UNIVERSIDADE FEDERAL DO CEARÁ
CENTRO DE CIÊNCIAS
DEPARTAMENTO DE MATEMÁTICA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA
DOUTORADO EM MATEMÁTICA

JOSÉ DANUSO ROCHA DE OLIVEIRA

MEAN CURVATURE FLOW SOLITONS IN THE HYPERBOLIC SPACE

FORTALEZA
2023

JOSÉ DANUSO ROCHA DE OLIVEIRA

MEAN CURVATURE FLOW SOLITONS IN THE HYPERBOLIC SPACE

Tese apresentada ao Programa de Pós-Graduação em Matemática do Centro de Ciências da Universidade Federal do Ceará, como requisito parcial à obtenção do título de doutor em Matemática. Área de Concentração: Matemática.

Orientador: Prof. Dr. Luciano Mari.

Coorientador: Prof. Dr. Andreas Savas-Halilaj.

FORTALEZA

2023

Dados Internacionais de Catalogação na Publicação
Universidade Federal do Ceará
Sistema de Bibliotecas
Gerada automaticamente pelo módulo Catalog, mediante os dados fornecidos pelo(a) autor(a)

O47m Oliveira, José Danuso Rocha de.
Mean curvature flow solitons in the hyperbolic space / José Danuso Rocha de Oliveira. – 2023.
83 f. : il.

Tese (doutorado) – Universidade Federal do Ceará, Centro de Ciências, Programa de Pós-Graduação em Matemática , Fortaleza, 2023.
Orientação: Prof. Dr. Luciano Mari.
Coorientação: Prof. Dr. Andreas Savas-Halilaj.

1. Mean curvature flow. 2. Solitons. 3. Hyperbolic space. I. Título.
CDD 510

JOSÉ DANUSO ROCHA DE OLIVEIRA

MEAN CURVATURE FLOW SOLITONS IN THE HYPERBOLIC SPACE

Tese apresentada ao Programa de Pós-Graduação em Matemática do Centro de Ciências da Universidade Federal do Ceará, como requisito parcial à obtenção do título de doutor em Matemática. Área de Concentração: Matemática.

Aprovada em: 26/06/2023

BANCA EXAMINADORA

Prof. Dr. Luciano Mari. (Orientador)
Università degli Studi di Torino

Prof. Dr. Andreas
Savas-Halilaj. (Coorientador)
University of Ioannina (UOI)

Prof. Dr. Marco Magliaro
Universidade Federal do Ceará

Prof. Dr. Eddygledson Souza Gama
Universidade Federal de Pernambuco

Prof. Dra. Fernanda Roing
Universidade Federal do Ceará

I dedicate this work to my family and friends
who gave me the emotional support to finish it.

ACKNOWLEGEDMENTS

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

Throughout this journey, I've had the support of various individuals who have contributed in both personal and professional capacities to my thesis. I would like to mention all of these people, but I don't have enough space for such a long list. It falls upon me to mention a small portion of those who have helped me, but if you are not mentioned, know that I hold you in great affection.

I am immensely grateful to my advisor Luciano Mari and my co-advisor Andreas Savas-Halilaj for the countless hours dedicated to the issues presented in this thesis. Without their patience, this work would not have the quality it exhibits. Thank you for being a compass in this stormy sea that is mathematical research.

I extend my deepest gratitude to the committee, Marco Magliaro, Eddygledson Gama, and Fernanda Roing, for their thorough reading and numerous corrections and suggestions that enriched the thesis.

I express my profound gratitude to the professors of the Mathematics department at UFC who provided me with a solid foundation of knowledge, especially those with whom I had more contact: professors Darlan, Jorge, Caminha, Fernanda, Diego and Daniel. Keep up the good work. I thank Andrea and Jessyca for their help with all the university bureaucracy.

To my friends inside or outside of Mathematics, thank you very much for your help!

I would like to thank my parents, Adriano and Rosa, for believing in the dream of the boy Danuso to become a doctor in Mathematics, which has now come true. Thank you for all support my brothers, Ary and Rafael, and nephew João. Without them, none of this would be possible.

Finally, my sincerest thanks to my beloved, Larissa Ribeiro, for the support and encouragement and for making my life more colorful and fun. I love you, my beautiful.

“Não nego meu sangue, não nego meu nome, Olho para fome e pergunto: o que há? Eu sou brasilêro, fio do Nordeste, Sou cabra da peste, sou do Ceará” (Assaré, 2013, p.67).

RESUMO

Nesta tese, nós estudamos soluções autosimilares para fluxo da curvatura média no espaço hiperbólico. Depois de relembrar alguns fatos gerais sobre solitons em ambientes gerais munidos com uma métrica produto torcido (warped product metric), nós focamos em solitons no espaço hiperbólico cujo fluxo, na direção de expansão, pelo campo conforme cujas trajetórias são ortogonais às horoesferas. Primeiramente, nós estudamos sua estabilidade, fornecendo uma condição suficiente. Em particular, solitons, que são (convenientemente) gráficos, são estáveis. Em seguida, nós investigamos solubilidade do Problema de Plateau no infinito. Por meio de técnicas de equações diferenciais ordinárias, nós caracterizamos exemplos cilíndricos e rotacionalmente simétricos, mostrando uma analogia estrita com solitons de translação (translating solitons ou translators) no espaço euclidiano. De fato, as soluções são os análogos apropriados do grim-reaper, bowl e winglike no espaço euclidiano. Por fim, sob algumas condições adicionais, nós caracterizamos o grim-reaper como o único soliton cuja a fronteira assintótica é dois planos paralelos. Um par de apêndices contém algum material auxiliar sobre varifolds e a fronteira assintótica de variedades Cartan-Hadamard.

Palavras-chave: soliton; fluxo pela curvatura média; espaço hiperbólico; translator; problema de Plateau assintótico; fronteira assíntotica.

ABSTRACT

In this thesis, we study self-similar solutions to the mean curvature flow in the hyperbolic space. After recalling some general facts about solitons in ambient spaces endowed with a warped product metric, we focus on solitons in hyperbolic space which flow, in the expanding direction, by the conformal field whose trajectories are orthogonal to horospheres. First, we study their stability, supplying a sufficient condition. In particular, solitons which are (suitably) graphical are stable. Next, we investigate the solvability of Plateau's problem at infinity. By means of ODE techniques, we then characterize cylindrical and rotationally symmetric examples, showing an analogy with translating solitons in Euclidean space. Indeed, the solutions are appropriate analogies of the grim-reaper, bowl, and winglike translators in Euclidean space. Eventually, under some additional conditions, we characterize the grim-reaper as the only soliton whose boundary at infinity are two parallel hyperplanes. A pair of appendices contain some auxiliary material about varifolds and the boundary at infinity of Cartan-Hadamard manifolds.

Keywords: soliton; mean curvature flow; hyperbolic Space; translator; asymptotic Plateau problem; boundary at infinity.

LIST OF FIGURES

CONTENTS

1 INTRODUCTION

In this thesis, we present the concept of mean curvature flow (MCF for short), in which a submanifold flows in the direction of its mean curvature vector field. MCF was vastly studied in the Euclidean space. An interesting problem related to MCF: given a vector field $X \in \mathfrak{X}(N)$ on the ambient space $(N, \overline{g})$, we find a submanifold $M \subset N$ such that, up a reparametrization of M, the MCF of M is moving along flow lines of the vector field X. A submanifold M with the property above is known as self-similar soliton (soliton for short).

In Euclidean space $\mathbb{R}^{m+1}$, there are symmetric solitons with respect to parallel vector field $X = \partial_0$. For instance, a grim-reaper cylinder $\mathcal{G}$ is a soliton given by the Cartesian product of a profile curve Γ by $\mathbb{R}^{m-1}$ (see Example 2.1.5). Other examples of solitons are bowl solitons and winglike solitons. Bowl solitons are rotationally symmetric solitons that can be written as an entire graph of a slice $\{0\} \times \mathbb{R}^m \subset \mathbb{R}^{m+1}$ (see Example 2.1.6). Winglike solitons are rotationally symmetric solitons that can be written as bigraph over the complement of a ball $\{0\} \times (\mathbb{R}^m \backslash B_R) \subset \mathbb{R}^{m+1}$ (see Example 2.1.7).

In this thesis, we find appropriated analogies to grim-reaper, winglike soliton and bowl soliton in the hyperbolic space with respect to the conformal field $X = \partial_0$ (see sections 4.2, 4.7 and 4.8). Eventually in Chapter 6, we prove that grim-reaper and vertical hyperplane are the only solitons that asymptote at infinity two parallel $(m-1)$-hyperplanes outside of a cylinder (see Theorem 6.3.1).

We organize the text in the following way: in Chapter 2, we present the definition of solitons in a more general context where the Riemannian manifold is equipped with a warped product metric and some results needed throughout the thesis. Among them, we define the Ilmanen space, where the solitons correspond to minimal manifolds (Equation (2.1.8)). In Chapter 3, we derive some basic formula when the ambient space is hyperbolic space and we prove stability for graphical soliton over the boundary at infinity. In Chapter 4, we study the solitons that can be written as graphs over some subset of the boundary at infinity $\partial_\infty \mathbb{H}^{m+1}$ (see appendix A). We compute the quasilinear equation arising from the soliton identity and recall the basic Comparison and Tangency Principle (Theorem 4.1.2 and Theorem 4.1.3). In Section 4.2, we examine solitons that can be written as the Cartesian product of a profile curve Γ by $\mathbb{R}^{m-1}$, $M = \Gamma \times \mathbb{R}^{m-1} \subset \mathbb{H}^{m+1}$ (grim-reaper soliton, Lemma 4.2.1) as in the Euclidean space. In Section 4.3, we construct subsolutions for Soliton Equation (SE_-) that act as barriers to prove some of our main results. In Section 4.4 through 4.8, we study rotationally symmetric solitons. The main outcome is that there exist only two families of this kind of solitons which we call bowl and winglike soliton in analogy with translators in the Euclidean space. In Chapter 5, as in the classical Plateau problem for minimal submanifolds, we prove that for a given compact subset of the boundary at infinity there exists a soliton with boundary at infinity equal to the given subset. We describe the geodesics of Ilmanen space. In chapter 6, we prove that the grim-reaper and the vertical hyperplane are the only solitons with respect to $X = -\partial_0$ that are asymptotic at infinity to parallel $(m-1)$-hyperplane outside a cylinder (GR property, Definition 6.1.5)

2 PRELIMINARIES

2.1 Solitons in warped product spaces

Throughout this thesis M will be a m-dimensional manifold and N an $(m+1)$-dimensional manifold equipped with a fixed Riemannian metric g. Let (ω_1, ω_2) be an interval containing 0 and $\Psi : (\omega_1, \omega_2) \times M \to N$ a smooth map. We say that Ψ is a *one-parameter family of immersions* if the map $\Psi_t : M \to N$ given by

$$\Psi_t(x) = \Psi(t, x)$$

for any $x \in M$, is an immersion. We often will denote by M_t the image of M in N via the immersion Ψ_t.

Definition 2.1.1. *An one-parameter family of immersions $\Psi : (\omega_1, \omega_2) \times M \to N$ is called solution to the* mean curvature flow (MCF for short) *if it satisfies the differential equation*

$$\partial_t \Psi(t, x) = \mathbf{H}(t, x)$$

for any $(t, x) \in (\omega_1, \omega_2) \times M$, where $\mathbf{H}(t, x)$ is the non-normalized mean curvature vector *of the immersion Ψ_t at the point $x \in M$.*

Let us discuss now a special class of solutions to the mean curvature flow, following the exposition in (Alías *et al.*, 2020).

Definition 2.1.2 (Self-similar Soliton)**.** *Let $X \in \mathfrak{X}(N)$ be a smooth vector field and $\Phi : (\sigma_1, \sigma_2) \times N \to N$ its associated flow, defined in the time-interval $(\sigma_1, \sigma_2) \subset \mathbb{R}$. A solution $\Psi : (\omega_1, \omega_2) \times M \to N$ to the mean curvature flow is called a* ***self-similar soliton with respect to the vector field*** *$X \in \mathfrak{X}(N)$ if there exists an immersion $\psi : M \to N$, a reparametrization $s : (\omega_1, \omega_2) \to (\sigma_1, \sigma_2)$ of the flow lines of X and a one-parameter family of diffeomorphisms $\eta : (\omega_1, \omega_2) \times M \to M$ such that*

$$\Psi(t, x) = \Phi\big(s(t), \psi(\eta(t, x))\big), \tag{2.1.1}$$

for any $(t, x) \in (\omega_1, \omega_2) \times M$.

Roughly speaking, such a solution $M_t \subset N$ to the MCF is moving along the flow lines of the vector field X. Differentiating the identity (2.1.1) with respect to t and estimating at

$t = 0$, we obtain the *soliton equation*

$$\mathbf{H} = s'(0)X^{\perp}$$

where $\{\cdot\}^{\perp}$ is the orthogonal projection on the normal bundle of ψ. Without loss of generality, we may assume that $s'(0) = 1$. Let $\psi : M \longrightarrow N$ be an immersion satisfying the partial differential equation

$$\mathbf{H} = X^{\perp}.$$

Definition 2.1.3. *An isometric immersion $\psi : M \longrightarrow N$ satisfying the differential equation*

$$\mathbf{H} = X^{\perp},$$

is called ***soliton solution to the MCF*** *(**soliton** for short) with respect to X.*

Although in this thesis we shall consider solitons in the hyperbolic space $\mathbb{H}^{m+1}$, it is useful to recall some properties that hold true for solitons in warped product spaces. Suppose that $I \subset \mathbb{R}$ is an open interval, $h : I \longrightarrow (0, +\infty)$ is a smooth function, P is a m-dimensional manifold equipped with a metric g_P and $N = I \times_h P$ the manifold equipped with the metric

$$\mathrm{g} = \mathrm{d}s^2 + h^2 \mathrm{g}_P. \tag{2.1.2}$$

Let $\pi_I : I \times_h P \longrightarrow I$ and $\pi_P : I \times_h P \longrightarrow P$ be the natural projections onto the first and second factor of N, respectively. Hence, any $Y \in \mathfrak{X}(N)$ can be decomposed in the form $Y_I + Y_P$, where $Y_I \in \mathfrak{X}(I)$ and $Y_P \in \mathfrak{X}(P)$. From the Koszul formula, it follows that the Levi-Civita connection D of N is given by

$$D_Y Z = D_Y^{I\times P} Z + Y_I(\log h) Z_I + Z_I(\log h) Y_I - \mathrm{g}_P(Y_P, Z_P) h h' \partial_s, \tag{2.1.3}$$

where $D^{I\times P}$ is the connection of the Riemannian product metric on $I \times P$. In warped product manifolds there exist two "canonical"vector fields which generate the tangent space of the fibers $\pi_I^{-1}(s)$, namely those defined by $X_{\pm} = \pm h\,\partial_s$. It turns out that the smooth function $f_{\pm} \in C^{\infty}(N)$ is the potential of $X_{\pm}$ given by

$$f_{\pm}(x) = \pm \int_{s_0}^{\pi_I(x)} h(\sigma)\mathrm{d}\sigma, \tag{2.1.4}$$

where $s_0 \in I$ is a fixed number. Indeed,

$$X_{\pm} = \pm h\,\partial_s = D f_{\pm}. \tag{2.1.5}$$

Moreover, by (2.1.3), the Hessian $D^2 f_\pm$ of $f_\pm$ is given by the formula

$$D^2 f_\pm(Y,Z) = \mathrm{g}(D_Y X_\pm, Z) = \pm h' \mathrm{g}(Y,Z), \tag{2.1.6}$$

for any $Y,Z \in \mathfrak{X}(N)$. Consequently, an immersion $\psi : M \to I \times_h P$ is a soliton of the MCF if its mean curvature satisfies the equation

$$\mathbf{H} = D f_\pm^\perp.$$

Example 2.1.4. *The warped product model $\mathbb{H}^{m+1} = \mathbb{R} \times_{e^{-s}} \mathbb{R}^m$ with Riemannian metric*

$$g_{\mathbb{H}} = \mathrm{d}s^2 + e^{-2s} \sum_{i=1}^{m} \mathrm{d}x_i^2,$$

where $(s; x_1, \ldots, x_m) \in \mathbb{R} \times \mathbb{R}^m$. Therefore, the associated conformal field is $X_\pm = \pm e^{-s} \partial_s$.

Soliton solutions to the mean curvature flow share many similarities with minimal submanifolds; see for example (Colding; Minicozzi, a), (Colding; Minicozzi, b), (Ilmanen, 1994) and (Smoczyk, 2001). According to ideas developed by Ilmanen in (Ilmanen, 1994, Chapter 2), there is a duality between solitons in $(I \times_h P, \mathrm{g})$ and minimal hypersurfaces in $(I \times_h P, \mathrm{g}_{f_\pm})$, where $\mathrm{g}_{f_\pm}$ is the metric given by

$$\mathrm{g}_{f_\pm} \doteq e^{\frac{2f_\pm}{m}} \mathrm{g}. \tag{2.1.7}$$

By a straightforward computation, it follows that the mean curvature $\widetilde{\mathbf{H}}$ of the isometric immersion $\psi : M \to (I \times_h P, \mathrm{g}_{f_\pm})$ relates to the mean curvature $\mathbf{H}$ of $\psi : M \to (I \times P, \mathrm{g})$ by the formula

$$\widetilde{\mathbf{H}} = e^{-\frac{2f_\pm}{m}} \left(\mathbf{H} - D f_\pm^\perp\right) = e^{-\frac{2f_\pm}{m}} \left(\mathbf{H} - X_\pm^\perp\right). \tag{2.1.8}$$

Consequently, solitons in the warped product manifold $(I \times_h P, \mathrm{g})$ correspond to minimal hypersurfaces in $(I \times P, \mathrm{g}_{f_\pm})$ and vice-versa. The Riemannian metric $\mathrm{g}_{f_\pm}$ is known in the literature as the **Ilmanen metric**.

There is another equivalent way to express a soliton as a minimal hypersurface. Consider the *weighted Riemannian manifold* $(I \times_h P, \mathrm{g}, e^{f_\pm} \mathrm{d}N)$ where $\mathrm{d}N$ is the volume form of $I \times_h P$ with respect to the metric g. Suppose that $M \subset I \times_h P$ is an immersed hypersurface. Then, the *weighted volume* of the hypersurface M is defined by

$$\mathrm{Vol}_{f_\pm}(M) \doteq \int_M e^{f_\pm} \mathrm{d}M$$

where dM is the volume form of M with respect to the volume induced from the Riemannian metric g. Following Gromov (Gromov, 2003), the vector field

$$\mathbf{H}_{f_\pm} \doteq \mathbf{H} - Df_\pm^\perp,$$

where $\mathbf{H}$ is the mean curvature of $M \subset (I \times_h P, \mathrm{g})$, is called the $f_\pm$-*mean curvature* of the hypersurface M.

For any normal variation M_t of M, with respect to a compactly supported variation normal along M with velocity vector field Z, the first variation formula for the weighted volume is given by

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Vol}_{f_\pm}(M_t) = -\int g(Z, \mathbf{H}_{f_\pm}) e^{f_\pm} \mathrm{d}M,$$

see for example (Ilmanen, 1994, Chapter 2). Consequently, M is a critical point of the weighted volume if

$$\mathbf{H}_{f_\pm} = \mathbf{H} - Df_\pm^\perp = 0.$$

Hypersurfaces of $(I \times_h P, \mathrm{g}, e^{f_\pm}\mathrm{d}N)$ with zero $f_\pm$-mean curvature are called $f_\pm$-*minimal hypersurfaces*. Therefore, there is a one-to-one correspondence between solitons and $f_\pm$-minimal hypersurfaces. Summarizing the notions of solitions, $f_\pm$-minimal hypersurfaces , and minimal hypersurfaces with respect to the Ilmanen metric are equivalent.

2.1.1 Solitons in the Euclidean space

In this subsection, we give some examples of solitons in the Euclidean space. For more details, we refer to (Martín *et al.*, 2019) and (Gama; Martín, 2020); see also (Alías *et al.*, 2020) for a more general setting.

Example 2.1.5 (Euclidean space as a Riemannian product)**.** *Using the same notation as above,* $N = \mathbb{R}^{m+1} = \mathbb{R} \times \mathbb{R}^m$*,* $I = \mathbb{R}$*,* $P = \mathbb{R}^m$*, and* $h \equiv 1$*. Hence, the parallel vector field* $X_\pm = \pm\partial_0$*, where* $x = (x_0, x_1, x_2, \ldots, x_m) \in \mathbb{R}^{m+1}$*. An example of soliton with respect to* $X_\pm = \pm\partial_0$ *is the vertical hyperplane* $\pi_v = \{x = (x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1} : x_1 = c\}$*, where* $c \in \mathbb{R}$ *is a constant. Any rotation of* π_v *in which* $\pm\partial_0$ *remains tangent to the hyperplane is another example of soliton with respect to* $\pm\partial_0$*. Another example of soliton with respect to* $X = \partial_0$ *is* ***the cylinder over the grim reaper***

curve *(or* ***grim reaper cylinder****):*

$$\mathcal{G} = \left\{(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1} : -\frac{\pi}{2} < x_1 < \frac{\pi}{2}, x_0 = -\log(\cos x_1)\right\}.$$

More generally, for $\theta \in \left[0, \frac{\pi}{2}\right)$, ***the tilted grim reaper*** *is given by:*

$$\mathcal{G}_\theta := \left\{(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1} : -\frac{\pi}{2\cos\theta} < x_1 < \frac{\pi}{2\cos\theta}, x_0 = -\sec^2(\theta)\log(x_1\cos\theta) + \tan(\theta)x_m\right\},$$

for more details, see (Gama; Martín, 2020). In this context, the soliton is called ***translating soliton*** *(or* ***translator****) because the mean curvature flow is given by a set-wise translation in the direction of $X_\pm = \pm\partial_0$.*

Example 2.1.6 (Bowl soliton)**.** *It was shown by (Altschuler; Wu, 1994) and (Clutterbuck et al., 2007) that there does exist an entire rotationally symmetric, strictly convex graphical soliton*

$$M = \{(u(\rho), x_1, \cdots, x_m) \in \mathbb{R}^{m+1} : \rho^2 = x_1^2 + \cdots + x_m^2\}$$

and the function $u : (0, \infty) \to \mathbb{R}$, $m \geq 2$, has the following asymptotic expansion at infinity

$$u(\rho) = \frac{\rho^2}{2(m-1)} - \frac{1}{2}\log\rho^2 + O\left(\frac{1}{\rho}\right)$$

for more detail see Lemma 2.2 in (Clutterbuck et al., 2007). This solution is called the translating paraboloid or the ***bowl soliton****.*

Example 2.1.7 (Winglike soliton)**.** *Given a radius R, we can construct an example of soliton that can be written as a bigraph over $\mathbb{R}^m \backslash B_R(0)$:*

$$M = \{(u_-(\rho), x_1, \cdots, x_m) \in \mathbb{R}^{m+1} : \rho \geq R\} \cup \{(u_+(\rho), x_1, \cdots, x_m) \in \mathbb{R}^{m+1} : \rho \geq R\}$$

where u_+, u_- are a solution to rotationally symmetric soliton equation in the Euclidean space and

$$u_\pm = \frac{\rho^2}{2(m-1)} - \ln\rho + O(\rho^{-1}) + C^\pm.$$

For more details, see Lemma 2.3 in (Clutterbuck et al., 2007). We call a soliton as M a ***winglike catenoid translator****.*

Example 2.1.8. *In general, for $N = \mathbb{R} \times_h P$, with $h \equiv 1$, $\{c\} \times P$ is a soliton with respect $X = \pm\partial_0$ (see Example 2.1 in (Alías et al., 2020)).*

Example 2.1.9 (Euclidean space in polar coordinates)**.** *Let $N = \mathbb{R}^{m+1} - \{0\}$, $I = (0,\infty)$, $P = \mathbb{S}^m$ with the standard metric, $h(s) = s$. Hence, $\mathrm{g} = ds^2 + s^2 g_P$. Consider the vector field $X\pm = \pm s\partial_s$. One example of a solution with respect to $X_\pm$ is the sphere called **self-expander** for X_+ and **self-shrinkers** for X_-.*

2.2 The geometric maximum principle

Since solitons can be regarded as minimal hypersurfaces, we can use the geometric maximum principle. According to this maximum principle, two different solitons of the MCF cannot "touch" each other at one interior or boundary point; for more details see for example (Eschenburg, 1989). More precisely the following holds true:

Theorem 2.2.1. *Let M_1 and M_2 be embedded oriented connected submanifolds of a manifold (possibly with boundary) $\mathcal{N}$ with unit normals ν_1 and ν_2, corresponding mean curvatures H_1 and H_2, and boundaries ∂M_1 and ∂M_2, respectively.*

(a) ***(Interior principle)*** *Suppose that there exists a common point x in the interior of M_1 and M_2 where $\nu_1(x) = \nu_2(x)$, M_1 lies above M_2 in a neighborhood U of x, and $H_1 \leq a \leq H_2$ therein, for a constant a. Then $M_1 \cap U = M_2 \cap U$.*

(b) ***(Boundary principle)*** *Let W_1 and W_2 be open domains with connected C^2-boundaries $\partial W_1 = M_1$ and $\partial W_2 = M_2$ intersecting $\partial\mathcal{N}$ transversally. Suppose that there exists a point x in $M_1 \cap M_2 \cap \partial\mathcal{N}$ such that $\nu_1(x) = \nu_2(x)$, M_1 lies above M_2 in a neighborhood U of x, and $H_1 \leq a \leq H_2$ therein, for a constant a. Then $M_1 \cap U = M_2 \cap U$.*

2.3 Second variation and stability

Let us suppose now that $M \subset N = I \times_h P$ is a two-sided soliton solution to the MCF. We denote by ν the oriented unit normal vector field along the hypersurface. The second variation formula is

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Vol}_{f_\pm}(M_t) = \int \left(|D^\perp Z|^2 - \left(|\mathrm{II}|^2|Z|^2 + \mathrm{Ric}^N(Z,Z) - D^2 f_\pm(Z,Z)\right)\right)e^{f_\pm}\mathrm{d}M,$$

where Z is a normal variational vector field along M, II is the second fundamental form of the immersion and Ric^N the Ricci curvature of the ambient space. Since M is assumed to be two-sided, any normal along M vector field Z can be written in the form $Z = \varphi\eta$, where φ is a smooth function and η is a globally defined unit normal vector field. Then, the right hand side of *the second variational formula gives rise to the quadratic form*

$$Q_{f_\pm}(\varphi,\varphi) \doteq \int \left(|\nabla\varphi|^2 - \left(|\mathrm{II}|^2\varphi^2 + \mathrm{Ric}^N(\nu,\nu) - D^2 f_\pm(\nu,\nu)\right)\varphi^2\right)e^{f_\pm}\mathrm{d}M,$$

where φ *is a compactly supported function. Integrating by parts, we obtain that*

$$Q_{f_\pm}(\varphi,\varphi) = -\int \varphi J\varphi e^{f_\pm}\mathrm{d}M,$$

where

$$J\varphi \doteq \Delta_{f_\pm}\varphi + \left(|\mathrm{II}|^2 + \mathrm{Ric}^N(\nu,\nu) - D^2 f_\pm(\nu,\nu)\right)\varphi, \tag{2.3.1}$$

and

$$\Delta_{f_\pm}\varphi \doteq \Delta\varphi + \mathrm{g}(Df_\pm,\nabla\varphi),$$

for any $\varphi \in C^2(M)$*; for details see for example (Barbosa et al., 2017). The form* $Q_{f_\pm}$ *is called the* stability operator*,* J *is called the* Jacobi operator *and* $\Delta_{f_\pm}$ *is called the* weighted Laplace operator.

Definition 2.3.1. *A* $f_\pm$*-minimal hypersurface* $M \subset N$ *is called* stable *if it holds* $Q_{f_\pm}(\varphi,\varphi) \geq 0$ *for all compactly supported functions* $\varphi \in C_c^\infty(M)$*. Otherwise, the hypersurface is called* unstable*. Similarly, a compactly supported domain* $\Omega \Subset M$ *is called* stable *if* $Q_{f_\pm}(\varphi,\varphi) \geq 0$ *for all* $\varphi \in C^\infty(\Omega)$*.*

Let us recall here the following stability result, proved independently by Fischer-Colbrie & Schoen (Fischer-Colbrie; Schoen, 1980, Section 1) and Allegretto (Allegretto, 1981), Moss & Piepenbrink (Moss; Piepenbrink, 1978).

Theorem 2.3.2. *Let* $\Omega \Subset M$ *be an open subset. Then, the following conditions are equivalent:*

- Ω *is stable,*
- *There exists* $w \in H^1_{\mathrm{loc}}(\Omega)$*,* $w > 0$ *such that* $Jw \leq 0$ *weakly on* Ω*;*
- *There exists* $w \in C^\infty(\Omega)$*,* $w > 0$ *such that* $Jw = 0$ *on* Ω*;*

In the next proposition, we compute the Jacobi operator of the non-normalized scalar mean curvature of a soliton of the mean curvature flow lying in an arbitrary warped product space $I \times_h P$*.*

Proposition 2.3.3. *Let $M \subset N = I \times_h P$ be a 2-sided soliton with respect to the vector field $X_\pm = \nabla f_\pm = \pm h\partial_s$. Then, the scalar mean curvature $H = \mathrm{g}(\mathbf{H}, \nu)$ satisfies*

$$JH = \pm h \operatorname{Ric}^N(\nu, \partial_s) \mp 2h'H, \tag{2.3.2}$$

where ν is the oriented unit normal of the hypersurface.

Proof. Let H be the scalar mean curvature of the soliton $M \subset N$, that is

$$H = \mathrm{g}(\mathbf{H}, \nu).$$

Then, from the soliton equation, we deduce that

$$H = \mathrm{g}(X_\pm, \nu) = \mathrm{g}(Df_\pm, \nu).$$

Furthermore, recall from the equation (2.1.6) that

$$D^2 f_\pm = \pm h' \,\mathrm{g}. \tag{2.3.3}$$

Let $\{e_1, \dots, e_m\}$ be a local orthonormal tangent frame, which is normal at a fixed point $p \in M$, and denote by b_{ij} the coefficients of $\mathrm{I\!I}$ with respect to the aforementioned frame field. From the Codazzi equation, we have

$$b_{iji} = b_{iij} + R^N_{\nu iij},$$

for any $i, j \in \{1, \dots, m\}$, where here R^N stands for the curvature operator of N. Let us compute now the gradient and the Laplacian of the scalar mean curvature H. We have,

$$e_i \mathrm{g}(X_\pm, \nu) = -b_{ij}\, \mathrm{g}(X_\pm, e_j),$$

for any $i \in \{1, \dots, m\}$. Moreover, by differentiating and estimating at p, we get

$$\begin{aligned}
e_i e_i \,\mathrm{g}(X_\pm, \nu) &= -b_{iji}\, \mathrm{g}(X_\pm, e_j) \mp b_{ij} h' \delta_{ij} - b_{ij}\, \mathrm{g}(X_\pm, D_{e_i} e_j) \\
&= -(b_{iij} + R^N_{\nu iij})\, \mathrm{g}(X_\pm^\top, e_j) + h'H - b_{ij} b_{ij}\, \mathrm{g}(X_\pm, \nu) \\
&= -\mathrm{g}(X_\pm^\top, \nabla H) \mp h'H + \operatorname{Ric}^N(\nu, X_\pm^\top) - |\mathrm{I\!I}|^2 \mathrm{g}(X_\pm, \nu) \\
&= -\mathrm{g}(Df_\pm, \nabla H) \mp h'H + \operatorname{Ric}^N(\nu, X_\pm - H\nu) - |\mathrm{I\!I}|^2 H \\
&= -\mathrm{g}(Df_\pm, \nabla H) \mp h'H \pm h \operatorname{Ric}^N(\nu, \partial_s) - \operatorname{Ric}^N(\nu, \nu) H - |\mathrm{I\!I}|^2 H,
\end{aligned}$$

whereby $\{\cdot\}^\top$ we denote the orthogonal projection on the tangent bundle of the hypersurface. From the last equality and (2.3.1) and (2.3.3), we immediately deduce

$$JH = \pm h \operatorname{Ric}^N(\nu, \partial_s) \mp 2h'H.$$

This completes the proof. ◻

3 SOLITONS IN THE HYPERBOLIC SPACE

We restrict ourselves now to the case of solitons in the hyperbolic space $(\mathbb{H}^{m+1}, g_{\mathbb{H}})$ of constant sectional curvature -1. We will occasionally use the following models of the hyperbolic space:

(a) The **warped product model** (Example 2.1.4).

