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ALEXANDRE AZEVEDO CEZAR

CIRCULAR BACKBONE COLORING FOR GRAPHS WITHOUT CYCLES OF SIZE FOUR

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Dissertation presented to the Mathematics Graduation Program at the Department of Mathematics, Federal University of Ceará, in partial fulfilment of the requirements for the degree of Master of Science in Mathematics, with focus in Combinatorics.
Advisor: Prof. Dr. Júlio César Silva Araújo.
Co-advisor: Prof. Dr. Ana Shirley Silva.

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RESUMO

Dado um grafo $G = (V(G), E(G))$ e um subgrafo $H = (V(H), E(H))$ de G, uma *k-coloração q-backbone* de (G,H) é uma função $\phi : V(G) \to \{1,2,3,...,k\}$ tal que, para toda aresta $uv \in E(G)$, temos $|\phi(u) - \phi(v)| \geq 1$ e, para toda aresta $uv \in E(H)$, temos $|\phi(u) - \phi(v)| \geq q$. O *número cromático q-backbone* de (G,H), denotado por $BBC_q(G,H)$, é o menor inteiro k tal que existe uma coloração ϕ como acima. Similarmente, uma *k-coloração q-backbone circular* de (G,H) é uma função $\phi : V(G) \to \{1,2,3,...,k\}$ tal que, para toda aresta $uv \in E(G)$, temos $|\phi(u) - \phi(v)| \geq 1$ e, para toda aresta $uv \in E(H)$, temos $k - q \geq |\phi(u) - \phi(v)| \geq q$. O *número cromático q-backbone circular* de (G,H), denotado por $CBC_q(G,H)$, é o menor inteiro k tal que existe uma coloração ϕ como acima. Nesta dissertação, primeiramente apresentamos um breve sumário dos resultados relacionados a Coloração Backbone. Após isto, mostramos que se G é um grafo planar sem ciclos de tamanho quatro e F é uma floresta geradora de caminhos induzidos de G, então $CBC_2(G,F) \leq 7$. Por fim, demonstramos o seguinte teorema: se G é um grafo conexo e $k \geq \max\{\chi(G), \lceil \chi(G)/2 \rceil + q\}$, então existe uma k-coloração c de G tal que $G_{c,q}$ é conexo, onde $G_{c,q}$ é o subgrafo de G tal que $V(G_{c,q})$ e $E(G_{c,q})$ é formado pelas arestas $vw \in E(G)$ que satisfazem $|c(v) - c(w)| \geq q$.

Palavras-chave: coloração de grafos; número cromático; coloração backbone circular; grafos planares sem C_4; árvore como backbone.

ABSTRACT

Given a graph $G = (V(G), E(G))$ and a subgraph $H = (V(H), E(H))$ of G, a *q-backbone k-coloring* of (G, H) is a function $\phi : V(G) \to \{1, 2, 3, \ldots, k\}$ such that, for every edge $uv \in E(G)$, we have $|\phi(u) - \phi(v)| \geq 1$ and, for every edge $uv \in E(H)$, we have $|\phi(u) - \phi(v)| \geq q$. The *q-backbone chromatic number* of (G, H), denoted by $BBC_q(G, H)$, is the smallest integer k such that there exists such coloring ϕ. Similarly, a *circular q-backbone k-coloring* of (G, H) is a function $\phi : V(G) \to \{1, 2, 3, \ldots, k\}$ such that, for every edge $uv \in E(G)$, we have $|\phi(u) - \phi(v)| \geq 1$ and, for every edge $uv \in E(H)$, we have $k - q \geq |\phi(u) - \phi(v)| \geq q$. The *circular q-backbone chromatic number* of (G, H), denoted by $CBC_q(G, H)$, is the smallest integer k such that there exists such coloring ϕ. In this dissertation, we firstly present a brief summary on the results found in literature regarding Backbone Coloring. Then, we prove that if G is a planar graph without cycles of size four and F is a spanning forest of induced paths of G, then $CBC_2(G, F) \leq 7$. Lastly, we show the following theorem : if G is a connected graph and $k \geq \max\{\chi(G), \lceil \chi(G)/2 \rceil + q\}$, then there exists a proper k-coloring c of G such that $G_{c,q}$ is connected, where $G_{c,q}$ is the subgraph of G such that $V(G_{c,q}) = V(G)$ and $E(G_{c,q})$ is the set of edges $vw \in E(G)$ that satisfy $|c(v) - c(w)| \geq q$.

Keywords: graph coloring; chromatic number; circular backbone coloring; planar graphs without C_4; tree backbone.

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1 INTRODUCTION

This dissertation presents a contribution to the well-known problem on Graph Theory called Graph Coloring. The first results involving graph coloring were mainly regarding the coloring of counties in a map and are dated as early as 1852 (KUBALE, 2004). By that time, it was observed that if each county was colored with a single color, then the map could be colored with four colors, in such a way that neighbouring counties are not colored with the same color. This observation was then published in the form of a puzzle and later became known as the *Four Color Theorem* (FRITSCH *et al.*, 1998). In telecommunications, Graph Coloring problems also appear in the form of *Frequency Assignment* problems (HALE, 1980). In one version of these problems, we consider a network of radio transmitters and if the frequencies assigned to some pairs of transmitters are equal, or too close, then they might interfere with one another. The problem is to assign frequencies to the transmitters in such way that the interference is at a minimum. In this case, each radio transmitter may be seen as a node, the frequencies available for each node as colors and if two transmitters may potentially interfere with one another, the corresponding nodes are joined by an edge.

In this dissertation, all graphs considered are simple and undirected. We are using the terminology and notations presented on (WEST, 2000). For more basic notions on Graph Theory and Graph Colorings, see (BONDY.; MURTY, 2007; JENSEN; TOFT, 2011; MOLLOY; REED, 2002). For more detailed definitions, see Section 2.

A *k-coloring* of a graph $G = (V(G), E(G))$, for some positive integer k, is any function $\phi : V(G) \rightarrow \{1, 2, 3, ..., k\}$. We say that a k-coloring ϕ of G is *proper* if for any $vw \in E(G)$ we have $|\phi(v) - \phi(w)| \geq 1$. If there exists a proper k-coloring $\phi : V(G) \rightarrow \{1, 2, 3, ..., k\}$, then we say that G is *k-colorable*. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the smallest positive integer k for which the graph is k-colorable. It is known that $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$, where $\omega(G)$ and $\Delta(G)$ are, respectively, the size of the largest clique in G and the maximum degree of a vertex in G.

Backbone Coloring is a special case of Graph Coloring where the colors $\phi(u), \phi(v)$ of the endpoints u, v of some special edges must satisfy $|\phi(u) - \phi(v)| \geq q$ for a given positive integer q. The subgraph obtained by these special edges is called *backbone* of G. Problems regarding backbone colorings were first introduced by Broersma et al. (BROERSMA *et al.*, 2007), based on coloring problems related to frequency assignment.

There are some known problems in the literature that model some variations of the

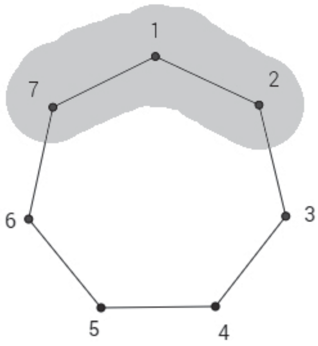
Frequency Assignment problem. A survey on such problems can be found in Hale (HALE, 1980). For instance, one of the most studied of these problems is the $L(2,1)$-labelling problem. An $L(2,1)$-*labelling* of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x)-f(y)| \geq 2$ if $d(x,y)=1$ and $|f(x)-f(y)| \geq 1$ if $d(x,y)=2$. The $L(2,1)$-labelling number $\lambda(G)$ of G, defined in Griggs et al. (GRIGGS; YEH, 1992), is the smallest number k such that G has an $L(2,1)$-labelling with $\max\{f(v) : v \in V(G)\} = k$. The square of G is the graph G^2, obtained from G by adding all edges between vertices that are at distance two. Observe that an $L(2,1)$-labelling of G may be seen as a coloring of the square of G, where the colors of the endpoints of edges of G must differ by at least two units and the colors of the remaining edges must be different. This corresponds to a backbone coloring of G^2, where G is the backbone subgraph and $q=2$.

More formally, consider a graph $G=(V(G),E(G))$ and a subgraph $H=(V(H),E(H))$ of G, we name (G,H) a *pair*. Given positive integers k and q, a *q-backbone k-coloring* of a pair (G,H) is a proper k-coloring $\phi : V(G) \to \{1,2,...,k\}$ satisfying $|\phi(u)-\phi(v)| \geq q$ for any $uv \in E(H)$. If such coloring exists, then we say that (G,H) is *q-backbone k-colorable*. The *q-backbone chromatic number* of (G,H), denoted by $BBC_q(G,H)$, is the smallest positive integer k such that (G,H) is q-backbone k-colorable. Similarly a *circular q-backbone k-coloring* of a pair (G,H) is a proper k-coloring $\phi : V(G) \to \{1,2,...,k\}$ satisfying $k-q \geq |\phi(u)-\phi(v)| \geq q$ for any $uv \in E(H)$. If such coloring exists, then we say that (G,H) is *circular q-backbone k-colorable*. The *circular q-backbone chromatic number* of (G,H), denoted by $CBC_q(G,H)$, is the smallest positive integer k such that (G,H) is circular q-backbone k-colorable.

In Figure 1a we see the colors fobidden for the neighbors of a vertex assigned with the color 1, when taking $q=2$ and $k=7$.

