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ALLEN ROOSSIM PASSOS IBIAPINA

MENGER'S THEOREM AND RELATED PROBLEMS ON TEMPORAL GRAPHS

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Thesis submitted to the Programa de Pós-Graduação em Matemática of the Centro de Ciências of the Universidade Federal do Ceará, as a partial requirement for obtaining the title of Doctor in Matemática. Concentration Area: Combinatória

Advisor: Prof. Dr. Ana Shirley Ferreira da Silva

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ABSTRACT

Temporal graphs are an important tool for modeling time-varying relationships in dynamic systems. Among the problems of interest, paths and connectivity problems have been the ones that have attracted the most attention of the community. The famous Menger's Theorem says that, for every pair of non-adjacent vertices s,z in a graph G, the maximum number of internally vertex disjoint s,z-paths in G is equal to the minimum number of vertices in $G-\{s,z\}$ whose removal destroys all s,z-paths (called and s,z-cut). This combined with flow techniques also leads to polynomial time algorithms to compute disjoint paths and cuts. A natural question is therefore whether such results carry over to temporal graphs, where the paths/walks between a pair of vertices must respect the flow of time. The first obstacle in such question is the definition of the robustness in context, i.e., what it means for paths/walks to be disjoint. In this thesis, we investigate three possible types of robustness, each of which leading to the definition of two optimization parameters, one concerning the maximum number of disjoint paths, and the other the minimum size of a cut. We give theoretical results about the validity of Menger's Theorem and computational complexity results for the decision problems related to each parameter.

Keywords: temporal graphs; Menger's theorem; connectivity; topological minors.

RESUMO

Grafos temporais são uma ferramenta importante para modelar relacionamentos que variam com o tempo em sistemas dinâmicos. Entre os problemas de interesse, os problemas de caminhos e conectividade têm sido os que mais atraíram a atenção da comunidade. O famoso Teorema de Menger diz que, para cada par de vértices não adjacentes s,z em um grafo G, o número máximo de caminhos internamente disjuntos entre s e z em G é igual ao número mínimo de vértices em $G-\{s,z\}$ cuja remoção destrói todos os caminhos entre s e z (chamado de corte s,z). Isso, combinado com técnicas de fluxo, também leva a algoritmos de tempo polinomial para calcular caminhos disjuntos e cortes. Uma questão natural é, portanto, se tais resultados se aplicam a grafos temporais, onde os caminhos/passeios entre um par de vértices devem respeitar o fluxo do tempo. O primeiro obstáculo nessa questão é a definição de robustez no contexto, ou seja, oque significa caminhos/passeios serem disjuntos. Nesta tese, investigamos três possíveis tipos de robustez, cada um dos quais leva à definição de dois parâmetros de otimização, um relacionado ao número máximo de caminhos disjuntos e o outro ao tamanho mínimo de um corte. Apresentamos resultados teóricos sobre a validade do Teorema de Menger e resultados de complexidade computacional para os problemas de decisão relacionados a cada parâmetro.

Palavras-chave: grafos temporais; teorema de Menger; menores topológicos; conectividade.

CONTENTS

1 INTRODUCTION

A temporal graph is a structure that captures the evolution of a graph over time. In a temporal graph, vertices represent entities, and edges represent relationships between these entities that change over time. Each edge has a timestamp associated with it, which indicates when the relationship between the corresponding vertices took place. Temporal graphs or similar structures have also been called temporal networks (1), time-varying graphs (2), edge-scheduled networks (3), among other names. Although many names have been used, there is little (if any) distinction between the various models. Such variety of names comes from the diversity of fields on which these structures can be applied. As examples, temporal graphs have been used to study spreading of diseases (4), interaction between species (5), development of cancer (6), robotics (7), etc. For other variety of applications of temporal graphs, we refer the reader to the work of Holme (1).

As a toy example, suppose we want to study the spread of a rumor in a social network. We could model the network as a temporal graph, where each vertex represents a person and each edge represents a communication event between two people. Each edge would have a timestamp associated with it, indicating the time at which the communication event occurred. As the rumor spreads, we could use the temporal graph to track its propagation over time. We could apply techniques from dynamic network analysis to identify the most influential vertex in the network, track the evolution of communities within the network, and study how the rumor spreads through different subgroups of people.

In particular, many of the problems investigated so far are related to temporal paths or walks. A temporal walk in a temporal graph is a walk in which the activation time of its edges is non-decreasing, and a temporal path is a temporal walk that does not repeat vertices. The study of temporal walks and paths in temporal graphs introduces additional challenges and complexities. Many problems that exist in static graphs can be translated to temporal graphs, and each translation can lead to many variations of the original problem. For example, in (8) they study, for many notions of minimality, temporal paths between a pair of vertices. In (9), they investigate the existence of a temporal path between a pair of vertices under constraints on the time each path can spend on each vertex. In (10), the interest is about a minimum subset of the edges such that every pair of vertices is connected through a temporal walk using such edges.

Other problems closely related to paths in graphs are those known as connectivity problems, or even robustness problems. In this context, Menger's Theorem is one of the most fundamental results in graph theory. It relates the maximum number of internally[1] vertex disjoint paths between a pair of vertices in a graph and the minimum number of vertices that must be removed to disconnect such pair. Formally, it can be stated as follows.

Theorem 1.0.1 (Menger, 1927(11)) *Let G be a graph and $s,z \in V(G)$ be a pair of non-adjacent vertices of G. Then the maximum number of vertex (edge) disjoint s,z-paths in G is equal to the minimum size of a subset $S \subseteq V(G) \setminus \{s,z\}$ ($S \subseteq E(G)$) that intersects every s,z-path.*

A natural question is how to adapt Menger's Theorem for temporal graphs. In this thesis we focus on three possible versions, which we discuss in Section 1.1. We have also worked on other problems not related to temporal graphs, which we discuss in Section 1.2.

1.1 Outline of Contributions

In this section, we will briefly describe versions of Menger's Theorem that we studied and provide an overview of our contribution to this field.

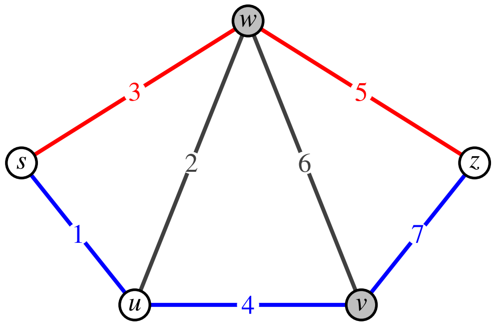
1.1.1 Vertex disjointness

Perhaps the most direct translation of the concepts in Menger's Theorem to the temporal context is to consider vertex disjoint temporal paths. Given a temporal graph and vertices s and z, we are interested in (1) finding the maximum number of vertex disjoint temporal paths from s to z, and (2) determining the minimum number of vertices that intersects every temporal path from s to z. In Figure 1 one can see examples of such sets. The graph depicted there is also called gem.

As stated previously, the question of interest is whether these two parameters are equal. As shown by Berman (3), this is not always the case. Knowing this, in their seminal work (12), Kempe, Kleinberg, and Kumar introduced the concept of *Mengerian Graphs*. These are graphs that satisfy the equality between the parameters (1) and (2), regardless of how the edges are timed. They then characterized Mengerian graphs restricted to functions where a pair of vertices is connected at most once through time. Such characterization is very elegant: a graph G is Mengerian under this constraint if and only if G has no gem as topological minor. This is a fascinating result, and one that even leads to a recognition algorithm thanks to a result

[1] In this thesis, all disjointness of paths and walks are considered only in the internal vertices, so from hereon we refrain from explicitly using the word "internally".

Figure 1 – Example of a temporal graph. Two vertex disjoint temporal paths from s to z are highlighted in red and blue. A set of vertices that intersect each temporal path from s to z is highlighted in lightgray



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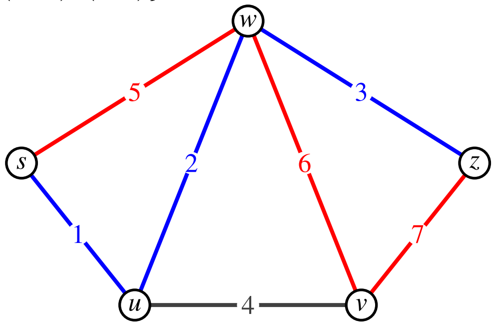
by Grohe, Kawarabayashi, Marx, and Wollen (13). However, the imposed constraint is quite a strong and unrealistic one, as indeed most real-life temporal graphs have multiple connections between the same pair of vertices through time. In this thesis, we provide a full characterization of Mengerian graphs, thus overcoming the limitation on the number of times a pair of vertices can be connected. For this, we introduce a new notion of minor, called m-topological minor, and we prove that Mengerian graphs are characterized by three forbidden m-topological minors. As this is a new concept, no general algorithm like the one presented in (13) is known to detect whether a graph has one of the forbidden m-topological minors. We then give a polynomial-time algorithm to decide whether a given graph G has any of the forbidden graph as m-topological minors, thus recognizing whether G is Mengerian. These, and other smaller results, can be found in Chapter 3. They appear at the Proceedings of the 23rd International Symposium on Fundamentals of Computation Theory - FCT 2021 (14).

1.1.2 Temporal vertex disjointness

In this version of the problem, we consider a temporal graph and a pair of non-adjacent vertices s, z, where two walks/paths from s to z can share a vertex, but not at the same time. We call these t-vertex disjoint walks/paths. In Figure 2, we have two temporal paths, highlighted in red and blue. These paths intersect in the vertex w, however, not at same time as the red one passes by w at times 5 and 6 and the blue at times 2 and 3. An interesting aspect of this version is that, unlike the other two, the parameters defined on disjoint walks differ from the parameters defined on disjoint paths. As before, given a temporal graph and non-adjacent vertices s and z, in each context we study two parameters: (1) the maximum number of t-vertex disjoint temporal walks/paths from s to z; and (2) finding a subset of elements of type (v,i), where v is a vertex and i is a time, $v \notin \{s,z\}$, that intersects every temporal walk/path. Such pairs of vertex and time are called temporal vertices.

Figure 2 – Example of a temporal graph. Two t-vertex disjoint temporal paths from s to z are highlighted in red and blue. A set of temporal vertices that intersect each temporal path from s to z is $\{(w,2),(v,6)\}$



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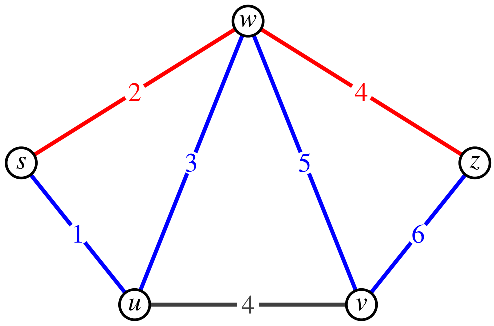
We show that, if walks are considered, then (1) and (2) are always equal and can be found in polynomial time. However, when defined in terms of paths, these problems are much more interesting. Not only (1) and (2) can be arbitrarily apart if paths are considered, but finding such values cannot be done in polynomial time, unless $\mathsf{P} = \mathsf{NP}$. We also provide an algorithm to decide whether (2) is at most h which runs in XP time when parameterized by h. We mention that such algorithm is not trivial, as we also prove that deciding whether a given set of temporal vertices intersects all temporal s,z-paths is co-NP-complete. These and other smaller results are presented in Chapter 4 and are currently submitted for publication. We also present the state of the art of the computational complexity of the decision problems related to (1) and (2) for vertex disjoint paths, t-vertex disjoint walks and t-vertex disjoint paths. We decided to present the computational complexity results for vertex disjoint paths in this chapter so we could better confront them with the computational complexity results for t-vertex disjoint walks/paths.

1.1.3 Snapshot disjointness

We introduce a new robustness concept that depends on the times, rather than on vertices and edges. Given a temporal graph and any pair of vertices s,z, we say that a pair of temporal s,z-paths are snapshot disjoint if, whenever one of them traverses an edge in a given moment of time, the other is not allowed to traverse any edge. In other wors, the sets of times in which each is active are disjoint. For an example, consider the red and blue paths in Figure 3.

The blue path is active in times $\{1,3,5,6\}$, and the red path in times $\{2,4\}$. Again, we are interested in (1) the maximum number of snapshot disjoint temporal paths between s and z; and (2) the minimum number of times that intersects all temporal paths between s and z. Figure 3 shows an example where these two parameters are equals to 2.

Figure 3 – Example of a temporal graph. Two snapshot disjoint temporal paths from s to z are highlighted in red and blue. Any temporal path from s to z intersect an edge active at time 4 or 5



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We show that, as happened with the vertex disjoint version, if defined in terms of walks, instead of paths, the values (1) and (2) do not change. Additionally, they are arbitrarily far apart, which inspired us to define the analogous notion of Mengerian graphs. A graph is s-Mengerian if there is equality between (1) and (2), regardless of how the edges are timed. We then characterize s-Mengerian graphs by a list of 5 forbidden m-topological minors, and we give a polynomial-time recognition algorithm for this class. Additionally, we give tight results for the parameterized complexity of the problems of deciding whether (1) is at least k, and whether (2) is at most k. We show that both are W[1]-hard and XP when parameterized by k. We highlight the fact that, while our XP algorithm for (2) is somewhat straightforward, the algorithm for (1) is much more involved and applies the technique used by Shiloach and Perl to solve the k-linkage problem on DAG's (15). These and other smaller results can be found in Chapter 5, and appear at the Proceedings of the 2nd Workshop of Algorithmic Foundations of Dynamic Networks - SAND 2023 (16). Our paper has won the best student paper award of the conference.

1.2 Other Contributions

Furthermore, two additional manuscripts were co-authored during the course of the PhD program. While these projects do not fit within the scope of this thesis, they represent important contributions to the field and deserve a brief highlight here.

The first work, published in the *Journal of Discrete Applied Mathematics* (17), investigated the *Nash Number* and *diminishing Grundy Number* of a graph, which are graph colouring parameters. The following is the abstract of such work.

> A *Nash k-colouring* is a k-colouring $(S_1,\dots,S_k)$ such that every vertex of S_i is adjacent to a vertex in S_j, whenever $|S_j| \geq |S_i|$. The *Nash Number* of G, denoted by $\mathrm{Nn}(G)$, is the largest k such that G admits a Nash k-colouring. A *diminishing greedy k-colouring* is a k-colouring $(S_1,\dots,S_k)$ such that, for all $1 \leq j < i \leq k$, $|S_i| \leq |S_j|$ and every vertex of S_i is adjacent to a vertex in S_j. The *diminishing Grundy Number* of G, denoted by $\Gamma^{\downarrow}(G)$, is the largest k such that G admits a diminishing greedy k-colouring. In this paper, we prove some properties of Nn and $\Gamma^{\downarrow}$. We mainly study the relations between them and other graph parameters such as the clique number ω, the chromatic number χ, the Grundy number Γ, and the maximum degree Δ. In particular we study the chain of inequalities $\omega(G) \leq \chi(G) \leq \mathrm{Nn}(G) \leq \Gamma^{\downarrow}(G) \leq \Gamma(G) \leq \Delta(G)+1$. We show each inequality $\gamma_1(G) \leq \gamma_2(G)$ of this chain is loose, that is that there is no function f such that $\gamma_2(G) \leq f(\gamma_1(G))$. We also prove the existence or non-existence of Reed's like inequality who proved that there exists $\varepsilon > 0$ such that $\chi(G) \leq \varepsilon\omega(G) + (1-\varepsilon)(\Delta(G)+1)$.
>
> We then study the Nash number and the diminishing Grundy number of trees and forests, and prove that $\Gamma(F) - 1 \leq \mathrm{Nn}(F) \leq \Gamma^{\downarrow}(F) \leq \Gamma(F)$.
>
> Finally we study the complexity of related problems. We show that computing the Nash number or the diminishing Grundy number is NP-hard even when the input graph is bipartite or chordal. We also show that deciding whether a graph satisfies $\gamma_1(G) = \gamma_2(G)$ is NP-hard for every pair (γ_1, γ_2) with $\gamma_1 \in \{\mathrm{Nn}, \Gamma^{\downarrow}\}$ and $\gamma_2 \in \{\omega, \chi, \Gamma, \Delta+1\}$.

The second one, published in the *Eletronic Journal of Combinatorics* (18), focuses on variations of the coloring problems constrained to bipartite graphs with bounded diameter. We give hardness results for k-List Coloring, k-Coloring, k-Precoloring Extension, Surjective C_6-Homomorphism and 3-Fall Coloring.

> We investigate a number of coloring problems restricted to bipartite graphs with bounded diameter. First, we investigate the k-LIST COLORING, LIST k-COLORING, and k-PRECOLORING EXTENSION problems on bipartite graphs with diameter at most d, proving NP-completeness in most cases, and leaving open only the LIST 3-COLORING and 3-PRECOLORING EXTENSION problems when $d = 3$.

Some of these results are obtained through a proof that the SURJECTIVE C_6-HOMOMORPHISM problem is NP-complete on bipartite graphs with diameter at most four. Although the latter result has been already proved [Vikas, 2017], we present ours as an alternative simpler one. As a byproduct, we also get that 3-BICLIQUE PARTITION is NP-complete. An attempt to prove this result was presented in [Fleischner, Mujuni, Paulusma, and Szeider, 2009], but there was a flaw in their proof, which we identify and discuss here.

Finally, we prove that the 3-FALL COLORING problem is NP-complete on bipartite graphs with diameter at most four, and prove that NP-completeness for diameter three would also imply NP-completeness of 3-PRECOLORING EXTENSION on diameter three, thus closing the previously mentioned open cases. This would also answer a question posed in [Kratochvíl, Tuza, and Voigt, 2002].

2 DEFINITIONS AND TERMINOLOGY

In this chapter we define almost all terminology used trough the thesis, with some occasional exceptions. In the first section, we define general objects that will be used through all the text. We also introduce a new notion, that of m-topological minors, which are our way of extending topological minors to take the multiplicity of an edge into consideration. Such notion will be used in Chapters 3 and 4. Then, we define temporal graphs and its terminology. Finally, we given some intuitive definitions of computational complexity.

2.1 Graph theory concepts

Given a natural number n, we denote by $[n]$ the set $\{1,2,\ldots,n\}$. Given a function $f\colon X \to Y$ and a subset $W \subseteq Y$, we denote by $f^{-1}(W)$ the set $\{x \in X \mid f(x) \in W\}$.

A *graph* is an ordered triple $G = (V,E,f)$ where V is a finite non-empty set (called *set of vertices*), E is a finite set (called the *set of edges*), and f is a function that associates to each $e \in E$ an unordered pair xy of vertices of V. The pair xy is called the *endpoints of e*, and if a pair xy has at least one related edge, then xy is also called *multiedge*. For simplicity, in what follows we omit the function f and simply refer to the endpoints of each edge instead. The *multiplicity* of xy is the number of edges that are associated with this pair. If the graph is denoted by G, we also use $V(G), E(G)$ to denote its vertex and edge sets, respectively. If G contains any edge with endpoints uv, then we say that u and v are *neighbors*, and that u is *adjacent* to v. The *neighborhood* of u is then the set of neighbors of u, and is denoted by $N_G(u)$. The *degree* of a vertex $v \in V(G)$ is equal to $|N_G(u)|$ and is denoted by $d_G(v)$. We omit the subscript when G is clear from the context. Notice that the degree of a vertex may be different from the number of edges incident to it. The *maximum degree* of G is then the maximum among the degree of vertices of G; it is denoted by $\Delta(G)$.

If a graph H is such that $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and each edge of H is associated to the same endpoints in G and H, then H is called a *subgraph of* G, and we write $H \subseteq G$. For each multiedge xy of multiplicity $k \geq 2$ of G, if we remove $k-1$ edges connecting the pair xy, we obtain a subgraph of G where all multiedges have multiplicity 1. Such subgraph is called *underlying simple graph* of G and is denoted by $U(G)$.

For a subset of vertices $S \subseteq V(G)$, $G - S$ is the subgraph of G such that the vertex set is $V(G) - S$ and the set of edges is the subset of edges in $E(G)$ composed by the edges with both endpoints in $V(G)-S$. For $T \subseteq E(G)$, the subgraph of G denoted by $G-T$ is $(V(G),E(G)-T)$. If x_1y_1 and x_2y_2 are multiedges, then $G-\{x_1y_1,x_2y_2\}$ is the graph $G-F$ where F is the set of edges with endpoints x_1y_1 or x_2y_2.

A *clique* is a subset of pairwise adjacent vertices, i.e., $C \subseteq V(G)$ such that for every $u,v \in C$, there is an edge connecting the pair uv. An *independent set* of G is a set $I \subseteq V(G)$ such that there are no edges connecting a pair uv for every $u,v \in I$. For any integer $j \geq 1$, we denote by K_j the complete graph on j vertices. That is, a graph in which every pair of distinct vertices is connected by exactly one edge (i.e., all edges have multiplicity 1), and the vertex set forms a clique.

Given two vertices $s,z \in V(G)$, an *s,z-walk* P in G is a sequence that alternates vertices and edges $(s=u_1,e_1,\ldots,e_{q-1},u_q=z)$ and it is such that e_i is an edge with endpoints u_iu_{i+1}, for every $i \in [q-1]$. We may denote such path by sPz, to emphasize the first and last vertices. If, additionally, no vertex appears more than once in P then P is called an *s,z-path*; and if $u_1=u_q$ and this is the only vertex appearing more than once in P, then P is called a *cycle*. A *subpath of P* is a path $(u_i,e_i,\ldots,e_{j-1},u_j)$ with $i,j \in [q-1]$ and $i \leq j$. We denote such subpath by u_iPu_j. If G is a simple graph, then we omit the edges in the sequence P. For subsets $X,Y \subseteq V(G)$, we say that P is an X,Y-path if P is an x,y-path for some $x \in X$ and $y \in Y$, and whose internal vertices do not belong to X nor Y. We denote by $V(P)$ and $E(P)$ the sets of vertices and edges appearing in P, respectively. The *size* of P is the number of vertices in $V(P)$ while the *length* is the number of edges in $E(P)$. For x,y-paths P,Q such that $V(P) \cap V(Q) = \{x,y\}$, we say that P and Q are *(internally vertex) disjoint*.

We say that G is *connected* if there is a path connecting u and v for every pair of vertices $u,v \in V(G)$. A *connected component* of G is a maximal connected subgraph of G. Given a graph G, a *(vertex) cut of G* is a subset $S \subseteq V(G)$ such that $G-S$ has more components than G. We call $u \in V(G)$ a *cut vertex* if $\{u\}$ is a cut. A graph G is *2-connected* if G is connected, has no cut vertex, and $|V(G)| \geq 3$.

If G is connected, for $S \subseteq V(G)$, we say that S *disconnects* G if $G-S$ is not connected. Additionally, for vertices $s,z \in V(G)$, if there are no paths from s to z in $G-S$, then we say that S is an *s,z-cut*. In a similar way, for $S \subseteq E(G)$, we say that S *disconnects* G if $G-S$ is not connected, and for vertices $s,z \in V(G)$, if there are no paths from s to z in $G-S$, then we say that S is an *edge s,z-cut*.

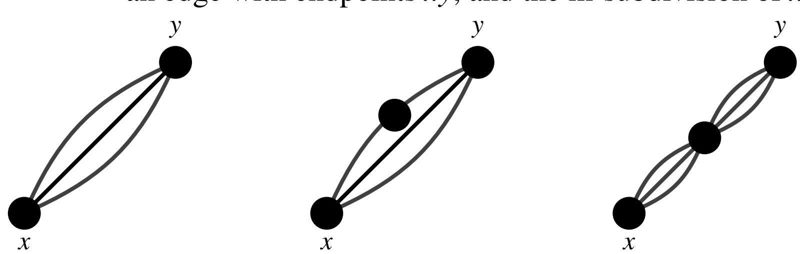
A *block* of G is a maximal connected subgraph of G that has no cut vertex. This is slightly different from being 2-connected as K_1 and K_2 are maximal with no cut vertex but are not 2-connected. Indeed, one can check that if B is a block of G, then either B is 2-connected, or B is an edge, or B is a component of G with only one vertex (see e.g. Section 4.1 of (19)). The *block-cutpoint graph* of G is the graph H having as vertex set the set of cut vertices of G and the set of blocks of G, and such that $uB \in E(H)$ if and only if $u \in V(B)$; it will be denoted by $\mathscr{B}(G)$. The following is largely known.

Theorem 2.1.1 (see e.g. Section 4.1 of (19)) *If G is 2-connected, then $\mathscr{B}(G)$ is a single vertex. And if G is connected but not 2-connected, then $\mathscr{B}(G)$ is a tree whose leaves are blocks of G.*

Observe that the above description tells us that the structure of G, in case G is connected, is that of edges and 2-connected components intersecting in cut vertices in a tree-like way.

Given a graph G and vertices $Z \subseteq V(G)$, the *identification of Z* is the graph obtained from $G - Z$ by adding a new vertex z and, for every edge e with endpoints $z'u$ where $z' \in Z$ and $u \notin Z$, add an edge e' with endpoints zu. Notice that in such operation we can create multiedges with multiplicity greater than one. The graph G' obtained from G by a *subdivision* of an edge e with endpoints uv is the graph having $V(G) \cup \{z_e\}$ as vertex set, and $E(G-e) \cup \{e', e''\}$ as edge set, where e' has endpoints uz_e and e'' has endpoints $z_e v$. Finally, the graph obtained from G by an *m-subdivision* of a multiedge xy is the graph obtained by subdividing all the edges with endpoints xy and then identifying the new vertices. Observe Figure 4 for an illustration of these definitions. Given a graph H, if G has a subgraph that can be obtained from m-subdivisions of H, then we say that H is an *m-topological minor* of G. And if G has a subgraph that can be obtained from subdivisions of H, then we say that H is a *topological minor* of G.

Figure 4 – From left to right: the multiedge xy, the subdivision of an edge with endpoints xy, and the m-subdivision of xy



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A *chain* in G is a path $(u_1, \ldots, u_q)$ in $U(G)$, $q \geq 2$, such that the multiplicity of $u_i u_{i+1}$ in G is at least two, for each $i \in [q-1]$.

2.1.1 Temporal Graphs

On the context of temporal graphs, sometimes we refer to an element of the set $\mathbb{N}\setminus\{0\}$ as a *timestep*.

A *temporal graph* is a pair (G,λ) where G is a graph and λ is a function that assigns to each edge $e\in E(G)$ a value in $\mathbb{N}\setminus\{0\}$. We say that e is *active in time* $\lambda(e)$, and call λ the *time function* of (G,λ). The *lifetime* of (G,λ), denoted by $\tau(G,\lambda)$ is the value $\max\{\lambda(e)\mid e\in E(G)\}$. A *temporal vertex* is an element $(v,t)\in V(G)\times[\tau(G,\lambda)]$, and the set of temporal vertices is denoted by $V^T(G,\lambda)$. For a pair of vertices uw of (G,λ), we denote by $\lambda(uw)$ the set $\{\lambda(e)\mid \text{e has endpoints } uw\}$; if such set is unitary we abuse the notation by saying that $\lambda(uw)$ is the value in such set. For a timestep $i\in[\tau(G,\lambda)]$, we define the *i-th snapshot of* (G,λ) as the subgraph of G defined as $(V(G),\lambda^{-1}(i))$.

Throughout this thesis, we can assume that for each pair of vertices and timestep, at most one edge connecting such a pair is active at such timestep. This allows us to assign for each edge e connecting the pair uv at timestep t the pair (uv,t). We call a pair (uv,t) a *temporal edge*. Observe that there is a bijection between temporal edges and edges of G. More formally, if e is an edge with endpoints uv and $\lambda(e)=t$, then (uv,t) is a temporal edge, and if (uv,t) is a temporal edge, then there is exactly one edge e with endpoints uv such that $\lambda(e)=t$. We denote the set of temporal edges of (G,λ) by $E^T(G,\lambda)$. Also, for each vertex $v\in V(G)$, the temporal degree $d^T(v)$ is the number of temporal edges incident to v.

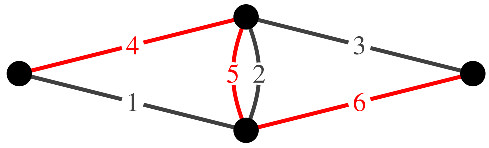
Given $s,z\in V(G)$, a *temporal s,z-walk* P is a walk $(s,e_1,v_2,e_2,\dots,e_{k-1},z)$ in G such that $\lambda(e_1)\le\cdots\le\lambda(e_{k-1})$; if P is also a path in G, then it is called a *temporal s,z-path*. To emphasize the time appearances of such edges, we may denote P by $(s,\lambda(e_1),v_2,\lambda(e_2),\dots,\lambda(e_{k-1}),z)$. In Figure 5 one can see a temporal path highlighted in red. The set of temporal vertices used by P is denoted by $V^T(P)$ and it is the set

$$(\{s\}\times[\lambda(e_1)])\cup(\{z\}\times\{\lambda(e_{k-1}),\dots,\tau(G,\lambda)\})\cup\left(\bigcup_{i=2}^{k-1}\{v_i\}\times\{\lambda(e_{i-1}),\dots,\lambda(e_i)\}\right).$$

For a subset S of vertices or edges, we denote by $(G-S,\lambda)$ the temporal graph $(G-S,\lambda')$, where λ' is the restriction of λ to $E(G-S)$. Similarly, for a subgraph $H\subseteq G$, the temporal graph (H,λ) denotes (H,λ'), where λ' is the restriction of λ to $E(H)$.

Let $Z\subseteq V(G)$ be any subset of vertices. The *identification of Z in* (G,λ) is the temporal graph (G^*,λ^*), where G^* is the graph obtained from G by the identification of Z into z,

Figure 5 – An example of temporal graph. The labels over the edges are the timesteps that they are active. In red is highlighted a temporal path



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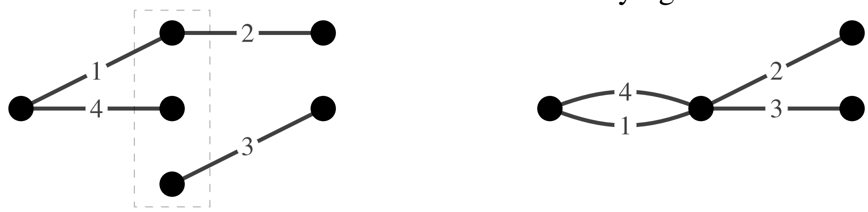
and λ^* is such that $\lambda^*(e) = \lambda(e)$ if $e \in E(G)$ has no endpoints in Z, and $\lambda^*(uz) = \bigcup_{x \in Z} \lambda(ux)$ for every $u \in N(Z) \setminus Z$. In Figure 6 we depict an example of identification.

Figure 6 – Example of identification in a temporal graph. The vertices inside the dashed rectangle are the ones in the set that we are identifying



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Similarly as before, if are given two temporal s,z-walks P and Q, we say that P and Q are *disjoint* if they do not share any internal vertex, i.e., if $V(P) \cap V(Q) = \{s,z\}$. Also, if s,z are non-adjacent, then a *vertex temporal s,z-cut* is a set $S \subseteq V(G) \setminus \{s,z\}$ such that there are no temporal s,z-walks in $(G - S, \lambda)$. Moreover if the first and last edges of P are e and f respectively, we say that P *starts* at timestep $\lambda(e)$ and *finishes* at timestep $\lambda(f)$. Throughout the text, whenever we talk about a vertex temporal s,z-cut we implicitly assume that it is well defined (in other words, that s,z are non-adjacent). Denote by $p_{G,\lambda}(s,z)$ the maximum number of vertex disjoint temporal s,z-walks and by $c_{G,\lambda}(s,z)$ the size of a minimum vertex temporal s,z-cut. If (G,λ) is clear from the context, we omit it from the subscript. The following proposition tells us that these definitions can be also made in terms of paths instead of walks, as it would give the same parameters.

Proposition 2.1.2 *Let (G,λ) be a temporal graph, and let $s,z \in V(G)$ be non-adjacent vertices. If P is a temporal s,z-walk in (G,λ), then there is a temporal s,z-path, say Q, such that $V(Q) \subseteq V(P)$.*

Proof. We use induction on the number of repetitions of vertices, if a temporal walk does not have vertex repetition, then it is a temporal path. Now, let $P = (s,t_1,v_1,\dots,t_\ell,v_\ell = z)$ be a temporal walk. If $v_i = v_j$ for some $i \neq j$, then we find a temporal s,z-path Q, such that $V(Q) \subseteq V(P)$. Suppose without loss of generality that $i < j$. One just needs to define $Q = (s,t_1,v_1,\dots,t_i,v_i,t_{j+1},v_{j+1},\dots,t_\ell,v_\ell)$. Observe that Q has fewer vertex repetitions than P, by applying induction hypothesis on Q, we obtain a temporal path Q' that satisfy the wanted properties. □

Other type of disjointness will be investigated in this thesis, but only this type will be referred in more than one chapter. Hence we postpone the presentation of other the definitions to the chapter they are used.

Throughout the text, we make concatenations of paths and of temporal paths and for this we need some further notation. If P is a u,v-path, Q is a v,w-path, and P finishes at most at the time when Q starts, then PQ denotes the temporal u,w-walk obtained from the concatenation of P and Q. Notice that PQ is a temporal u,w-path if and only if $V(P) \cap V(Q) = \{v\}$.

2.2 Computational Complexity

In this section, we give only intuition about computational complexity notions. The interested reader is referred to (20, 21).

A *decision problem* is a type of computational problem that asks whether a given input satisfies a particular property or condition, with the answer being either "yes" or "no". In other words, it is a problem that can be formulated as a question that has a binary (yes/no) answer. All decision problems are classified according to how hard it is to solve them. Solving a problem consists in giving an *algorithm* that correctly answers the problem's question. As here we only mean to give intuition behind such classification, we refrain from formally defining what an algorithm is, but simply say that it is a set of simple instructions to be followed by a machine. The *running time* of an algorithm is equal to the number of steps required for the algorithm to solve a problem, and is measured in terms of the size of the input. The running time is typically expressed using the *big O notation*, which provides an upper bound on the algorithm's running time as a function of the input size. For example, an algorithm with a time complexity of $O(p(n))$ would take $f(n)$ steps to solve a problem on an input of size n, where $f(n)$ is a function asymptotically bounded by $p(n)$. When p is a polynomial function, we say that the algorithm *runs in polynomial time*.

A decision problem is said to be *polynomial-time solvable* if there exists an algorithm to solve it which runs in polynomial time. The class of polynomial-time solvable problems is denoted by P. Not all problems are in P, as can be expected, and some of the remaining problems can be classified into classes that are interpreted as not being polynomial-time solvable. The class NP (which stands for "nondeterministic polynomial time") contains all decision problems that can be verified by an algorithm in polynomial time, i.e, the complexity of such algorithm is $O(p(n))$ where p is a polynomial function. More specifically, a decision problem is in NP if, given a "yes" instance of the problem together with a *certificate*, there is an algorithm that runs in polynomial time that can verify whether such certificate is indeed a proof for the input to be positive. This means that, for any given "yes" instance, there exists a short proof that can be checked in polynomial time. A decision problem is in co-NP if its *complement* (i.e., the set of "no" instances) is in NP. In other words, a problem is in co-NP if it can be verified in polynomial time that a given input is a "no" instance.