(b) The **half-space model** $\mathbb{H}^{m+1} = \mathbb{R}^+ \times \mathbb{R}^m$ with metric

$$g_{\mathbb{H}} = \frac{1}{x_0^2} \sum_{i=0}^{m} \mathrm{d}x_i^2,$$

where $(x_0; x_1, \dots, x_m) \in \mathbb{R}^+ \times \mathbb{R}^m$.

Note that the map $F : \mathbb{R} \times_{e^{-s}} \mathbb{R}^m \to \mathbb{R}^+ \times \mathbb{R}^m$ given by

$$F(s; x_1, \dots, x_m) = (e^s; x_1, \dots, x_m)$$

is an isometry from the warped product model to the half-space model, so a soliton with respect to the direction $X = \pm e^{-s}\partial_s$ in the warped product model is isometric to a soliton with respect to the direction $X = \pm\partial_0$ in the half-space model.

We give some important relations between the scalar mean curvature H and the coordinate functions of the soliton $M \subset \mathbb{H}^{m+1}$.

Lemma 3.0.1. *Let $M \subset \mathbb{H}^{m+1}$ be a two-sided hypersurface of the hyperbolic space, where $\mathbb{H}^{m+1}$ is modelled via the half-space model $\mathbb{R}^+ \times \mathbb{R}^m$. Assume that M is a soliton of the mean curvature flow, with respect to the vector field $X_\pm = \pm\partial_0$, and denote by $x_k : M \to \mathbb{R}$, $k \in \{0, 1, \dots, m\}$, also the restriction of the coordinate function x_k to M. Then, the following formulas hold true:*

(a) The coordinate function $x_0 : M \to \mathbb{R}$ satisfies the following differential equations

$$\nabla x_0 = x_0^2 \partial_0^\top \quad \textit{and} \quad |\nabla x_0|^2 + x_0^4 H^2 = x_0^2.$$

Moreover,

$$\nabla^2 x_0(e_i, e_j) = \pm x_0^2 H b_{ij} + 2x_0^{-1}\,\mathrm{g}(e_i, \nabla x_0)\,\mathrm{g}(e_j, \nabla x_0) - x_0 \delta_{ij},$$

where $\{e_1, \dots, e_m\}$ is a tangent local orthonormal frame on M and b_{ij} are the components of the second fundamental form. Additionally,

$$\Delta x_0 = \pm(1 \mp 2x_0) x_0^2 H^2 - (m-2) x_0.$$

Furthermore, the other coordinate functions $x_k : M \to \mathbb{R}$, $k \in \{1, \ldots, m\}$, satisfy the equations

$$\nabla x_k = x_0^2 \partial_k^\top \quad \text{and} \quad \Delta x_k = x_0^{-2}(2x_0 \mp 1)\mathrm{g}(\nabla x_0, \nabla x_k).$$

(b) The scalar mean curvature H satisfies the differential equations

$$\mathrm{g}(\nabla H, v) = \mp \mathrm{II}(\partial_0^\top, v) = \mp x_0^{-2} \mathrm{II}(\nabla x_0, v),$$

and

$$\Delta H = \mp x_0^{-2}\, \mathrm{g}(\nabla x_0, \nabla H) \pm (x_0^{-1} \mp |\mathrm{II}|^2) H.$$

for any tangent vector field $v \in \mathfrak{X}(M)$.

Proof. Let us start with some general computations exploiting the conformally flat structure of the hyperbolic space. Let us denote by D the Levi-Civita connection of the hyperbolic space $\mathbb{H}^{m+1}$, by $\langle \cdot, \cdot \rangle$ the standard inner product in $\mathbb{R}^{m+1}$ and by $D^{\mathbb{R}}$ the Euclidean Levi-Civita connection. From the Koszul formula, we have that

$$D_{v_1} v_2 = D^{\mathbb{R}}_{v_1} v_2 - x_0^{-1} \langle \partial_0, v_1 \rangle v_2 - x_0^{-1} \langle \partial_0, v_2 \rangle v_1 + x_0^{-1} \langle v_1, v_2 \rangle \partial_0, \tag{3.0.1}$$

for any $v_1, v_2 \in \mathfrak{X}(\mathbb{H}^{m+1})$. If $u \in C^\infty(\mathbb{H}^{m+1})$ is a smooth function then, from the formula (3.0.1), we easily get that

$$\begin{aligned} D^2 u(v_1, v_2) \;\; &= \;\; Hess(u)(v_1, v_2) + x_0^{-1} v_2(u) \langle \partial_0, v_1 \rangle \\ &\qquad + x_0^{-1} v_1(u) \langle \partial_0, v_2 \rangle - x_0^{-1} \partial_0(u) \langle v_1, v_2 \rangle, \end{aligned} \tag{3.0.2}$$

for any $v_1, v_2 \in \mathfrak{X}(\mathbb{H}^{m+1})$, where *Hess* stands for the Hessian operator with respect to the Euclidean metric. Consider now the restriction of the function u on M, which for simplicity we denote again by the letter u. Suppose that $\{e_1, \ldots, e_m\}$ is a local orthonormal frame M which is normal at a fixed point $p \in M$. Let us also denote by ν the unit normal along M. Then, ∇u is the orthogonal projection, with respect to the metric of the $\mathbb{H}^{m+1}$, of Du on the tangent bundle of M, i.e.,

$$\nabla u = \mathrm{g}(Du, e_i) e_i.$$

Differentiating, we get that, for any $i, j \in \{1, \ldots, m\}$, it holds

$$\nabla^2 u(e_i, e_j) = D^2 u(e_i, e_j) + \mathrm{g}(Du, \mathrm{II}(e_i, e_j)). \tag{3.0.3}$$

(a) In the case where u is the k^{th}-coordinate function $x_k : \mathbb{H}^{m+1} \to \mathbb{R}$, we have

$$Dx_k = x_0^2 \partial_k \quad \text{and} \quad \nabla x_k = x_0^2 \partial_k^\top, \tag{3.0.4}$$

for any $k \in \{0, 1, \ldots, m\}$. Using the fact that M is a soliton, from the second identity of (3.0.4), we deduce that

$$|\nabla x_0|^2 = x_0^4 \, \mathrm{g}(\partial_0^\top, \partial_0^\top) = x_0^4 \, \mathrm{g}(\partial_0, \partial_0) - x_0^4 \, \mathrm{g}(\partial_0^\perp, \partial_0^\perp) = x_0^2 - x_0^4 H^2.$$

Since the Euclidean Hessian of each coordinate function $x_k : \mathbb{H}^{m+1} \to \mathbb{R}$ is zero, we obtain from (3.0.2), (3.0.3) and (3.0.4) that

$$\begin{aligned} \nabla^2 x_0(e_i, e_j) &= x_0 \, \mathrm{g}(e_i, Dx_0) \, \mathrm{g}(\partial_0, e_j) \\ &\quad + x_0 \, \mathrm{g}(e_j, Dx_0) \, \mathrm{g}(\partial_0, e_i) - x_0 \delta_{ij} + b_{ij} \, \mathrm{g}(Dx_0, \nu) \\ &= 2x_0^3 \, \mathrm{g}(e_i, \partial_0^\top) \, \mathrm{g}(\partial_0^\top, e_j) - x_0 \delta_{ij} + b_{ij} x_0^2 \, \mathrm{g}(\partial_0^\perp, \nu) \\ &= 2x_0^{-1} \, \mathrm{g}(e_i, \nabla x_0) \, \mathrm{g}(\nabla x_0, e_j) - x_0 \delta_{ij} \pm b_{ij} x_0^2 H. \end{aligned}$$

Taking the trace with respect to the induced metric, we get

$$\begin{aligned} \Delta x_0 &= 2x_0^3 \, \mathrm{g}(\partial_0^\top, \partial_0^\top) - m x_0 \pm x_0^2 H^2 \\ &= 2x_0^3 \big(\mathrm{g}(\partial_0, \partial_0) - \mathrm{g}(\partial_0^\perp, \partial_0^\perp) \big) - m x_0 \pm x_0^2 H^2 \\ &= 2x_0^3 \big(x_0^{-2} - H^2 \big) - m x_0 \pm x_0^2 H^2 \\ &= \pm (1 \mp 2x_0) x_0^2 H^2 - (m-2) x_0. \end{aligned}$$

Moreover, again from (3.0.2), (3.0.3) and (3.0.4), we find that for $k \geq 1$ it holds

$$\begin{aligned} \Delta x_k &= 2x_0 \, \mathrm{g}(e_i, Dx_k) \, \mathrm{g}(\partial_0, e_i) + \mathrm{g}(Dx_k, H\nu) \\ &= 2x_0^3 \, \mathrm{g}(e_i, \partial_k) \, \mathrm{g}(\partial_0, e_i) \pm x_0^2 \mathrm{g}(\partial_k, \partial_0^\perp) \\ &= 2x_0^3 \, \mathrm{g}(\partial_k^\top, \partial_0^\top) \pm x_0^2 \mathrm{g}(\partial_k, \partial_0 - \partial_0^\top) \\ &= 2x_0^3 \, \mathrm{g}(\partial_k^\top, \partial_0^\top) \mp x_0^2 \mathrm{g}(\partial_k^\top, \partial_0^\top) \\ &= x_0^{-2} (2x_0 \mp 1) \, \mathrm{g}(\nabla x_0, \nabla x_k). \end{aligned}$$

(b) The proof follows the same lines as in Proposition 2.3.3. Differentiating with respect to e_i, $i \in \{1, \ldots, m\}$, we get that

$$e_i H = e_i \, \mathrm{g}(\pm \partial_0, \nu) = \pm \, \mathrm{g}(D_{e_i} \partial_0, \nu) \pm \mathrm{g}(\partial_0, D_{e_i} \nu).$$

From the formula (3.0.1) we see that

$$D_{e_i}\partial_0 = -x_0^{-1}e_i,$$

for any $i \in \{1,\dots,m\}$. Hence, keeping in mind (3.0.4), we have that

$$e_iH = \pm\,\mathrm{g}(\partial_0^\top, D_{e_i}\nu) = \mp\mathrm{II}(\partial_0^\top, e_i) = \mp x_0^{-2}\mathrm{II}(\nabla x_0, e_i).$$

Differentiating once more, using Codazzi equations and (3.0.4), we deduce that at $p \in M$ it holds

$$\begin{aligned}
\Delta H &= e_ie_iH = e_i\big(\mp b_{ij}\,\mathrm{g}(\partial_0, e_j)\big)\\
&= \mp b_{iji}\,\mathrm{g}(\partial_0, e_j) \mp b_{ij}\,\mathrm{g}(D_{e_i}\partial_0, e_j) \mp b_{ij}\,\mathrm{g}(\pm\partial_0, D_{e_i}e_j)\\
&= \mp\,\mathrm{g}(\partial_0^\top, b_{iij}e_j) \mp b_{ij}\,\mathrm{g}(D_{e_i}\partial_0, e_j) - b_{ij}b_{ij}\,\mathrm{g}(\pm\partial_0, \nu)\\
&= \mp\,\mathrm{g}(\partial_0^\top, \nabla H) \pm x_0^{-1}H - H|\mathrm{II}|^2\\
&= \mp x_0^{-2}\,\mathrm{g}(\nabla x_0, \nabla H) \pm x_0^{-1}H - H|\mathrm{II}|^2.
\end{aligned}$$

This completes the proof of lemma. □

Theorem 3.0.2. *Let $M \subset \mathbb{H}^{m+1}$ be a 2-sided hypersurface whose scalar mean curvature does not change sign. If M is a soliton with respect to $-\partial_0$, then M is stable. If M is a soliton with respect to ∂_0, then M is stable in the region*

$$S = \{p \in M : x_0(p) \geq 2/m\}.$$

Proof. Let us compute the Jacobi operator of H. Let ν be the unit normal along the soliton M. Up to changing sign to the unit normal vector, we can assume that $H \geq 0$. From (2.3.1), (3.0.1) and Lemma 3.0.1(b), we have

$$\begin{aligned}
JH &= \Delta H + \mathrm{g}(\pm\partial_0^\top, \nabla H) + H|\mathrm{II}|^2\\
&\quad + \mathrm{Ric}^{\mathbb{H}^{m+1}}(\nu,\nu)H - \mathrm{g}(\pm D_\nu\partial_0, \nu)H\\
&= \Delta H \pm \mathrm{g}(\partial_0^\top, \nabla H) + H|\mathrm{II}|^2 - mH \pm x_0^{-1}H\\
&= -(m \mp 2x_0^{-1})H.
\end{aligned}$$

Hence,

$$JH + (m \mp 2x_0^{-1})H = 0.$$

By the strong maximum principle, $H > 0$ on M. The result follows from the result in Theorem 2.3.2. □

4 BARRIERS AND SYMMETRIC EXAMPLES OF SOLITONS

4.1 Graphical solitons and their stability

Let us consider graphical solitons in $\mathbb{H}^{m+1}$, that is, solitons that can be written in the form

$$\Gamma(u) = \left\{(u(x);x) \in \mathbb{H}^{m+1} = \mathbb{R}^+ \times \mathbb{R}^m : x \in \Omega \subset \mathbb{R}^m\right\},$$

where Ω is an open subset of $\mathbb{R}^m$ and $u : \Omega \to \mathbb{R}$ a smooth function.

Proposition 4.1.1. *The graph $\Gamma(u) \subset \mathbb{H}^{m+1}$ is a soliton of the hyperbolic space, with respect to the vector field $X = \pm\partial_0$, if and only if u satisfies the following equation:*

$$\operatorname{div}^{\mathbb{R}}\left(\frac{\nabla^{\mathbb{R}} u}{\sqrt{1+|\nabla^{\mathbb{R}} u|^2}}\right) = \frac{-mu \pm 1}{u^2\sqrt{1+|\nabla^{\mathbb{R}} u|^2}}, \tag{SE$_\pm$}$$

where $\operatorname{div}^{\mathbb{R}}$ *is the Euclidean divergence, $\nabla^{\mathbb{R}}$ denotes the Euclidean gradient and $|\cdot|$ the Euclidean norm. In particular, solitons with respect to $-\partial_0$ have negative mean curvature in upward direction and $\Gamma(u)$ has nowhere zero mean curvature.*

Proof. Observe first that the graph $\Gamma(u) \subset \mathbb{H}^{m+1}$ is the image of the embedding $\psi : \Omega \to \mathbb{H}^{m+1}$ given by $\psi(x) = (u(x);x)$, for any $x \in \Omega$. One can readily check that the components of the induced metric $g_{\mathbb{H}}$ on the graph are

$$(g_{\mathbb{H}})_{ij} = \frac{u_i u_j + \delta_{ij}}{u^2}$$

where $i, j \in \{1,\dots,m\}$. The components $(g_{\mathbb{H}})^{ij}$ of the inverse of the induced metric $g_{\mathbb{H}}$ are given by

$$(g_{\mathbb{H}})^{ij} = u^2\left(\delta_{ij} - \frac{u_i u_j}{1+|\nabla^{\mathbb{R}} u|^2}\right) \tag{4.1.1}$$

for $i, j \in \{1,\dots,m\}$; see for example (Osserman, 1969, page 1101), Moreover, the unit $g_{\mathbb{H}}$-normal $\nu_{g_{\mathbb{H}}}$ along the graph is given by the expression

$$\nu_{g_{\mathbb{H}}} = \frac{u\,\partial_0 - u\nabla^{\mathbb{R}} u}{\sqrt{1+|\nabla^{\mathbb{R}} u|^2}} = \frac{u\,\partial_0 - u\sum_{j=1}^m u_j \partial_j}{\sqrt{1+\sum_{j=1}^m u_j^2}}. \tag{4.1.2}$$

Denoting $\psi_{ij} := \nabla^{\mathbb{R}}{}_{d\psi(\partial_i)} d\psi(\partial_j)$ and $\psi_i := d\psi(\partial_i)$ and making use of (3.0.1) and (4.1.2), the components of the second fundamental form of the graph are given by

$$\begin{aligned}(b_{g_{\mathbb{H}}})_{ij} =& g_{\mathbb{H}}\big(\nabla^{\mathbb{H}}{}_{d\psi(\partial_i)} d\psi(\partial_j), \nu_{g_{\mathbb{H}}}\big) = g_{\mathbb{H}}(\psi_{ij}, \nu_{g_{\mathbb{H}}}) + u^{-1}\langle \psi_i, \psi_j\rangle g_{\mathbb{H}}(\partial_0, \nu_{g_{\mathbb{H}}}) \\ =& \frac{1}{\sqrt{1+|\nabla^{\mathbb{R}} u|^2}}\left(\frac{u_{ij}}{u} + \frac{\delta_{ij} + u_i u_j}{u^2}\right),\end{aligned} \tag{4.1.3}$$

for any $i, j \in \{1, \ldots, m\}$. Raising one index utilizing the graph metric, the shape operator satisfies

$$(b_{g_{\mathbb{H}}})^k_j = (g_{\mathbb{H}})^{ki}(b_{g_{\mathbb{H}}})_{ij} = \frac{1}{\sqrt{1+|\nabla^{\mathbb{R}} u|^2}}\left[u\left(u^k_j - \frac{u^k u^i u_{ij}}{1+|\nabla^{\mathbb{R}} u|^2}\right) + \delta^k_j\right]. \tag{4.1.4}$$

Therefore, the scalar mean curvature is

$$\begin{aligned} H &= (g_{\mathbb{H}})^{ij}(b_{g_{\mathbb{H}}})_{ij} = \frac{1}{\sqrt{1+|\nabla^{\mathbb{R}} u|^2}}\left\{u u_{ij}\left(\delta_{ij} - \frac{u_i u_j}{1+|\nabla^{\mathbb{R}} u|^2}\right) + m\right\} \\ &= u\,\mathrm{div}^{\mathbb{R}}\left(\frac{\nabla^{\mathbb{R}} u}{\sqrt{1+|\nabla^{\mathbb{R}} u|^2}}\right) + \frac{m}{\sqrt{1+|\nabla^{\mathbb{R}} u|^2}} \end{aligned} \tag{4.1.5}$$

On the other hand, $\Gamma(u)$ is a soliton with respect to $X = \pm\partial_0$ only if

$$H = \pm g_{\mathbb{H}}(\partial_0, \nu_{g_{\mathbb{H}}}) = \frac{\pm 1}{u\sqrt{1+|\nabla^{\mathbb{R}} u|^2}}. \tag{4.1.6}$$

Combining (4.1.5) with (4.1.6) we obtain the desired result. □

Remark 4.1.1. *Let us mention here that Serrin considered quasilinear equations quite similar to those of the form* ($\mathrm{SE}_\pm$)*; see (Serrin, 1967) and (Serrin, 1969, Chapter IV, pages 477-478). In particular, he studied equations of the form*

$$\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f(x,u).$$

However, a gradient term on the right-hand side was not considered.

We recall the Comparison and Maximum Principle theorems by Pucci and Serrin (Pucci; Serrin, 2007, Theorem 2.1.3 & 2.1.4).

Theorem 4.1.2 (Maximum Principle)**.** *Let Ω be a connected bounded open domain of $\mathbb{R}^m$ with boundary $\partial\Omega$ and $u, v \in C^2(\Omega)$ solutions of the nonlinear differential inequality*

$$\mathcal{F}\big(x; u; Du; D^2u\big) \geq \mathcal{F}\big(x, v; Dv; D^2v\big),$$

where the function $\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{m^2} \to \mathbb{R}$ is continuously differentiable. Suppose also that the matrix $Q = [Q_{ij}]$ given by

$$Q_{ij} = \mathcal{F}_{u_{ij}}\big(x, u(x), Du(x), \theta D^2u(x) + (1-\theta)D^2v(x)\big)$$

is positive definite for any $x \in \Omega$ and any $\theta \in [0,1]$. If $u \leq v$ in Ω and $u = v$ at some point $x_0 \in \Omega$, then $u \equiv v$ in Ω.

Theorem 4.1.3 (Comparison Principle)**.** *Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be solutions of the nonlinear differential inequality given in Theorem 4.1.2. Suppose that the matrix $Q = [Q_{ij}]$ is positive definite in Ω and that for every fixed $x \in \Omega$ the function*

$$t \mapsto \mathcal{F}\big(x, t, Dv(x), D^2v(x)\big) \tag{4.1.7}$$

is non-increasing on the half-line $[v(x), \infty)$-but not necessarily differentiable. If $u \leq v$ in $\partial\Omega$, then $u \leq v$ in Ω. The terms u, Du in Q can be replaced by v, Dv if at the same time the terms Dv, D^2v in (4.1.7) *are replaced by Du, D^2u and the semi-line $[v(x), \infty)$ is replaced by $(-\infty, u(x)]$.*

Definition 4.1.4. *Let Ω be an open subset of $\mathbb{R}^m$. A positive C^2-smooth function $\varphi : \Omega \to (0, \infty)$ is called:*

1. ***Subsolution*** *to the quasilinear differential equation* (SE$_\pm$)*, if it satisfies the inequality*

$$\operatorname{div}^{\mathbb{R}}\left(\frac{\nabla^{\mathbb{R}}\varphi}{\sqrt{1+|\nabla^{\mathbb{R}}\varphi|^2}}\right) \geq \frac{-m\varphi \pm 1}{\varphi^2\sqrt{1+|\nabla^{\mathbb{R}}\varphi|^2}}, \tag{4.1.8}$$

2. ***Supersolution*** *to the quasilinear differential equation* (SE$_\pm$)*, if if it satisfies the inequality*

$$\operatorname{div}^{\mathbb{R}}\left(\frac{\nabla^{\mathbb{R}}\varphi}{\sqrt{1+|\nabla^{\mathbb{R}}\varphi|^2}}\right) \leq \frac{-m\varphi \pm 1}{\varphi^2\sqrt{1+|\nabla^{\mathbb{R}}\varphi|^2}}. \tag{4.1.9}$$

Proposition 4.1.5. *If u is a subsolution (supersolution) to*

$$\operatorname{div}^{\mathbb{R}}\left(\frac{\nabla^{\mathbb{R}}u}{\sqrt{1+|\nabla^{\mathbb{R}}u|^2}}\right) = \frac{-mu - 1}{u^2\sqrt{1+|\nabla^{\mathbb{R}}u|^2}}, \tag{SE$_-$}$$

then the region above (below) the graph of u is $g_{\mathbb{I}}$-mean-convex.

Proof. Assume that u is a subsolution. Notice that the scalar mean curvature with respect to upward direction $\nu_{g_{\mathbb{I}}}$ of the graph $\Gamma(u) \hookrightarrow (\mathbb{H}^{m+1}, g_{\mathbb{I}})$ is positive. Therefore, the region above $\Gamma(u)$ is $g_{\mathbb{I}}$-mean-convex. Similarly when u is a supersolution.

□

Proposition 4.1.6. *If u_2 is a subsolution to* (SE$_-$) *and $u_1 = u_2 - \varepsilon$ for some $\varepsilon > 0$, then u_1 is a subsolution too. Similarly, if v_1 is a supersolution to* (SE$_-$) *and $v_2 = v_1 + \varepsilon$ for some $\varepsilon > 0$, then v_2 is a supersolution too.*

Proof. Notice that

$$\operatorname{div}^{\mathbb{R}}\left(\frac{\nabla^{\mathbb{R}}u_2}{\sqrt{1+|\nabla^{\mathbb{R}}u_2|^2}}\right)=\operatorname{div}^{\mathbb{R}}\left(\frac{\nabla^{\mathbb{R}}u_1}{\sqrt{1+|\nabla^{\mathbb{R}}u_1|^2}}\right)$$

and

$$\frac{-mu_2-1}{u_2^2\sqrt{1+|\nabla^{\mathbb{R}}u_2|^2}}>\frac{-mu_1-1}{u_1^2\sqrt{1+|\nabla^{\mathbb{R}}u_1|^2}}$$

. Therefore,

$$\operatorname{div}^{\mathbb{R}}\left(\frac{\nabla^{\mathbb{R}}u_1}{\sqrt{1+|\nabla^{\mathbb{R}}u_1|^2}}\right)\geq\frac{-mu_1-1}{u_1^2\sqrt{1+|\nabla^{\mathbb{R}}u_1|^2}}.$$

Similarly, for v_1 and v_2. □

Combining Theorem 3.0.2 and Proposition 4.1.1 we obtain the following result.

Theorem 4.1.7. *A graphical soliton, with respect to the vector field $X=-\partial_0$, in the hyperbolic space, is always stable. Moreover, a graphical soliton with respect to the vector field $X=\partial_0$ in the hyperbolic space is stable if it is contained in the region $S=\{(x_0;x)\in\mathbb{R}^+\times\mathbb{R}^m : x_0\geq 2/m\}$.*

4.2 Cylindrical solitons

Let us describe here cylindrical solitons, that is, solitons which can be written in the form $\Gamma\times\mathbb{R}^{m-1}\subset\mathbb{H}^{m+1}$, where here Γ is a curve in the x_0x_1-plane. For simplicity, let us work in regions where Γ can be represented as the graph of a smooth positive function $u:(\varepsilon_1,\varepsilon_2)\subset\mathbb{R}\longrightarrow(0,\infty)$ with respect to the direction ∂_0. So we assume that Γ can be written as the image of $\gamma(t)=(u(t),t)$ in the x_0x_1-plane. In this case, the equation (SE$_\pm$) becomes:

$$\begin{aligned}
\operatorname{div}^{\mathbb{R}}\left(\frac{\nabla^{\mathbb{R}}u}{\sqrt{1+|\nabla^{\mathbb{R}}u|^2}}\right) &= \frac{-mu\pm 1}{u^2\sqrt{1+|\nabla^{\mathbb{R}}u|^2}}\\
\frac{u_{tt}}{\sqrt{1+u_t^2}}+u_t\left(-\frac{1}{2}\frac{1}{(1+u_t^2)^{3/2}}2u_tu_{tt}\right) &= \frac{-mu\pm 1}{u^2\sqrt{1+u_t^2}}\\
u_{tt}-\frac{u_t^2}{1+u_t^2}u_{tt} &= \frac{-mu\pm 1}{u^2}\\
\frac{u_{tt}}{1+u_t^2} &= \frac{-mu\pm 1}{u^2}.
\end{aligned}$$

CASE A: Let us first consider one dimensional solitons with respect to $X = \partial_0$. Here we are interested in positive solutions to the ODE:

$$\frac{u_{tt}}{1+u_t^2} = \frac{-mu+1}{u^2}. \tag{SE$_+$}$$

One obvious solution is the constant one given by

$$u(t) = \frac{1}{m},$$

for any $t \in \mathbb{R}$. Let us examine non trivial solutions now. Consider the function $v : (-\varepsilon_1, \varepsilon_2) \to \mathbb{R}$ given by $v = u_t$. Then, we get the following system of first order ODEs:

$$\begin{cases} u_t = v, \\ v_t = u^{-2}(1-mu)(1+v^2), \end{cases}$$

Let us compute at first the integral curves of the vector field $Z : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2$ given by

$$Z(u,v) = \big(v, u^{-2}(1-mu)(1+v^2)\big).$$

In order to analyse the integral curves, let us consider the function $G : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ given by

$$G(u,v) = u^{-1} + m\log u + \log\sqrt{1+v^2}.$$

Therefore,

$$\operatorname{grad} G = \left(-\frac{1}{u^2} + \frac{m}{u}, \frac{v}{1+v^2}\right).$$

Note that the only critical point of G is the point $(1/m, 0)$ and that $\langle \operatorname{grad} G, Z\rangle = 0$. Therefore if $\alpha = (\alpha_1, \alpha_2) : (-\varepsilon_1, \varepsilon_2) \to \mathbb{R}^+ \times \mathbb{R}$ is an integral curve of Z, then

$$(G \circ \alpha)' = \langle \operatorname{grad} G(\alpha_1, \alpha_2), \alpha' \rangle = \langle \operatorname{grad} G(\alpha_1,\alpha_2), Z(\alpha_1,\alpha_2)\rangle = 0.$$

Hence, there exists a constant c such that $G \circ \alpha = c$. Consequently, the level sets of G are the integral curves of Z. Putting everything together, we see that

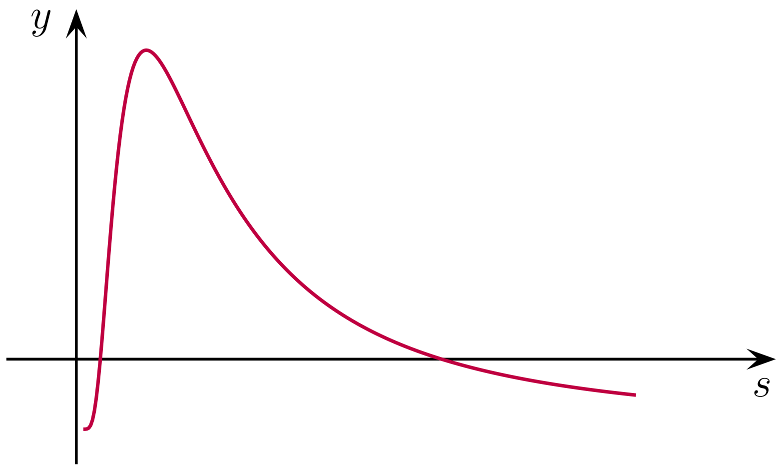
$$\begin{aligned} c &= \frac{1}{u} + m\log u + \frac{1}{2}\log(1+u_t^2) \\ e^c &= e^{\frac{1}{u} + m\log u + \frac{1}{2}\log(1+u_t^2)} \\ e^{2c} &= e^{\frac{2}{u}} u^{2m}(1+u_t^2) \\ u_t^2 &= e^{2c} e^{-2/u} u^{-2m} - 1 \geq 0, \end{aligned}$$

where c is a positive constant. One can easily check that, for $c > m(1-\log m)$, the function $f : (0,\infty) \to \mathbb{R}$ given by

$$f(s) = e^{2c}e^{-2/s}s^{-2m} - 1,$$

takes non-negative values only if the variable s takes values in a suitable closed interval $[a,b] \subset \mathbb{R}^+$ with $a > 0$ and b depending on c. Moreover, $f(a) = f(b) = 0$ and $f(s) > 0$ for any $s \in (a,b)$. Hence, any solution to (SE_+) must be bounded above and below away from zero; see Figure 1.