Some graph coloring problems involve considering a graph G that belongs to a certain graph class and evaluating the smallest number of colors needed to color G. These problems have a Backbone Coloring version, where we want to color a pair (G,H), with G and H belonging to certain graph classes. That is the case of problems similar to the Steinberg's conjecture. The Steinberg's conjecture (STEINBERG, 1993) states that every planar graph G that does not contain a C_4 neither a C_5 as a subgraph is 3-colorable. Araujo et al. (ARAUJO *et al.*, 2018) tackled a Backbone Coloring version of this problem where the backbone H is a spanning forest of the graph G and a version where the backbone is a spanning forest of paths. Following Broersma et al. (BROERSMA *et al.*, 2007), several other authors studied Backbone

Figure 1 – Colors firbidden by the color 1



Fonte: elaborado pelo autor.

Coloring, considering either the graph G or the subgraph H to belong to a certain class. Some of which are presented in Chapter 3.

In this text, we study Circular Backbone Coloring of graphs that do not contain a C_4 and taking a forest of induced paths as backbone. We prove the following theorem:

Theorem 1.1 *Let G be a planar graph without cycles of size four. If F is an spanning induced path forest of G, then* $CBC_2(G,F) \leq 7$.

We emphasize that Theorem 1.1 holds for *every* spanning induced path forest F of a given planar graph G having no cycles of length four. There are several results in literature (BU; ZHANG, 2011; BU; LI, 2011; BU; BAO, 2015) showing that for a given connected graph G, there *exists* a spanning connected subgraph $H \subseteq G$ such that the q-backbone chromatic number of (G,H) is upper bounded. We generalize all these results in the following theorem:

Theorem 1.2 *If G is a connected graph and* $k \geq \max\{\chi(G), \lceil \chi(G)/2 \rceil + q\}$*, then there exists a spanning connected subgraph H of G and a proper q-backbone k-coloring c of* (G,H).

This result shows us that, for every graph G, there exists a spanning tree T such that the pair (G,T) can be q-backbone colored with few colors beyond $\chi(G)$. Furthermore, we prove this result is tight.

This text is organized as follows: in Chapter 2, we briefly present some definitions used throughout the text. Chapter 3 is dedicated to a review on some previously known results regarding Backbone Coloring, listed by backbone graph class and the nature of the results. In Chapter 4, we present the proof of Theorem 1.1. This proof, however, is extensive, therefore it is split into two sections. In Section 4.1, we present the structural lemmas, that allow us study a special counterexample and reduce it to smaller and easier cases. In Section 4.2, we use the structural lemmas and prove the most important lemmas needed to conclude our proof. In Chapter 5 we present a conclusion of this text and some questions for further research. Finally, in the appendix we show a brief proof of Theorem 1.2. We emphasize that the content in the appendix is as it has been submitted to the Journal of Graph Theory.

2 DEFINITIONS

In this chapter we present some important notions and definitions that are used throughout this dissertation.

A *simple graph*, or just *graph*, is a pair $G = (V(G), E(G))$ where the set $V(G)$ is the set of *vertices* and $E(G)$, satisfying $E(G) \subseteq \binom{V(G)}{2}$, is named the set of *edges*. To avoid excessive notation, an edge $u, v \in E(G)$ is simply denoted by uv. Let uv be an edge of G, the vertices u and v are named the *endpoints* of the edge uv.

Let $G = (V(G), E(G))$ be a graph. We say a graph $H = (V(H), E(H))$ is a *subgraph* of G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph H of G is said to be *proper* if either $V(H)$ or $E(H)$ is a proper subset of $V(G)$ or $E(G)$, respectively. The subgraph H of G is said to be *vertex-induced*, or simply *induced*, if for every $v, w \in V(H)$ satisfying $vw \in E(G)$ we have $vw \in E(H)$. Furthermore, a subgraph is said to be a *spanning subgraph* if $V(H) = V(G)$.

Let G be a graph, H be a subgraph of G and $v \in V(G)$, we define the *neighbourhood* of v in H to be the set $N_H(v) = \{w \in V(H) \mid vw \in E(G)\}$. The *degree* of v in H, denoted by $d_H(v)$, is given by $d_H(v) = |N_H(v)|$. We omit the subscript when $H = G$. The maximum and minimum degree of G are, respectively, denoted by $\Delta(G)$ and $\delta(G)$. A graph G is *d-degenerated* if every subgraph $H \subset G$ satisfy $\delta(H) \leq d$.

We say that *there is a triangle in* G if there is a subgraph H of G such that $H = (\{u, v, w\}, \{uv, vw, uw\})$, for some $\{u, v, w\} \subseteq V(G)$.

Let $S = \{v_1, \cdots, v_j\}$ be a proper subset of $V(G)$. We denote by $G - S$ the induced subgraph G' of G such that $V(G') = V(G) \setminus S$.

A *complete graph* is a graph which every two distinct vertices are connected by an edge. An *independent set* is a set of vertices in which no two distinct vertices are joined by an edge. A graph G is said to be a *bipartite graph* if $V(G)$ can be partitioned into two independent sets.

A *path* is a graph whose vertices can be ordered so that two vertices are adjacent if, and only if, they are consecutive in this order. A graph G is said to be *connected* if, for any vertices $u, v \in V(G)$, there exists a path $P \subseteq G$ such that $u, v \in V(P)$. For any disconnected graph G, each maximal connected subgraph $H \subseteq G$ is named a *connected component*, or simply *component*, of G. A *cycle* is a graph whose vertices can be placed in a circle so that two vertices are consecutive on that circle if, and only if, they are adjacent. A graph that has a cycle as subgraph is named *cyclic*.

Let $P_1 = (v_1, \cdots, v_p), P_2 = (w_1, \cdots, w_q)$ be two paths in G such that $V(P_1) \cap V(P_2) = \emptyset$ and $v_p w_1 \in E(G)$. We say that the path $(v_1, \cdots, v_p, w_1, \cdots, w_q) \subseteq G$ is a *concatenation* of P_1 and P_2, and is denoted by $P_1 \circ P_2$.

A *forest* is an acyclic graph. If a forest G is a connected graph, then G is named *tree*. A *star* is a tree in which all but one vertex have degree 1. A *constellation* is a graph in which all components are stars. A graph is called a *path forest*, or *linear forest*, if all its components are paths. A *Hamiltonian path* of a graph G is a spanning path of G.

A graph is said to be a *split graph* if its vertices can be partitioned into two sets such that one set forms a complete subgraph whereas the other set forms an independent set.

A graph G is said to be *planar* if it can be embedded on a plane in such a way that each vertex is represented by a dot and each edge vw is represented by a line connecting the vertices v and w such that no two lines cross each other. A planar graph together with a specific embedding is named a *plane graph*. Each maximal connected region of the plane is a *face* of the graph G.

Given a plane graph $G = (V(G), E(G))$, we define the set $F(G)$ as the *set of the faces* of the given plane embedding of G. Let $f \in F(G)$, the *degree* of the face f, denoted by $d_G(f)$, is the size of the smallest closed walk that contains every edge adjacent to f. For more details on these concepts we refer to (WEST, 2000), but we highlight the following two propositions which we use throughout our proofs:

Proposition 2.1 *Let $G = (V(G), E(G))$ be a plane graph and $F(G)$ be its set of the faces. The following holds:*

$$2|E(G)| = \sum_{f \in F(G)} d_G(f). \tag{2.1}$$

Proposition 2.2 *Let $G = (V(G), E(G))$ be a plane graph and $F(G)$ be its set of the faces. The following holds:*

$$|V(G)| - |E(G)| + |F(G)| = 2. \tag{2.2}$$

Proposition 2.2 is known as the *Euler's Formula*.

The number of faces of degree $i \in \mathbb{N}$ in $F(G)$ is denoted by F_i. Similarly, the number of faces of degree at least $i \in \mathbb{N}$ is denoted by F_{i+}.

Given a plane graph G, its *dual graph*, or simply its *dual*, is a graph $G^* = (F(G), E^*)$, where $ff' \in E^*$ if the faces f and f' share an edge in G.

2.1 Backbone Coloring Definitions

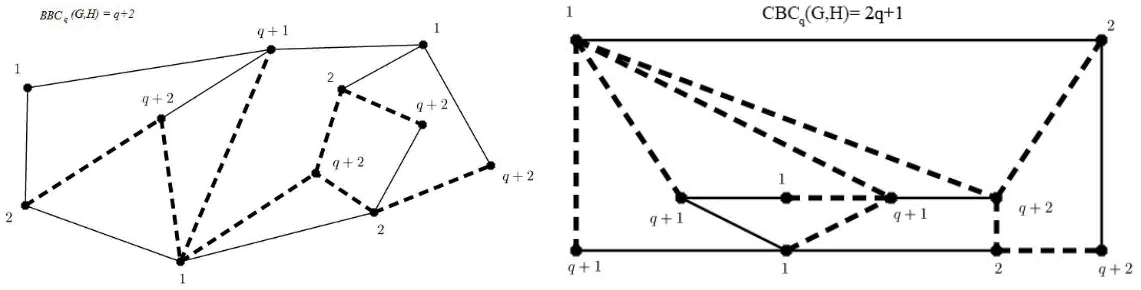
A *pair* is an ordered pair (G,H), where G is a graph and H is a subgraph of G. In a pair (G,H), the subgraph H is said to be the *backbone* of the graph G. Given a pair (G,H), a *subpair* (G',H') of (G,H) is a pair such that $G' \subseteq G$ and $H' \subseteq H$, also denoted $(G',H') \subseteq (G,H)$. We say a subpair $(G',H') \subseteq (G,H)$ is *proper* if either H' or G' is a proper subgraph of H and G, respectively. A subpair (G',H') is said to be *induced* if G' is an induced subgraph of G and H' is the subgraph of H induced by $V(H) \cap V(G')$. Notice that $(G-S, H-S)$ is an induced subpair of (G,H), for any $S \subset V(G)$.