Before defining other classes of interest, we need to define reductions. A reduction is a technique used to show that one problem is at least as hard as another problem. Formally, a *reduction* from problem A to problem B is a *polynomial-time algorithm* that transforms instances of problem A into instances of problem B in polynomial time, such that the solution to the transformed instance of problem B can be used to solve the original instance of problem A in polynomial time. In other words, if we have an algorithm that can solve problem B in polynomial time, we can use it to solve problem A by transforming the instance of problem A into an instance of problem B using the reduction algorithm, solving the instance of problem B using the algorithm for problem B, and then translating the solution back into a solution for problem A using the same reduction algorithm.

A decision problem is said to be NP-*hard* every other problem in NP can be reduced to it in polynomial time. In other words, an NP-hard problem is one that is at least as hard as any other problem in NP. It is NP-*complete* if it is NP-hard and belongs to NP. Similarly, a decision problem is said to be co-NP-*hard* if its complement is NP-hard. In other words, a co-NP-hard problem is one that is at least as hard as any other problem in co-NP. It is *co*-NP-*complete* if it is co-NP-hard and belongs to co-NP.

Parameterized complexity theory refines the classification previously presented. It relies on the intuitive notion that an algorithm that solves a problem in time $O(f(k) \cdot n)$, where k is small when compared to n, is inherently better than one that runs in time $O(n^{f(k)})$.

A *parameterized problem* is a decision problem along with a *parameter* k. A parameterized problem is said to be *fixed-parameter tractable* (FPT) if there exists an algorithm that solves it in time $f(k) \cdot n^c$ for some computable function f and some constant c, where n is the size of the input. It can be thought of as the set of "easy" problems in the parameterized complexity. On the other hand, the class of *slice-wise polynomial* problems contains all the problems that can be solved in time $O(n^{f(k)})$; it is denoted by XP.

Parameterized complexity theory also includes the study of parameterized hardness, which is concerned with the identification of parameterized problems that are not FPT. A parameterized problem is said to be *para*-NP-*complete* if it is NP-complete for fixed values of the parameter. Observe that this means that it is also not in XP. On the other hand, a notion that intuitively means that an XP algorithm is the best one can hope for (i.e., it is not FPT) is that of W[1]-hardness. To give a better intuition into this class, we need another reduction concept.

Given two parameterized problems Π, Π', and instances (x,k) and (x',k') of Π and Π', respectively, it is said that they are *equivalent* if (x,k) is a "yes" instance of Π if and only if (x',k') is a "yes" instance of Π'. A *parameterized reduction* from Π to Π' is a function that, given an instance (x,k) of Π, computes an equivalent instance (x',k') of Π' such that $k' \leq g(k)$ in time $f(k) \cdot |x|^{O(1)}$, where f and g are computable functions. This means that if problem Π' is FPT, then so is Π. Let $\textsc{Clique}_k$ denote the problem of deciding whether a given graph has a clique of size at least k, parameterized by k. We say that a parameterized problem Π is W[1]-*hard* if there is a parameterized reduction from $\textsc{Clique}_k$ to Π. As $\textsc{Clique}_k$ is not believed to be FPT, and parameterized reductions preserve solvability in parameterized time, intuitively this means that Π is also not believed to be FPT. As it happens with polynomial time reductions, parameterized reductions can be composed. So, in order to prove W[1]-hardness, if suffices to provide a parameterized reduction from a problem known to be W[1]-hard.

3 VERTEX DISJOINT PATHS AND CUTS

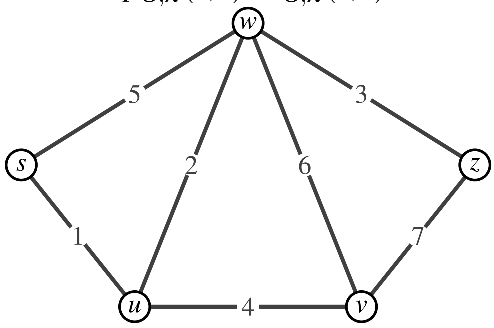
In this chapter, we begin our discussion of vertex connectivity in temporal graphs.. As we will see shortly, Menger's Theorem adapted to this context does not hold. This was first observed by Berman (3), and inspired Kempe, Kleinberg, and Kumar (12) to defined what they called Mengerian graphs. Informally, a Mengerian graph can be seen as a graph G that cannot be used as a counter-example for such adaptation of Menger. They then characterize simple Mengerian graphs. Our main result in this chapter is a characterization of general Mengerian graphs, as well as a polynomial-time recognition algorithm. In Section 3.1, we contextualize the problem and state our main result; in Section 3.2, we give some tool propositions and lemmas; in Section 3.3, we prove the necessity part of our characterization theorem; in Section 3.4 we prove the sufficiency part; in Section 3.5, we present the recognition algorithm; finally, in Section 3.6 we make some concluding remarks.

3.1 Definitions and Main theorem statement

We ask that the reader recalls the definitions made in Section 2.1.1. Observe that, by Proposition 2.1.2, there is no difference in considering paths or walks in the definitions of $c_{G,\lambda}(s,z)$ or $p_{G,\lambda}(s,z)$. This also occurs for almost all arguments in this chapter. Given a temporal graph (G,λ), a set $\mathscr{P}$ of vertex disjoint temporal s,z-paths and a vertex temporal s,z-cut S, we know that every path in $\mathscr{P}$ has a vertex in S. Furthermore, as the paths in $\mathscr{P}$ are vertex disjoint, a function f that takes an element $P \in \mathscr{P}$ and returns an element $f(P) \in S \cap V(P)$ is injective. Thus, $|\mathscr{P}| \leq |S|$ and the inequality $p_{G,\lambda}(s,z) \leq c_{G,\lambda}(s,z)$ follows naturally. In possession of the last inequality and knowing Menger's Theorem, the question that arises immediately is if equality always occurs. This was first answered negatively in (3). Here, we exhibit the example of (12) that, as we will see later, is the example with the minimum number of vertices and edges.

Observe the underlying graph of the temporal graph depicted in Figure 7. It is a very important graph for the results in this chapter and it is also known as *gem*. We prove that for such a temporal graph depicted in the figure it holds that $1 = p_{G,\lambda}(s,z) < c_{G,\lambda}(s,z) = 2$. Observe that, as the only edge connecting sw is active at time 5, there is only one temporal s,z-path using such edge. This path, can only finish using edges active at time 6 and 7, so such path also uses v, and since $\{w,v\}$ disconnects s from z, we get that there are no two vertex disjoint temporal s,z-paths. At the same time, no single vertex in $\{u,v,w\}$ breaks all temporal s,z-paths, i.e., there is no

Figure 7 – A temporal graph (G,λ) whose time function λ and pair of non-adjacent vertices $s,z \in V(G)$ are such that $p_{G,\lambda}(s,z) < c_{G,\lambda}(s,z)$



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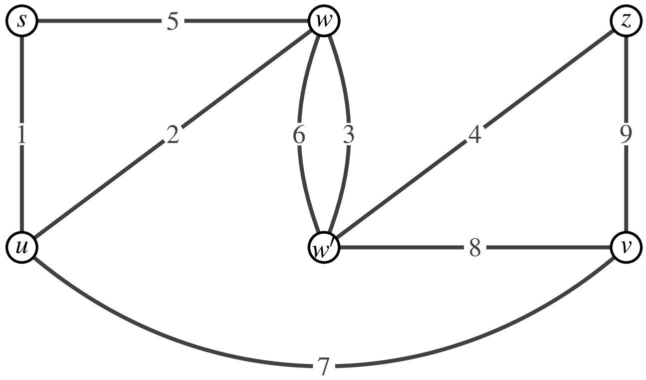
vertex temporal s,z-cut of size 1. In (12) they also define a graph G to be *Mengerian* if and only if there is no time function λ on G such that $p_{G,\lambda}(s,z) < c_{G,\lambda}(s,z)$. They then give the following characterization for the case where G is a simple graph.

Theorem 3.1.1 *(12)* Let G be a simple temporal graph. Then G is Mengerian if and only if G has no gem as topological minor.

Despite being a very nice result, the hypothesis of G being simple implies that each pair of vertices is connected only once in the lifetime of the temporal graph. This is a quite strong constraint which might be even unrealistic, depending on the application. Nevertheless, this hypothesis is necessary as one can see in Figure 8, where we present a temporal graph (G,λ) such that G has no gem as topological minor and $p_{G,\lambda}(s,z) < c_{G,\lambda}(s,z)$. To see that G has no gem as topological minor, notice that the maximum degree of G is 3, while the gem has a vertex of degree 4. The same argument used to prove that the temporal graph depicted in Figure 7 is non-Mengerian can be used here. Any temporal path that starts by using the multiedge sw also uses the vertex v. As $\{w,v\}$ disconnects s and z, then any temporal s,z-path uses one of these vertices. Thus, $p_{G,\lambda}(s,z) = 1$. To see that $c_{G,\lambda}(s,z) = 2$, it is enough to verify that the removal of each vertex does not kill all temporal s,z-paths and that $\{w,v\}$ is a vertex temporal s,z-cut. This indicates that a generalization of Theorem 3.1.1 needs others graphs. The main result of this chapter, Theorem 3.1.2, is the generalization of such result to any graph.

Before diving into the main result, we observe that the distance between parameters $c_{G,\lambda}(s,z)$ and $p_{G,\lambda}(s,z)$ can be arbitrarily large. In (12) they have shown that for each $\ell \in \mathbb{N} \setminus \{0\}$, there is a temporal graph (G_ℓ, λ_ℓ) and non-adjacent vertices s,z such that $p_{G_\ell,\lambda_\ell}(s,z) = 1$ and

Figure 8 – A temporal graph (G,λ) where the pair of non-adjacent vertices $s,z \in V(G)$ is such that $p_{G,\lambda}(s,z) < c_{G,\lambda}(s,z)$. Moreover, G has no gem as topological minor



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$c_{G_\ell,\lambda_\ell}(s,z) = \ell$. Their example has $\Theta(\ell^3)$ vertices and is such that $\Delta(G_\ell - \{s,z\}) = \Theta(\ell)$. In Section 3.2, we present an example also satisfying $p_{G_\ell,\lambda_\ell}(s,z) = 1$ and $c_{G_\ell,\lambda_\ell}(s,z) = \ell$, but where the number of vertices is $\Theta(\ell^2)$ and the maximum degree of the vertices excluding s and z is 4. An important corollary we draw attention to is that there is no function f such that $c_{G,\lambda}(s,z) \leq f(p_{G,\lambda}(s,z))$ for every temporal graph (G,λ) and non-adjacent pair of vertices $s,z \in V(G)$.

Now that we have observed that these parameters can be far apart, we characterize all Mengerian graphs G. We say in advance that this characterization is made in terms of forbidden structures. More precisely, we use the definition of m-topolgical minor introduced in Section 2.1.1. The graphs we use in our characterization are the ones depicted in Figure 9; we denote the set of such graphs by $\mathscr{F} = \{F_1, F_2, F_3\}$. Observe that each m-subdivision H of graphs F_1 or F_2 in such figure, contains exactly one chain P, which we call the *chain of* H.

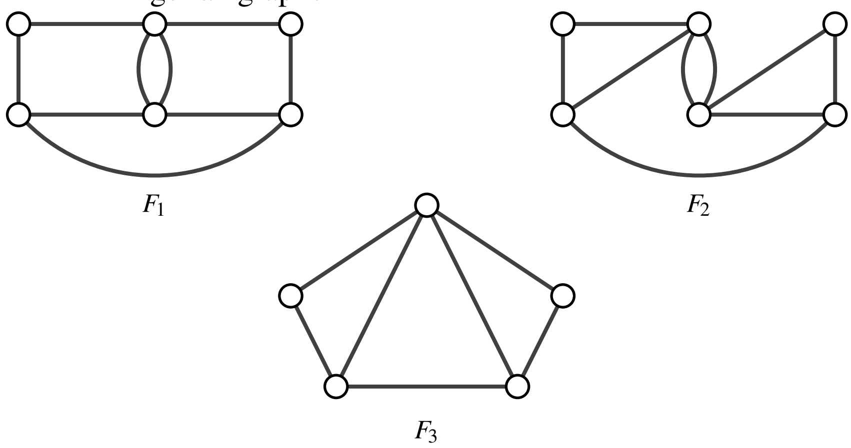
Theorem 3.1.2 *A graph G is Mengerian if and only if G has no graph in $\mathscr{F}$ as m-topological minor.*

The proof of this theorem is presented in Sections 3.3 and 3.4. In the next section, we see some useful preliminary results.

3.2 Preliminary results

The next lemma allows us to avoid explicitly analyzing some cases in parts of our proof. Let (M,λ) be a temporal graph and τ be its lifetime. The *reverse* of (M,λ) is the temporal

Figure 9 – Graphs in the family $\mathscr{F}$ used to characterize the Mengerian graphs



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graph (M,λ^-) where $\lambda^-\colon E(M) \to \mathbb{N} - \{0\}$ is such that $\lambda^-(e) = \tau + 1 - \lambda(e)$.

Lemma 3.2.1 *Let (M,λ) be a temporal graph and $s,z \in V(G)$. Then,*

$c_{M,\lambda}(s,z) = c_{M,\lambda^-}(z,s)$ *and* $p_{M,\lambda}(s,z) = p_{M,\lambda^-}(z,s)$.

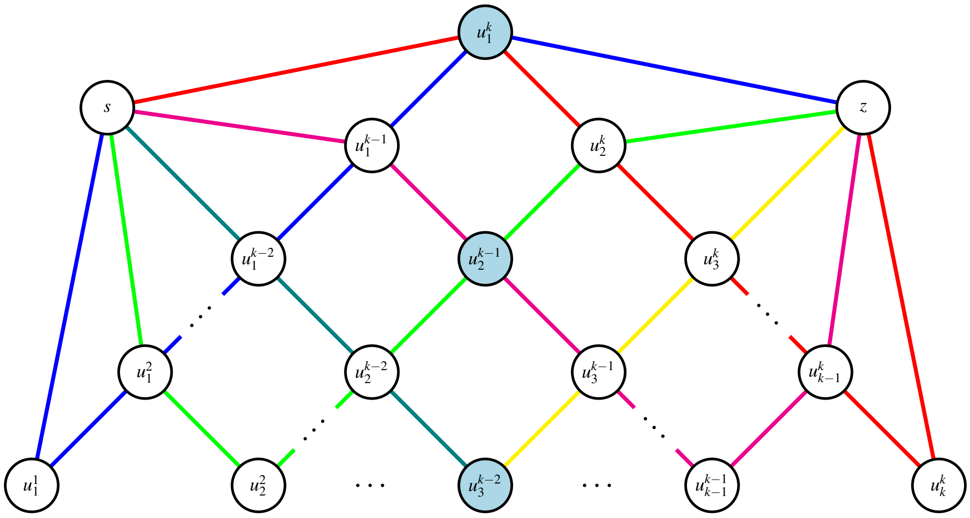
Proof. It suffices to see that if $J = (s = v_1, e_1, \ldots, e_k, v_{k+1} = z)$ is a temporal s,z-path in (M,λ), then $J^- = (z = v_{k+1}, e_{k+1}, \ldots, e_1, v_1 = s)$ is a temporal z,s-path in (M,λ^-), and vice-versa. As $V(J) = V(J^-)$, the result holds. □

Next proposition tells us that the distance between $p_{G,\lambda}(s,z)$ and $c_{G,\lambda}(s,z)$ can be arbitrarily large, as claimed in the previous section.

Proposition 3.2.1 *For each $\ell \in \mathbb{N} \setminus \{0\}$, there is a temporal graph (G,λ) and non-adjacent vertices $s,z \in V(G)$ such that $\Delta(G - \{s,z\}) \leq 4$, $c_{G,\lambda}(s,z) = \ell$ and $p_{G,\lambda}(s,z) = 1$. Moreover, each edge of G is active at most once.*

Proof. One can see Figure 10 to understand the temporal graph we construct. First, let $k = 2\ell$. Consider a set V with $\frac{k(k+1)}{2} + 2$ elements. Two elements of V are s and z, we call the other by u^i_j for $i \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,i\}$. In the figure, one can see that the elements $u^i_1,\ldots,u^i_i$ are disposed in a diagonal, with the upper diagonal being realted to the value $i = k$. For each $i \in \{1,\ldots,k\}$ we add edges between the pairs u^i_j and u^i_{j+1} for $j \in \{1,\ldots,i-1\}$; we denote the set of such edges by H_i. For each $j \in \{1,\ldots,k\}$, we add edges between the vertices u^i_j and u^{i+1}_j for $i \in \{j,\ldots,k-1\}$ and denote the set of such edges by P_j. Finally, we make s adjacent to all vertices in $\{u^1_1, u^2_1, \ldots, u^k_1\}$ and z adjacent to all vertices in in $\{u^k_1, u^k_2 \ldots, u^k_k\}$. Denote by $G = (V,E)$ such graph. Now we define a time function λ in such graph. For an edge $e \in E$ let:

Figure 10 – An example that shows a temporal graph $\mathscr{G}$ where $p_{\mathscr{G}}(s,z) = 1$ and $c_{\mathscr{G}}(s,z) = \ell$. Edges highlighted with the same color appear at the same time. Light blue vertices form a vertex temporal s,z-cut



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$$\lambda(e) = \begin{cases} i & \text{, if } e \in H_i \text{ or } e \text{ has endpoints } su_1^i\text{, and} \\ j & \text{, if } e \in P_j \text{ or } e \text{ has endpoints } su_j^k. \end{cases}$$

Therefore, in the figure, one can see that the edges of the same color all appear in the same snapshot. Define $H_i' = (s, i, u_1^i, i, u_2^i \dots, i, u_i^i)$. As the only edges H_i' uses are su_1^i and edges in H_i, we know that H_i' is a temporal s, u_i^i-path in (G, λ). Notice now that $P_i' = (u_i^i, i, u_i^{i+1}, \dots, u_i^k, i, z)$ is a temporal u_i^i, z-path using only edges $u_i^k z$ and edges in P_i that we know that are active at time i. Denote by J_i the temporal s,z-path obtained by the concatenation of H_i' and P_i'. Notice now that for $i_1, i_2 \in \{1, \dots, k\}$ such that $i_1 < i_2$, there is only one vertex $v \in V \setminus \{s,z\}$ in the intersection of $V(J_{i_1})$ and $V(J_{i_2})$, namely $u_{i_1}^{i_2}$. Given the set of temporal s,z-paths $\{J_1, \dots, J_k\}$ each vertex intersect at most 2 of these paths. So, as vertex temporal s,z-cut, say S, intersect all temporal paths, we have that $|S| \geq k/2 = \ell$. This implies that $c_{G,\lambda}(s,z) \geq \ell$. To see that the equality occurs, one just needs to consider the vertex temporal s,z-cut $\{u_1^k, u_2^{k-1}, \dots, u_\ell^\ell\}$. Notice that such set disconnects s and z even in the underlying graph. Therefore there is no temporal s,z-path that does not intersect such set. We then get that $c_{G,\lambda}(s,z) = \ell$.

To show that $p_{G,\lambda}(s,z) = 1$, suppose otherwise and let P_1, P_2 be two vertex disjoint-temporal s,z-paths and suppose that P_1 starts using the edge $su_1^{i_1}$, while P_2 uses the edge $su_1^{i_2}$ with $i_2 > i_1$. Now, if we consider the graph formed only by the edges with timesteps at least i_2, we have that $u_1^{i_2}, \dots, u_{i_2}^{i_2}$ induces a path in such graph; indeed, note that all the other edges incident in $u^{i_2}_j$, for $j < i_2$, are active in time j. We then have that P_2 must start with such path, i.e., $P_2 = (s, i_2, u^{i_2}_1, \ldots, i_2, u^{i_2}_{i_2}) i_2 P' z$, for some P'. Then, $V(P)$ contains the set $S = \{u^{i_2}_1, \ldots, u^{i_2}_{i_2}\}$ (these are highlighted in light blue in Figure 10). Now, observe that if we exclude s from the underlying graph, then S disconnects $u^{i_1}_1$ from z. Thus, $V(P_1) \cap S \neq \emptyset$. This fact together with $S \subseteq V(P_2)$ implies that $V(P_1) \cap V(P_2)$ contains other vertex than s and z. As P_1 and P_2 are taken to be vertex disjoint, this is a contradiction. □

Proposition 3.2.2 *Consider graphs M and N. If N is m-topological minor of M, then $U(N)$ is topological minor of $U(M)$.*

Proof. Let H be the graph obtained from N by the m-subdivision of a multiedge xy of N. Then, it is clear from the definition of m-subdivision that $U(H)$ is the graph obtained from the subdivision of the edge xy of $U(N)$. If H' is an m-subdivision of N, then we can apply the previous argument successively to conclude that $U(H')$ is a subdivision of $U(N)$. The proposition follows. □

Later on, we only use a simpler version of the next proposition.

Proposition 3.2.3 *Let $(G,\lambda), (H,\beta)$ be two temporal graphs such that $s,z \in V(G) \cap V(H)$ are non-adjacent vertices in G and in H. Denote by L_1 the set of temporal s,z-paths in (G,λ) and by L_2 the set of temporal s,z-paths in (H,β). Suppose that there is a function $f\colon L_1 \to L_2$ that is surjective and such that for any set of temporal s,z-paths in L_1, say $\{P_1,\ldots,P_k\}$, we have that:*

$$(\star)\ \bigcap_{i=1}^{k} V(P_i) \neq \{s,z\} \text{ if and only if } \bigcap_{i=1}^{k} V(f(P_i)) \neq \{s,z\}$$

Then, $c_{G,\lambda}(s,z) = c_{H,\beta}(s,z)$ and $p_{G,\lambda}(s,z) = p_{H,\beta}(s,z)$.

Proof. Notice that if P_1 and P_2 are vertex disjoint temporal s,z-paths in (G,λ), then by $(\star)$ we have that $f(P_1)$ and $f(P_2)$ are vertex disjoint temporal s,z-paths in (H,β). Thus if we consider a maximum set of vertex disjoint temporal s,z-paths in (G,λ), then its image is a set of vertex disjoint temporal s,z-paths in (H,β) and therefore $p_{H,\beta}(s,z) \geq p_{G,\lambda}(s,z)$. To obtain equality, take a set of vertex disjoint temporal s,z-paths in (H,β), $Q_1,\ldots,Q_\ell$. As f is surjective, we know that there are $P_1,\ldots,P_\ell$ such that $f(P_i) = Q_i$ for $i \in \{1,\ldots,\ell\}$. Moreover, Property $(\star)$ ensures us that $P_1,\ldots,P_\ell$ are vertex disjoint. Thus, $p_{G,\lambda}(s,z) = p_{H,\beta}(s,z)$.

Now, let S be a vertex temporal s,z-cut in (G,λ). For each $v \in S$, define L_v the set of temporal s,z-paths using vertex v. As we have that $v \in \bigcap_{P \in L_v} V(P)$, by $(\star)$ there is

$w_v \in \bigcap_{P \in L_v} V(f(P))$ different from s and z. Now we show that the set $W = \{w_v \mid v \in S\}$ is a vertex temporal s,z-cut in (H,β). Suppose otherwise and let Q be a temporal s,z-path in (H,β) not intersecting W. Using that f is surjective, let P be a path such that $f(P) = Q$. As S is a vertex temporal s,z-cut in (G,λ), then there is $v \in S$ such that $v \in V(P)$, in other words $P \in L_v$. However, recall that for each $v \in S$, we choose $w_v \in W$, and by its choice, we have that $w_v \in V(f(P)) = V(Q)$. A contradiction as Q does not intersect W. So, for every vertex temporal s,z-cut, S, in (G,λ), we can find a vertex temporal s,z-cut, W, in (H,β) such that $|W| \leq |S|$. Thus, $c_{H,\beta}(s,z) \leq c_{G,\lambda}(s,z)$. Now, take a vertex temporal s,z-cut W in (H,β). Let L_w be the set of temporal s,z-paths in L_2 that pass by w for each $w \in W$. Thus, by $(\star)$ and the fact $w \in \bigcap_{Q \in L_w} V(Q)$, we have that there is $v_w \in \bigcap_{P \in f^{-1}(L_w)} V(P)$ different from s and z. We prove that $S = \{v_w \mid w \in W\}$ is a vertex temporal s,z-cut in (G,λ). Let P be a temporal s,z-path in (G,λ). We know that there is a vertex $w \in V(f(P)) \cap W$ as W is a vertex temporal s,z-cut in (H,β). As $f(P) \in L_w$, by construction we know that some $v_w \in V(P)$ has been chosen to be put in S. Therefore, S is a cut and $|S| \leq |W|$. We conclude that $c_{G,\lambda}(s,z) = c_{H,\beta}(s,z)$. □

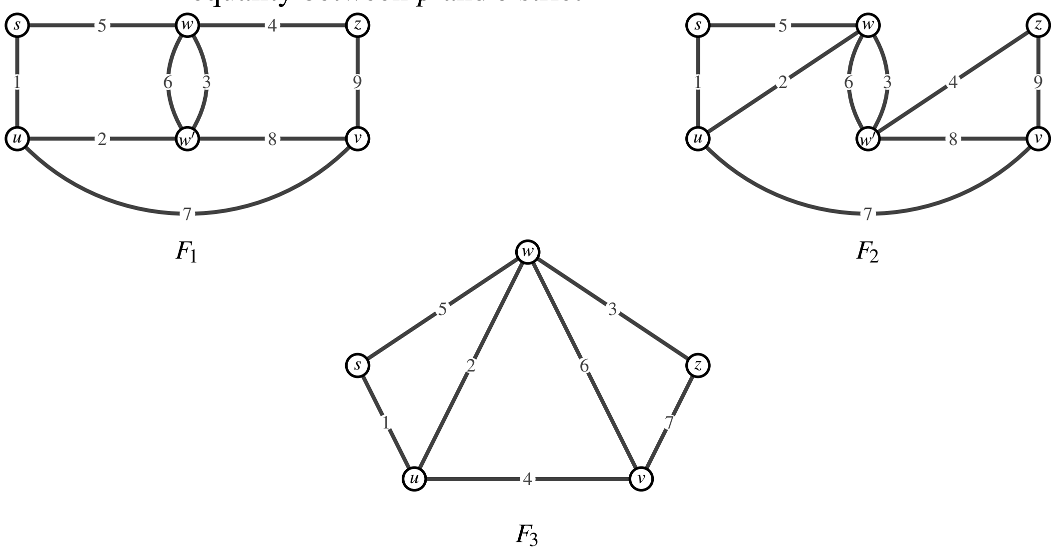
3.3 Proof of Necessity of Theorem 3.1.2

In this section, we prove that if M has an m-subdivision of F_1, F_2 or F_3, then M is non-Mengerian. We start by proving that those graphs are non-Mengerian. We do it by proving 3 lemmas. Lemma 3.3.1 states that F_1, F_2 and F_3 are non-Mengerian, Lemma 3.3.2 that any m-subdivision of a non-Mengerian graph is still non-Mengerian, and finally, Lemma 3.3.3 states that if a graph contains a non-Mengerian as subgraph, then it is non-Mengerian. Therefore, if a graph M contains F_j as an m-topological minor for some $j \in \{1,2,3\}$, then it contains an m-subdivision of a non-Mengerian graph by the first lemma. The second result gives us that M contains a non-Mengerian graph as subgraph and the final result gives us that M is non-Mengerian.

Lemma 3.3.1 *Graphs F_1, F_2, and F_3 are non-Mengerian.*

Proof. We use time functions represented in Figure 11. It was proved in (12) that F_3 is non-Mengerian. So, we only show that F_1 and F_2 are non-Mengerian. Let (G,λ) be any of the temporal graphs depicted in the figure. To show that $c_{G,\lambda}(s,z) \geq 2$ in F_1 and F_2, we show that for each vertex x different from s,z there is a temporal s,z-path not passing by x. For w and w' we take the temporal path $(s,1,u,7,v,9,z)$. A temporal s,z-path not passing by u is $(s,5,w,6,w',8,v,9,z)$. Finally, a temporal s,z-path not passing by v in F_1 is $(s,1,u,2,w',3,w,4,z)$, while such a path in F_2 is $(s,1,u,2,w,3,w',4,z)$. Now we show that $p_{G,\lambda}(s,z)$= 1 in both F_1 and F_2. Suppose otherwise and observe that $\{v,w\}$ is a vertex temporal s,z-cut. Note also that any 2 disjoint temporal s,z-paths must include a path that starts with the edge labelled with 5; let P be such path. Notice, however, that the subgraph formed by the edges active at time at least 5 is a tree, and hence contains exactly one s,z-path. It thus follows that P must be equal to the path contained in such tree, in which case $\{v,w\} \subseteq V(P)$. As $\{v,w\}$ is a vertex temporal s,z-cut, any other temporal s,z-path must intersect P, a contradiction. □

Figure 11 – Graphs in the family F with time functions that makes the inequality between p and c strict



Source: elaborated by the author.

Lemma 3.3.2 *Any m-subdivision of a non-Mengerian graph N is non-Mengerian.*

Proof. Let N be a non-Mengerian graph and consider $\lambda : E(N) \to \mathbb{N}$ and $s,z \in V(N)$ to be such that $p_{N,\lambda}(s,z)$<$c_{N,\lambda}(s,,z)$. Also, suppose that N' is obtained from N by m-subdividing a multiedge, say xy. We construct a function λ' from λ that proves that N is also non-Mengerian.

Let $D \subseteq E(N)$ be the set of edges of N with endpoints xy, and denote by v_{xy} the vertex of N' created by the m-subdivision of xy. Moreover, denote by D_x and D_y the sets of edges of N' with endpoints xv_{xy} and $v_{xy}y$, respectively. Finally, define λ' to be such that $\lambda'(e) = \lambda(e)$, for every $e \in E(N) \setminus D$, and $\lambda'(D_x) = \lambda'(D_y) = \lambda(D)$. We show that we can apply Proposition 3.2.3 to prove that $p_{N,\lambda}(s,z)$=$p_{N',\lambda'}(s,z)$ and $c_{N,\lambda}(s,z)$=$c_{N',\lambda'}(s,z)$ and conclude that N' is non-Mengerian.

Let L and L' be the set of temporal s,z-paths of (N,λ) and (N',λ') respectively. Define $f : L' \to L$ as follows: $f(P) = P$ if P does not use the vertex v_{xy}, otherwise if P uses the vertex v_{xy} then, for $u \in \{x,y\}$ it uses edge f_u between the pair $v_{xy}u$. Suppose that P uses first x and then v_{xy}, then $P = (s,t_1,v_1,\ldots,t_k,x = v_k,t_{k+1},v_{xy} = v_{k+1},t_{k+2},y = v_{k+2},\ldots,v_{k'} = z)$. In this case define $f(P) = (s,t_1,v_1,\ldots,t_k,x,t_{k+2},y = v_{k+2},t_{k+3}\ldots,v_{k'} = z)$. If P uses y before v_{xy} we can define analogously. Notice that if P uses v_{xy}, then it also uses x and y. Thus, for any temporal s,z-path P, we have $V(f(P)) = V(P) \setminus \{v_x\}$. Now, let $P_1,\ldots,P_k$ be a set of temporal s,z-paths in (N',λ'). Suppose that $\bigcap_{i=1}^{k} V(P) \supseteq \{s,z,w\}$ for some $w \neq s,z$. If $w \neq v_{xy}$, then $w \in V(f(P_i))$ for every $i \in \{1,\ldots,k\}$, and therefore $\bigcap_{i=1}^{k} V(f(P_i)) \supseteq \{s,z,w\}$. And if $w = v_{xy}$, then $\{x,y\} \subseteq V(f(P_i))$. As s and z are not adjacent and there is an edge connecting xy, at least one of x and y are different from s and z, say x. Therefore $\bigcap_{i=1}^{k} V(f(P_i)) \supseteq \{s,z,x\}$. So, to prove that f has the property $(\star)$ we only need to show that if $\bigcap_{i=1}^{k} V(P) = \{s,z\}$, then $\bigcap_{i=1}^{k} V(f(P)) = \{s,z\}$. This is clear as $\bigcap_{i=1}^{k} V(f(P)) = \bigcap_{i=1}^{k} V(P) \setminus \{v_{xy}\}$. Then f has Property $(\star)$.

Finally, we now show that f is surjective. Take Q a temporal s,z-path in (N,λ). If Q does not use an edge connecting the pair xy, then Q is also a temporal s,z-path in (N',λ') not using v_{xy}. Thus, $f(Q) = Q$. If Q uses an edge between xy, say that such edge is active at timestep t, then we substitute such edge by edges between xv_{xy} and $v_{xy}y$, both active also at time t, and call Q' the obtained temporal s,z-path. From the definition of Q' we have that $f(Q') = Q$. □

The last lemma of this section is the following:

Lemma 3.3.3 *Let M be a graph and N be a subgraph of M. If N is non-Mengerian, then M also is non-Mengerian,*

Proof. Let s,z,λ be such that $p_{N,\lambda}(s,z) < c_{N,\lambda}(s,z)$. Consider the time function λ' on $E(M)$ defined as follows.

$$\lambda(e) = \begin{cases} \lambda(e)+1 & \text{, for every } e \in E(N), \\ 1 & \text{, for every } e \in E(M) \setminus E(N) \text{ having } z \text{ as endpoint, and} \\ \max \lambda(E(N))+2 & \text{, otherwise.} \end{cases}$$

Because $N \subseteq M$ and $\lambda(e) < \lambda(g)$ (respec. $\lambda(e) = \lambda(g)$) if and only if $\lambda'(e) < \lambda'(g)$ (respec. $\lambda'(e) = \lambda'(g)$) for every $e,g \in E(N)$, note that we get that every temporal path of (N,λ) is a temporal path of (M,λ'). Therefore it suffices to prove that all temporal s,z-paths of (M,λ') are the temporal paths of (N,λ). This implies that the set of temporal paths of these two temporal graphs coincides and, applying Proposition 3.2.3 using the identity function, the lemma follows. Indeed, suppose a temporal s,z-path of (M,λ') that uses an edge that is not in N.