Fig. 1 – Graph of f



Source: elaborated by the author.

Indeed, the graph of f is given by Figure 1 because:

1. $f'(s) = e^{2c}2e^{-\frac{2}{s}}s^{-2m}\left(\frac{1}{s^2} - \frac{m}{s}\right)$. Therefore $f'(s) > 0$ for $0 < s < \frac{1}{m}$, $f'(s) < 0$ for $s > \frac{1}{m}$ and $f'(\frac{1}{m}) = 0$.
2. $\lim_{s\to 0} f(s) = -1$
3. $\lim_{s\to\infty} f(s) = -1$

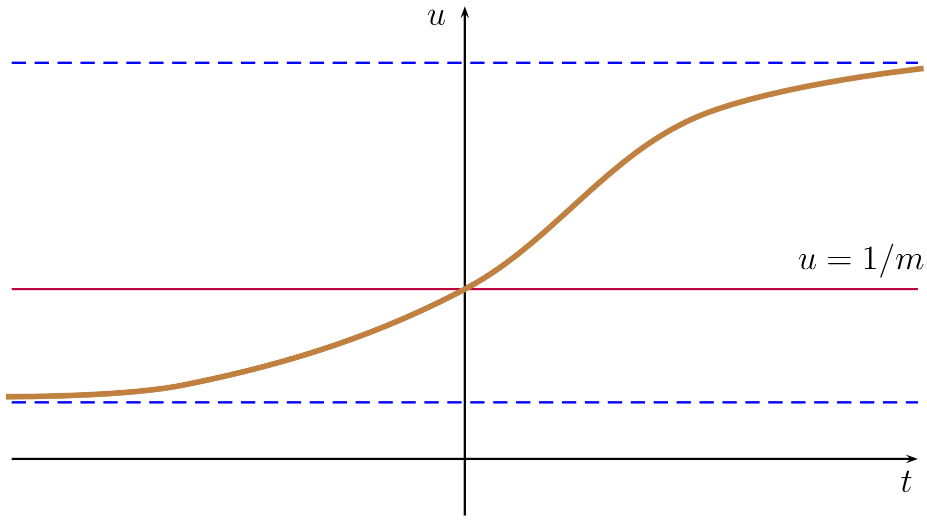
Moreover, notice from (4.2.1) that the part of the curve γ above the line $x_0 = 1/m$ is concave and the part below $x_0 = 1/m$ is convex. Consequently, the solutions exist for all values of the parameter t; see Figure 2.

CASE B: Let us examine now cylindrical solitons with respect to $X = -\partial_0$. In this case we have to deal with the following ODE:

$$\frac{u_{tt}}{1+u_t^2} = \frac{-mu-1}{u^2}. \qquad (SE_-)$$

Observe that the solutions to such an equation must be concave. As in the previous case, consider the function $v : (-\varepsilon,\varepsilon) \to \mathbb{R}$ given by $v = u_t$ and reduce the second order ODE into the following

Fig. 2 – Graph of u



Source: elaborated by the author.

system:

$$\begin{cases} u_t = v, \\ v_t = -u^{-2}(1+mu)(1+v^2). \end{cases}$$

Hence, the solutions to the above system are precisely the integral curves of the vector field $Z : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2$ given by

$$Z(u,v) = \left(v, -u^{-2}(1+mu)(1+v^2)\right).$$

Consider the potential $G : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is given by

$$G(u,v) = \log\sqrt{1+v^2} + m\log u - u^{-1}$$

Observe that

$$\operatorname{grad} G = \left(\frac{m}{u} + \frac{1}{u^2}, \frac{v}{1+v^2}\right).$$

Moreover, $\operatorname{grad} G$ is nowhere zero and perpendicular to Z. Consequently, the level sets of G are precisely the integral curves of Z. As a matter of fact, we get that

$$\begin{aligned} c &= \log\sqrt{1+u_t^2} + m\log u - \frac{1}{u} \\ e^{2c} &= (1+u_t^2)u^{2m}e^{-\frac{2}{u}} \\ u_t^2 &= e^{2c}e^{2/u}u^{-2m} - 1 \geq 0, \end{aligned}$$

where c is a positive constant. One can easily check that, for a fixed number c and natural number $m \in \mathbb{N}$, the function $g : (0,\infty) \to \mathbb{R}$ given by

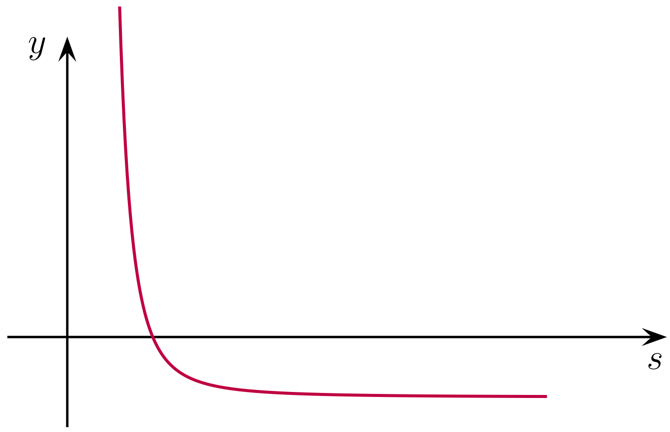
$$g(s) = e^{2c}e^{2/s}s^{-2m} - 1,$$

takes positive values only if s lies in an interval of the form $(0,b]$. Indeed, the graph of g is as in Figure 3 because:

1. $g'(s) = -2e^{2c}e^{\frac{2}{s}}s^{-2m}(\frac{1}{s^2}+\frac{1}{s}) < 0 \quad \forall s > 0.$.
2. $\lim_{s\to 0} g(s) = \infty$.
3. $\lim_{s\to\infty} g(s) = -1$.

Observe now that since the equation (SE$_-$) is autonomous, if u is a solution then for any fixed $a \in \mathbb{R}$, $u_a(t) = u(t-a)$ is again a solution to (SE$_-$). This means that the solutions are invariant under translations which keep fixed the ∂_0 direction. Thus, without loss of generality, we may assume that the interval of definition of u contains 0.

Fig. 3 – Graph of g



Source: elaborated by the author.

Lemma 4.2.1 (Grim-Reaper)**.** *Let u be a solution to the differential equation* (SE$_-$) *satisfying the initial conditions*

$$u(0) = h > 0 \quad \textit{and} \quad u_t(0) = 0.$$

Then, the following hold true:

(a) The function u is even and defined on a maximal bounded interval $(-T,T)$, where $T = T(h) = T_h$ is a positive number; see Figure 4.

(b) The function u and its derivative u_t satisfy

$$\lim_{t\to\pm T} u(t) = 0 \quad \textit{and} \quad \lim_{t\to\pm T} u_t(t) = \mp\infty.$$

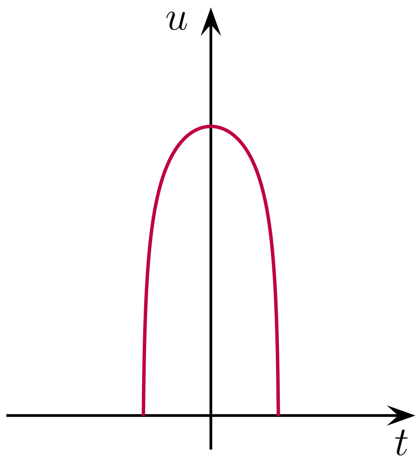
(c) The height h and the length $\ell(h) = 2T(h)$ of the domain of definition of u are related by

$$\ell(h) = 2\int_0^h \frac{ds}{\sqrt{e^{(2/s-2/h)}(h/s)^{2m}-1}}.$$

(d) $\lim_{h\to\infty} \ell(h) = \infty$.

(e) $\ell(h) = 2T(h)$ *is increasing in h.*

Fig. 4 – Graph of u



Source: elaborated by the author.

Proof. (a) We will show at first that the maximal domain of definition of u is bounded. To achieve this, let us suppose to the contrary that there exists a solution u defined on an interval of the form $(-a,\infty)$, where $a > 0$. Fix some point $t_1 > 0$. Since $u_{tt} < 0$ by (SE_-), it follows that u has at most one maximum point. Hence, from our initial conditions, u attains at $t = 0$ its global maximum. Moreover, u_t is strictly decreasing and $u_t(t) < 0$ for any $t > 0$. Then, for any $t > t_1$ we have that

$$u(t) - u(t_1) = \int_{t_1}^t u_t(s)ds \le \int_{t_1}^t u_t(t_1)ds = u_t(t_1)(t-t_1).$$

Hence,

$$0 < u(t) < u(t_1) + u_t(t_1)(t-t_1).$$

On the other hand, since $u_t(t_1) < 0$, we obtain that

$$0 \le \lim_{t\to+\infty} u(t) \le \lim_{t\to+\infty} \big(u(t_1) + u_t(t_1)(t-t_1)\big) = -\infty,$$

which leads to a contradiction. Hence, t cannot tend to $+\infty$. In the same way, we prove that there is no solution defined in an interval of the form $(-\infty, b)$ with $b > 0$. Consequently, the maximal domain of definition of such a solution must be a bounded maximal time interval of the form $(-a,b)$, where a and b are positive numbers. Note now that for small values of t, the function $\tilde{u}$ given by $\tilde{u}(t)=u(-t)$ is again a solution to (SE$_-$). Since, $\tilde{u}(0)=u(0)$ and $\tilde{u}_t(0)=0=u_t(0)$, from the uniqueness, we get that $\tilde{u}=u$ which implies that u is even. Hence, the maximal time of solution is of the form $(-T,T)$, where T is a positive number.

(b) We will show now that u tends to zero as t tends to $\pm T$ and that u_t tends to $\mp\infty$ as time tends to $\pm T$. Recall that in the interval $[0,T)$ the function u is decreasing. Suppose to the contrary that

$$\lim_{t\to T} u(t)=l,$$

where $l>0$. Then it is possible to extend the solution to the first order ODE

$$u_t=-\sqrt{e^{2c}e^{2/u}u^{-2m}-1} \tag{4.2.1}$$

in an interval $T+\varepsilon$, for some $\varepsilon>0$. This contradicts the fact that T is maximal. From (4.2.1), we get that $u_t\to-\infty$ as t approaches T. Analogously we treat the behaviour when t approaches $-T$.

(c) Recall that on the interval $[0,T)$, the function u satisfies the first order ODE

$$u_t=-\sqrt{e^{2c}e^{2/u}u^{-2m}-1},$$

where c is the constant given by $e^{2c}=h^{2m}e^{-2/h}$. In this particular interval, u is strictly decreasing and its inverse $t:(0,h)\to(0,T)$ satisfies the equation

$$t_u=-\frac{1}{\sqrt{e^{2c}e^{2/u}u^{-2m}-1}}.$$

After an integration we get that

$$T=\int_0^h \frac{ds}{\sqrt{e^{2c}e^{2/s}s^{-2m}-1}}.$$

This completes the proof.

(d) For $s\in(\frac{h}{2},h)$

$$\frac{1}{\sqrt{e^{2/s-2/h}\left(h/s\right)^{2m}-1}}>\frac{1}{\sqrt{e^{2/s-2/h}\left(h/s\right)^{2m}}}=\frac{1}{e^{1/s-1/h}\left(h/s\right)^{m}}>\frac{1}{e^{1/\frac{h}{2}-1/h}\left(h/s\right)^{m}}=\frac{s^m}{e^{\frac{1}{h}}h^m}.$$

Hence, $T>\int_{\frac{h}{2}}^{h}\frac{s^m}{e^{\frac{1}{h}}h^m}ds=\frac{1}{e^{\frac{1}{h}}h^m(m+1)}\left(h^{m+1}-\left(h/2\right)^{m+1}\right)=\frac{h}{(m+1)e^{\frac{1}{h}}}\left(1-\frac{1}{2^{m+1}}\right)$. Therefore, $T\to\infty$ as $h\to\infty$.

(e) We prove it in two steps:

(a) Suppose, by contradiction, that there exist $h_2, h_1 > 0$ with $h_2 > h_1$ such that $T(h_2) < T(h_1)$. Denote by $u_{h_i} : (-T(h_i), T(h_i)) \to \mathbb{R}$ a solution to the differential equation (SE$_-$) with initial conditions $u_{h_i}(0) = h_i$ and ${u_{h_i}}'(0) = 0$.
Notice that $(u_{h_2} - u_{h_1})(0) > 0$ and $(u_{h_2} - u_{h_1})(T(h_2)) < 0$. By continuity, there exists $\delta \in (0, T(h_2))$ such that $u_{h_2}(\delta) = u_{h_1}(\delta)$ and $u_{h_2}(-\delta) = u_{h_1}(-\delta)$. By Comparison Theorem (Theorem 4.1.3)), $u_{h_1} = u_{h_2}$ in $(-\delta, \delta)$. This is a contradiction because $u_{h_2}(0) = h_2 > h_1 = u_{h_1}(0)$.

(b) Suppose, by contradiction, that there exist $h_2, h_1 > 0$ with $h_2 > h_1$ such that $T(h_2) = T(h_1)$. Fix a constant $\varepsilon \in (0, h_2 - h_1)$, hence there exists a constant $a \in (0, T(h_2))$ such that $u_{h_2} > \varepsilon$ in $(-a, a)$. Set $v_{h_2,\varepsilon} = u_{h_2} - \varepsilon$ in $(-a, a)$. Notice that $v_{h_2,\varepsilon}$ is a subsolution to (SE$_-$). $(v_{h_2,\varepsilon} - u_{h_1})(0) > 0$ and $(v_{h_2,\varepsilon} - u_{h_1})(a) < 0$. By continuity, there exists $\delta \in (0, a)$ such that $v_{h_2,\varepsilon}(\delta) = u_{h_1}(\delta)$ and $v_{h_2,\varepsilon}(-\delta) = u_{h_1}(-\delta)$. By Comparison Theorem (Theorem 4.1.3) , $u_{h_1} \geq v_{h_2,\varepsilon}$ in $(-\delta, \delta)$, contradicting $v_{h_2,\varepsilon}(0) = h_2 - \varepsilon > h_1 = u_{h_1}(0)$.

Thus, ℓ and T are increasing in h.

□

Remark 4.2.1. *Similarly to (Martín et al., 2019) and (Gama; Martín, 2020), we call the graph of u a* ***grim-reaper*** *with maximum height h and centered in 0, symbolically $\mathcal{G}_{h,0}$. Furthermore, $\mathcal{G}_{h,[H_*,H^*]}$ is the grim-reaper with maximum height h and*

$$\partial_\infty \mathcal{G}_{h,[H_*,H^*]} = \{x_1 = H_*\} \cup \{x_1 = H^*\} \subset \partial_\infty \mathbb{H}^{m+1}.$$

4.3 Barriers

Now we construct some examples of barriers. To simplify the computation, we use the relation between the Levi-Civita connections of conformal metrics.

Lemma 4.3.1. *Let N be a manifold equipped with two conformal metric $\overline{g} = e^{2\mathsf{w}}\tilde{g}$.*

(a) The relation between the Levi-Civita connections is:

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y + (X(\mathsf{w}))Y + (Y\mathsf{w})X - \tilde{g}(X,Y)\widetilde{\nabla}\mathsf{w},$$

for all $X, Y \in \mathfrak{X}(N)$

(b) The relation between the gradients of a function $u : N \to \mathbb{R}$ is:

$$\widetilde{\nabla} u = \frac{1}{e^{2\mathrm{w}}}\overline{\nabla} u.$$

Proof. (a) For coordinate frame $\{\partial_i\}$, by Koszul formula:

$$\begin{aligned}
\bar{g}(\partial_k, \overline{\nabla}_{\partial_j}\partial_i) &= \frac{1}{2}\{\partial_i \bar{g}_{jk} + \partial_j \bar{g}_{ik} - \partial_k \bar{g}_{ij}\} \\
&= \frac{1}{2}\{\partial_i(e^{2\mathrm{w}}\tilde{g}_{jk}) + \partial_j(e^{2\mathrm{w}}\tilde{g}_{ik}) - \partial_k(e^{2\mathrm{w}}\tilde{g}_{ij})\} \\
&= e^{2\mathrm{w}}\frac{1}{2}\{\partial_i(\tilde{g}_{jk}) + \partial_j(\tilde{g}_{ik}) - \partial_k(\tilde{g}_{ij})\} + \\
&\quad + \frac{1}{2}e^{2\mathrm{w}}\{2\partial_i(\mathrm{w})\tilde{g}_{jk} + 2\partial_j(\mathrm{w})\tilde{g}_{ik} - 2\partial_k(\mathrm{w})\tilde{g}_{ij}\} \\
&= \bar{g}(\partial_k, \widetilde{\nabla}_{\partial_j}\partial_i) + \bar{g}(\partial_k, \partial_i(\mathrm{w})\partial_j) + \bar{g}(\partial_k, \partial_j(\mathrm{w})\partial_i) - \\
&\quad - \bar{g}(\partial_k, \tilde{g}_{ij}\widetilde{\nabla}\mathrm{w}) \\
&= \bar{g}(\partial_k, \widetilde{\nabla}_{\partial_j}\partial_i + \partial_i(\mathrm{w})\partial_j + \partial_j(\mathrm{w})\partial_i - \tilde{g}(\partial_j, \partial_i)\widetilde{\nabla}\mathrm{w}).
\end{aligned}$$

(b) Now the gradient in coordinate is given by $\widetilde{\nabla} u = \tilde{g}^{ij}\partial_j u \partial_i = \frac{1}{e^{2\mathrm{w}}}\bar{g}^{ij}\partial_j u \partial_i = \frac{1}{e^{2\mathrm{w}}}\overline{\nabla} u$.

□

In the following lemma, we will give the relation between the second fundamental forms in the hyperbolic metric and Ilmanen's metric of a submanifold.

Lemma 4.3.2 (Conformal second fundamental forms)**.** *Let $\mathcal{S}$ be a hypersurface in $\mathbb{H}^{m+1}$.*

$$\mathrm{II}_{g_{\mathbb{I}}}(v_1, v_2) = e^{\frac{1}{mx_0}}\left\{\mathrm{II}_{g_{\mathbb{H}}}(v_1, v_2) - \nu_{g_{\mathbb{H}}}\left(\frac{1}{mx_0}\right) g_{\mathbb{H}}(v_1, v_2)\right\}, \tag{4.3.1}$$

$\forall p \in \mathcal{S}$ and $\forall v_1, v_2 \in T_p\mathcal{S}$, where $\mathrm{II}_{g_{\mathbb{I}}}$ and $\mathrm{II}_{g_{\mathbb{H}}}$ are the second fundamental form with respect to $g_{\mathbb{I}}$ and $g_{\mathbb{H}}$ respectively, $\nu_{g_{\mathbb{H}}}$ is the $g_{\mathbb{H}}$-normal along $\mathcal{S}$, where $\nu_{g_{\mathbb{I}}} = \frac{\nu_{g_{\mathbb{H}}}}{e^{\frac{1}{mx_0}}}$.

Proof. Let $\{\bar{E}_i\}$ be a $g_{\mathbb{H}}$-orthonormal frame along $\mathcal{S}$. Define the $g_{\mathbb{I}}$-orthonormal frame by $\tilde{E}_i := \frac{\bar{E}_i}{e^{\frac{1}{mx_0}}}$. Similarly the $g_{\mathbb{I}}$-normal $\nu_{g_{\mathbb{I}}}$ along $\mathcal{S}$ is given by $\frac{\nu_{g_{\mathbb{H}}}}{e^{\frac{1}{mx_0}}}$. Using Lemma 4.3.1,

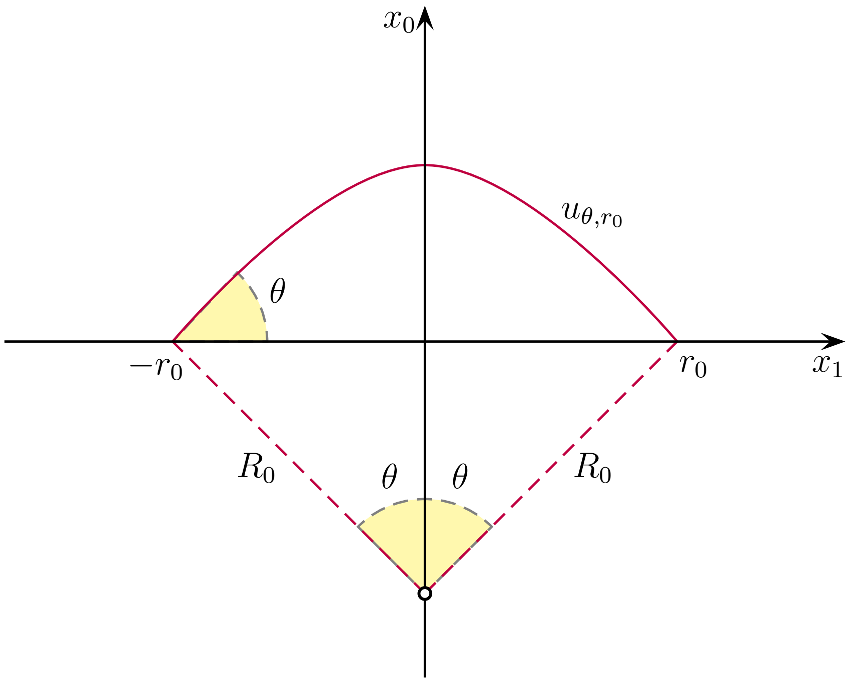
$$\nabla^{\mathbb{I}}{}_X Y = \nabla^{\mathbb{H}}{}_X Y + X\left(\frac{1}{mx_0}\right) Y + Y\left(\frac{1}{mx_0}\right) X - g_{\mathbb{H}}(X, Y)\nabla^{\mathbb{H}}\left(\frac{1}{mx_0}\right),$$

where $\nabla^{\mathbb{I}}$ and $\nabla^{\mathbb{H}}$ are the Levi-Civita connection in relation to $g_{\mathbb{I}}$ and $g_{\mathbb{H}}$ respectively. Therefore,

$$\begin{aligned} g_{\mathbb{I}}(\nabla^{\mathbb{I}}{}_{\tilde{E}_i}\nu_{g_{\mathbb{I}}},\tilde{E}_j) &= e^{\frac{2}{mx_0}} g_{\mathbb{H}}\left(\nabla^{\mathbb{H}}{}_{\tilde{E}_i}\nu_{g_{\mathbb{I}}}+\tilde{E}_i\left(\frac{1}{mx_0}\right)\nu_{g_{\mathbb{I}}}+\nu_{g_{\mathbb{I}}}\left(\frac{1}{mx_0}\right)\tilde{E}_i-g_{\mathbb{H}}(\tilde{E}_i,\nu_{g_{\mathbb{I}}})\nabla^{\mathbb{H}}\left(\frac{1}{mx_0}\right),\tilde{E}_j\right)\\ &= e^{\frac{2}{mx_0}}\left\{g_{\mathbb{H}}\left(\tilde{E}_i\left(e^{-\frac{1}{mx_0}}\right)\nu_{g_{\mathbb{H}}},\tilde{E}_j\right)+g_{\mathbb{H}}(e^{-\frac{1}{mx_0}}\nabla^{\mathbb{H}}{}_{\tilde{E}_i}\nu_{g_{\mathbb{H}}},\tilde{E}_j)+\nu_{g_{\mathbb{I}}}\left(\frac{1}{mx_0}\right)g_{\mathbb{H}}(\tilde{E}_i,\tilde{E}_j)\right\}\\ &= e^{\frac{2}{mx_0}}\left\{e^{-\frac{1}{mx_0}}g_{\mathbb{H}}(\nabla^{\mathbb{H}}{}_{\tilde{E}_i}\nu_{g_{\mathbb{H}}},\tilde{E}_j)+e^{-\frac{1}{mx_0}}\nu_{g_{\mathbb{H}}}\left(\frac{1}{mx_0}\right)g_{\mathbb{H}}(\tilde{E}_i,\tilde{E}_j)\right\}\\ \mathrm{II}_{g_{\mathbb{I}}}(\tilde{E}_i,\tilde{E}_j) &= e^{\frac{1}{mx_0}}\left\{\mathrm{II}_{g_{\mathbb{H}}}(\tilde{E}_i,\tilde{E}_j)-\nu_{g_{\mathbb{H}}}\left(\frac{1}{mx_0}\right)g_{\mathbb{H}}(\tilde{E}_i,\tilde{E}_j)\right\}. \end{aligned}$$

□

Fig. 5 – u_{θ,r_0}



Source: elaborated by the author.

To define a spherical barrier(see Figure 5), we need to set some notation. Given a angle $\theta \in \left(0,\frac{\pi}{2}\right]$, a radius $r_0 > 0$ and a origin $o \in \partial_\infty \mathbb{H}^{m+1}$, we set

$$R_0 := r_0 \csc\theta \quad B_{r_0}(o) = \{x \in \partial_\infty \mathbb{H}^{m+1} : |x-o| \le r_0\},$$

$$u_{\theta,r_0} : B_{r_0}(o) \subset \partial_\infty \mathbb{H}^{m+1} \to \mathbb{R}, \qquad u_{\theta,r_0}(x) := \sqrt{R_0^2 - |x-o|^2} - R_0\cos\theta,$$

$$\mathcal{B}_\theta(o,r_0) = \left\{(x_0,x) \in \mathbb{H}^{m+1} \ : \ x \in B_{r_0}(o),\ x_0 = u_{\theta,r_0}(x)\right\}.$$

Namely, $\mathcal{B}_\theta(o,r_0)$ is the intersection of $\mathbb{H}^{m+1}$ with an Euclidean sphere of radius R_0 that makes an angle θ with $\partial_\infty \mathbb{H}^{m+1}$.

Proposition 4.3.3 (Spherical barrier)**.** *For all $\theta \in \left(0, \frac{\pi}{2}\right]$ and $r_0 \in \mathbb{R}^+$, u_{θ,r_0} is a subsolution to* SE$_-$.

Proof. Using that $\mathcal{B}_{\frac{\pi}{2}}(o,r_0))$ is totally geodesic with respect to $g_{\mathbb{H}}$, by Lemma 4.3.2,

$$\mathrm{II}_{g_{\mathbb{I}}}(v_1,v_2) = e^{\frac{1}{mx_0}}\left\{-\nu_{g_{\mathbb{H}}}\left(\frac{1}{mx_0}\right)g_{\mathbb{H}}(v_1,v_2)\right\},$$

for all $v_1, v_2 \in T_p\mathcal{B}_{\frac{\pi}{2}}(o,r_0)$, where $\nu_{g_{\mathbb{H}}}$ is the $g_{\mathbb{H}}$-normal field pointing upward. By $-\nu_{g_{\mathbb{H}}}\left(\frac{1}{mx_0}\right) > 0$, $\mathrm{II}_{g_{\mathbb{I}}}$ is positive definite. Therefore, $H > 0$ and $\mathcal{B}_{\frac{\pi}{2}}(o,r_0)$. Hence $\mathcal{B}_{\frac{\pi}{2}}(o,r_0)$ is a subsolution. For $\theta \neq \pi/2$, set $u_2 := u_{\pi/2,r_0}$ and $u_1 := u_{\theta,r_0}$. Therefore, $u_1 = u_2 - \varepsilon$ for suitable $\varepsilon > 0$. By Proposition 4.1.6, u_1 is a subsolution. □

4.4 Rotationally symmetric graph over an annulus

Definition 4.4.1 (Rotationally symmetric graph soliton over an annulus)**.** *Let $u : (T_1,T_2) \to \mathbb{R}$, $\rho \mapsto u(\rho)$, be a function such that $T_2 > T_1 > 0$ and the m-submanifold generated by rotating $x_0 = u(x_1)$ about the x_0-axis be a soliton with respect to $X = -\partial_0$, that is,*

$$\begin{aligned} R_{x_0}(u) : A \subset \partial_\infty \mathbb{H}^{m+1} &\to \mathbb{R} \\ (x_1, x_2, \cdots, x_m) &\mapsto u\left(\sqrt{x_1^2 + \cdots + x_m^2}\right), \end{aligned}$$

*where $A := \left\{(x_1,\cdots,x_m) \in \partial_\infty\mathbb{H}^{m+1} : T_1 < \sqrt{x_1^2+\cdots+x_m^2} < T_2\right\}$ is an annulus. The m-submanifold graph of $R_{x_0}(u)$, $M := \Gamma(R_{x_0}(u))$, is a soliton with respect to $X = -\partial_0$. M is called **rotationally symmetric graph soliton over an annulus** with respect to $X = -\partial_0$.*

Proposition 4.4.2 (ODE for a rotationally symmetric graph soliton over an annulus)**.** *Let $M = \Gamma(R_{x_0}(u))$ be a rotationally graph soliton over an annulus. Therefore, the C^2-function $u : (T_1,T_2) \to \mathbb{R}$ is a solution for:*

$$\frac{u''}{\left(1+u'^2\right)} + \frac{m-1}{\rho}u' = -\frac{1+mu}{u^2}. \tag{4.4.1}$$

Proof. Note that $\rho^2 = x_1^2 + \cdots x_m^2$, hence:

$$
\begin{aligned}
2\rho\nabla^{\mathbb{R}}\rho &= 2x_1\partial_1 + \cdots + 2x_m\partial_m,\\
\nabla^{\mathbb{R}}\rho &= \frac{x_1\partial_1 + \cdots + \partial_m}{\rho},\\
|\nabla^{\mathbb{R}}\rho|_{\mathbb{R}} &= 1,
\end{aligned}
$$

and

$$
\mathrm{div}^{\mathbb{R}}(\nabla^{\mathbb{R}}\rho) = \frac{1}{\rho}\mathrm{div}^{\mathbb{R}}(x_1\partial_1 + \cdots + x_m\partial_m) + g_{\mathbb{R}}(-\frac{1}{\rho^2}\nabla^{\mathbb{R}}\rho, x_1\partial_1 + \cdots + x_m\partial_m) = \frac{m}{\rho} - \frac{1}{\rho} = \frac{m-1}{\rho}
$$

By equation ($SE_{\pm}$),

$$
\begin{aligned}
\mathrm{div}^{\mathbb{R}}\left(\frac{\nabla^{\mathbb{R}}(R_{x_0}(u))}{\sqrt{1+|\nabla^{\mathbb{R}}R_{x_0}(u)|^2_{\mathbb{R}}}}\right) &= \frac{-mR_{x_0}(u)-1}{R_{x_0}(u)^2\sqrt{1+|\nabla^{\mathbb{R}}R_{x_0}(u)|^2_{\mathbb{R}}}}\\
\mathrm{div}^{\mathbb{R}}\left(\frac{u'(\rho)\nabla^{\mathbb{R}}\rho}{\sqrt{1+|u'(\rho)\nabla^{\mathbb{R}}\rho|^2_{\mathbb{R}}}}\right) &= \frac{-mu(\rho)-1}{u(\rho)^2\sqrt{1+|u'(\rho)\nabla^{\mathbb{R}}\rho|^2_{\mathbb{R}}}}\\
\mathrm{div}^{\mathbb{R}}\left(\frac{u'(\rho)\nabla^{\mathbb{R}}\rho}{\sqrt{1+(u'(\rho))^2}}\right) &= \frac{-mu(\rho)-1}{u(\rho)^2\sqrt{1+(u'(\rho))^2}}
\end{aligned}
$$

Define the auxiliar function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = \frac{t}{\sqrt{1+t^2}}$,

$$
\varphi'(t) = \frac{1}{(1+t^2)^{\frac{3}{2}}}
$$

. Therefore,

$$
\begin{aligned}
\mathrm{div}^{\mathbb{R}}(\varphi(u'(\rho))\nabla^{\mathbb{R}}\rho) &= -\frac{1+mu(\rho)}{u(\rho)^2\sqrt{1+u'(\rho)^2}}\\
g_{\mathbb{R}}(\varphi'(u'(\rho))u''(\rho)\nabla^{\mathbb{R}}\rho, \nabla^{\mathbb{R}}\rho) + \varphi(u'(\rho))\mathrm{div}^{\mathbb{R}}(\nabla^{\mathbb{R}}\rho) &= -\frac{1+mu(\rho)}{u(\rho)^2\sqrt{1+u'(\rho)^2}}\\
\varphi'(u'(\rho))u''(\rho) + \varphi(u'(\rho))\frac{m-1}{\rho} &= -\frac{1+mu(\rho)}{u(\rho)^2\sqrt{1+u'(\rho)^2}}\\
\frac{u''(\rho)}{(1+u'(\rho)^2)^{\frac{3}{2}}} + \frac{m-1}{\rho}\frac{u'(\rho)}{\sqrt{1+u'(\rho)^2}} &= -\frac{1+mu(\rho)}{u(\rho)^2\sqrt{1+u'(\rho)^2}}\\
\frac{u''(\rho)}{(1+u'(\rho)^2)} + \frac{m-1}{\rho}u'(\rho) &= -\frac{1+mu(\rho)}{u(\rho)^2}.
\end{aligned}
$$

□

Lemma 4.4.3. *Let u_2 and u_1 be solutions to* (4.4.1) *on an interval* (r_1, r_2). *Then, either* $u_2 \equiv u_1$ *or* $u_2 - u_1$ *does not have any non-negative local maximum on* (r_1, r_2).

