory pair. Then, the derivability of the nonempty sequent* $A \Rightarrow B$ *implies that either* $\alpha \notin \Sigma_1$ *for some* $\alpha \in A$*, or* $\beta \notin \Sigma_0$ *for some* $\beta \in B$.

Proof. Consider a derivable sequent of the form $\alpha_1, \alpha_2, \ldots, \alpha_m \Rightarrow \beta_1, \beta_2, \ldots, \beta_n$ where $m + n > 0$. Using rules $(\wedge/)$ and $(/\vee)$ we may derive the sequent $\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_m \Rightarrow \beta_1 \vee \beta_2 \vee \ldots \vee \beta_n$. Call the latter sequent *Seq*. Suppose $\alpha_i \in \Sigma_1$ for every $1 \leq i \leq m$, and $\beta_j \in \Sigma_0$ for every $1 \leq j \leq n$. By Corol. 4.2.1 and in view of the fact that $\wedge$ is a $\vdash$-ordinary conjunction and that $\vee$ is a $\vdash$-ordinary disjunction, it follows that (i) $\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_m \in \Sigma_1$ and that (ii) $\beta_1 \vee \beta_2 \vee \ldots \vee \beta_n \in \Sigma_0$. Given that Σ_1 is a $\vdash$-theory, from (i) and the derivability of *Seq* it follows that (iii) $\beta_1 \vee \beta_2 \vee \ldots \vee \beta_n \in \Sigma_1$; given that Σ_0 is a $\vdash$-cotheory, from (ii) and the derivability of *Seq* it follows that (iv) $\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_m \in \Sigma_0$. Now one may use the sequent axiom (id) to conclude from (i) and (iv), in case $m \neq 0$, that the pair $\langle\Sigma_1, \Sigma_0\rangle$ is not unconnected; the same may be concluded from (ii) and (iii) in case $n \neq 0$. *QED*

The next section will introduce an adequate Kripke semantics for $\mathcal{K}^{\mathrm{n}}$.

4.4 Kripke Semantics

Here, as usual, a **frame** $\mathcal{F} = \langle W, R\rangle$ will be a structure containing a nonempty set W and a relation $R \subseteq W \times W$ — members of W are often called **worlds** and R is said to be an **accessibility relation** between these worlds. A **state-of-affairs** s on the frame $\mathcal{F}$ is a mapping $s : \mathcal{P} \to \mathbf{2}^W$. A **valuation** is defined as the recursive extension of a given state-of-affairs s into a mapping $V^s : \mathcal{L} \to \mathbf{2}^W$, as follows:

$$\begin{array}{rcl} V^s(p) & = & s(p), \text{ where } p \in \mathcal{P} \\ V^s(\varphi_1 \wedge \varphi_2) & = & V^s(\varphi_1) \cap V^s(\varphi_2) \\ V^s(\varphi_1 \vee \varphi_2) & = & V^s(\varphi_1) \cup V^s(\varphi_2) \end{array}$$

$$
\begin{array}{rcl}
V^s(\frown\varphi) & = & \{w \in W : \forall v \in W(wRv \text{ implies } v \notin V^s(\varphi))\} \\
V^s(\smile\varphi) & = & \{w \in W : \exists v \in W(wRv \text{ and } v \notin V^s(\varphi))\}
\end{array}
$$

$$
\begin{array}{rcl}
V^s(\ominus\varphi) & = & \{w \in W : w \notin V^s(\varphi) \text{ and } w \notin V^s(\frown\varphi)\} \\
V^s(\circleddash\varphi) & = & \{w \in W : w \notin V^s(\varphi) \text{ or } w \notin V^s(\smile\varphi)\}
\end{array}
$$

As there is thus a unique valuation V^s associated to each given state-of-affairs s, we will in what follows simply omit the index s from V^s. It is helpful to fix at this point the reading of the statement '$w \in V(\circleddash\varphi)$' as guaranteeing the consistency of φ at w, and to fix the reading of the statement '$w \notin V(\ominus\varphi)$' as guaranteeing the determinedness of φ at w.

Given the definitions of $\bot$ and $\top$ as abbreviations (Section 4.2), it is easy to check from the above notion of valuation that $V(\bot) = \varnothing$ and $V(\top) = W$. A **model** $\mathcal{M} = \langle \mathcal{F}, V\rangle$ is a structure where $\mathcal{F}$ is a frame and V is a valuation on $\mathcal{F}$. Given a class of frames $\mathbb{F}$, with the above definitions we may immediately consider the class $\mathbb{M}$ of all models based on such frames. We say that $\varphi \in \mathcal{L}$ is **satisfied** at a state $w \in W$ of a model $\mathcal{M} = \langle W, R, V\rangle$ if $w \in V(\varphi)$; this is denoted by $\mathcal{M}, w \Vdash \varphi$. When $w \notin V(\varphi)$ we write $\mathcal{M}, w \nVdash \varphi$ and say that $\mathcal{M}$ **falsifies** φ at w. Given two sets of sentences, Γ and Δ, we say that **Γ entails Δ**, denoted by $\Gamma \models \Delta$, when at every world of every model either some sentence in Γ is falsified or some sentence in Δ is satisfied; sometimes this definition is relativized to some given class of frames on which the relevant models are to be based. It is not hard to check that $\models$ is a gcr. As usual, the failure of $\Gamma \models \Delta$ will be denoted by $\Gamma \not\models \Delta$. When φ is satisfied at all states of all models of a given frame $\mathcal{F}$ we say that φ is **valid** in $\mathcal{F}$, in symbols $\mathcal{F} \Vdash \varphi$. The definition of satisfaction is extended to sequents by writing $\mathcal{M}, w \Vdash A \Rightarrow B$ if $\mathcal{M}$ falsifies some $\varphi \in A$ at w or $\mathcal{M}$ satisfies some $\varphi \in B$ at w. Moreover, on what concerns the other definitions, for any given model $\mathcal{M}$ and any given frame $\mathcal{F}$ we write $\mathcal{M} \Vdash A \Rightarrow B$ to say that $\mathcal{M}, w \Vdash A \Rightarrow B$ at every state w in $\mathcal{M}$, and write $\mathcal{F} \Vdash A \Rightarrow B$ to say that $\mathcal{M} \Vdash A \Rightarrow B$ for every model $\mathcal{M}$ of $\mathcal{F}$.

Using the above semantics, and taking into account the definitions in Section 4.2, is not hard to check that:

Proposition 4.4.1. *Both $\frown$ and $\smile$ enjoy the minimal conditions expected of a negation. Indeed, for any atomic variables p and q:*

(1) $\frown p \not\models p$ (2) $q \not\models \frown q$

(3) $\smile p \not\models p$ (4) $q \not\models \smile q$

Proof. Consider the frame in which $W = \{w\}$, $R = \varnothing$ and, based on this frame, consider a model such that $V(p) = \varnothing$ and $V(q) = W$. It is easy to see that this is a counter-model that bears witness to (1) and (4). From that a counter-model witnessing assertions (2) and (3) is built by simply replacing $R = \varnothing$ by its complement $R = W \times W$. *QED*

Proposition 4.4.2. *The entailment* $\models$ *is* $\smallfrown$*-undetermined as well as* $\smallsmile$*-inconsistent.*

Proof. Consider a frame $\mathcal{F}$ such that $W = \{u, v\}$ and R is the total relation $W \times W$, and consider a model $\mathcal{M}$ such that $V(p) = \{u\}$ for an atomic variable p. It follows that $V(\smallsmile p) = \{u\}$ and $V(\smallfrown p) = \varnothing$, thus both p and $\smallsmile p$ are satisfied at u, and both p and $\smallfrown p$ are falsified at v. *QED*

Our present semantical framework allows us also to provide straightforward verifications for many inferences which would give rise to long derivations. The following statements that guarantee that consistency propagates through conjunction and that determinedness propagates through disjunction may indeed very easily be verified.

Proposition 4.4.3. $\textcircled{\smile}\varphi, \textcircled{\smile}\psi \models \textcircled{\smile}(\varphi \wedge \psi)$ *and* $\textcircled{\frown}(\varphi \vee \psi) \models \textcircled{\frown}\varphi, \textcircled{\frown}\psi$

Proof. Suppose that there is a model $\mathcal{M}$ and a world w such that (i) $\mathcal{M}$ satisfies $\textcircled{\smile}\varphi$ and $\textcircled{\smile}\psi$ at w and (ii) $\mathcal{M}$ falsifies $\textcircled{\smile}(\varphi \wedge \psi)$ at w. From the last it follows that (iii) $\smallsmile(\varphi \wedge \psi)$ is satisfied at w and (iv) $\varphi \wedge \psi$ is satisfied at w. From (i) and (iv) it follows that (v) $\mathcal{M}$ falsifies both $\smallsmile\varphi$ and $\smallsmile\psi$ at w. By (iii) there is a world w' such that $\varphi \wedge \psi$ is falsified at w'. But by (v), both φ and ψ must be satisfied at w'. Which is a contradiction.

Suppose again by contradiction that there is a model $\mathcal{M}_1$ and a world w in $\mathcal{M}_1$ such that (i) $\mathcal{M}_1$ satisfies $\textcircled{\frown}(\varphi \vee \psi)$ at w and (ii) $\mathcal{M}_1$ falsifies $\textcircled{\frown}\varphi$ and $\textcircled{\frown}\psi$ at w. By (i) it follows that both (iii) $\varphi \vee \psi$ and (iv) $\smallfrown(\varphi \vee \psi)$ are falsified at w. By (iii) it follows that (v) φ is falsified at w and (vi) ψ is falsified at w. From (ii), (v) and (vi) it follows that (vii) $\smallfrown\varphi$ is satisfied at w and (viii) $\smallfrown\psi$ is satisfied at w. From (iv) there is a world u such that (x) $\mathcal{M}_1$ satisfies $\varphi \vee \psi$ at u. By (vii) and (viii), φ and ψ are satisfied at u, which contradicts (x). *QED*

More importantly, the usual inductive reasoning allows us to establish that any derivable inference can be checked semantically:

Proposition 4.4.4. [Soundness] *All rules of* $\mathcal{K}^{n}$ *are sound for frame validity, for arbitrary frames, that is, the conclusion of each given rule is valid on all frames that validate the premisses of that rule.*

Proof. Let $\mathcal{F}$ be some fixed arbitrary frame. We will skip the proof of frame validity for the standard structural rules and for the standard rules for conjunction and disjunction, and concentrate below on the distinctive rules of $\mathcal{K}^{\mathrm{n}}$.