Let e be the first edge in such path that is not in $E(N)$. Notice that e is not adjacent to z, as all edges incident to z that are not in $E(N)$ are active at time 1 and all edges incident to s are active at time greater than 1. So, as e is not in $E(N)$ and it is not incident to z, then it is active at time $\tau = \max \lambda(E(N)) + 2$. However, all edges incident to z are active at timesteps smaller than τ, a contradiction as such path could not end at z. □

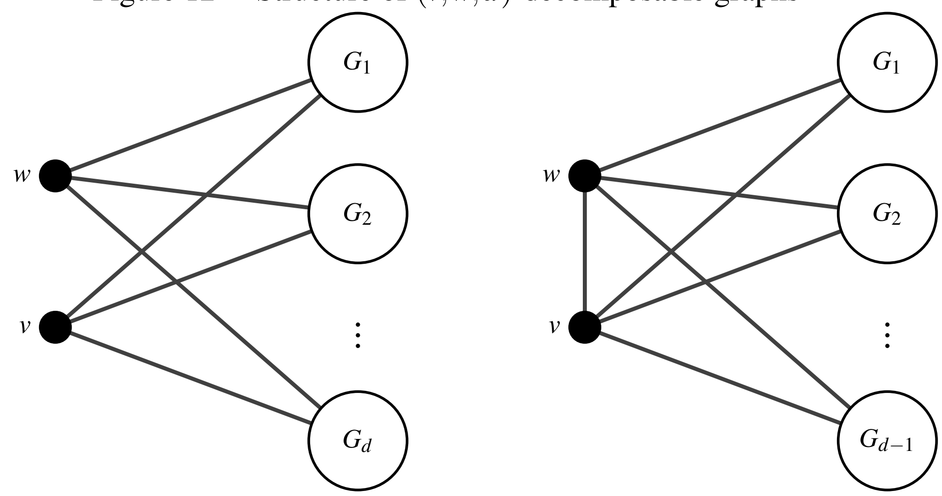
3.4 Proof of Sufficiency of Theorem 3.1.2

In this section we prove that if a graph M is non-Mengerian, then for some $j \in \{1,2,3\}$ we have that F_j is m-topological minor of M. We do it by contradiction. For this, we say that a graph M, together with a pair of non-adjacent vertices, s,z, and a time function λ, is a counter-example if $p_{M,\lambda}(s,z) < c_{M,\lambda}(s,z)$ and M has no F_j as m-topological minor for every $j \in \{1,2,3\}$. If M is a counter-example such that $|V(M)| + |E(M)|$ is minimum, then we call M, λ, s, z a minimum counter-example.

The following has been defined and proved in (12) and will be useful in our proof. It is depicted in Figure 12. Given a graph G, vertices $v, w \in V(G)$, and a positive integer d, a graph G is called (v,w,d)-*decomposable* if:

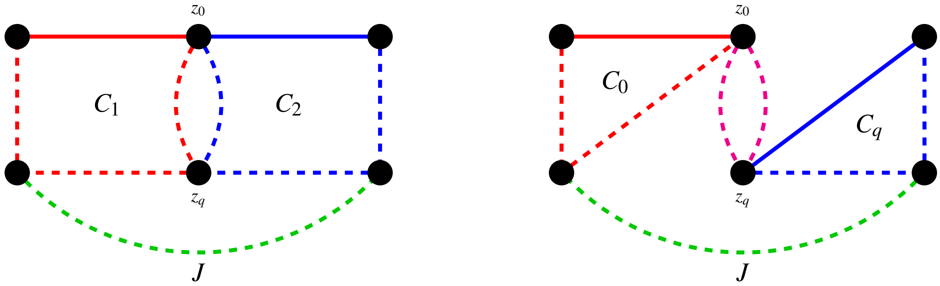
- Both v and w have degree d.
- Either $G - \{v,w\}$ consists of d components or $vw \in E(G)$ and $G - \{v,w\}$ has $d-1$ components.

Figure 12 – Structure of (v,w,d)-decomposable graphs



Source: elaborated by the author.

Figure 13 – Cycle C_1 and C_0 are highlighted in red, C_2 and C_q in blue, and path J in green. Chain $(z_0,\ldots,z_q)$ is represented only as a multiedge. Dotted lines represent paths, while solid lines represent edges



Source: elaborated by the author.

Lemma 3.4.1 ((12)) *Let G be a 2-connected simple graph with $\Delta(G) \geq 4$. If F_3 is not a topological minor of G, then G is (v,w,d)-decomposable for some $v,w \in V(G)$ and integer $d \geq 4$.*

As we suppose a counter-example exists, we should provide a contradiction. Often, the obtained contradiction is an m-subdivision H of F_j for some $j \in \{1,2,3\}$. In order to simplify the arguments about a graph containing such a subgraph H, we use the next two propositions. Observe that the propositions simply describe the structure of an m-subdivision of F_1 and F_2. Figure 13 helps verification.

Proposition 3.4.1 *Let M be a graph, $L = (z_0,\cdots,z_q)$ be a chain in M, C_1 and C_2 be edge disjoint cycles and J be a C_1,C_2-path. If all the items below hold, then the set of vertices and edges of J,L,C_1,C_2 form a subgraph of M that is an m-subdivision of F_1.*

- *$V(C_1) \cap V(C_2) = V(L)$.*
- *$V(L) \cap V(J) = \emptyset$.*
- *Letting w_1,w_2 be the extremities of J, we have that $\{w_1,w_2\} \cap N(z_i) = \emptyset$ for some $i \in \{0,q\}$.*

Reciprocally, if H is an m-subdivision of F_1 in G, then such J,L,C_1,C_2 exist that form H.

Proposition 3.4.2 *Let M be a graph, $L = (z_0,\cdots,z_q)$ be a chain in G, C_0 and C_q be vertex disjoint cycles in M such that, for each $i \in \{0,q\}$, $V(C_i) \cap V(L) = \{z_i\}$, and J be a C_0,C_q-path such that $V(L) \cap V(J) = \emptyset$. Then the set of vertices and edges of L,J,C_0,C_q form a subgraph of M that is an m-subdivision of F_2. Reciprocally, if H is an m-subdivision of F_2 in M, then such J,L,C_0,C_q exist that form H.*

We do not always we find F_1 or F_2 as m-topological minors. However, in such cases we can say enough about the structure of the graph in order to still arrive to a contradiction.

Figures 14 and 15 helps to understand the following definition. For a graph M that contains a chain $L = (z_0, \dots, z_q)$, we say that M has *crossed structure around* L, or simply *crossed structure* when L is clear from the context, when we can partition $V(G) \setminus V(L)$ into sets H, A_1, B_1, A_2, B_2 such that:

- $H = \{h_1, h_2, h_3, h_4\}$ is such that h_1 and h_3 are adjacent to z_0, and h_2 and h_4 are adjacent to z_q.
- Either $A_1 = \emptyset$ and there is an edge with endpoints h_1h_2, or $N(A_1) = \{h_1, h_2\}$.
- Either $A_2 = \emptyset$ and there is an edge with endpoints h_3h_4, or $N(A_2) = \{h_3, h_4\}$.
- Either $B_1 = \emptyset$, in which case there might be an edge with endpoints h_1h_4 or not, or $N(B_1) = \{h_1, h_4\}$.
- Either $B_2 = \emptyset$ and there is an edge with endpoints h_2h_3, or $N(B_2) = \{h_3, h_4\}$.

Figure 14 – Structure of a crossed graph

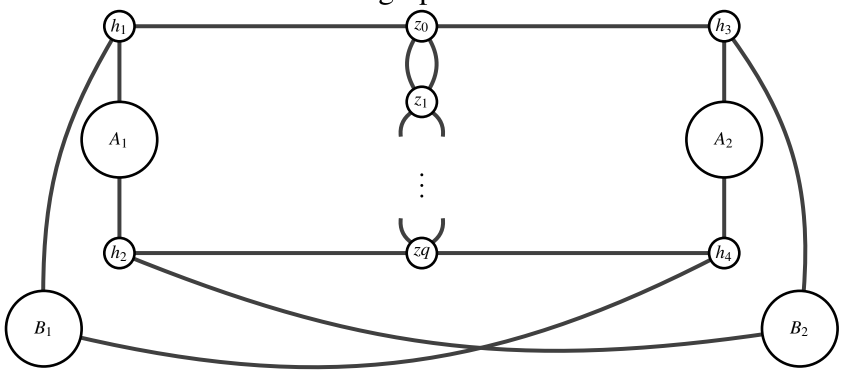


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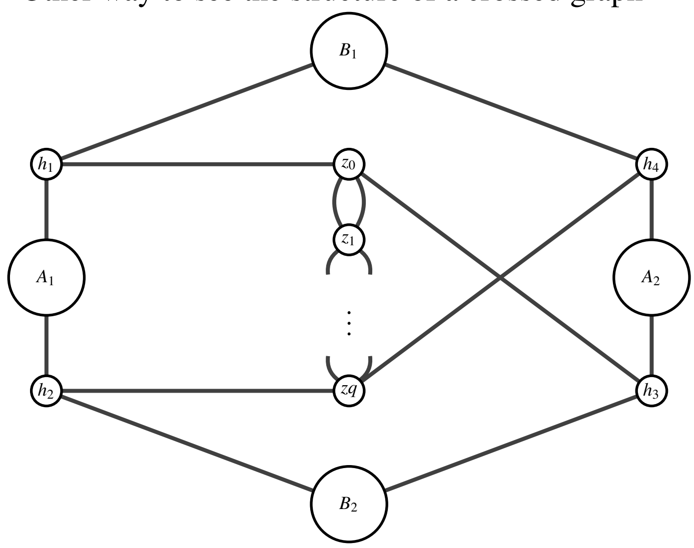
Figure 15 – Other way to see the structure of a crossed graph



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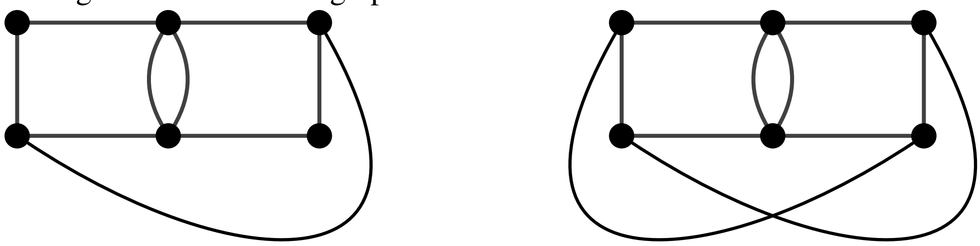
We call an h_2,h_3-path either using vertices of B_2, or containing just one edge with endpoints h_2h_3, a *crossing path*. Similarly, if either there is an edge with endpoints h_1h_4, or $B_1 \neq \emptyset$, then we call an h_1,h_4-path either using vertices of B_1, or containing just one edge with endpoints h_1h_4, a *crossing path*. If $B_1 = \emptyset$ and there is no edge with endpoints h_1h_4, then we say that M is *1-crossed*, otherwise, we say that M is *2-crossed*. Figure 16 represents the smallest 1-crossed and 2-crossed graphs.

Figure 16 – Smallest graphs with 1-crossed and 2-crossed structure



Source: elaborated by the author.

Next lemma is important in the proof of Theorem 3.1.2 and also later on, when we study the recognition of Mengerian graph.

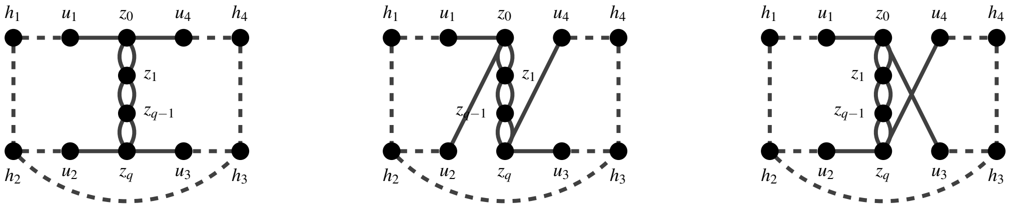
Lemma 3.4.2 *Let M be a 2-connected graph that has no F_3 as m-topological minor, and let L be a chain of M for which all internal vertices have degree 2 in M. If the graph obtained from M by identifying the vertices of L has an m-subdivision of F_3, then one of the following holds, and we can decide which one in time $O(n^3)$, where $n = |V(M)|$:*

- *F_1 or F_2 is an m-topological minor of M.*
- *M has a crossed structure.*

Proof. Write L as $(z_0,\dots,z_q)$. First, call M_L the graph obtained from M by identifying the vertices of L, and denote by ℓ the vertex obtained by such identification. By hypothesis, M_L has a subdivision of F_3. So, apply the algorithm presented in (13) to find such a subdivision, H. This takes time $O(n^3)$. Note that $\ell \in V(H)$ as otherwise we would have that H is contained in M, a contradiction to the fact that M has no F_3 as topological minor. Note also that $N_H(\ell) = (N_M(z_0) \cup N_M(z_q)) \setminus V(L)$ as all internal vertices of L have degree 2. First we prove that $|N_H(\ell) \cap N_M(z_0)| = 2$. For this, we analyse the cases:

- If $|N_H(\ell) \cap N_M(z_0)| \in \{0,4\}$, then either $N_H(\ell) \subseteq N_M(z_0)$ or $N_H(\ell) \subseteq N_M(z_q)$. Therefore the subgraph contained in M obtained from H by replacing ℓ by either z_0 or z_q is isomorphic to H, a contradiction as M has no subdivision of F_3.

Figure 17 – Possible adjacencies of $u_1, \cdots, u_4$ in $\{z_0, z_q\}$. Dashed lines denote paths, while solid lines denote edges. The cases u_1, u_2 incident to z_q, u_1, u_3 incident to z_q, and u_1, u_4 incident to z_q are analogous



(a) u_1, u_4 incident to z_0. (b) u_1, u_2 incident to z_0. (c) u_1, u_3 incident to z_0.

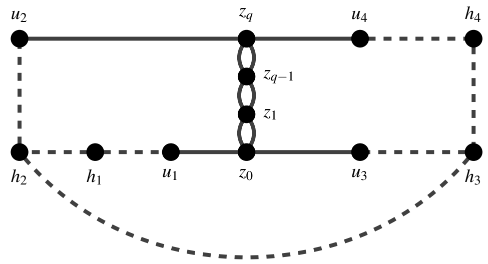
Source: elaborated by the author.

- If $|N_H(\ell) \cap N_M(z_0)| = 1$, then let Q be a z_0, z_q-path in M contained in L and $a \in N_H(\ell) \cap N_M(z_0)$. Then, if we substitute the edge $a\ell$ in H by the a, z_q-path az_0Qz_q, we would obtain a subgraph of M that is a subdivision of H, again a contradiction.
- If $|N_H(\ell) \cap N_M(z_0)| = 3$. Let $a \in N_H(\ell) \setminus N_M(z_0)$; because $N_H(\ell) \subseteq N_M(z_0) \cup N_M(z_q)$, we know that $a \in N_M(z_q)$. By applying an argument similar to the previous one, i.e., replacing $a\ell$ in H by z_0Qz_qa, where Q is a z_0, z_q-path contained in L, we get a contradiction again.

Therefore $|N_H(\ell) \cap N_M(z_0)| = 2$. Note that we can apply the same arguments to get $|N_H(\ell) \cap N_M(z_q)| = 2$. Since $N_H(\ell) \subseteq N_M(z_0) \cup N_M(z_q)$ and $d_H(\ell) = 4$, we also have that $(N_M(z_0) \cap N_M(z_q)) \setminus V(L) = \emptyset$. Now let H^* be the subgraph of M formed by L and $V(H) \cap V(M)$. Let h_1, h_2, h_3, h_4 be the vertices of H^* corresponding to the path of F_3 minus the vertex of degree 4. Formally, h_2, h_3 correspond to the vertices of degree 3 in F_3, h_1 and h_4 have degree 2, and there is a path in H^* not using vertices of $V(L)$ passing by h_1, h_2, h_3, h_4, in this order. Also, for each $i \in [4]$, let P_i be the path in H related to the edge linking h_i to the vertex of degree 4 in F_3. Formally, P_i is the path in H between h_i and $N_H(\ell)$ not passing through the vertices in $\{h_1, \ldots, h_4\} \setminus \{h_i\}$; call by u_i the extremity of P_i distinct from h_i. We can find these vertices and paths by running a search on H^*, which takes $O(m+n)$ time where $m = |E(M)|$. As we have proved, in M two of the vertices in $\{u_1, \ldots, u_4\}$ are incident to z_0 and the other two are incident to z_q. All the possible adjacencies are depicted in Figure 17.

Observe that if the situations in cases (a) or (b) of Figure 17 occur, then H^* is an m-subdivision of F_1 or F_2. Additionally, if situation (c) occurs, note that if $u_2 \neq h_2$, then H^* is also an m-subdivision of F_1. To see this, draw h_4 inside the cycle formed by h_2, u_2, L, u_3, h_3; Figure 18 helps one to see that. A similar argument can be applied in case $u_3 \neq h_3$. Since all these tests can be done in constant time, we can now consider that the situation (c) occurs and h_2z_q and h_3z_0 are edges of H^*.

Figure 18 – Other drawing of H^*



Source: elaborated by the author.

Define A_1 to be the set of internal vertices in the h_2, z_0-path in H^* passing by h_1; A_2 to be the set of internal vertices in the h_3, z_q-path in H^* passing by h_4; and B_2 the set of internal vertices in the h_2, h_3-path not passing by h_1 or h_4. Also, for each $X \in \{A_1, A_2, B_1\}$, denote by P_X the path formed by X. Notice that all vertices in $V(H^*) \setminus (A_1 \cup A_2 \cup B_1)$ either have degree 3 in H^* or are internal vertices of the chain, which we know that have degree 2 in M. Now we show that either both items below hold, or we can find an m-subdivision of F_1. Note that this finishes our proof since this test can be done in time $O(m+n)$, and these items imply that M has a crossed structure.

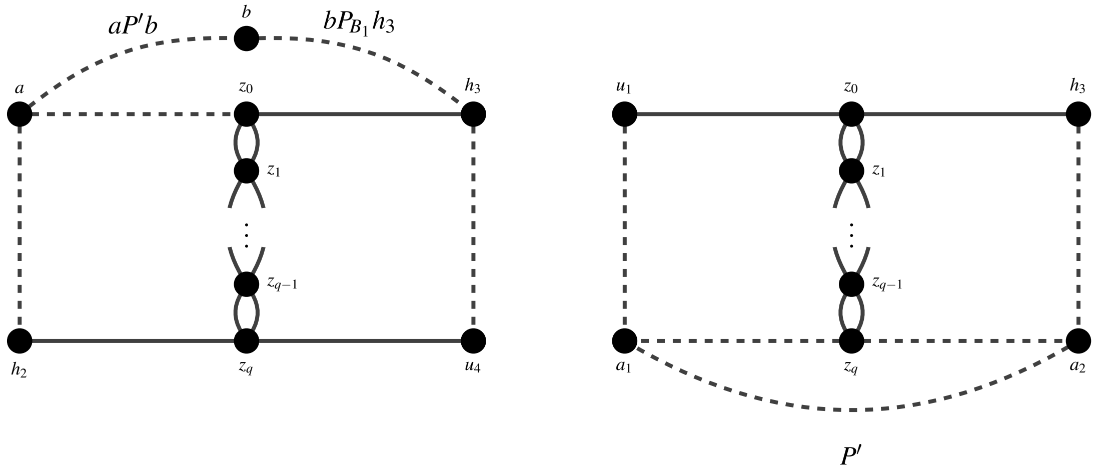
- All paths between B_2 and $A_1 \cup A_2$ in M are contained in H^*.
- If there is a path between A_1 and A_2 in M that is not contained in H^*, then it must be a path between u_1 and u_4.

Suppose first that there exists a path P' in M between $a \in A_1$ and $b \in B_1$ not contained in H^*. We obtain the elements necessary to apply Proposition 3.4.1; observe the graph depicted at the left in Figure 19 to follow the argument. Let $P'' = aP'bP_{B_1}h_3$. Also, let C_1, C_2 be two edge-disjoint cycles formed by $A_1 \cup \{h_2\}$ and $A_2 \cup \{h_3\}$ together with the chain L. We can apply Proposition 3.4.1 to L, P'', C_1, C_2 to obtain an m-subdivision of F_1. An analogous argument can be applied to a path between B_1 and A_2.

Now, suppose that there is a path P' between $a_1 \in A_1$ and $a_2 \in A_2$ in M not contained in H^*, and suppose that $a_1 \neq u_1$. Observe the graph at right in Figure 19 to see that, again, we can apply Proposition 3.4.1 to obtain an m-subdivision of F_1. The same argument can be applied in case $a_2 \neq u_4$ and therefore the second item above follows. □

In what follows, we want to prove that if M, λ, s, z is a minimum counter-example, then either $\Delta(M) \geq 4$, or M has a crossed structure. For this, the following two lemmas will be useful.

Figure 19 – At left, we depict a subdivision of F_1 when there exists P' between B_1 and A_1 not in H^*. At right, a subdivision of F_1 when there exists P' between $A_1 \setminus \{u_1\}$ and A_2 not in H^*. Dashed lines represent paths, while solid lines represent edges



Source: elaborated by the author.

Lemma 3.4.3 *Let M, λ, s, z be a minimum counter-example. Then, M is 2-connected and every multiedge incident in s or z has multiplicity 1.*

Proof. We first observe that M is connected. If s and z are in distinct components, then $p_{M,\lambda}(s,z)$=$c_{M,\lambda}(s,z)$= 0. Then s,z are in the same component and as M, λ, s, z is a minimum counter-example, it follows that M is connected. Now, we prove that M is 2-connected. By contradiction, suppose that $\{v\}$ is a cut of M, and let $M_1, \cdots, M_d$ be the subgraphs of M induced by the vertex set of each component of $M - v$ and v. If s and z belong to M_i for some $i \in [d]$, then every temporal s,z-path is contained in M_i, making M_i a non-Mengerian graph properly contained in M, contradicting the minimality of M, λ, s, z. So suppose that $s \in M_i$ and $z \in M_j$ for $i, j \in [d]$, with $i \neq j$. Observe that in this case, either there is no temporal s,z-path, in which case the empty set is a vertex temporal s,z-cut (i.e. $p_{M,\lambda}(s,z)$=$c_{M,\lambda}(s,z)$= 0), or $p_{M,\lambda}(s,z)$=$c_{M,\lambda}(s,z)$= 1 as every temporal s,z-path goes through v. This contradicts the definition of minimum counter-example.

Now, by contradiction, suppose that $sy \in E(U(M))$ has multiplicity at least 2 in M. Let $e \in E(M)$ be the edge with endpoints sy that minimizes $\lambda(e)$, and let $S = \{e' \in E(M) \mid e' \neq e$ and e' has endpoints $sy\}$. Denote by (M^*, λ) the temporal graph $(M - S, \lambda)$. Because (M^*, λ) is contained in (M, λ), observe that we get that $p_{M^*,\lambda}(s,z) \leq p_{M,\lambda}(s,z)$ and $c_{M^*,\lambda}(s,z) \leq c_{M,\lambda}(s,z)$. In addition, notice that a set $\mathscr{P}$ of vertex disjoint temporal paths in (M, λ) also corresponds to a set of vertex disjoint temporal paths in (M^*, λ) of equal size, since it suffices to replace, if needed, the first edge $e' \in S$ with endpoints sy by e. This gives us that $p_{M^*,\lambda}(s,z)=p_{M,\lambda}(s,z)$ and $c_{M^*,\lambda}(s,z)=c_{M,\lambda}(s,z)$, contradicting the minimality of M,λ,s,z. A similar argument can be applied if some edge incident in z has multiplicity at least 2, with the difference that we take the one maximizing λ. □

In order to treat the case where $\Delta(M) \leq 3$, we prove that a graph M that has a crossed structure is in fact Mengerian, and hence cannot compose a minimum counter-example. For this, the following structural lemma will be useful.

Lemma 3.4.4 *Let M,λ, s,z be a minimum counter-example. Suppose that $S=\{uu',vv'\} \subseteq E(U(M))$ is such that $U(M)-S$ is disconnected, and consider that u',v' are in the same component M' of $U(M)-S$. If $\min\{d(u),d(v)\} \geq 3$, then:*

1. *At least one of s,z is not in M'.*
2. *If $x \in V(M') \cap \{s,z\}$, then $V(M')=\{x\}$.*
3. *If $s,z \notin V(M')$, then $U(M')$ is an u',v'-path.*

Proof. Let M'' be the component of $U(M)-S$ containing u,v. Note that if $u'=v'$, then we get that $V(M')=\{u'\}$ as otherwise u' would disconnect M, contradicting Lemma 3.4.3. This is used throughout the proof to conclude $V(M')=\{u'\}$. Note additionally that $u \neq v$ since otherwise we would have $\min\{d(u),d(v)\}=2$, contradicting the hypothesis. We divide the proof into cases.

Case 1. uu' and vv' have multiplicity 1.

If $\{s,z\} \subseteq V(M')$, then let (M^*,λ^*) be obtained by identifying all vertices in M'' and let v^* be the vertex obtained by such identification. We want to show that M^*,λ^*,s,z is a counter-example smaller than M,λ,s,z. As this is a contradiction, we cannot have $\{s,z\} \subseteq V(M')$, and thus we conclude the first item of the lemma. Observe that $|V(M^*)|<|V(M)|$, as $V(M)\setminus V(M')$ contains at least two vertices, u and v. Additionally, since uu' and vv' have multiplicity 1 and every temporal s,z-path not contained in M' must use both uu' and vv', observe that this operation preserves the temporal s,z-paths not contained in (M',λ). Note that this also cannot create new m-topological minors as the vertex obtained by the contraction has degree 2 in M^*. More formally, let H be a graph contained in M^*. If such graph does not contain vertex v^*, then H is contained in M. Otherwise, as $d(v^*)=2$ and both multiedges incident to v^* have multiplicity 1, by removing v^* and adding uu',vv' together with a u,v-path contained in $V(M)\setminus V(M')$, we obtain an m-subdivision of H that is contained in M. Thus, any m-topological minor of M^* is also an m-topological minor of M. This means that to conclude that M^*,λ^*,s,z is a smaller counter-example, it is enough to show that $p_{M,\lambda}(s,z)$=$p_{M^*,\lambda^*}(s,z)$ and $c_{M,\lambda}(s,z)$=$c_{M^*,\lambda^*}(s,z)$. We do it using Proposition 3.2.3. Let $\mathscr{L}$ and $\mathscr{L}'$ be the set of temporal paths of (M,λ) and (M^*,λ^*) respectively. For each $P \in \mathscr{L}$, define $f(P)$ as follows: If P does not use vertices in M'', then $f(P) = P$, otherwise, we remove the parts of P passing by M'' and replace it by vertex v^*. Observe that $f\colon \mathscr{L} \to \mathscr{L}'$ is surjective. Moreover, we have that $V(f(P)) = V(P)$ or $V(f(P)) = (V(P) \cup \{v^*\}) \setminus V(M'')$ depending if P uses vertices in M'' or not. Because every path passing by M'' must contain u and v, observe that we get Property $(\star)$. Therefore, we get that M^*,λ^*,s,z is a smaller counter-example, a contradiction. Observe that, in case $\{s,z\} \cap V(M') = \emptyset$, then either $u' = v'$ and Item 3 follows, or a similar argument can be applied by identifying all vertices of $V(M')$. We therefore get that $|\{s,z\} \cap V(M')| = 1$.

Now, suppose that $s \in V(M')$ and $z \notin V(M')$. In what follows, we apply again identification operations that produce vertices of degree 2; hence no m-topological minors are created by the arguments presented in the previous paragraph, and we consider this implicitly. Observe that, as $\{u,v\}$ is a vertex temporal s,z-cut, we have $1 = p_{M,\lambda}(s,z) < c_{M,\lambda}(s,z) = 2$. Call the only edge with endpoints uu' by e and the only edge with endpoints vv' by f. First, suppose that (*) every two temporal s,z-paths Q_1,Q_2 intersect in $V(M') \setminus \{s\}$, and let (M^*,λ^*) be the temporal graph obtained by identifying all vertices in $V(M) \setminus V(M')$ into a single vertex v^*. Observe that by (*) we get $p_{M^*,\lambda^*}(s,z) = 1$, and since M^* is smaller than M, we then must have $c_{M^*,\lambda^*}(s,z) = 1$. This is a contradiction as any vertex separating s from z in (M^*,λ^*) would also disconnect s from z in (M,λ). Thus there are two temporal s,z-paths in (M,λ), Q_1 and Q_2, such that $V(Q_1) \cap V(Q_2) \cap V(M') = \{s\}$. Finally, suppose that $|V(M')| > 1$, and now let (M^*,λ^*) be obtained from (M,λ) by identifying all vertices in $V(M')$ into a single vertex x. From the existence of Q_1,Q_2 and the fact that any temporal s,z-path must use e or f, observe that we must have $p_{M^*,\lambda^*}(x,z) = 1$. Again because M^* is smaller than M, we get $c_{M^*,\lambda^*}(x,z) = 1$, a contradiction as any such vertex would also disconnect s from z in (M,λ). Therefore, $V(M') = \{s\}$. The case $V(M') \cap \{s,z\} = \{z\}$ is analogous.

Case 2. uu' has multiplicity at least 2.

As M is 2-connected (Lemma 3.4.3) and $d(u) \geq 3$, then for $x,y \in N(u) \setminus \{u'\}$ there is a path P that does no pass by u; notice that P does not intersect $V(M')$ as u,v disconnects M' from x,y. Therefore, the cycle C_u formed by P and u does not intersect $V(M')$. Now, if $d(u') \geq 3$, then we can proceed in the same way to obtain a cycle $C_{u'}$ contained in M'. Consider then a path J_1 in $M-u$ from $C_u - u$ to v', and a path J_2 in $M'-u'$ from v' to $C_{u'} - u'$, and observe that we can apply Proposition 3.4.2 to get an m-subdivision of F_3, a contradiction. Therefore $d(u') = 2$. Call u_2 the vertex in $N(u') \setminus \{u\}$. Observe that a similar argument can be applied to $u'u_2$ in case it has multiplicity at least 2, as long as we do not arrive to v. In other words, by letting $(u = u_0, u' = u_1, \dots, u_q)$ be maximal such that $d(u_i) = 2$ for every $i \in [q-1]$, we get that:

(a) Either $(u_0, u_1, \dots, u_{q-1})$ is a chain and $u_{q-1}u_q$ has multiplicity 1;

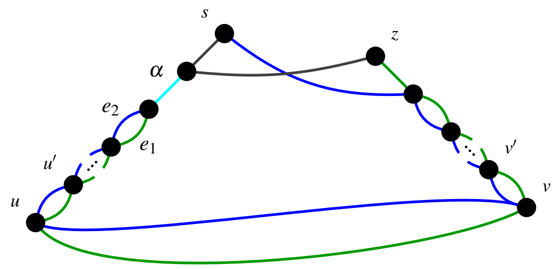
(b) Or $(u_0, u_1, \dots, u_q)$ is a chain and $u_q = v$.

Observe that if vv' also has multiplicity at least 2, then we can also obtain a maximal chain $(v = v_0, v' = v_1, \dots, v_p)$ such that $d(v_i) = 2$ for every $i \in [p]$.

We now prove that $s \notin V(M')$. By contradiction, suppose otherwise, and note that, since the edges incident in s have multiplicity 1 (Lemma 3.4.3), we get Item (a) above. Let $\alpha = \lambda(u_{q-1}u_q)$, and denote by E' the set of edges with endpoints $u_{q-2}u_{q-1}$; we know that there are at least two such edges. Also observe that $\lambda(e) \neq \lambda(e')$ for every $e, e' \in E'$ with $e \neq e'$ as we are in a minimum counter-example. Over all $e \in E'$, let e_1 be the edge minimizing $\lambda(e)$, and e_2 the one maximizing $\lambda(e)$. If $\lambda(e_1) \geq \alpha$, then $\lambda(e_2) > \alpha$ and hence all the paths passing by e_2 must traverse $u_q u_{q-1}$ from u_q to u_{q-1} before using e_2; therefore, every such path could use the edge e_1 instead. In other words, $(M - \{e_2\}, \lambda), s, z$ is still a counter-example, contradicting the minimality of M, λ, s, z. We then get $\lambda(e_1) < \alpha$. A similar argument can be applied to conclude that $\alpha < \lambda(e_2)$.

Now, note that every temporal s,z-path using e_1 must use it right before $u_{q-1}u_q$, while one that uses e_2 must do so right after using $u_{q-1}u_q$. We claim that z must also be in M'. Suppose otherwise and let P be a temporal s,z-path passing by e_1. Then P must first use some edge with endpoints vv', then use some path contained in the chain $(u, u_1, \cdots, u_{q-1})$. But observe that in this case, sPv_q contains the cut $\{u, v\}$ and therefore the other endpoint of P cannot be in M''. Therefore, we must have $z \in V(M')$. Let J_1 be a temporal s,z-path using e_1 and J_2 a temporal s,z-path using e_2. Observe Figure 20 to follow the arguments; we will see shortly that vv' must have multiplicity at least 2, as depicted in the figure. Also, let E_1 be the set of edges in J_1 appearing before time α and E_2 be the set of edges in J_2 appearing after time α. Because $\max \lambda(E_1) = \lambda(e_1) < \alpha < \min \lambda(E_2) = \lambda(e_2)$, these sets are disjoint, which immediatly implies that vv' has multiplicity at least 2. Now observe that $u, u_1, \cdots, u_q$ are all contained in both J_1 and J_2, and this implies that J_1, J_2 must also contain v and v'. Note additionally that the s,v-path contained in J_1 uses only edges in E_1, while the v,z-path contained in J_2 uses only edges in E_2. Since E_1, E_2 are disjoint, observe that, by picking a maximal chain $(v = v_0, v' = v_1, \dots, v_p)$ such

Figure 20 – Paths J_1 and J_2. Subset E_1 is highlighted in green, and E_2, in blue. Edge labeled α, highlighted in teal, has multiplicity 1 and is contained in both paths



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that $\{v_0, v_1, \ldots, v_p\} \subseteq V(J_1) \cap V(J_2)$, we get that v_p must have degree at least 3, contradicting what is said in the paragraph after Item (b). We then conclude that $s \notin V(M')$ and an analogous approach can be used to conclude that $z \notin V(M')$; observe that this gives us Items 1 and 2, and hence it just remains to prove Item 3.