Proof. Assume to the contrary that $u_2 - u_1$ attains a local maximum $c_0 \geq 0$ at a point r_0. Then, $u_2 \leq u_1 + c_0$ near r_0 with equality attained at the point r_0. From $c_0 \geq 0$ and Proposition 4.1.6, $u_1 + c_0$ is a supersolution to (4.4.1) and this contradicts the Maximum Principle (Theorem 4.1.2). □

4.5 Rotationally symmetric graph over cylinder

We are going to analyze solitons which are radially symmetric graphs over a cylinder in $\mathbb{H}^{m+1}$. We are going to use the following coordinates in $\mathbb{H}^{m+1}$:

$$x_0 = z,$$
$$x_1^2 + \cdots + x_m^2 = \rho^2,$$
$$\frac{(x_1, \ldots, x_m)}{\rho} = \omega \in \mathbb{S}^{m-1} \subset \mathbb{R}^m \approx \partial_\infty \mathbb{H}^{m+1}.$$

$(z, \rho, \omega) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{S}^{m-1} = \mathbb{H}^{m+1} - \{x_1^2 + \cdots + x_m^2 = 0\}$ and the Riemannian metric

$$\bar{g} = \frac{1}{x_0^2}(dx_0^2 + \cdots + dx_m^2) = \frac{1}{z^2}(dz^2 + d\rho^2 + \rho^2 g^2_{\mathbb{S}^{m-1}}),$$
$$\bar{g}^{-1} = z^2(\partial_z \otimes \partial_z + \partial_\rho \otimes \partial_\rho + \frac{1}{\rho^2} g^{-1}_{\mathbb{S}^{m-1}}).$$

Definition 4.5.1. *A **radially symmetric graph over the cylinder** is given by:*

$$\mathfrak{G}_\phi : A \subset \mathbb{R}_+ \times \mathbb{S}^{m-1} \to \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{S}^{m-1}$$
$$(h, \alpha) \mapsto (h, \phi(h), \alpha).$$

Now let us compute some quantities to find the mean curvature of $\mathfrak{G}_\phi$. Define the function $\mathfrak{F}_\phi : \mathbb{H}^{m+1} \to \mathbb{R}$ given by $\mathfrak{F}_\phi(z, \rho, \omega) = \rho - \phi(z)$. Hence $d\mathfrak{F}_\phi = d\rho - \phi'(z)dz$ and $\mathfrak{G}_\phi(\mathbb{R}_+ \times \mathbb{S}^{m-1}) = \mathfrak{F}_\phi{}^{-1}(0)$. The gradient of $\mathfrak{F}_\phi$ is given by $\overline{\nabla}\mathfrak{F}_\phi = i_{d\mathfrak{F}_\phi}\bar{g}^{-1} = z^2(-\phi'(z)\partial_z + \partial_\rho)$. The normal vector field ν is $\frac{\overline{\nabla}\mathfrak{F}_\phi}{|\overline{\nabla}\mathfrak{F}_\phi|_{\bar{g}}} = \frac{1}{z\sqrt{1+(\phi')^2}}(z^2(-\phi'\partial_z + \partial_\rho))$.

Using Lemma 4.3.1 for $g_{\mathbb{H}} = \frac{1}{z^2} g_{\mathbb{R}}$, where $\mathrm{w} = -\ln z$. It follows that:

Lemma 4.5.2. *Let* $\{\partial_z, \partial_\rho, \partial_{\theta 1}, \ldots, \partial_{\theta m-1}\}$ *coordinate frame for* $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{S}^{m-1}$ *with the hyperbolic metric* $g_{\mathbb{H}}$. *Then, the following equations hold:*

(a) $\nabla^{\mathbb{H}}{}_X\partial_z = -\frac{1}{z}X, \quad \forall X \in \mathfrak{X}(N)$.

(b) $\nabla^{\mathbb{H}}{}_{\partial_\rho}\partial_\rho = \frac{1}{z}\partial_z$.

(c) $\nabla^{\mathbb{H}}{}_{\partial_z}\partial_\rho = -\frac{1}{z}\partial_\rho$.

(d) $\nabla^{\mathbb{H}}{}_{\partial_{\theta i}}\partial_\rho = \nabla^{\mathbb{H}}{}_{\partial_\rho}\partial_{\theta i} = 0$.

(e) $\nabla^{\mathbb{H}}{}_{\partial_{\theta i}}\partial_{\theta j} = \nabla^{\mathbb{R}}{}_{\partial_{\theta i}}\partial_{\theta j} + g_{\mathbb{R}}(\partial_{\theta i}, \partial_{\theta j})\left(\frac{1}{z}\partial_z\right)$.

(f) $\nabla^{\mathbb{H}}{}_{\partial_z}\partial_{\theta i} = \nabla^{\mathbb{H}}{}_{\partial_{\theta i}}\partial_z = -\frac{1}{z}\partial_{\theta i}$.

(g) $\nabla^{\mathbb{H}}{}_{\partial_\rho}\partial_{\theta i} = \nabla^{\mathbb{H}}{}_{\partial_{\theta i}}\partial_\rho = 0$.

Using Lemma 4.5.2 we can compute the second fundamental form and mean curvature of the graph of $\mathfrak{G}_\phi$.

Lemma 4.5.3. *Let $\{\partial_h, \partial_{\xi_1}, \dots, \partial_{\xi_{m-1}}\}$ be a coordinate frame for $\mathbb{R}_+ \times \mathbb{S}^{m-1}$ such that the set of last vector fields $\{\partial_{\xi_1}, \dots, \partial_{\xi_{m-1}}\}$ is coordinate frame for $\mathbb{S}^{m-1}$. As before let the set $\{\partial_z, \partial_\rho, \partial_{\theta 1}, \dots, \partial_{\theta m-1}\}$ be coordinate frame for the codomain $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{S}^{m-1}$ with the hyperbolic metric $g_{\mathbb{H}}$. Define $E_0 := \mathfrak{G}_{\phi_*}(\partial_h) = \partial_z + \phi'\partial_\rho$ and $E_\alpha := \mathfrak{G}_{\phi_*}(\partial_{\xi_\alpha}) = \partial_{\theta\alpha}$ for $1 \le \alpha \le m-1$. Denote by H the (unnormalized) mean curvature of $\mathfrak{G}_\phi$. Then the following equations hold:*

(a) $\nabla^{\mathbb{H}}{}_{E_0}E_0 = \left(-\frac{1}{z} + \frac{\phi'^2}{z}\right)\partial_z + \left(-\frac{2\phi'}{z} + \phi''\right)\partial_\rho$.

(b) $g_{\mathbb{H}}\left(\nabla^{\mathbb{H}}{}_{E_0}E_0, \nu\right) = \frac{1}{z^2}\frac{1}{\sqrt{1+(\phi')^2}}[-\phi' - (\phi')^3 + \phi'' z]$.

(c) $\nabla^{\mathbb{H}}{}_{E_\alpha}E_\beta = \nabla^{\mathbb{H}}{}_{\partial_{\theta\alpha}}\partial_{\theta\beta} = \nabla^{\mathbb{R}}{}_{\partial_{\theta\alpha}}\partial_{\theta\beta} + g_{\mathbb{R}}(\partial_{\theta\alpha}, \partial_{\theta\beta})\left(\frac{1}{z}\partial_z\right)$.

(d) $g_{\mathbb{H}}\left(\nabla^{\mathbb{H}}{}_{E_\alpha}E_\beta, \nu\right) = g_{\mathbb{H}}(\partial_{\theta\alpha}, \partial_{\theta\beta})\left(-\frac{\phi'}{\sqrt{1+(\phi')^2}}\right) + g_{\mathbb{H}}\left(\nabla^{\mathbb{R}}{}_{\partial_{\theta\alpha}}\partial_{\theta\beta}, \frac{z}{\sqrt{1+(\phi')^2}}\partial_\rho\right)$.

(e) $H = \frac{1}{\sqrt{1+(\phi')^2}}\left(\frac{\phi'' z}{1+(\phi)^2} - m\phi'\right) - \frac{(m-1)z}{\phi\sqrt{1+(\phi')^2}}$.

(f) $g_{\mathbb{H}}(-\partial_z, \nu) = \frac{\phi'}{z\sqrt{1+(\phi')^2}}$.

Proof. (a)

$$\begin{aligned}
\nabla^{\mathbb{H}}{}_{E_0}E_0 &= \nabla^{\mathbb{H}}{}_{E_0}(\partial_z + \phi'\partial_\rho) \\
&= \nabla^{\mathbb{H}}{}_{E_0}\partial_z + \nabla^{\mathbb{H}}{}_{E_0}(\phi'\partial_\rho) \\
&= \nabla^{\mathbb{H}}{}_{\partial_z+\phi'\partial_\rho}\partial_z + \phi''\partial_\rho + \phi'\nabla^{\mathbb{H}}{}_{\partial_z+\phi'\partial_\rho}\partial_\rho \\
&= \nabla^{\mathbb{H}}{}_{\partial_z}\partial_z + \phi'\nabla^{\mathbb{H}}{}_{\partial_\rho}\partial_z + \phi''\partial_\rho + \phi'\nabla^{\mathbb{H}}{}_{\partial_z}\partial_\rho + (\phi')^2\nabla^{\mathbb{H}}{}_{\partial_\rho}\partial_\rho \\
&= -\frac{1}{z}\partial_z + \phi'\left(-\frac{1}{z}\partial_\rho\right) + \phi''\partial_\rho + \phi'\left(-\frac{1}{z}\partial_\rho\right) + (\phi')^2\frac{1}{z}\partial_z \\
&= \left(-\frac{1}{z} + \frac{\phi'^2}{z}\right)\partial_z + \left(-\frac{2\phi'}{z} + \phi''\right)\partial_\rho.
\end{aligned}$$

(b)

$$
\begin{aligned}
&g_{\mathbb{H}}(\nabla^{\mathbb{H}}{}_{E_0}E_0,\nu) = \\
&= g_{\mathbb{H}}\left(\left(\frac{-1+(\phi')^2}{z}\right)\partial_z + \left(\frac{-2\phi'+\phi'' z}{z}\right)\partial_\rho, \frac{z}{\sqrt{1+(\phi')^2}}\left(-\phi'\partial_z+\partial_\rho\right)\right) \\
&= (-1+(\phi')^2)\frac{1}{\sqrt{1+(\phi')^2}}(-\phi)|\partial_z|^2_{g_{\mathbb{H}}} + (-2\phi'+\phi'' z)\frac{1}{\sqrt{1+(\phi')^2}}|\partial_\rho|^2_{g_{\mathbb{H}}} \\
&= \frac{1}{z^2}\frac{1}{\sqrt{1+(\phi')^2}}(+\phi' - (\phi')^3 - 2\phi' + \phi'' z) = \\
&= \frac{1}{z^2}\frac{1}{\sqrt{1+(\phi')^2}}(-\phi' - (\phi')^3 + \phi'' z).
\end{aligned}
$$

(c) It follows by (e) of lemma 4.5.2.

(d)

$$
\begin{aligned}
&g_{\mathbb{H}}\left(\nabla^{\mathbb{H}}{}_{E_\alpha}E_\beta,\nu\right) = \\
&= g_{\mathbb{H}}\left(\nabla^{\mathbb{R}}{}_{\partial_{\theta\alpha}}\partial_{\theta\beta} + \mathrm{g}_{\mathbb{R}^{m+1}}(\partial_{\theta\alpha},\partial_{\theta\beta})\left(\frac{1}{z}\partial_z\right), \frac{z}{\sqrt{1+(\phi')^2}}(-\phi'\partial_z+\partial_\rho)\right) \\
&= g_{\mathbb{H}}(\partial_{\theta\alpha},\partial_{\theta\beta})\left(-\frac{\phi'}{\sqrt{1+(\phi')^2}}\right) + g_{\mathbb{H}}\left(\nabla^{\mathbb{R}}{}_{\partial_{\theta\alpha}}\partial_{\theta\beta}, \frac{z}{\sqrt{1+(\phi')^2}}\partial_\rho\right).
\end{aligned}
$$

(e)

$$
\begin{aligned}
H &= (\mathfrak{G}_\phi{}^* g_{\mathbb{H}})^{00} g_{\mathbb{H}}(\nabla^{\mathbb{H}}{}_{E_0}E_0,\nu) + \sum_{\alpha,\beta}(\mathfrak{G}_\phi{}^* g_{\mathbb{H}})^{\alpha\beta} g_{\mathbb{H}}(\nabla^{\mathbb{H}}{}_{E_\alpha}E_\beta,\nu) \\
&= \frac{z^2}{1+(\phi')^2}\left(\frac{1}{z^2}\frac{1}{\sqrt{1+(\phi')^2}}(-\phi' - (\phi')^3 + \phi'' z)\right) + \\
&+ \sum_{\alpha,\beta}(\mathfrak{G}_\phi{}^* g_{\mathbb{H}})^{\alpha\beta}\left((\mathfrak{G}_\phi{}^* g_{\mathbb{H}})_{\alpha\beta}\left(-\frac{\phi'}{\sqrt{1+(\phi')^2}}\right) + g_{\mathbb{H}}\left(\nabla^{\mathbb{R}}{}_{\partial_{\theta\alpha}}\partial_{\theta\beta}, \frac{z}{\sqrt{1+(\phi')^2}}\partial_\rho\right)\right) \\
&= \frac{1}{\sqrt{1+(\phi')^2}}\left(\frac{-\phi' - (\phi')^3 + \phi'' z}{1+(\phi')^2} - (m-1)\phi'\right) - \frac{(m-1)z}{\phi\sqrt{1+(\phi')^2}}.
\end{aligned}
$$

It was used the Euclidean mean curvature of the sphere of radius ϕ and dimension $m-1$ is $\frac{m-1}{\phi}$.

(f)

$$
\begin{aligned}
g_{\mathbb{H}}(-\partial_z,\nu) &= g_{\mathbb{H}}\left(-\partial_z, \frac{z}{\sqrt{1+(\phi')^2}}(-\phi'\partial_z+\partial_\rho)\right) \\
&= \frac{\phi'}{z\sqrt{1+(\phi')^2}}.
\end{aligned}
$$

□

Now it follows that the cylindrical graph of $\mathfrak{G}_\phi$ is a soliton with respect to $-\partial_0 = -\partial_z$ if and only if $\frac{1}{\sqrt{1+(\phi')^2}}\left(\frac{\phi'' z}{1+(\phi)^2} - m\phi'\right) - \frac{(m-1)z}{\phi\sqrt{1+(\phi')^2}} = H = \bar{g}(-\partial_z, \nu) = \frac{\phi'}{z\sqrt{1+(\phi')^2}}$. Equivalently,

$$\frac{\phi'' z}{1+(\phi')^2} - m\phi' - \frac{(m-1)z}{\phi} = \frac{\phi'}{z}. \tag{4.5.1}$$

Proposition 4.5.4 (ODE for cylindrical graph)**.** *A cylindrical graph* $\mathrm{Im}(\mathfrak{G}_\phi)$ *is a soliton with respect to* $-\partial_0$ *if and only if*

$$\frac{\phi''}{1+(\phi')^2} - \frac{1+mz}{z^2}\phi' - \frac{(m-1)}{\phi} = 0. \tag{4.5.2}$$

In order to understand the rotationally symmetric graph over a cylinder, we need to study the previous ODE.

4.6 Energy Method

In this section, we define an energy F to help us to analyze the qualitative behaviour of cylindrical graphs $\mathfrak{G}_\phi$. By (4.5.2),

$$\frac{\phi'' z}{1+(\phi')^2} - m\phi' - \frac{(m-1)z}{\phi} = \frac{\phi'}{z}.$$

Define $F(z) := z^{-m}\frac{\phi'}{\sqrt{1+(\phi')^2}}$. Hence,

$$F'(z) = \frac{1}{z^{m+1}\sqrt{1+\phi'^2}}\left[\frac{z\phi''}{1+(\phi')^2} - m\phi'\right].$$

By (4.5.2), $\frac{z\phi''}{1+\phi'^2} - m\phi' = \frac{\phi'}{z} + (m-1)\frac{z}{\phi}$. Therefore,

$$F'(z) = \frac{1}{z^{m+1}\sqrt{1+\phi'^2}}\left[\frac{\phi'}{z} + (m-1)\frac{z}{\phi}\right] = \frac{1}{z^2}F + \frac{m-1}{z^m\phi\sqrt{1+\phi'^2}}.$$

Thus, $F' - \frac{1}{z^2}F = \frac{m-1}{z^m\phi\sqrt{1+\phi'^2}}$. Multiplying by $e^{\int_z^h \frac{1}{r^2}dr}$,

$$(Fe^{\int_z^h \frac{1}{r^2}dr})' = e^{\int_z^h \frac{1}{r^2}dr}\frac{m-1}{z^m\phi\sqrt{1+\phi'^2}}.$$

Integrating,

$$F(h) - F(z)e^{\int_z^h \frac{1}{r^2}dr} = \int_z^h \frac{m-1}{r^m\phi(r)\sqrt{1+\phi'(r)^2}}e^{\int_r^h \frac{1}{s^2}ds}\, dr \tag{4.6.1}$$

$$F(h) - F(z)e^{-\frac{1}{h}+\frac{1}{z}} = \int_z^h \frac{m-1}{r^m\phi(r)\sqrt{1+\phi'(r)^2}}e^{-\frac{1}{h}+\frac{1}{r}}\, dr \tag{4.6.2}$$

To understand the behaviour of the solution to (4.5.2), it is necessary to prove some technical lemmas.

Lemma 4.6.1 (Behaviour of concave branch)**.** *Given* $z_0 > 0$*,* $\tau_0 > 0$ *and* $\phi_0' > 0$*. Let* $\phi : (h_*, h^*) \to \mathbb{R}$ *be the maximal solution to:*

$$\begin{cases} \dfrac{\phi''}{1+(\phi')^2} - \dfrac{1+mz}{z^2}\phi' - \dfrac{(m-1)}{\phi} = 0, \\ \phi(z_0) = \tau_0, \\ \phi'(z_0) = -\phi_0'. \end{cases} \tag{4.6.3}$$

where $z_0 \in (h_*, h^*)$*. If* $\phi''(z_0) \le 0$*, then the following statements hold true:*

(a) $\phi' < 0$*,* $\phi'' < 0$ *on* (h_*, z_0)*.*

(b) $h_* = 0$*.*

(c) $\lim_{z\to 0^+} \phi(z) = \phi_0$ *for some* $\phi_0 > 0$*.*

(d) $\lim_{z\to 0^+} \phi'(z) = 0$*.*

Proof. (a) We claim that $\phi'(z) < 0 \quad \forall z \in (h_*, z_0)$. Otherwise, we would have a $z_c \in (h_*, z_0)$ such that $\phi'(z_c) = 0$ and $\phi' < 0$ on (z_c, z_0). By (4.6.3), $\phi''(z_c) > 0$, therefore ϕ' is increasing on a neighborhood of z_c, contradicting that $\phi' < 0$ on (z_c, z_0). Thus, $\phi' < 0$ on (h_*, z_0).

We claim that $\phi'' \le 0$ on (h_*, z_0). Otherwise, we would have a interval $(a, b] \subset (h_*, z_0]$ such that $\phi'' > 0$ on (a, b) and $\phi''(b) = 0$. By (4.6.3),

$$-\phi'(a)\left(m + \frac{1}{a}\right) \le (m-1)\frac{a}{\phi(a)} \text{ and } -\phi'(b)\left(m + \frac{1}{b}\right) = (m-1)\frac{b}{\phi(b)}.$$

Using that ϕ is decreasing on (h_*, z_0), we have:

$$0 < \phi(b) < \phi(a) \tag{4.6.4}$$

$$\frac{1}{\phi(b)} > \frac{1}{\phi(a)} \tag{4.6.5}$$

$$(m-1)\frac{b}{\phi(b)} > (m-1)\frac{a}{\phi(a)} \tag{4.6.6}$$

$$-\phi'(b)\left(m+\frac{1}{b}\right) > -\phi'(a)\left(m+\frac{1}{a}\right) \tag{4.6.7}$$

$$-\phi'(b) > -\phi'(a)\frac{\left(m+\frac{1}{a}\right)}{\left(m+\frac{1}{b}\right)} > -\phi'(a) \tag{4.6.8}$$

$$\phi'(b) < \phi'(a) < 0. \tag{4.6.9}$$

Contradicting that ϕ' is increasing on $(a,b]$. Thus, $\phi'' \leq 0$ on (h_*, z_0). Suppose by contradiction that $\phi''(z_i) = 0$ for some $z_i \in (h_*, z_0)$. Therefore, z_i is a critical point for ϕ'' on (h_*, z_0), hence $\phi'''(z_i) = 0$. Differentianting (4.6.3),

$$\left[\frac{\phi'''}{1+(\phi')^2} - \frac{2(\phi'')^2\phi'}{(1+(\phi')^2)^2}\right] - \left[\left(\frac{-2-mz}{z^3}\right)\phi' + \left(\frac{1+mz}{z^2}\right)\phi''\right] + \frac{m-1}{\phi^2}\phi' = 0 \tag{4.6.10}$$

Therefore, $\left(\left(\frac{2+mz}{z^3}\right) + \frac{m-1}{\phi^2}\right)\phi' = 0$ at $z = z_i$, contradicting that $\left(\left(\frac{2+mz}{z^3}\right) + \frac{m-1}{\phi^2}\right) > 0$ and $\phi'(z_i) < 0$. Thus, $\phi'' < 0$ on (h_*, z_0).

(b) Suppose by contradiction that $h_* > 0$. We claim that:

(i) $\lim_{z\to {h_*}^+} \phi(z) = \phi_{h_*}$ for some $\phi_{h_*} > 0$.

(ii) $\lim_{z\to {h_*}^+} \phi'(z) = -\phi'_{h_*}$ for some $\phi'_{h_*} \geq 0$.

If (i) and (ii) are true, then we could extend solution $\phi : (h_*, h^*) \to \mathbb{R}$, a contradiction by maximality of (h_*, h^*). Namely,

(i) As ϕ is decreasing on (h_*, z_0), either $\lim_{z\to {h_*}^+} \phi(z) = \infty$ or $\lim_{z\to {h_*}^+} \phi(z) = \phi_{h_*}$ for some $\phi_{h_*} > 0$. If $\lim_{z\to {h_*}^+} \phi(z) = \infty$, we find a contradiction by Maximum Principle (Proposition C.0.1). Indeed, we put a small grim-reaper $\mathcal{G}_{h,[H_*,H^*]}$ (Remark 4.2.1) below $\mathrm{Im}(\mathfrak{G}_\phi)$, that is, $h < h_*$ and $H_* > \phi(z_0)$. Setting $\mathcal{G}_t := \mathcal{G}_{h_t,[H_*(h_t),H^*(h_t)]}$ where $h_t := h + t$ and $H_*(h_t) := H_*(h)$. Increasing t, there exist a $\mathcal{G}_{t_0}$ such that $\mathcal{G}_{t_0} \cap \mathrm{Im}\,\mathfrak{G}_\phi \neq \emptyset$, contradiction by the Maximum Principle. Thus, the only possibility remained is $\lim_{z\to {h_*}^+} \phi(z) = \phi_{h_*}$ for some $\phi_{h_*} > 0$.

(ii) As $\phi'' < 0$ on (h_*, z_0), ϕ' is decreasing on (h_*, z_0). As $\phi' < 0$ on (h_*, z_0), ϕ' is bounded therefore $\lim_{z\to h_*} \phi'(z)$ does exist.

Thus, h_* must be equal to zero.

(c) As ϕ is decreasing, we just have to exclude the possibility that $\lim_{z\to 0^+} \phi(z) = \infty$. Again, we use the Maximum Principle to arrive at a contradiction. Namely, put a small grim-reaper below $\mathrm{Im}(\mathfrak{G}_\phi)$ and slide the grim-reaper in the direction x_1 up to touching $\mathrm{Im}(\mathfrak{G}_\phi)$, a contradiction by Maximum Principle.

(d) As $\phi'' < 0$ on $(0, z_0)$, ϕ' is decreasing. As $\phi' < 0$ on $(0, z_0)$, $\lim_{z\to 0^+} \phi'(z) = -\phi'_0$ for some $\phi'_0 \geq 0$. By argument using a spherical barrier, ϕ'_0 cannot be positive. Thus, $\phi'_0 = 0$.

□

Lemma 4.6.2 (Behaviour of a cylindrical graph with a critical point)**.** *Given $z_1 > 0$ and $\tau > 0$. Let $\phi : (h_*, h^*) \to \mathbb{R}$ be the maximal solution to:*

$$\begin{cases} \dfrac{\phi''}{1+(\phi')^2} - \dfrac{1+mz}{z^2}\phi' - \dfrac{(m-1)}{\phi} = 0, \\ \phi(z_1) = \tau, \\ \phi'(z_1) = 0. \end{cases} \tag{4.6.11}$$

Then the following statements hold:

(a) Every critical point of ϕ is a local minimum point.

(b) The only critical point of ϕ is $z = z_1$ and ϕ attains global minimum at $z = z_1$.

(c) ϕ is strictly decreasing on (h_*, z_1) and strictly increasing and convex on (z_1, h^*).

(d) $h_* = 0$.

(e) $\lim_{z\to 0^+} \phi(z) = \tau_*$ for some $\tau_* > 0$.

(f) h^* is finite, $\lim_{z\to h^{*-}} \phi(z) = \tau^*$ for some $\tau^* > 0$ and $\lim_{z\to h^{*-}} \phi'(z) = \infty$.

(g) There exists $\lambda_0 \in (0, z_1)$ such that $\phi''(\lambda_0) = 0$, $\phi'(\lambda_0) < 0$ and $\phi''(z) > 0, \forall z \in (\lambda_0, z_1)$.

(h) $\lim_{z\to 0^+} \phi'(z) = 0$.

(i) $\phi''(z) < 0 \quad \forall z \in (0, \lambda_0)$ and $\phi''(z) > 0, \forall z \in (\lambda_0, z_1)$.

(j) ϕ' is bounded on $(0, z_1)$.

Proof. (a) By (4.6.11), $\phi''(z) > 0$ whenever $\phi'(z) = 0$.

(b) Suppose by contradiction that there exists another critical point $z = z_c$ for ϕ. By (a), $z = z_c$ is a local minimum. We must have a local maximal point between the two local minimum points $z = z_c$ and $z = z_1$, however, this is not possible by (a). Thus, $z = z_1$ is the only minimum point in the maximal interval (h_*, h^*).

(c) Suppose by contradiction that there exists $z_+ \in (h_*, z_1)$ such that $\phi'(z_+) > 0$. There exists a interval $(z_1 - \epsilon, z_1)$ where $\phi' < 0$ by minimality at $z = z_1$. Therefore, by continuity of ϕ', there exists a critical point on (z_+, z_1) contradicting (b). Similarly , we prove that $\phi' > 0$ on (z_1, h^*). Therefore, by (4.6.11), $\phi'' > 0$ on (z_1, h^*).

(d) By (c), there exists $z_0 \in (h_*, z_0)$ such that $\phi'(z_0) < 0$. Therefore, by Lemma 4.6.1.(b), $h_* = 0$.

(e) Suppose by contradiction that $\lim_{z\to 0^+} \phi(z) = \infty$. We can put a small spherical barrier $\mathcal{B}_{\frac{\pi}{2}}(o, r_0)$ (see Proposition 4.3.3) below the cylindrical graph $\mathrm{Im}(\mathfrak{G}_\phi)$ and we slide $\mathcal{B}_{\frac{\pi}{2}}(o, r_0)$ in the direction of x_0 until some part of the spherical barrier has x_0-coordinate greater than x_0-coordinate of some part of $\mathrm{Im}(\mathfrak{G}_\phi)$ gives a contradiction by Comparison Principle (Proposition 4.1.3). Therefore, $\lim_{z\to 0^+} \phi(z)$ is finite.

(f) Suppose by contradiction that $h^* = \infty$. As ϕ is increasing on (z_1, h^*), there are two situations:

  (i) $\lim_{z\to\infty} \phi(z) = \infty$.

  (ii) $\lim_{z\to\infty} \phi(z) = \tau_0$ for some $\tau_0 > 0$.