Rule ($\circledsmile$/) : Assume that (a) $\mathcal{F} \Vdash (A \Rightarrow \varphi, B)$ and (b) $\mathcal{F} \Vdash (A \Rightarrow \smile\varphi, B)$. Suppose that $\mathcal{F} \nVdash A, \circledsmile\varphi \Rightarrow B$. Then, there are a model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ and a world w in $\mathcal{M}$ such that $\mathcal{M}, w \nVdash A, \circledsmile\varphi \Rightarrow B$. From this we have that (c) $\mathcal{M}, w \Vdash \alpha$ for every $\alpha \in A$, (d) $\mathcal{M}, w \Vdash \circledsmile\varphi$ and (e) $\mathcal{M}, w \nVdash \beta$ for every $\beta \in B$. By (c), (e) and (a) it follows that (f) $\mathcal{M}, w \Vdash \varphi$. Now from (c), (e) and (b) it follows that (g) $\mathcal{M}, w \Vdash \smile\varphi$. By the definition of valuation, (f) and (g), we conclude that $\mathcal{M}, w \nVdash \circledsmile\varphi$. This contradicts (d).

Rule (/$\circledsmile$) : Assume that (a) $\mathcal{F} \Vdash (A, \varphi, \smile\varphi \Rightarrow B)$ and suppose that (b) $\mathcal{F} \nVdash (A \Rightarrow \circledsmile\varphi, B)$. By (b) there are a model $\mathcal{M}$ and a world w such that (c) $\mathcal{M}, w \Vdash \alpha$ for every $\alpha \in A$, (d) $\mathcal{M}, w \nVdash \circledsmile\varphi$ and (e) $\mathcal{M}, w \nVdash \beta$ for every $\beta \in B$. By (a), (c) and (e) we have that $\mathcal{M}, w \nVdash \varphi$ or $\mathcal{M}, w \nVdash \smile\varphi$, and so $\mathcal{M}, w \Vdash \circledsmile\varphi$, which contradicts (d).

Rule ($\circledfrown$/) : Assume that (a) $\mathcal{F} \Vdash (A \Rightarrow \varphi, \frown\varphi, B)$ and suppose that (b) $\mathcal{F} \nVdash (A, \circledfrown\varphi \Rightarrow B)$. By (b) there are a model $\mathcal{M}$ and a world w such that (c) $\mathcal{M}, w \Vdash \alpha$ for every $\alpha \in A$, (d) $\mathcal{M}, w \Vdash \circledfrown\varphi$ and (e) $\mathcal{M}, w \nVdash \beta$ for every $\beta \in B$. By (a), (c) and (e) we have that $\mathcal{M}, w \Vdash \varphi$ or $\mathcal{M}, w \Vdash \frown\varphi$, and so $\mathcal{M}, w \nVdash \circledfrown\varphi$, contradicting (d).

Rule (/$\circledfrown$) : Assume that (a) $\mathcal{F} \Vdash (A, \varphi \Rightarrow B)$ and (b) $\mathcal{F} \Vdash (A, \frown\varphi \Rightarrow B)$. Suppose that $\mathcal{F} \nVdash A \Rightarrow \circledfrown\varphi, B$. Then, there are a model $\mathcal{M}$ and a world w in $\mathcal{M}$ such that $\mathcal{M}, w \nVdash A \Rightarrow \circledfrown\varphi, B$. From this we have that (c) $\mathcal{M}, w \Vdash \alpha$ for every $\alpha \in A$, (d) $\mathcal{M}, w \nVdash \circledfrown\varphi$ and (e) $\mathcal{M}, w \nVdash \beta$ for every $\beta \in B$. By (c), (e) and (a) we have that (f) $\mathcal{M}, w \nVdash \varphi$, and from (c), (e) and (b) it follows that (g) $\mathcal{M}, w \nVdash \frown\varphi$. By (f) and (g) we have $\mathcal{M}, w \Vdash \circledfrown\varphi$, which contradicts (d).

Rule ($\smile\frown$) : By contraposition assume that $\mathcal{F} \nVdash (\frown[B], \smile\varphi \Rightarrow \smile[A])$. Then, there are a model $\mathcal{M}$ and a state u in $\mathcal{M}$ such that (i) $\mathcal{M}, u \Vdash \frown\beta$ for every $\beta \in B$, (ii) $\mathcal{M}, u \Vdash \smile\varphi$ and (iii) $\mathcal{M}, u \nVdash \smile\alpha$ for every $\alpha \in A$. By (ii) there exists a world v in $\mathcal{M}$ such that (iv) uRv and (v) $\mathcal{M}, v \nVdash \varphi$. It follows, by (iii) and (iv), that (vi) $\mathcal{M}, v \Vdash \alpha$ for every $\alpha \in A$. From (i) and (iv) it follows that (vii) $\mathcal{M}, v \nVdash \beta$ for every $\beta \in B$. By (v), (vi) and (vii) we conclude that $\mathcal{M}, v \nVdash A \Rightarrow \varphi, B$, therefore $\mathcal{F} \nVdash A \Rightarrow \varphi, B$.

Rule ($\frown\smile$) : By contraposition assume that $\mathcal{F} \nVdash (\frown[B] \Rightarrow \frown\varphi, \smile[A])$. Then, there are a model $\mathcal{M}$ and a state w in $\mathcal{M}$ such that (i) $\mathcal{M}, w \Vdash \frown\beta$ for every $\beta \in B$, (ii) $\mathcal{M}, w \nVdash \frown\varphi$ and (iii) $\mathcal{M}, w \nVdash \smile\alpha$ for every $\alpha \in A$. By (ii) there exists a world z in $\mathcal{M}$ such that (iv)

wRz and (v) $\mathcal{M}, z \Vdash \varphi$. It follows, by (iii) and (iv), that $\mathcal{M}, z \Vdash \alpha$ for every $\alpha \in A$. From (i) and (iv) it follows that (vii) $\mathcal{M}, z \nVdash \beta$ for each $\beta \in B$. By (v), (vi) and (vii) we conclude that $\mathcal{M}, z \nVdash A, \varphi \Rightarrow B$, therefore $\mathcal{F} \nVdash A, \varphi \Rightarrow B$. *QED*

As an immediate application of the above soundness result, we may transfer the results in Prop. 4.4.2 to our sequent system, and conclude that the consequence relation $\vdash_{\mathrm{n}}$ associated to $\mathcal{K}^{\mathrm{n}}$ is in fact $\frown$-undetermined and $\smile$-inconsistent. For the same reason, the results in Prop. 4.3.1 may be transferred to our semantics. With little effort, results analogous to those in Prop. 4.3.2 concerning the $\rhd$-ordinary connectives originally characterized by way of our sequent system may also be restated in our present modal semantical framework, in which those connectives are conveniently interpreted. The connections between the two previous approaches will in fact be strengthened by the completeness result to be proven in the next section.

4.5 Completeness

Recall from Section 4.2 that a theory Σ_1 and a cotheory Σ_0 define a closed theory pair if $\Sigma_1 \cup \Sigma_0 = \mathcal{L}$. For closed theory pairs it will often be simpler thus to refer to the cotheory Σ_0 as $\overline{\Sigma_1}$, and we shall follow such policy from this point on, calling the single theory Π **saturated** if $\langle \Pi, \overline{\Pi} \rangle$ forms a closed (and obviously unconnected) theory pair. Following the definition of gcr from Section 4.3, we will concentrate below on the gcr $\vdash$ defined by the deductive system for $\mathcal{K}^{\mathrm{n}}$. Given a set of sentences Ψ, by $\lceil \Psi \rceil$ we will denote the theory $\{\psi : \Psi \vdash \psi\}$, and by $\lfloor \Psi \rfloor$ we will denote the cotheory $\{\psi : \psi \vdash \Psi\}$.

The interaction rules of our system $\mathcal{K}^{\mathrm{n}}$ allow us to prove some useful properties of saturated theories:

Lemma 4.5.1. *For any saturated theory Σ:*

(i) $\overline{\smile^{-1}[\Sigma]}$ *is a theory* (ii) $\frown^{-1}[\Sigma]$ *is a cotheory*

Item (i). Assume that $\overline{\smile^{-1}[\Sigma]} \vdash \varphi$ and suppose by reductio that $\varphi \notin \overline{\smile^{-1}[\Sigma]}$, that is, $\smile\varphi \in \Sigma$. By the assumption we know that there is some derivable sequent $\varphi_1, \varphi_2, \ldots, \varphi_n \Rightarrow \varphi$ in $\mathcal{K}^{\mathrm{n}}$ where $\{\varphi_1, \varphi_2, \ldots, \varphi_n\} \subseteq \overline{\smile^{-1}[\Sigma]}$. From this sequent, using rule $(\smile\frown)$ it follows that $\smile\varphi \Rightarrow \smile\varphi_1, \smile\varphi_2, \ldots, \smile\varphi_n$ is derivable in $\mathcal{K}^{\mathrm{n}}$. Given that Σ is a theory, $\vee$ is $\vdash$-ordinary, and $\smile\varphi \in \Sigma$, then $\smile\varphi_1 \vee \smile\varphi_2 \vee \ldots \vee \smile\varphi_n \in \Sigma$. But Σ is also saturated, thus $\smile\varphi_i \in \Sigma$ for some $1 \leq i \leq n$, by Prop. 4.2.3(1). It follows that $\varphi_i \notin \overline{\smile^{-1}[\Sigma]}$. Absurd.

[Item (ii)] This is analogous to the previous item, but we now use rule $(\frown\smile)$, the fact that $\wedge$ is $\vdash$-ordinary and Prop. 4.2.3(2). Details are safely left to the reader. *QED*

Let W_S be the set of all saturated theories of $\mathcal{K}^{\text{n}}$. Define over W_S the following binary relation R_S:

$$\Gamma\, R_S\, \Delta \;\text{ iff }\; \overline{\smile^{-1}[\Gamma]} \subseteq \Delta \subseteq \overline{\frown^{-1}[\Gamma]}$$

The **canonical frame** is defined as the structure $\mathcal{F}_S = \langle W_S, R_S \rangle$.