Suppose by contradiction that Item 3 does not hold, i.e., that M' is not a u', v'-path. Observe that this implies that Item (a) occurs as Item (b) would imply that M' is a u', v'-path. We can apply the same argument to find a path $P = (v = v_1, v' = v_2, \ldots, v_{p-1}, v_p)$ such that $(v_1, \ldots, v_{p-1})$ is either a chain or a single vertex, $d(v_i) = 2$ for every $i \in \{2, \ldots, p-1\}$, and $v_{p-1}v_p$ has multiplicity 1. The possibility of $(v_1, \ldots, v_{p-1})$ being a single vertex occurs because vv' might have multiplicity 1 (and therefore we will get $p = 2$). Let e denote $u_{q-1}u_q$ and f denote $v_{p-1}v_p$. Define $X = V(M') \setminus (\{u_1, \ldots, u_q\} \cup V(P))$; as M' is not a $u'v'$-path, we get $X \neq \emptyset$. Let (M^*, λ^*) be the temporal graph obtained by identifying $X \cup \{u_q, v_p\}$ into a single vertex, W. As W is incident only to the edges corresponding to e and f in M^*, we get that this cannot have created an m-topological minor. Finally, observe that because e and f have multiplicity 1, we get that $p_{M^*,\lambda^*}(s,z) = p_{M,\lambda}(s,z)$ and $c_{M^*,\lambda^*}(s,z) = c_{M,\lambda}(s,z)$, a contradiction as $(M, \lambda), s, z$ is a minimum counter-example. □

The next lemma treats the case where $\Delta(M) \leq 3$ by showing that we must have a crossed structure.

Lemma 3.4.5 *Let M, λ, s, z be a minimum counter-example. If $\Delta(M) \leq 3$, then M has a crossed structure.*

Proof. We start by finding the chain of M. Denote by $p'_{M,\lambda}(s,z)$ the maximum number of edge disjoint temporal s,z-paths, i.e., the maximum k such that there are temporal s,z-paths $P_1,\dots,P_k$ with $E(P_i)\cap E(P_j)=\emptyset$ for every $i,j\in[k]$, $i\neq j$. Define also $c'_{M,\lambda}(s,z)$ as the minimum cardinality of a subset $S\subseteq E(M)$ such that $(M-S,\lambda)$ has no temporal s,z-paths. Note that, since $sz\notin E(U(M))$, then removing one endpoint of each $e\in S$ also destroys every temporal s,z-path; hence $c'_{M,\lambda}(s,z)\geq c_{M,\lambda}(s,z)>p_{M,\lambda}(s,z)$. Using the equality $c'_{M,\lambda}(s,z)=p'_{M,\lambda}(s,z)$ proved in (3), one can find at least $c_{M,\lambda}(s,z)$ edge disjoint temporal s,z-paths. Because $c_{M,\lambda}(s,z)>p_{M,\lambda}(s,z)$, at least two of these paths, say J_1,J_2, must intersect in an internal vertex. In what follows, we extract from these paths a chain $(z_0,\dots,z_q)$ such that $(N(z_0)\cup N(z_q))\setminus\{z_0,\dots,z_q\}$ has cardinality 4.

We denote the vertices of J_1 by $s=x_0,x_1,\dots,x_{q_1}=z$ and the vertices of J_2 by $s=y_0,y_1,\dots,y_{q_2}=z$, and suppose these vertices appear in J_1 and J_2 in these orders. Let $i>0$ be minimum such that $x_i\in V(J_2)$; since J_1,J_2 intersect in an internal vertex, we know that $i<q_1$. Let $j>0$ be the index of x_i in J_2, i.e., $j\in[q_2-1]$ is such that $x_i=y_j$. Then we have $N(x_i)\supseteq\{x_{i-1},x_{i+1},y_{j-1},y_{j+1}\}$. By the choice of x_i we have that $x_{i-1}\notin\{x_{i+1},y_{j-1},y_{j+1}\}$. Moreover, as $y_{j-1}\neq y_{j+1}$ and $d_M(x_0)\leq 3$, we must have $N(x_i)=\{x_{i-1},y_{j-1},y_{j+1}\}$, this implies $x_{i+1}\in\{y_{j-1},y_{j+1}\}$. Now we can define k to be the maximum integer such that $x_i,x_{i+1},\dots,x_{i+k}$ are also consecutive vertices of J_2. Observe that such vertices might appear in the same order in J_2, i.e., these are vertices $y_j,y_{j+1},\dots,y_{j+k}$, or in the reverse order, i.e., these are vertices $y_j,y_{j-1},\dots,y_{j-k}$. Even if both cases are symmetrical, we prove each separately for the sake of completeness.

- $L=(x_i,x_{i+1},\dots,x_{i+k})=(y_j,y_{j+1},\dots,y_{j+k})$. In this case, let $i^*=i+k$ and $j^*=j+k$. First, observe that, because $x_i=y_j$ and $x_{i^*}=y_{j^*}$, we get $(N(x_i)\cup N(x_{i^*}))\setminus V(L)=\{x_{i-1},y_{j-1},x_{i^*+1},y_{j^*+1}\}$. Hence, if we prove that these vertices are all distinct, then L is the desired chain. In what follows, we argue that either this is the case, or we find another appropriate chain.

  Because J_1 and J_2 are paths, we know that $x_{i-1}\neq x_{i^*+1}$ and $y_{j-1}\neq y_{j^*+1}$. By the maximality of k and because there are no edges of multiplicity at least 2 incident to z (Lemma 3.4.3), we get $x_{i^*+1}\neq y_{j^*+1}$. Recall that $x_{i-1}\notin V(J_2)$ by the choice of x_i; hence $x_{i-1}\neq y_{j-1}$ and $x_{i-1}\neq y_{j^*+1}$. It thus remains to prove that $y_{j-1}\neq x_{i^*+1}$. Suppose by contradiction that $y_{j-1}=x_{i^*+1}$. Such assumption and the fact that the degree of y_{j-1} is at most 3 imply that the sets $\{y_{j-2},y_j\}$ and $\{x_{i^*},x_{i^*+2}\}$ must intersect. Recall that $x_{i^*}=y_{j^*}$. Since J_2 is a path and $j-2<j<j^*$, we get that $x_{i^*}\notin$

$\{y_{j-2}, y_j\}$. And because J_1 is a path and $i^* < i^* + 1$, we also get $x_{i^*} \neq x_{i^*+2}$. By analogous arguments and the fact that $y_j = x_j$ we get that $y_j \notin \{y_{j-2}, x_{i^*+2}\}$. It thus follows that $x_{i^*+2} = y_{j-2}$. Analogously to the definition of k, we can define $k' \geq 2$ to be maximum such that $L' = (x_{i^*+1}, x_{i^*+2}, \ldots, x_{i^*+k'}) = (y_{j-1}, y_{j-2}, \ldots, y_{j-k'})$. We have $(N(x_{i^*+1}) \cup N(x_{i^*+k'})) \setminus V(L') = \{x_{i^*}, y_j, x_{i^*+k'+1}, y_{j-k'-1}\}$. We now argue that these vertices are all distinct, and hence L' is the desired chain. Observe that $x_{i^*+k'+1} \neq y_{j-k'-1}$ by the maximality of k' and the fact that there are no edges of multiplicity at least 2 incident to z (Lemma 3.4.3). Moreover, as J_2 is a path and $x_{i^*} = y_{j^*}$, we get that x_{i^*}, y_j and $y_{j-k'-1}$ are all distinct. Similarly, as J_1 is a path and $y_j = x_i$, we get that $x_{i^*+k'+1}$ is also different from x_{i^*} and y_j, and we are done.

- $L = (x_i, x_{i+1}, \ldots, x_{i+k}) = (y_j, y_{j-1}, \ldots, y_{j-k})$. In this case, let $i^* = i + k$ and $j^* = j - k$. Notice that $(N(x_i) \cup N(x_{i^*})) \setminus V(L) = \{x_{i-1}, y_{j+1}, x_{i^*+1}, y_{j^*-1}\}$. Again we want to argue that these vertices are all distinct. Because J_1 and J_2 are paths, it follows that $x_{i-1} \neq x_{i^*+1}$ and $y_{j+1} \neq y_{j^*-1}$. By the maximality of k and by Lemma 3.4.3, we have that $x_{i^*+1} \neq y_{j^*-1}$. By the choice of x_i, we have that $x_{i-1} \neq y_{j+1}$ and $x_{i-1} \neq y_{j^*-1}$. It hence remains to show that $y_{j+1} \neq x_{i^*+1}$. Suppose by contradiction that $y_{j+1} = x_{i^*+1}$. We proceed in a similar way as in the previous item. Because y_{j+1} has degree at most 3, we get that $\{y_j, y_{j+2}\} \cap \{x_{i^*}, x_{i^*+2}\} \neq \emptyset$. But since J_1 and J_2 are paths, $y_j = x_i$ and $x_{i^*} = y_{j^*}$, it follows that $y_{j+2} = x_{i^*+2}$. Therefore, let $k' \geq 2$ be maximum such that $L' = (x_{i^*+1}, x_{i^*+2}, \ldots, x_{i^*+k'}) = (y_{j+1}, y_{j+2}, \ldots, y_{j+k'})$. As we have that $(N(x_{i^*+1}) \cup N(x_{i^*+k'})) \setminus V(L') = \{x_i, y_{j^*}, x_{i^*+k'+1}, y_{j+k'+1}\}$, we finish the proof by arguing that these are all distinct vertices, and therefore L' is the desired chain. Since J_1 and J_2 are paths, $x_i = y_j$ and $y_{j-k} = x_{i^*}$, we get that $x_i \notin \{x_{i^*+k'+1}, y_{j-k}, y_{j+k'+1}\}$ and $y_{j-k} \notin \{y_{j+k'+1}, x_{i^*+k'+1}\}$. Finally, by the maximality of k' and by Lemma 3.4.3, we get $x_{i^*+k'+1} \neq y_{j+k'+1}$.

Now let $L = (z_0, \cdots, z_q)$ be the chain previously obtained. We first show that, by decreasing the size of such chain, if necessary, we can suppose that each internal vertex of L has degree 2, while maintaining the property that $(N(z_0) \cup N(z_q)) \setminus V(L)$ has cardinality 4. Observe that this property applied to L and the fact that $\Delta(M) \leq 3$ implies that $N(z_0) \setminus V(L)$ has exactly 2 vertices, say a and b, the same holding for $N(z_q) \setminus V(L)$, call them c and d. First suppose that there exists an edge with endpoints $z_i z_j$ for some $i, j \in [q-1]$, $j > i+1$. Then, since $N(z_0) \setminus V(L) = \{a, b\}$, we get that $(z_0, \ldots, z_i)$ is a smaller chain such that $(N(z_0) \cup N(z_i)) \setminus \{z_0, \ldots, z_i\} = \{a, b, z_{i+1}, z_j\}$, with all such vertices being distinct. So we can suppose that $N(z_i) \cap V(L) = \{z_{i-1}, z_{i+1}\}$ for every $i \in [q-1]$. Now, suppose that, for some $i \in [q-1]$, we have $N(z_i) = \{z_{i-1}, z_{i+1}, e\}$. As a,b,c,d are all distinct, we get that either $e \notin \{a,b\}$ or $e \notin \{c,d\}$; suppose, without loss of generality, that the former happens. Thus $L = (z_0, \dots, z_i)$ is a smaller chain having the required property. By iteratively applying this procedure, we eventually arrive to the desired chain.

We now proceed to prove that M has crossed structure. Let M' be obtained from M by the identification of L, and let ℓ denote the new vertex. We argue that M' is 2-connected; suppose otherwise. By Lemma 3.4.3, we know that M is 2-connected, and hence the only possibility is that $M' - \{\ell\}$ is disconnected. Let H, H' be any two components of $M' - \{\ell\}$. Observe that H, H' are also components of $M - V(L)$. By the fact that $N_{M'}(\ell) = (N_M(z_0) \cup N_M(z_q)) \setminus V(L)$ and that $d(z_i) = 2$ for every $i \in [q-1]$, we get that if z_0 has no neighbors in H, then z_q disconnects $V(H') \cup \{z_0, \dots, z_{q-1}\}$ from $V(H)$ in M, contradicting Lemma 3.4.3. Hence z_0 must be adjacent to some vertex in H, say x, and analogously to some vertex in H', say x'. The same holds for z_q, so let $y \in V(H)$ and $y' \in V(H')$ being adjacent to z_q. It therefore follows that $S = \{xz_0, yz_q\} \subseteq E(U(M))$ (resp. $S' = \{x'z_0, y'z_q\} \subseteq E(U(M))$) is such that $U(M) - S$ (resp. $U(M) - S'$) is disconnected. Observe that, as x, y are in the same component of $M - V(L)$ and because z_0 and z_q are linked by L in $U(M) - S$, one can see that the vertex set of the components of $U(M) - S$ are exactly $V(H)$ and $V(H') \cup V(L)$. Similarly, the vertex set of the components of $U(M) - S'$ are exactly $V(H')$ and $V(H) \cup V(L)$. Note that, together with the fact that $d_M(z_0) = d_M(z_q) = 3$ and applying Lemma 3.4.4, we can conclude that either $V(H') = \{s\}$ and $V(H) = \{z\}$ or $V(H') = \{z\}$ and $V(H) = \{s\}$. In either case, we get a contradiction to the fact that $(N(z_0) \cup N(z_q)) \setminus V(L)$ has cardinality 4.

Coming back to our main argument, we now have that M' is 2-connected. Also, since $\Delta(M) = 3$ and $(N(z_0) \cup N(z_q)) \setminus V(L)$ has cardinality 4, we get that $U(M')$ has exactly one vertex of degree 4, namely ℓ. Observe that Lemma 3.4.1 tells us that every 2-connected graph with exactly one vertex of degree at least 4 must have a subdivision of F_3. Recall that M does not have F_i as m-topological minor for each $i \in [3]$, as $(M, \lambda), s, z$ is a counter-example. It follows from Lemma 3.4.2 that M has a crossed structure. □

Recall that, in order to conclude the case $\Delta(M) \leq 3$, we want to prove that if M has a crossed structure, then M is Mengerian. This together with Lemma 5.1 gives us that, in fact, a minimum counter-example cannot satisfy $\Delta(M) \leq 3$. This is the idea of the proof of our final tool lemma.

Lemma 3.4.6 *If $(M,\lambda),s,z$ is a minimum counter-example, then $\Delta(M) \geq 4$.*

Proof. By contradiction, suppose that $\Delta(M) \leq 3$. By Lemma 5.1, we get that M has a crossed structure. We use the same terminology used in Figure 14, i.e., $(z_0,\ldots,z_q)$ is the chain, $h_1,h_3 \in N(z_0)$, $h_2,h_4 \in N(z_q)$, h_1,h_2 are linked through A_1, h_3,h_4 are linked through A_2, h_1,h_4 are possibly linked through B_1 (recall that M might be 1-crossed), and h_2,h_3 are linked through B_2, in case G is 2-crossed.

We want to arrive to a contradiction by showing that $p_{M,\lambda}(s,z) = c_{M,\lambda}(s,z)$. Observe that if $p_{M,\lambda}(s,z) = 0$, then it follows directly. So suppose that $p_{M,\lambda}(s,z) \geq 1$ (and hence $c_{M,\lambda}(s,z) \geq 2$). Let D be the set of vertices of M with degree 3, and observe that D contains $\{h_1,h_2,h_3,h_4,z_0,z_q\}$. We first prove some further structural properties.

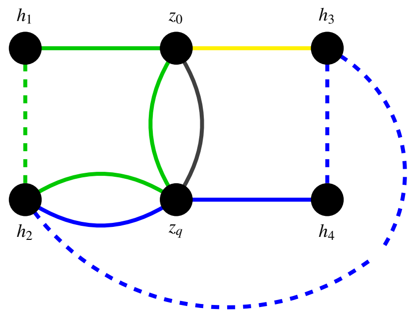
1. $N(s) \cup N(z) \subseteq D$;
2. For each edge $f \in E(M)$, there is a temporal s,z-path P using f and not containing vertices in D;
3. Multiedges z_qh_2 and z_0h_3 have multiplicity 1. Additionally, if M is 2-crossed, then also z_0h_1 and z_qh_4 have multiplicity 1.
4. If $c_{M,\lambda}(s,z) = 2$ and there is a vertex $v \in N(s) \cap N(z)$, then the multiedge incident to v and $w \notin \{s,z\}$ has multiplicity at least 2.

For Item 1, suppose that s has a neighbor x such that $N(x) = \{s,y\}$ (i.e., $d(x) = 2$). By Lemma 3.4.3, we know that sx has multiplicity 1. Then every $e \in E(M)$ with endpoints xy is such that $\lambda(e) \geq \lambda(sx)$, and we can remove x and make s directly adjacent to y through an edge active at time $\lambda(sx)$ to obtain a smaller counter-example, a contradiction. Item 1 follows by applying a similar argument to z.

Now, for Item 2, suppose by contradiction that $f \in E(M)$ is such that every temporal s,z-path using f contains D. Observe that the previous item gives us that every such path contains all neighbors of s. Note that if there are no temporal s,z-paths in $(M-f,\lambda)$, then $p_{G,\lambda}(s,z) = c_{G,\lambda}(s,z) = 1$ as we could pick any endpoint of f not in $\{s,z\}$ to be a vertex temporal s,z-cut. This, along with the fact that every path using f contains a vertex temporal s,z-cut, implies that a maximum set of vertex disjoint temporal s,z-paths cannot contain a path passing by f, i.e., we get that $p_{(M-f,\lambda)}(s,z) = p_{(M,\lambda)}(s,z)$. Since $(M-f,\lambda)$ cannot be a counter-example, we get that there exists a vertex temporal s,z-cut S of $(M-f,\lambda)$ with less than $c_{M,\lambda}(s,z)$ vertices. If S contains some vertex of D, then S also intersects every path using f, a contradiction as S would be a vertex temporal s,z-cut in (M,λ). Hence $S \subseteq V(M) \setminus D$; consider then $u \in S$. We argue that we can obtain a vertex temporal s,z-cut in (M,λ) of same size as S containing some vertex of D. This gives us a contradiction by the previous argument. If u lies in a temporal s,z-path P whose internal vertices have degree 2, then by removing all internal vertices of P from (M,λ) we decrease $p_{M,\lambda}(s,z)$ and $c_{M,\lambda}(s,z)$ by exactly one, a contradiction with the minimality of $(M,\lambda),s,z$. Therefore, every such path containing u must contain some vertex of D different from s and z; let v be such a vertex. Note that in fact every temporal s,z-path containing u must also contain v. This means that $T = (S\setminus\{u\})\cup\{v\}$ is also a vertex temporal s,z-cut in $(M-f,\lambda)$.

For Item 3, recall Proposition 3.4.1 and observe Figure 21 to see that if z_qh_2 has multiplicity at least 2, then we can find an m-subdivision of F_1. Indeed, we have a cycle C_1, highlighted in green, that is edge disjoint of C_2, highlighted in blue. Additionally C_1,C_2 are connected by a path, highlighted in yellow, and intersect in the chain (h_2,z_q). Note also that the analogous holds for z_0h_3, and for z_0h_1 and z_qh_4 as well in case M is 2-crossed.

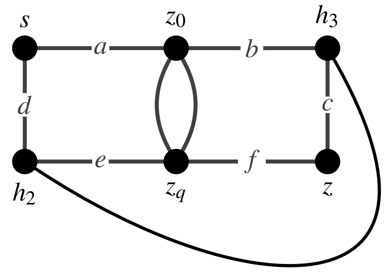
Figure 21 – Structure of M in case h_2z_q has multiplicity at least 2. The chain $(z_0,\ldots,z_q)$ is simplified to two parallel edges. Dashed lines denote paths and solid lines denote edges



Source: elaborated by the author.

Finally, for Item 4 suppose by contradiction that $c_{G,\lambda}(s,z)=2$, there is a vertex $v\in N(s)\cap N(z)$ and the multiplicity of vw is 1, where w is the neighbor of v different fom s and z. Observe that w is well defined by Item 1 and the fact that $\Delta(M)=3$. Let $\alpha=\lambda(sv)$, $\beta=\lambda(vz)$ and $\gamma=\lambda(vw)$. If $\beta<\alpha$, then every path using sv must use vw and we have $\alpha\le\gamma$; in the same way every path using vz must use vw and we have $\gamma\le\beta$, a contradiction. Therefore we must have $\alpha\le\beta$. Observe that, because $\{v\}$ is not a vertex temporal s,z-cut, there exists a temporal s,z-path in $(M-v,\lambda)$. We get a contradiction as such temporal path is vertex disjoint from (s,α,v,β,z), giving us that $p_{M,\lambda}(s,z)=c_{M,\lambda}(s,z)=2$.

Figure 22 – Structure of M with time function labels on specific edges. Only the labeled lines represent edge, the others represent paths with the two lines connecting z_0z_q representing a chain



Source: elaborated by the author.

Now that we have proved Items 1-4, we proceed with the proof of the lemma. First note that Lemma 3.4.3 gives us that $\{s,z\} \cap \{z_0,\ldots,z_q\} = \emptyset$. We now analyse the cases.

- M is 1-crossed. By the definition of crossed structure we have that, if $A_1 \neq \emptyset$, then $N(A_1) = \{h_1,h_2\}$. Then we can apply Lemma 3.4.4 for multiedges connecting h_1 and h_2 and the vertices of A_1. Using the same for A_2 we have either the multiedge h_3h_4 or that A_2 is an h_3,h_4-path. Hence both h_2 and h_3 have neighbors whose degree equals 2, and by Item 1 we get $\{s,z\} \cap \{h_2,h_3\} = \emptyset$. In a similar way, if $s \in V(A_1)$, then Lemma 3.4.4 gives us that $V(A_1) = \{s\}$. This is a contradiction to Item 1, as in this case s has a neighbor of degree 2, namely h_1. Therefore, if $s \in V(A_1) \cup \{h_1\}$ we must have $s = h_1$. We can apply the same argument to conclude that A_1 must be empty. Indeed, if this is not the case, then Lemma 3.4.4 gives us that A_1 is an h_1,h_2-path, again contradicting Item 1 as in such case s has a neighbor of degree 2, namely its neighbor in A_1. Thus we must have the multiedge h_1h_2. Because the analogous holds for z, we have the following possible cases:
  - $s = h_1$ and $z = h_4$. By Lemma 3.4.3 and Item 3, we have that $sh_2, h_2z_q, sz_0, z_0h_3, h_3z, z_qz$ have multiplicity 1. Define: $a = \lambda(sz_0)$, $b = \lambda(z_0h_3)$, $c = \lambda(h_3z)$, $d = \lambda(sh_2)$, $e = \lambda(h_2z_q)$ and $f = \lambda(z_qz)$. Figure 22 help one to understand the labels.

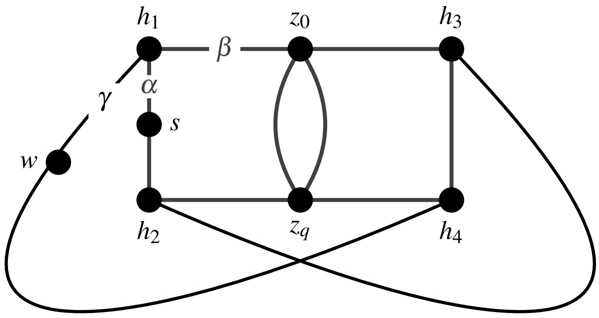
    By Item 2, there exists a temporal s,z-path using z_0h_3 that does not contain D (recall that D contains every vertex of degree 3). Observe that such path must be $P_1 = (s,a,z_0,b,h_3,c,z)$. Similarly, observe that the only temporal s,z-path using h_2z_q and not containing D is $P_2 = (s,d,h_2,e,z_q,f,z)$. This is a contradiction as these paths are disjoint and $\{z_0,h_2\}$ is a vertex temporal s,z-cut (i.e. $c_{M,\lambda}(s,z) = 2$).
  - $s = h_1$ and $z \in V(B_1)$. By Lemma 3.4.4, we know that $V(B_1) = \{z\}$, and by Lemma 3.4.3 and Item 3, that sh_2 and h_2z have multiplicity 1 and z_qh_2 has mul-

tiplicity 1. Therefore, $c_{M,\lambda}(s,z) = 2$ and h_2 is a common neighbor of s and z having degree 3 and such that every edge incident to h_2 has multiplicity 1. This contradicts Item 4.

- M is 2-crossed. By the definition of crossed structure we have that, if $A_1 \neq \emptyset$, then $N(A_1) = \{h_1, h_2\}$. Then we can apply Lemma 3.4.4 for multiedges connecting h_1 and h_2 and the vertices of A_1 to obtain that all vertices of A_1 have degree 2. Using the same for A_2 we have either the multiedge h_3h_4 or that A_2 is an h_3,h_4-path. Notice that the same can be applied to B_1 and B_2. We then subdivide in a case analysis.
  - $c_{M,\lambda}(s,z) = 3$: note that $d(s) \geq 3$ and $d(z) \geq 3$; hence $\{s,z\} \subseteq \{h_1,h_2,h_3,h_4\}$. Because these vertices are all symmetrical, suppose without loss of generality that $s = h_1$. By Item 1 and Lemma 3.4.3, we get sh_2 and sh_4 are multiedges of M of multiplicity 1. Additionally, since $sz \notin E(M)$, we get that z must be equal to h_3, which analogously as before gives us that zh_2 and zh_4 are multiedges of M of multiplicity 1. Now define the cut $S = \{h_2, z_0, h_4\}$, and observe that, for each $x \in S$, there is exactly only one s,z-path in $M - (S \setminus \{x\})$; denote such path by P_x. Observe that the uniqueness of such paths imply that each of them must be a temporal path, as otherwise we could obtain a smaller cut. We get a contradiction as P_{z_0}, P_{h_2} and P_{h_4} are 3 vertex disjoint temporal s,z-paths.
  - $c_{M,\lambda}(s,z) = 2$ and $\{s,z\} \subseteq \{h_1,h_2,h_3,h_4\}$. As in the previous case, we can suppose, without loss of generality, that $s = h_1$, which in turn gives us that $z = h_3$ and that sh_2, sh_4, zh_2, zh_4 are multiedges of M of multiplicity 1. This and Item 3 contradict Item 4.
  - $c_{M,\lambda}(s,z) = 2$ and $\{s,z\} \nsubseteq \{h_1,h_2,h_3,h_4\}$. First suppose that $s \in V(A_1)$. Then, by Lemma 3.4.4 we have $A_1 = \{s\}$ and that sh_1 and sh_2 have multiplicity 1. Now, if $z \in B_1 \cup B_2 \cup \{h_3,h_4\}$, then again by Lemmas 3.4.3 and 3.4.4 and Item 3 we get that there is $h_i \in N(s) \cap N(z)$ such that all edges incident to h_i are simple and $i \in \{1,2\}$. This situation that contradicts Item 4. And because $sz \notin E(M)$, we then get that $z \in V(A_2)$. Let $\alpha = \lambda(sh_1), \beta = \lambda(h_1z_0)$ and $\gamma = \lambda(f)$ where f is an edge with endpoints h_1w for $w \in N(h_1) \setminus \{s, z_0\}$ (notice that f can have endpoints h_1h_4) Figure 23 shows where these labels are in the graph.

    First we prove that $\alpha \leq \gamma$. Suppose otherwise, then all temporal s,z-paths using f must also use h_1z_0 and must start by using sh_2. If one of these paths uses h_1z_0 before

Figure 23 – Labeled lines represent edges and its times. Unlabeled lines represent paths



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f, then we have $\beta \leq \gamma$. With the fact that $\alpha \leq \max\{\beta, \gamma\}$ (as otherwise no path uses sh_1), we get $\alpha \leq \gamma$, a contradiction. Therefore, every temporal s,z-path that uses f must also use, in this order, the vertices h_4, h_1, z_0 and starts by sh_2. Let P be such a path; we analyze the sequence of vertices of P between the vertices h_2 and h_4. If P visits h_3 after h_2, then as every h_3,h_4-paths that does not pass by h_2 must pass by z_0 or z, we know that z_0 is visited by P before h_4, a contradiction. Therefore P does not visit h_3 after h_2, which means that P must start with s, h_2, z_q. Now observe that, because P must visit h_4, h_1, z_0 in this order, we can conclude that P uses, in this order, $s, h_2, z_q, h_4, h_1, z_0, h_3, z$. Therefore P uses all vertices of degree 3, and since P is a generic temporal s,z-path using f, we know that this holds for all such paths, which contradicts Item 2. We can thus conclude $\alpha \leq \gamma$, as we wanted to prove. Observe that this also gives us that there must exist a path passing by f that visits h_1 before h_4, and since f was an arbitrary edge with endpoints h_1w, we get that this holds for all such edges. Then, there is a temporal h_1,h_4-path Q that is contained in $\{h_1, h_4\} \cup B_1$ Because the graph is symmetric, the same argument can be applied to ensure the following:

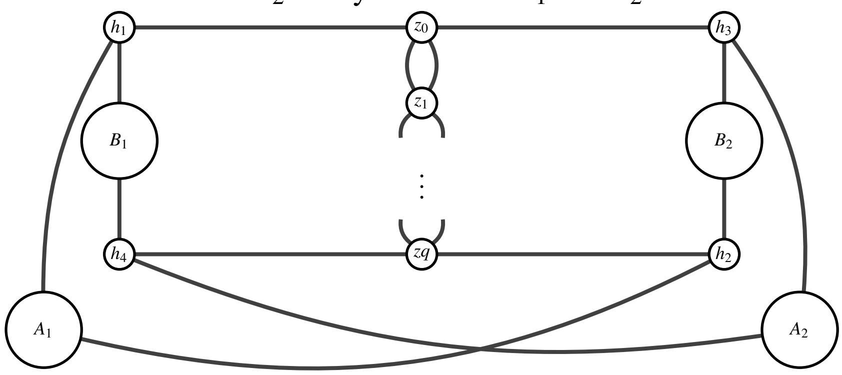
* $\lambda(sh_1) \leq \lambda(f)$ where f has endpoints h_1w_1 and w_1 is the neighbor of h_1 different from s and z_0;
* $\lambda(sh_2) \leq \lambda(f)$ where f has endpoints h_2w_2 and w_2 is the neighbor of h_2 different from s and z_q;
* $\lambda(zh_3) \geq \lambda(f)$ where f has endpoints h_3w_3 and w_3 is the neighbor of h_3 different from z and z_0;
* $\lambda(zh_4) \geq \lambda(f)$ where f has endpoints h_4w_4 and w_4 is the neighbor of h_4 different

from z and z_q.

In the same way there is Q' a temporal h_2,h_3-path with vertices in $\{h_2,h_3\} \cup B_2$. Since $\lambda(sh_1) \leq \lambda(f)$ for every f with endpoints h_1w_1, and $\lambda(zh_4) \geq \lambda(f)$ for every f with endpoints h_4w_4, we get that $P_1 = sQz$ is a temporal s,z-path. Similarly we get that $P_2 = sQ'z$ is a temporal s,z-path, a contradiction as P_1 and P_2 are disjoint and $\{h_1,h_2\}$ is a vertex temporal s,z-cut (i.e. $c_{M,\lambda}(s,z) = 2$).

Finally, observe that the case where $s \in V(A_2)$ is analogous, as in this case we get $z \in V(A_1)$. Observe Figure 24 to see that the cases $s \in V(B_1)$ or $s \in V(B_2)$ are also analogous. Additionally, if instead z is within some of these subgraphs, then we can apply Lemma 3.2.1. □

Figure 24 – A redrawing of M that shows how the sets A_1 and A_2 are symmetric to B_1 and B_2



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We have finally all ingredients to finish the proof of Theorem 3.1.2.

Proof of Sufficiency of Theorem 3.1.2. We want to prove that if F is not an m-topological minor of M for every $F \in \{F_1,F_2,F_3\}$, then M is Mengerian.

Suppose otherwise and let $(M,\lambda),s,z$ be a minimum counter-example. Applying Lemma 3.4.3, we know that M is 2-connected; hence so is $U(M)$. Observe that if $U(M)$ contains a subdivision of F_3, then so does M and we are done. Hence suppose that $U(M)$ has no F_3 as topological minor. By Lemma 3.4.6 we get that $\Delta(M) \geq 4$, and by Lemma 3.4.1 we get that there are $v,w \in V(M)$ and $d \geq 4$ for which $U(M)$ is (v,w,d)-decomposable. Observe that the components of $U(M) - \{v,w\}$ and of $M - \{v,w\}$ differ only on the multiplicity of the edges. So let $M_1,\cdots,M_p$ be the components of $M - \{v,w\}$, and for each $i \in [p]$, let v_i (resp. w_i) be the vertex in $V(M_i) \cap N(v)$ (resp. $V(M_i) \cap N(w)$). Since $p \in \{d-1,d\}$ and $d \geq 4$, we have $p \geq 3$.

We now analyse the following cases.

1. $\{s,z\} = \{v,w\}$. Without loss of generality, suppose $s = v$ and $w = z$, and let $\mathscr{P}$ be a maximum set of vertex disjoint temporal s,z-paths (i.e., $|\mathscr{P}| = p(s,z)$). Since s,z are non-adjacent, each path in $\mathscr{P}$ has at least one internal vertex. So for each $P \in \mathscr{P}$, let $v_{i_P} = V(P) \cap \{v_1, \cdots, v_p\}$ be the vertex in the neighborhood of s used by P; also, let $S = \{v_{i_P} \ : \ P \in \mathscr{P}\}$. Because the paths in $\mathscr{P}$ are internally disjoint, we know that $|S| = |\mathscr{P}|$. And since $|\mathscr{P}| = p(s,z) < c(s,z)$, we get that S cannot be a vertex temporal s,z-cut. Therefore, there exists a temporal path P' in $(M - S, \lambda)$. Observe that $P' - \{s,z\}$ must be contained in $V(M_j)$ with $v_j \notin S$, and that in this case P' intersects the paths in $\mathscr{P}$ only in $\{s,z\}$. This contradicts the maximality of $\mathscr{P}$.
2. There exist $i, j \in [p]$ with $i \neq j$ such that $s \in V(M_i)$ and $z \in V(M_j)$. Observe that both $\{vv_i, ww_i\}$ and $\{vv_j, ww_j\}$ satisfy Lemma 3.4.4. Therefore, we have that $V(M_i) = \{s\}$ and $V(M_j) = \{z\}$. By Lemma 3.4.3, all multiedges incident to s and z have multiplicity 1. Hence, let e_1 be the edge with endpoints sv and e_2 be the edge with endpoints vz. Because w is not a vertex temporal s,z-cut, there must be a temporal s,z-path in $(M - w, \lambda)$. Additionally, because M is (v,w,d)-decomposable, observe that such temporal path must be $(s, \lambda(e_1), v, \lambda(e_2), z)$. Applying a similar argument to $(M - v, \lambda)$, we get 2 disjoint temporal s,z-paths in M, a contradiction as $\{v,w\}$ is a vertex temporal s,z-cut.
3. $\{s,z\} \cap \{v,w\}$ is a unitary set. Suppose that $s = v$ and let $i \in [p]$ be such that $z \in V(M_i)$. In this case, again applying Lemma 3.4.4 to $\{sv_i, ww_i\}$, we get that $V(M_i) = \{z\}$, and hence s and z are adjacent, a contradiction. The other cases are clearly analogous.
4. There exists $i \in [p]$ such that $\{s,z\} \subseteq V(M_i)$. This contradicts Lemma 3.4.4 applied to $\{vv_i, ww_i\}$.

3.5 Mengerian graphs Recognition

In this section, we focus on proving the following theorem.