We will show that (i) and (ii) are impossible:

  (i) In this case, we can put a grim-reaper $\mathcal{G}_{h,[H_*,H^*]}$ below the cylindrical graph $\mathrm{Im}(\mathfrak{G}_\phi)$ and increasing h until $\mathcal{G}_{h,[H_*,H^*]}$ touches $\mathrm{Im}(\mathfrak{G}_\phi)$ in $\mathrm{Im}(\mathfrak{G}_{\phi|_{(z_1,\infty)}})$ finding a contradiction by Maximum Principle (Proposition C.0.1). Namely, there exists a small grim-reaper $\mathcal{G}_{h_0,[H_*(h_0),H^*(h_0)]}$ such that $x_1(\mathcal{G}_{h_0,[H_*(h_0),H^*(h_0)]}) > \tau_*$ and $h_0 < z_1$, that is, $\mathcal{G}_{h_0,[H_*(h_0),H^*(h_0)]} \subset \{x_1 > \tau_*\} \cap \{x_0 < z_1\}$. Set $\mathcal{G}_t := \mathcal{G}_{h_t,[H_*(h_t),H^*(h_t)]}$ where $t > 0$, $h_t := h_0 + t$ and $H_*(h_t) := H_*(h_0)$. Notice that $\mathcal{G}_t \subset \{x_1 > \tau_*\}$. Set $t_s := \sup\{t > 0 : \mathcal{G}_r \cap \mathrm{Im}(\mathfrak{G}_\phi) = \emptyset \quad \forall 0 < r < t\}$. Therefore, $\mathcal{G}_{t_s} \cap \mathrm{Im}(\mathfrak{G}_{\phi|_{(z_1,\infty)}}) \neq \emptyset$ contradicting the Maximum Principle.

  (ii) In this case, there exists a inflection point $r_i \in (z_1, \infty)$ such that $\phi''(r_i) = 0$. By (4.6.11), $\phi'(r_i) < 0$, a contradiction by (c).

Thus, h^* is finite. Now suppose by contradiction that $\lim_{z\to h^{*-}} \phi(z) = \infty$. We use a similar argument as in (a). Therefore, there exists $\tau^* > 0$ such that $\lim_{z\to h^{*-}} \phi(z) = \tau^*$. By (III), $\phi' > 0$ on (z_1, h^*). Hence, by (4.6.11), $\phi'' > 0$ on (z_1, h^*) and ϕ' is strictly increasing on (z_1, h^*). The limit of $\phi'(z)$ as $z \to h^{*-}$ cannot be finite because we could extend the solution ϕ contradicting the maximality of h^*. Thus, $\lim_{z\to h^{*-}} \phi'(z) = \infty$.

(g) Define λ_0 as the smallest number in $[0, z_1]$ such that $\phi''(z) > 0 \quad \forall z \in (\lambda_0, z_1)$. We claim that:

(i) $\lambda_0 > 0$

(ii) $\lim_{z\to\lambda_0^+}\phi(z)$ is finite.

(iii) $\phi'(\lambda_0) < 0$

(iv) $\phi''(\lambda_0) = 0$

Indeed, we prove this in the following way:

(i) Let us assume the opposite, $\lambda_0 = 0$, and see if it leads to a contradiction. As $\phi'' > 0$ on $(0, z_1)$, ϕ' is increasing on $(0, z_1)$. Therefore, either $\lim_{z\to 0^+}\phi'(z) = -\infty$ or $\lim_{z\to 0^+}\phi'(z) = -\phi'_{\lambda_0}$ for some positive number $\phi'_{\lambda_0} > 0$. In any case, by (4.6.11) and (e), $\lim_{z\to 0^+}\phi''(z) = -\infty$, contradicting that $\phi'' > 0$ on $(0, z_1)$. Thus, λ_0 must be positive.

(ii) $\lim_{z\to\lambda_0^+}\phi(z)$ cannot be infinity because we can touch $\mathrm{Im}(\mathfrak{G}_{\phi|_{(\lambda_0,z_1)}})$ with a grim-reaper from below finding a contradiction by Maximum Principle as in the proof of (d).

(iii) Suppose by contradiction that $\lim_{z\to\lambda_0}\phi'(z) = -\infty$. Therefore, changing coordinates for a rotationally symmetric graph over an annulus (Definition 4.4.1), $(z, \phi(z)) = (u(\rho), \rho)$. Setting $\rho_0 := \phi(\lambda_0{}^+)$, $u(\rho_0) = \lambda_0$ and $u'(\rho_0) = 0$. Therefore, by Equation (4.4.1), $u''(\rho_0) < 0$, a contradiction by the behaviour of $\mathrm{Im}(\phi|_{(\lambda_0,z_1)})$. Thus, we conclude that $\lim_{z\to\lambda_0^+}\phi'(z) = -\phi'_{\lambda_0}$ for some positive number $\phi'_{\lambda_0} > 0$.

(iv) By continuity of ϕ'' and $\phi'' > 0$ on (λ_0, z_1), $\phi''(\lambda_0) \geq 0$. Notice that $\phi''(\lambda_0)$ cannot be positive because we otherwise could extend on the left side the interval where $\phi'' > 0$ beyond (λ_0, z_1). Thus, $\phi''(\lambda_0) = 0$.

(h) Using spherical barrier, we can assure that the graph of ϕ in x_0x_1-plane meets orthogonally the x_1-axis, $\{x_0 = 0\}$.

(i) Differentianting (4.6.11),

$$\left[\frac{\phi'''}{1+(\phi')^2} - \frac{2(\phi'')^2\phi'}{(1+(\phi')^2)^2}\right] - \left[\left(\frac{-2-mz}{z^3}\right)\phi' + \left(\frac{1+mz}{z^2}\right)\phi''\right] + \frac{m-1}{\phi^2}\phi' = 0 \qquad (4.6.12)$$

Therefore, as $\phi' < 0$ on $(0, z_1)$, if $\phi''(z) = 0$ for $z \in (0, z_1)$, then, by Equation (4.6.12), $\phi'''(z) > 0$. Thus, every critical point of ϕ' on $(0, z_1)$ is a local minimum point. As in (II), $\phi'|_{(0,z_1)}$ attains global minimum at $z = \lambda_0$ and $\phi'' < 0$ on $(0, \lambda_0)$ and $\phi'' > 0$ on (λ_0, z_1).

(j) By (g) and (h), $\mathrm{Im}\,\phi'|_{(0,z_1)} \subset (0, \phi'(\lambda_0)]$. Thus, $\phi'|_{(0,z_1)}$ is bounded.

□

Lemma 4.6.3. *For $a \in (0,h^*)$, the quantity $\mathcal{I}(z) := \int_z^a \frac{m-1}{r^m\phi(r)\sqrt{1+\phi'(r)^2}} e^{-\frac{1}{a}+\frac{1}{r}}\, dr \to \infty$ as $z \to 0$.*

Proof. Noticing that $|\phi'|$ is bounded on $(0,a)$ and setting $K := \max_{r\in(0,a)} \phi(r)\sqrt{1+\phi'(r)^2}$,

$$\int_z^a \frac{m-1}{r^m\phi(r)\sqrt{1+\phi'(r)^2}} e^{-1/a+1/r} dr > \frac{(m-1)e^{-1/a}}{K}\int_z^a \frac{e^{1/r}}{r^m} dr$$
$$> \frac{(m-1)e^{-1/a}}{K}\int_z^a \frac{1}{r^m} dr = \frac{(m-1)e^{-1/a}}{K}(-m+1)\left(\frac{1}{a^{m+1}} - \frac{1}{z^{m+1}}\right).$$

Therefore, $\lim_{z\to 0} \mathcal{I}(z) = \infty$.

□

Proposition 4.6.4 (Behaviour of a solution for a symmetric graph soliton over an anullus)**.** *Given a radius $R > 0$ and a height $h > 0$. If $u : (T_1,T_2) \to \mathbb{R}$ is a maximal solution to:*

$$\begin{cases} \dfrac{u''}{(1+u'^2)} + \dfrac{m-1}{\rho}u' = -\dfrac{1+mu}{u^2}, \\ u(R) = h, \\ u'(R) = 0. \end{cases} \tag{4.6.13}$$

Then it follows that:

(a) $\rho = R$ is a global maximum point for u.

(b) T_2 is finite.

(c) $T_1 > 0$.

(d) $u'' < 0$ on (T_1,T_2).

(e) $\lim_{\rho\to T_1^+} u(\rho) = z_1 > 0$ and $\lim_{\rho\to T_1^+} u'(\rho) = +\infty$

(f) $\lim_{\rho\to T_2^-} u(\rho) = 0$ and $\lim_{\rho\to T_2^-} u'(\rho) = -\infty$

Proof. (a) We claim that the only critical point is $\rho = R$. Namely, by (4.4.1), $\frac{u''}{(1+u'^2)} = -\frac{1+mu}{u^2}$ at $\rho = R$. Hence $u''(R) < 0$. Therefore $\rho = R$ is a strict local maximum point for u. There is no other critical point besides $\rho = R$. Indeed, suppose, by contradiction, that there exists another critical point $R_1 \in (T_1,T_2)$, $R_1 \neq R$. Again by (4.4.1), $\rho = R_1$ is a strict local maximum point. This is a contradiction by continuity of u'. Therefore, $u' > 0$ in (T_1,R) and $u' < 0$ in (R,T_2). Thus, $\rho = R$ is the global maximum point.

(b) Suppose, by contradiction, that the solution u is defined (T_1,∞), that is, $T_2 = \infty$. We know that u is decreasing in (R,∞). Hence, because u is decreasing and bounded below in

(R,∞), there exists $z_1 \geq 0$ such that $\lim_{\rho\to\infty} u(\rho) = z_1$. Then we can find a small spherical barrier $\mathcal{B}_{\frac{\pi}{2}}(o,r_0)$ (see Proposition 4.3.3) with radius $r_0 > z_1$ and below $M = \Gamma(R_{x_0}(u))$. To use the Maximum Principle (Proposition C.0.1), we increase the radius r_0 until the spherical barrier touches M. Namely, set $I = \{r > 0 : \mathcal{B}(o,r_1) \cap M = \emptyset \quad \forall r_1 \leq r\}$ and $r_S = \sup I$. $\mathcal{B}(o,r_S) \cap M \neq \emptyset$ and $\mathcal{B}(o,r_S)$ touches M inside $\mathbb{H}^{m+1}$. This is a contradiction by Proposition C.0.1. Thus, T_2 is finite as claimed.

(c) Set the auxiliary function $\varphi : \mathbb{R} \to (-1,1)$, $\varphi(t) = \frac{t}{\sqrt{1+t^2}}$, $\varphi'(t) = \frac{1}{(1+t^2)^{\frac{3}{2}}}$

By Equation 4.4.1,

$$\frac{u''}{1+(u'^2)} + \frac{m-1}{\rho}u' = -\frac{1+mu}{u^2} \tag{4.6.14}$$

$$\frac{u''}{1+(u'^2)}\frac{\rho^{m-1}}{\sqrt{1+(u')^2}} + \frac{m-1}{\rho}u'\frac{\rho^{m-1}}{\sqrt{1+(u')^2}} = -\frac{1+mu}{u^2}\frac{\rho^{m-1}}{\sqrt{1+(u')^2}} \tag{4.6.15}$$

$$\varphi'(u')u''\rho^{m-1} + \varphi(u')(\rho^{m-1})' = -\frac{1+mu}{u^2}\frac{\rho^{m-1}}{\sqrt{1+(u')^2}} \tag{4.6.16}$$

$$(\varphi(u')\rho^{m-1})' = -\frac{1+mu}{u^2}\frac{\rho^{m-1}}{\sqrt{1+(u')^2}} < 0. \tag{4.6.17}$$

Set the auxiliary function $\Phi(\rho) := \varphi(u'(\rho))\rho^{m-1}$. Hence Φ is strictly decreasing, $\Phi' < 0$. $\varphi(u'(\rho))\rho^{m-1} = \Phi(\rho) > \Phi(R-\epsilon) = \varphi(u'(R-\epsilon))(R-\epsilon)^{m-1}$, for a small $\epsilon > 0$, and $\rho \in (T_1, R-\epsilon)$. As $\operatorname{Im}\varphi = (-1,1)$, $\rho^{m-1} > \varphi(u'(R-\epsilon))(R-\epsilon)^{m-1} > 0$. Hence, as $\rho \to T_1$, $T_1^{m-1} \geq \varphi(u'(R-\epsilon))(R-\epsilon)^{m-1} > 0$. Thus, T_1 cannot be equal to zero as claimed.

(d) By (4.6.13) and $u' \geq 0$ on $(T_1,R]$, $u'' < 0$ on (T_1,R). We only need to prove it for (R,T_2). The solution u is defined in some neighborhood of R, say $[R-2\delta, R+2\delta]$ for some small $\delta > 0$. As $u''(R+\delta) < 0$ and $u'(R+\delta) < 0$, we can change the coordinates for a rotationally symmetric graph over the cylinder on the left side of R, that is, $(u(\rho),\rho) = (z,\phi(z))$ on $\{(u(\rho),\rho) : R < \rho < R+2\delta\}$. Setting $z_0 := u(R+\delta)$, $\tau_0 := R+\delta$ and $\phi_0' := \frac{1}{\frac{du}{d\rho}(R+\delta)}$. $\frac{d^2\phi}{dz^2}(z_0) < 0$ and Lemma 4.6.1, $\phi'' < 0$ on $(0,z_0)$ therefore $u'' < 0$ on (R,T_2).

(e) As $u' > 0$ and $u > 0$ in (T_1,R), the limit $\lim_{\rho\to T_1} u(\rho)$ exists. As $u'' < 0$ in (T_1,T_2), the limit $\lim_{\rho\to T_1} u'(\rho)$ exists or is $+\infty$. Therefore, we have the following possibilities:

(i) $\lim_{\rho\to T_1} u(\rho) = z_1 > 0$ and $\lim_{\rho\to T_1} u'(\rho) = z_1' > 0$.
(ii) $\lim_{\rho\to T_1} u(\rho) = 0$ and $\lim_{\rho\to T_1} u'(\rho) = z_1' > 0$.
(iii) $\lim_{\rho\to T_1} u(\rho) = 0$ and $\lim_{\rho\to T_1} u'(\rho) = \infty$.
(iv) $\lim_{\rho\to T_1} u(\rho) = z_1 > 0$ and $\lim_{\rho\to T_1} u'(\rho) = \infty$.

(i) It is not possible because we could extend the solution contradicting the maximality of T_1.

(ii) It is not possible because because we could touch $M = \Gamma(R_{x_0}(u))$ from below with a small spherical barrier $\mathcal{B}_{\frac{\pi}{2}}(o, r_0)$ (see Proposition 4.3.3) contradicting the Maximum Principle (Proposition C.0.1).

(iii) Here we need to change the coordinates and see part of the rotationally symmetric graph as a radially symmetric over a cylinder. Set $M_1 := \Gamma(R_{x_0}(u|_{(T_1,R)})$. M_1 can be viewed as a radially symmetric graph over a cylinder. Hence we find a $\phi : (0,h) \longrightarrow \mathbb{R}_+$ such that the image of $\mathfrak{G}_\phi$ is M_1. By the Lemma 4.6.3 and the Equation 4.6.1, $F(z)e^{-\frac{1}{h}+\frac{1}{z}} \longrightarrow -\infty$ as $z \longrightarrow 0$. Hence $F(z) < 0$ in some interval $(0, z_0)$. As F and ϕ' have the same sign, $\phi'(z) < 0$ in $(0, z_0)$. Contradicting the fact that $\phi' > 0$ in $(0, h)$. Therefore case (iii) is not possible.

(iv) This is the only remained possibility.

(f) Setting the constants as in the proof of (d) and by $\frac{d^2\phi}{dz^2}(z_0) < 0$ and Lemma 4.6.1, we conclude that $\lim_{\rho \to T_2^-} u(\rho) = 0$ and $\lim_{\rho \to T_2^-} u'(\rho) = -\infty$.

□

4.7 Winglike solitons

In this part, we use propositions 4.6.1, 4.6.2, and 4.6.4 to describe a winglike soliton.

Definition 4.7.1. *A* ***winglike soliton*** *is a rotational symmetric soliton with respect to $-\partial_0$ that can be generated by a rotation of a smooth curve γ in the first quadrant of the x_0x_1-plane with two ends at the boundary at infinity $\partial_\infty \mathbb{H}^{m+1}$ where $x_0(\gamma)$ is bounded.*

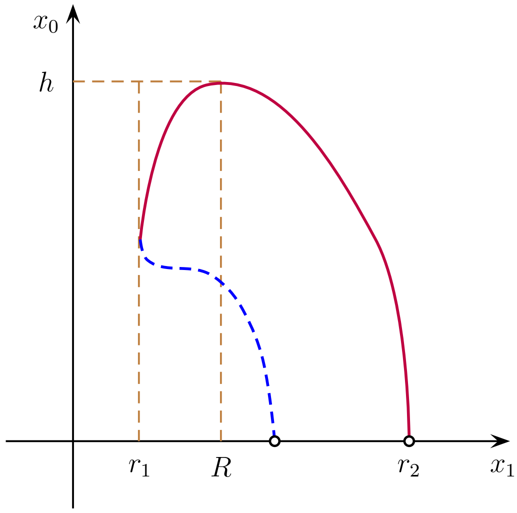
Theorem 4.7.2 (Winglike soliton's behaviour)**.** *Suppose that $x_0 \circ \gamma$ has a stationary point at t_0 and let $\gamma(t_0) = (h, R)$. Then γ can be written as the bi-graph over the x_0 axis of $\phi_1, \phi_2 : (0, h] \longrightarrow (0, \infty)$ satisfying the following properties (Figure 7):*

(a) It holds $\phi_1 < \phi_2$ on $(0, h)$ and $\phi_1(h) = \phi_2(h) = R$. Additionally, $\phi_1(0^+) < \phi_2(0^+)$, namely, γ cannot have the same end-points;

(b) the graph of ϕ_2 is a concave branch on $(0, h)$;

(c) there exists $\lambda_0 \in (0, h)$ such that ϕ_1 is the union of a concave branch on $(0, \lambda_0)$ and a convex branch on (λ_0, h);

Fig. 6 – Behaviour of a winglike soliton



Source: elaborated by the author.

Proof. In some neighborhood of $\gamma(t_0) = (h, R)$, the image of γ is the graph of a function u solution for (4.6.13). By Proposition 4.6.4, we can define $\phi_2 : (0,h) \to (0,\infty)$ by $(u(\rho), \rho) = (z, \phi_2(z))$ for $\rho \in (R, T_2)$. Define $z_1 := \lim_{\rho \to T_1^+} u(\rho)$ and $\phi_1|_{(z_1,h)}$ by $(u(\rho), \rho) = (z, \phi_1(z))$ for $\rho \in (T_1, R)$. By Lemma 4.6.2 with $\tau = T_1 = \phi_1(z_1)$, we can extend ϕ_1 to $(0,h)$. By contruction of ϕ_1, $T_1 = \min_{\rho \in (0,h)} \phi(\rho)$ and $\phi_1 < \phi_2$ on (z_1, h).

(a) Suppose by contradiction that $\phi_1(z_i) = \phi_2(z_i)$ for some $z_i \in (0, z_1)$ or $\phi_1(0^+) = \phi_2(0^+) = T_2$. In first case, the image of γ intersects itself at $(z_i, \phi_1(z_i))$. In the second case, the image of γ does not intersect itself but has the same ends in $\partial_\infty \mathbb{H}^{m+1}$. In any case, we can write a part of the image of γ as two graphs of functions $u_2, u_1 : (T_1, T_3) \to (0, \infty)$ over x_1-axis solutions to (4.4.1), where $T_3 := \phi_1(z_i) = \phi_2(z_i)$ in the first case or $T_3 := T_2$ in the second case with $u_2 \geq u_1$. As $u_2 \neq u_1$, $u_2 - u_1$ attains a maximum value, a contradiction by Lemma 4.4.3. Therefore, (a) is true as claimed.

(b) It follows by (d) of Proposition 4.6.4.

(c) It follows by (c) and (g) of Lemma 4.6.2.

□

4.8 Bowl soliton

In this section, we study the existence and behaviour of bowl soliton.

Definition 4.8.1. *Given a height $h > 0$, a **bowl soliton** M is a soliton with respect to $-\partial_0$ obtained by rotating a curve $(z, \phi(z))$ as in Definition 4.5.1 such that $\phi(h^-) = 0$.*

Theorem 4.8.2 (Existence of a bowl soliton)**.** *Given a height* $h > 0$*, there exists a bowl soliton* $M = \operatorname{Im}\mathfrak{G}_\phi$*, for some* $\phi : (0,h) \to (0,\infty)$*. Futhermore,* ϕ *is a concave branch on* $(0,h)$*, that is,* $\phi'' < 0$ *on* $(0,h)$ *and* $\phi'(h^-) = -\infty$*.*

Proof. We use a sequence of winglike soliton (M_i) to converge to a bowl soliton. Namely, by Theorem 4.7.2 and Proposition 4.6.4, for a sequence of positive number (ε_i), $\varepsilon_i \searrow 0$, there exists a sequence of function $u_i : (T_1(\varepsilon_i), T_2(\varepsilon_i)) \to (0,\infty)$ solution to:

$$\begin{cases} \dfrac{u_i''}{(1+u_i'^2)} + \dfrac{m-1}{\rho}u_i' = -\dfrac{1+mu_i}{u_i^2} & \rho > 0, \\ u_i(\varepsilon_i) = h, \\ u_i'(\varepsilon_i) = 0, \end{cases} \tag{4.8.1}$$

where $\Gamma(R_{x_0}(u_i)) \subset M_i$.

Claim 1. $\{T_2(\varepsilon_i)\}$ is bounded from below and away from zero.

Indeed, suppose the contrary that there is a subsequence $T_2(\varepsilon_j) \searrow 0$. Fix a grim-reaper $\mathcal{G}_h := \mathcal{G}_{h,0} = \mathcal{G}_{h,[H_*,H^*]}$ (see Remark 4.2.1). Therefore, there exists j_0 such that $(T_1(\varepsilon_{j_0}), T_2(\varepsilon_{j_0})) \subset (0, H^*)$. By the behaviour of the grim-reaper (Lemma 4.2.1), $\max\{x_0(\mathcal{G}_h)\} = h$. By behaviour of winglike soliton (Theorem 4.7.2), $\max\{x_0(M_{j_0})\} = h = u_{j_0}(\varepsilon_{j_0})$. Hence, $(M_{j_0} \cap \mathcal{G}_h) \cap \{x_1 > 0\} \neq \emptyset$. Now we increase the height of the grim-reaper up until it touches M_{j_0} just at one point in $\{x_1 > 0\}$. Namely, set $h_S := \sup\{z \in (h,\infty) : \mathcal{G}_z \cap M_{j_0} \neq \emptyset\}$. $(\mathcal{G}_{h_S} \cap M_{j_0}) \cap \{x_1 > 0\}$ is just one point and M_{j_0} is below G_{h_S}, a contradiction by Maximum Principle (Theorem C.0.1). Therefore, this proves Claim 1.

Define $R := \inf\{T_2(\varepsilon_i)\}$ and $v_i := u_i|_{[\varepsilon_i, R)}$. $\{v_i\}$ has uniformly bounded C^2-norm on any fixed compact set of $(0,R)$. Therefore, up to subquence, $\{v_i\}$ converge to a solution u to a solution to (4.4.1) with $\lim_{\rho\to 0^+} u(\rho) = h$ and $\lim_{\rho \to 0^+} u'(\rho) = 0$. Futhermore, $u'' < 0$ on $(0,R)$ because $v_i'' < 0$ and $u'(0^+) = 0$. Therefore, for corresponding function ϕ, such that $(u(\rho), \rho) = (z, \phi(z))$, $\phi'' < 0$ on $(0,h)$ and $\phi'(h^-) = -\infty$.

□

5 PLATEAU'S PROBLEM

In this section we will prove the existence of solitons $M \subset N$ with respect to $-\partial_0$ with given asymptotic boundary $\Sigma = \partial_\infty M$. In order to accomplish this goal, the Theorem 1.5 in (Castéras *et al.*, 2018) (Theorem 5.0.2) will be adapted for our purposes.

Recall that a Cartan-Hadamard manifold is a complete, connected and simply connected Riemannian $(m+1)$-manifold of non-positive sectional curvature.

Definition 5.0.1 (SC condition). *Let $(N^{m+1}, \overline{g})$ be a Cartan-Hadamard manifold. We say that N satifies the strict convexity condition (**SC condition**) if given $x \in \partial_\infty N$ and a relatively open subset $W \subset \partial_\infty N$ containing x, there exists a C^2 open subset $\Omega \subset N$ such that $x \in int\, \partial_\infty \Omega \subset W$ and $N \backslash \Omega$ is convex.*

Theorem 5.0.2 (Theorem 1.5, (Castéras *et al.*, 2018)). *Let N^{m+1}, $m+1 \geq 3$ be a Cartan-Hadamard manifold satisfying the SC condition and let $\Sigma \subset \partial_\infty N^{m+1}$ be a (topologically) embedded closed $(k-1)$-dimensional submanifold, with $2 \leq k \leq m$. Then there exists a complete, absolutely area minimizing, locally rectifiable k-current M modulo 2 in N^{m+1} asymptotic to Σ at infinity, i.e., $\partial_\infty M = \Sigma$*

Our main theorem in this chapter is the following one:

Theorem 5.0.3 (Plateau's problem). *Let $\Sigma \subset \partial_\infty \mathbb{H}^{n+1}$ be the boundary of a relatively compact subset $A \subset \partial_\infty \mathbb{H}^{n+1}$ with $A = \overline{\mathrm{int}(A)}$. Then, there exists a closed set W of local finite perimeter in $\mathbb{H}^{n+1}$ with $\partial_\infty W = A$ such that $M = \partial[W]$ is a conformal soliton for $-\partial_0$ on the complement of a closed set S of Hausdorff dimension $\dim_{\mathscr{H}}(\mathrm{S}) \leq n-7$, and that $\partial_\infty \mathrm{spt}(M) = \Sigma$. Furthermore, when $n < 7$, then M is a properly embedded smooth hypersurface of $\mathbb{H}^{n+1}$.*

Theorem 5.0.4 (Hopf and Rinow). *Let N be a Riemannian manifold and let $p \in N$. The following assertions are equivalent:*

(a) $\exp_p$ *is defined on all of T_pN.*

(b) The closed and bounded sets on N are compact.

(c) N is complete as metric space.

(d) N is geodesically complete.

(e) There exists a sequence of compact subsets $K_n \subset N$, $K_n \subset K_{n+1}$ and $\bigcup_n K_n = M$, such that if $q_n \notin K_n$ then $\mathrm{dist}(p, q_n) \to \infty$.

For a proof of Theorem 5.0.4, see Theorem 2.7, Chapter 7 in (do Carmo, 1992).

In our particular case in Theorem 5.0.2, we need to show that Ilmanen's space $(\mathbb{H}^{m+1}, g_{\mathbb{I}})$ is a Cartan-Hadamard and the SC condition holds. By $g_{\mathbb{I}} = e^{\frac{2}{mx_0}} g_{\mathbb{H}} > g_{\mathbb{H}}$, $\text{dist}_{g_{\mathbb{I}}}(p,q) > \text{dist}_{g_{\mathbb{H}}}(p,q)$. By the completeness of $(\mathbb{H}^{m+1}, g_{\mathbb{H}})$ and Theorem 5.0.4.(e), $(\mathbb{H}^{m+1}, g_{\mathbb{I}})$ is complete.

5.1 Sectional Curvature of Ilmanen's Space

By the following proposition, the sectional curvature is non-positive for any pair of vectors.

Proposition 5.1.1 (Ilmanen's sectional curvature)**.** *The sectional curvatures of Ilmanen's metric are given by:*

(a) $\sec_{g_{\mathbb{I}}}(\partial_i, \partial_0) = \frac{-\left(2\left(\frac{1}{mx_0}\right)+\frac{1}{x_0}\right)}{e^{\frac{2}{mx_0}}} \quad \forall i \neq 0$

(b) $\sec_{g_{\mathbb{I}}}(\partial_i, \partial_j) = -\frac{\frac{1}{m}+x_0}{e^{\frac{2}{mx_0}}} \quad \forall i, j \neq 0$ *and* $i \neq j$

(c) $\sec_{g_{\mathbb{I}}}(\sin\theta\partial_0 + \cos\theta\partial_i, \partial_j) = \sin^2\theta \sec_{g_{\mathbb{I}}}(\partial_0, \partial_i) + \cos^2\theta \sec_{g_{\mathbb{I}}}(\partial_i, \partial_j), \quad \forall i, j \neq 0, i \neq j$ *and* $\theta \in (0, 2\pi)$

Proof. By exercise 4.7.14 of (Petersen, 2016), using the same notation $g_{\mathbb{I}} = \frac{e^{\frac{2}{mx_0}}}{x_0} g_{\mathbb{R}} = e^{2\psi} g_{\mathbb{R}}$, where $\psi = \frac{1}{mx_0} - \ln x_0$. Hence, $\partial_0(\psi) = -\frac{1}{mx_0^2} - \frac{1}{x_0}$ and $\text{Hess}_{\mathbb{R}}\psi(\partial_0, \partial_0) = \frac{2}{mx_0^3} + \frac{1}{x_0^2}$. Therefore the sectional curvature in relation to Ilmanen's metric is given by:

$$\begin{aligned} &e^{2\psi}\sec_{g_{\mathbb{I}}}(X,Y) = \\ &= \sec_{\mathbb{R}}(X,Y) - \text{Hess}_{\mathbb{R}}\psi(X,X) - \text{Hess}_{\mathbb{R}}\psi(Y,Y) + (X(\psi))^2 + (Y(\psi))^2 - |d\psi|^2_{\mathbb{R}}. \end{aligned} \tag{5.1.1}$$

Applying for $X = \partial_i$ and $Y = \partial_0$, (a) follows. For $X = \partial_i$ and $Y = \partial_j$, (b) follows. And for $X = \sin\theta\partial_0 + \cos\theta\partial_i$ and $Y = \partial_j$, (c) follows. □

Proposition 5.1.2. *Let p be a point in $\mathbb{H}^{m+1}$, π be a 2-plane contained in $T_p\mathbb{H}^{m+1}$ and* $\sec_{g_{\mathbb{I}}}(\pi)$ *be the sectional curvature with respect to $g_{\mathbb{I}}$. Then* $\sec_{g_{\mathbb{I}}}(\pi) \leq 0$.