The two following auxiliary results will be helpful in establishing the proof of the Canonical Model Lemma, further on.

Lemma 4.5.2. *Let Σ be a saturated theory. Then $\smile\varphi \in \Sigma$ if and only if there is a saturated theory Π such that $\Sigma\, R_S\, \Pi$ and $\varphi \notin \Pi$.*

Proof. Assume first that there is some Π such that $\Sigma\, R_S\, \Pi$ and $\varphi \notin \Pi$. Since $\varphi \notin \Pi$ and $\overline{\smile^{-1}[\Sigma]} \subseteq \Pi$ it follows that $\varphi \notin \overline{\smile^{-1}[\Sigma]}$. From this we conclude that $\smile\varphi \in \Sigma$.

Conversely, assume that $\smile\varphi \in \Sigma$. Suppose $\alpha \in \overline{\smile^{-1}[\Sigma]}$ and $\beta \notin \overline{\frown^{-1}[\Sigma]}$, and thus $\smile\alpha \notin \Sigma$ and $\frown\beta \in \Sigma$. Recall, by Lemma 4.5.1, that $\overline{\smile^{-1}[\Sigma]}$ is a theory and $\frown^{-1}[\Sigma]$ is a cotheory. We need to show that there is some saturated theory Π such that $\alpha \in \Pi$ and $\beta \in \overline{\Pi}$, from which it will follow that $\Sigma\, R_S\, \Pi$, and such that $\varphi \notin \Pi$. We claim that the pair $P = \langle P_1, P_0 \rangle = \left\langle \overline{\smile^{-1}[\Sigma]}, \left\lfloor \frown^{-1}[\Sigma] \cup \{\varphi\} \right\rfloor \right\rangle$ is $\vdash$-unconnected. Suppose instead, by reductio, that $\overline{\smile^{-1}[\Sigma]} \vdash \varphi, \frown^{-1}[\Sigma]$. It follows that there are finite sequences of sentences $\alpha_1, \ldots, \alpha_m \notin \smile^{-1}[\Sigma]$ and $\beta_1, \ldots, \beta_n \in \frown^{-1}[\Sigma]$ such that $\alpha_1, \ldots, \alpha_m \Rightarrow \varphi, \beta_1, \ldots, \beta_n$ is derivable. Call such sequent *Seq*. Notice that $\alpha_i \notin \smile^{-1}[\Sigma]$ means that (a) $\smile\alpha_i \in \overline{\Sigma}$, for every $1 \leq i \leq m$, and $\beta_1, \ldots, \beta_n \in \frown^{-1}[\Sigma]$ means that (b) $\frown\beta_j \in \Sigma$, for every $1 \leq j \leq n$. From *Seq*, using rule $(\smile\frown)$ it follows that $\frown\beta_1, \ldots, \frown\beta_n, \smile\varphi \Rightarrow \smile\alpha_1, \ldots, \smile\alpha_m$ is also derivable. In view of Prop. 4.3.4, from the latter sequent and facts (a) and (b) we may conclude that $\smile\varphi \notin \Sigma$, which conflicts with our initial assumption.

Now that we know that the pair P is unconnected, we may use Prop. 4.2.1 to extend it to a saturated unconnected pair $P^\star = \langle \Pi, \overline{\Pi} \rangle$. By construction, $\alpha \in P_1$ and $\beta, \varphi \in P_0$, so it follows that $\alpha \in \Pi$ and $\beta \in \overline{\Pi}$, and also that $\varphi \notin \Pi$. *QED*

Lemma 4.5.3. *Let Σ be a saturated theory. Then $\frown\varphi \in \Sigma$ if and only if $\varphi \notin \Pi$ for every saturated theory Π such that $\Sigma\, R_S\, \Pi$.*

Proof. For the left to right direction assume that $\frown\varphi \in \Sigma$, that is (i) $\varphi \notin \overline{\frown^{-1}[\Sigma]}$. Suppose for an arbitrary saturated theory Π that (ii) $\Sigma R_S \Pi$. By (ii), (i) and definition of R_S we conclude that $\varphi \notin \Pi$.

For the right to left direction assume, by contraposition, that (a) $\frown\varphi \notin \Sigma$. We claim that the pair $P = \langle P_1, P_0 \rangle = \left\langle \left\lceil \{\varphi\} \cup \overline{\smile^{-1}[\Sigma]} \right\rceil, \frown^{-1}[\Sigma] \right\rangle$ is a $\vdash$-disconnected theory pair — remember, from Lemma 4.5.1, that $\frown^{-1}[\Sigma]$ is a cotheory. Otherwise we have that $\overline{\smile^{-1}[\Sigma]}, \varphi \vdash \frown^{-1}[\Sigma]$. It follows that there are finite sequences of sentences $\alpha_1, \ldots, \alpha_m \notin \smile^{-1}[\Sigma]$ and $\beta_1, \ldots, \beta_n \in \frown^{-1}[\Sigma]$ such that $\alpha_1, \ldots, \alpha_m, \varphi \Rightarrow \beta_1, \ldots, \beta_n$ is derivable. Notice that $\alpha_i \notin \smile^{-1}[\Sigma]$ means that (b) $\smile\alpha_i \in \overline{\Sigma}$, for every $1 \leq i \leq m$, and $\beta_1, \ldots, \beta_n \in \frown^{-1}[\Sigma]$ means that (c) $\frown\beta_j \in \Sigma$, for every $1 \leq j \leq n$. Using the rule $(\frown\smile)$ in the latter sequent it follows that $\frown\beta_1, \ldots, \frown\beta_n, \frown\varphi \vdash \smile\alpha_1, \ldots, \smile\alpha_m$ is derivable. By this sequent, Prop. 4.3.4, facts (b) and (c) we may conclude that $\frown\varphi \in \Sigma$, which contradicts (a).

By Prop. 4.2.1 there is thus a closed disconnected theory pair $\langle \Pi, \overline{\Pi} \rangle$ that extends P. Since $\frown^{-1}[\Sigma] \subseteq \overline{\Pi}$, then $\Pi \subseteq \overline{\frown^{-1}[\Sigma]}$. We also have, by construction, that $\overline{\smile^{-1}[\Sigma]} \subseteq \Pi$ and $\varphi \in \Pi$. In sum, for this particular Π, we have that $\Sigma R_S \Pi$ and $\varphi \in \Pi$. *QED*

We define the canonical model $\mathcal{M}_S$ as the structure $\langle \mathcal{F}_S, V_S \rangle$ where $\mathcal{F}_S$ is the canonical frame and V_S is the valuation defined by:

$$V_S(p) = \{\Gamma \in W_S : p \in \Gamma\}$$

In the proof of the following result it will help to work with the non-canonical measure of sentence complexity given by the function $\ell : \mathcal{L} \to \mathbb{N}$, recursively defined as follows:

$$\begin{array}{rcll} \ell(p) & = & 0 & \text{if } p \in \mathcal{P} \\ \ell(\varphi \circledast \psi) & = & 1 + \max\{\ell(\varphi), \ell(\psi)\} & \text{if } \circledast \in \{\wedge, \vee\} \\ \ell(\#\varphi) & = & 1 + \ell(\varphi) & \text{if } \# \in \{\smile, \frown\} \\ \ell(\circledhash\varphi) & = & 2 + \ell(\varphi) & \text{if } \circledhash \in \{\circledsmile, \circledfrown\} \end{array}$$

Lemma 4.5.4. [Canonical Model]
In the canonical model, for any saturated theory Γ and any sentence φ:

$$\mathcal{M}_S, \Gamma \Vdash \varphi \text{ if and only if } \varphi \in \Gamma$$

Proof. The proof is by induction over $\ell(\varphi)$. The base case ($\ell(\varphi) = 0$) is trivial, using (id) and the definition of the canonical model. Assume now, by Induction Hypothesis, that $\mathcal{M}_S, \Gamma \Vdash \varphi$ iff $\varphi \in \Gamma$, for any saturated theory Γ and for every sentence φ such that $\ell(\varphi) < k$. We will detail below the 'non-local' cases involving one of the modal negations and one of the restoration connectives.

$[\ell(\smallsmile\psi) = k]$ By the definition of satisfaction, we have $\mathcal{M}_S, \Gamma \Vdash \smallsmile\psi$ iff $\mathcal{M}_S, \Delta \nVdash \psi$ for some saturated theory Δ such that $\Gamma\, R_S\, \Delta$. Since $\ell(\smallsmile\psi) = \ell(\psi) + 1$, then $\ell(\psi) < k$, thus the Induction Hypothesis applies and allows us to conclude that $\mathcal{M}_S, \Delta \nVdash \psi$ iff $\psi \notin \Delta$. From Lemma 4.5.2 we know that there is a saturated theory Δ such that $\Gamma\, R_S\, \Delta$ and $\psi \notin \Delta$ if and only if $\smallsmile\psi \in \Gamma$. Summing up, we may conclude that $\mathcal{M}_S, \Gamma \Vdash \smallsmile\psi$ iff $\smallsmile\psi \in \Gamma$.

$[\ell(\ominus\psi) = k]$ Suppose first that $\ominus\psi \in \Gamma$. In view of the derivability of $\ominus\psi, \smallsmile\psi, \psi \Rightarrow$ (Prop. 4.3.1), from Prop. 4.3.4 we conclude that either $\psi \notin \Gamma$ or $\smallsmile\psi \notin \Gamma$. Given that both $\ell(\psi) < \ell(\ominus\psi)$ and $\ell(\smallsmile\psi) < \ell(\ominus\psi)$, the Induction Hypothesis guarantees that $\psi \notin \Gamma$ iff $\mathcal{M}_S, \Gamma \nVdash \psi$, and also that $\smallsmile\psi \notin \Gamma$ iff $\mathcal{M}_S, \Gamma \nVdash \smallsmile\psi$. By the definition of satisfaction we know that $\mathcal{M}_S, \Gamma \nVdash \psi$ or $\mathcal{M}_S, \Gamma \nVdash \smallsmile\psi$ if and only if $\mathcal{M}_S, \Gamma \Vdash \ominus\psi$. It follows from $\ominus\psi \in \Gamma$, thus, that $\mathcal{M}_S, \Gamma \Vdash \ominus\psi$. For the converse, suppose now that $\mathcal{M}_S, \Gamma \Vdash \ominus\psi$. By the definition of satisfaction, the definition of ℓ and the Induction Hypothesis, we know that (a) $\smallsmile\psi \notin \Gamma$ or (b) $\psi \notin \Gamma$. In case (b), in view of the derivability of $\Rightarrow \psi, \ominus\psi$ (Prop. 4.3.1), from Prop. 4.3.4 we conclude that $\ominus\psi \notin \overline{\Gamma}$, that is, $\ominus\psi \in \Gamma$; in case (a) the same conclusion follows in view of the derivability of $\Rightarrow \smallsmile\psi, \ominus\psi$. *QED*

As usual, from the above lemma we immediately conclude the following:

Proposition 4.5.1. [Completeness] *If* $\Gamma \models \Delta$ *then* $\Gamma \vdash \Delta$.