Theorem 3.5.1 *Let M be a graph. Then, one can decide whether M is a Mengerian graph in time $O(mn^3)$, where $n = |V(M)|$ and $m = |E(U(M))|$.*

In (13), it is shown that given two simple graphs G and H, one can check if H is a topological minor of G in time $O(f(|V(H)|)|V(G)|^3)$. Thus, for a finite family of graphs $\mathscr{H} = \{H_1, \ldots, H_k\}$, each H_i of constant size and k also constant, the problem of deciding whether

H is a topological minor of G for some $H \in \mathscr{H}$ can be solved in polynomial time. If the same holds for m-topological minor, then Theorem 3.5.1 implies that we can recognize Mengerian graphs in polynomial time. We prove that this is the case for our family of graphs, i.e., that deciding whether a given graph M has some $F \in \{F_1, F_2, F_3\}$ as m-topological minor can be done in polynomial time. Our algorithm makes use of the one presented in (13). Observe that we only need to recognize m-subdivisions of F_1 and F_2, since any m-subdivision of F_3 is also a subdivision of F_3 (i.e., we can recognize it by applying the algorithm in (13)). The following lemma is the key for reaching polynomial time.

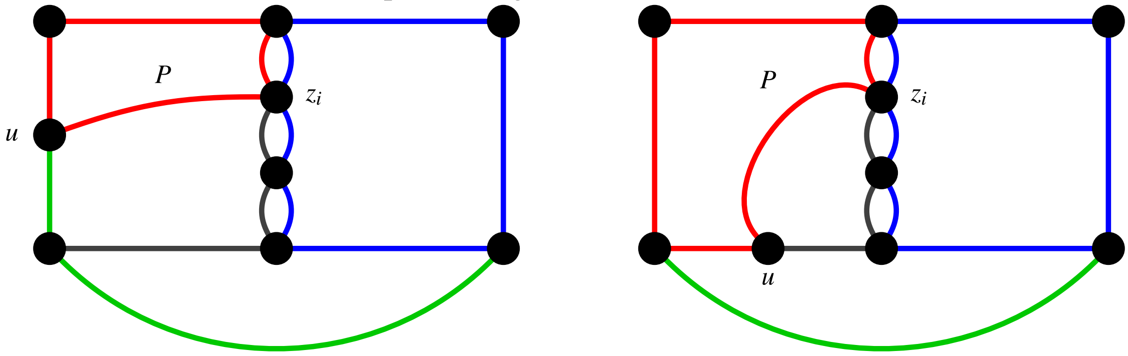
Lemma 3.5.1 *Let M be a graph that has no F_3 as topological minor and has some graph in $\{F_1, F_2\}$ as m-topological minor. Then M has an m-subdivision H of some graph in $\{F_1, F_2\}$ such that $d(v) = 2$ for every $v \in V(H)$ that is an internal vertex of the chain of H.*

Proof. We can suppose that M is 2-connected as otherwise we can constrain ourselves to the 2-connected components of M. Let $F \preceq_m M$, where $F \in \{F_1, F_2\}$, and let $H \subseteq M$ be an m-subdivision of F in M. Let $L = (z_0, \dots, z_q)$ be the chain of H. We prove that if $d(z_i) > 2$ for some $i \in [q-1]$, then we can find an m-subdivision $H' \subseteq M$ of $F' \in \{F_1, F_2\}$ whose chain is smaller than the chain L. This means that if H is an m-subdivision of a graph in $\{F_1, F_2\}$ with minimum chain size, then $d(z_i) = 2$ for every $i \in [q-1]$ and the lemma follows.

So let $z_i \in V(L)$ with $i \in [q-1]$ be such that $d(z_i) > 2$, and let $v \in N(z_i) \setminus \{z_{i-1}, z_{i+1}\}$. Because M is 2-connected as well, we get that z_i cannot disconnect v from H, in case $v \notin V(H)$. Let P be a path between v and H not passing through z_i, and let $u \in V(H)$ be the other endpoint of P; it might happen that P is a single vertex in case $v = u \in V(H)$. The proof consists of analyzing all possible choices of F, and where u might lie in H.

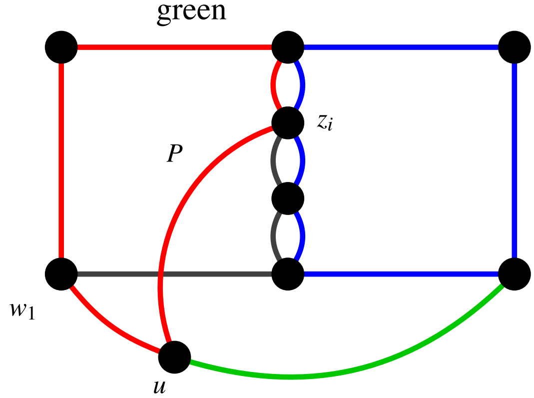
- $F = F_1$. We use Proposition 3.4.1 in H to find subgraphs C_1, C_2, J, L as indicated in Figure 13. Let $w_1 \in V(J) \cap V(C_1)$ and $w_2 \in V(J) \cap V(C_2)$. As the ordering of L is arbitrary, we can suppose that $\{w_1, w_2\} \cap N(z_0) = \emptyset$.
  First, consider that $u \in V(C_1) \setminus (N(z_0) \cup V(L))$, and observe Figure 25 to follow the construction. Let Q be the u, z_i-path contained in C_1 not passing through z_{i+1}, and let C_1' be the cycle formed by P and Q. Let J' be either equal to J, in case $w_1 \in V(C_1')$, or be the C_1', C_2-path defined by J together with the w_1, u-path contained in $C_1 - V(L)$. Finally, let $L' = (z_0, \dots, z_i)$, and observe that C_1', C_2, J', L' satisfy the first two conditions in Proposition 3.4.1. The third condition also holds as $N_H(z_0) \cap V(C_1') = N_H(z_0) \cap N(Q) = N_H(z_0) \cap N(C_1)$; thus we get a subdivision of F_1 with a smaller chain, as desired.

Figure 25 – Case where $u \in V(C_1) \setminus (J \cup V(L))$. Cycle C_1' is highlighted in red, cycle C_2 in blue, and path J' in green



Source: elaborated by the author.

Figure 26 – Case where $u \in V(J) \setminus \{w_1, w_2\}$. Cycle C_1' is highlighted in red, cycle C_2 in blue, and path J' in green



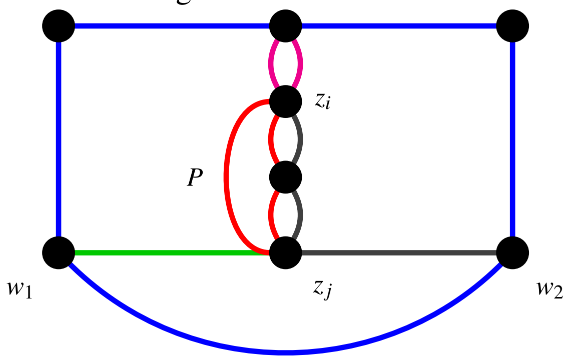
Source: elaborated by the author.

Now suppose that $u \in (V(C_1) \cap N(z_0)) \setminus V(L)$. Let C_1' be the cycle defined by path P plus the u,z_i-path contained in C_1 not passing through z_0, and let $L' = (z_i, \ldots, z_q)$. Again observe that C_1', C_2, L', J satisfy the first two conditions of Proposition 3.4.1. For the third condition, observe that the neighbors of z_i are contained in $V(P) \cup \{z_{i-1}\}$, which is disjoint from $\{w_1, w_2\}$.

Consider now the case $u \in V(J) \setminus (V(C_1) \cup V(C_2))$. Observe Figure 26 to follow the construction. Let C_1' be the cycle formed by P together with uJw_1 and the w_1,z_i-path contained in C_1 not passing by z_q. Also, let $L' = (z_0, \ldots, z_i)$, and J' be equal to uJw_2. One can again check that the conditions in Proposition 3.4.1 hold for C_1', C_2, L', J'.

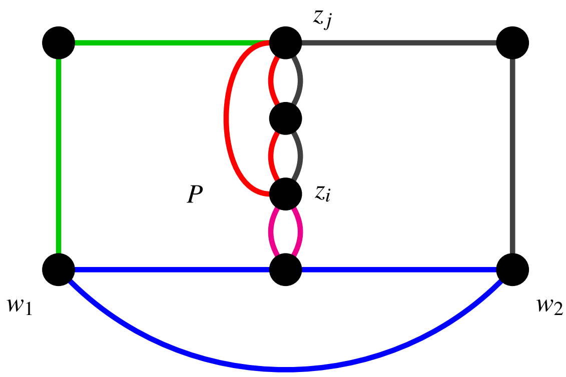
Finally, suppose $u \in L$, i.e., $u = z_j$ for some $j \in \{0, \ldots, q\}$. Now we find a subdivision of F_2 with a smaller chain by using Proposition 3.4.2. Observe Figure 27 to follow the construction. Recall that $i > 0$ and suppose first that $j > i$. Let C_1' be the cycle formed

Figure 27 – Case where $u = v_j \in V(L)$ for some $j \in \{i+1, \ldots, q\}$. Cycle C_1' is highlighted in red, cycle C_2' in blue, path J' in green, and the new chain L' in magenta



Source: elaborated by the author.

Figure 28 – Case where $u = v_j \in V(L)$ for some $j \in \{0, \ldots, i-1\}$. Cycle C_1' is highlighted in red, cycle C_2' in blue, path J' in green, and the new chain L' in magenta



Source: elaborated by the author.

by P together with a z_i, z_j-path contained in L. For each $\ell \in [2]$, let J_ℓ be the w_ℓ, z_0-path contained in C_ℓ that does not pass through z_q. Then define C_2' as the cycle formed by the paths J, J_1 and J_2. Finally, let $L' = (z_0, \ldots, z_i)$, and J' be the z_j, w_1-path contained in C_1 that does not use vertex z_{j-1}. Recall that P is a u, v-path, where $v \in N(z_i) \setminus \{z_{i-1}, z_{i+1}\}$ to see that C_1' is a cycle on at least 3 vertices. The other desired properties of C_2', L', J' can be seen to hold from the structure of H. Additionally, L' is smaller than L, as desired.

For the case that $j < i$ we can apply a similar argument(see Figure 28).

- $F = F_2$. Similarly as the last item, we start by using Proposition 3.4.2, to find C_1, C_2, linked by a chain L and a path J, with L and J being disjoint (observe again Figure 13 to see the related structures). Call w_1 the vertex in $V(J) \cap V(C_1)$ and w_2 the vertex in $V(J) \cap V(C_2)$. Additionally, let $V(C_1) \cap V(L) = \{z_0\}$ and $V(C_2) \cap V(L) = \{z_q\}$.

Notice that the case where $u \in V(C_1) \setminus V(L)$ is analogous to the case that $u \in V(C_2) \setminus V(L)$; so suppose $u \in V(C_1) \setminus V(L)$. Let C_1' be the cycle formed by P, together with the u, z_0-path contained in C_1 that also contains w_1, and a z_0, z_i-path contained in L. Also, let $L' = (z_i, \ldots, z_q)$. We have that C_1', C_2, L', J satisfy the properties in Proposition 3.4.2 and $|V(L')| \leq |V(L)|$.

Now suppose $u \in V(J) \setminus \{w_1, w_2\}$, and let C_1' be the cycle formed by P, together with $w_1 J u$, any w_1, z_0-path contained in C_1, and a z_0, z_i-path contained in L. Again, let $L' = (z_i, \ldots, z_q)$, and observe that C_1', C_2, L', uJw_2 satisfy Proposition 3.4.2.

Finally, suppose $u \in V(L)$, i.e., $u = z_j$ for some $j \in \{0, \ldots, q\}$. Observe that cycles C_1 and C_2 are symmetrical in the structure of H; hence, we can suppose, without loss of generality, that $j > i$. In this case, let C_2' be the cycle formed by P and a z_i, z_j-path contained in L; J' be a z_j, w_1-path contained in $(L - \{z_0, \ldots, z_{j-1}\}) \cup C_2 \cup J$; and $L' = (z_0, \ldots, z_i)$. One can see that C_1, C_2', L', J' satisfy Proposition 3.4.2.

□

Proof of Theorem 3.5.1. We need to prove that we can decide whether M has one of the graphs in $\{F_1, F_2, F_3\}$ as an m-topological minor in time $O(mn^3)$ where $n = |V(M)|$ and $m = |E(U(M))|$. First, we test if M has F_3 as topological minor, which can be done in time $O(n^3)$ according to the algorithm presented in (13). If the answer is positive, we are done as the concepts of m-topological minor and topological minor coincide because F_3 is a simple graph. Hence, suppose that F_3 is not a topological minor of M. We first describe a procedure that will be iteratively applied to some chains of M. For a chain L, such procedure returns either an m-subdivision of a graph in $\{F_1, F_2\}$ having chain L, or tells us that no m-subdivision of F_1 or F_2 having L as chain exists. After presenting such procedure, we argue that only a polynomial number of chains need to be investigated.

For a maximal chain L such that the internal vertices of L have degree 2, we identify all the vertices of L into a single vertex, and test if the obtained graph, M_L, has a subdivision of F_3. Observe that this operation applied to the chain of an m-subdivision of F_1 or F_2 would create a subdivision of F_3. Therefore, if the answer is negative, then we can conclude that there is no m-subdivision of F_1 or F_2 whose chain is L. Otherwise, as M itself has no subdivision of F_3, while M_L does, then we can apply Lemma 3.4.2 to either conclude that M has F_1 or F_2 as m-topological minors, or that M has a crossed structure. If the former occurs, we stop our algorithm. Otherwise, we apply the same procedure to another such chain. Finally, if the answer is negative for all such chains, then we can conclude from Lemma 3.5.1 that M has no m-subdivisions of F_1 or F_2, and therefore must be a Mengerian graph.

Now, we argue that each multiedge xy of multiplicity at least two is contained in at most one maximal chain L whose internal vertices of L have degree 2. Observe that we then get a total number of at most m such chains, and hence our algorithm runs in time $O(mn^3)$. Our argument also leads to a linear algorithm to find such chain, if it exists. So denote by $P = (u_1, \ldots, u_q)$ the chain constructed so far; hence P is initially equal to (x, y). If u_1 and u_q have degree bigger than 2, then P is the searched chain. Otherwise, suppose without loss of generality that u_q has degree 2 and let u_{q+1} be the neighbor of u_q distinct from u_{q-1}. If $u_q u_{q+1}$ has multiplicity 1, then we can conclude that xy is not contained in any such chain. Otherwise, increase the chain by adding u_{q+1}. □

3.6 Concluding remarks

In this chapter, we give a full characterization of Mengerian graphs in terms of m-topological minors, generalizing Kempe et al.'s result (12). We also provide a polynomial-time recognition algorithm that tests whether a given graph G has an m-subdivision of F_i, for some $i \in \{1, 2, 3\}$. In order to search for such an m-subdivision, and denoting by m the number of edges in $U(G)$, we prove in Lemma 3.5.1 that we need only to concentrate on $O(m)$ possible chains of G. If this was not the case, then the approach taken by us, of contracting a chain and testing for the existence of $F3$, would not work as G might have an exponential number of chains. Indeed, letting $n = |V(G)|$, if G has for instance a clique on $O(n)$ vertices whose incident edges all have multiplicity 2, then all choices of $O(2^n)$ paths should be considered. Therefore our approach is highly dependent on the structure of $F1$ and $F2$. We then ask:

Question 1 *Consider a fixed graph H. Can we decide in polynomial time whether a given graph G has an m-subdivision of H?*

In particular, a good question is whether the algorithm in (13) can be generalized to find also m-subdivisions. As their algorithm is extremely technical and is designed for simple graphs, this does not seem to be a trivial task. Additionally, observe that choosing the candidate images of $V(H)$ and applying Robertson and Seymour's (22) algorithm for k-Linkage is also not a trivial option since such algorithm considers only simple graphs too.

In the literature a common approach is to constrain hard problems to temporal graph classes (see e.g. (2, 23, 24)). We call $\mathcal{G} = (G, \lambda)$ a *Mengerian temporal graph* if G is Mengerian. It could be the case that Mengerian temporal graphs are "easier" when solving certain problems. They for instance give trivial "yes" instances to the following problem, which up to our knowledge is open in general.

Question 2 *Given a temporal graph $\mathcal{G} = (G, \lambda)$, can one decide in polynomial time whether $p(s,z) = c(s,z)$ for every pair $s,z \in V(G)$? What about whether $p(s,z) = c(s,z)$ for a given pair $s,z \in V(G)$?*

As will be noted in Section 4.5, finding k vertex disjoint temporal s,z-paths is NP-complete (12), and deciding the existence of a temporal s,z-cut of size at most k is W[1]-hard when parameterized by k (25). Note that, given a Mengerian temporal graph $\mathcal{G} = (G, \lambda)$, even if we can be sure that $p(s,z) = c(s,z)$, it is not clear whether these values can be easily computed. Hence, we ask whether Mengerian temporal graphs are also easy instances for the following problem.

Question 3 *Given a Mengerian temporal graph $\mathcal{G} = (G, \lambda)$ and $s,z \in V(G)$, can one compute $p(s,z)$ (and hence $c(s,z)$) in polynomial time?*

Finally, some papers also study network design problems (see e.g. (26, 27)) and inference problems (see (12)). In this kind of problems we are given a graph G, and we want to assign a time function to the edges of G in order to satisfy some given constraint. In this case, the graph G is a simple graph and assigning multiple labels $t_1, \ldots, t_\ell$ to the same edge uv translates, in our notation, into picking the multigraph G' where uv has multiplicity ℓ, and assigning the labels $t_1, \ldots, t_\ell$ to the edges with endpoints uv in a way that all these labels appear; denote by $\ell(uv)$ the number of labels related to uv. In (26, 27) the authors consider strictly increasing paths and investigate assignments that minimize $\max_{uv \in E(G)} \ell(uv)$ and such that all pairs of vertices are linked through a temporal path. And in (12), they consider partial knowledge about a temporal graph $\mathcal{G} = (G, \lambda)$, namely, only an interval $[s(e), z(e)]$ containing $\lambda(e)$ for each $e \in E(G)$, and they want to infer λ knowing that only a given set of pairs of vertices should be linked through temporal paths in $\mathcal{G}$. Similar design problems could be considered taking Mengerian graphs into account. In particular, we ask the following.

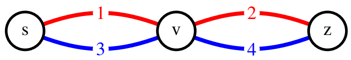
Question 4 *Given a simple graph G having no F_3 as m-topological minor, can we compute in polynomial a function $\mu : E(G) \to \mathbb{N}$ such that the multigraph obtained from G by making each $uv \in E(G)$ have multiplicity $\mu(G)$ is Mengerian, and such that $|\sum_{e\in E(G)}|\mu(e)|$ is maximum?*

Additionally, it would be interesting to investigate design and inference problems when constrained to Mengerian graphs.

4 TEMPORAL VERTEX DISJOINT WALKS AND CUTS

In this chapter, we continue to investigate vertex connectivity concepts on temporal graphs, except that now we allow walks/paths to share a vertex, as long as the sharing occurs in different timesteps. As an example, consider the temporal graph in Figure 29, where two temporal paths, highlighted in red and blue, intersect in v. However, while the red path is in v in timesteps 1 and 2, the blue only arrives in v at timestep 3. In this chapter we consider that these temporal paths are disjoint.

Figure 29 – Two paths that use the same vertex at different timesteps



Source: elaborated by the author.

One interesting aspect of this type of intersection is the fact that, unlike the previous section, defining parameters in terms of paths or in terms of walks leads to distinct values. In this chapter, we investigate mostly the problems related to t-vertex disjoint temporal paths, not walks.

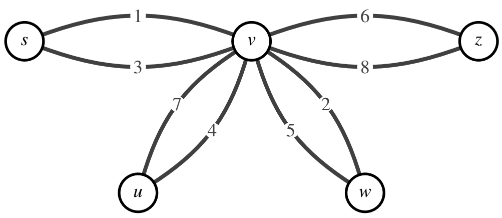
In Section 4.1, we give formal definitions and some preliminary results. In Section 4.2, we present in more details our findings about the computational complexity of (temporal) vertex connectivity problems and contrast them with previously known results. We postponed the presentation of the state of the art for complexity results on vertex connectivity because we wanted to contrast them with complexity results on t-vertex connectivity. In Section 4.3, we present our results for t-vertex connectivity when walks are considered. And in Section 4.4, we present our results for t-vertex connectivity when paths are considered. In particular, we prove that Menger's Theorem always holds for walks, which is not true for paths. Additionally, the related problems on walks are polynoimally-time solvable, while the ones for paths are hard. Finally we present our concluding remarks in Section 4.5.

4.1 Terminology and preliminary results

Let (G,λ) be a temporal graph and be $s,z \in V(G)$ two non-adjacent vertices. For temporal s,z-walks P and Q, we say that such paths are *t-vertex disjoint* if $V^T(P) \cap V^T(Q) \subseteq$

$\{s,z\} \times [\tau(G,\lambda)]$. We denote the maximum number of t-vertex disjoint temporal s,z-walks by $wtp_{G,\lambda}(s,z)$ and the size of a minimum set $S \subseteq V^T(G,\lambda)$ such that S intersects all temporal s,z-walks is denoted by $wtc_{G,\lambda}(s,z)$. In this chapter, we distinguish between paths and walks, which is different from the previous chapter where Proposition 2.1.2 gave us that, if the problems investigated there were defined in terms of walks, instead of paths, then the related metrics would not differ. These concepts can be defined to paths instead of walks and we denote the parameters by $tp_{G,\lambda}(s,z)$ and $tc_{G,\lambda}(s,z)$. Figure 30 provides an example of a graph where these parameter are different. In such temporal graph, every temporal s,z-path uses the temporal vertices $(v,3),(v,4),(v,5),(v,6)$, so there are no two temporal vertices from s to z that are disjoint in temporal vertices, while we have the temporal s,z-walks $W_1 = (s,1,v,2,w,5,v,6,z)$ and $W_2 = (s,3,v,4,u,7,v,8,z)$ that are t-vertex disjoint.

Figure 30 – Example that shows a temporal graph where $tp_{G,\lambda}(s,z) = 1$ and $wtp_{G,\lambda}(s,z) = 2$



Source: elaborated by the author.

As every temporal s,z-path is a temporal s,z-walk, we have the inequalities $tp_{G,\lambda}(s,z) \leq wtp_{G,\lambda}(s,z)$ and $tc_{G,\lambda}(s,z) \leq wtc_{G,\lambda}(s,z)$. However, the difference between these parameters can be arbitrarily large, as stated in the next proposition.

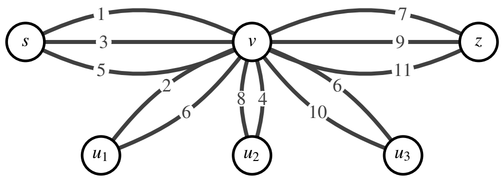
Proposition 4.1.1 *For positive integer $k \in \mathbb{N} \setminus \{0\}$, there is a temporal graph (G,λ) and a pair of non-adjacent vertices s,z of G such that $tp_{G,\lambda}(s,z) = 1$ and $wtp_{G,\lambda}(s,z) = k$. The same holds for cuts, i.e, $tc_{G,\lambda}(s,z) = 1$ and $wtc_{G,\lambda}(s,z) = k$.*

Proof. We can construct a temporal graph (G,λ) satisfying all the desired properties, i.e., such that $tc_{G,\lambda}(s,z)$=$tp_{G,\lambda}(s,z) = 1$ and $wtc_{G,\lambda}(s,z)$=$wtp_{G,\lambda}(s,z) = k$. To help readability, we provide an example in Figure 31. Begin with a path between vertices s and z using only one other vertex, v, and add k additional vertices, which we label $u_1,\dots,u_k$. Then add, for each $i \in [k]$, two edges connecting the pair $u_i v$. We define λ in such a way that:

- $\lambda(sv) = \{1,3\dots,2k-1\}$ and $\lambda(vz) = \{2k+1,2k+3,\dots,4k-1\}$,

- For each $i \in [k]$, $\lambda(vu_i) = \{2i, 2(k-1+i)\}$.

Figure 31 – Example that shows a temporal graph where $wtp_{G,\lambda}(s,z) = 3$ and $tp_{G,\lambda}(s,z) = 1$



Source: elaborated by the author.

Notice that every temporal s,z-path in (G,λ) is of the form (s,t_1,v,t_2,z) for $t_1,t_2 \in [4k-1]$. Moreover, the maximum time an edge connecting sv is active is at time $2k-1$ while the minimum time an edge connecting vz is active is $2k+1$. Therefore, any temporal s,z-path contains the temporal vertex $(v,2k)$, which implies $tp_{G,\lambda}(s,z)$=1.

Now, for $i \in [k]$, consider the following temporal s,z-walk:

$$P_i = (s, 2i-1, v, 2i, u_i, 2(k-1+i), v, 2(k+i)-1, z).$$

Define $R = \{s,z\} \times \{\tau(G,\lambda)\}$. We now observe that for each $i \in [k]$, we have that $V^T(P_i) \setminus R \subseteq \{(v,2i-1),(v,2i),(v,2(k-1+i)),(v,2(k+i)-1)\} \cup (u_i \times [4k-1])$. Therefore, one can notice that $V^t(P_{i_1}) \cap V^t(P_{i_2}) \subseteq R$. Thus, $wtp_{G,\lambda}(s,z)$= k.

Now, for the parameters $tc_{G,\lambda}(s,z)$ and $wtc_{G,\lambda}(s,z)$, we prove shortly (Theorem 4.3.1) that equality always holds; hence k =$wtp_{G,\lambda}(s,z) = wtc_{G,\lambda}(s,z)$. Finally, observe that $\{(v,2k)\}$ is a t-vertex s,z-cut, as all edges connecting sv are active before such time while all edges connecting vz are active after such time. □

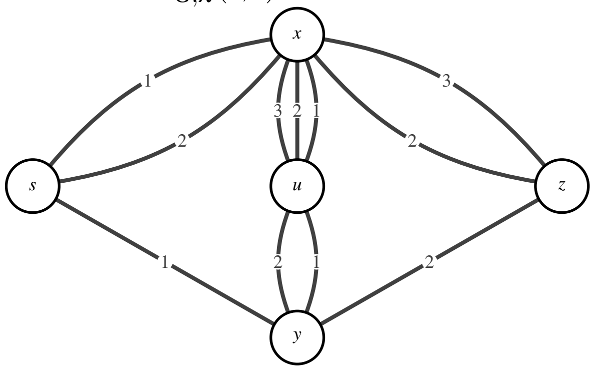
Now, we prove that the distance between parameters $tp_{G,\lambda}(s,z)$ and $tc_{G,\lambda}(s,z)$ can be arbitrarily high.

Proposition 4.1.2 *For every integer $k \geq 2$, there is a temporal graph (G,λ) and $s,z \in V(G)$ non-adjacent vertices such that $tc_{G,\lambda}(s,z)$–$tp_{G,\lambda}(s,z)$≥ k.*

Proof. We begin by presenting the temporal graph (G,λ) of Figure 32, that contains two vertices $s,z \in V(G)$ that are not adjacent and we have that $tp_{(G,\lambda)}(s,z)$= 2 and $tc_{(G,\lambda)}(s,z)$= 3.

We first show that $tc_{G,\lambda}(s,z)$= 3. For a contradiction, suppose otherwise. Notice that $(s,2,x,2,z)$ is a temporal s,z-path using only the temporal vertex $(x,2)$. Therefore, such temporal vertex is in any t-vertex s,z-cut. Considering the temporal path $(s,1,y,2,z)$ we have that at least one of $(y,1)$ or $(y,2)$ is in a t-vertex s,z-cut. Hence, a t-vertex s,z-cut of size 2 must be one of the following sets: $\{(x,2),(y,1)\}$ or $\{(x,2),(y,2)\}$. However, the first set does not form a t-vertex s,z-cut because the temporal path $(s,1,x,1,u,2,y,2,z)$ does not have any temporal vertices on it. Similarly, the second set also fails to form a t-vertex s,z-cut since the temporal path $(s,1,y,1,u,3,x,3,z)$ does not intersect with the set. Moreover, as all temporal edges incident to s are $(sx,1),(sx,2),(sy,1)$ we have that $\{(x,1),(x,2),(y,1)\}$ is a t-vertex s,z-cut. It follows that $tc_{G,\lambda}(s,z){=}\ 3$.

Figure 32 – Example that shows a temporal graph where $tp_{G,\lambda}(s,z) = 2$ and $tc_{G,\lambda}(s,z) = 3$



Source: elaborated by the author.

Now we show that $tp_{G,\lambda}(s,z){=}\ 2$. One can see that the temporal s,z-paths $P_1 = (s,2,x,2,z)$ and $P_2 = (s,1,y,2,z)$ are t-vertex disjoint. So, we only need to show that there are no 3 t-vertex disjoint temporal s,z-paths. Suppose otherwise; as we have exactly 3 temporal edges incident to s, each of these 3 paths uses exactly one these 3 temporal edges. Let P_1 be the temporal path using temporal edge $(sx,1)$, P_2 the one using the temporal edge $(sx,2)$ and P_3 the temporal path using the temporal edge $(sy,1)$. Notice that $(x,2) \in V^T(P_2)$. So the path P_1 using the temporal edge $(sx,1)$ must leave x at time 1, i.e, P_1 starts with the subpath $(s,1,x,1,u)$. As P_1 is a path and hence cannot repeat x, it must finish with the temporal edge $(yz,2)$. Therefore, $(y,2),(u,1) \in V^T(P_1)$. This implies that the path P_3, that starts by using the temporal edge $(sy,1)$, must leave y before time 2. However the only temporal edge incident to y at time 1 different from $(sy,1)$ is $(yu,1)$. Such temporal edge is not available as $(u,1) \in V^T(P_1)$. A contradiction as we suppose P_1,P_2,P_3 t-vertex disjoint temporal s,z-paths.

Now, for an integer $k \geq 3$, let $G_1,\dots,G_k$ be disjoint copies of G. For each $i \in [k]$ let λ_i be a time function in G_i behaving as λ in G. For each $i \in \{1,\dots,k\}$, we denote $V(G_i) = \{s_i,z_i,x_i,y_i,u_i\}$. Let H be the graph obtained by the disjoint union of $G_1,\dots,G_k$ and then, the identification of $s = \{s_1, \ldots, s_k\}$ and $z = \{z_1, \ldots, z_k\}$. To define a time function λ' on H, let (ab,t) be an element of $E^T(G_i, \lambda_i)$, define $\lambda'(ab,t) = t$. It is straightforward to see that $tp_{G,\lambda}(s,z) = 2k$ while $tc_{G,\lambda}(s,z) = 3k$, as we wanted. □

Now, we define the computational problems we deal with in this chapter. Recall that in the last chapter we define, for each temporal graph (G,λ) and pair of non-adjacent vertices $s,z \in V(G)$, the parameters $p_{G,\lambda}(s,z)$ and $c_{G,\lambda}(s,z)$, which are, respectively the maximum number of temporal paths from s to z that are vertex disjoint and the minimum number of vertices in a set $S \subseteq V(G) - \{s,z\}$ such every temporal s,z-path intersects S. We now formally state the related decision problems.

$\geq k$ VERTEX DISJOINT PATHS
Input. A temporal graph (G,λ), vertices $s,z \in V(G)$, and an integer k.
Question. $p_{G,\lambda}(s,z) \geq k$?

$\leq h$ VERTEX CUT
Input. A temporal graph (G,λ), a pair of non-adjacent vertices $s,z \in V(G)$, and a positive integer h.
Question. $c_{G,\lambda}(s,z) \leq h$?

Observe that such problems can easily be adapted to consider t-vertex disjoint of paths and walks. We then define:

$\geq k$ T-VERTEX DISJOINT PATHS
Input. A temporal graph (G,λ), non-adjacent vertices $s,z \in V(G)$, and an integer k.
Question. $tp_{G,\lambda}(s,z) \geq k$?

$\leq h$ T-VERTEX PATH CUT
Input. A temporal graph (G,λ), a pair of non-adjacent vertices $s,z \in V(G)$, and a positive integer h.
Question. $tc_{G,\lambda}(s,z) \leq h$?

$\geq k$ T-VERTEX DISJOINT WALKS
Input. A temporal graph (G,λ), a pair of non-adjacent of vertices $s,z \in V(G)$, and a positive integer k.

Question. $wtp_{G,\lambda}(s,z) \geq k$?

$\leq h$ T-VERTEX WALK CUT

Input. A temporal graph (G,λ), a pair of non-adjacent vertices $s,z \in V(G)$, and a positive integer h.

Question. $wtc_{G,\lambda}(s,z) \leq h$?

Finally, in Section 4.3 we use the notion of *digraph*, here a digraph is a pair (V,E) such that $E \subseteq V \times V \setminus \{(v,v) \mid x \in V\}$. The elements $e \in E$ are called *arcs* and the elements of V are called vertices and we omit the parenthesis when referring to them.

4.2 Main results and related works

As previously said, in this section we formally state our results, at the same time that we present existing results on vertex disjoint paths problems. All these results are summarized in Table 1.

In (3), Berman proved that $\geq k$ VERTEX DISJOINT PATHS is NP-complete. This was improved for fixed $k = 2$ in (12), where Kempe, Kleinberg, and Kumar also proved that deciding whether there is a vertex temporal s,z-cut of size at most k is NP-complete, for given k. Observe that the latter problem can be easily solved in time $O(|V(G)|^k)$, which raises the question about whether it is FPT when parameterized by k. This is answered negatively in (25), where Zschoche, Fluschnik, Molter, and Niedermeier proved that this is W[1]-hard. In (24), Fluschnik, Molter, Niedermeier, Renken, and Zschoche further investigate $\leq h$ VERTEX CUT, giving a number of hardness results (e.g. that the problem is hard even if G is a line graph), as well as some positive ones (e.g., that the problem is polynomial-time solvable when G has bounded treewidth).

More recently, vertex disjoint *paths* between a given set of terminal vertices has been studied (**?**, 28, 29). This is a generalization to the temporal context of the k-LINKAGE problem. In the latter, we are given a graph G and a set of k pairs $(s_1,z_1),\dots,(s_k,z_k)$, and we want to decide whether there are vertex disjoint paths $P_1,\dots,P_k$ such that each P_i is a path from s_i to z_i. This problem is known to be hard on static graphs (30). In this sense, the results in (**?**, 28, 29) are not so surprising as they generalize an already hard problem. The easier related problem would then be when the paths have equal starting point and equal target point, which is known to be polynomial-time solvable on static graphs thanks to the famous Menger's Theorem and flow

Table 1 – "NP-c"=NP-complete."W[1] for h"= W[1]-hard when parameterized by h (similar for "XP for h"). All the results related to t-vertex disjointness are presented here.