Proof. Let $\tau \leq T_p\mathbb{H}^{m+1}$ be a hyperplane parallel to $\partial_1, \partial_2, \ldots, \partial_{m-1}$ and ∂_m. Then $\pi \cap \tau \neq \emptyset$. Either $\pi \subset \tau$ (In this first case, up to rotation, we can choose $\pi = \partial_1 \wedge \partial_2$) or $\pi \cap \tau$ is a line (In this second case, up to rotation, we can choose $\pi = \partial_1 \wedge v$, where $v = \cos\theta\partial_0 + \sin\theta\partial_2$ for $\theta \neq \frac{\pi}{2}$). The first case follows by Proposition 5.1.1.(b). The second case follows by Proposition 5.1.1.(c). □

5.2 Geodesics of Ilmanen's space

In this section, we study the qualitative behaviour of geodesics of $(\mathbb{H}^{m+1}, g_{\mathbb{I}})$. We just need to focus on the geodesics in the x_0x_1-plane by the symmetries of $(\mathbb{H}^{m+1}, g_{\mathbb{I}})$

Let γ be a geodesic in the x_0x_1-plane, $\gamma(t) = (x_0(t), x_1(t))$. The coordinates of γ obey the geodesic equations:

$$\begin{cases} \frac{d^2x_0}{dt^2} + \Gamma^0_{ij}\frac{dx_i}{dt}\frac{dx_j}{dt} = 0, \\ \frac{d^2x_1}{dt^2} + \Gamma^1_{ij}\frac{dx_i}{dt}\frac{dx_j}{dt} = 0. \end{cases} \tag{5.2.1}$$

where Γ^k_{ij} are the Christhoffel symbols of $g_{\mathbb{I}}$ with respect to coordinate frame $\{\partial_i\}$. By the Koszul formula,

$$\begin{aligned} \Gamma^0_{10} &= \frac{1}{2}{g_{\mathbb{I}}}^{m0}\left(\frac{\partial g_{\mathbb{I}0m}}{\partial x_1} + \frac{\partial g_{\mathbb{I}1m}}{\partial x_0} - \frac{\partial g_{\mathbb{I}10}}{\partial x_m}\right) \\ &= \frac{1}{2}\frac{x_0^2}{e^{\frac{2}{mx_0}}}\left(\frac{\partial}{\partial x_1}\left(\frac{e^{\frac{2}{mx_0}}}{x_0^2}\right)\right) = 0, \end{aligned}$$

$$\begin{aligned} \Gamma^0_{11} &= \frac{1}{2}{g_{\mathbb{I}}}^{m0}\left(\frac{\partial g_{\mathbb{I}1m}}{\partial x_1} + \frac{\partial g_{\mathbb{I}1m}}{\partial x_0} - \frac{\partial g_{\mathbb{I}11}}{\partial x_m}\right) \\ &= \frac{1}{2}\frac{x_0^2}{e^{\frac{2}{mx_0}}}\left(-\frac{\partial}{\partial x_0}\left(\frac{e^{\frac{2}{mx_0}}}{x_0^2}\right)\right) \\ &= \left(\frac{1}{mx_0^2} + \frac{1}{x_0}\right), \end{aligned}$$

$$\begin{aligned} \Gamma^0_{00} &= \frac{1}{2}{g_{\mathbb{I}}}^{00}\left(\frac{\partial g_{\mathbb{I}00}}{\partial x_0} + \frac{\partial g_{\mathbb{I}00}}{\partial x_0} - \frac{\partial g_{\mathbb{I}00}}{\partial x_0}\right) \\ &= \frac{1}{2}{g_{\mathbb{I}}}^{00}\left(\frac{\partial g_{\mathbb{I}00}}{\partial x_0}\right) \\ &= -\left(\frac{1}{mx_0^2} + \frac{1}{x_0}\right), \end{aligned}$$

$$\begin{aligned} \Gamma^1_{00} &= \frac{1}{2}{g_{\mathbb{I}}}^{11}\left(\frac{\partial g_{\mathbb{I}01}}{\partial x_0} + \frac{\partial g_{\mathbb{I}01}}{\partial x_0} - \frac{\partial g_{\mathbb{I}00}}{\partial x_1}\right) \\ &= 0, \end{aligned}$$

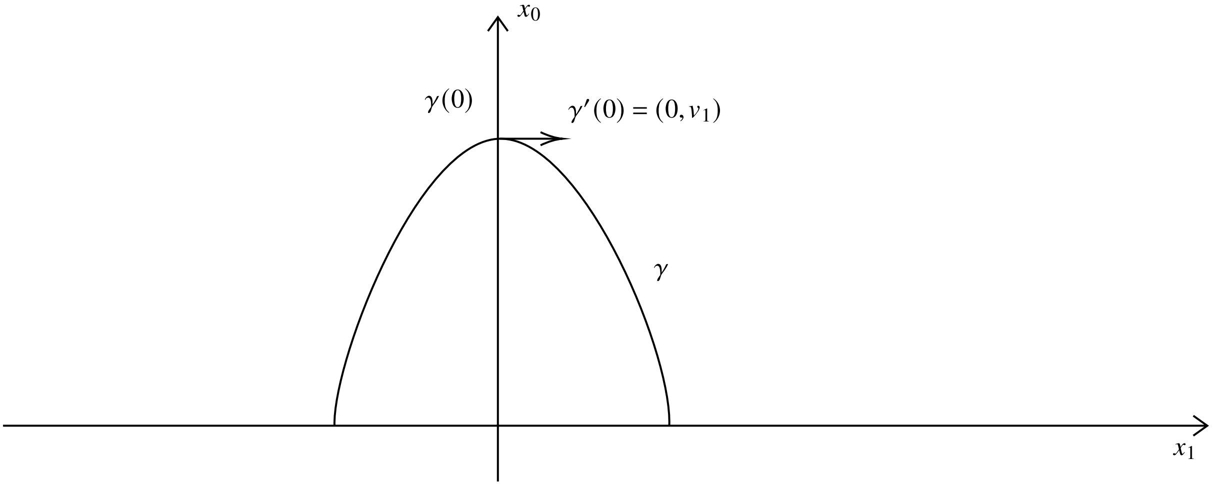
$$
\begin{aligned}
\Gamma^1_{10} &= \frac{1}{2} {g_{\mathbb{I}}}^{11} \left(\frac{\partial g_{\mathbb{I}01}}{\partial x_1} + \frac{\partial g_{\mathbb{I}11}}{\partial x_0} - \frac{\partial g_{\mathbb{I}10}}{\partial x_1} \right) \\
&= - \left(\frac{1}{m x_0^2} + \frac{1}{x_0} \right),
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2} {g_{\mathbb{I}}}^{11} \left(\frac{\partial g_{\mathbb{I}11}}{\partial x_1} + \frac{\partial g_{\mathbb{I}11}}{\partial x_1} - \frac{\partial g_{\mathbb{I}11}}{\partial x_1} \right) \\
&= 0,
\end{aligned}
$$

plugging them in (5.2.1),

$$
\begin{cases}
\frac{d^2 x_0}{dt^2} - \left(\frac{1}{m x_0^2} + \frac{1}{x_0} \right) \left(\left(\frac{dx_0}{dt} \right)^2 - \left(\frac{dx_1}{dt} \right)^2 \right) = 0, \\
\frac{d^2 x_1}{dt^2} - \left(\frac{1}{m x_0^2} + \frac{1}{x_0} \right) \left(2 \frac{dx_0}{dt} \frac{dx_1}{dt} \right) = 0.
\end{cases}
\tag{5.2.2}
$$

Fig. 7 – Behaviour of geodesic



Source: elaborated by the author.

Proposition 5.2.1 (Behaviour of geodesics)**.** *Let γ be a geodesic of $(\mathbb{H}^{m+1}, g_{\mathbb{I}})$ in the x_0x_1-plane defined in a maximal interval $\mathbb{R}$ with initial conditions $\gamma(0) = (\lambda, 0, \ldots, 0)$ and $\frac{d\gamma}{dt}(0) = v_1 \partial_1$ for some $\lambda, v_1 \in \mathbb{R} \quad \lambda, v_1 > 0$. Then:*

(a) $\frac{dx_1}{dt} > 0, \quad \forall t \in \mathbb{R}$.

(b) $\frac{dx_0}{dt} < 0, \quad \forall t > 0$. *and* $\frac{dx_0}{dt} > 0, \quad \forall t < 0$.

(c)

$$\lim_{t \to \infty} x_0(t) = 0 \text{ and } \lim_{t \to -\infty} x_0(t) = 0$$

Proof. By the initial conditions, $\frac{dx_1}{dt} > 0$ for small enough t. Suppose by contradiction that $\frac{dx_1}{dt}(t_0) = 0$ for some $t_0 \in \mathbb{R}$. The ordinary differential equation in (5.2.2) involving x_1 is:

$$\frac{d^2x_1}{dt^2} + F(t)\frac{dx_1}{dt} = 0 \tag{5.2.3}$$

where $F(t) = -\left(\frac{1}{mx_0^2} + \frac{1}{x_0}\right)\left(2\frac{dx_0}{dt}\right)$. Setting $y_1 = \frac{dx_1}{dt}$,

$$\frac{dy_1}{dt} + F(t)y_1 = 0. \tag{5.2.4}$$

By linearity of (5.2.4), the solution is unique, and $y_1(t) = 0 \quad \forall t \in \mathbb{R}$ is a solution with initial condition $y_1(t_0) = 0$. This is a contradiction because $y_1(0) = v_1 \neq 0$. Therefore, $\frac{dx_1}{dt} > 0 \quad \forall t \in \mathbb{R}$. Dividing the second equation in (5.2.2) by $\frac{dx_1}{dt}$,

$$\frac{\frac{d^2x_1}{dt^2}}{\frac{dx_1}{dt}} = \left(\frac{1}{mx_0^2} + \frac{1}{x_0}\right)\left(2\frac{dx_0}{dt}\right). \tag{5.2.5}$$

Integrating in t,

$$\ln\left(\frac{dx_1}{dt}\right) = -\frac{2}{mx_0} + 2\ln(x_0) + C_1 \tag{5.2.6}$$

$$\frac{dx_1}{dt} = e^{2\left(-\frac{1}{mx_0}+\ln(x_0)\right)}e^{C_1} \tag{5.2.7}$$

$$\frac{dx_1}{dt} = e^{\frac{-2}{mx_0}}x_0^2 e^{C_1} \tag{5.2.8}$$

$$x_1 = e^{C_1}\int e^{2\left(-\frac{1}{mx_0}\right)}x_0^2 dt + C_2. \tag{5.2.9}$$

Plugging (5.2.7) in (5.2.2),

$$\frac{d^2x_0}{dt^2} - \left(\frac{1}{mx_0^2} + \frac{1}{x_0}\right)\left(\frac{dx_0}{dt}\right)^2 + \left(\frac{1}{mx_0^2} + \frac{1}{x_0}\right)\left(e^{2\left(-\frac{1}{mx_0^2}+\ln x_0\right)}e^{C_1}\right)^2 = 0 \tag{5.2.10}$$

$$\frac{d^2x_0}{dt^2} - \left(\frac{1}{mx_0^2} + \frac{1}{x_0}\right)\left(\frac{dx_0}{dt}\right)^2 + \left(\frac{x_0^2}{m} + x_0^3\right)e^{\frac{-4}{mx_0}}e^{2C_1} = 0 \tag{5.2.11}$$

By the initial conditions and equation (5.2.7), $e^{C_1} = \frac{v_1}{e^{\frac{2}{m\lambda}}\lambda^2}$.

(a) The equation (5.2.8) proves that $\frac{dx_1}{dt} > 0 \quad \forall t \in \mathbb{R}$.

(b) To prove that γ behaves as the figure above, that is, γ is concave in the Euclidean sense, we need to demonstrate that $g_{\mathbb{R}}(\nabla^{\mathbb{R}}{}_{\gamma'}\gamma', \nu_{\mathbb{R}})$ is positive, where $\nu_{\mathbb{R}}$ is the Euclidean normal vector field along γ in the x_0x_1-plane pointing downwards, i.e., $g_{\mathbb{R}}(\partial_0, \nu_{\mathbb{R}}) < 0$. Notice that $\nu_{\mathbb{R}} = \alpha(t)(-\frac{dx_1}{dt}, \frac{dx_0}{dt})$ for positive function $\alpha = \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_0}{dt}\right)^2}$ and $\nabla^{\mathbb{R}}{}_{\gamma'}\gamma' = (\frac{d^2x_0}{dt^2}, \frac{d^2x_1}{dt^2})$. Therefore,

$$g_{\mathbb{R}}(\nabla^{\mathbb{R}}{}_{\gamma'}\gamma', \nu_{\mathbb{R}}) = \alpha\left(-\frac{d^2x_0}{dt^2}\frac{dx_1}{dt} + \frac{d^2x_1}{dt^2}\frac{dx_0}{dt}\right). \tag{5.2.12}$$

By (5.2.11) and (5.2.2),

$$g_{\mathbb{R}}(\nabla^{\mathbb{R}}{}_{\gamma'}\gamma', \nu_{\mathbb{R}}) =$$
$$= \alpha\left(-\left(\frac{1}{mx_0^2}+\frac{1}{x_0}\right)\left(\frac{dx_0}{dt}\right)^2\frac{dx_1}{dt}+\left(\frac{x_0}{m}+x_0^3\right)e^{\frac{-4}{mx_0}}e^{2C_1}\frac{dx_1}{dt}+\left(\frac{1}{mx_0^2}+\frac{1}{x_0}\right)2\left(\frac{dx_0}{dt}\right)^2\frac{dx_1}{dt}\right)$$
$$= \alpha\left(\left(\frac{1}{mx_0^2}+\frac{1}{x_0}\right)\left(\frac{dx_0}{dt}\right)^2\left(\frac{dx_1}{dt}\right)+\left(\frac{x_0^2}{m}+x_0^3\right)e^{\frac{-4}{mx_0}}e^{2C_1}\frac{dx_1}{dt}\right) > 0,$$

where we are using $\frac{dx_1}{dt} > 0$. By $g_{\mathbb{R}}(\nabla^{\mathbb{R}}{}_{\gamma'}\gamma', \nu_{\mathbb{R}}) > 0$, the image of γ is concave in the Euclidean sense as in the figure above. Therefore, $\frac{dx_0}{dt} < 0, \forall t > 0$, and $\frac{dx_0}{dt} > 0, \forall t < 0$.

(c) By concavity and $\frac{dx_1}{dt} > 0$, $\lim_{t\to\infty} x_1(t)$ must be finite (say, $\lim_{t\to\infty} x_1(t) = H^*$). By concavity, $\lim_{t\to\infty} x_0(t)$ does exist. We prove that it is zero. Now Suppose by contradiction that $\lim_{t\to\infty} x_0(t) = V_* > 0$, for some $V_* \in \mathbb{R}$. Observe that:

$$\text{length}_{g_{\mathbb{I}}}(\gamma|_{[0,\infty]}) \leq \frac{e^{\frac{1}{mV_*}}}{V_*}\text{length}_{g_{\mathbb{R}}}(\gamma|_{[0,\infty]}) < \infty,$$

where $\text{length}_{g_{\mathbb{R}}}(\gamma) < \infty$ since $\gamma|_{[0,\infty]}$ is a graph of a concave function on $[0, H^*]$. However, $\text{length}_{g_{\mathbb{I}}}(\gamma|_{[0,\infty]}) = |\dot{\gamma}(0)|_{g_{\mathbb{I}}}(\infty - 0)$ since γ is $g_{\mathbb{I}}$-geodesic.

□

In the next lemma, we prove that in fact the behaviour of a geodesic with initial velocity pointing upwards (but not vertically) still behaves as the geodesics in lemma 5.2.1.

Definition 5.2.2. *A geodesic γ is of right-hand* ***grim-reaper type*** *if and only if: for some $t_m \in \mathbb{R}$*

(a) The geodesic attains a maximum height, i.e,

$$\sup_{t\in\mathbb{R}} x_0(t) = x_0(t_m).$$

(b) $\frac{dx_1}{dt} > 0 \quad \forall t \in \mathbb{R}$.

(c) $\frac{dx_0}{dt} > 0 \quad \forall t < t_m$ *and* $\frac{dx_0}{dt} < 0 \quad \forall t > t_m$.

(d) The geodesic γ goes to $\partial_\infty \mathbb{H}^{m+1}$, i.e.,

$$\lim_{t\to\pm\infty} x_0(t) = 0.$$

(e) The geodesic γ is symmetric, i.e., $x_0(t_m + t) = x_0(t_m - t)$ and $x_1(t_m + t) - x_1(t_m) = x_1(t_m) - x_1(t_m - t)$, $\forall t \in \mathbb{R}$.

The definition of the left-hand grim-reaper type is symmetrically defined.

Proposition 5.2.3. *Let γ be a geodesic with initial velocity $\gamma'(0) = v_0\partial_0 + v_1\partial_1$ with $v_0, v_1 > 0$. Then the following claims are true:*

1. *The height is bounded, that is, $M_0 := \sup_{t\in\mathbb{R}} x_0(t) < \infty$.*
2. *The maximum height is attained, that is, there exists t_m such that $x_0(t_m) = M_0$.*
3. *γ is of grim-reaper type.*

Proof. If $\frac{dx_0}{dt} = 0$ for some t_m, we can reason as in Proposition 5.2.1 and the result is proven. Therefore our goal is to prove that there exists such t_m such that $\frac{dx_0}{dt}|_{t=t_m} = 0$. Suppose by contradiction that $\frac{dx_0}{dt} > 0 \quad \forall t \in \mathbb{R}$.

We first examine the case that $x_0 \to \infty$ as $t \to \infty$. For x_0-coordinate of γ, the differential equation (5.2.11) gives:

$$\frac{d^2x_0}{dt^2} - \left(\frac{1}{mx_0^2} + \frac{1}{x_0}\right)\left(\frac{dx_0}{dt}\right)^2 + \left(\frac{x_0^2}{m} + x_0^3\right) e^{\frac{-4}{mx_0}} e^{2C_1} = 0.$$

Notice that $\left(\frac{dx_0}{dt}\right)^2 \le |\gamma'|_I^2 \cdot \frac{x_0^2}{e^{\frac{2}{mx_0}}}$, where $|\gamma'|_I$ is the constant Ilmanen's norm because γ is a geodesic. Hence,

$$\left|\left(\frac{1}{mx_0^2} + \frac{1}{x_0}\right)\left(\frac{dx_0}{dt}\right)^2\right| \le \left(\frac{1}{mx_0^2} + \frac{1}{x_0}\right)|\gamma'|_I^2 \frac{x_0^2}{e^{\frac{2}{mx_0}}} \tag{5.2.13}$$

$$\le \frac{2x_0}{e^{\frac{2}{mx_0}}}|\gamma'|_I^2, \tag{5.2.14}$$

for large enough x_0. Estimating the third term, for x_0 large enough:

$$\left|\left(\frac{x_0^2}{m} + x_0^3\right) e^{\frac{-4}{mx_0}} e^{2C_1}\right| \le 2x_0^3 e^{\frac{-4}{mx_0}} e^{2C_1}.$$

Thus, $\frac{d^2x_0}{dt^2} = \left(\frac{1}{mx_0^2} + \frac{1}{x_0}\right)\left(\frac{dx_0}{dt}\right)^2 - \left(\frac{x_0^2}{m} + x_0^3\right) e^{\frac{-4}{mx_0}} e^{2C_1} \to -\infty$ as $x_0 \to \infty$. This is a contradiction. Indeed, given a large $a > 0$. Then $\frac{d^2x_0}{dt^2} < -a$ for $t > t_0$ and a large enough t_0. Then, integrating the inequality:

$$\frac{dx_0}{dt} - \frac{dx_0}{dt}|_{t=t_0} < -a(t - t_0) \tag{5.2.15}$$

$$x_0(t) < -a\left(\frac{t^2}{2} - t_0 t\right) + \frac{dx_0}{dt}|_{t=t_0} \cdot t + x_0(t_0) \tag{5.2.16}$$

$$x_0(t) < F(t) \tag{5.2.17}$$

, where F is a quadratic function, $F(t) = -a\left(\frac{t^2}{2} - t_0 t\right) + \frac{dx_0}{dt}|_{t=t_0} \cdot t + x_0(t_0)$. However, F has a maximum and $x_0(t) \to \infty$ as $t \to \infty$. Therefore $\sup_{t\in\mathbb{R}} x_0(t) < \infty$.

Now we prove that the maximum is attained. Suppose by contradiction that $x_0(t) < \sup_{t\in\mathbb{R}} x_0(t) =: M_0 \quad \forall t \in \mathbb{R}$. By the same reasoning in the proof of item 3) of Proposition 5.2.1, the image of γ is Euclidean concave. Since $\frac{dx_0}{dt} > 0$ and $x_0 \to M_0$ as $t \to \infty$, $\frac{dx_0}{dt} \to 0$ as $t \to \infty$. Therefore,

$$\left(\frac{1}{mx_0^2} + \frac{1}{x_0}\right)\left(\frac{dx_0}{dt}\right)^2 \to 0 \text{ as } t \to \infty \tag{5.2.18}$$

$$\underset{\text{by } 5.2.11}{\Longrightarrow} \frac{d^2x_0}{dt^2} \to -\left(\frac{M_0^2}{m} + M_0^3\right) e^{\frac{-4}{mM_0}} < 0. \tag{5.2.19}$$

This is a contradiction. Indeed, define $A_0 = \left(\frac{M_0^2}{m} + M_0^3\right) e^{\frac{-4}{mM_0}} e^{2C_1}$ and for $t > t_0$ and some large enough t_0,

$$\frac{d^2x_0}{dt^2} < -\frac{A_0}{2} \tag{5.2.20}$$

$$\frac{dx_0}{dt} - \left.\frac{dx_0}{dt}\right|_{t=t_0} < \frac{-A_0}{2}(t - t_0) \tag{5.2.21}$$

$$\frac{dx_0}{dt} < -\frac{A_0}{2}(t - t_0) + \left.\frac{dx_0}{dt}\right|_{t=t_0} \tag{5.2.22}$$

for a fixed $t_0 \in \mathbb{R}$ and $t > t_0$. As $t \to \infty$, the right-hand side goes to $-\infty$, but the left-hand side goes to 0.

We conclude that there exist a $t_m \in \mathbb{R}$ such that $x_0(t_m) = M_0$. Since $\left.\frac{dx_0}{dt}\right|_{t=t_m} = 0$, we can argument as in proposition 5.2.1 and conclude that γ is of grim-reaper type. □

5.3 Proof of Theorem 5.0.3

In the Ilmanen space $(\mathbb{H}^{m+1}, g_{\mathbb{I}})$, we will denote by $\mathcal{P}_\infty \in \partial_\infty\mathbb{H}^{m+1}$ the asymptote class of the vertical geodesic and $\partial'_\infty\mathbb{H}^{m+1} := \partial_\infty\mathbb{H}^{m+1} - \{\mathcal{P}_\infty\}$. Notice that we can identify $\partial'_\infty\mathbb{H}^{m+1}$ with the set $\{x_0 = 0\}$.

Proposition 5.3.1 (SC condition on $\partial'_\infty\mathbb{H}^{m+1}$)**.** *Let $\mathcal{S} \subset \mathbb{H}^{m+1}$ be a hyperbolic totally geodesic sphere and $\mathcal{N}$ be the upper connected component of $\mathbb{H}^{m+1}\backslash\mathcal{S}$, that is,* $\sup x_0(\mathcal{N}) = \infty$ *and $\Omega := \mathbb{H}^{m+1}\backslash\mathcal{N}$. Then $\mathcal{N}$ is strictly $g_{\mathbb{I}}$-convex in the upward direction.*

Proof. By the lemma 4.3.2,

$$\mathrm{II}_{g_{\mathbb{I}}}(v_1, v_2) = e^{\frac{1}{mx_0}} \left\{\mathrm{II}_{g_{\mathbb{H}}}(v_1, v_2) - \nu_{g_{\mathbb{H}}}\left(\frac{1}{mx_0}\right) g_{\mathbb{H}}(v_1, v_2)\right\}.$$

As $\mathcal{S}$ is totally $g_{\mathbb{H}}$-geodesic, $\mathrm{II}_{g_{\mathbb{H}}} \equiv 0$ and $\nu_{g_{\mathbb{H}}}\left(\frac{1}{mx_0}\right) < 0$, $\mathrm{II}_{g_{\mathbb{I}}}$ is positive definitive. Therefore, the scalar mean curvature of $\mathcal{S} \hookrightarrow (\mathbb{H}^{m+1}, g_{\mathbb{I}})$ with respect to the upward direction $\nu_{g_{\mathbb{I}}}$ is positive, and $\mathcal{N}$ is $g_{\mathbb{I}}$-convex. Thus, for any point in $\partial_\infty \mathbb{H}^{m+1}$ different from $\mathcal{P}_\infty$, the SC condition holds using a suitable Ω. □

Proof of Theorem 5.0.3. By completeness and Lemma 5.1.1, the Ilmanen space $(\mathbb{H}^{m+1}, g_{\mathbb{I}})$ is a Cartan-Hadamard manifold. SC condition holds for points at $\partial'_\infty \mathbb{H}^{m+1}$ by Lemma 5.3.1. However, SC condition may fail at $\mathcal{P}_\infty$. To overcome this difficulty, let us proceed as in (Castéras *et al.*, 2018) and (Lang, 1995). Fix a point $o \in \mathbb{H}^{m+1}$. Define the cone

$$C(o, A) := \{\gamma^{o,x}(t); t \geq 0, x \in A\}$$

, where $\gamma^{o,x}$ is the $g_{\mathbb{I}}$-geodesic joining o and $x \in A$. And denote by $B_r(o)$ the $g_{\mathbb{I}}$-geodesic ball with center in o and radius r. For each $i \in \mathbb{N}$, set

$$T_i = \partial B_i(o) \cap C(o, A)$$

with orientation pointing outside of $B_i(o)$ and denote by $[T_i]$ its associated n-rectifiable current. $\partial[T_i]$ is supported in $C(o, \Sigma)$. Since A relativily compact in $\partial'_\infty \mathbb{H}^{m+1}$, we can find a big enough bowl soliton $\mathscr{B}$ such that $\overline{C(o, A)}$ lies in the open subgraph U of $\mathscr{B}$. According to a result of (Lang, 1995), for each $i \in \mathbb{N}$, there exists a set $W_i \subset B_i(o)$ of finite perimeter such that $M_i = \partial[W_i] - [T_i]$ is area minimizing in $B_i(o)$. Note that $\partial M_i = -\partial[T_i]$ is supported in U. Moreover, since $B_i(o)$ is strictly convex, by Strong Maximum Principle of White (White, 2010) we deduce that

$$\operatorname{spt} M_i \cap \partial B_i(o) = \operatorname{spt} \partial M_i, \qquad i \in \mathbb{N}.$$

Claim 1. $\operatorname{spt} M_i \subset U$. In order to prove Claim 1, suppose to the contrary that this is not true and consider the foliation of $\mathbb{H}^{m+1}$ determined by bowl soliton. Then we could find a large bowl soliton $\mathscr{B}'$ lying above $\mathscr{B}$ and touching $\operatorname{spt} M_i$ from above at some point $p \notin \operatorname{spt} \partial M_i$. Let U' be the open set below $\mathscr{B}'$, and consider the manifold with boundary

$$N' = \overline{U'} \cap B_i(o).$$

Let $v(M'_i)$ be the stationary integral varifold obtained, by forgetting orientations, from the connected component of M_i whose support contains p; see (Simon, 1983, Section 27). The strong maximum principle of White (White, 2010, Theorem 4), guarantees that $\operatorname{spt} v(M'_i) \cap N'$

contains a connected component of $\mathscr{B} \cap B_i(o)$. In particular, $\operatorname{spt} \partial M_i'$ contains a piece of $\mathscr{B} \cap \partial B_i(o)$. This however contradicts $\partial M_i' \subset \operatorname{spt} \partial M_i \subset U$. Having observed that each W_i is contained in U and is therefore separated from $\mathcal{P}_\infty$, the rest of the argument follows verbatim as in (Lang, 1995; Castéras *et al.*, 2018).

$\square$

6 UNIQUENESS THEOREM

The main goal of this chapter is to prove the Uniqueness Theorem (Theorem 6.3.1).

6.1 C^k-asymptotic to Euclidean half-spaces

As in (Gama; Martín, 2020) and (Martín *et al.*, 2019), we will define an asymptotic graph over a half-hyperplane outside a cylinder, and then we prove that the only solitons with such behaviour at infinity are grim-reapers (Lemma 4.2.1) and vertical Euclidean hyperplane.

Definition 6.1.1 (Cylinder in the halfspace model)**.** *In the halfspace model, a cylinder $\mathcal{C}(c,r)$ of center $c = (c_0, c_1, 0, \ldots, 0) \in \mathbb{H}^{m+1}$ and radius $r > 0$ is*

$$\mathcal{C}(c,r) = \{x = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1} : (x_0 - c_0)^2 + (x_1 - c_1)^2 < r^2\}$$

For convenience, a cylinder $\mathcal{C}(c,r)$ will be denoted by $\mathcal{C}$ omitting the center c and radius r where r is small enough that $\mathcal{C}(c,r) \subset \mathbb{H}^{m+1} = \mathbb{R}^+_* \times \mathbb{R}^m$

Definition 6.1.2 (Half-hyperplane with respect to a horosphere)**.** *Let $H_{\mathbb{H}} = \Pi \cap \mathbb{H}^{m+1}$ be a half Euclidean hyperplane in the half space model of $\mathbb{H}^{m+1}$, where $\Pi \subset \mathbb{R}^{m+1}$ is an Euclidean hyperplane not parallel to $\partial_\infty \mathbb{H}^{m+1}$. Suppose that $\sigma = H_{\mathbb{H}} \cap \{x_0 = c_0\}$, for some constant $c_0 \in \mathbb{R}$.(Note that σ is a horosphere). An $\mathbb{H}$-half-hyperplane $\mathcal{H}_\sigma(\delta)$ with respect a horosphere $\sigma \subset \Pi_E$ and distance δ is the following set:*

$$\mathcal{H}^{\pm}_\sigma(\delta) := \{x \in H_{\mathbb{H}} : \pm d_{H_{\mathbb{H}}}(\sigma, x) > \delta\}$$

where $d_{H_{\mathbb{H}}}(\sigma, \bullet)$ is the signed hyperbolic distance from σ with the agreement that $d_{H_{\mathbb{H}}}(\sigma, \bullet)$ is positive in the direction of mean curvature of σ. For symplicity, we will denote $\mathcal{H}^+_\sigma(0)$ by $\mathcal{H}^+_\sigma$. (Similarly, $\mathcal{H}^-_\sigma$).

Definition 6.1.3 (Euclidean graph over half hyperplane)**.** *Let A be a subset of $H_{\mathbb{H}}$ and $\varphi : A \to \mathbb{R}$ be a real valued function. The* ***Euclidean graph of*** *φ* ***over*** *A is given by*

$$\mathfrak{G}_\varphi = \{p + \varphi(p)\nu_{\mathbb{R}} : p \in A\}$$

where $\nu_{\mathbb{R}}$ is the Euclidean normal to $H_{\mathbb{H}}$.

Definition 6.1.4 (Euclidean C^k-$(H_{\mathbb{H}}, \sigma, \pm)$-asymptotic)**.** *Suppose that $H_{\mathbb{H}}$ is a half-hyperplane, let σ be a horosphere and define $\mathcal{H}_\sigma^\pm$ as in Definition 6.1.3. an embedded submanifold $M \subset \mathbb{H}^{m+1}$ is C^k**-asymptotic** to $\mathcal{H}_\sigma^\pm$ if:*

1. *M can be represented as an Euclidean graph of a C^k-function $\varphi : \mathcal{H}_\sigma^\pm \to \mathbb{R}$*
2. *$\forall \epsilon > 0$, $\exists \delta > 0$ such that:*
   $\sup_{p \in \mathcal{H}_\sigma^\pm(\delta)} |\varphi(p)| < \epsilon,$
   $\sup_{p \in \mathcal{H}_\sigma^\pm(\delta)} |\nabla^{\mathbb{R}^{(l)}} \varphi_p|_{\mathbb{R}} < \epsilon$ *for any* $1 \le l \le k,$
   where $\nabla^{\mathbb{R}^{(l)}}$ is the l-th Euclidean derivative of φ.