Proof. Suppose by contraposition that $\Gamma \nvdash \Delta$. By Prop. 4.2.1 there is a closed unconnected pair $\langle \Gamma^\star, \Delta^\star \rangle$ that extends $\langle \lceil\Gamma\rceil, \lfloor\Delta\rfloor \rangle$. It follows that $\Gamma^\star$ is a saturated theory and that $\Delta^\star = \overline{\Gamma^\star}$. By the Canonical Model Lemma, we have $\mathcal{M}_S, \Gamma^\star \Vdash \varphi$ iff $\varphi \in \Gamma^\star$. Thus, we conclude that $\Gamma^\star \not\models \Delta^\star$, and by monotonicity it follows that $\Gamma \not\models \Delta$. *QED*

4.6 Extensions of $\mathcal{K}^{\mathrm{n}}$

In the literature it is common to find the minimal system of normal modal logic extended by adding new axioms and to see the resulting system shown to be sound and complete with respect to a class of frames in which the accessibility relation enjoys certain appropriate properties. As an illustration of how this strategy may be applied to our systems with negative modalities, we introduce in this section two systems that extend $\mathcal{K}^{\mathrm{n}}$. The system $\mathcal{T}^{\mathrm{n}}$ extends $\mathcal{K}^{\mathrm{n}}$ by adding the dual axiomatic rules $\overline{\Rightarrow \varphi, \smallsmile\varphi}^{\,(\mathrm{rf1})}$ and $\overline{\smallfrown\varphi, \varphi \Rightarrow}^{\,(\mathrm{rf2})}$. By adding the axiomatic rules $\overline{\smallsmile\smallsmile\varphi \Rightarrow \varphi}^{\,(\mathrm{sm1})}$ and $\overline{\varphi \Rightarrow \smallfrown\smallfrown\varphi}^{\,(\mathrm{sm2})}$ to $\mathcal{K}^{\mathrm{n}}$, we define the system $\mathcal{B}^{\mathrm{n}}$. The gcrs $\vdash^{\mathcal{T}}_{\mathrm{n}}$ and $\vdash^{\mathcal{B}}_{\mathrm{n}}$ correspond, respectively, to the deductive systems associated to $\mathcal{T}^{\mathrm{n}}$ and $\mathcal{B}^{\mathrm{n}}$.

Recall that a binary relation R is called **reflexive** if xRx holds for every x, and is called **symmetric** if xRy implies yRx. In what follows we will show that $\mathcal{T}^n$ is sound and complete with respect to the class of reflexive frames (i.e., the class of frames with a reflexive accessibility relation), and similarly for $\mathcal{B}^n$ and the class of symmetric frames.

Proposition 4.6.1. [Correspondence] *Let* $\mathcal{F} = \langle W, R\rangle$ *be a frame. Then:*

(1.1)	*R is reflexive*	*only if*	$\mathcal{F} \Vdash\ \Rightarrow \varphi, \smile\varphi$ *and* $\mathcal{F} \Vdash \frown\varphi, \varphi \Rightarrow$
(1.2)	*R is reflexive*	*if*	$\mathcal{F} \Vdash\ \Rightarrow \varphi, \smile\varphi$ *or* $\mathcal{F} \Vdash \frown\varphi, \varphi \Rightarrow$
(2.1)	*R is symmetric*	*only if*	$\mathcal{F} \Vdash \smile\smile\varphi \Rightarrow \varphi$ *and* $\mathcal{F} \Vdash \varphi \Rightarrow \frown\frown\varphi$
(2.2)	*R is symmetric*	*if*	$\mathcal{F} \Vdash \smile\smile\varphi \Rightarrow \varphi$ *or* $\mathcal{F} \Vdash \varphi \Rightarrow \frown\frown\varphi$

Proof. (1.1) Assume that (i) R is reflexive and suppose that (ii) $\mathcal{F} \nVdash\ \Rightarrow \varphi, \smile\varphi$, for some sentence φ. It follows from (ii) that (iii) $\mathcal{M}_0, w \nVdash \varphi$ and (iv) $\mathcal{M}_0, w \nVdash \smile\varphi$ for some model $\mathcal{M}_0$ of $\mathcal{F}$ and some w in $\mathcal{M}_0$. From (i) and (iv), the definition of valuation gives us (v) $\mathcal{M}_0, w \Vdash \varphi$. This contradicts (iii). Suppose now that $\mathcal{F} \nVdash \frown\varphi, \varphi \Rightarrow$. Then, there are a model $\mathcal{M}_1$ and a world u such that $\mathcal{M}_1, u \Vdash \varphi$ and $\mathcal{M}_1, u \Vdash \frown\varphi$. From the latter, invoking the reflexivity of R, we conclude that $\mathcal{M}_1, u \nVdash \varphi$. Contradiction.

(1.2) Suppose that $\mathcal{F}$ is not reflexive. Then, there is a world m such that $\langle m, m\rangle \notin R$. Let C be the set $\{z : \langle m, z\rangle \in R\}$. Let p be a propositional variable and let $\mathcal{M}_2 = \langle \mathcal{F}, V\rangle$ be a model such that $V(p) = C$. Obviously $m \notin V(p)$, thus $\mathcal{M}_2$ falsifies p at m. Moreover, by construction of C, we have $x \in V(p)$ for every x such that $\langle m, x\rangle \in R$. By the definition of valuation, $\mathcal{M}_2$ falsifies $\smile p$ at m. Thus, $\mathcal{M}_2$ falsifies $\Rightarrow p, \smile p$ at m. If we enrich $\mathcal{M}_2$ by a propositional variable q such that $V(q) = \overline{C}$, we see that $\mathcal{M}_2$ falsifies $\frown q, q \Rightarrow$ at m.

(2.1) Assume that R is symmetric. Suppose that $\mathcal{F} \nVdash \smile\smile\varphi \Rightarrow \varphi$. There is thus a model $\mathcal{M}_0$ and a state w such that (i) $\mathcal{M}_0, w \Vdash \smile\smile\varphi$ and (ii) $\mathcal{M}_0, w \nVdash \varphi$. From (i), there must be some z such that (iii) $\langle w, z\rangle \in R$ and (iv) $\mathcal{M}_0, z \nVdash \smile\varphi$. The symmetry of R allows us to conclude (v) $\langle z, w\rangle \in R$ from (iii), and from (iv) and (v) it follows that $\mathcal{M}_0, w \Vdash \varphi$. This contradicts (ii). If we suppose that $\mathcal{F} \nVdash \varphi \Rightarrow \frown\frown\varphi$ we reach a contradiction through a similar line of reasoning.

(2.2) Suppose that R is not symmetric. Then there are m, n such that $\langle m, n\rangle \in R$ yet $\langle n, m\rangle \notin R$. Let C be the set $\{z : \langle n, z\rangle \in R\}$. Let p, q be propositional variables and let $\mathcal{M}_2 = \langle \mathcal{F}, V\rangle$ be a model where $V(p) = C$ and $V(q) = \overline{C}$. Since $m \notin V(p)$, then $\mathcal{M}_2$ falsifies p at m. Given that, for arbitrary z, we have that $\langle n, z\rangle \in R$ implies $z \in V(p)$, we conclude by the definition of valuation that $\mathcal{M}_2$ falsifies $\smile p$ at n. Once $\langle m, n\rangle \in R$, then $\mathcal{M}_2$ satisfies $\smile\smile p$ at m. Thus, $\mathcal{M}_2$ falsifies $\smile\smile p \Rightarrow p$ at m. Similarly, $\mathcal{M}_2$ also falsifies

$q \Rightarrow {\frown}{\frown}q$ at m. *QED*

Soundness of $\mathcal{T}^{\mathrm{n}}$ and $\mathcal{B}^{\mathrm{n}}$ are corollaries of the 'only-if' part of Prop. 4.6.1. To illustrate some differences between those systems we invite the reader to use rules (sm1) and (sm2) of $\mathcal{B}^{\mathrm{n}}$, on the one hand, and the soundness of $\mathcal{T}^{\mathrm{n}}$, on the other hand, to check that:

Proposition 4.6.2. *Sequents $\ominus\varphi \Rightarrow \ominus{\smile}\varphi$ and $\circleddash{\frown}\varphi \Rightarrow \circleddash\varphi$ are derivable in $\mathcal{B}^{\mathrm{n}}$ but not in $\mathcal{T}^{\mathrm{n}}$.*

Proof. Assume that there is a model $\mathcal{M} = \langle W, R, V\rangle$, where R is symmetric, and w is a world in $\mathcal{M}$ such that (i) $\mathcal{M}, w \Vdash \ominus\varphi$ and (ii) $\mathcal{M}, w \Vdash \ominus\psi$. By (ii) we have that (iii) $\mathcal{M}, w \nVdash {\smile}\varphi$ and (iv) $\mathcal{M}, w \nVdash {\smile}{\smile}\varphi$. By (ii) and (iv) it follows that (v) $\mathcal{M}, w \nVdash \varphi$. By (iii) there is a world w_2 such that wRw_2 and (vi) $\mathcal{M}, w_2 \nVdash {\smile}\varphi$. Since R is symmetric w_2Rw, by (vi) we have that $\mathcal{M}, w \Vdash \varphi$. It is an absurd.