X	Y	$\geq k$ X DISJOINT YS	$\leq h$ X Y-CUT
vertex	walk/ path	NP-c (3) even if $k = 2$ (12)	NP-c even if $\tau = 2$, W[1] for h (25)
t-vertex	walk	Poly	Poly
	path	NP-c	co-NP-hard for given h XP for h

Source: elaborated by the author.

techniques. We then study all problems related to t-vertices defined in Section 4.1. Our results then strengthen the previously cited results, as we prove hardness even when the pairs of vertices are all the same.

In this chapter, we show that $\geq k$ T-VERTEX DISJOINT WALKS and $\leq h$ T-VERTEX WALK CUT both can be solved in polynomial time. We prove something even stronger than that, namely that $wtp_{G,\lambda}(s,z)$=$wtp_{G,\lambda}(s,z)$ (or, in other words, Menger's Theorem holds for t-vertex disjoint walks); additionally such value can be found in polynomial time. This is proved in Section 4.3. For t-vertex disjoint temporal paths, we prove that $\geq k$ VERTEX DISJOINT PATHS is NP-complete and $\leq h$ T-VERTEX PATH CUT is co-NP-complete. Moreover, we also show a non-trivial XP algorithm for $\leq h$ T-VERTEX PATH CUT when parameterized by h. These results are presented in Section 4.4. We close this chapter with some concluding remarks.

4.3 Temporal Walks

In this section, we prove the problems defined on walks are easy.

Theorem 4.3.1 *Let (G,λ) be a temporal graph with lifetime τ and $s,z \in V(G)$ be non-adjacent vertices. Then, $wtp_{G,\lambda}(s,z)$=$wtc_{G,\lambda}(s,z)$. Moreover, the value of such parameters can be found in polynomial time.*

Proof. We use what is known by the *static expansion* of $\mathcal{G}$ (see e.g. (31)). This is the digraph having vertex set $V(G) \times [\tau]$, and edge set $E' \cup E''$, where E' and E'' are defined below.

$E' = \{(u,i)(u,i+1) \mid u \in V(G), i \in [\tau - 1]\}$.

$E'' = \{(u,i)(v,i) \mid (u,v) \in V(G)^2, i \in \lambda(uv)\}$.

Let D denote the digraph obtained from the static expansion of $\mathcal{G}$ by identifying all vertices $\{(s,i) \mid i \in [\tau]\}$ into a single vertex, s, and all vertices $\{(z,i) \mid i \in [\tau]\}$ into a single vertex, z. We prove that the maximum number of t-vertex disjoint temporal s,z-walks in $\mathscr{G}$ is equal to the maximum number of vertex disjoint s,z-paths in D, while the minimum size of a set $S \subseteq V^t(G,\lambda) \setminus (\{s,z\} \times [\tau(G,\lambda)])$ that intersects every temporal s,z-walk in $\mathscr{G}$ is equal to the minimum size of an s,t-cut in D, i.e., minimum $S \subseteq V(D) \setminus \{s,z\}$ intersecting all s,z-paths in D. The theorem thus follows by Menger's Theorem on static digraphs, and the fact that computing these parameters in a static digraph is largely known to be polynomial-time solvable (see e.g. (19)).

First, given an s,z-path P in D, we explain how to construct a temporal s,z-walk P' in $\mathscr{G}$. So let $P = (\alpha_0 = s, e_1, \dots, e_q, \alpha_q = z)$ be an s,z-path in D. We start with P' being equal to P, and replace objects in P' until we obtain a temporal s,z-walk in $\mathscr{G}$. So for each $i \in [q-1]$, write α_i as (u_i,t_i). Also, for simplicity of notation, we consider α_0 to be equal to (s,t_1) and α_q to be equal to (z,t_{q-1}). Observe that $e_1 = s(u_1,t_1)$ and $e_q = (u_{q-1},z_{q-1})t$, i.e., t_1 and t_{q-1} are the starting and finishing times of P, respectively. Now, for each $i \in [q]$, if $u_{i-1} = u_i$ (and hence $t_i = t_{i-1}+1$), remove e_i and α_i from P'. And if $u_{i-1} \neq u_i$ (and hence $t_i = t_{i-1}$), then replace e_i in P' by t_i. Note that in the latter case, we get that $(u_{i-1}u_i,t_i)$ is a temporal edge of $\mathscr{G}$; call such fact (*). Observe that we are now left with a sequence that alternates temporal vertices and timesteps. Because of (*), and since $t_1 \le t_2 \le \dots \le t_q$ as $(u,i)(v,j)$ is not an edge of D whenever $i > j$, it suffices to replace each temporal vertex (u_i,t_i) in the sequence by simply u_i in order to obtain a temporal walk in $\mathscr{G}$. One can verify that $V^T(P') \setminus (\{s,z\} \times [\tau]) = \{\alpha_1, \dots, \alpha_{q-1}\} = V(P) \setminus \{s,z\}$. Additionally, observe that the backward transformation satisfying the same property can also be defined, i.e., given a temporal s,z-walk P', we can construct an s,z-path P in D such that $V^T(P') \setminus (\{s,z\} \times [\tau]) = V(P) \setminus \{s,z\}$. This directly implies that maximum number of t-vertex disjoint temporal s,z-walks in $\mathscr{G}$ is equal to the maximum number of vertex disjoint s,z-paths in D.

So now suppose that $X \subseteq (V(G) \setminus \{s,z\}) \times [\tau]$ is a minimum set intersecting every temporal s,z-walk in $\mathscr{G}$. Note that $X \subseteq V(D)$ and that, if there is an s,z-path not intersecting X in D, then there is a temporal s,z-walk not intersecting X in $\mathscr{G}$ by the previous paragraph, a contradiction. On the other hand, consider $X \subseteq V(D) \setminus \{s,z\}$ to be an s,z-cut in D. By construction, $X \subseteq (V(G) \setminus \{s,z\}) \times [\tau]$. And, again by the previous paragraph, there cannot be a temporal s,z-walk in $\mathscr{G}$ not intersecting X. □

Such result also implies that the problem $\ge k$ T-VERTEX DISJOINT WALKS and $\le h$ T-VERTEX WALK CUT can be solved in polynomial time.

4.4 Temporal Paths

In this section, we present the computational complexity results on the problems $\geq k$ T-VERTEX DISJOINT PATHS and $\leq h$ T-VERTEX PATH CUT. We also prove that, unlike the other defined parameters, just testing whether a set of temporal vertices is a cut in this context is already hard. The results presented here are summarized in Table 1.

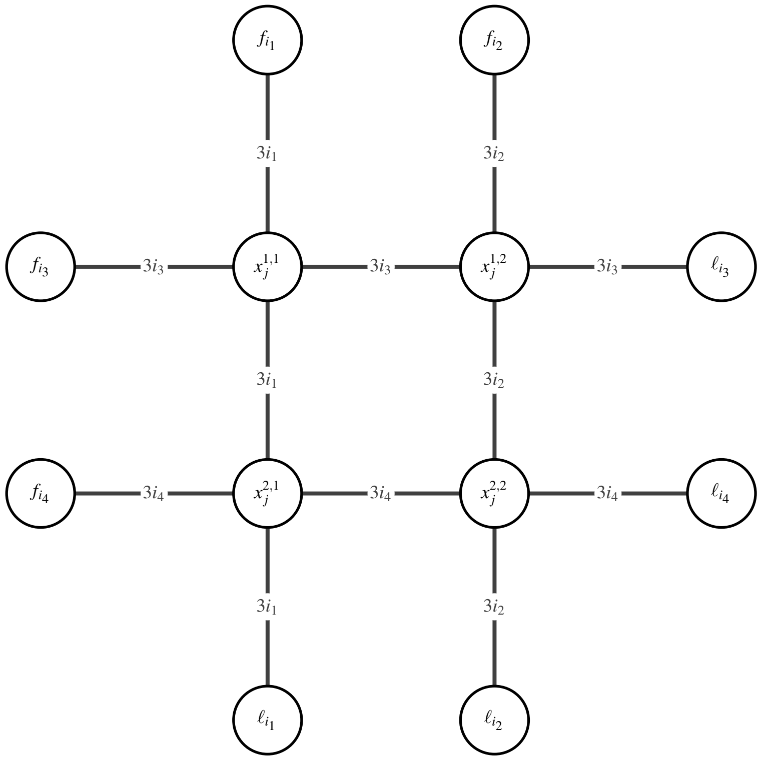
Theorem 4.4.1 $\geq k$ T-VERTEX DISJOINT PATHS *is* NP*-complete.*

Proof. First, observe that given a set of paths $P_1, \ldots, P_k$, we can can compute the set of temporal vertices that each path uses in polynomial time. If such sets are disjoint, we can conclude that the paths form a set of t-vertex disjoint temporal s,z-paths. Consequently, the problem is in NP.

To establish the hardness of the problem, we employ a reduction from $(2,2,3)$-SAT. In this problem, we are given a Boolean formula in conjunctive normal form (CNF) where each variable occurs exactly twice positive and twice negative. The question is whether there exists an assignment of truth values to the variables that satisfies all the clauses. This is known to be NP-complete (32).

Given a formula ϕ on variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$, we construct a related graph and simultaneously define a time function on it. For each $j \in \{1, \ldots, n\}$, we construct a cycle Q_j with vertices $x_j^{r,c}$ such that $r, c \in \{1,2\}$ and two vertices $x_j^{r,c}, x_j^{r',c'}$ are adjacent if $r = r'$ or $c = c'$. Such vertices are called of *variable type*. We then add more vertices in the neighborhood of Q_j and define the time labeling in the defined edges. To aid following the construction, Figure 33 may be helpful. The columns of Q_j will represent the positive occurrences of x_j. Formally, let C_{i_1}, C_{i_2} be the clauses where x_j appears positively and suppose $i_1 < i_2$, then we define $\lambda(x_j^{1,1}x_j^{2,1}) = \{3i_1\}$ and $\lambda(x_j^{1,2}x_j^{2,2}) = \{3i_2\}$. Similarly, the rows represent the negative occurrences. Hence, if C_{i_3} and C_{i_4} are the clauses where the negative occurrences of x_j appears and $i_3 < i_4$, then we define $\lambda(x_j^{1,1}x_j^{1,2}) = \{3i_3\}$ and $\lambda(x_j^{2,1}x_j^{2,2}) = \{3i_4\}$. For each $i \in \{1, \ldots, m\}$ we add vertices f_i and ℓ_i. Now for each $j \in \{1, \ldots, n\}$, $i \in \{1, \ldots, m\}$ and $h \in \{1,2\}$, if the h-th appearance of the variable x_j is in the clause C_i, then we add edges connecting the pairs $f_i x_j^{1,h}$ and $x_j^{2,h}\ell_i$; if, instead, the h-th occurrence of $\neg x_j$ appears in C_i, then we add edges connecting the pairs $f_i x_j^{h,1}$ and $\ell_i x_j^{h,2}$. For all these edges we give time $3i$. We also add an edge between ℓ_i and f_{i+1} and define $\lambda(\ell_i f_{i+1}) = 3i+2$. Finally, for each $i \in \{1, \ldots, m-1\}$, we have that f_i has three vertices of variable type in its neighborhood and the same is true for ℓ_i. Call the set of these six vertices plus f_i by S_i. We have that $|S_i| = 7$. For each vertex $v \in S_i$, we add edges sv and vz an define these edges to be active at time $3i+1$. Finally, we add edges sf_1, $\ell_m z$ such that $\lambda(sf_1) = 1$ and $\lambda(\ell_m z) = 3m+3$. Looking at Figure 34, one can have a better understanding of how are the snapshots of the constructed temporal graph.

Figure 33 – A representation of the neighborhood of Q_j for a variable x_j that appears positively in C_{i_1} and C_{i_2} and negatively in C_{i_3} and C_{i_4}



Source: elaborated by the author.

Now, we show that there is a satisfying assignment to ϕ if and only if there are $P_1, \ldots, P_{N+1}$ t-vertex disjoint temporal s,z-paths in (G,λ), where $N = 7m$. Suppose a satisfying assignment to ϕ. For each $i \in [m]$ and $v \in S_i$, let $P_v = (s, 3i+1, v, 3i+1, z)$. Observe that as $\bigcup_{i=1}^{m}\{P_v \colon v \in S_i\}$ is a set of t-vertex disjoint temporal s,z-paths. Hence we just need to construct one more temporal s,z-path that is t-vertex disjoint from the ones already defined.

For each $i \in \{1, \ldots, m-1\}$, we construct a temporal path from f_i to f_{i+1} using vertices of variable type of a variable that satisfies the clause C_i. First we suppose that the variable x_j assigned true value appears positively in C_i; suppose also that it is the h-th appearance of such literal. Then we define the temporal path $P_i = (f_i, 3i, x_j^{1,h}, 3i, x_j^{2,h}, \ell_i)$. Now suppose that the variable that satisfies clause C_i appears negatively in such clause, and suppose that this is the h-th appearance of such literal. Then we define the temporal path $P_i = (f_i, 3i, x_j^{h,1}, 3i, x_j^{h,2}, \ell_i)$. As we are supposing a satisfying assignment, we know that each P_i is defined. Now for each P_i

different from P_m, we can extend it to a temporal path between f_i and f_{i+1}, call it Q_i, by adding the edge $\ell_i f_{i+1}$ appearing at time $3i+2$. Note that Q_i starts at $3i$ and finishes at $3i+2$. Then, we can call P the temporal f_1, ℓ_m-walk obtained by concatenating $Q_1, \dots, Q_{m-1}, P_m$. Finally, we extend P to a temporal s,z-walk, Q, by adding the edges sf_1 and $\ell_m f$. To observe that Q is a path, one just needs to recall that, by the definition of P_i, a vertex $x_j^{r,c}$ is used in P_i if and only if x_j appears positively in C_i for the c-th time, or x_j appears negatively in C_i for the r-th time. As it occurs at most once, Q is a temporal s,z-path.

Now, let $P_1, \dots, P_{N+1}$ b a set of t-vertex disjoint temporal s,z-paths in (G,λ). Notice that $d^T(s) = d^T(z) = N+1$. Moreover, $\bigcup_{i=1}^m S_i = N(s) \cap N(z)$ and, for each $i \in \{1, \dots, m\}$ and $v \in S_i$, as we have that both edges sv and vz are only active at time $3i+1$, one of the paths $P_1, \dots, P_{N+1}$ must be $(s, 3i+1, v, 3i+1, z)$. As it holds for every $v \in N(s) \cap N(z)$, we can suppose, without loss of generality, that all paths $P_1, \dots, P_N$ are of this type. Next, we construct our truth assignment to ϕ based on the temporal path P_{N+1}. For each $i \in \{1, \dots, m\}$, we observe that P_{N+1} goes from f_i to ℓ_i using edges active at time $3i$. Observe that P_{N+1} must start with the edge sf_1, as this is the only edge incident in s that is not used by a path in $P_1, \cdots, P_N$. We know that the temporal path P_{N+1} finishes at timestep $3(m+1)$ and the only edge active at timestep at least $3i+2$ incident to some vertex in $S_1 \cup \{\ell_1\}, \dots, S_i \cup \{\ell_i\}$ is $\ell_i f_{i+1}$. This implies that P_{N+1} uses f_i and then goes to ℓ_i. We can write $f_i P_{N+1} \ell_i$ as $(f_i, 3i, x_{j_i}^{r_i, c_i}, 3i, x_{j_i}^{r'_i, c'_i}, 3i, \ell_i)$. If $c_i = c'_i$, then we assign true to the variable x_j; otherwise we have $r_i = r'_i$ and then we assign false. Notice that, as P_{N+1} is a path, we are not assigning different values to the same variable. If a variable is not assigned any value, we can just set it to true. For each clause C_i, since $f_i P_{N+1} \ell_i = (f_i, 3i, x_{j_i}^{r_i, c_i}, 3i, x_{j_i}^{r'_i, c'_i}, 3i, \ell_i)$ is a temporal path, the value assigned to the variable x_{j_i} satisfy the clause C_i. □

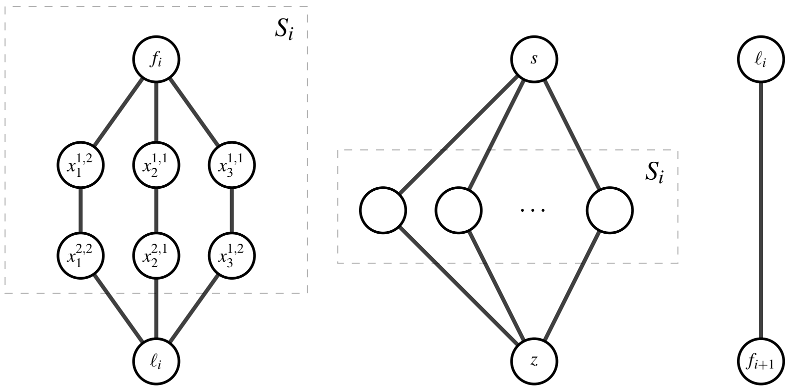
We now see that the same construction can also be used to prove hardness for the cut problems.

Theorem 4.4.2 *Given a temporal graph (G,λ) with lifetime τ, non-adjacent vertices $s,z \in V(G)$, and $S \subseteq (V(G) \setminus \{s,z\}) \times [\tau]$.*

- *Deciding whether S is a t-vertex s,z-cut in (G,λ) is co-*NP*-complete.*
- $\leq h$ T-VERTEX PATH CUT *is co-*NP*-hard.*

Proof. Observe that any temporal s,z-path P in (G,λ) can be checked to intersect S in polynomial time. In other words, there is a polynomial-time-checkable certificate for a "no" answer to the problem and hence it is in co-NP.

Figure 34 – From left to right snapshots of timesteps $3i, 3i+1, 3i+2$ of a temporal graph generated for an instance whose i-th clause $C_i = (x_1 \vee x_2 \vee \neg x_3)$, where x_1 and $\neg x_3$ are in its second appearance and x_2, its first



Source: elaborated by the author.

To prove hardness we use the same construction as used in Theorem 4.4.1. So, let ϕ be an instance of $(2,2,3)$-SAT and (G,λ) constructed from it like in the last theorem. We already know that any t-vertex s,z-cut contains $S = \bigcup_{i=1}^{m}(S_i \times \{3i+1\})$. Therefore, as in the last theorem, S is a t-vertex s,z-cut if and only if ϕ is not satisfiable. This proves the first item.

For the second item, one just needs to observe that for $h = 7m$, the answer of $\leq h$ T-VERTEX PATH CUT is "yes" if and only if S is a t-vertex s,z-cut. □

Now, we prove that given a temporal graph $\mathscr{G} = (G,\lambda)$ of lifetime τ, a pair of non-adjacent vertices s,z and a set $S \subseteq V(G) \times [\tau]$ of size h, one can decide in time FPT when parameterized by h if S is a t-vertex s,z-cut. Remember that Theorem 4.4.2 tells us that the problem is co-NP-complete. Observe that this immediately gives us an algorithm for $\leq h$ T-VERTEX PATH CUT which runs in XP time, namely time $O((hn\tau)^h m)$ where $n = |V(G)|$, when parameterized by h; it suffices to test all possible subsets of temporal vertices.

Theorem 4.4.3 *Given a temporal graph $\mathscr{G} = (G,\lambda)$ with lifetime τ, non-adjacent vertices $s,z \in V(G)$, and $S \subseteq (V(G) \setminus \{s,z\}) \times [\tau]$, deciding whether S is a t-vertex s,z-cut in $\mathscr{G}$ can be done in time $O(h^h m)$ where $h = |S|$ and $m = |E(G)|$.*

Proof. For each $u \in V(G) \setminus \{s,z\}$ appearing in S, denote by $\mathscr{L}_u = \{L_0, \cdots, L_{t_u}\}$ the interval windows defined by the appearances of u. More formally, suppose that $\{(u,i_1),\ldots,(u,i_{t_u})\}$ are all the occurrences of u in S, with $i_1 < \ldots < i_{t_u}$. Then L_0 is equal to the interval $\{1,2,\ldots,i_1\}$, L_{t_u} is equal to the interval $\{i_{t_u}+1, i_{t_u}+2,\ldots,\tau\}$, and L_j is equal to the interval $\{i_{j-1}+1,\ldots,i_{j+1}-1\}$ for each $j \in [t_{u-1}]$. Now, let $U = \{u_1,\ldots,u_q\} \subseteq V(G)$ be the set of all vertices that appear at least once in S; denote t_{u_i} by t_i for each $u_i \in U$. For every choice of values $I = \{j_1,\ldots,j_q\}$ where

$j_i \in \{0,\ldots,t_i\}$, let $\mathcal{G}_I$ be the temporal graph obtained from $\mathcal{G}$ by removing all the temporal edges incident in (u_i,t) in a timestamp not in L_{j_i} for every $i \in [q]$. More formally, $\mathcal{G}_I = (G,\lambda_I)$, where

$$\lambda_I(uv) = \begin{cases} \lambda(e) & \text{, if } u,v \notin U, \\ \lambda(e) \cap L_{j_i} & \text{, if } u = u_i \in U \text{ and } v \notin U, \text{ and} \\ \lambda(e) \cap L_{j_i} \cap L_{j_h} & \text{, if } u = u_i \in U \text{ and } v = u_h \in U. \end{cases}$$

We claim that S is not a t-vertex s,z-cut in $\mathcal{G}$ if and only if there exists a choice of I for which $\mathcal{G}_I$ contains a temporal s,z-walk. Observe that the problem therefore reduces to testing, for every choice of I, whether there exists a temporal s,z-walk in $\mathcal{G}_I$, which can be done in time $O(m)$ using the algorithm proposed in (33). Now, observe that there are $\prod_{i=1}^{q}(t_i+1)$ possible choices of I. Since each t_i is at most h, as well as q, it follows that $\prod_{i=1}^{q}(t_i+1) \le (h+1)^h = O(h^h)$, and the theorem follows.

Now to prove our claim, first consider that S is not a t-vertex s,z-cut in $\mathcal{G}$, and let P be a temporal s,z-path in $\mathcal{G}$ not intersecting S. For each $u_i \in V(P) \cap U$, let ℓ_i^1, ℓ_i^2 be the times of the temporal edges of P incident to u_i. We know that there are exactly 2 such edges since P is a path and $u_i \notin \{s,z\}$. Also, because P does not intersect S, we get that $(u_i,t) \notin S$ for every $t \in [\ell_i^1, \ell_i^2]$. This means that $[\ell_i^1, \ell_i^2] \subseteq L_{j_i}$ for some $j_i \in \{0,\ldots,t_i\}$. By choosing such j_i for every $u_i \in V(P) \cap U$, and letting $j_i = 0$ for every $u_i \in U \setminus V(P)$, we get that P is contained in $\mathcal{G}_I$, where $I = \{j_1,\ldots,j_q\}$, as we wanted to show.

Finally, suppose that there exists $I = \{j_1,\ldots,j_q\}$, and a temporal s,z-walk W in $\mathcal{G}_I$. Observe that, because $\mathcal{G}_I \subseteq \mathcal{G}$, the temporal s,z-path P contained in W is a path in $\mathcal{G}$. It remains to argue that P does not intersect S, which implies that S is not a t-vertex s,z-cut in $\mathcal{G}$. Suppose otherwise and let $(u_i,t) \in S \cap V^T(P)$; also let (e_1,ℓ_1) and (e_2,ℓ_2) be the two temporal edges of P incident to u_i. Observe that t is not within any of the defined intervals for u_i, which means that, regardless of the choice of j_i, there are no temporal edges in $\mathcal{G}_I$ incident in u_i; hence $\ell_1 < t < \ell_2$. Note also that the interval chosen for u_i is either to the left or to the right of t, i.e., L_{j_i} is either contained in $\{1,\ldots,t-1\}$ or in $\{t+1,\ldots,\tau\}$. This is a contradiction as in this case either $\ell_1 \notin L_{j_i}$ (and hence $(e_1,\ell_1) \notin E^T(\mathcal{G}_I)$) or $\ell_2 \notin L_{j_i}$ (and hence $(e_2,\ell_2) \notin E^T(\mathcal{G}_I)$). □

4.5 Concluding remarks

Given a temporal graph $\mathcal{G} = (G,\lambda)$ with lifetime τ and non-adjacent vertices $s,z \in V(G)$, in this chapter we have investigated the problems related to the existence of t-vertex disjoint temporal s,z-walks/paths. We have shown that a version of Menger's Theorem holds for walks, while the parameters for paths can be arbitrarily far apart. We have also given hardness results for the related paths and cuts problems, which are summarized in Table 1. Moreover, we have shown that testing whether a given set $(S \subseteq V(G) \setminus \{s,z\}) \times [\tau]$ intersects every temporal s,z-path is co-NP-hard, which is in stark contrast with other disjointness notions. Nevertheless, we show that an XP algorithm parameterized by h for $\leq h$ T-VERTEX PATH-CUT is still possible. In Table 1, we identify the following open questions.

Question 5 *Is $\geq k$* T-VERTEX DISJOINT PATHS FPT *when parameterized by k?*

Question 6 *Is $\leq h$* T-VERTEX PATH-CUT FPT *when parameterized by h?*

Recall that in Proposition 3.2.1, we present a construction of a temporal graph $\mathcal{G}$ containing a pair of non-adjacent vertices s,z such that the size of a vertex temporal s,z-cut is arbitrarily large, while the maximum number of vertex disjoint temporal s,z-paths remains 1. As for t-vertex disjointness, we prove in Theorem 4.3.1 that there is always equality between parameters when temporal walks are considered, while in Proposition 4.1.2 we see that the difference between the parameters can be arbitrarily large when temporal paths are considered. However, in order to increase such difference we must also increase the maximum number of t-vertex disjoint temporal s,z-paths. Therefore, we ask:

Question 7 *Does there exist a function f such that $tp_{\mathcal{G}}(s,z) \geq f(tc_{\mathcal{G}}(s,z))$ for every temporal graph $\mathcal{G}$ and pair of non-adjacent vertices s,z?*

Finally, observe that the time functions presented in Figure 11 to show that F_1, F_2, F_3 are not Mengerian (Proposition 3.3.1) cannot be used to show the existence of $\mathcal{G}, s, z$ such that $tp_{\mathcal{G}}(s,z)$ is smaller than $tc_{\mathcal{G}}(s,z)$. Indeed, in each of those temporal graphs we have that both parameters are equal to 2. We then say that a graph G is a *t-Mengerian graph* if $tp_{G,\lambda}(s,z) = tc_{G,\lambda}(s,z)$ for every time function λ and every pair of non-adjacent vertices $s,z \in V(G)$. We ask:

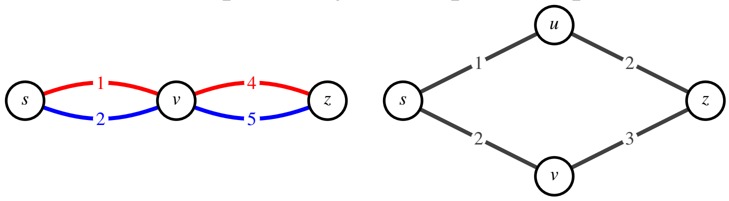
Question 8 *Is there a relation between the classes of t-Mengerian graphs and Mengerian graphs?*

Question 9 *Can t-Mengerian graphs be characterized by a family of forbidden m-topological minors? Can they be recognized in polynomial time?*

5 SNAPSHOT DISJOINT PATHS AND CUTS

In this chapter, we investigate a different concept of connectivity in temporal graphs. We allow temporal walks/paths to share any vertex, however the set of timesteps of each temporal walk/path must not intersect with each other. As example in Figure 35, we have two temporal graphs, the one on the left has two snapshot disjoint temporal s,z-paths that are highlighted while the one on the right does not have this property as there is only two temporal paths from s to z, while one of them starts by using edge active at timestep 2, the other finishes at such timestep.

Figure 35 – On the left a temporal graph with two snapshot disjoint temporal s,z-paths. On the right a graph that does not have two snapshot disjoint temporal s,z-paths



Source: elaborated by the author.

An interesting aspect of this type of problem is that it relies on time more than the others concepts. Up to our knowledge, this is the first time this type of robustness has been investigated. To better understand these concepts, consider the following scenario. Suppose a temporal graph (G,λ) models a communication network. Such network might be prone to interruptions of all communications at a given timestep due to attacks, blackouts, maintenance, etc. A good measure of robustness of such networks could then be the minimum number of timesteps in which the communications must get interrupted in order to break all possible connections between a pair of vertices. A network with higher measure means that it is less susceptible to failing under such interruptions and hence is considered more robust. In Figure 35, for instance, if there is an interruption on timesteps 2, then for the temporal graph to the right, vertex s cannot relay a message to z anymore, while for the one to left it would be possible.

This chapter follows similarly as the others. In Section 5.1, we introduce the terminology and give basic results. In Section 5.2 we state our results and compare them with some existing related work. In Section 5.3 we present our results about the computational complexity of the snapshot disjoint related problems; this section is divided to present algorithms and hardness results. In Section 5.4, we present two versions of Menger's Theorem for snapshot disjointness. In Section 5.4, we discuss two versions of Menger's Theorem. In Section 5.5, we fill a gap in the knowledge about multiedge disjoint cuts, proving it to be hard. Finally, Section 5.6 is reserved for the final considerations and present open problems.

5.1 Terminology and Preliminary Results

Let (G,λ) be a temporal graph with lifetime τ, also let $s,z \in V(G)$ be vertices in G and J,Q be temporal s,z-walks. We say that Q and J are *snapshot disjoint* if $\lambda(E(Q)) \cap \lambda(E(J)) = \emptyset$. A subset $S \subseteq [\tau]$ is a *snapshot s,z-cut* if every temporal s,z-walk uses an edge active at some time in S, i.e., if $\lambda(E(P)) \cap S \neq \emptyset$ for every temporal s,z-walk P. We denote by $sp_{G,\lambda}(s,z)$ the maximum number of snapshot disjoint temporal s,z-walks and by $sc_{G,\lambda}(s,z)$ the minimum size of a snapshot s,z-cut. Then, if $\mathscr{P}$ is a set of snapshot disjoint temporal s,z-walks and S a snapshot s,z-cut, each walk $P \in \mathscr{P}$ has an edge active at a time $\alpha_P \in \lambda(E(P)) \cap S$. As the walks in $\mathscr{P}$ are snapshot disjoint we have that $\alpha_P \neq \alpha_Q$ for every $Q \in \mathscr{P}$ different of P. Therefore $|\mathscr{P}| \leq |S|$. In particular, we have the inequality $sp_{G,\lambda}(s,z) \leq sc_{G,\lambda}(s,z)$.

We show a notion of equivalence between paths and walks for such concept.

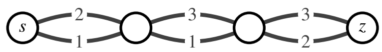
Proposition 5.1.1 *Let (G,λ) be a temporal graph and $s,z \in V(G)$. If P is a temporal s,z-walk in (G,λ), then there is a temporal s,z-path, say Q, such that $E(Q) \subseteq E(P)$, in particular $\lambda(E(Q)) \subseteq \lambda(E(P))$.*

Proof. Let $P = (s,e_1,v_1,\ldots,e_\ell,v_\ell = z)$ be a temporal walk. If $v_i = v_j$ for some $i \neq j$, then we find a temporal s,z-path Q, such that $E(Q) \subseteq E(P)$. Suppose without loss of generality that $i < j$. One just needs to define $Q = (s,e_1,v_1,\ldots,e_i,v_i,e_{j+1},v_{j+1},\ldots,e_\ell,v_\ell)$. We can apply the same procedure if there is some vertex repeated in Q. By repeating this process we can decrease the number of vertex repetitions until we obtain a temporal s,z-path. □

Observe that, as it happens with vertex disjoint paths and walks and unlike what happens with t-vertex disjoint paths and walks, the above proposition immediately implies that defining the parameters in terms of one or the other would lead to the same values.

Again the natural question about the inequality $sp_{G,\lambda}(s,z) \leq sc_{G,\lambda}(s,z)$ arises. We show that this inequality can be strict. Let (G,λ) be the temporal graph in Figure 36. Notice that any temporal s,z-path must use at least 2 timesteps, then $sp_{G,\lambda}(s,z) = 1$. To observe that $sc_{G,\lambda}(s,z) = 2$, one just needs to test removing each timestep. Such example is a particular case of the next proposition.

Figure 36 – Example of a temporal graph where $sp_{G,\lambda}(s,z) < sc_{G,\lambda}(s,z)$



Source: elaborated by the author.

Proposition 5.1.2 *For any positive integer $k \geq 1$, there is a temporal graph (G,λ) such that $sp_{G,\lambda}(s,z) = 1$ and $sc_{G,\lambda}(s,z) = k+1$.*

Proof. First, let $\ell = \binom{2k+1}{k}$. We add to G vertices s,z and then, for each $S \subseteq [2k+1]$ of size k, we link s and z through a path P_S of length $k+1$. Finally, our time function λ is defined in such way that $\lambda(E(P_S)) = [2k+1] \setminus S$ and P_S is a temporal path from s to z.

By construction, any temporal path from s to z has the form P_S for some $S \subseteq [2k+1]$ of size k. Moreover, as $\lambda(E(P_S)) \subseteq [2k+1]$ has size $k+1$ and by the Pigeon Hole Principle, we have that $\lambda(E(P_{S_1})) \cap \lambda(E(P_{S_2}) \neq \emptyset$ for any $S_1, S_2 \subseteq [2k+1]$ of size k. Thus, $sp_{G,\lambda}(s,z) = 1$. To prove that $sc_{G,\lambda}(s,z) \geq k+1$, let $S \subseteq [2k+1]$ be a set of size at most k. Observe that P_S is a temporal path from s to z such that $\lambda(E(P_S)) \cap S = \emptyset$, and hence S is not a snapshot s,z-cut. Notice that by the Pigeon Hole Principle any set $S \subseteq [2k+1]$ of size $k+1$ is a snapshot s,z-cut. Thus, $sc_{G,\lambda}(s,z) = k+1$. □

In this chapter we investigate the analogous notion of Mengerian graphs introduced by Kempe, Kleinberg, and Kumar (12). A graph G is *s-Mengerian* if $sp_{G,\lambda}(s,z) = sc_{G,\lambda}(s,z)$, for every pair $s,z \in V(G)$, for every time function λ. By definition, the graphs constructed in the above proposition are non-s-Mengerian.

Apart from s-Mengerian graphs, in this chapter we investigate the decision problems related to parameters $sp_{G,\lambda}(s,z)$ and $sc_{G,\lambda}(s,z)$, formally defined below.

$\leq h$-SNAPSHOT s,z-CUT

Input. A temporal graph (G,λ), a pair of vertices $s,z \in V(G)$, and an integer h.

Question. Is there a snapshot s,z-cut in (G,λ) of size at most h?

$\geq k$-SNAPSHOT DISJOINT TEMPORAL s,z-PATHS

Input. A temporal graph (G,λ), a pair of vertices $s,z \in V(G)$, and a positive integer k.

Question. Is there a set of snapshot disjoint temporal s,z-paths in (G,λ) of size at least k?