For some positive integers k and q, a *q-backbone k-coloring* of a pair (G,H) is a proper coloring $\phi : V(G) \to \{1, \cdots, k\}$ such that, for all $vw \in E(H)$, we have $|\phi(v) - \phi(w)| \geq q$. If such coloring exists, then we say that (G,H) is *q-backbone k-colorable*. The *q-backbone chromatic number* of (G,H), $BBC_q(G,H)$, is the smallest positive integer k such that the graph is q-backbone k-colorable. Similarly, for some positive integers k and q, a *circular q-backbone k-coloring* of a pair (G,H) is a proper coloring $\phi : V(G) \to \{1, \cdots, k\}$ such that for all $vw \in E(H)$ we have $k - q \geq |\phi(v) - \phi(w)| \geq q$. If such coloring exists, then we say that (G,H) is *circular q-backbone k-colorable*. The *circular q-backbone chromatic number* of (G,H), $CBC_q(G,H)$, is the smallest positive integer k such that the graph is circular q-backbone k-colorable. A *partial (circular) q-backbone k-coloring* of a pair (G,H) is a (circular) q-backbone k-coloring of a subpair $(G',H') \subseteq (G,H)$. Notice that it is not important whether the backbone H is a spanning subgraph of G or not, as only the edges in $E(H)$ are considered on the coloring. Therefore, unless stated otherwise, we treat H to be a spanning subgraph of G.

In Figure 2 on the left, an example of a a graph G and its backbone H, whose edges are shown as the dashed lines. In this case we have $BBC_q(G,H) = q+2$. On the right, a graph G and its backbone H, whose edges are shown as the dashed lines. In this case we have $CBC_q(G,H) = 2q+1$

Let $S = \{1, \cdots, k\}$ and $\mathscr{P}(S)$ be the set of all subsets of S. Let (G,H) be a pair and $\mathscr{L} : V(G) \to \mathscr{P}(S)$ such that $\mathscr{L}(v) \neq \emptyset$ for all v in $V(G)$. We say (G,H) is circular q-backbone $\mathscr{L}$-colorable if there is a circular q-backbone k-coloring ϕ of (G,H) such that $\phi(v) \in \mathscr{L}(v)$. Even more, let $l : V(G) \to \{1, \cdots, k\}$. We say (G,H) is circular q-backbone l-colorable if

Figure 2 – Examples of (circular) backbone colorings



Fonte: elaborado pelo autor.

for every $\mathscr{L} : V(G) \rightarrow \mathscr{P}(S)$ such that $|\mathscr{L}(v)| = l(v)$, the pair (G,H) is circular q-backbone $\mathscr{L}$-colorable.

3 STATE OF THE ART

In this chapter, we make a brief exposition of some results found in the literature regarding the Backbone Coloring problem. The sections are organized according to the graph classes being analysed, whether they are: any subgraph, a path forest, a forest, a constellation or a matching. For each of these topics, we consider the parameters *BBC* and *CBC*, then we show complexity results that we found and we end each section with open questions.

3.1 General Results

In Havet et al. (HAVET *et al.*, 2014), three general bounds are presented for *BBC* and *CBC*, for any pair (G,H). The first one is shown in the following theorem:

Theorem 3.1 *For any pair* (G,H) *and an integer* $q \geq 2$*, the following inequalities hold:*

$$q\chi(H) - (q-1) = BBC_q(H,H) \leq BBC_q(G,H) \leq BBC_q(G,G) = q\chi(G) - (q-1).$$

This theorem is a direct consequence of the definition of backbone coloring. They were also able to show the following upper bound:

Theorem 3.2 *For any pair* (G,H) *and* $q \geq 2$*, it holds:*

$$BBC_q(G,H) \leq (\chi(G) + q - 2)\chi(H) - (q-2).$$

Furthermore, for $q \geq 4$ *the upper bound is tight.*

Finally, regarding the parameter *CBC* we have the following theorem:

Theorem 3.3 *For any graphs* $H \subseteq G$ *such that* $2 \leq \chi(H) \leq \chi(G)$ *and* $q \geq 2$*, it holds:*

$$CBC_q(G,H) \leq (\chi(G) + q - 2)\chi(H).$$

One interesting result also proven by Havet et al. is that if H is a connected spanning subgraph of G, then $CBC_q(G,H) = 2q$ if, and only if, G is bipartite.

Miškuf et al. (MIŠKUF *et al.*, 2010) showed that:

Theorem 3.4 *Given a graph* G *and a* d*-degenerated subgraph* T *of* G*, the* 2*-backbone chromatic number satisfy* $BBC_2(G,T) \leq \Delta(G) + d + 1$.

Along with that result, they also showed that for a given $\Delta(G)$, there exists a graph G with maximum degree $\Delta(G)$ and a tree backbone T such that $BBC_2(G,T) = \Delta(G)+2$. They extend this result by showing that $\Delta(G)$ is strictly greater than the maximum degree of the tree T.

Regarding complexity results, it is presented in Havet et al. (HAVET *et al.*, 2014) that given an integer $q \geq 2$, a connected graph G and a connected spanning subgraph H of G, then deciding whether $BBC_q(G,H) \leq q+2$ can be done in polynomial time.

3.2 Forest Backbone

Given a positive integer k and an integer q such that $q \geq 2$, Broersma et al. (BROERSMA *et al.*, 2007) defined the parameter $\mathscr{T}_q(k)$ to be the maximum possible value of $BBC_q(G,T)$ for any graph G such that $\chi(G) = k$ and any forest T subgraph of G. Concerning this parameter, they showed that $\mathscr{T}_2(k) = 2k-1$, for every $k \geq 1$.

In Zhu et al. (BU *et al.*, 2013) the following theorem is proven:

Theorem 3.5 *For any positive integers k and ℓ, there exists a graph G with girth greater than ℓ and $\chi(G) = k$, and a spanning forest T of G such that $BBC_2(G,T) = 2k-1$.*

This result is interesting because it provides additional information to the one presented by Broersma et al. Since not only there exists a graph G with a tree backbone such that $BBC_2(G,T) = 2\chi(G)-1$, but also that there exist infinitely many graphs such that this holds, even with the additional property that these graphs have high girth. Furthermore, the proof of this theorem provides a method for constructing such graphs.

Bu et al. (BU; ZHANG, 2011; BU; LI, 2011; BU; BAO, 2015) studied the coloring of planar graphs without special cycles. In Bu (BU; ZHANG, 2011), it is proven that:

Theorem 3.6 *If a planar graph G without C_4 cycles, then there exists a spanning tree T of G such that $BBC_2(G,T) \leq 4$.*

In Bu et al. (BU; LI, 2011) it is shown that the following theorems hold:

Theorem 3.7 *If a planar graph G without C_6 cycles and such that no two C_3 cycles share an edge, then there exists a spanning tree T of G such that $BBC_2(G,T) \leq 4$.*

Theorem 3.8 *If a planar graph G without C_7 cycles and such that no two C_3 cycles share an edge, then there exists a spanning tree T of G such that $BBC_2(G,T) \leq 4$.*

And finally, in Bu et al. (BU; BAO, 2015), the following theorems are proved:

Theorem 3.9 *If a planar graph G without C_8 cycles and such that no two C_4 cycles share an edge, then there exists a spanning tree T of G such that $BBC_2(G,T) \leq 4$.*

Theorem 3.10 *If a planar graph G without C_9 cycles and such that no two C_4 cycles share an edge, then there exists a spanning tree T of G such that $BBC_2(G,T) \leq 4$.*

These theorems are proven by assuming the existence of a structure called a *minimal counterexample*, studying its properties and then using the discharging method to find a contradiction to the existence of such counterexample.

Araujo et al. (ARAUJO *et al.*, 2018) also studied backbone coloring of planar graphs with tree backbones, which resulted in the following theorem:

Theorem 3.11 *If G is a planar graph with no cycles of length 4 or 5 and H is a forest in G, then $CBC_2(G,H) \leq 7$.*

Broersma et al. (BROERSMA *et al.*, 2007) also proved two complexity results, showing that, given a positive integer ℓ, the following results hold:

Theorem 3.12 *Deciding whether $BBC_2(G,T) \leq \ell$ is solved in polynomial time, for any $\ell \leq 4$, where T is a spanning tree of G.*

Theorem 3.13 *Deciding whether $BBC_2(G,P) \leq \ell$ is NP-complete, for any $\ell \geq 5$, where P is a Hamiltonian path of G.*

Havet et al. (HAVET *et al.*, 2014) studied the complexity of some decision problems regarding graphs with tree backbones. The following results hold:

Theorem 3.14

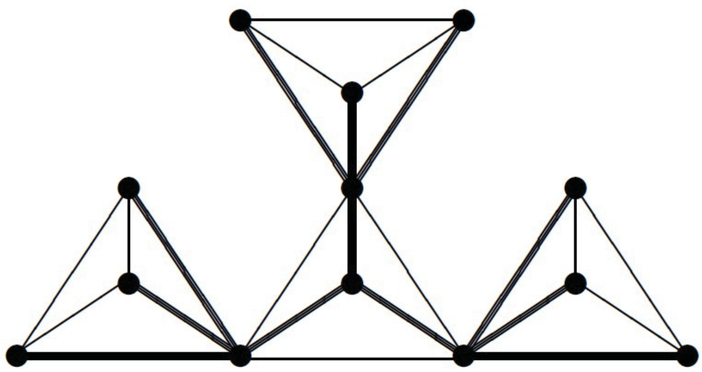
- *Deciding whether $BBC_2(G,T) \leq 5$ is an NP-complete problem, where T is a spanning tree of G.*
- *Deciding whether $BBC_q(G,T) \leq q+5$ is an NP-complete problem, for any $q \geq 5$, where T is a spanning tree of G.*

When studying the coloring bounds on forest backbones, many questions arise. Broersma et al. stated the following open problems: Is there any bound on $\mathscr{T}_q(k)$, for $q \geq 3$? Is there any constant c such that $BBC_2(G,T) \leq \chi(G)+c$, for chordal graphs G and any tree? Is there any constant c such that $BBC_2(G,T) \leq \chi(G) + c$ for graphs G without triangles, where T is a spanning tree?