Definition 6.1.5 (GR Property)**.** *We say that a hypersurface $M^m \subset \mathbb{H}^{m+1}$ has the **(GR) Property** if M is a complete, connected, properly immersed soliton with respect to $-\partial_0$ that, outside a cylinder C, is C^1-asymptotic to two $\mathbb{H}$-half-hyperplanes $\mathcal{H}_{\sigma_1}$ and $\mathcal{H}_{\sigma_2}$, where σ_1 is a horosphere such that σ_1 is one of the connected components of $\Pi_1 \cap \partial C$ and $\mathcal{H}_{\sigma_1}$ can be either $\mathcal{H}_{\sigma_1}^+$ or $\mathcal{H}_{\sigma_1}^-$(similarly, for σ_2).*

Definition 6.1.6 (Wings)**.** *Let M be a hypersurface with the (GR) property. We call **wings** of M the two parts that are C^1-asymptotic to $\mathcal{H}_{\sigma_1}$ and $\mathcal{H}_{\sigma_2}$.*

6.2 Hyperbolic Dynamic Lemma

The main goal of this section is the Hyperbolic Dynamic Lemma (Lemma 6.2.3). That lemma will be a crucial tool to obtain the Uniqueness Theorem (Theorem 6.3.1). For the Hyperbolic Dynamic Lemma (Lemma 6.2.3), we recall some definitions:

Definition 6.2.1 (Locally bounded area)**.** *Let $\Omega \subset (\mathbb{H}^{m+1}, g_{\mathbb{I}})$ be a open set. We say that a sequence of smooth manifolds $\{\mathcal{M}_i\}$ has **locally bounded area** if for any relatively compact open subset $B \Subset \Omega$ there exists a constant $K = K(B)$ such that:*

$$\text{area}_{g_{\mathbb{I}}}(B \cap \mathcal{M}_i) < K \quad \forall i \in \mathbb{N}$$

Definition 6.2.2 (Singular set)**.** *Let us denote by*

$$\mathcal{Z} := \left\{p \in \Omega : \limsup_{i\to\infty} \text{area}\{M_i \cap \mathbb{B}_r(p)\} = \infty \text{ for every } r > 0\right\},$$

*the set where the area blows up. $\mathcal{Z}$ is called the **singular set**. Clearly, $\mathcal{Z}$ is a closed set.*

To recall the definitions concerning varifolds, read appendix **??**.

Lemma 6.2.3 (Hyperbolic Dynamic Lemma)**.** *Let M have the (GR) property. Suppose that $\{v_i\} \subset (span\{\partial_0, \partial_1\})^{\perp}$ and define $M_i := M + v_i$. Then, after passing to a subsequence, $\{M_i\}$ weakly converges to a connected stationary integral varifold M_∞. Moreover, $\partial_\infty M_\infty \subset \partial_\infty M_i$.*

Proof. As in Lemma 3.1 of (Gama; Martín, 2020), we use Theorem C.0.2. Summarizing the proof:

First step: we prove that sequence $\{M_i\}$ has locally bounded area outside of the cylinder C with respect to the Ilmanen's metric $g_{\mathbb{I}}$. Then the singular set $\mathcal{Z}$ is inside C.

Second step: By Theorem C.0.3, we prove that the singular set $\mathcal{Z}$ is empty inside cylinder C.

Third step: By Theorem C.0.2, there exists a limit varifold M_∞ for a subsquence of $\{M_i\}$. Furthermore, outside of cylinder C, M_∞ is the limit of a sequence of graphs that are stable and therefore they satisfy curvature bounds. Therefore, the convergence is smooth with multiplicity 1 outside of C.

Fourth step: We prove that $\partial_\infty M_\infty \subset \partial_\infty M_i$.

First step: For a fixed point $P = (p_0, \ldots, p_m) \in \mathcal{H}_{\sigma_1}$(Respectively $P \in \mathcal{H}_{\sigma_2}$) and a Euclidean normal $\nu_{\mathbb{R}}$=$(a_0, \ldots, a_m)$ to $\mathcal{H}_{\sigma_1}$ with $|\nu_{\mathbb{R}}|_{\mathbb{R}} = 1$. Hence there exists a $\theta \in (0, \pi)$ such that $\nu_{\mathbb{R}} = \cos\theta\partial_1 + \sin\theta\partial_0$. We use the following change of coordinates:

$$\begin{cases} y_0 &= \cos\theta x_1 + \sin\theta x_0, \\ y_1 &= -\sin\theta x_1 + \cos\theta x_0, \\ y_k &= x_k \quad \forall k \in \{2, \ldots, m\}. \end{cases}$$

For constants $s_j > 0 \; \forall j \in \{0, \ldots, m\}$, we define the box B centered in $P = (p_0, \ldots, p_m)$ with size lenghts s_j.

$$B := \{X = (x_0, x_1, \ldots, x_m) \in \mathbb{H}^{m+1} | -s_0 < a_0(x_0 - p_0) + \cdots + a_m(x_m - p_m) < s_0 \quad |x_j - p_j| < s_j \quad \forall j \in \{2, \ldots, m\}$$
$$|\cos\theta(x_1 - p_1) + \sin\theta(x_0 - p_0)| < s_0 \text{ and } |-\sin\theta(x_1 - p_1) + \cos\theta(x_0 - p_0)| < s_1\}$$

Let $\mathcal{W}_{\sigma_1,i}$ be the wing of M_i asymptotic to $\mathcal{H}_{\sigma_1}$(see Definition 6.1.6). We claim there exists a constant $K > 0$ (depending only on B) such that, for all $i \in \mathbb{N}$, $\text{area}_{g_{\mathbb{I}}}(B \cap \mathcal{W}_{\sigma_1,i}) < K$. Similarly to $\mathcal{W}_{\sigma_2,i}$. Indeed, by Definition 6.1.4, $\mathcal{W}_{\sigma_1,i}$ can be written as a graph over $\mathcal{H}_{\sigma_1}$. Therefore, for each i, there exists a map $\varphi_i : B \cap \mathcal{H}_{\sigma_1} \to B$ such that $\varphi_i(B \cap \mathcal{H}_{\sigma_1}) = B \cap \mathcal{W}_{\sigma_1,i}$.

Hence there exists a function $f_i : B \cap \mathcal{H}_{\sigma_1} \to \mathbb{R}$ such that $\varphi_i(p) = p + f_i(p)\nu_{\mathbb{R}}$. Therefore,

$$\begin{aligned} \text{area}_{g_{\mathbb{I}}}(B \cap \mathcal{W}_{\sigma_1,i}) &= \int\limits_{B\cap\mathcal{W}_{\sigma_1,i}} dV_{g_{\mathbb{I}}} \\ &= \int\limits_{B\cap\mathcal{H}_{\sigma_1}} \sqrt{\det g_{\mathbb{I}}(M_i)}\, dy_1 \wedge \cdots \wedge dy_m, \end{aligned}$$

where $\det g_{\mathbb{I}}(M_i) = \det\left[g_{\mathbb{I}}\left(\varphi_{i*}\left(\frac{\partial}{\partial y_k}\right), \varphi_{i*}\left(\frac{\partial}{\partial y_l}\right)\right)\right]_{k,l}$.

By $x_0(B \cap \mathcal{W}_{\sigma_1,i})$ is bounded and $|\nabla^{\mathbb{R}} f_i|_{\mathbb{R}} < \epsilon$ for some $\epsilon > 0$ and Lemma 6.2.4 below, there exists a constant $C > 0$ such that $\sqrt{\det g_{\mathbb{I}}(M_i)} < C$ in $B \cap \mathcal{H}_{\sigma_1}$. Therefore,

$$\int\limits_{B\cap\mathcal{H}_{\sigma_1}} \sqrt{\det g_{\mathbb{I}}(M_i)}\, dy_1 \wedge \cdots \wedge dy_m < C \int\limits_{B\cap\mathcal{H}_{\sigma_1}} dy_1 \wedge \cdots \wedge dy_m = C\, \text{area}_{g_{\mathbb{R}}}(B \cap \mathcal{H}_{\sigma_1})$$

Define $K := C\, \text{area}_{g_{\mathbb{R}}}(B \cap \mathcal{H}_{\sigma_1})$. Thus, $\text{area}_{g_{\mathbb{I}}}(B \cap \mathcal{W}_{\sigma_1,i}) < K$, i. e., $\{M_i\}$ has locally bounded $g_{\mathbb{I}}$-area outside of C.

Second step: By the first step, $\mathcal{Z} \subset C$. In order to use Theorem C.0.3, we choose a open set $\mathcal{N}$ above $\mathcal{B}_{\frac{\pi}{2}}(o, r_0)$ such that $\partial\mathcal{N} = \mathcal{B}_{\frac{\pi}{2}}(o, r_0)$ a spherical barrier (See Proposition 4.3.3) with $r_0 > 0$ small enough such that $\mathcal{B}_{\frac{\pi}{2}}(o, r_0) \cap C = \emptyset$. By Proposition **??**, the second fundamental form $\mathrm{II}_{g_{\mathbb{I}}}$ of $\mathcal{B}_{\frac{\pi}{2}}(o, r_0)$ in the upward direction is positive definite and $g_{\mathbb{I}}(H_{\partial\mathcal{N}}, \xi) = g_{\mathbb{I}}\left(\sum_{i=1}^m \mathrm{II}_{g_{\mathbb{I}}}(\tilde{E}_i, \tilde{E}_i)\nu_{g_{\mathbb{I}}}, \nu_{g_{\mathbb{I}}}\right) \geq 0$ where $\{\tilde{E}_i\}$ is a $g_{\mathbb{I}}$-orthonormal frame. Now we increase the radius r_0 and guarantee that $\mathcal{Z} = \emptyset$. Namely, It is enough to prove that $\sup\{r_0 \in \mathbb{R}^+ : \mathcal{B}_{\frac{\pi}{2}}(o, r_0) \cap C = \emptyset\} = \infty$. Suppose, by contradition, that the supremum is finite, $r_M := \sup\{r_0 \in \mathbb{R}^+ : \mathcal{B}_{\frac{\pi}{2}}(o, r_0) \cap C = \emptyset\}$. By definition, $\mathcal{B}_{\frac{\pi}{2}}(o, r_M) \cap \mathcal{Z} \neq \emptyset$ and $\mathcal{Z} \subset \mathcal{N}$. Therefore, by Theorem C.0.3, $\partial\mathcal{N} \subset \mathcal{Z}$ contradicting the fact that $\mathcal{Z} \subset C$ by the asymptotic behaviour of $\mathcal{B}_{\frac{\pi}{2}}(o, r_M)$. Thus $\mathcal{Z} = \emptyset$.

Third step: By first and second steps, $\{M_i\}$ has locally bounded $g_{\mathbb{I}}$-area. Thus, by Theorem C.0.2, $\{M_i\}$ converges weakly to a stationary integral varifold M_∞.

Fourth step: By (GR) property, $\partial_\infty M_i = \partial_\infty M_j\ \forall i, j \in \mathbb{N}$. Take a point P_∞ that is not in the boundary at infinity of M_i, $P_\infty \in \partial_\infty \mathbb{H}^{m+1} \backslash \partial_\infty M_i$. By the uniform asymptotic behaviour of $\{M_i\}$, there exists a spherical barrier $\mathcal{B}_{\frac{\pi}{2}}(o, r_0)$ (see Proposition 4.3.3) such that $o = P_\infty$, $\partial_\infty \mathcal{B}_{\frac{\pi}{2}}(o, r_0) \cap \partial_\infty M_i = \emptyset$ and $\mathcal{B}_{\frac{\pi}{2}}(o, r_0) \cap M_i = \emptyset \quad \forall i \in \mathbb{N}$. Therefore, there is no point of M_∞ in the connected component of $\mathbb{H}^{m+1} \backslash \mathcal{B}_{\frac{\pi}{2}}(o, r_0)$ whose boundary at infinity contains P_∞. Therefore, P_∞ is not in the boundary at infinity of M_∞. Thus, $\partial_\infty M_\infty \subset \partial_\infty M_i$.

□

Lemma 6.2.4. *It holds that:*

$$\det g_{\mathbb{I}} = (e^{2\mathsf{w}})^m (1 + |\nabla^{\mathbb{R}} f_i|^2),$$

where $\mathsf{w} = \frac{1}{mx_0} - \ln x_0$

Proof. Recall that $\varphi_i : B \cap \mathcal{H}_{\sigma_1} \to B$ such that $\varphi_i(B \cap \mathcal{H}_{\sigma_1}) = B \cap \mathcal{W}_{\sigma_1,i}$. And $f_i : B \cap \mathcal{H}_{\sigma_1} \to \mathbb{R}$ such that $\varphi_i(X) = \exp^{\mathbb{R}}_X(f_i(X)\nu_{\mathbb{R}})$.

$$g_{\mathbb{I}ab} := g_{\mathbb{I}}\left(\varphi_{i*}\left(\frac{\partial}{\partial y_a}\right), \varphi_{i*}\left(\frac{\partial}{\partial y_b}\right)\right).$$

Notice that $\varphi_{i*}\left(\frac{\partial}{\partial y_a}\right) = \frac{\partial}{\partial y_a} + \frac{\partial f_i}{\partial y_a}\frac{\partial}{\partial y_0}$. Hence,

$$\begin{aligned} g_{\mathbb{I}ab} &= e^{2\mathsf{w}} g_{\mathbb{R}}\left(\frac{\partial}{\partial y_a} + \frac{\partial f_i}{\partial y_a}\frac{\partial}{\partial y_0}, \frac{\partial}{\partial y_b} + \frac{\partial f_i}{\partial y_b}\frac{\partial}{\partial y_0}\right) \\ &= e^{2\mathsf{w}}\left(\delta_{ab} + \frac{\partial f_i}{\partial y_a}\frac{\partial f_i}{\partial y_b}\right) \end{aligned}$$

Thus,

$$\det g_{\mathbb{I}} = (e^{2\mathsf{w}})^m \det\left(\delta_{ab} + \frac{\partial f_i}{\partial y_a}\frac{\partial f_i}{\partial y_b}\right)_{1 \le a,b \le m}$$

Define $D := \det\left(\delta_{ab} + \frac{\partial f_i}{\partial y_a}\frac{\partial f_i}{\partial y_b}\right)_{1 \le a,b \le m}$. We claim that $D = (1 + |\nabla^{\mathbb{R}} f_i|^2_{\mathbb{R}})$ proving the lemma. Indeed, define the matrix $(A_{ab}) := \left(\delta_{ab} + \frac{\partial f_i}{\partial y_a}\frac{\partial f_i}{\partial y_b}\right)_{1 \le a,b \le m}$. If we prove that A has eigenvalues 1 with multiplicity $m-1$ and $1 + |\nabla^{\mathbb{R}} f_i|^2_{\mathbb{R}}$ with multiplicity 1, then $D = \det A = 1 + |\nabla^{\mathbb{R}} f_i|^2_{\mathbb{R}}$. Namely, take $v = (v_1, \ldots, v_m)$, with $g_{\mathbb{R}}\left(\nabla^{\mathbb{R}} f_i, v\right) = 0$. Let us set $w = Av$:

$$\begin{aligned} w_a &= \sum_{1 \le j \le m} A_{aj} v_j = \sum_{1 \le j \le m} \delta_{aj} v_j + \frac{\partial f_i}{\partial y_a}\frac{\partial f_i}{\partial y_j} v_j \\ w_a &= v_a + \frac{\partial f_i}{\partial y_a} g_{\mathbb{R}}\left(\nabla^{\mathbb{R}} f_i, v\right) = v_a \end{aligned}$$

Therefore, $w = v$, $Av = v$ and 1 is eigenvalue with multiplicity $m-1$. Let us set $u = A(\nabla^{\mathbb{R}} f_i)$.

$$\begin{aligned} u_a &= \sum_{1 \le j \le m} A_{aj}\frac{\partial f_i}{\partial y_j} = \sum_{1 \le j \le m} \delta_{aj}\frac{\partial f_i}{\partial y_j} + \frac{\partial f_i}{\partial y_a}\frac{\partial f_i}{\partial y_j}\frac{\partial f_i}{\partial y_j} \\ u_a &= \frac{\partial f_i}{\partial y_a} + \frac{\partial f_i}{\partial y_a}|\nabla^{\mathbb{R}} f_i|^2_{\mathbb{R}} \\ u_a &= (1 + |\nabla^{\mathbb{R}} f_i|^2_{\mathbb{R}})\frac{\partial f_i}{\partial y_a} \end{aligned}$$

Therefore, $A(\nabla^{\mathbb{R}} f_i) = (1 + |\nabla^{\mathbb{R}} f_i|^2_{\mathbb{R}})\nabla^{\mathbb{R}} f_i$ and $(1 + |\nabla^{\mathbb{R}} f_i|^2_{\mathbb{R}})$ is eigenvalue with multiplicity 1. $\square$

6.3 Uniqueness Theorem

In this section we will prove the Uniqueness Theorem (Theorem 6.3.1). By matter of organization, we separate the proof in two cases, when $x_0(M)$ is bounded above and when it is not.

Theorem 6.3.1 (Uniqueness Theorem)**.** *Let M^m be a mean curvature flow soliton with respect to $-\partial_0$ in $(\mathbb{H}^{m+1}, g_{\mathbb{H}})$ with the (GR) property (see Definition 6.1.5). Then M is a grim reaper or a vertical totally $g_{\mathbb{H}}$-geodesic hypersurface.*

In the case where $x_0(M)$ is bounded above, observe that every hyperplane at infinity is downward pointing. We need the following lemmas.

Lemma 6.3.2 (Vertical half hyperplanes of the type $-$)**.** *Suppose that M has the (GR) property (Definition 6.1.5) and $\mathcal{H}^-_{\sigma_i}$ is one of the half hyperplanes. Then $\mathcal{H}^-_{\sigma_i}$ is vertical, that is, $x_1(\mathcal{H}^-_{\sigma_i}) = \{c_1\}$, where $c_1 \in \mathbb{R}$ is a constant.*

Proof. Suppose by contradiction that $\mathcal{H}^-_{\sigma_i}$ is not vertical, that is, $x_1\left(\mathcal{H}^-_{\sigma_i}\right) \neq x_1\left(\partial_\infty \mathcal{H}^-_{\sigma_i}\right)$. Up to rotations, we can assume that $x_1\left(\partial_\infty \mathcal{H}^-_{\sigma_i}\right) = \{c_1\}$ for a constant $c_1 \in \mathbb{R}$ and $x_1\left(\mathcal{H}^-_{\sigma_i}\right) \subset (-\infty, c_1]$ (when $x_1\left(\mathcal{H}^-_{\sigma_i}\right) \subset [c_1, \infty)$ the reasoning is similar). By the asymptotic behaviour of M, there exists a small spherical barrier $\mathcal{B}_{\frac{\pi}{2}}(o, r_0)$ such that:

(i) r_0 is small enough such that $\mathcal{B}_{\frac{\pi}{2}}(o, r_0) \cap C = \emptyset$.

(ii) $\mathcal{B}_{\frac{\pi}{2}}(o, r_0) \cap M = \emptyset$

(iii) $x_1\left(\mathcal{B}_{\frac{\pi}{2}}(o, r_0)\right) \subset (-\infty, c_1)$

In order to use the Maximum Principle (Proposition C.0.1), we have to find a suitable spherical barrier. We move the center $o = (q_0, q_1, q_2, \ldots, q_m)$ in the x_1-direction until the spherical barrier touches M in the wing. Namely, define $\mathcal{B}_\mu := \mathcal{B}(o_\mu, r_0)$, where $o_\mu = (q_0, q_1 + \mu, q_2, \ldots, q_m)$. Define the interval $I := \{\mu_1 \in [0, \infty) : \mathcal{B}_\mu \cap M = \emptyset \quad \forall 0 \leq \mu < \mu_1\}$ and $\mu_S := \sup I$.

We claim that $\sup x_1(\mathcal{B}_{\mu_S}) < c_1$. In fact, for all μ such that $\sup x_1(\mathcal{B}_\mu) > c_1$, $\mathcal{B}_\mu \cap M \neq \emptyset$ by the asymptotic behaviour of M. And for μ such that $\sup x_1(\mathcal{B}_\mu) = c_1$, $\mathcal{B}_\mu$ must touch M in the wing, because $\mathcal{B}_\mu$ is not C^1-asymptotic to any subset of non-vertical half hyperplane $\mathcal{H}^-_{\sigma_1}$.

We claim that $\mathcal{B}_{\mu_S} \cap M \neq \emptyset$. Indeed, by definition of supremum, there exists a sequence of points (P_j) in M such that $\mathrm{dist}_{\mathbb{H}}(P_j, \mathcal{B}_{\mu_S}) \to 0$ as $j \to \infty$. By the asymptotic behaviours of M and $\mathcal{B}_{\mu_S}$, $\inf\{x_0(P_j)\} > 0$. By the boundedness of coordinates of $\{P_j\}$, up a subsequence, (P_j) converges to a point P_∞ in $\mathbb{H}^{m+1}$. $P_\infty \in M$ because M is closed. Hence $\mathrm{dist}_{\mathbb{H}}(P_\infty, \mathcal{B}_{\mu_S}) = 0$. Therefore, P_∞ belongs to $\mathcal{B}_{\mu_S}$ too because $\mathcal{B}_{\mu_S}$ is closed and we have a sequence of point in $\mathcal{B}_{\mu_S}$ converging to P_∞.

Thus, by the Maximum Principle, $\mathcal{B}_{\mu_S}$ must be a stationary hypersurface, a contradiction. Therefore, $x_1(\mathcal{H}^-_{\sigma_1})$ is not contained in $(-\infty, c_1]$. Similarly for $[c_1, \infty)$.

We conclude that $x_1(\mathcal{H}^-_{\sigma_1}) = \{c_1\}$, that is, $\mathcal{H}^-_{\sigma_1}$ is vertical. □

Lemma 6.3.3. *Let M^m be a mean curvature flow soliton with respect to $-\partial_0$ in $(\mathbb{H}^{m+1}, g_{\mathbb{H}})$ with the (GR) property (see Definition 6.1.5) and* $\sup x_0(M) < \infty$. *Then M is contained in a chimney $C := \{x_1 > b_1\} \cap \{x_1 < e_1\} \cap \{x_0 < c_0\}$ for some constants b_1, e_1 and c_0.*

Proof. By Lemma 6.3.2, $\mathcal{H}^-_{\sigma_1} = \{x_1 = c_1\}$ and $\mathcal{H}^-_{\sigma_2} = \{x_1 = d_1\}$, say, $c_1 < d_1$. There exists a small spherical barrier $\mathcal{B}_{\frac{\pi}{2}}(o, r_0)$ such that:

(i) $x_1\left(\mathcal{B}_{\frac{\pi}{2}}(o, r_0)\right) < c_1$,

(ii) $\mathcal{B}_{\frac{\pi}{2}}(o, r_0) \cap M = \emptyset$.

We move the center $o = (q_0, q_1, \ldots, q_m)$ in x_1-direction and increase the radius r_0 in such a way that the family of spherical barrier converges to a vertical hyperplane. Namely, define $\mathcal{B}_\lambda := \mathcal{B}_{\frac{\pi}{2}}(o_\lambda, r_\lambda)$ where $o_\lambda = (q_0, q_1 - \lambda, q_2, q_m)$ and $r_\lambda = r_0 + \lambda$. Notice that $P = (q_0, q_1 + r_0, q_2, \ldots, q_m) \in \partial_\infty \mathcal{B}_\lambda \quad \forall \lambda \in \mathbb{R}$. By the Maximum Principle (Proposition C.0.1), $B_\lambda \cap M = \emptyset \quad \forall \lambda > 0$. Therefore, $\{x_1 < q_1 + r_0\} \cap M = \emptyset$. Now define $b_1 := q_1 + r_0$. Similarly we can find a constant e_1. Take a constant c_0 greater than $\sup x_0(M)$. Thus, $M \subset \{x_1 > b_1\} \cap \{x_1 < e_1\} \cap \{x_0 < c_0\}$. □

Lemma 6.3.4. *Let M^m be a mean curvature flow soliton with respect to $-\partial_0$ in $(\mathbb{H}^{m+1}, g_{\mathbb{H}})$ with the (GR) property (see Definition 6.1.5) and* $\sup x_0(M) < \infty$ *with $\mathcal{H}^-_{\sigma_1} = \{x_1 = c_1\}$ and $\mathcal{H}^-_{\sigma_2} = \{x_1 = d_1\}$ where $c_1 < d_1$. Then there exists a small grim-reaper $\mathcal{G}_{h,[H_*,H^*]}$ below M centered in $[c_1, d_1]$, that is,*

(i) $\mathcal{G}_{h,[H_*,H^*]} \cap M = \emptyset$.

(ii) $c_1 < x_1\left(\mathcal{G}_{h,[H_*,H^*]}\right) < d_1$.

(iii) $\frac{H_* + H^*}{2} = \frac{c_1 + d_1}{2}$.

Proof. By the asymptotic behaviour of M, there exists a small spherical barrier fitting between $\mathcal{H}^-_{\sigma_1}$ and $\mathcal{H}^-_{\sigma_2}$ and not touching M, that is, there exists $\mathcal{B}_{\frac{\pi}{2}}(o, r_0)$ with center $o = (q_0, q_1, \ldots, q_m) \in \partial_\infty \mathbb{H}^{m+1}$ such that:

(i) $c_1 < x_1\left(\mathcal{B}_{\frac{\pi}{2}}(o, r_0)\right) < d_1$.

(ii) $\mathcal{B}_{\frac{\pi}{2}}(o,r_0)\cap M=\emptyset$.

Define a half tube $T:=\{P\in\mathbb{H}^{m+1}|q_1-r_0<x_1(P)<q_1+r_0$ and $|x_0(P)-q_0|<r_0\}$. Moving the spherical barrier below M and between $\mathcal{H}^-_{\sigma_1}$ and $\mathcal{H}^-_{\sigma_2}$, we can garantee that, by the Maximum Principle (Proposition C.0.1), $T\cap M=\emptyset$. Indeed, define $\mathcal{B}_{\mu_2,\mu_3,\dots,\mu_m}:=\mathcal{B}\left(o_{\mu_1,\mu_2,\dots,\mu_m},r_0\right)$ where $o_{\mu_2,\dots,\mu_m}=(q_0,q_1,q_2+\mu_2,q_3+\mu_3,\dots,q_m+\mu_m)$. By the Maximum Principle, varying $\mu_2,\mu_3,\dots,\mu_{m-1}$ and μ_m, $\mathcal{B}_{\mu_2,\mu_3,\dots,\mu_m}\cap M=\emptyset$. Therefore,

$$\left(\bigcup_{\mu_2,\dots,\mu_m\in\mathbb{R}^+}\mathcal{B}_{\mu_2,\dots,\mu_m}\right)\cap M=\emptyset.$$

Thus, $T\cap M=\emptyset$ as claimed. Now there exists a small grim-reaper $\mathcal{G}_{h_1,[H_*(h_1),H^*(h_1)]}$ contained in T. By the Maximum Principle using $\mathcal{G}_{h_1,[H_*(h_1),H^*(h_1)]}$ as barrier, we can center a grim reaper in $[c_1,d_1]$.

□

Proof of Uniqueness Theorem when $\sup x_0(M)<\infty$. Suppose that M is not a grim-reaper. In this case, the two hyperplanes point downward, that is, M is C^1-asymptotic to $\mathcal{H}^-_{\sigma_1}$ and $\mathcal{H}^-_{\sigma_2}$.

First step: Now we prove that M is above a grim reapers with the same boundary at infinity. Indeed, by Lemma 6.3.2, $\mathcal{H}^-_{\sigma_1}$ and $\mathcal{H}^-_{\sigma_2}$ are vertical. Hence $\mathcal{H}^-_{\sigma_1}=\{x_1=c_1\}$ and $\mathcal{H}^-_{\sigma_2}=\{x_1=d_1\}$ with $c_1<d_1$. By Lemma 6.3.4, there exists a grim-reaper $\mathcal{G}_{h,[H_*,H^*]}$ below M such that $\frac{(H_*+H^*)}{2}=\frac{(c_1+d_1)}{2}$. In order to use Maximum Principle (Proposition C.0.1), we define a family of grim reaper $\mathcal{G}_\lambda:=\mathcal{G}_{h_\lambda,[H_*(h_\lambda),H^*(h_\lambda)]}$ where $h_\lambda=h+\lambda$ and $[H_*(h_\lambda)\,,\,H^*(h_\lambda)]$ is the correspondent interval in the x_1-axis with the same center of $[H_*(h),H^*(h)]$. We can increase this grim reaper without contact with M until it has the same boundary at infinity of M. Namely, define the interval $I:=\{\lambda_0>0:\mathcal{G}_\lambda\cap M=\emptyset\quad\forall 0<\lambda<\lambda_0\}$ and $\lambda_S:=\sup I$. We claim that $\mathcal{G}_{\lambda_S}\cap M=\emptyset$. Otherwise M must coincide with $\mathcal{G}_{\lambda_S}$ by Maximum Principle and we are assuming that M is not a grim reaper. By definition of λ_S, we have $\mathrm{dist}_{\mathbb{R}}(\mathcal{G}_{\lambda_S},M)=0$. There exists a sequence of points $(Q_i)_{i\in\mathbb{N}}$ in M such that $\mathrm{dist}_{\mathbb{R}}(Q_i,\mathcal{G}_{\lambda_S})\to 0$ as $i\to\infty$. In order to use the Hyperbolic Dynamic Lemma (Lemma 6.2.3), define $M_i=M+v_i$, where $v_i:=(0,0,-x_2(Q_i),\dots,-x_m(Q_i))$. By the Hyperbolic Dynamic Lemma, up a subsequence, $\{M_i\}$ weakly converges to a connected stationary integral varifold M_∞. Notice that $R_i:=(x_0(Q_i),x_1(Q_i),0,\dots,0)\in M_i$, up a to subsequence, converges to a point R_∞ and $R_\infty\in M_\infty\cup\partial_\infty M_\infty$. The grim reaper $\mathcal{G}_{\lambda_S}$ has the same boundary at infinity as M_i and M_∞. Otherwise, by the asymptotic behaviours of $\mathcal{G}_{\lambda_S}$ and M_∞, R_∞ would not belong to $\partial_\infty\mathcal{G}_{\lambda_S}\cup\partial_\infty M_\infty$ therefore R_∞ would belong to $\mathcal{G}_{\lambda_S}\cap M_\infty$ and , by the Maximum Principle,

$M_\infty = \mathcal{G}_{\lambda_S}$. Therefore, as claimed $\partial_\infty \mathcal{H}^-_{\sigma_1} \cup \partial_\infty \mathcal{H}^-_{\sigma_2} = \partial_\infty M_i = \partial_\infty \mathcal{G}_{\lambda_S}$. Thus, $\mathcal{G}_{\lambda_S}$ is the unique grim reaper with boundary at infinity equal to $\partial_\infty \mathcal{H}^-_{\sigma_1} \cup \partial_\infty \mathcal{H}^-_{\sigma_2}$.