The sequent $\ominus\varphi \Rightarrow \ominus{\smile}\varphi$ is not derivable in $\mathcal{T}^{\mathrm{n}}$. Consider the reflexive frame $\mathcal{F}$ such that $W = \{u, v\}$ and $R = \{\langle u, u\rangle, \langle u, w\rangle, \langle w, w\rangle\}$. Let $\mathcal{M}$ be a model based on $\mathcal{F}$ such that $V(p) = \varnothing$, for some atomic p. It follows that $\mathcal{M}, u \Vdash {\smile}p$ and $\mathcal{M}, u \Vdash {\smile}{\smile}p$. Thus, $\mathcal{M}, u \nVdash \ominus{\smile}p$. Since that $\mathcal{M}, u \nVdash p$, then $\mathcal{M}, w \Vdash \ominus p$. We conclude from the above that $\mathcal{M}$ is a counter-model for $\ominus\varphi \Rightarrow \ominus{\smile}\varphi$. *QED*

It might be interesting to contrast the latter result concerning the propagation of consistency through the paraconsistent negation and the dual propagation of determinedness through the paracomplete negation to the earlier general propagation results in Prop. 4.4.3.

Completeness will be attained next with the help of the following auxiliary results.

Lemma 4.6.1. *Assume the theories Γ_a and Γ_b to be closed with respect to $\vdash_{\mathrm{n}}^{\mathcal{T}}$ and $\vdash_{\mathrm{n}}^{\mathcal{B}}$ respectively. Then:*

1. *In $\mathcal{T}^{\mathrm{n}}$ we have that $\varphi \vee {\smile}\varphi \in \Gamma_a$.*
2. *In $\mathcal{T}^{\mathrm{n}}$ we have that $\varphi \wedge {\frown}\varphi \notin \Gamma_a$.*
3. *In $\mathcal{B}^{\mathrm{n}}$ we have that ${\smile}{\smile}\varphi \in \Gamma_b$ implies $\varphi \in \Gamma_b$.*
4. *In $\mathcal{B}^{\mathrm{n}}$ we have that $\varphi \in \Gamma_b$ implies ${\frown}{\frown}\varphi \in \Gamma_b$.*

Proof. The first two facts follow from closure of Γ_a and the obvious derivability of $\Rightarrow \varphi \vee \smile\varphi$ and $\varphi \wedge \frown\varphi \Rightarrow$ in $\mathcal{T}^{\mathrm{n}}$, in view of axiomatic rules (rf1) and (rf2). The remaining facts are easy consequences of closure of Γ_b and the axiomatic rules (sm1) and (sm2). *QED*

We should guarantee that the canonical construction yields the appropriate properties:

Proposition 4.6.3. [Canonical Systems] *The systems $\mathcal{T}^{\mathrm{n}}$ and $\mathcal{B}^{\mathrm{n}}$ are canonical.*

Proof. For $\mathcal{T}^{\mathrm{n}}$ we have to show, for the canonical frame $\mathcal{F}_S$, that $\langle \Gamma, \Gamma \rangle \in R_S$ for all $\Gamma \in W_S$, that is, $\overline{\smile^{-1}[\Gamma]} \subseteq \Gamma \subseteq \overline{\frown^{-1}[\Gamma]}$. Suppose that $\varphi \in \overline{\smile^{-1}[\Gamma]}$. Then, $\smile\varphi \notin \Gamma$. Since Γ is a closed theory and, by Lemma 4.6.1(1) and Prop. 4.2.3(1), $\varphi \vee \smile\varphi \in \Gamma$, it follows that $\varphi \in \Gamma$. To show that $\Gamma \subseteq \overline{\frown^{-1}[\Gamma]}$ the reasoning is similar, in view of Lemma 4.6.1(2) and Prop. 4.2.3(2).

For $\mathcal{B}^{\mathrm{n}}$ assume that Γ, Δ are closed theories such that $\langle \Gamma, \Delta \rangle \in R_S$, that is, $\overline{\smile^{-1}[\Gamma]} \subseteq \Delta \subseteq \overline{\frown^{-1}[\Gamma]}$. If $\varphi \in \overline{\smile^{-1}[\Delta]}$, then $\smile\varphi \notin \Delta$. From this we have that $\smile\varphi \notin \overline{\smile^{-1}[\Gamma]}$, therefore $\smile\smile\varphi \in \Gamma$. By Lemma 4.6.1(3) we conclude that $\varphi \in \Gamma$. Assume now that $\varphi \in \Gamma$. By Lemma 4.6.1(4), $\frown\frown\varphi \in \Gamma$. It follows that $\frown\varphi \notin \overline{\frown[\Gamma]}$. Since $\Delta \subseteq \overline{\frown^{-1}[\Gamma]}$, then $\frown\varphi \notin \Delta$, that is, $\varphi \in \overline{\frown^{-1}[\Delta]}$. Thus, $\overline{\smile^{-1}[\Delta]} \subseteq \Gamma \subseteq \overline{\frown^{-1}[\Delta]}$, that is $\langle \Delta, \Gamma \rangle \in R_S$. *QED*

Let $\models^{\mathcal{T}}$ be the entailment relation defined with respect to the class of all reflexive frames, and $\models^{\mathcal{B}}$ be defined for the class of all symmetric frames. An immediate consequence of Prop. 4.6.3, proven exactly as in Prop. 4.5.1, is:

Corollary 4.6.1. [Completeness for $\mathcal{X} \in \{\mathcal{T}, \mathcal{B}\}$] *For every* $\Gamma \cup \Delta \subseteq \mathcal{L}$*:*

$$\Gamma \models^{\mathcal{X}} \Delta \textit{ implies } \Gamma \vdash^{\mathcal{X}}_{\mathrm{n}} \Delta$$

Having established, for both $\mathcal{T}^{\mathrm{n}}$ and $\mathcal{B}^{\mathrm{n}}$, that all inferences verified semantically are also derivable in the next section we will study the role of these stronger modal systems in helping to more naturally restore inferences of some standard logical systems by means of Derivability Adjustment Theorems.

4.7 Recovering the Lost Perfection

Let $\sim$ be a unary negation symbol. Some standard rules for negation that could be added to the system $\mathcal{DL}$ are:

$$\frac{A \Rightarrow \varphi, B}{A, {\sim}\varphi \Rightarrow B}\ ({\sim}/)$$

It is easy to see that such rules would characterize $\sim$ as what we have, in Section 4.2, called an ordinary negation, respecting both statements (Cns) and (Dtm). Legitimate non-classical negations, nonetheless, while obviously failing either consistency or determinedness, may still respect other typical rules of negation. We list below, in particular, some standard sequent rules involving negation and the standard connectives modeled by a bounded distributive lattice:

$$\frac{A, {\sim}\varphi, {\sim}\psi \Rightarrow B}{A, {\sim}(\varphi \vee \psi) \Rightarrow B}\ (\text{dm1.1}) \qquad \frac{A \Rightarrow {\sim}\varphi, B \qquad A \Rightarrow {\sim}\psi, B}{A \Rightarrow {\sim}(\varphi \vee \psi), B}\ (\text{dm1.2})$$

$$\frac{A \Rightarrow {\sim}\varphi, {\sim}\psi, B}{A \Rightarrow {\sim}(\varphi \wedge \psi), B}\ (\text{dm2.1}) \qquad \frac{A, {\sim}\varphi \Rightarrow B \qquad A, {\sim}\psi \Rightarrow B}{A, {\sim}(\varphi \wedge \psi) \Rightarrow B}\ (\text{dm2.2})$$

$$\frac{A \Rightarrow \varphi, B}{A \Rightarrow {\sim}{\sim}\varphi, B}\ (\text{dm3.1}) \qquad \frac{A, \varphi \Rightarrow B}{A, {\sim}{\sim}\varphi \Rightarrow B}\ (\text{dm3.2})$$

$$\frac{}{A, {\sim}\top \Rightarrow B}\ (\text{dm4.1}) \qquad \frac{}{A \Rightarrow {\sim}\bot, B}\ (\text{dm4.2})$$

We will discuss in this section which of the above rules are derivable and which of them may be somehow *recovered* from the viewpoint of each of the sequent systems studied in the previous sections.

Recall that our language $\mathcal{L}$ contains two indigenous symbols for negation, namely, $\frown$ and $\smile$. For those negations it is not hard to check that:

Proposition 4.7.1. *In $\mathcal{K}^{\mathrm{n}}$:*

1. *Rules* (dm1.1) *and* (dm2.1) *are derivable for both* $\frown$ *and* $\smile$.
2. *Rules* (dm1.2) *and* (dm4.1) *are derivable for* $\frown$.
3. *Rules* (dm2.2) *and* (dm4.2) *are derivable for* $\smile$.
4. *Rule* (dm1.2) *fails for* $\smile$, *and rule* (dm2.2) *fails for* $\frown$.
5. *Rules* (dm3.1) *fails for* $\frown$ *and rule* (dm3.2) *fails for* $\smile$.

In $\mathcal{T}^{\mathrm{n}}$:

6. *Rule* $(/\sim)$ *is derivable for* $\smile$ *and rule* $(\sim/)$ *is derivable for* $\frown$.

In $\mathcal{B}^{\mathrm{n}}$*:*

7. *Rule* (dm3.1) *is derivable for* $\frown$ *and rule* (dm3.2) *is derivable for* $\smile$.

Moreover, in either $\mathcal{T}^{\mathrm{n}}$ *or* $\mathcal{B}^{\mathrm{n}}$ *(thus, also in* $\mathcal{K}^{\mathrm{n}}$*):*

8. *Rule* (dm3.2) *fails for* $\frown$ *and rule* (dm3.1) *fails for* $\smile$.

9. *Rule* $(\sim/)$ *fails for* $\smile$ *and rule* $(/\sim)$ *fails for* $\frown$.

Proof. Items (1), (2) and (3) follow directly from Prop. 4.3.2(2) and the second half of Prop. 4.2.4. Items (6) and (7) follow from the characterizing axioms of $\mathcal{T}^{\mathrm{n}}$ and $\mathcal{B}^{\mathrm{n}}$.