For this last question, Miškuf et al. (MIŠKUF *et al.*, 2009) proved it to be false by modifying the Mycielski construction and creating sequences of graphs (G_n) and (T_n), such that G_n is a triangle-free graph and (T_n) is a spanning tree backbone of G_n, satisfying $BBC_2(G_n, T_n) = 2\chi(G_n) - 1$.

We know from Havet et al. (HAVET *et al.*, 2014) that if G is a planar graph with spanning tree T, then $BBC_2(G,T) \leq 7$. Is any way to prove that without relying on the Four Color Theorem? Furthermore, can this upper bound be lowered? In Figure 3 there is an example where $BBC_2(G,T) = 6$, for some graph G with a forest T, so we know that the right-hand side of the inequality cannot be lowered, but we do not know yet if there is an example where $BBC_2(G,T) = 7$, so the question is still left open.

Figure 3 – Graph such that $BBC_2(G,T) = 6$



Fonte: Broersma et al. (BROERSMA *et al.*, 2007) p.16.

3.3 Path Forest Backbone

After considering forest backbones, we study the bounds on simpler backbones. The first one we consider is path forest backbones.

Given a positive integer k and an integer q such that $q \geq 2$, Broersma et al. (BROERSMA *et al.*, 2007) defined the parameter $\mathscr{P}_q(k)$ to be the maximum possible value of $BBC_q(G,P)$ for any graph G such that $\chi(G) = k$ and any Hamiltonian path P of G. Concerning this parameter, they showed that for any $k \geq 1$;

$$\mathscr{P}_2(k) = \begin{cases} 2k-1, & \text{if } k \leq 4; \\ 2k-2, & \text{if } k = 5 \text{ or } k = 6; \\ 6t, & \text{if } k \geq 7 \text{ and } k = 4t, \text{ for some integer } t; \\ 6t+1, & \text{if } k \geq 7 \text{ and } k = 4t+1 \text{ for some integer } t; \\ 6t+3, & \text{if } k \geq 7 \text{ and } k = 4t+2 \text{ for some integer } t; \\ 6t+5, & \text{if } k \geq 7 \text{ and } k = 4t+3 \text{ for some integer } t. \end{cases}$$

Regarding the parameter CBC, it is known some upper bounds. For instance, from Araujo et al. (ARAUJO *et al.*, 2018), we get the following result:

Theorem 3.15 *If G is a planar graph without cycles of length 4 and 5, and H is a path forest, then $CBC_2(G,H) \leq 6$.*

Note that this theorem would be implied by Steinberg's conjecture (STEINBERG, 1993), which says that if G is planar with no cycles of length four or five, then $\chi(G) \leq 3$. Broersma el al.(BROERSMA *et al.*, 2007) believe that if we drop the restriction on cycles of length five, then we get an upper bound of 7 for $CBC_2(G,H)$. They, then, proposed the following conjecture:

Conjecture 3.1 *If G is a planar graph without cycles of length 4 and H is a path forest, then $CBC_2(G,H) \leq 7$.*

As we stated in Chapter 1, in this dissertation we prove a variant of this conjecture where each path of the backbone is an induced subgraph of G.

Broersma et al. (BROERSMA *et al.*, 2007) presented a question regarding the parameter $\mathscr{P}_q(k)$. What are the bounds for $\mathscr{P}_q(k)$ when $q \geq 3$? It was left as further research and, to the best of our knowledge, no one has tackled this problem yet.

3.4 Constellation Backbone

In this section, we consider the cases where the backbone is a constellation. A constellation, or forest of stars, is a simple case for those groups.

Given a positive integer k and an integer q such that $q \geq 2$, Broersma et al. (BROERSMA *et al.*, 2007) defined the parameter $\mathscr{S}_q(k)$ to be the maximum possible value of $BBC_q(G,S)$ for any graph G such that $\chi(G) = k$ and any constellation S subgraph of G. Concerning this parameter, they showed that:

$$\mathscr{S}_q(k) = \begin{cases} q+1, & \text{if } k = 2; \\ \left\lceil \frac{3}{2}k \right\rceil + q - 2, & \text{if } 3 \leq k \leq 2q-3; \\ k+2q-2, & \text{if } 2q-1 \leq k \leq 2q \text{ where } q = 2; \\ k+2q-2, & \text{if } 2q-2 \leq k \leq 2q-1 \text{ where } q \geq 3; \\ 2k-1, & \text{if } k = 2q \text{ where } q \geq 3; \\ 2k - \left\lfloor \frac{k}{q} \right\rfloor, & \text{if } k \geq 2q+1. \end{cases}$$

Interestingly, in another paper, Broersma et al. (BROERSMA *et al.*, 2009a) showed that for split graphs G with chromatic number $\chi(G) = k$, we have:

$$BBC_q(G,S) \leq \begin{cases} k+q, & \text{if } k = 3 \text{ and } q \geq 2 \text{ or } k \geq 4 \text{ and } q = 2; \\ k+q-1, & \text{on the other cases.} \end{cases}$$

which are fairly good upper bounds for the q-backbone coloring of (G,S), as they are not that far from $\chi(G)$.

Concerning complexity results, Broersma et al. (BROERSMA *et al.*, 2009b) also proved that, for a graph G and a constellation S, determining whether $BBC_q(G,S) \leq \ell$ is a problem solved in polynomial time for $\ell \leq q+1$, and it is an NP-complete problem if $\ell \geq q+2$.

In their studies of complexity, Havet et al. (HAVET *et al.*, 2014) worked with stars with bounded degree. Some of the results they obtained are:

Theorem 3.16 *Given a planar graph G and a constellation S with maximum degree* 3*, deciding whether $BBC_q(G,S) \leq q+3$ is an NP-complete problem.*

Theorem 3.17 *Given a planar graph G and a constellation S with maximum degree* 2*, deciding whether $BBC_2(G,S) \leq 5$ is an NP-complete problem.*

3.5 Matching Backbone

To end our chapter about the State of the Art, we present a few results on the cases where the edges of the backbone form a matching.

Given a positive integer k and an integer q such that $q \geq 2$, Broersma et al. (BROERSMA *et al.*, 2007) defined the parameter $\mathscr{M}_q(k)$ to be the maximum possible value of $BBC_q(G,M)$ for any graph G such that $\chi(G) = k$ and any matching M of G. Concerning that parameter, they showed that:

$$\mathscr{M}_q(k) = \begin{cases} k+q-1, & \text{if } 2 \le k \le q; \\ 2k-2, & \text{if } q+1 \le k \le 2q; \\ 2k-3, & \text{if } k = 2q+1; \\ 2tq, & \text{if } k = t(q+1) \text{ where } t \ge 2; \\ 2tq+2c-1, & \text{if } k = t(q+1)+c \text{ where } t \ge 2 \text{ and } 1 \le c \le \frac{q+3}{2}; \\ 2tq+2c-2, & \text{if } k = t(q+1)+c \text{ where } t \ge 2 \text{ and } \frac{q+3}{2} \le c \le q. \end{cases}$$

They also studied the cases where G is a split graph with $\chi(G) = k$, which resulted in:

$$BBC_q(G,M) \le \begin{cases} k+1, & \text{if } k \ge 4 \text{ and } q \le \min\left\{\frac{k}{2}, \frac{k+5}{3}\right\}; \\ k+2, & \text{if } k = 9 \text{ or } k \ge 11 \text{ and } \frac{k+6}{3} \le q \le \left\lceil \frac{1}{2}k \right\rceil; \\ \left\lceil \frac{1}{2}k \right\rceil + q, & \text{if } k = 3,5,7 \text{ and } q \ge \left\lceil \frac{1}{2}k \right\rceil; \\ \left\lceil \frac{1}{2}k \right\rceil + q + 1, & \text{if } k = 4,6 \text{ or } k \ge 8 \text{ and } q \ge \left\lceil \frac{1}{2}k \right\rceil + 1. \end{cases}$$

Miškuf et al. (MIŠKUF *et al.*, 2010) studied the properties of graphs with bounded degree. In their work, the following results have been shown:

Theorem 3.18 *Let M be a matching in a graph G. If G is a C_n cycle, then $BBC_2(G,M) = 3$. Also, if $G = K_n$ for some $n \ge 3$, then $BBC_2(G,M) = n$.*

Theorem 3.19 *Let G be a graph with maximum degree Δ. If M is a matching in G, then $BBC_2(G,M) \le \Delta + 1$.*

Regarding some complexity results, in Broersma et al.(BROERSMA *et al.*, 2003), it is shown that:

Theorem 3.20 *Let M be a perfect matching of a planar graph G. Deciding whether $BBC_2(G,M) \le \ell$, for $\ell \le 3$ is a polynomial problem. For $\ell \ge 4$ it is an NP-complete problem.*

Havet et al. (HAVET *et al.*, 2014) also studied some complexity results regarding matching backbones, both in backbone coloring and in circular backbone coloring. The following theorems hold:

Theorem 3.21 *Let M be a perfect matching of a planar graph G.*

- *Deciding whether $BBC_q(G,M) \leq q+2$ is an NP-complete problem.*
- *Deciding whether $CBC_2(G,M) \leq 4$ is an NP-complete problem.*
- *Deciding whether $CBC_2(G,M) \leq 5$ is an NP-complete problem.*

Broersma et al. (BROERSMA *et al.*, 2007) present a few open problems as well. As their results are just upper bounds, it is left unknown whether for any planar graph G and a perfect matching M of G, the inequality $BBC_2(G,M) \leq 5$ holds. Furthermore, their proof of the inequality $BBC_2(G,M) \leq 6$ uses the Four Color Theorem and, to the best of our knowledge, it is still unknown if there is a way to prove that $BBC_2(G,M) \leq 6$ without using this strong result.