Observe that snapshot disjoint paths are more related to edge disjointness. Indeed, the two paths in the example on the left of Figure 35 share vertex v and temporal vertex $(v,3)$.

This is why in the next section we compare the complexity results obtained here with the known complexity results for the edge disjoint versions of these problems. We also fill a gap in the knowledge about edge disjointness. But first, we need to formally define such concetps. Two temporal s,z-paths are *edge disjoint* if they do not share any edge, and an *edge temporal s,z-cut* is a set S of edges of G intersecting every temporal s,z-path. For instance, the example on the left of Figure 35 the blue and red paths are edge disjoint, and the set of edges incident to s form an edge temporal s,z-cut. The concept of edge disjoint has been previously presented in the proof of Lemma . We say that two temporal s,z-paths are *multiedge disjoint* if they do not share any multiedge, and a *multiedge temporal s,z-cut* is a set S of multiedges of G intersecting every temporal s,z-path. For instance, the example on the right of Figure 35 has the multiedge disjoint paths $(s,1,u,2,z)$ and $(s,2,v,3,z)$, and the set $\{su,sv\}$ is a multiedge temporal s,z-cut. Note that Proposition 5.1.1 tells us that considering walks in these definitions would again lead to the same related parameters. We finally present the optimization related problems.

$\leq h$-EDGE TEMPORAL S,Z-CUT
Input. A temporal graph (G,λ), a pair of vertices $s,z \in V(G)$, and an integer h.
Question. Is there an edge temporal s,z-cut in (G,λ) of size at most h?

$\geq k$-EDGE DISJOINT TEMPORAL S,Z-PATHS
Input. A temporal graph (G,λ), a pair of vertices $s,z \in V(G)$, and an integer k.
Question. Is there a set of edge disjoint temporal s,z-paths in (G,λ) of size at least k?

$\leq h$-MULTIEDGE TEMPORAL S,Z-CUT
Input. A temporal graph (G,λ), a pair of vertices $s,z \in V(G)$, and an integer h.
Question. Is there a multiedge temporal s,z-cut in (G,λ) of size at most h?

$\geq k$-MULTIEDGE DISJOINT TEMPORAL S,Z-PATHS
Input. A temporal graph (G,λ), a pair of vertices $s,z \in V(G)$, and an integer k.
Question. Is there a set of multiedge disjoint temporal s,z-paths in (G,λ) of size at least k?

5.2 Main results and related works

As already mentioned, snapshot disjointness is a newly introduced concept, therefore no previous results exist. So, we present the edge-related concepts and existing results. We therefore compare our results with results obtained for (multi)edge disjoint paths. For any of these concepts, the notions of disjoint temporal walks and temporal paths lead to equivalent parameters, as a consequence of Proposition 5.1.1. Our results and related ones are summarized in Table 2. In (3), Berman has shown that $\leq h$-EDGE TEMPORAL S,Z-CUT and $\geq k$-EDGE DISJOINT TEMPORAL S,Z-PATHS are polynomial-time solvable, and that $\geq k$-MULTIEDGE DISJOINT TEMPORAL S,Z-PATHS is NP-complete. Up to our knowledge, no proof of hardness for $\leq h$-MULTIEDGE TEMPORAL S,Z-CUT is known. Even though multiedge disjointness was not our main problem of interest, we present in Section 5.5 a proof of NP-completness for $\leq h$-MULTIEDGE TEMPORAL S,Z-CUT in order to close the gap in knowledge.

As for the snapshot disjoint related problems, we prove that, when parameterized by h and k respectively, $\leq h$-SNAPSHOT s,z-CUT and $\geq k$-SNAPSHOT DISJOINT TEMPORAL s,z-PATHS are both W[1]-hard, and that this is best possible, i.e., that they are also XP. While the XP algorithm for $\leq h$-SNAPSHOT s,z-CUT follows easily from the definition and the fact that we can test all possible cuts in XP time (namely, $O(\tau^h)$ time), the algorithm for $\geq k$-SNAPSHOT DISJOINT TEMPORAL s,z-PATHS is much more involved and uses a technique similar to the one applied to find k vertex disjoint paths between k given pairs of vertices (also known as the k-LINKAGE PROBLEM) in a DAG (15).

X	$\geq k$ X DISJOINT PATHS	$\leq h$ X CUT
multiedge	NP-c (3)	NP-c even if $\tau = 2$
edge	Poly (3)	Poly (3)
snapshot	W[1] for k XP for k	W[1] for h XP for h

Table 2 – "NP-c"=NP-complete."W[1] for h"= W[1]-hard when parameterized by h (similar for "XP for h"). All the results related to snapshot disjointness are presented here.

5.3 Complexity Results

In this section, we present our complexity results for the problems presented in Section 5.1. We start by presenting our positive results, then we move on to the negative results.

5.3.1 Algorithms

We start with a simple case where we get equality between the parameters, and where we can compute them in polynomial time. Such case happens when λ is injective, in other words, when each snapshot has at most one edge.

Proposition 5.3.1 *Let (G,λ) be a temporal graph such that λ is injective. Then $sp_{G,\lambda}(s,z) = sc_{G,\lambda}(s,z)$ for every $s,z \in V(G)$. Moreover, we can compute such value in polynomial time.*

Proof. If P and Q are snapshot disjoint temporal s,z-paths, then, for $e \in E(P)$ and $f \in E(Q)$, we have that $\lambda(e) \neq \lambda(f)$, therefore $e \neq f$. On other hand, if P and Q are such that $E(P) \cap E(Q) = \emptyset$, then we have that $\lambda(E(P)) \cap \lambda(E(Q)) = \emptyset$. Thus, P and Q are snapshot disjoint if and only if they are edge disjoint. Therefore, the maximum number of edge disjoint temporal s,z-paths is equal to $sp_{G,\lambda}(s,z)$. Moreover notice that if $S \subseteq E(G)$ is a set such that every temporal s,z-path uses an edge of S, then every temporal s,z-path uses an edge active at timestep $\{\lambda(e) \mid e \in S\}$. If $S^T \subseteq [\tau]$ is such that every temporal s,z-path uses an edge active at timestep $\alpha \in S^T$, then it uses the only edge in $\lambda^{-1}(\alpha)$. Therefore, the size of a minimum set of edges such that every temporal s,z-path intersects such set is equal to $sc_{G,\lambda}(s,z)$. Using a result proved in (3) we conclude that $sp_{G,\lambda}(s,z)$=$sc_{G,\lambda}(s,z)$, and that both parameters can be found in polynomial time. □

Now we give XP algorithms for both problems. The first algorithm we are going to present is the simplest one and consists in the usual approach for XP algorithm, which is to test all possible solutions. It relies on the fact that such test can be done in polynomial time (see (23)).

Theorem 5.3.2 *Given a temporal graph (G,λ) of lifetime τ, a positive integer h and $s,z \in V(G)$, we can solve $\leq h$-SNAPSHOT s,z-CUT in $O(\tau^h(m+n))$ time, where $n = |V(G)|$ and $m = |E(G)|$.*

Proof. Let $(G,\lambda), s, z, h$ be as in the hypothesis of the theorem. For each subset $S \subseteq [\tau]$ of size h, define G_S such that $V(G_S) = V(G)$ and $E(G_S) = \{e \in E(G) \mid \lambda(e) \notin S\}$. Define also $\lambda_S(e) = \lambda(e)$ for all $e \in E(G_S)$. Now, by the definition of G_S, any temporal s,z-path in (G_S,λ_S) is a temporal s,z-path in (G,λ) that does not use edges active at timesteps in S. Reciprocally, every temporal s,z-path in G that does not use edges active at timesteps in S is a temporal s,z-path in (G_S,λ_S). As testing if there is a temporal s,z-path in (G_S,λ_S) can be done in time $O(m)$ (23) and (G_S,λ_S) can be constructed in $O(n+m)$ time, it suffices to apply this test to (G_S,λ_S) for every $S \subseteq [\tau]$ of size h. Since there are at most τ^h such sets, the theorem follows. □

The next algorithm is much more involved, and uses a technique similar to the one used to find disjoint paths between given pairs of vertices in a DAG (15).

Theorem 5.3.3 *Given a temporal graph (G,λ), vertices $s,z \in V(G)$ and a positive integer k, we can solve $\geq k$-SNAPSHOT DISJOINT TEMPORAL s,z-PATHS in time $O(m^k)$, where $m = |E(G)|$.*

Proof. We construct a digraph D with vertices s^* and z^* such that $|V(D)| = O(m^k)$ and there is an s^*,z^*-path in D if and only if there are k snapshot disjoint temporal s,z-paths in (G,λ).

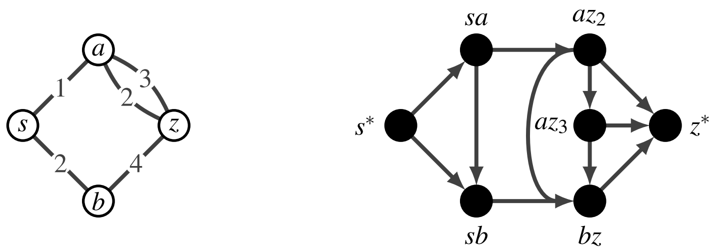
The vertices of digraph D are k-tuples formed by edges of G, together with vertices s and z; formally $V(D) \subseteq F^k$, where $F = E(G) \cup \{s,z\}$. Vertex s^* is set to be equal to $(s,\dots,s)$, while vertex z^* is set to be equal to $(z,\dots,z)$. Each dimension of $V(D)$ represents one of the desired k disjoint paths, and a set of snapshot disjoint temporal s,z-paths $P_1,\dots,P_k$ will be represented by an s^*,z^*-path P in D, as previously said. So s^* represents the starting point, and z^* represents the finish point of every temporal s,z-path. Then, when an edge of D is traversed by P, we want that one of the k paths also traverses an edge. Because we want to allow that only one of the paths gets closer to z with each step of P, there will be an edge from $\alpha \in V(D)$ to $\beta \in V(D)$ only if exactly one position of α and β differ. Not only this, but we want that, at each step of P, the path P_i that gets closer to z is the one whose last traversed edge occurs the earliest among all the P_i's. In the next paragraph, we formally construct digraph D.

As previously said, let $F = E(G) \cup \{s,z\}$. Along the construction, we will be referring to Figure 38. Because we want to avoid simultaneous traversal of paths that intersect in a snapshot, we only consider elements of F^k whose edges of G all appear in distinct snapshots. For instance, if $k = 2$ and we allow the existence of vertex (e,e') such that $t = \lambda(e) = \lambda(e')$, then this would mean that the constructed paths P_1 and P_2 intersect in timestep t. Therefore, we define $V(D)$ as formalized below. Observe that this implies, in Figure 38 on the left, that vertices $\{(e,e) \mid e \in E(G)\} \cup \{(az_2,sb),(sb,az_2)\}$ do not exist in $V(D)$, where az_2 denotes the edge with endpoints az active in timestep 2.

$$V(D) = \{(u_1,\dots,u_k) \in F^k \mid \forall i,j \in [k] \text{ with } i \neq j, \text{ we have } \lambda(u_i) \neq \lambda(u_j) \text{ or } u_i = u_j \in \{s,z\}\}$$

Now, we define the edge set of D. For this, we first construct an auxiliary graph M whose vertex set is equal to F; observe Figure 37 to follow the construction. First of all, we want that a traversal of an edge in D translates into a valid traversal in (G,λ). Therefore, for every pair $e,f \in F$, add to M an edge from e to f only if e can be followed by f in a temporal s,z-path in (G,λ). Formally, add ef in the following cases: for every $e \in E(G)$, and every $f \in E(G)$

Figure 37 – Example of construction in Theorem 5.3.3. On the left, we have the temporal graph, and on the right, the auxiliary directed graph M



Source: elaborated by the author.

adjacent to e and such that $\lambda(e) \leq \lambda(f)$; for $e = s$ and every $f \in E(G)$ incident to s; and for every $e \in E(G)$ incident to z and $f = z$.

Finally, as previously said, we want that at each edge traversal of an s^*, z^*-path in D, the path in (G, λ) that is getting closer to z is that one whose last used edge is the earliest among all the other paths. To help with this, we also define $\lambda(s)$ to be equal to 0, and $\lambda(z)$ to be equal to $\tau + 1$, where τ is equal to the lifetime of (G, λ). This means intuitively that we give always priority to leave s, and that, once we reach z in any dimension, then we cannot depart from z anymore. So, given a vertex $\alpha = (u_1, \ldots, u_k) \in V(D)$, we add an edge from α to $\beta \in V(D)$ if and only if:

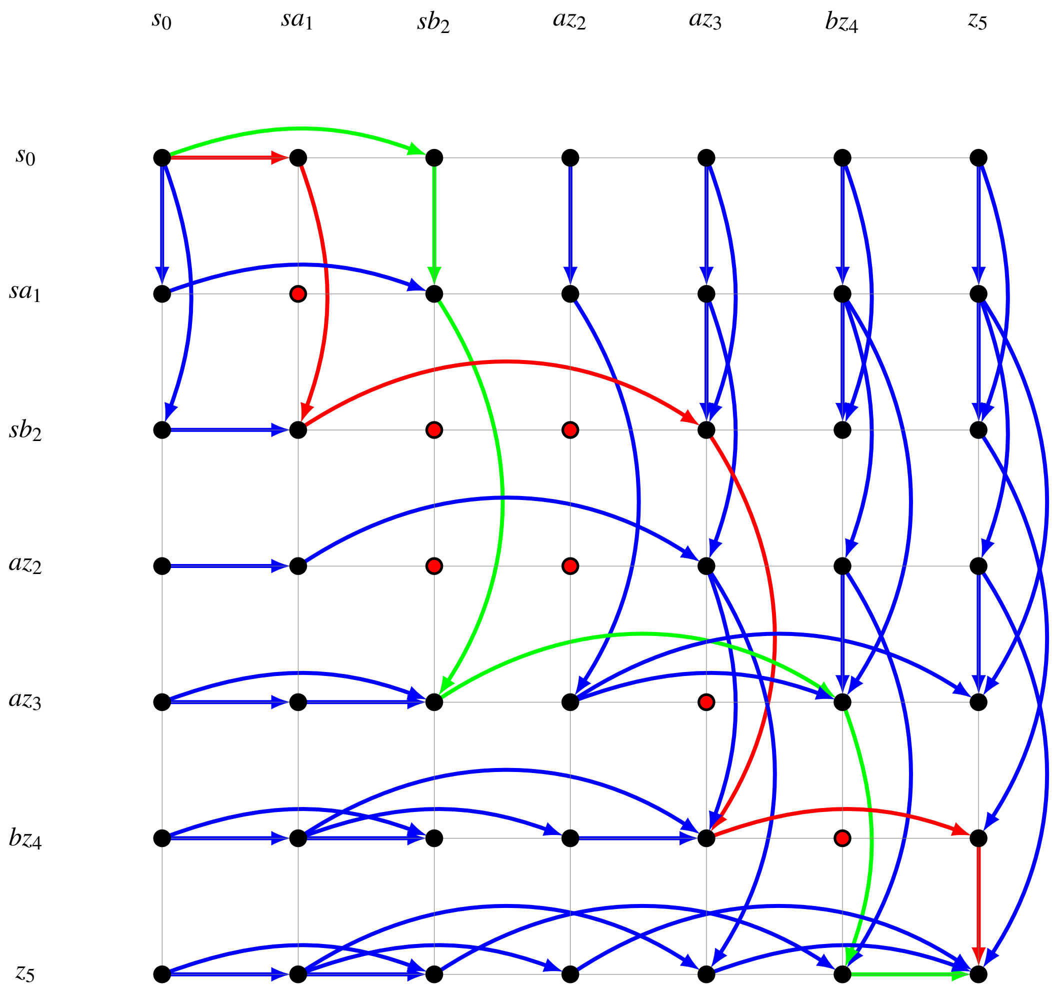
- β differs from α in exactly one position, i;
- i is such that $\lambda(u_i) \leq \min_{j \in [k]} \lambda(u_j)$; and
- By letting u'_i be the value in the i-th position of β, we have that $u_i u'_i$ is a valid move, i.e., that $u_i u'_i \in E(M)$.

Observe Figure 37 on the left. Another way of seeing this construction is by starting with copies of M on each row and column of D, then removing the vertices that do not belong to D, and finally removing from row/column e any edge leaving f with $\lambda(f) > \lambda(e)$.

We now prove that there are k snapshot disjoint temporal s, z-paths in (G, λ) if and only if there is an s^*, z^*-path in D. In what follows, given a vertex $\alpha \in V(D)$, we denote by $(a_1, \ldots, a_k)$ the tuple related to α. Recall that $s^* = (s, \ldots, s)$ and $z^* = (z, \ldots, z)$. Suppose $P_1, \ldots, P_k$ is a set of snapshot disjoint temporal s, z-paths in (G, λ). For each $i \in [k]$, let $e_i^1, \ldots, e_i^{p_i}$ be the sequence of edges used in P_i in order of traversal and define $e_i^0 = s$ and $e_i^{p_i+1} = z$. By induction, we define a sequence of vertices of D, $(s^* = \alpha^1, \ldots, \alpha^q = \alpha)$, that forms an s^*, α-path for some α with the following property:

(P) For each dimension $i \in [k]$ and each $\ell \in [q]$, the sequence of edges traversed in dimension

Figure 38 – Digraph D related to the temporal graph in Figure 37; value $k = 2$ is being used, which means that $V(D) \subseteq F^2$. Each row and column is labeled with an element e of F, together with the value $\lambda(e)$; this will help in the construction. A vertex (e, f) of D is represented in the intersection of row e and column f. Red dots in the figure represent the fact that the related pair $(row, column)$ is not a vertex in D. The snapshot disjoint temporal s,z-paths $P_1 = (s,1,a,3,z)$ and $P_2 = (s,2,b,4,z)$ can be obtained either through the red or the green s^*,z^*-path.



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i is a subpath of P_i. Formally, by removing s and repetitions of edges from the sequence $(a_i^1,\dots,a_i^q)$, we obtain a subsequence of $e_i^1,\dots,e_i^{p_i}$.

First, we define $\alpha_1 = s^*$; clearly property (P) holds as all paths start in s. Now suppose that sequence $\alpha_1,\dots,\alpha_q$ satisfying Property (P) is obtained, $q \geq 1$. Let $i \in [k]$ be such that $\lambda(a_i^q) = \min_{j\in[k]} \lambda(a_j^q)$. By Property (P), observe that either $a_i^q = s$, or a_i^q is an edge of P, or $a_i^q = z$. If the latter occurs, then we get that P is an s^*,z^*-path in D, since $\lambda(z) > \lambda(e)$ for every $e \in F\setminus\{z\}$, i.e., the only way $\lambda(z)$ is minimum is if all other positions are also equal to z. So suppose one of the other cases occurs. Note that it means that there is some edge following a_i^q in P, say e_i^ℓ. By definition of temporal path, $\lambda(e_i^\ell) \geq \lambda(a_i^q)$; hence $a_i^q e_i^\ell \in E(M)$. Define α^{q+1} to be equal to α^q except that in position i we have e_i^ℓ instead of a_i^q, and note that $(\alpha^1,\dots,\alpha^{q+1})$ is a path in D that satisfies Property (P).

Now suppose the existence of an s^*,z^*-path in D, $(\alpha^1,\dots,\alpha^q)$. We construct a set of k snapshot disjoint temporal s,z-paths in (G,λ). For this, for each $i \in [k]$, let P_i be a sequence of edges obtained from dimension i, i.e., from $(a_i^1,\dots,a_i^q)$ by removing occurrences of s and z, and repetitions of edges. Because each transition respects M, we trivially get that P_i defines a temporal s,z-path in (G,λ). It remains to show that such paths are snapshot disjoint. Suppose otherwise, and let i,j be such that there are edges e_i in P_i and e_j in P_j such that $\lambda(e_i) = \lambda(e_j) = \ell$. Let ℓ_i be the smallest index such that $a_i^{\ell_i} = e_i$, and ℓ_j be the smallest index such that $a_j^{\ell_j} = e_j$. By the definition of $V(D)$, we have that $\ell_i \neq \ell_j$. Indeed no vertex of D can contain two elements of F with same value of λ, and recall that $i \neq j$ as P_i,P_j are distinct paths. So, we can suppose, without loss of generality, that $\ell_i < \ell_j$. Observe that this means that α^{ℓ_i-1} differ from α^{ℓ_i} in exactly position i; additionally, it means that $\lambda(e_i) \leq \min_{h\in[k]} \lambda(a_h^{\ell_i-1})$. In particular, we have that $\ell = \lambda(e_i) \leq \lambda(a_j^{\ell_i-1})$. But observe that, in a fixed dimension, the values of λ can only increase, i.e., since $\ell_i < \ell_j$, we get $\lambda(a_j^{\ell_i-1}) \leq \lambda(a_j^{\ell_j}) = \lambda(e_j) = \ell$. We get a contradiction as in this case vertex α^{ℓ_i-1} should not be defined as it contains two elements with the same value of λ, namely $a_j^{\ell_i-1}$ and e_i.

Finally, recall that $|V(D)| \leq (m+2)^k$, where $m = |E(G)|$, and that deciding if there is a path between two vertices in D can be made in time $O(|V(D)|^2)$ using Dijkstra's algorithm. So, deciding if there are k snapshot disjoint temporal s,z-paths in (G,λ) can be done in time $O(m^k)$. □

5.3.2 Hardness results

Now that we have XP algorithms for both problems, in this section we show that these algorithms are optimal in the sense that both problems are W[1]-hard when parameterized by the solution size. We start with the problems of finding paths.

Theorem 5.3.4 $\geq k$-SNAPSHOT DISJOINT TEMPORAL s,z-PATHS *is* W[1]*-hard when parameterized by k.*

Proof. We make a parameterized reduction from $\geq k$-INDEPENDENT SET when parameterized by k. The input of such problem is a simple graph G and an integer k, and the question is whether G has an independent set of size at least k. This is known to be W[1]-hard (see e.g. (20)).

So consider an instance G,k of $\geq k$-INDEPENDENT SET and let $|V(G)| = n$. Observe Figure 39 to follow the construction. First, add to G' vertices s and z. Then, for each $u \in V(G)$, add to G' an s,z-path on $d(u)$ edges; denote such path by Q_u. Now, consider any ordering $e_1,\ldots,e_m$ of $E(G)$, and denote the edges incident to a vertex $u \in V(G)$ by $\delta(u)$. We can define $\lambda : E(G') \to \mathbb{N} \setminus \{0\}$ in a way that each Q_u is a temporal s,z-path using the orders of the edges in $\delta(u)$. Formally, for each $u \in V(G)$, let $\delta(u) = \{e_{i_1},\ldots,e_{i_q}\}$ with $i_1 < \ldots < i_q$, and define $\lambda(E(Q_u))$ to be equal to $\{i_1,\ldots,i_q\}$ in a way that Q_u is a temporal path.

We show that (G',λ) has k snapshot disjoint temporal s,z-paths if and only if G has an independent set of size at least k. By the definition of G', all temporal s,z-paths are of type Q_u for some $u \in V(G)$. Therefore, it suffices to show that a subset $S \subseteq V(G)$ is an independent set of G if and only if $\{Q_u \mid u \in S\}$ is a set of snapshot disjoint temporal s,z-paths in (G',λ). Suppose first that S is an independent set, and suppose by contradiction that $u_1,u_2 \in S$ are such that Q_{u_1} and Q_{u_2} are not snapshot disjoint. Then there exists $e_1 \in E(Q_1)$ and $e_2 \in E(Q_2)$ such that $\lambda(e_1) = \lambda(e_2)$. By construction, this means that $e_1 = e_2$, and since $u_1 \neq u_2$, we get that actually this edge has endpoints u_1u_2, a contradiction as S is an independent set. Thus, $\{Q_u \mid u \in S\}$ is a set of snapshot disjoint temporal s,z-paths. Finally, observe that if e_i has endpoints vw, then $\lambda(Q_v) \cap \lambda(Q_w) = \{i\}$, which directly implies that if $\{Q_u \mid u \in S\}$ is a set of snapshot disjoint temporal s,z-paths, then S cannot contain any pair of adjacent vertices. □

Now, we prove the analogous result for the cut problem.

Theorem 5.3.5 $\leq h$-SNAPSHOT DISJOINT TEMPORAL s,z-PATHS *is* W[1]*-hard when parameterized by h.*

Figure 39 – On the left, graph G, and on the right, the constructed temporal graph (G', λ). In G', paths $Q_a, \dots, Q_e$ are depicted from top to bottom, in this order

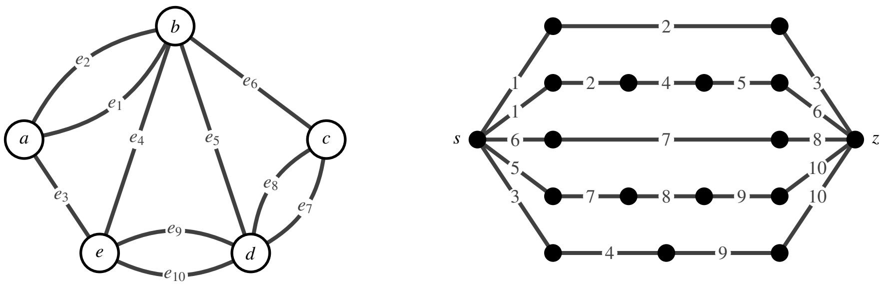

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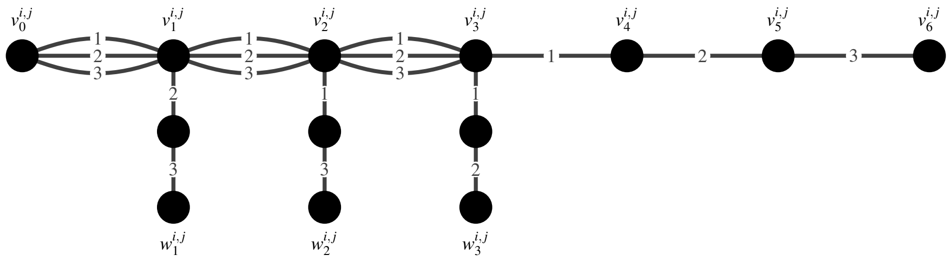
Figure 40 – A representation of $F_{i,j}$ with labels of λ where $m = 3$ and $\Delta_{i,j} = \{1, 2, 3\}$



Source: elaborated by the author.

Proof. We make a reduction from MULTICOLORED k-CLIQUE, when parameterized by k, known to be W[1]-hard (20). Such problem has as input a simple graph G, an integer k, and a partition of $V(G)$ into k independent sets (alternatively, a proper k-coloring), and the question is whether G has a (multicolored) clique of size k. So let G be a graph and $\{X_1, \dots, X_k\}$ be a proper k-coloring of G. By adding artificial vertices and edges if necessary, we can suppose that the number of edges between X_i and X_j is equal to a value m, for every pair $i, j \in [k]$. So, for $i, j \in [k]$, $i \neq j$, denote the set of such edges by $E_{i,j} = \{e_1^{i,j}, \dots, e_m^{i,j}\}$. We make this assumption in order to make presentation simpler.

Now, for each $i, j \in [k]$, $i \neq j$, we construct a gadget denoted by $F_{i,j}$. Observe Figure 40 to follow the construction. First add to $F_{i,j}$ the set of vertices $V_{i,j} = \{v_0^{i,j}, \dots, v_{2m}^{i,j}\}$, making the first $m+1$ of them form a path of multiplicity m, and the latter $m+1$ form a path of multiplicity 1. Formally, for each $\ell \in \{0, \dots, m-1\}$, add m edges with endpoints $v_\ell^{i,j} v_{\ell+1}^{i,j}$. Also, for each $\ell \in \{m, \dots, 2m-1\}$, add 1 edge with endpoints $v_\ell^{i,j} v_{\ell+1}^{i,j}$. Now, for each $\ell \in [m]$, add vertex $w_\ell^{i,j}$ and join such vertex with $v_\ell^{i,j}$ by a path with $m-1$ edges and denote such path by $P_\ell^{i,j}$. We say that vertex $w_\ell^{i,j}$ of our gadget is *associated with edge* $e_\ell^{i,j}$ of $E_{i,j}$. The time function is defined only later.

Now, we finish the construction of our temporal graph. For this, take the union of all graphs $F_{i,j}$ and identify all vertices $v_0^{i,j}$, calling the obtained vertex s, and identify all vertices $v_{2m}^{i,j}$, calling the obtained vertex z. Also, for each $i,j \in [k]$, $i \neq j$, and $\ell \in [m]$, we add two edges between $w_\ell^{i,j}$ and z. Denote by G' the obtained graph, and by W the set $\{w_\ell^{i,j} \mid i,j \in [k], i \neq j, \ell \in [m]\}$. Observe that G' contains $O(k^2 \cdot m)$ vertices and edges.

Now we define λ. The idea is that each $F_{i,j}$ will be active during its own dedicated time window. Formally, define $\Delta_{i,j} = [f_{i,j}+1, f_{i,j}+m]$, for each pair $i,j \in [k]$, $i \neq j$, in a way that $\Delta_{i,j} \cap \Delta_{i',j'} = \emptyset$ whenever $\{i,j\} \neq \{i',j'\}$. Now, consider $i,j \in [k]$ with $i \neq j$. For each $\ell \in \{0,\dots,m-1\}$, we define λ in a way that every value in $\Delta_{i,j}$ appears in some edge with endpoints $v_\ell^{i,j} v_{\ell+1}^{i,j}$. Also, for each $\ell \in [m]$, we let $\lambda(v_{m-1+\ell}^{i,j} v_{m+\ell}^{i,j}) = \{f_i + \ell\}$, and we define $\lambda(E(P_\ell^{i,j}))$ to be equal to $\Delta_{i,j} \setminus \{f_{i,j}+\ell\}$ and in a way that $P_\ell^{i,j}$ is a temporal $v_\ell^{i,j}, w_\ell^{i,j}$-path. Finally, the only edges that remain unlabelled are the edges between z and vertices of type w. For such edges, we reserve a time window of size $n = |V(G)|$, that we denote by Δ_V, where any timestep in such set is greater that any timestep we used to define λ so far. Moreover, we associate each vertex $v \in V(G)$ with a timestep $t_v \in \Delta_V$. Let $w_\ell^{i,j} \in W$ and recall that such vertex is associated with $e_\ell^{i,j} \in E(G)$. Suppose $e_\ell^{i,j}$ have endpoints xy, and let the two edges of G' with endpoints $w_\ell^{i,j} z$ be active in timesteps $\{t_x, t_y\}$.

Now, we prove that G has a clique of size k if and only if (G', λ) has a snapshot s,z-cut of size at most $\binom{k}{2} + k$. Consider first a clique C of G of size k, and let $\{e_{\ell_1}^{i_1,j_1}, \dots, e_{\ell_a}^{i_a,j_a}\}$ be the set of edges of G between vertices of C. Notice that, because C has a vertex from each part, we get that $a = \binom{k}{2}$. Define $S = \{f_{i_b,j_b} + \ell_b \mid b \in \{1,\dots,a\}\} \cup \{t_v \mid v \in C\}$. We prove that S is a snapshot s,z-cut. By contradiction, suppose that P is a temporal s,z-path not passing by S, i.e., such that $\lambda(E(P)) \cap S = \emptyset$. Since $a = \binom{k}{2}$ and all edges incident to s are active in timesteps $\bigcup_{i,j \in [k], i \neq j} \Delta_{i,j}$, we can define $b \in [a]$ to be the index related to the first edge in P, i.e., P starts in an edge of F_{i_b,j_b}, say the one active in timestep $f_{i_b,j_b} + \ell_b$. Observe that the value $f_{i_b,j_b} + \ell_b$ is within the temporal s,z-path contained in F_{i_b,j_b}, and that it also disconnects s and $w_\ell^{i_b,j_b}$ for every $\ell \in [m] \setminus \{\ell_b\}$. Hence, P must start with the temporal $s, w_{\ell_b}^{i_b,j_b}$-path contained in F_{i_b,j_b}. However, as $e_{\ell_b}^{i_b,j_b}$ is incident to vertices of the clique, say x and y, then we have that P uses timestep t_x or t_y, a contradiction as $\{t_x, t_y\} \subseteq S$.

Now, suppose that S is a minimum snapshot s,z-cut in (G', λ) and that it has size at most $\binom{k}{2} + k$. Let $V_S = \{x \in V(G) \mid t_x \in S\}$. We prove that V_S is a clique of G of size k. Denote by O the set of pairs $\{(i,j) \mid i,j \in [k], i < j\}$. We say that $(i,j) \in O$ is *open* if $\Delta_{i,j} \cap S = \{f_{i,j} + \ell\}$

for some $\ell \in [m]$, and we say that $e_\ell^{i,j}$ is the *open edge of* (i, j). The following simple facts will be useful:

1. For every $i, j \in [k]$, $i \neq j$, we have $\Delta_{i,j} \cap S \neq \emptyset$: this is due to the fact that there is a temporal s, z-path using only timesteps in $\Delta_{i,j}$, namely the path $(s, t_1, v_1^{i,j}, t_1, v_2^{i,j}, \ldots, v_m^{i,j}, t_1, v_{m+1}^{i,j}, t_1 + 1, \ldots, t_1 + m - 1, t)$, where $t_1 = f_{i,j} + 1$;
2. If $\ell \in [m]$ is such that $\Delta_{i,j} \cap S = \{f_{i,j} + \ell\}$, then $\{x, y\} \subseteq V_S$, where xy are the endpoints of $e_\ell^{i,j}$: this is because there exists a temporal $s, w_\ell^{i,j}$-path not using any timestep in S, namely $P_\ell^{i,j}$, and hence such path can be extended to a temporal s, z-path by using an edge with endpoints $w_\ell^{i,j} z$ either in timestep t_x or in timestep t_y;
3. For every $i, j \in [k]$, $i \neq j$, we have $|\Delta_{i,j} \cap S| \leq 2$: it suffices to see that any two timesteps in $\Delta_{i,j}$ intersects all temporal paths between s and any vertex in $\{w_\ell^{i,j} \mid \ell \in [m]\} \cup \{z\}$;
4. If $x \in V_S$, then x is incident to some open edge: indeed, if x is not incident to any open edge, then $w_\ell^{i,j}$ is disconnected from s by $S \setminus \{t_x\}$ for every edge $e_\ell^{i,j}$ incident in x, and since timestep t_x contains only edges incident to some such $w_\ell^{i,j}$. It follows that $S \setminus \{t_x\}$ is also a snapshot s, z-cut, contradicting the minimality of S.

By Fact 3, if (i, j) is not open, then $\Delta_{i,j} \cap S = \{f_{i,j} + \ell_1, f_{i,j} + \ell_2\}$ for some pair of values $\ell_1, \ell_2 \in [m]$. In such case, we say that edges $e_{\ell_1}^{i,j}$ and $e_{\ell_2}^{i,j}$ are *chosen* for (i, j). We show how to modify S in order to decrease the number of chosen edges.