To check the remaining items, the completeness results in Prop. 4.5.1 and Corol. 4.6.1 come in handy. A simple strategy to show that some instance of a given schematic rule must fail involves falsifying some sequent that is derivable from that rule. On what concerns item (8), for example, notice that $\sim\sim p \Rightarrow p$ would obviously be derivable from (dm3.2), for any atomic sentence p. Yet, to falsify the sequent $\frown\frown p \Rightarrow p$ it suffices to consider a frame $\mathcal{F}_1$ such that $W_1 = \{u, v\}$ and R_1 is the total (thus reflexive and symmetric) relation $W_1 \times W_1$, and consider a model $\mathcal{M}$ such that $V(p) = \{v\}$: note indeed that $\mathcal{M}, v \Vdash p$, and uR_1v and vR_1v imply $\mathcal{M}, u \nVdash \frown p$ and $\mathcal{M}, v \nVdash \frown p$, and thus $\mathcal{M}, u \Vdash \frown\frown p$ given that uR_1x implies $x \in \{u, v\}$, while obviously $\mathcal{M}, u \nVdash p$. Analogously, $p \Rightarrow \sim\sim p$ would be derivable from (dm3.1), yet in the model just considered we have $\mathcal{M}, v \Vdash p$ and $\mathcal{M}, v \nVdash \smile\smile p$, thus falsifying the sequent $p \Rightarrow \smile\smile p$.

For item (4), consider a frame $\mathcal{F}_2$ where $W_2 = \{u, v, w\}$ and $R_2 = \{\langle u, v\rangle, \langle u, w\rangle\}$, and a model $\mathcal{M}'$ in which $V'(p) = \{v\}$ and $V'(q) = \{w\}$, for atomic sentences p and q; this is indeed a model that witnesses the failure of $\frown(p \wedge q) \models (\frown p \vee \frown q)$ and the failure of $\smile p \wedge \smile q \models \smile(p \vee q)$. For item (5) one might consider a frame $\mathcal{F}_2$ such that $W_2 = \{u, v\}$ and $R_2 = \{\langle u, v\rangle\}$, and consider a model $\mathcal{M}''$ such that $V''(p) = \{u\}$ and $V''(q) = \varnothing$.

At last, on what concerns item (9), note that the proof of Prop. 4.4.2 still applies unchanged. *QED*

The result in Prop. 4.7.1(9) should come as no surprise: As shown in (Marcos, 2005a), with the exception of degenerate cases, normal modal logics based on $\frown$ are paracomplete and modal logics based on $\smile$ are paraconsistent. It is interesting to call attention, though, to a particular byproduct of the proof of Prop. 4.7.1(8): the counter-models presented to $\frown\frown p \models p$ and to $p \models \smile\smile p$ are based on equivalence relations, and so one should not expect these two inferences to be valid for any of the usual classes of frames characterizing modal logics weaker than $S5$ — in other words, one might say that the intuitionistic-like negation has indeed a good reason to fail double negation elimination, and analogously the paraconsistent negation may reasonably be expected to fail double negation introduction.

Notice now that the rules that are shown to fail in the previous proposition may often be restored in one way or another, with the help of the connectives expressing consistency and determinedness in our rich modal language. If, for instance, the following restored versions of our missing sequent rules turn out to be derivable, this will help us in finding conditions under which one can recover some of the lost inferences:

$$\frac{A, \varphi \Rightarrow B}{A \Rightarrow \frown\varphi, \circleddash\varphi, B}\,(/\frown)^\circ \qquad \frac{A \Rightarrow \varphi, B}{A, \ominus\varphi, \smile\varphi \Rightarrow B}\,(\smile/)^\circ$$

$$\frac{A \Rightarrow \smile\varphi, B \qquad A \Rightarrow \smile\psi, B}{A, \ominus\varphi, \ominus\psi \Rightarrow \smile(\varphi \vee \psi), \circleddash(\varphi \vee \psi), B}\,(\text{dm1.2})^\circ$$

$$\frac{A, \frown\varphi \Rightarrow B \qquad A, \frown\psi \Rightarrow B}{A, \ominus(\varphi \wedge \psi), \frown(\varphi \wedge \psi) \Rightarrow \circleddash\varphi, \circleddash\psi, B}\,(\text{dm2.2})^\circ$$

$$\frac{A \Rightarrow \varphi, B}{A, \ominus\varphi \Rightarrow \frown\frown\varphi, \circleddash\frown\varphi, B}\,(\text{dm3.1})^\circ \qquad \frac{A, \varphi \Rightarrow B}{A, \ominus\smile\varphi, \smile\smile\varphi \Rightarrow \circleddash\varphi, B}\,(\text{dm3.2})^\circ$$

Rules $(/\frown)^\circ$ and $(\smile/)^\circ$ are obviously derivable from the basic rules $(/\circleddash)$ and $(\ominus/)$. The remaining rules above may be checked with the help of the following sequents:

Proposition 4.7.2. *In $\mathcal{K}^{\mathrm{n}}$ the following are derivable:*

(SD12) $\ominus\varphi, \ominus\psi, \smile\varphi, \smile\psi \Rightarrow \smile(\varphi \vee \psi), \circleddash(\varphi \vee \psi)$
(SD22) $\ominus(\varphi \wedge \psi), \frown(\varphi \wedge \psi) \Rightarrow \frown\varphi, \frown\psi, \circleddash\varphi, \circleddash\psi$
(SD31) $\ominus\varphi, \varphi \Rightarrow \frown\frown\varphi, \circleddash\frown\varphi$
(SD32) $\ominus\smile\varphi, \smile\smile\varphi \Rightarrow \varphi, \circleddash\varphi$

Proof. For (SD12), suppose by *reductio* that there is a model $\mathcal{M}$ with a world w in which $\ominus\varphi, \ominus\psi, \smile\varphi, \smile\psi$ are all satisfied and $\smile(\varphi \vee \psi), \circleddash(\varphi \vee \psi)$ are both falsified. It follows from the joint satisfaction of $\ominus\varphi$ and $\smile\varphi$ at w that φ must by falsified at w. The same reasoning applies to ψ, and thus we may conclude that $\varphi \vee \psi$ is falsified at w. From the latter, given that $\circleddash(\varphi \vee \psi)$ is also falsified at w, we conclude that $\varphi \vee \psi$ is falsified indeed at every world accessible to w. Note now that the satisfaction of $\smile\varphi$ at w demands in particular the existence of a world w' accessible to w. Given that $\smile(\varphi \vee \psi)$ is falsified at w, we must also conclude that $\varphi \vee \psi$ is satisfied at w'. We reach thus a contradiction.

For (SD31), suppose by *reductio* that in the world w of a model $\mathcal{M}$ the sentences $\ominus\varphi$ and φ are both satisfied (forcing thereby φ to be satisfied at any world accessible to w), while the sentences $\frown\frown\varphi$ and $\ominus\frown\varphi$ are both falsified (forcing φ to be falsified at any world accessible to w). But to falsify $\frown\frown\varphi$ at w there must first of all exist some world w' accessible from w. Contradiction.

Items (SD22) and (SD32) are proved similarly. In all cases, completeness may be used in the end to transfer the semantically verified results to facts about the proof formalism. *QED*

It is instructive to contrast the latter result to what we had learned from items (4) and (5) from Prop. 4.7.1.

Instead of axiomatizing Classical Logic ($\mathcal{CL}$) simply by adding rules $(/\sim)$ and $(\sim/)$ to $\mathcal{DL}$, we will here axiomatize it in the language $\mathcal{L}$ by adding the restored rules $(/\frown)^\circ$ and $(\smile/)^\circ$ to $\mathcal{DL}$, plus the two following rules: $\overline{\Rightarrow \ominus\varphi}^{(\text{cns})}$ and $\overline{\ominus\varphi \Rightarrow}^{(\text{dtm})}$. The associated gcr will be referred to as $\vdash_{\text{cl}}$. The intuition behind such system is precisely that $\mathcal{CL}$ is to be obtained by explicitly imposing a universal consistency assumption as well as a universal determinedness assumption.

At this point we can finally state:

Proposition 4.7.3. [Derivability Adjustment Theorem] *Let* $\Pi^{\#}_{\sim}$ *be the result of uniformly substituting each occurrence of the symbol* $\sim$ *in each sentence of* Π *by an occurrence of a unary symbol* $\# \in \{\frown, \smile\}$. *Then, inferences from* $\mathcal{CL}$ *may be recovered from* $\mathcal{T}^{\text{n}}$ *in the following way:*

$$\Gamma^{\#}_{\sim} \vdash_{\text{cl}} \Delta^{\#}_{\sim} \text{ iff there are finite sets } \Sigma_c, \Sigma_d \subseteq \mathcal{L} \text{ such that } \ominus[\Sigma_c], \Gamma \vdash^{\mathcal{T}}_{\text{n}} \Delta, \ominus[\Sigma_d]$$

Furthermore, Σ_c *may be constrained above to a finite collection of sub-sentences of* Γ, *and* Σ_d *may be constrained to a finite collection of sub-sentences of* Δ.

Proof. For the right-to-left direction, first one should notice that all the rules of $\mathcal{T}^{\text{n}}$ are classically valid. Any derivation constructed in $\mathcal{T}^{\text{n}}$ may then in principle be reproduced as a derivation associated to the gcr $\vdash_{\text{cl}}$, any occurrence of a sentence of the form $\ominus\varphi$ on the left-hand side of a given sequent may be eliminated by cut using the axiomatic rule (cns), and any occurrence of a sentence of the form $\circleddash\varphi$ on the right-hand side of a given sequent may be eliminated by cut using the axiomatic rule (dtm).