4 ON PLANAR GRAPHS WITHOUT C4

In this chapter we present the proof of Theorem 1.1, that we restate below:

Theorem 4.1 *Let G be a planar graph without cycles of size four and let F be an spanning induced path forest of G, then $CBC_2(G,F) \leq 7$.*

The proof for this theorem is done by assuming the existence of a minimal pair (G,F) such that $CBC_2(G,F) > 7$. A *minimal pair* is a pair (G,F) , where F is a spanning induced path forest of G, such that every proper subpair $(G-G', F-F') \subset (G,F)$ is circular 2-backbone 7-colorable. Throughout the proof, we obtain contradictions to the existence of (G,F) by extending a coloring φ of $(G-G', F-F')$ to a circular 2-backbone 7-coloring ϕ of (G,F).

In order to give an example of such argument, consider v to be a leaf of G. If (G,F) is a minimal pair, then $(G-v, F-v)$ is circular 2-backbone 7-colorable. Let φ be this coloring. As v has only a single neighbour w in G, then either $\varphi(w)+2$ or $\varphi(w)-2$ belongs to $\{1,\cdots,7\}$, therefore, assuming $\phi(v)$ to be any of these two values and $\phi(u) = \varphi(u)$, for every other vertex u of $V(G-v)$, then ϕ is a circular 2-backbone 7-coloring of (G,F). Consequently we assume every graph G in a minimal pair (G,F), from now on, does not have a leaf.

In the next section, we present the first tools we need in order to prove the theorem.

4.1 List Backbone Coloring

We emphasize that we use List Coloring concepts merely as a tool, but we do not prove any profound result concerning list-chromatic number or any other parameter adapted to the backbone coloring paradigm.

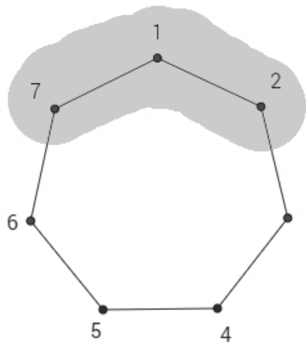
Let (G', F') be an induced proper subpair of (G,F). Let φ be a circular 2-backbone 7-coloring of $(G-G', F-F')$. For each $v \in V(G')$, we define the set $A_\varphi(v)$ of its *available colors* as the subset of elements $c \in \{1,\cdots,7\}$ such that:

$$\begin{array}{lll} (i) & 1 \leq |\phi(u)-c|, & \text{for all } u \in N_{G-G'}(v) \setminus N_F(v), \\ (ii) & 2 \leq |\phi(w)-c| \leq 5, & \text{for all } w \in N_{F-F'}(v). \end{array}$$

We also define $a_\varphi(v) = |A_\varphi(v)|$.

Let vw be an edge of $E(G) \setminus E(G')$, such that $w \in V(G')$. We say *the vertex w forbids a color c to be assigned to the vertex v* either if $\phi(w) = c$ or if $vw \in E(F)$ and $|\phi(w)-c| \in \{1,6\}$. Also, we say that *a color $c_f \in \{1,\cdots,7\}$ forbids a color c* if $|c_f - c| \in \{0,1,6\}$. We denote forb(c_f) the set of colors forbidden by c_f. Observe that $|\text{forb}(c_f)| = 3$, for any $c_f \in \{1,\cdots,7\}$. Figure 4a depicts an example of some forbidden colors. Note that the colors 1, 2 and 7 are forbidden by the color 1.

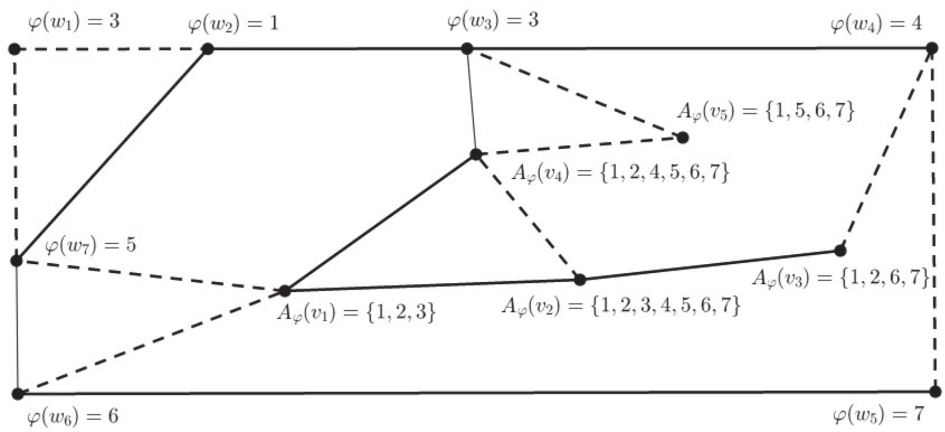
Figure 4 – Colors forbidden by the color 1



Fonte: elaborado pelo autor.

In the graph shown in Figure 5a, we color the vertices $\{w_1, w_2, \cdots, w_7\}$ as shown. For the remaining vertices $\{v_1, v_2, v_3, v_4, v_5\}$ we show their respective available sets

Figure 5 – Sets of available colors



Fonte: elaborado pelo autor.

Let (G', F') be an induced proper subpair of the minimal pair (G, F) and let $v \in V(G')$. The *total degree* of v in $(G - G', F - F')$, denoted by $d^t_{(G-G',F-F')}(v)$, is given by the following equation:

$$d^t_{(G-G',F-F')}(v) = d_{G-G'}(v) + 2d_{F-F'}(v). \tag{4.1}$$

To avoid excessive notation, we write $d^t_{G-G'}(v)$ instead of $d^t_{(G-G',F-F')}(v)$, whenever there is no ambiguity.

Proposition 4.2 *Let (G',F') be an induced proper subpair of a minimal pair (G,F). Let φ be a circular 2-backbone 7-coloring of $(G-G',F-F')$. For every $v \in V(G')$, the following inequality holds:*

$$a_\varphi(v) \geq 7 - d^t_{G-G'}(v). \tag{4.2}$$

Demonstration. Let $w_1, \cdots, w_k$ be the neighbours of v in G that lie in $G-G'$. If w_iv is not an edge of F, then w_i forbids only a single color to be assigned to v. On the other hand, if w_iv is an edge of F, then w_i forbids three colors to be assigned to v. Therefore, the number of forbidden colors to be assigned to v is, at most, the number of vertices that forbid a single color plus three times the number of vertices that forbid three colors. One can notice that this is exactly $d_{G-G'}(v) + 2d_{F-F'}(v) = d^t_{G-G'}(v)$. Since the number of used colors is at most 7, then we get $a_\varphi(v) \geq 7 - d^t_{G-G'}(v)$. ■

As we may not know which colors are actually forbidden or available for a vertex, it is more desirable to work with the number of forbidden or available colors for it and study its implications in the worst case scenario. The concept of available sets helps us turn a Circular Backbone Coloring problem into a List Coloring problem that should agree with the backbone.

Let (G,H) be a pair and consider a function $\mathscr{L} : V(G) \to \mathscr{P}(\{1,\cdots,k\})$ such that $\mathscr{L}(v)$ is not empty for every $v \in V(G)$. We say (G,H) is *circular q-backbone $\mathscr{L}$-colorable* if there exists a circular q-backbone k-coloring ϕ of (G,H) such that $\phi(v) \in \mathscr{L}(v)$, for every v in $V(G)$. Furthermore, let $\ell : V(G) \to \{1,\cdots,k\}$, we say (G,H) is *circular q-backbone ℓ-colorable* if, for every list assignment $\mathscr{L} : V(G) \to \mathscr{P}(\{1,\cdots,k\})$ satisfying $|\mathscr{L}(v)| = \ell(v)$, for every v in $V(G)$, the pair (G,H) is circular q-backbone $\mathscr{L}$-colorable. To avoid excessive notation, we say (G,H) is (ℓ,q)-colorable or $(\mathscr{L},q)$-colorable if it is, respectively, circular q-backbone ℓ-colorable or circular q-backbone $\mathscr{L}$-colorable. Whenever $q = 2$ we simply use ℓ-colorable or $\mathscr{L}$-colorable instead of $(\ell,2)$-colorable or $(\mathscr{L},2)$-colorable, respectively.

Consider (G_1,H_1) a subpair of (G_2,H_2) with list assignments, respectively, $\mathscr{L}_1$ and $\mathscr{L}_2$ such that $\mathscr{L}_1(v) \subseteq \mathscr{L}_2(v)$ for every $v \in V(G_1)$ and (G_1,H_1) admits a list circular backbone coloring ϕ_1 with respect to $\mathscr{L}_1$. If there exists a list circular backbone coloring ϕ_2 of (G_2,H_2), with respect to $\mathscr{L}_2$, such that $\phi_2(v) = \phi_1(v)$ for every v, then we say that ϕ_1 can be $\mathscr{L}_2$-extended to ϕ_2.