Claim 5.3.1 *If (i, j) is not open, then we can suppose that $V_S \cap (X_i \cup X_j) = \emptyset$.*

Let $e_{\ell_1}^{i,j}$ and $e_{\ell_2}^{i,j}$ be the chosen edges for (i, j). Denote by $x_1 y_1$ and $x_2 y_2$ the endpoints of $e_{\ell_1}^{i,j}$ and $e_{\ell_2}^{i,j}$, respectively. Suppose without loss of generality that $\{x_1, x_2\} \subseteq X_i$ and $\{y_1, y_2\} \subseteq X_j$. Now suppose that there exists $x \in X_i \cap S$. If $x \in \{x_1, x_2\}$, say $x = x_1$, then replace $f_{i,j} + \ell_2$ in S by y_1, obtaining S'. Observe that in this case $e_{\ell_1}^{i,j}$ becomes an open edge, but such that $\{t_{x_1}, t_{y_1}\} \subseteq S'$. Hence S' is still a snapshot s, z-cut containing fewer chosen edges. And if $x \notin \{x_1, x_2\}$, then let xy be any edge incident in x such that $y \in X_j$. We can suppose that such edge exists as otherwise x cannot be in any multicolored k-clique and then we can remove it from G. Let $\ell \in [m]$ be such that $e_\ell^{i,j}$ has endpoints xy. Observe that Fact 3 also tells us that $S' = (S \setminus \{f_{i,j} + \ell_2\}) \cup \{f_{i,j} + \ell\}$ is a snapshot s, z-cut. We can then apply the previous argument to replace $f_{i,j} + \ell_1$ by t_y to again obtain a snapshot s, z-cut with fewer chosen edges.

Claim 5.3.2 *We can suppose that every pair is open.*

We can suppose that G is connected, as otherwise the answer to MULTICOLORED k-CLIQUE is trivially "no". Now, let I_1 be the set of indices $\{i \in [k] \mid V_S \cap X_i = \emptyset\}$ and $I_2 = S \setminus I_1$. Suppose first that $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$, and consider $i_1 \in I_1$ and $i_2 \in I_2$. By Claim 5.3.1 and because $i_2 \in I_2$, we know that (i_1, i_2) is open. So let $e_\ell^{i_1,i_2}$ be the open edge for (i_1, i_2), and let xy be its endpoints, with $x \in X_{i_1}$ and $y \in X_{i_2}$. By Fact 2, we get that $\{t_x, t_y\} \subseteq S$, i.e., $\{x, y\} \subseteq V_S$, contradicting Claim 5.3.1 as $x \in X_{i_1}$ and $i_1 \in I_1$.

So either $I_1 = \emptyset$ or $I_2 = \emptyset$. Observe that if $I_1 = \emptyset$, then $V_S \cap X_i \neq \emptyset$, for every $i \in [k]$, and the claim follows from Claim 5.3.1. And if $I_2 = \emptyset$, then by Fact 3 we get that S contains two edges for every pair i, j, totalling $|S| = 2\binom{k}{2}$. Since $|S| \leq \binom{k}{2} + 2$, we get that this happens only if $k \leq 3$, in which case MULTICOLORED k-CLIQUE is polynomial-time solvable.

Finally, observe that the set of open edges, E^*, contains exactly $\binom{k}{2}$ edges, by definition of open edge and by Claim 5.3.2. We then get that $|S| = |E^*| + |V_S| = \binom{k}{2} + |V_S|$. It follows that $|V_S| \leq k$. Additionally, by Fact 2 we know that E^* forms a subgraph of G with vertex set V_S. Because G is a simple graph, E^* contains $\binom{k}{2}$ edges, and V_S contains at most k vertices, the only way this can be possible is if V_S contains exactly k pairwise adjacent vertices, i.e., V_S is a clique of size k, as we wanted to prove. □

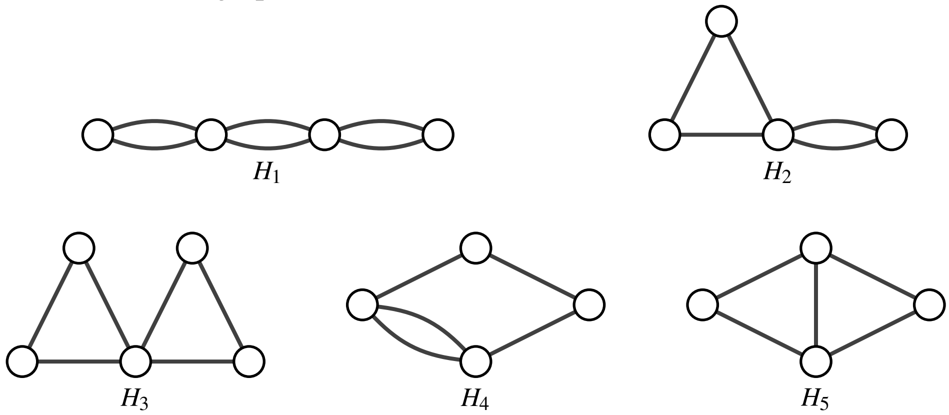
5.4 Characterization and Recognition of s-Mengerian Graphs

On this section we focus on characterizing s-Mengerian graphs. We also present a polynomial-time algorithm for the recognition of such graphs. As in Chapter 3, we use forbidden m-topological minors to characterize these graphs. The 5 forbidden structures are the graphs $H_1, \ldots, H_5$ represented in Figure 41. Let $\mathscr{H}$ be the set of such graphs. We first prove that these graphs are not s-Mengerian.

Proposition 5.4.1 *Let (G, λ) be one of the temporal graphs depicted in Figure 42. Then $sp_{G,\lambda}(s,z) < sc_{G,\lambda}(s,z)$.*

Proof. To observe that $sc_{G,\lambda}(s,z) > 1$, one just needs do verify that for each timestep, there is a temporal s,z-path not using this timestep. Now, for the cases $G \in \{H_1, H_2, H_3, H_4\}$, suppose $sp_{G,\lambda}(s,z) = sc_{G,\lambda}(s,z)$ and let Q, J be snapshot disjoint temporal s,z-paths. In each of those cases, there are only two edges incident to s, one active at timestep 1 and other at timestep 2. Hence, one of these paths, say Q, starts at timestep 2. Observe that in this case, Q can only finish through edges active at timestep 3. Therefore, J cannot use timestep 2 nor 3, however, all

Figure 41 – Forbidden m-topological minors for s-Mengerian graphs



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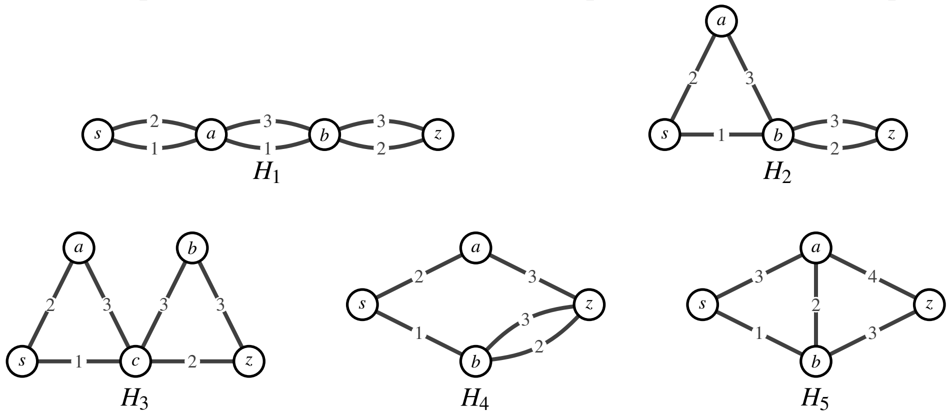
edges incident to z are active at timesteps 2 or 3, a contradiction as there is no temporal s,z-path contained in the first snapshot. Finally, suppose $G = H_5$; we will apply a similar argument. So, let Q,J be snapshot disjoint temporal s,z-paths. Note that one of them, say Q, must use the edge incident to s active at timestep 3 and, therefore, finishes using the edge incident to z active at timestep 4. It follows that J is not allowed to use timestep 3 nor 4, a contradiction as all edges incident to z are only active at such timesteps. □

Figure 42 – Graphs in the set $\mathscr{H}$ with time function that turns the inequality between the maximum quantity of snapshot disjoint temporal s,z-paths and the size of a minimum snapshot s,z-cut a strict inequality



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Now, we prove that for graphs G,H, if H is m-topological minor of G and H is non-s-Mengerian, then G also is non-s-Mengerian. The next two propositions do this work.

Proposition 5.4.2 *If G is non-s-Mengerian, then an m-subdivision of G is also non-s-Mengerian.*

Proof. Let G be a non-s-Mengerian graph and consider $\lambda : E(G) \to \mathbb{N} \setminus \{0\}$ and $s,z \in V(G)$ to be such that $sp_{G,\lambda}(s,z) < sc_{G,\lambda}(s,z)$. Also, suppose that H is obtained from G by m-subdividing a multiedge, say xy. We construct a function λ' from λ that proves that H is also non-s-Mengerian.

Let $D \subseteq E(G)$ be the set of edges of G with endpoints xy, and denote by v_{xy} the vertex of H created by the m-subdivision of xy. Moreover, denote by D_x and D_y the sets of edges of H with endpoints xv_{xy} and $v_{xy}y$, respectively. Finally, define λ' to be such that $\lambda'(e) = \lambda(e)$, for every $e \in E(G) \setminus D$, and $\lambda'(D_x) = \lambda'(D_y) = \lambda(D)$. We show that $sp_{G,\lambda}(s,z) = sp_{H,\lambda'}(s,z)$ and $sc_{G,\lambda}(s,z) = sc_{H,\lambda'}(s,z)$, which finishes our proof.

Given a set of snapshot disjoint temporal s,z-paths in (G,λ), if some of these paths, say P, uses the edge xy, then in (H,λ') we can substitute such edge by an edge in D_x and another in D_y active at the same timestep to obtain a temporal path P' such that $V(P') = V(P) \cup \{v_{xy}\}$. This gives us a set of snapshot disjoint temporal s,z-paths in (H,λ'). In the other direction, if it is given a set of snapshot disjoint temporal s,z-paths in (H,λ'), if some of them uses the vertex v_{xy}, let f_j be the edge used in D_j for $j \in \{x,y\}$. Suppose without loss of generality that $\lambda(f_x) \geq \lambda(f_y)$. Then we substitute both edges incident to v_{xy} by an edge in D appearing at time $\lambda(f_x)$. This implies that $sp_{G,\lambda}(s,z) = sp_{G,\lambda'}(s,z)$.

To see that $sc_{G,\lambda}(s,z) = sc_{H,\lambda'}(s,z)$, let $S \subseteq [\tau(G,\lambda)]$ be a snapshot s,z-cut and P be a temporal s,z-path in (H,λ'). We show that $\lambda(E(P)) \cap S \neq \emptyset$. If P does not use vertex v_{xy}, then it is a temporal path of (G,λ) and then $\lambda(E(P)) \cap S \neq \emptyset$ follows. If P uses the vertex v_{xy}, then, in the same way as before, we can find a temporal s,z-path P' in (G,λ) such that $\lambda(E(P')) \subseteq \lambda(E(P))$ as $\lambda(E(P'))$ intersects S, we know that the same happens for P'. Thus S is a snapshot s,z-cut cut in (H,λ'). Now, suppose that S is snapshot s,z-cut in (H,λ'), and let P a temporal s,z-path in (G,λ). If P does not use any edge with endpoints xy, then P is also a temporal s,z-path in (H,λ'), and hence $\lambda(E(P)) \cap S \neq \emptyset$. And if P uses an edge with endpoints xy, then, in the same way as we did before, we construct a temporal s,z-path P' in (H,λ') such that $\lambda(E(P')) = \lambda(E(P))$. The rest of proof follows as in the last case. $\square$

Proposition 5.4.3 *G is s-Mengerian if and only if H is s-Mengerian, for every $H \subseteq G$.*

Proof. To prove necessity, suppose that $H \subseteq G$ is non-Mengerian, and let s,z,λ be such that $sp_{H,\lambda}(s,z) < sc_{H,\lambda}(s,z)$. Consider the time function λ' in $E(G)$ defined as follows.

$$\lambda'(e) = \begin{cases} \lambda(e)+1 & \text{, for every } e \in E(H), \\ 1 & \text{, for every } e \in E(G) \setminus E(H) \text{ with endpoints } yz \text{, and} \\ \max \lambda(E(H))+2 & \text{, otherwise.} \end{cases}$$

Notice that if there is an edge $e \in E(G) \setminus E(H)$ connecting sz, then such edge defines a temporal path in (G, λ') using only the timestep $\lambda'(e) = 1$. Therefore, removing all edges of snapshot 1 decreases the parameters $sp_{G,\lambda}(s,z)$ and $sp_{G,\lambda}(s,z)$ in exactly 1. Then the obtained graph is s-Mengerian if and only if G is s-Mengerian. Suppose then that there are no edges connecting sz.

Because $H \subseteq G$ and $\lambda \subseteq \lambda'$, note that we get $sp_{H,\lambda}(s,z) \leq sp_{G,\lambda'}(s,z)$ and $sc_{H,\lambda}(s,z) \leq sc_{G,\lambda'}(s,z)$. Therefore it suffices to prove $sp_{G,\lambda'}(s,z) \leq sp_{H,\lambda}(s,z)$ and $sc_{G,\lambda'}(s,z) \leq sc_{H,\lambda}(s,z)$. As we show next, these inequalitis hold because the time function λ' does not allow for the existence of a temporal s,z-path not contained in H. By contradiction, suppose that P is such a path, and let $M = \max \lambda(E(H))$. Let $e \in E(P) \setminus E(H)$. If $\lambda(e) = 1$, then we have that e is incident to z, this is a contradiction as there is no edge connecting sz and all the edges incident to s are active at time at least 2. Thus, we have $\lambda(e) = M+2$, in this case notice that every edge incident to z is active at time at most $M+1$. This is a contradiction as the last edge of P is incident to z and it has activation time at least M since $\lambda(e) = M+2$. □

Now we can state the main result of this section.

Theorem 5.4.4 *Let G be a graph. Then G is s-Mengerian if and only if G does not have any of the graphs in $\mathscr{H}$ as m-topological minor.*

Proof. First we prove necessity. Suppose that H_i is an m-topological minor of G for some $i \in [5]$. This means that G contains a subgraph H with H being an m-subdivision of H_i. By Propositions 5.4.1 and 5.4.2, we get that H is non-s-Mengerian. And by Proposition 5.4.3, that also G is non-s-Mengerian.

Now, we prove the sufficiency. Suppose otherwise and let $(G, \lambda), s, z$ be a counter-example that minimizes $|V(G)| + |E(G)| + \tau(G, \lambda)$. In other words, $(G, \lambda), s, z$ is such that G does not have any of the graphs in Figure 41 as m-topological minor, G is non-s-Mengerian, and among all possible such graphs and all possible time functions, (G, λ) is the one that minimizes $|V(G)| + |E(G)| + \tau(G, \lambda)$.

First suppose that there is a cycle C containing s, z. As $sz \notin E(G)$, we have that such cycle has size at least 4. As G has no H_5 as m-topological minor, there is no path between the vertices of C that is not contained in C. Also, as G has no H_4 as m-topological minor, all the multiedges of C have multiplicity 1. Therefore, there are only two paths between s and z in G. If these two paths are snapshot disjoint temporal s,z-paths, then we take the times of the two edges incident to z, as this is a snapshot s,z-cut. We get a contradiction as in this case $sp_{G,\lambda}(s,z)$=$sc_{G,\lambda}(s,z)$. So, if both paths between s and z are temporal paths, they must intersect at a time α. But now $\{\alpha\}$ is a snapshot s,z-cut, again leading to a contradiction as in this $sp_{G,\lambda}(s,z)$=$sc_{G,\lambda}(s,z)$. Therefore, we can suppose that there is no cycle containing s and z. So, if we consider the decomposition of G in 2-connected components $G_1,\ldots,G_k$ we have that s and z are in different components. Moreover, as we are supposing G,λ,s,z minimum, the graph induced by the decomposition is a path and we can suppose that $s \in V(G_1), z \in V(G_k)$. Suppose that, for some i, G_i contains has more than 3 vertices and let $v \in V(G_i) \cap V(G_j)$ for some $j \in \{i-1,i+1\}$. Let C_1 be a cycle of G_i containing v. If G_j has more than 3 vertices, then we can find another cycle C_2 contained in G_j such that $V(C_1) \cap V(C_2) = \{v\}$, this contradicts the fact that G has no H_3 as m-topological minor. So, we can suppose that $|V(G_j)| = 2$. If the multiedge contained in G has multiplicity 1, then let α be the time of such edge. We have that $\{\alpha\}$ is a snapshot s,z-cut, and then $sp_{G,\lambda}(s,z)$ = $sc_{G,\lambda}(s,z)$ a contradiction as (G,λ) is a counter-example. Therefore we can assume that the multiedge of G_j has multiplicity at least 2, however the graph induced by such multiedge and the cycle contained in G_i containing v is an m-subdivision of the graph H_2, contradicting again the fact the (G,λ) is a counter-example. Therefore we can assume that no graph G_i contains a cycle for every $i \in \{1,\ldots,k\}$. Therefore, $U(G)$ is a s,z-path. If some of the multiedge of G has multiplicity 1, then $sc_{G,\lambda}(s,z) \leq 1$ and we have the equality what contradicts that $(G,\lambda),s,z$ is counter-example. We can say also that $|U(G)| = 3$, as otherwise G would have H_1 as m-topological minor. Now, we show that this is also a contradiction and it ends the proof.

Now we show that for a graph H such that $|U(H)|$ is a path of size 3 and any time function λ in H, we have that $sp_{G,\lambda}(s,z)$ = $sc_{G,\lambda}(s,z)$. We do it by induction in the number of edges of H. The base of induction is trivial. By Proposition 5.4.1, we can suppose that there are two edges appearing at same time, say f and g. As $U(H)$ is a path of size 3, these two edges forms a temporal s,z-path. Let $H' = H - \{f,g\}$. By induction hypothesis, we have that $sp_{H',\lambda}(s,z)$ = $sc_{H',\lambda}(s,z)$. Now let $\mathscr{P}$ be a set of snapshot disjoint temporal s,z-paths and S be a snapshot s,z-cut in (H',λ). Then, $\mathscr{P} \cup \{(s,f,w,g,z)\}$ is a set of snapshot disjoint temporal s,z-path in (H,λ) and $S \cup \{\lambda(f)\}$ is a snapshot s,z-cut in (H,λ). Therefore, $sc_{H,\lambda}(s,z)$ = $sc_{H',\lambda}(s,z)$ + 1= $sp_{H',\lambda}(s,z)$+ 1 = $sp_{H,\lambda}(s,z)$. □

We turn our attention to the recognition of s-Mengerian graphs, showing that it can be done in polynomial time. We observe that the proof of characterization of s-Mengerian graphs helps us to construct an algorithm of recognition of s-Mengerian graphs. We make the proper adaptation and prove the next theorem.

Theorem 5.4.5 *One can decide in polynomial time whether a graph G has a graph in $\mathscr{H}$ as m-topological minor.*

Proof. Let G be an instance and $n = |V(G)|$ and $m = |E(G)|$. First, we find the block-cutpoint graph of G; denote by $B_1, \ldots, B_k$ the set of blocks of G. This can be done in $O(m+n)$. Then, we apply the following algorithm, which returns either "yes", in case G has a graph in $\mathscr{H}$ as m-topological minor, or "no" otherwise:

1. Test whether the subgraph of G consisting of the multiedges with multiplicity at least 2 has a path of size at least 4. If so, then answer "yes";
2. If any block B_i with at least 4 vertices is such that $|E(B_i)| > |V(B_i)|$, then answer "yes";
3. If there are two blocks B_i and B_j such that both have at least 3 vertices and they intersect in a cut vertex, then answer "yes";
4. If there are two blocks B_i and B_j such that B_i has at least 3 vertices, and B_j is a multiedge with multiplicity at least 2, then answer "yes";
5. If none of the previous cases applies, then answer "no".

Observe that each step of the algorithm can be done in linear time if the appropriate data structure is used. We now prove that G has a graph in $\mathscr{H}$ as m-topological minor if and only if the algorithm answers "yes". We first argue that the sufficient part holds. If the algorithm answers "yes" in Step 1, then G contains a path on at least 4 vertices, all of whose edges have multiplicity at least 2; this is exactly an m-subdivision of H_1. If it returns "yes" in Step 2, then there is a cycle C in B_i of size at least 4 and an edge $e \in E(B_i) \setminus E(C)$. If such edge connects a pair of vertices in the cycle, then $(V(C), E(C) \cup \{e\})$ is an m-subdivision of either H_4 or H_5. Otherwise, let uv be the endpoints of e and suppose that u is not in $V(C)$. As B_i is 2-conneccted, we have that there is a path P between u and C that does not contain v. We have that $(V(P) \cup V(C), E(P) \cup E(C))$ is an m-subdivision of H_5. If the algorithm returns "yes" in Step 3 or Step 4, one can easily see that G has either H_2 or H_3 as m-topological minor

Now, suppose that G has an m-topological minor H in $\mathscr{H}$. If $H = H_1$, then clearly a subdivision of H will be found in Step 1. If $H \in \{H_4, H_5\}$, then note that the m-subdivision of H must be contained in a 2-connected block, B_i, and that such block cannot be a simple cycle (i.e., a cycle where $|E(B_i)| = |V(B_i)|$). Therefore, the algorithm returns "yes" in Step 2. Observe now that, after Step 2, all 2-connected blocks are either simple cycles, or cycles of size 3. Therefore, if we arrived to this point without returning "yes", then we must have $H \in \{H_2, H_3\}$ and the vertex of each cycle of length 3 in H must form a block in G. One can then see that Steps 3 and 4 will detect H.

□

5.5 Multiedge Cut

Finally we prove hardness of $\leq h$-MULTIEDGE TEMPORAL S,Z-CUT. This is to close a gap in Table 2.

Theorem 5.5.1 $\leq h$-MULTIEDGE TEMPORAL S,Z-CUT *is* NP*-complete, even if* $\tau = 2$.

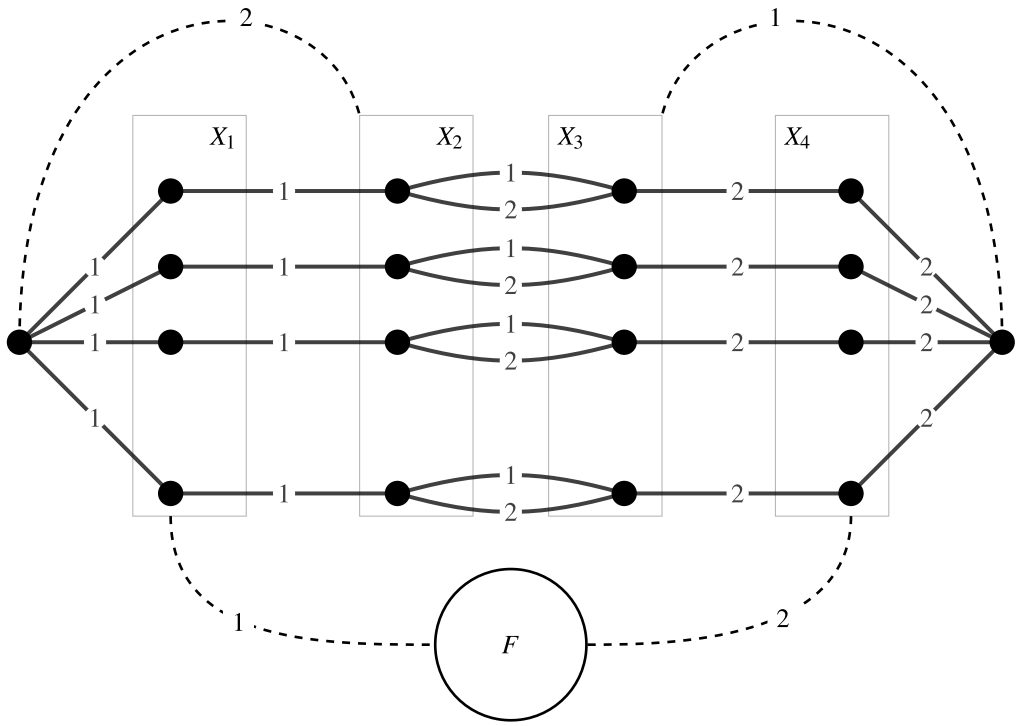
Proof. We make a reduction from Vertex Cover, which consists of, given a simple graph G and a positive integer k, deciding whether there exists a subset $S \subseteq V(G)$ such that $|S| \leq k$ and every $e \in E(G)$ is incident to some $u \in S$; such a set is called a *vertex cover* (of size at most k). So, consider $\mathscr{I} = (G,k)$ an instance of Vertex Cover. We construct a graph G' with vertex set $V(G') = \{s,z\} \cup \{x_v^1, x_v^2, x_v^3, x_v^4 \mid v \in V(G)\} \cup \{f_{vw} \mid vw \in E(G)\}$. One can use Figure 43 to follow the construction. Then, we add edges from s to x_v^1 and x_v^2, and from x_v^3 and x_v^4 to z, for every $v \in V(G)$. Also, let $(x_v^1, x_v^2, x_v^3, x_v^4)$ be a path, and add, for each edge $vw \in E(G)$, edges $x_v^1 f_{vw}$ and $f_{vw} x_w^4$. More formally, we have:

$$\begin{aligned} E(G') = \quad & \{sx_v^1, sx_v^2, x_v^3 z, x_v^4 z \mid v \in V(G)\} \\ & \cup \{x_v^i x_v^{i+1} \mid i \in \{1,2,3\}, v \in V(G)\} \\ & \cup \{x_v^1 f_{vw}, f_{vw} x_w^4 \mid vw \in E(G)\}. \end{aligned}$$

Finally, add a second edge with endpoints $x_v^2 x_v^3$, for each $v \in V(G)$. Since these are the only edges with multiplicity greater than 1, we will generally denote an edge by its endpoints, with the exception of these, which we denote by e_v^1, e_v^2. Now for each $i \in \{1,\dots,4\}$ define $X_i = \{x_v^i \colon v \in V\}$, let $F = \{f_{vw} \mid vw \in E(G)\}$ and consider a timefunction λ such that:

$$\lambda(e) = \begin{cases} 1 & \text{, if } e \in (\{s\} \times X_1) \cup (X_1 \times X_2) \cup (X_3 \times \{z\}) \cup (X_1 \times F) \\ 2 & \text{, if } e \in (\{s\} \times X_2) \cup (X_3 \times X_4) \cup (X_4 \times \{z\}) \cup (F \times X_4) \text{, and} \\ i & \text{, if } e = e_v^i \text{ for some } v \in V(G). \end{cases}$$

Figure 43 – A representation of temporal graph (G', λ). The dotted lines represent all the edges between the corresponding set of vertices



Source: elaborated by the author.

We prove that G has a vertex cover of size at most k if and only if (G', λ) has a multiedge temporal s,z-cut of size at most $n+k$. Given a solution S of Vertex Cover, we can define $S' = \{sx_v^1, x_v^4 z \colon v \in S\} \cup \{x_v^2 x_v^3 \colon v \notin S\}$. It remains to argue that S' disconnects s from z, or more formally, that there is no temporal s,z-path in $(G' - S^*, \lambda)$, where S^* contains every $e \in E(G')$ such that the endpoints of e are in S'. Notice that if a temporal s,z-path does not use edges between X_1 and F, then it contains one of the following paths: $(s, x_v^1, x_v^2, x_v^3, x_v^4, z)$, $(s, x_v^2, x_v^3, x_v^4, z)$, or $(s, x_v^1, x_v^2, x_v^3, z)$. In any case such path intersects S'. Now suppose that a temporal s,z-path, P, uses an edge $x_v^1 f_{vw}$, which implies that it also uses $f_{vw} x_w^4$ and therefore it arrives in z at timestep 2. Note that the only edges active at timeste 2 incident to x_w^4 are incident either to F, or to x_w^3, or to z. In the former case, we hit a dead end because f_{uw} has only one edge incident to it in timestep 2, for every $u \in N(w)$. This also happens in the second case, since from x_w^3 one can only go to x_w^2, hitting again a dead end. We then get that if P contains $x_v^1 f_{vw}$, then it must also contain $x_w^4 z$. By a similar argument one can also show that it must contain sx_v^1 too. As $vw \in E(G)$, at least one of v and w are in S, and by construction we get that P uses some multiedge of S', as we wanted to show.

Now let S' be a multiedge temporal s,z-cut of size at most $n+k$. For each $v \in V(G)$, let the set of multiedges $\{sx_v^1, x_v^1x_v^2, x_v^2x_v^3, x_v^3x_v^4, x_v^4z\}$ be denoted by A_v, and let $S'_v = S' \cap A_v$. Because A_v forms a temporal s,z-path, we know that $S'_v \neq \emptyset$. One can notice that every temporal s,z-path using $x_v^1x_v^2$ also uses sx_v^1, and in the same way every temporal s,z-path using $x_v^3x_v^4$ uses x_v^4t. Then by changing S' if necessary, we can suppose that S'_v is a non-empty subset of $\{sx_v^1, x_v^2x_v^3, x_v^4z\}$. Now we show that we can actually suppose that $S' \cap A_v$ is either $\{sx_v^1, x_v^4t\}$ or $\{x_v^2x_v^3\}$. We do it by analysing the cases where this does not happen.

- $S'_v = \{sx_v^1\}$ or $S'_v = \{x_v^4z\}$. We just solve the first subcase as the second is similar. Notice that $(s, x_v^2, x_v^3, x_v^4, z)$ is a temporal s,z-path, and that the only edge in this path that can be in S' is sx_v^2 because of the case being analyzed. So, removing sx_v^2 and adding x_v^4 to S' maintains the property of being a multiedge temporal s,z-cut and turns S'_v into $\{sx_v^1, x_v^4z\}$.
- $S'_v = \{sx_v^1, x_v^2x_v^3\}$ or $S'_v = \{x_v^2x_v^3, x_v^4z\}$. In both subcases we can remove $x_v^2x_v^3$ and add either x^4z (in the first case) or sx_v^1 (in the second one).

Now we can define $S = \{v \in V \colon S_v = \{sx_v^1, x_v^4t\}\}$. The desired property $|S| \leq k$ follows from the fact that $1 \leq |S'_v| \leq 2$ for every $v \in V(G)$, and that $|S'| \leq n+k$. Finally, suppose that $vw \in E(G)$ is such that $S \cap \{v,w\} = \emptyset$. Then (s, x_v^1, x_w^4, z) is a temporal s,z-path not passing through the edges in S', a contradiction. □

5.6 Concluding remarks

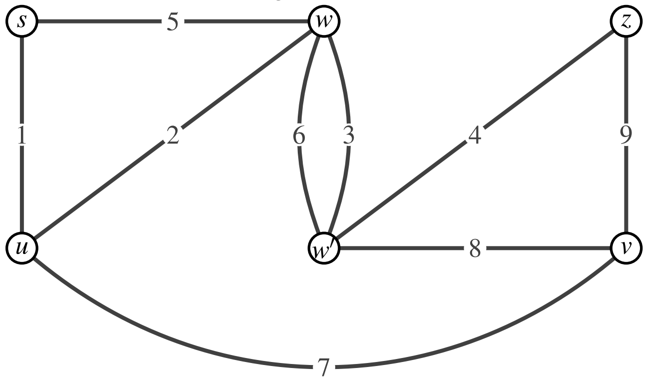
In this chapter, we have introduced the concept of snapshot disjointness and proved that the related paths and cut problems, when parameterized by the size of the solution, are both W[1]-hard and XP-time solvable. We then adapted to our context the definition of Mengerian graphs introduced by Kempe, Kleinberg and Kumar (12), giving also a characterization in terms of forbidden m-topological minors, as well as a polynomial-time recognition algorithm. From Table 2, we extract the following open problem.

Question 10 *Is* $\geq k$-MULTIEDGE DISJOINT TEMPORAL S,Z-PATHS FPT *when parameterized by k?*

Question 11 *Is* $\leq h$-MULTIEDGE TEMPORAL S,Z-CUT FPT *when parameterized by h?*

In (27), the authors present a disjointness concept that can be viewed as a kind of edge disjointness too. In this concept, two temporal paths are considered *node departure disjoint* if they do not leave the same vertex at the same time. Additionally, a *node departure s,z-cut* is a

Figure 44 – A temporal graph (G,λ) on which the multiedge version of Menger's Theorem does not hold



Source: elaborated by the author.

set S of temporal vertices, such that for every temporal path from s to z, there is a temporal vertex $(u,t) \in S$ such that the temporal path leaves vertex u at time t. They then prove that a version of Menger's Theorem holds in this context, i.e., that the related maximum number of paths and minimum cut size are always equal. However, they work only on strictly increasing temporal paths. We then ask whether the same holds with the temporal paths investigated in this work.

Question 12 *Given a temporal graph $\mathcal{G}$ and vertices s,z, if non-strict temporal paths are considered, is the maximum number of node-departure disjoint s,z-paths always equal to the minimum size of a node-departure cut?*

Finally, while Menger's Theorem is known to hold for edge disjoint paths (3, 27), it is also known not to hold for multiedge disjoint paths as seen in Section 5.5. Therefore, we can again define Mengerian graphs in such context and ask whether such graphs can be characterized. We can do the same for node-departure time disjointness.

Question 13 *Can Mengerian graphs in the context of multiedge disjointness and in the context of node-departure time disjointness be characterized by forbidden m-topological minors? Can they be recognized in polynomial time? Are all the different types of Mengerian graphs related?*

Finally, we comment about the edge version of Menger. In (3), Berman proves that the temporal version of Menger's Theorem applied to edges always holds. It should be noted that Berman uses the same model as us, i.e., an edge has always a single appearance and multiple edges are allowed. If, instead, the paths are not allowed to share distinct edges that have the same endpoints (i.e., they are multiedge disjoint), then Menger's Theorem also does not necessarily hold. To see this, observe again Figure 44. Note that every temporal s,z-path passing by sw must use ww' (at time 6) and vz; and every temporal s,z-path passing by su uses either ww' (at any time) or vz. Therefore there are no two multiedge disjoint temporal s,z-paths. Because there is no $xy \in E(U(G))$ such that the removal of every edge with endpoints xy breaks all the temporal s,z-paths, we get that the multiedge version of Menger's Theorem also does not hold. Then it is natural to ask whether there is a characterization of Mengerian graphs on the multiedge context.

Question 14 *Let $\mathscr{G}$ contain every graph G for which the multiedge version of Menger's Theorem holds on (G,λ) for every timefunction λ. Can $\mathscr{G}$ be characterized in terms of forbidden structures?*

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