For the left-to-right direction, one may proceed by induction on the structure of the derivations. The base case (0-step derivations) is trivial, and it suffices to take $\Sigma_c = \Sigma_d = \varnothing$. The idea for the remainder of the construction is to collect consistency assumptions and determinedness assumptions on the fly: for each further step of a $\mathcal{CL}$-derivation intended to witness the fact that $A^{\#}_{\sim} \vdash_{\text{cl}} B^{\#}_{\sim}$, for appropriate finite sets $A \subseteq \Gamma$ and $B \subseteq \Delta$, check whether a rule has been used that does not belong to the common core of the sequent systems for $\mathcal{CL}$ and for $\mathcal{T}^{\text{n}}$, in that case, construct the corresponding step in the $\mathcal{T}^{\text{n}}$-derivation by using the qualified versions of the same rules (taking into account Prop. 4.7.1 and the rules derived with the help of Prop. 4.7.2). For a bit more of detail, suppose the construction of the classical derivation has proceeded by applying rules (dm1.1) or rule (dm2.1) at a given derivation step. Then, according to Prop. 4.7.1(1), exactly the same derivation step may be taken in $\mathcal{K}^{\text{n}}$ (thus also in $\mathcal{T}^{\text{n}}$). Similarly, according to items (2) and (3) of Prop. 4.7.1, the same steps may be taken in $\mathcal{K}^{\text{n}}$ (or in $\mathcal{T}^{\text{n}}$) in case (dm1.2) is used with respect to $\frown$ or in case (dm2.2) is used with respect to $\smile$. Now, if (dm1.2) is expected to be used with respect to $\smile$, then (dm1.2)$^\circ$ should be used instead, and if (dm2.2) is expected to be used with respect to $\frown$, then (dm2.2)$^\circ$ should be used instead — notice that in both cases there will be consistency and determinedness assumptions added to the contexts at the root of the derivation, that is, there will be sentences added to Σ_c and to Σ_d. Finally, notice that any derivation step using rule $(/\!\sim)$ in a classical derivation may still be taken in $\mathcal{T}^{\text{n}}$ with respect to $\smile$, in view of Prop. 4.7.1(6); with respect to $\frown$ one should use the derivable rule $(/\!\frown)^\circ$ instead — and in this case an appropriate sentence will be added to Σ_d. Dually, any classical derivation step using rule $(\sim\!/)$ may be reproduced in $\mathcal{T}^{\text{n}}$ with respect to $\frown$, or be replaced, with respect to $\smile$, by a step making use of rule $(\smile\!/)^\circ$, demanding the addition of an appropriate sentence to Σ_c. *QED*

The above result could alternatively be checked by using the appropriate consistency and determinedness assumptions to semantically constrain the $\mathcal{T}^{\text{n}}$-models in order to emulate the corresponding $\mathcal{CL}$-models.

In the case of our basic system $\mathcal{K}^{\text{n}}$, a counterpart for the above result would not try to recover all classical inferences. The natural candidate, in that case, would be a weaker system, which we briefly mention. Let $\mathcal{DM}$ be the system obtained by adding rules (dm1.1), (dm1.2), (dm2.1) and (dm2.2) to system $\mathcal{DL}$, let $\mathcal{DM}^{i}$ be $\mathcal{DM}$ plus (dm3.1), and let $\mathcal{DM}^e$ be $\mathcal{DM}$ plus (dm3.2). Adding both (dm3.1) and (dm3.2) to $\mathcal{DM}$ characterizes the so-called De Morgan Logic (cf. (Font, 1997)). Now, other Derivability Adjustment Theorems are to be expected if we fix our attention on the relation between $\mathcal{DM}^i$ and the paracomplete fragment of $\mathcal{K}^n$, or on the relation between $\mathcal{DM}^e$ and the paraconsistent fragment of $\mathcal{K}^n$. Furthermore, if $\mathcal{B}^n$ is used instead of $\mathcal{T}^n$ then less consistency and determinedness assumptions will need to be collected, as iterated negation is more well-behaved by the very design of $\mathcal{B}^n$.

A fully detailed exploration of the latter results on derivability adjustment is left as matter for a future study.

4.8 Closing Remarks

We have started our study from the logic underlying bounded distributive lattices and investigated in this chapter the logic $\mathcal{K}^n$ that upgrades the former by adding a modal paraconsistent negation and a modal paracomplete negation, and also adds modal operators internalizing appropriate notions of consistency and determinedness into the object-language level. We have characterized the properties of our connectives from an abstract viewpoint, proposed a sequent-style proof formalism for the minimal normal system enjoying such properties in our chosen language, and proven its completeness with respect to the expected standard Kripke-like semantics. We have also considered two extensions of our basic system, adding axioms connected to versions of excluded middle, pseudo-scotus and forms of double negation manipulation, and we have discussed how these systems allow one to recover the inferences of some logics lying in between De Morgan Logic and Classical Logic. Studying other extensions should be instigating inasmuch as they are attained by adding axioms that express intuitively important properties of *negation*, such as the 'controllable forms' of consistency and of determinedness expressed by $\smile\varphi, \smile(\smile\varphi) \Rightarrow$ and $\Rightarrow \frown(\frown\varphi), \frown\varphi$, which are valid in euclidean frames. Axioms that involve the interaction between the two non-classical negations are also attractive, such as $\smile\varphi \Rightarrow \frown\varphi$, valid in functional frames, or as $\frown\varphi, \smile(\frown\varphi) \Rightarrow$ and $\Rightarrow \frown(\smile\varphi), \smile\varphi$, valid in transitive frames, or as $\smile\smile\varphi \Rightarrow \frown\frown\varphi$, valid in confluent (a.k.a. Church-Rosser) frames.

Nonetheless, in producing deductive extensions of the basic system without extending its language, the standard Kripke semantics which we have employed has a somewhat serious shortcoming. Indeed, even though we have thought of our paracomplete negation as independent of our paraconsistent negation, both $\mathcal{T}^{\mathrm{n}}$ and $\mathcal{B}^{\mathrm{n}}$ were built by adding not just one but two 'dual' axioms. It would have seemed more appropriate, however, to devise complete systems in which each one of those axioms could be introduced in separate. An obvious alternative to deal with such difficulty related to frame incompleteness is simply to change the semantical framework. Such a strategy is common in the literature on systems of intuitionistic modal logics, in which a second relation (a quasi ordering) is added to the frame, coupled with the consideration of truth-increasing valuations. This seems very well-motivated, and would allow one to prove in particular that truth is hereditarily preserved towards the future, according to the order introduced by the second relation, and falsity is hereditarily preserved towards the past, according to the same order (for the positive case, cf. (Celani; Jansana, 1997); for an application to the case of our modal negations, cf. (Dunn; Zhou, 2005)). The additional advantage of this alternative framework, besides bringing the heredity conditions to the fore, is that it allows one to add each axiom in separate, and continue thinking thus about the two non-classical negations as really independent of each other. However, that strategy cannot be extended without modification to our richer language. The reason is simple: the restoration connectives were in a sense designed to *fail* the heredity conditions, as they allow one to recover standard classic-like models when they are applied to sentences of a given theory. It rests as a challenge, thus, to identify the right semantic framework in which the study of extensions of our system $\mathcal{K}^{\mathrm{n}}$ should be done. For one thing, from (Marcos, 2005a) we already know that if we add a classical implication connective $\rightarrow$ to $\mathcal{K}^{\mathrm{n}}$, *any* normal modal logic may be rewritten in the minimal language containing just such $\rightarrow$ and the paraconsistent negation $\smile$; in this case indeed the usual classical connectives, the usual box-plus and diamond-plus connectives, the dual paracomplete negation $\frown$, and the restoration connectives dealing with $\smile$-consistency and with $\frown$-determinedness may all be explicitly *defined*. There is also a rich literature (important references include (Došen, 1984; Vakarelov, 1989)) concerning the systems obtained by the addition of an intuitionistic implication instead of a classical implication — for those systems it is customary to consider interpretation structures containing two accessibility relations, one to deal with implication and another one to deal with the non-classical negations. In the present study we have concentrated however on the implicationless fragment of these logics, to which the restoration connectives were explicitly added in order to internalize the corresponding useful meta-theoretical concepts.

Another line of research that we see as potentially fruitful is the investigation of matters related to variegated versions of our Derivability Adjustment Theorems, especially from a semantical perspective. We note that there is a modular way of connecting 'quasi canonical sequent rules' such as the main ones we have proposed in this chapter to restrictions concerning the so-called 'non-deterministic semantics' (cf. (Avron, 2005; Avron; Konikowska; Zamansky, 2012)). From that viewpoint, one may see how De Morgan Logic gets associated to four truth-values, where conjunction and disjunction are interpreted as in Dunn-Belnap matrices, and its negation (both paraconsistent and paracomplete) is defined according to the so-called truth-order. Furthermore, by adding rule $(/\sim)$ a further determinization is produced, and only three truth-values are left, as negation ceases to be paracomplete; an analogous phenomenon happens if $(\sim/)$ is added, and negation ceases to be paraconsistent; if both rules are added, Classical Logic is obtained. Now, if one considers $\mathcal{DM}$ from the start, a four-valued semantics is still available, but negation is non-deterministic: there are two possibilities of output for each of the four inputs. Such negation may be partially determinized by adding rules (dm3.1) or (dm3.2); adding both rules would result in the full determinization that corresponds to De Morgan Logic. Our non-classical negations go the other way round, by deleting some De Morgan rules, (dm1.2) or (dm2.2). The result of performing this deletion over $\mathcal{DM}$ is that disjunction will also start to behave non-deterministically. Such modular approach may be easily extended to include the consistency and the determinedness operators, which will also be (non-deterministically) interpretable over the already mentioned four truth-values (an automated mechanism for uncovering the semantic aspects of such paraconsistent fragments of $\mathcal{DM}$ was launched in (Ciabattoni et al., 2013)). Our Derivability Adjustment Theorems could then be thought of as ways of taming non-classicality and controlling non-determinism from a logical viewpoint.

Some of the sequent rules that we have studied are more important than others. Such is the case of the interaction rules $(\frown\smile)$ and $(\smile\frown)$, which could be thought of as a sort of multiple-conclusion sequent calculus contextual generalization of the so-called 'Becker's Rule', from the traditional modal literature, adapted to the case of negative normal modalities. To the best of our knowledge, they seem not to have been proposed before. It is worth noting that by the addition of the usual sequent rules for classical implication, our system $\mathcal{K}^n$ is upgraded into a modal version of the logic of formal inconsistency $\mathcal{BK}$ (see (Avron; Konikowska; Zamansky, 2013)), obtained precisely by the addition of the already mentioned interaction rules (so, to be sure, $\mathcal{K}^n$ plus classical implication coincides with $\mathcal{BK}$ plus interaction rules). Such interaction rules are indeed absolutely instrumental in warranting the modal character of our systems (and, in particular, in guaranteeing that we are dealing with systems respecting the standard *replacement property*), and it seems worth studying the classes of paraconsistent and paracomplete logics that lend themselves in a natural way to reasonable extensions obtained by the addition of such rules. In a future study we will also show how sequent systems such as those studied in the present chapter may be seen as particular examples of the 'Basic Sequent Systems' studied in (Lahav; Avron, 2013). In that paper, the authors have shown how to provide Kripke semantics to such kinds of systems in a way so as to allow one to semantically obtain confirmations of important proof-theoretic properties such as cut-admissibility and analyticity. In showing that the mentioned approach indeed applies to our systems, we will guarantee that one can count on such proof-theoretic properties, provide alternative completeness proofs and allow for a smoother extension of our systems to normal systems characterized by other important classes of frames.