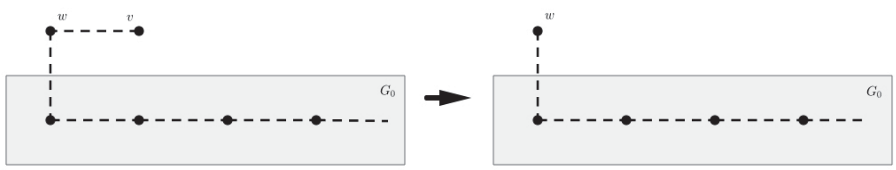
In the following subsections, we focus on subpairs (G',F') of a minimal pair (G,F). Consider a circular 2-backbone 7-coloring φ of $(G-V(G'),F-V(F'))$ and the corresponding sets of available colors $A_\varphi : V(G') \to \mathscr{P}(\{1,2,...,7\})$. Notice that if (G',F') is A_φ-colorable, then (G,F) is circular 2-backbone 7-colorable.

4.1.1 Leaf Reduction Propositions

The reduction propositions we present here are useful to extend a coloring of a subpair into a coloring of the pair itself. Such extension is constructed by using list backbone coloring, but since we do not know which elements are in each list, we mostly work with the number of colors in each list.

For Propositions 4.3, 4.4 and 4.5, let (G',F') be a subpair of (G,F) and consider $vw \in E(G')$. Let (G_0,F_0) be an induced subpair of (G',F') such that $V(G_0) = V(G') \setminus \{v,w\}$, $V(F_0) = V(F') \setminus \{v,w\}$ and $N_{G_0}(v) = \emptyset$. Consider the functions $\ell_w : V(G_0) \cup \{w\} \to \{1,\cdots,7\}$ and $\ell_{wv} : V(G_0) \cup \{v,w\} \to \{1,\cdots,7\}$ such that $\ell_{wv}(u) \geq \ell_w(u)$, for every u in $V(G_0) \cup \{w\}$, and assume $(G[V(G_0) \cup \{w\}], F[V(F_0) \cup \{w\}])$ is ℓ_w-colorable. The idea is to reduce the problem of list coloring (G',F') to the problem of list coloring $(G'-v,F'-v)$, where v is a leaf of F'. The graph shown in Figure 6a depicts an example we use throughout this entire section.

Figure 6 – Example of the leaf propositions



Fonte: elaborado pelo autor.

The following propositions are stated under the above conditions:

Proposition 4.3 *If $\ell_{wv}(v) \geq 4$, then $(G_0 \cup \{w,v\}, F_0 \cup \{w,v\})$ is ℓ_{wv}-colorable.*

Demonstration. As it is not specified whether $vw \in E(F)$, we assume that this is the case. Consider a list assignment $\mathscr{L}_{wv} : V(G_0) \cup \{v,w\} \to \mathscr{P}(\{1,\cdots,7\})$ such that $|\mathscr{L}_{wv}(u)| = \ell_{wv}(u)$, for every u in $V(G_0) \cup \{v,w\}$.

Define $\sigma_w(v) \subseteq \mathscr{L}_{wv}(w)$ as the set of colors in the list of w that forbid all the colors in $\mathscr{L}_{wv}(v)$, that is $\sigma_w(v) = \{c_f \in \mathscr{L}_{wv}(w) \mid \mathscr{L}_{wv}(v) \subseteq forb(c_f)\}$. If $c_f \in \sigma_w(v)$, then c_f forbids at most three colors to v. However $\ell_{wv}(v) = |\mathscr{L}_{wv}(v)| \geq 4$, which means that $\sigma_f(v) = \emptyset$. Therefore, any circular coloring φ of $(G[V(G_0) \cup \{w\}], F[V(F_0) \cup \{w\}])$ can be $\mathscr{L}_{wv}$-extended to a circular coloring ϕ of $(G[V(G_0) \cup \{w,v\}], F[V(F_0) \cup \{w,v\}])$ by taking $\phi(v) \in \mathscr{L}_{wv}(v) \setminus forb(\varphi(w))$. Such coloring ϕ exists since $(G[V(G_0) \cup \{w\}], F[V(F_0) \cup \{w\}])$ is ℓ_w-colorable. ■

Proposition 4.4 *If $\ell_{wv}(v) \geq 3$ and $\ell_{wv}(w) - 1 \geq \ell_w(w)$, then $(G_0 \cup \{w,v\}, F_0 \cup \{w,v\})$ is ℓ_{wv}-colorable.*

Demonstration. If $\ell_{wv}(v) \geq 4$, then it follows directly by Proposition 4.3. We only focus on the case where $\ell_{wv}(v) = 3$.

Consider a list $\mathscr{L}_{wv} : V(G_0) \cup \{v,w\} \to \mathscr{P}(\{1,\cdots,7\})$ such that $|\mathscr{L}_{wv}(u)| = \ell_{wv}(u)$ for every u in $V(G_0) \cup \{v,w\}$.

Define $\sigma_w(v) = \{c_f \in \mathscr{L}_{wv}(w) \mid \mathscr{L}_{wv}(v) \subseteq forb(c_f)\}$ as before. Because $\ell_{wv}(v) = 3$, there is at most one color c in $\sigma_w(v)$, so we can assume $|\sigma_w(v)| \leq 1$. Therefore, for any list $\mathscr{L}_w : V(G_0) \cup \{w\} \to \mathscr{P}(\{1,\cdots,7\})$ satisfying:

$$(i) \quad \mathscr{L}_w(u) = \mathscr{L}_{wv}(u), \text{ for every } u \in V(G_0),$$
$$(ii) \quad \mathscr{L}_w(w) = \mathscr{L}_{wv}(w) \setminus \sigma_w(v),$$

There is a list circular coloring φ of $(G[V(G_0) \cup \{w\}], F[V(F_0) \cup \{w\}])$, and it can be $\mathscr{L}_{wv}$-extended to a list circular coloring of $(G[V(G_0) \cup \{w,v\}], F[V(F_0) \cup \{w,v\}])$ by taking $\phi(v) \in \mathscr{L}_{wv}(v) \setminus forb(\varphi(w))$. In fact, as $\varphi(w) \notin \sigma_w(v)$, such coloring φ exists since $(G[V(G_0) \cup \{w\}], F[V(F_0) \cup \{w\}])$ is ℓ_w-colorable. ■

Proposition 4.5 *If $\ell_{wv}(v) \geq 2$ and $\ell_{wv}(w) - 2 \geq \ell_w(w)$, then $(G_0 \cup \{w,v\}, F_0 \cup \{w,v\})$ is ℓ_{wv}-colorable.*

Demonstration. If $\ell_{wv}(v) \geq 3$, then this holds true by Proposition 4.4. We only focus on the case where $\ell_{wv}(v) = 2$.

Consider a list $\mathscr{L}_{wv} : V(G_0) \cup \{v,w\} \to \mathscr{P}(\{1,\cdots,7\})$ such that $|\mathscr{L}_{wv}(u)| = \ell_{wv}(u)$ for every u in $V(G_0) \cup \{v,w\}$.

Define $\sigma_w(v) = \{c_f \in \mathscr{L}_{wv}(w) \mid \mathscr{L}_{wv}(v) \subseteq forb(c_f)\}$ as before. Observe that because $\ell_{wv}(v) = 2$, there is at most two colors in $\sigma_w(v)$, so we can assume $|\sigma_w(v)| \leq 2$. Therefore, for any list $\mathscr{L}_w : V(G_0) \cup \{w\} \to \mathscr{P}(\{1, \cdots, 7\})$ satisfying:

$$(i) \quad \mathscr{L}_w(u) = \mathscr{L}_{wv}(u), \text{ for every } u \in V(G_0),$$
$$(ii) \quad \mathscr{L}_w(w) = \mathscr{L}_{wv}(w) \setminus \sigma_w(v),$$

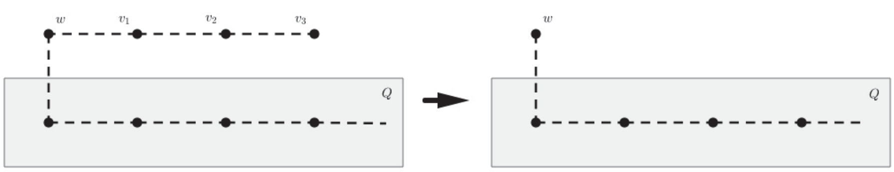
There is a list circular coloring φ of $(G[V(G_0) \cup \{w\}], F[V(F_0) \cup \{w\}])$, and it can be $\mathscr{L}_{wv}$-extended to a list circular coloring of $(G[V(G_0) \cup \{w, v\}], F[V(F_0) \cup \{w, v\}])$ by taking $\phi(v) \in \mathscr{L}_{wv}(v) \setminus forb(\varphi(w))$. In fact, as $\varphi(w) \notin \sigma_w(v)$, such coloring φ exists since $(G[V(G_0) \cup \{w\}], F[V(F_0) \cup \{w\}])$ is ℓ_w-colorable. ■

4.1.2 Path Reduction Propositions

Propositions 4.3, 4.4 and 4.5 are used to reduce a graph by one vertex at a time. We dedicate this subsection to prove some propositions that reduce the graph by one path at a time.

For Propositions 4.6 and 4.7, let $P = Q \circ (w) \circ (v_1, \cdots, v_j)$ be a path, where $(w) \circ (v_1, \cdots, v_j)$ is an induced subpath of P, for some $j \geq 1$. Consider the functions $\ell_w : V(Q) \cup \{w\} \to \{1, \cdots, 7\}$ and $\ell_{wv} : V(Q) \cup \{w, v_1, \cdots, v_j\} \to \{1, \cdots, 7\}$ such that $\ell_{wv}(u) \geq \ell_w(u)$, for every u in $V(Q) \cup \{w\}$, and assume $(Q \circ (w), Q \circ (w))$ is ℓ_w-colorable. The graph shown in Figure 11a depicts an example that can used throughout this entire section.

Figure 7 – Example of the path propositions



Fonte: elaborado pelo autor.