Second step: Now we prove that M is below a grim reaper with the same boundary at infinity. In fact, by Lemma 6.3.3, M is inside a chimney $C := \{x_1 > b_1\} \cap \{x_1 < e_1\} \cap \{x_0 < c_0\}$. There exists a big grim reaper $\mathcal{G}_{h_0,[H_*(h_0),H^*(h_0)]}$ above C and with the same center as $[c_1, d_1]$, that is, $h_0 > c_0$, $H_*(h_0) < b_1$, $H^*(h_0) > e_1$ and $\frac{c_1+d_1}{2} = \frac{H_*(h_0)+H^*(h_0)}{2}$. We decrease this grim reaper without contact with M until it has the same boundary at infinity as M. Namely, in order to use the Maximum Principle again, we define a family of grim reapers $\mathcal{G}_\mu := \mathcal{G}_{h_\mu,[H_*(h_\mu),H^*(h_\mu)]}$, where $h_\mu := h_0 + \mu$ and define the interval $J := \{\mu_0 \in (-\infty, 0] : \mathcal{G}_\mu \cap M = \emptyset \quad \forall \mu > \mu_0\}$ and $\mu_I = \inf J$. We claim that $\mathcal{G}_{\mu_I} \cap M = \emptyset$. Otherwise M must coincide with $\mathcal{G}_{\mu_I}$ by the Maximum Principle and we are assumiming that M is not a grim reaper. By definition of μ_I, $\text{dist}_{\mathbb{R}}(\mathcal{G}_{\mu_I}, M) = 0$. There exists a sequence of points $(S_i)_{i\in\mathbb{N}}$ in M such that $\text{dist}_{\mathbb{R}}(S_i, \mathcal{G}_{\mu_I}) \to 0$ as $i \to \infty$. Like in the last step, in order to use the Hyperbolic Dynamic Lemma (Lemma 6.2.3), define $M_i = M + v_i$, where $v_i := (0, 0, -x_2(S_i), \ldots, -x_m(S_i))$. By Hyperbolic Dynamic Lemma, up to a subsequence, $\{M_i\}$ weakly converges to a connected stationary integral varifold M_∞. Notice that $T_i := (x_0(S_i), x_1(S_i), 0, \ldots, 0) \in M_i$, up a subsequence, converges to a point T_∞ and $T_\infty \in M_\infty \cup \partial_\infty M_\infty$. We claim that the grim reaper $\mathcal{G}_{\mu_I}$ has the same boundary at infinity as M_i. Otherwise, by the asymptotic behaviours of $\mathcal{G}_{\mu_I}$ and M_∞, T_∞ would not belong to $\partial_\infty \mathcal{G}_{\mu_I} \cup \partial_\infty M_\infty$ therefore R_∞ would belong to $\mathcal{G}_{\mu_I} \cap M_\infty$ and , by the Maximum Principle, $M_\infty = \mathcal{G}_{\mu_I}$. Therefore as claimed $\partial_\infty \mathcal{H}^-_{\sigma_1} \cup \partial_\infty \mathcal{H}^-_{\sigma_2} = \partial_\infty M_i = \partial_\infty \mathcal{G}_{\mu_I}$. Therefore $\mathcal{G}_{\mu_I}$ is the unique grim reaper with boundary at infinity equal to $\partial_\infty \mathcal{H}^-_{\sigma_1} \cup \partial_\infty \mathcal{H}^-_{\sigma_2}$.

Third step: By the first and second steps, M is between two grim reapers, $\mathcal{G}_{\lambda_S}$ and $\mathcal{G}_{\mu_I}$, with the same boundary at infinity, $\partial_\infty \mathcal{H}^-_{\sigma_1} \cup \partial_\infty \mathcal{H}^-_{\sigma_2}$. However, there is only one grim reaper with this boundary at infinity. Therefore $\mathcal{G}_{\lambda_S}$ and $\mathcal{G}_{\mu_I}$ must coincide. Hence, M must coincide with $\mathcal{G}_{\lambda_S}$ and $\mathcal{G}_{\mu_I}$. This contradicts our assumption that M is not a grim reaper. □

In this second part of this section, we prove the Uniqueness Theorem when $x_0(M)$ is unbounded above.

Proof of Uniqueness Theorem when $\sup x_0(M) = \infty$. We are assuming that $\sup x_0(M) = \infty$.

First case: Suppose that the two half hyperplanes point upward, that is, $\mathcal{H}_{\sigma_1}$ and $\mathcal{H}_{\sigma_2}$ are of type +. In this case, $\inf x_0(M) > 0$. Therefore, we can put a small grim reaper $\mathcal{G}_{h,[H_*,H^*]}$ below M, that is, there is a grim reaper $\mathcal{G}_{h,[H_*,H^*]}$ such that $h < \inf x_0(M)$. Hence

$\mathcal{G}_{h,[H_*,H^*]} \cap M = \emptyset$. We increase the height of $\mathcal{G}_{h,[H_*,H^*]}$ until it touches M and we use the Maximum Principle (Theorem C.0.1) to arrive at a contradiction. Namely, define

$$\mathcal{G}_\lambda := \mathcal{G}_{h_\lambda,[H_*(h_\lambda),H^*(h_\lambda)]},$$

where $h_\lambda := h+\lambda$

Define $I := \{\lambda_0 > 0 : \mathcal{G}_\lambda \cap M = \emptyset \quad \forall \lambda_0 > \lambda > 0\}$ and $\lambda_S := \sup I$. If $\mathcal{G}_{\lambda_S} \cap M \neq \emptyset$, by Maximum Principle, $\mathcal{G}_{\lambda_S} = M$, contradiction. Therefore, $\mathcal{G}_{\lambda_S} \cap M = \emptyset$ and, by definition of λ_S, $\mathrm{dist}_{\mathbb{R}}(\mathcal{G}_{\lambda_S}, M) = 0$, that is, there exists a sequence of points (Q_i) in M such that $\mathrm{dist}_{\mathbb{R}}(\mathcal{G}_{\lambda_S}, Q_i) \to 0$ as $i \to \infty$. In order to use the Hyperbolic Dynamic Lemma (Lemma 6.2.3), define

$$M_i := M + v_i,$$

where $v_i := (0,0,-x_3(Q_i),\ldots,-x_m(Q_m))$. By the Hyperbolic Dynamic Lemma, up to a subsequence, $\{M_i\}$ weakly converges to a varifold M_∞. Notice that the sequence of points (R_i), $R_i := (x_0(Q_i), x_1(Q_i), 0, \ldots, 0) \in M_i$, up to a subsequence, converges to a point R_∞ and R_∞ belongs to $\mathcal{G}_{\lambda_S}$ because:

(i) $\{x_0(R_i)\}$ and $\{x_1(R_i)\}$ are bounded.

(ii) $\inf\{x_0(R_i)\} > 0$.

(iii) $\mathrm{dist}_{\mathbb{R}}(\mathcal{G}_{\lambda_S}, R_\infty) = \lim_{i\to\infty} \mathrm{dist}_{\mathbb{R}}(\mathcal{G}_{\lambda_S}, R_i) = 0$.

(iv) $\mathcal{G}_{\lambda_S}$ is closed.

As R_∞ is a limit of points $R_i \in M_i$, R_∞ belongs to M_∞. Hence,

$$\mathcal{G}_{\lambda_S} \cap M_\infty \neq \emptyset.$$

Therefore, $\mathcal{G}_{\lambda_S} = M_\infty$ by Maximum Principle (Theorem C.0.1). This is a contradiction.

Second case: Now we suppose that one of the half hyperplanes points downward and the other points upward, say, M is C^1-asymptotic to $\mathcal{H}^-_{\sigma_1}$ and $\mathcal{H}^+_{\sigma_2}$. By Lemma 6.3.2, $\mathcal{H}^-_{\sigma_1}$ is vertical, that is, $x_1(\mathcal{H}^-_{\sigma_1}) = \{c_1\}$ for a constant c_1. We claim that for every $\epsilon > 0$,

$$\{x_1 > c_1 + \epsilon\} \cap M = \emptyset.$$

Indeed, by asymptotic behaviour of M, there exists a small spherical barrier $\mathcal{B}_{\frac{\pi}{2}}(o, r_0)$ with center $o = (q_0, q_1, \ldots, q_m)$ and radius r_0 such that:

(i) $\sup x_1(\mathcal{B}_{\frac{\pi}{2}}(o, r_0)) = c_1 + \epsilon$.

(ii) $\mathcal{B}_{\frac{\pi}{2}}(o, r_0) \cap M = \emptyset$.

We increase the radius of r_0 and move the center $o = (q_0, q_1, \ldots, q_m)$ in the x_1-direction in such a way that the spherical barriers converge to the half hyperplane $\{x_1 = c_1 + \epsilon\}$ in order to use the Maximum Principle. Namely, define

$$\mathcal{B}_\lambda := \mathcal{B}(o_\lambda, r_\lambda).$$

where $o_\lambda := (q_0, q_1 + \lambda, q_2, \ldots, q_m)$ and $r_\lambda := r_0 + \lambda$. Define $I := \{\lambda_0 > 0 : \mathcal{B}_\lambda \cap M = \emptyset \quad \forall \lambda_0 > \lambda > 0\}$. We claim that $\sup I = \infty$. Indeed, if $\lambda_S = \sup I$, $\mathcal{B}_{\lambda_S} \cap M \neq \emptyset$ therefore, by the Maximum Principle, $\mathcal{B}_{\lambda_S} = M$, contradiction. Thus, $\{x_1 > c_1 + \epsilon\} \cap M = \emptyset$ as claimed. Similarly by symmetry $\{x_1 < c_1 - \epsilon\} \cap M = \emptyset$ for all $\epsilon > 0$. We conclude that $M \subset \{x_1 = c_1\}$ and $M = \{x_1 = c_1\}$ by Maximum Principle. □

REFERENCES

ALÍAS, L.; LIRA, J. de; RIGOLI, M. Mean curvature flow solitons in the presence of conformal vector fields. **J. Geom. Anal.**, v. 30, n. 4, p. 1466–1529, 2020.

ALLEGRETTO, L. Positive solutions and spectral properties of second order elliptic operators. **Pacific J. Math.**, v. 92, p. 15–25, 1981.

ALTSCHULER, S. J.; WU, L. F. Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle. **Calculus of Variations and Partial Differential Equations**, Springer, v. 2, n. 1, p. 101–111, 1994.

BARBOSA, E.; SHARP, B.; WEI, Y. Smooth compactness of f-minimal hypersurfaces with bounded f-index. **Proc. Amer. Math. Soc.**, v. 145, p. 4945–4961, 2017.

CARMO, M. P. D. **Riemannian geometry**. [S.l.]: Springer, 1992. v. 6.

CASTÉRAS, J.-B.; HOLOPAINEN, I.; RIPOLL, J. Convexity at infinity in cartan-hadamard manifolds and applications to the asymptotic dirichlet and plateau problems. **Mathematische Zeitschrift**, v. 290, 2018.

CLUTTERBUCK, J.; SCHNÜRER, O. C.; SCHULZE, F. Stability of translating solutions to mean curvature flow. **Calculus of Variations and Partial Differential Equations**, Springer, v. 29, p. 281–293, 2007.

COLDING, T.; II, W. M. Generic mean curvature flow i; generic singularities. **Ann. of Math.**, v. 175, p. 755–833. 2012a.

COLDING, T.; II, W. M. Smooth compactness of self-shrinkers. **Comment. Math. Helv.**, v. 87, p. 463–475. 2012b.

COLDING, T. H.; MINICOZZI, W. P. **A course in minimal surfaces**. [S.l.]: American Mathematical Soc., 2011. v. 121.

EBERLEIN, P.; O'NEILL, B. Visibility manifolds. **Pacific Journal of Mathematics**, Pacific Journal of Mathematics, A Non-profit Corporation, v. 46, n. 1, p. 45 – 109, 1973. Disponível em: https://projecteuclid.org/journals/pacific-journal-of-mathematics/volume-46/issue-1/Visibility-mani.

ESCHENBURG, J.-H. Maximum principle for hypersurfaces. **Manuscripta Math.**, v. 64, n. 1, p. 55–75, 1989.

FISCHER-COLBRIE, D.; SCHOEN, R. The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature. **Comm. Pure Appl. Math.**, v. 33, p. 199–211, 1980.

GAMA, E.-S.; MARTÍN, F. Translating solitons of the mean curvature flow asymptotic to hyperplanes in $\mathbb{R}$n+1. **Int. Math. Res. Not. IMRN**, n. 24, p. 10114–10153, 2020. ISSN 1073-7928.

GROMOV, M. Isoperimetric of waists and concentration of maps. **Geom. Funct. Anal.**, v. 13, p. 178–205, 2003.

ILMANEN, T. Elliptic regularization and partial regularity for motion by mean curvature. **Mem. Amer. Math. Soc.**, v. 108, n. 520, p. x+90, 1994.

JOST, J. **Riemannian geometry and geometric analysis**. [S.l.]: Springer. v. 42005.

MARTÍN, F.; GARCÍA, J. P.; SAVAS-HALILAJ, A.; SMOCZYK, K. A characterization of the grim reaper cylinder. **J. reine angew. Math.**, v. 746, p. 1–28, 2019.

MOSS, W.; PIEPENBRINK, J. Positive solutions of elliptic equations. **Pacific J. Math.**, v. 75, p. 219–226, 1978.

OSSERMAN, R. Minimal varieties. **Bull. Amer. Math. Soc.**, v. 75, p. 1092–1120, 1969.

PETERSEN, P. **Riemannian Geometry**. [S.l.]: Springer, 2016. v. 171.

PUCCI, P.; SERRIN, J. **The maximum principle**. [S.l.]: Birkhäuser Verlag, Basel, 2007. v. 73. x+235 p. (Progress in Nonlinear Differential Equations and their Applications, v. 73). ISBN 978-3-7643-8144-8.

SERRIN, J. The dirichlet problem for quasilinear equations with many independent variables. **Proc. Nat. Acad. Sci. U.S.A.**, v. 58, p. 1829–1835, 1967. ISSN 0027-8424.

SERRIN, J. The problem of dirichlet for quasilinear elliptic differential equations with many independent variables. **Philos. Trans. Roy. Soc. London Ser. A**, v. 264, p. 413–496, 1969. ISSN 0080-4614.

SIMON, L. Introduction to geometric measure theory. **Tsinghua Lectures**, v. 2, n. 2, p. 1–229, 2014.

SMOCZYK, K. A relation between mean curvature flow solitons and minimal submanifolds. **Math. Nachr.**, v. 229, p. 175–186, 2001.

WHITE, B. Which ambient spaces admit isoperimetric inequalities for submanifolds? **Journal of Differential Geometry**, Lehigh University, v. 83, n. 1, p. 213 – 228, 2009. Disponível em: https://doi.org/10.4310/jdg/1253804356.

WHITE, B. Controlling area blow-up in minimal or bounded mean curvature varieties. **J. Differential Geom.**, v. 102, p. 501–535, 2016.

APPENDIX A – BOUNDARY AT INFINITY

In this part of the text we give a precise definition of boundary at infinity. For more details about the subject, read (Eberlein; O'Neill, 1973).

Points at infinity

In this section, we provide an intrinsic definition for the boundary at infinity by utilizing geodesics. To achieve this, we recall some basic definitions of Riemannian Geometry:

Definition A.0.1 (Unit tangent bundle)**.** *For a Riemannian Manifold* $(N, \overline{g})$*. The* ***unit tangent bundle*** UTN *of* N *is the subset of unit tangent vectors of* TN*, i.e.,*

$$UTN := \coprod_{p \in N} \{v \in T_pN : \overline{g}_p(v, v) = 1\}$$

The projection $\mu : UTN \longrightarrow N$ *given by* $\mu(v_p) = p$

Definition A.0.2. *Given* $v, w \in UT_pN$*, the* ***angle*** $\theta = \measuredangle(v, w)$ *between* v *and* w *is the unique number* $0 \leq \theta \leq \pi$ *such that* $\overline{g}(v, w) = \cos\theta$*.*

Definition A.0.3. *A* ***Cartan-Hadamard manifold*** $(N, \overline{g})$ *is a complete, simply connected Riemannian manifold of dimension* $n \geq 2$ *and having sectional curvature*

$$sec_{\overline{g}}(v, w) \leq 0 \quad \forall p \in N, \quad \forall v, w \in T_pN$$

For any two points $p \neq q$ in a Cartan-Hadamard manifold, there exists a unique geodesic γ_{pq} such that $\gamma_{pq}(0) = p$ and $\gamma_{pq}(t) = q$ where $t = \text{dist}_{\overline{g}}(p, q)$ (see section 6.9 of (Jost, 2008)).

Definition A.0.4. *Given* $p \neq q$ *in a Cartan-Hadamard manifold* $(N, \overline{g})$*, let* γ_{pq} *be the unique (unit speed) geodesic such that* $\gamma_{pq}(0) = p$ *and* $\gamma_{pq}(t) = q$ *where* $t = \text{dist}_{\overline{g}}(p, q)$*. The* ***angle*** $\measuredangle_p(q_1, q_2)$ *subtended by points* $q_1, q_2 \in N$ *and a distinct point* $p \in N$ *is defined by* $\measuredangle(\gamma'_{pq_1}(0), \gamma'_{pq_2}(0))$*.*

Now, we are able to classify the geodesics that asymptotically approach at infinity.

Definition A.0.5. *The unit speed geodesics* $\alpha : (-\infty, \infty) \longrightarrow N$ *and* $\beta : (-\infty, \infty) \longrightarrow N$ *in a Cartan-Hadamard manifold* N *are* ***asymptotic*** *provided there exists a number* $c > 0$ *such that* $\text{dist}_{\overline{g}}(\alpha(t), \beta(t)) \leq c \quad \forall t \geq 0$

Proposition A.0.6. *The following statements are true:*

(I) *If α and β are asymptotic, then so are orientation-preserving unit speed reparametrizations of α and β.*

(II) *The asymptote relation is a equivalence relation on the set of all geodesics in N; The equivalence classes are called **asymptote classes**.*

(III) *If asymptotic geodesics in N have a point in commom, then they are the same up to parametrization.*

(IV) *Given a geodesic α and a point $p \in N$ there exists a unique geodesic β such that $\beta(0) = p$ and β is asymptotic to α*

Proof. (I), (II), (III) are straightforward and (IV) is proved in Proposition 1.2 of (Eberlein; O'Neill, 1973). □

Hence, we can consider the asymptote classes as points at infinity and define the boundary at infinity as follows:

Definition A.0.7 (Asymptote class)**.** *Let $\alpha : (-\infty, \infty) \to N$ be a unit speed geodesic. We denote the asymptote class of α by $\alpha(\infty)$ and the asymptote class of the reverse curve $t \mapsto \alpha(-t)$ by $\alpha(-\infty)$.*

Definition A.0.8 (Boundary at infinity)**.** *Let $(N, \overline{g})$ be a Cartan-Hadamard manifold. A **point at infinity** of N is a asymptote class of geodesics of N. The **boundary at infinity** of N, denoted by $\partial_\infty N$, is the set of points at infinity of N. And $\overline{N} := N \cup \partial_\infty N$. If $P \in \partial_\infty N$ we write either $\alpha(\infty) = P$ or $\alpha \in P$ depending upon context.*

Example A.0.9. *For $N = \mathbb{H}^{m+1} = \{(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1} : x_0 > 0\}$ with $\overline{g} = g_{\mathbb{H}} = \frac{1}{x_0^2} g_{\mathbb{R}}$, the boundary at infinity $\partial_\infty \mathbb{H}^{m+1}$ can be identified with $\{(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1} : x_0 = 0\} \cup \{\mathcal{P}_\infty\}$ where $\mathcal{P}_\infty$ is the asymptote class of unit speed vertical geodesics pointing upward.*

Cone topology

In this section, we define the cone topology in $\overline{N} = N \cup \partial_\infty N$ to define a boundary at infinity for a general subset of N. For more details, read section 2 of (Eberlein; O'Neill, 1973). For this purpose, we need to endow $\overline{N} = N \cup \partial_\infty N$ with some topology such that it preserves the topology of N and some natural assumptions are required.

Definition A.0.10 (Admissible topology)**.** *A topology τ on $\overline{N}$ is **admissible** if it satifies the following four conditions:*

(1) ***Closure property***: *the topology on N induced by τ is the original topology of N, and N is a dense open set of $\overline{N}$*

(2) ***Geodecic extension property***: *if α is any geodesic of N then its asymptotic extension is continuous.*

(3) ***Isometric extension property***: *if φ is any isometry of N, then its asymptotic extension is continuous (and hence a homeomorphism by a functorial argument)*

(4) ***Intensive property***: *if $x \in \partial_\infty N$, V is a neighborhood of x in $\overline{N}$, and $r > 0$ is any positive number then there exists a neighborhood U of x such that $N_r(U) := \{q \in \overline{N} : d(q,U) < r\} \subset V$. Here we have extended the metric trivially so that $d(a,b) = \infty$ if $a \neq b$ and either points lies in $\partial_\infty N$*

Now, we construct the cones that serve as a basis for an admissible topology known as the cone topology.

Definition A.0.11 (Angle for points at infinity)**.** *Let p be a point of N distinct from points $a, b \in \overline{N}$. The* ***angle*** *subtended by a, b at p is $\sphericalangle_p(a,b) := \sphericalangle(\gamma'_{pa}(0), \gamma'_{pb}(0))$*

Definition A.0.12 (Cone)**.** *Let $v_p \in UT_pN$ and let ε be a number, $0 < \varepsilon < \pi$. Then the set $C(v,\varepsilon) := \{b \in \overline{N} : \sphericalangle_p(\gamma_v(\infty), b) < \varepsilon\}$ is called* ***cone*** *of vertex $p = \mu(v_p)$, axis v_p a angle ε.*

Proposition A.0.13 (Cone topology)**.** *If N is a Hadamard manifold, there is a unique topology κ on $\overline{N}$ such that:*

(1) *κ has the closure property*

(2) *For each $x \in \partial_\infty N$ the set of cones containing x is a local basis for κ at x.*

We call κ the ***cone topology*** *on $\overline{N}$*

Proof. See proposition 2.3 in (Eberlein; O'Neill, 1973).

□

Proposition A.0.14. *The cone topology κ for $\overline{N}$ is admissible.*

Proof. See proposition 2.9 in (Eberlein; O'Neill, 1973).

□

Definition A.0.15 (Boundary at infinity of a subset of N)**.** *Given a subset A of N, the* ***boundary at infinity*** *of A, denoted by $\partial_\infty A$, is the intersection between the clousure of A in $\overline{N}$ and $\partial_\infty N$.*

APPENDIX B – VARIFOLD

In this appendix, we give some definition about varifolds that we will need throughout the text. This appendix is mostly based on (Simon, 2014) and chapter 3 of (Colding; Minicozzi, 2011).

The concept of varifold is a generalization of manifold. A varifold is a measure in the space of ℓ-dimensional tangent spaces of ambient space. Having this concept at our disposal, we can take advantage of the power of Geometric Measure Theory, including Compactness Theorem (Theorem C.0.2).

Definition B.0.1 (Varifold)**.** *An ℓ-dimensional* ***varifold*** *V in the Riemannian manifold $(N, \overline{g})$ is a Radon measure on the Grassiamannian $G_\ell(N)$ of ℓ-planes on N.*

Definition B.0.2 (Weight of V)**.** *Let $\pi : G_\ell(N) \to N, \quad \pi(p, \mathcal{W}) = p$ be the projection. The Radon measure μ_V (called the* ***weight of*** *V) in N is given by the the pushforward of Radon measure V by π, that is,*

$$\mu_V(B) := V(\pi^{-1}(B)) = V(G_\ell(B)) \quad \forall B \subset N \text{ Borel set.}$$

The ***support of*** *V is support of μ_V and the* ***mass*** *of V on a set $U \subset N$ is just $\mu_V(U)$*

Without loss of generality, we can assume that N is isometrically embbeded in $\mathbb{R}^{n+k}$ for some $k \geq 0$ by Nash Theorem. Let $\mathcal{H}^m$ denote the m-Hausdorff measure in $\mathbb{R}^{n+k}$

Notice that we can associate an embedded submanifold M^m in N with a Radon measure V_M in $G_m(N)$ given by:

$$V_M(B) = \mathcal{H}^m(\pi(B \cap TM)) \quad \forall B \subset G_m(N) \text{ Borel set .}$$

Therefore, the weight of V is given by $\mu_{V_M}(U) = \mathcal{H}^m(U \cap M)$ for all Borel set $U \subset N \subset \mathbb{R}^{n+k}$. Thus, M can be viewed as a varifold V_M.

Now, we define an ℓ-rectifiable set, which can be described, in general terms, as a set that exhibits as piece-wise smooth set.

Definition B.0.3 (ℓ-Rectifiable Set)**.** *A set $S \subset \mathbb{R}^{n+k}$ is said to be ℓ-**rectifiable** if $S \subset S_0 \cup S_1$, where $\mathcal{H}^\ell(S_0) = 0$, where $\mathcal{H}^\ell$ is ℓ-dimensional Hausdorff measure of the ambient space $\mathbb{R}^{n+k}$, and S_1 is the image of $\mathbb{R}^\ell$ under a Lipschitz map. More generally, S is said to be **countably** ℓ-**rectifiable** if $S \subset \cup_{i \geq 0} S_i$, where $\mathcal{H}^\ell(S_0) = 0$ and for $i \geq 1$ each S_i is the image of $\mathbb{R}^\ell$ under a Lipschitz map.*

Definition B.0.4 (Rectifiable Varifold)**.** *Let S be a countably ℓ-rectifiable set of $\mathbb{R}^{n+k}$ with $\mathcal{H}^\ell(S) < \infty$ and let θ be a positive locally $\mathcal{H}^\ell$-integrable function on S. Set V equal to the varifold associated to the set S (exactly as if S were a smooth submanifold). The associated varifold $V' = \theta V$ is called a* ***rectifiable varifold****. If θ is integer-valued, then V' is an* ***integral varifold****.*

The next definition gives us a way to push the varifold forward by a map f.

Definition B.0.5 (Image varifold)**.** *Suppose that U and $\tilde{U}$ are open subsets of $\mathbb{R}^{n+k}$ and $f : U \to \tilde{U}$ is C^1 with $f|_{\mathrm{spt}\,\mu_V \cap U}$ proper. We define the* ***image varifold*** *$f_\sharp V$ on $\tilde{U}$ by*

$$f_\sharp V(A) = \int_{F^{-1}(A)} J_{\mathcal{W}} f(p) dV(p, \mathcal{W}), \quad A \text{ Borel, } A \subset G_\ell(\tilde{U}),$$

where $F : G^+_\ell(U) \to G_\ell(\tilde{U})$ is defined by $F(p, \mathcal{W}) = (f(p), f_*(\mathcal{W}))$, $f_*(\mathcal{W})$ is the pushforward of $\mathcal{W}$ by f, and

$$J_{\mathcal{W}} f(p) := (\det((df_p|_{\mathcal{W}})^* \circ (df_p|_{\mathcal{W}})))^{\frac{1}{2}}, \quad (p, \mathcal{W}) \in G_\ell(N)$$
$$G^+_\ell(U) := \{(p, \mathcal{W}) \in G_\ell(U) : J_{\mathcal{W}} f(p) \neq 0\}$$

Now, we can pushforward the varifold using the flow of a compactly supported vector field, and subsequently measure the variations in mass, as in the case of smooth manifolds. Then, we can find the right analogy to a stationary submanifold.

Definition B.0.6 (First variation)**.** *Given a C^1 vector field Z compactly supported in an open set $\Omega \Subset N$, the* ***first variation*** *is defined as*

$$\delta V(Z) := \frac{d}{dt}\bigg|_{t=0} ((\Phi_t)_\sharp V)(\Omega) = \int_{G_\ell(\Omega)} (\mathrm{div}^{\mathcal{W}} Z) dV(p, \mathcal{W}),$$

where $\Phi_t : G_\ell(\Omega) \to G_\ell(\Omega)$ is induced by the flow ϕ_t of Z by $\Phi_t(p, \mathcal{W}) = (\phi_t(p), (\phi_t)_*(\mathcal{W}))$, and $\mathrm{div}^{\mathcal{W}} Z = \sum_{i=1}^{\ell} \overline{g}(\overline{\nabla}_{e_i} Z, e_i)$ with $\{e_i\}$ an orthonormal basis of $\mathcal{W}$. If V has locally bounded first variation, that is,

$$|\delta V(Z)| \leq C \sup_N |Z| \quad \text{for all } Z \text{ compactly supported on } \Omega,$$

then the ***total variation measure*** *$||\delta V||$ is a Radon measure on N, where $||\delta V||$ is characterized by*

$$||\delta V||(\Omega) = \sup_{Z, |Z| \leq 1, \operatorname{spt} Z \Subset \Omega} |\delta V(Z)|.$$

*A ℓ-varifold V is called **stationary** provided $\delta V = 0$.*

APPENDIX C – WHITE'S COMPACTNESS THEOREM

In this appendix we introduce the main tools that we will use in the proofs.

Maximum Principle

The theorem below is the version of the Maximum Principle (Theorem 7.6 in (White, 2009)) for $\mathbb{H}^{m+1}$

Theorem C.0.1 (**Maximum Principle in $\mathbb{H}^{m+1}$**). *Let B be a open set in $\mathbb{H}^{m+1}$. Let M be a smooth, connected hypersurface properly embedded in B and dividing B in two components. Let Ω be one of the two components of $B\backslash M$. Suppose that Ω is mean concave along M, i.e., that at each point of M, the mean curvature is a nonnegative multiple of the outward unit normal to Ω. Suppose S is a spatial support of a nonzero stationary m-varifold in $\mathbb{H}^{m+1}$ such that S is disjoint from Ω. If S contains any point of M, then it must contain all of M, and M must be a minimal surface.*

A compactness theorem for minimal surfaces

White (White, 2016, Theorem 2.6 and Theorem 7.4) shows that under some natural conditions the singular set $\mathcal{Z}$ satisfies the same maximum principle as properly embedded minimal surfaces without boundary.

Theorem C.0.2 (**Compactness Theorem for Integral Varifold**). *Let $\{M_i\}$ be a sequence of minimal hypersurfaces in $\mathbb{R}^{n+1}$, with not necessary the canonical metric, whose area is locally bounded, then a subsequence of $\{M_i\}$ converges weakly to a stationary integral varifold M_∞.*

Theorem C.0.3 (**White's strong barrier principle Theorem 7.3 in (White, 2016)**). *Let (Ω, g) be a Riemannian $(m+1)$-manifold and $\{M_i\}_{i\in\mathbb{N}}$ a sequence of properly embedded minimal hypersurfaces in (Ω, g). Suppose that the set $\mathcal{Z}$ of $\{M_i\}_{i\in\mathbb{N}}$ is contained in a closed region $\mathcal{N}$ of Ω with smooth, connected boundary $\partial\mathcal{N}$ such that $g\left(H_{\partial\mathcal{N}}, \xi\right) \geq 0$, at every point of $\partial\mathcal{N}$, where $H_{\partial\mathcal{N}}(p)$ is the mean curvature vector of $\partial\mathcal{N}$ at p and $\xi(p)$ is the unit normal at p to the surface ∂N that points into $\mathcal{N}$. If the set $\mathcal{Z}$ contains any point of $\partial\mathcal{N}$, then it contains all of $\partial\mathcal{N}$.*

Remark C.0.1. *The above theorem is a sub-case of a more general result of White. In fact the strong barrier principle of White holds for sequences of embedded hypersurfaces of n-dimensional Riemannian manifolds which are not necessarily minimal but have uniformly bounded mean curvatures. For more details we refer to (White, 2016).*