5 Conclusion

We studied in this Thesis distinct ways of enriching some modal logics. First, in the context of fuzzy modal logics, we realized this task semantically. The partition $\{[0, i), [i, 1]\}$ is the base for the construction of a fuzzy Kripke semantics that characterize the modal system K and its extensions with multiple instances of the axiom $G^{k,l,m,n}$.

In Chapter 3 we present a Hilbert axiomatization for the regular *LAB* logics, logics that contains the positive fragment of classical logic plus one paraconsistent negation. After that we prove a completeness result for the minimal logic $K_{\smile}$ and for some extensions of it, like the systems $S4_{\smile}$ and $S5_{\smile}$, systems that characterize the same class of frames that the systems $S4$ and $S5$, which are well studied in the literature, see((Cresswell; Hughes, 2003), (Chellas, 1980)). In this logic the positive modalities and all modalities studied in Chapter 4 can be defined. In Chapter 4 another form of enrichment of modal logics is studied. We add to a fragment of positive classical logic, without implication, the connectives $\smile$ and $\frown$ that intend to represent a paraconsistent negation and a paracomplete negation respectively, and its associated restoration connectives $\ominus$ and $\circleddash$. Here our minimal logic $\mathcal{K}^{\mathrm{n}}$ is presented in terms of a sequent calculus, that contains rules of introduction and elimination for the connectives $\wedge, \vee, \ominus$ and $\circleddash$ and rules $(\smile\frown)$, $(\frown\smile)$ that express appropriate interactions between our modal negations. The systems $\mathcal{T}^{\mathrm{n}}$ and $\mathcal{B}^{\mathrm{n}}$ extend $\mathcal{K}^{\mathrm{n}}$ by adding dual axiomatic rules, and they are sound and complete with respect to the class of reflexive frames and symmetric frames, respectively.

By means of adequate DATs, we also see that the extensions of $\mathcal{K}^{\mathrm{n}}$, and even extensions of the simple system $\mathcal{DL}$, may be used to talk about more expressive systems. The logics $\mathcal{T}^{\mathrm{n}}$ and $\mathcal{B}^{\mathrm{n}}$, for example, allow Classical Logic to be recovered, while the logics $\mathcal{DM}^{i}$ and $\mathcal{DM}^{e}$ recover the DeMorgan Logic. These informations are summarized in the Table 3.

We would like to suggest three more lines of research for future investigations, besides those cited in the final sections of Chapters 2 and 4. The first of them is related

Deductive System	Recovered Inferences	Sequent Rules
$\mathcal{T}^{n}$	Classical Logic	$\mathcal{K}^{n} \cup \{\overline{\Rightarrow \varphi, \smile\varphi}^{(rf1)}, \overline{\frown\varphi, \varphi \Rightarrow}^{(rf2)}\}$
$\mathcal{B}^{n}$	Classical Logic	$\mathcal{K}^{n} \cup \{\overline{\smile\smile\varphi \Rightarrow \varphi}^{(sm1)}, \overline{\varphi \Rightarrow \frown\frown\varphi}^{(sm2)}\}$
$\mathcal{DM}^{i}$	DeMorgan Logic	$\mathcal{DM} \cup \{$dm(3.1)$\}$
$\mathcal{DM}^{e}$	DeMorgan Logic	$\mathcal{DM} \cup \{$dm(3.2)$\}$

Table 3: Rich logics

to the property of confluence, that is mentioned in different places in this thesis. In Chapter 2 frames with this property are used to characterize the so called fuzzy modal logics of confluence. In Chapter 4 we saw that extending $\mathcal{K}^{n}$ with the sequent rule $\overline{\smile\smile\varphi \Rightarrow \frown\frown\varphi}^{(cnf)}$ produces a system characterized by the class of confluent frames. The version of confluence axiom studied in Chapter 2 is formulated in terms of the modalities $\Box$ and $\Diamond$, the instances $\Box\varphi \supset \varphi$ and $\varphi \supset \Box\Diamond\varphi$, that express in first-order logic reflexivity and symmetry respectively, have equivalent negative versions, namely, $\smile\varphi \vee \varphi$ and $\smile\smile\varphi \supset \varphi$, that were studied in Chapter 3. What about the general case? Can we formulate an axiom-schema, using the negative modalities $\smile$ and $\frown$, similar to $\Diamond^{k}\Box^{l}\varphi \supset \Box^{m}\Diamond^{n}\varphi$?

Other investigation that deserves our attention is one that considers multiple paraconsistent negations inside a logic. Restall studied in (Restall, 1997) a logic with multiple positive and negative diamond-like modal operators, that are paracomplete negations. The author did not study the case where there are interactions among his negations. We think that it is possible to propose a logic with two paraconsistent negations by choosing, in the proof-theoretical presentation, adequate axioms of interaction. Another challenge is the case of a logic with more than two paraconsistent negations: how we can formulate its interaction axioms?

Another interesting work is to investigate if it is possible to propose a semantic enrichment, similar the one proposed in Chapter 2, to *LAB* logics and to logics with negative modalities studied respectively in Chapters 3 and 4.

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APPENDIX A – MTD in LAB Logics

In what follows let (Ax1) and (Ax2) denote the classical theorems $p \supset (q \supset p)$ and $(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$ respectively.

Proposition 3.1.2 *Let α and β be formulae of $S_{\wedge\vee\supset\smile}$ and $\Gamma \subseteq S_{\wedge\vee\supset\smile}$. Given a derivation for $\Gamma, \alpha \vdash \beta$, it is possible to build a derivation for $\Gamma \vdash \alpha \supset \beta$.*

Proof. The proof is an induction on the length of the derivation $\beta_1, \beta_2, \ldots, \beta_n$ of β_n from $\Gamma \cup \{\alpha\}$, where β_n is β.
[Basis] 1-step derivations. There are two cases to consider:

Case 1 β is an axiom

Consider the following derivation

1. β [Axiom]
2. $\beta \supset (\alpha \supset \beta)$ [Ax1]
3. $\alpha \supset \beta$ [MP 1,2]

Case 2 $\beta \in \Gamma \cup \{\alpha\}$

(a) $\beta \in \Gamma$

1. β [Hyp]
2. $\beta \supset (\alpha \supset \beta)$ [Ax1]
3. $\alpha \supset \beta$ [MP 1,2]

Since $\beta \vdash \alpha \supset \beta$, by monotonicity, $\Gamma \vdash \alpha \supset \beta$.

(b) β is α

Note that $\beta \supset \beta$ is a theorem of LAB Logics. In fact, consider the following derivation:

1.	$\beta \supset ((\beta \supset \beta) \supset \beta)$	[Ax1]
2.	$\beta \supset (\beta \supset \beta)$	[Ax1]
3.	$(\beta \supset ((\beta \supset \beta) \supset \beta)) \supset ((\beta \supset (\beta \supset \beta)) \supset (\beta \supset \beta))$	[Ax2]
4.	$(\beta \supset (\beta \supset \beta)) \supset (\beta \supset \beta)$	[MP 1,3]
5.	$\beta \supset \beta$	[MP 2,4]

[Step] (k+1)-step derivations.

Let $\mathcal{D}$ be a derivation for $\Gamma, \alpha \vdash \beta$ with $k+1$ steps. Assume by Induction Hypothesis that there is a derivation for $\Gamma \vdash \alpha \supset \beta_i$ for every $1 \leq i \leq k$. We shall show that $\Gamma \vdash \alpha \supset \beta$. There are three cases to consider:

Case 1 The last step in $\mathcal{D}$ is an application of MP on lines $i < k+1$ and $j < k+1$.

By I.H. we can build the following two derivations:

Derivation $\mathcal{D}_1$:

$\vdots$

p. $\alpha \supset \beta_i$ [Justification]

Derivation $\mathcal{D}_2$:

$\vdots$

q. $\alpha \supset (\beta_i \supset \beta)$ [Justification]

Let the Derivation $\mathcal{D}_3$ be the concatenation of $\mathcal{D}_1$ and $\mathcal{D}_2$:

	$\vdots$	
p.	$\alpha \supset \beta_i$	[Justification]
	$\vdots$	
p+q.	$\alpha \supset (\beta_i \supset \beta)$	[Justification]

To conclude the proof add to $\mathcal{D}_3$ the following three lines:

p+q+1.	$(\alpha \supset (\beta_i \supset \beta)) \supset ((\alpha \supset \beta_i) \supset (\alpha \supset \beta))$	[Ax2]
p+q+2.	$(\alpha \supset \beta_i) \supset (\alpha \supset \beta)$	[MP p+q, p+q+1]
p+q+3.	$\alpha \supset \beta$	[MP p, p+q+2]

Case 2 The formula β is $\smile\psi \supset \smile\varphi$, that is obtained by an application of (R1) to the previous theorem $\vdash \varphi \supset \psi$. Thus, $\vdash \beta$. From the latter there is a derivation $\mathcal{D}_3$, with j steps, that ends with β. Add to $\mathcal{D}_3$ the lines $j+1$ and $j+2$, as illustrated below, to obtain $\alpha \supset \beta$.

	$\vdots$	
j.	β	[Justification]
j+1.	$\beta \supset (\alpha \supset \beta)$	[Ax1]
j+2.	$\alpha \supset \beta$	[MP j, j+1]

Case 3 The formula β is $\smile\varphi \supset \psi$ that is obtained by an application of (R2) to the previous theorem $\vdash \varphi$. Since $\vdash \beta$, we can build a derivation for $\alpha \supset \beta$ similarly to what has been done in the Case 2.

QED