Proposition 4.6 *If $\ell_{vw}(v_j) \geq 3$, $\ell_{vw}(v_p) \geq 4$ for all $p \in \{1, \cdots, j-1\}$, and $\ell_{vw}(w) - 1 \geq \ell_w(w)$, then $(Q \circ (w) \circ (v_1, \cdots, v_j), Q \circ (w) \circ (v_1, \cdots, v_j))$ is ℓ_{wv}-colorable.*

Demonstration. We prove this by induction on j. For $j = 1$, by Proposition 4.4, taking $G_0 = F_0 = Q$ and $v = v_1$, $(Q \cup \{w, v_1\}, Q \cup \{w, v_1\})$ is ℓ_{wv}-colorable.

Now, assume it is true for some $j \geq 1$. For $j+1$ we have $P' = Q \circ (w) \circ (v_1, \cdots, v_j, v_{j+1}) = P \circ (v_{j+1})$. Consider the list assignment $\ell : V(Q) \cup \{w, v_1, \cdots, v_j\}$ such that $\ell(w) = \ell_{wv}(w) \geq \ell_w(w) + 1$, $\ell(v_j) = \ell_{vw}(v_j) - 1 \geq 3$ and, for all $p \in \{1, \cdots, j-1\}, \ell(v_p) = \ell_{vw}(v_p) \geq 4$. By induction hypothesis, (P, P) is ℓ-colorable. Finally, observe that by Proposition 4.4, taking $v = v_{j+1}$, $(P \cup \{v_{j+1}\}, P \cup \{v_{j+1}\})$ is ℓ_{wv}-colorable. ■

Proposition 4.7 *If* $\ell_{vw}(v_j) \geq 2$, $\ell_{vw}(v_p) \geq 4$ *for all* $p \in \{1, \cdots, j-1\}$, *and* $\ell_{vw}(w) - 2 \geq \ell_w(w)$, *then* $(Q \circ (w) \circ (v_1, \cdots, v_j), Q \circ (w) \circ (v_1, \cdots, v_j))$ *is* ℓ_{wv}*-colorable.*

Demonstration. We prove this by induction on j. For $j = 1$, by Proposition 4.5, taking $G_0 = F_0 = Q$ and $v = v_1$, $(Q \cup \{w, v_1\}, Q \cup \{w, v_1\})$ is ℓ_{wv}-colorable.

Now, assume that it is true for some $j \geq 1$. For $j+1$, we have $P' = Q \circ (w) \circ (v_1, \cdots, v_j, v_{j+1}) = P \circ (v_{j+1})$. Consider the list assignment $\ell : V(Q) \cup \{w, v_1, \cdots, v_j\}$ such that $\ell(w) = \ell_{wv}(w) \geq \ell_w(w) + 2$, $\ell(v_j) = \ell_{vw}(v_j) - 2 \geq 2$ and, for all $p \in \{1, \cdots, j-1\}, \ell(v_p) = \ell_{vw}(v_p) \geq 4$. By induction hypothesis, (P, P) is ℓ-colorable. Finally, observe that, by Proposition 4.5, taking $v = v_{j+1}$, $(P \cup \{v_{j+1}\}, P \cup \{v_{j+1}\})$ is ℓ_{wv}-colorable. ■

4.1.3 Structural Lemmas

Let us move towards a minimal pair (G, F), where F is a spanning induced path forest of G. Let P be a connected component of F. We call a subpath of P such that every vertex has degree in G at most five a *strong subpath*. In this section, we analyse the possible structures of P, eliminating those for which a coloring of $(G - P, F - P)$ can be extended to a coloring of (G, F).

Given any $v \in V(G)$, we say that v is a *type* (p, q) *vertex*, if $p = d_G(v)$ and $q = d_F(v)$.

For a minimal pair (G, F), consider $v \in V(G)$ and $(G', F') = (G - \{v\}, F - \{v\})$. By the definition of minimal pair, (G, F) is not circular 2-backbone 7-colorable whereas (G', F') is. Therefore $a_\varphi(v) = 0$, where φ is any circular 2-backbone 7-coloring of (G', F'). By Corollary 4.2, we have that $0 \geq 7 - d^t_{G'}(v)$, that is, $d^t_{G'}(v) \geq 7$.

Now, if v belongs to a strong subpath P_1 of P in a minimal pair (G, F), then because $d_G(v) \leq 5$ and $d^t_{G'}(v) \geq 7$, we have that v can only be of type $(3, 2)$, $(4, 2)$, $(5, 1)$ or $(5, 2)$.

The notion of vertex type is useful for the next lemmas.

Lemma 4.1 *Let P_1 be a strong subpath of P. If there is a type $(5,1)$ vertex in P_1, then all other vertices of P_1 are of type $(5,2)$.*

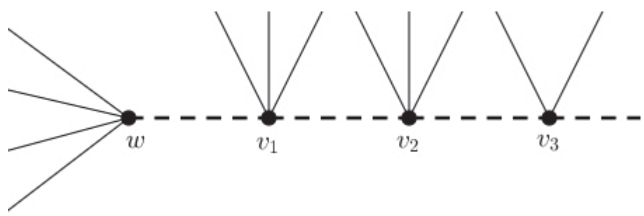
Demonstration. Suppose, by contradiction, that P_1 has either a $(3,2)$ vertex, or a $(4,2)$ vertex or a second $(5,1)$ vertex. Without loss of generality, assume that $P_1 = (w) \circ (v_1, \cdots, v_j)$, where w is a type $(5,1)$ vertex and v_j is the closest one to w that is not a type $(5,2)$ vertex. Therefore, for all $p \in \{1,2,3,...,j-1\}$, v_p is a $(5,2)$ vertex.

Consider $(G',F') = (P_1,P_1)$ and φ be a circular 2-backbone 7-coloring of $(G-V(P_1), F-V(P_1))$. Notice that $d^t_{G-V(P_1)}(w) = 4$, $d^t_{G-V(P_1)}(v_j) \leq 5$ and for all $p \in \{1,2,3,...,j-1\}$, $d^t_{G-V(P_1)}(v_p) = 3$. Therefore $a_\varphi(w) \geq 3$, $a_\varphi(v_j) \geq 2$ and for all $p \in \{1,2,3,...,j-1\}, a_\varphi(v_p) \geq 4$. We want to prove that (G',F') is *av*-colorable, which implies that (G,F) is circular 2-backbone 7-colorable, a contradiction.

Let $\ell_{wv} : V(G') \to \{1, \cdots, 7\}$ with $\ell_{vw}(u) = a_\varphi(u)$, for every u in $V(G')$.

Let Q be the empty set, we see that $Q \circ (w)$ is ℓ_w-colorable, being $\ell_w : V(Q) \cup \{w\} \to \{1, \cdots, 7\}$ such that $\ell_w(w) \leq \ell_{wv}(w) - 2$. Therefore, by Proposition 4.7, $(Q \circ (w) \circ (v_1, \cdots, v_j), Q \circ (w) \circ (v_1, \cdots, v_j))$ is ℓ_{wv}-colorable. ■

Figure 8 – Structure forbidden from the minimal pair due to Lemma 4.1



Fonte: elaborado pelo autor.

Lemma 4.2 *Let P_1 be a strong subpath of P. If there is a vertex w in P_1 of type $(3,2)$, then all other vertices of P_1 are of type $(5,2)$.*

Demonstration. By Lemma 4.1, P_1 does not contain a type $(5,1)$ vertex. Suppose by contradiction that there exists some vertex v distinct of w of type $(3,2)$ or $(4,2)$. Choose a vertex v closest to w in P_1. Name the vertices of P_1 such that $P_1 = (w) \circ (v_1, \cdots, v_j)$, where $v = v_j$.

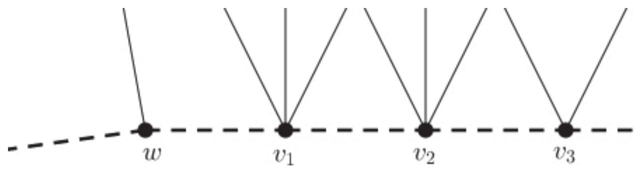
Let $(G',F') = (P_1,P_1)$ and φ be a circular 2-backbone 7-coloring of $(G-V(P_1), F-V(P_1))$. Notice that $d^t_{G-V(P_1)}(w) = 4$, $d^t_{G-V(P_1)}(v_j) \leq 5$ and, for every $p \in \{1,2,3,...,j-

$1\}$, $d^t_{G-V(P_1)}(v_p) = 3$. Therefore $a_\varphi(w) \geq 3$, $a_\varphi(v_j) \geq 2$ and for every $p \in \{1,2,3,...,j-1\}, a_\varphi(v_p) \geq 4$. We want to prove that (G',F') is *av*-colorable, which implies that (G,F) is circular 2-backbone 7-colorable, a contradiction.

Let $\ell_{wv} : V(G') \to \{1,\cdots,7\}$ with $\ell_{vw}(u) = a_\varphi(u)$, for every u in $V(G')$.

Let Q be the empty set, we see that $Q \circ (w)$ is ℓ_w-colorable, where $\ell_w : V(Q) \cup \{w\} \to \{1,\cdots,7\}$ such that $\ell_w(w) \leq \ell_{wv}(w) - 2$, as $\ell_w(w) \geq 1$. Therefore, by Proposition 4.7, $(Q \circ (w) \circ (v_1,\cdots,v_j), Q \circ (w) \circ (v_1,\cdots,v_j))$ is ℓ_{wv}-colorable. Which means (G',F') is *av*-colorable. Therefore, (G,F) is circular 2-backbone 7-colorable. ■

Figure 9 – Structure forbidden from the minimal pair due to Lemma 4.2



Fonte: elaborado pelo autor.

Lemma 4.3 *Let P_1 be a strong subpath of P. If there is a type $(4,2)$ vertex in P_1, then there is at most one other type $(4,2)$ vertex in P_1 and all other vertices of P_1 are of type $(5,2)$.*

Demonstration. By Lemmas 4.1 and 4.2, P_1 does not contain a type $(5,1)$ vertex nor a type $(3,2)$.

Suppose by contradiction that P_1 has three type $(4,2)$ vertices. Name the vertices of P_1 such that $P_1 = (w) \circ (v_1,\cdots,v_m,...,v_j)$ and let m, j be the smallest integers such that w, v_m and v_j are type $(4,2)$ vertices. Therefore, for all $p \in \{1,2,3,...,j-1\} \setminus \{m\}$, v_p is a type $(5,2)$ vertex.

Let $(G',F') = (P_1,P_1)$ and φ be a circular 2-backbone 7-coloring of $(G-V(P_1), F-V(P_1))$. Notice that $d^t_{G-V(P_1)}(w) = 5 = d^t_{G-V(P_1)}(v_j)$, $d^t_{G-V(P_1)}(v_m) = 2$ and, for every $p \in \{1,2,3,...,j-1\} \setminus \{m\}$, $d^t_{G-V(P_1)}(v_p) = 3$. Therefore, $a_\varphi(w) \geq 2$, $a_\varphi(v_j) \geq 2$, $a_\varphi(v_m) \geq 5$ and, for every $p \in \{1,2,3,...,j-1\} \setminus \{m\}, a_\varphi(v_p) \geq 4$. Let $\ell_{wv} : V(G') \to \{1,\cdots,7\}$ with $\ell_{vw}(u) = a_\varphi(u)$ for every u in $V(G')$.

Let $Q = (w,v_1,\cdots,v_{m-1})$ and $\ell^* : V(Q) \cup \{v_m\} \to \{1,\cdots,7\}$ such that $\ell^*(u) = \ell_{wv}(u)$, for every $u \in \{w,v_1,\cdots,v_{m-1}\}$, and $\ell^*(v_m) = 3$. Also, let $\ell_w : \{w\} \to \{1,\cdots,7\}$ such that $\ell_w(w) = 1 = \ell^*(w) - 1$. The pair $(\{w\},\{w\})$ is ℓ_w-colorable. Therefore, by Proposition 4.6, $(Q \cup \{v_m\}, Q \cup \{v_m\})$ is ℓ^*-colorable. On the other hand, by Proposition 4.7, $(Q \cup \{v_m, v_{m+1}, ..., v_j\}, Q \cup \{v_m, v_{m+1}, ..., v_j\})$ is ℓ_{wv}-colorable. ■

4.1.4 Pair Equivalence

Let (G_1, F_1) and (G_2, F_2) be two pairs and consider P to be a path such that $P_1 = P \circ (w_1) \subseteq F_1$ and $P_2 = P \circ (w_2, v_1, \cdots, v_j) \subseteq F_2$, for $j \geq 0$. For $j = 0$, consider that $(w_2, v_1, \cdots, v_j) = (w_2)$. Let ϕ_1 and ϕ_2 be circular 2-backbone 7-colorings of $(G_1 - P_1, F_1 - P_1)$ and $(G_2 - P_2, F_2 - P_2)$ respectively. We say that P_1 and P_2 are *equivalent* if (P_1, P_1) being a_{ϕ_1}-colorable implies that (P_2, P_2) is a_{ϕ_2}-colorable.

We emphasize that this definition of pair equivalence does not rely too much on the pairs (G_1, F_1) and (G_2, F_2), but on the list assignments a_{ϕ_1} and a_{ϕ_2}. For the following propositions, we consider $a_{\phi_1}(u) = a_{\phi_2}(u)$, for every $u \in V(P)$ and omit the pairs (G_1, F_1) and (G_2, F_2).

Proposition 4.8 *If w_1 is a type $(3,2)$ vertex and w_2 is a type $(5,1)$, then $(P \circ (w_1), P \circ (w_1))$ and $(P \circ (w_2), P \circ (w_2))$ are equivalent.*

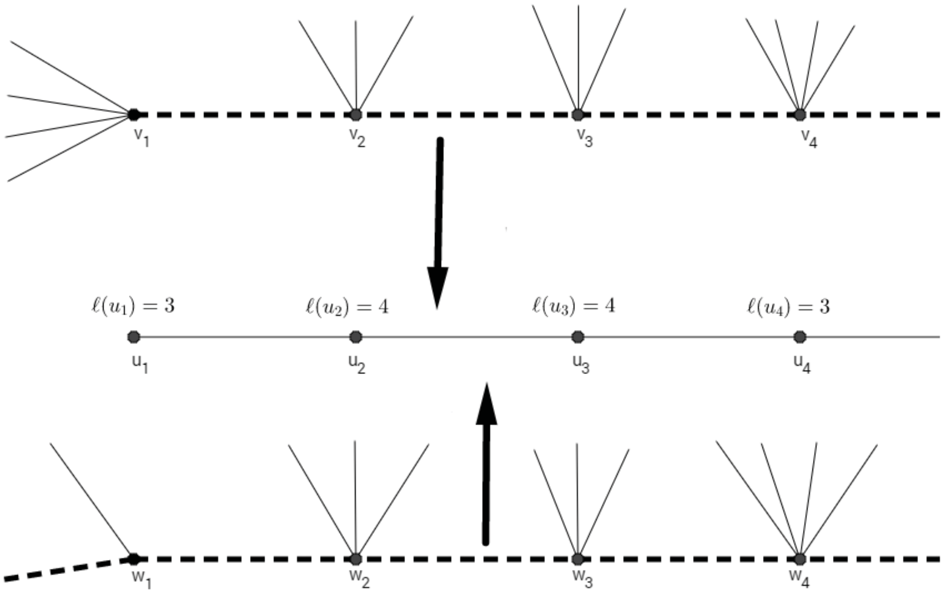
Demonstration. Notice that $a_{\phi_1}(w_1) = a_{\phi_2}(w_2) = 3$. Therefore they are equivalent. ■

Proposition 4.9 *If w_1 is a type $(3,2)$ vertex, w_2 and v_j, for $j \geq 1$, are type $(4,2)$ and v_p is type $(5,2)$ vertex, for $p \in \{1, \cdots, j\}$, then $(P \circ (w_1), P \circ (w_1))$ and $(P \circ (w_2, v_1, \cdots, v_j), P \circ (w_2, v_1, \cdots, v_j))$ are equivalent.*

Demonstration. Notice that $a_{\phi_1}(w_1) = 3$, $a_{\phi_2}(w_2) = 5$, $a_{\phi_2}(v_j) = 2$ and, for every $p \in \{1, \cdots, j-1\}$, $a_{\phi_2}(v_p) = 4$. Consider $\ell_w : V(P) \cup \{w_2\} \to \{1, \cdots, 7\}$ such that $\ell_w(w_2) = 3$ and $\ell_w(v) = a_{\phi_2}(v)$ for every v in $V(P)$. Applying Proposition 4.7 on the pair $(P \circ (w_2, v_1, \cdots, v_j), P \circ (w_2, v_1, \cdots, v_j))$, the conclusion follows. ■

Proposition 4.10 *If w_1 is a type $(6,1)$ vertex, w_2 is a type $(6,2)$, v_j, for $j \geq 1$, is a type $(5,1)$ and v_p is type $(5,2)$ vertex, for $p \in \{1, \cdots, j\}$, then $(P \circ (w_1), P \circ (w_1))$ and $(P \circ (w_2, v_1, \cdots, v_j), P \circ (w_2, v_1, \cdots, v_j))$ are equivalent.*

Figure 10 – Example of equivalent paths



Fonte: elaborado pelo autor.

Demonstration. Notice that $a_{\phi_1}(w_1) = 2$, $a_{\phi_2}(w_2) = 3$, $a_{\phi_2}(v_j) = 3$ and, for every $p \in \{1, \cdots, j-1\}$, $a_{\phi_2}(v_p) = 4$. Consider $\ell_w : V(P) \cup \{w_2\} \to \{1, \cdots, 7\}$ such that $\ell_w(w_2) = 2$ and $\ell_w(v) = a_{\phi_2}(v)$ for every v in $V(P)$. Applying Proposition 4.6 on the pair $(P \circ (w_2, v_1, \cdots, v_j), P \circ (w_2, v_1, \cdots, v_j))$, the conclusion follows. ■

The notion of equivalence of the pairs is useful since we are interested not in the vertices themselves, but in their available list.

4.2 Main result

In this section, we prove Theorem 1.1, which we restate in the sequel:

Theorem 4.11 *Let G be a planar graph without cycles of size four and let F be an spanning induced path forest of G, then* $CBC_2(G,F) \leq 7$.

This theorem gives us the following corollary:

Corollary 4.1 *Let G be a planar graph without cycles of size four and let M be a matching in G, then* $CBC_2(G,M) \leq 7$.

We recall that in Broersma et al. (BROERSMA *et al.*, 2007), the authors prove that $BBC_2(G,M) \leq 6$, which implies that $CBC_2(G,M) \leq 7$. However, their proof requires the Four

Color Theorem, which is not needed here.

We prove Theorem 1.1 by contradiction, double counting edges and evaluating the average degree of the graph G in a minimal pair (G,F). These evaluations are done in Section 4.2.3. Before we do so, the reader should be familiar with the concept of islands, that we present in Section 4.2.2. Furthermore, one of the lemmas requires a key lemma that can be proved using the propositions presented in Section 4.1. We present and prove that key lemma in the next section.

4.2.1 Average Degree Lemma

Lemma 4.6, a key lemma used to prove Theorem 1.1, evaluates the average degree of the graph G of the minimal pair (G,F). The proof of Lemma 4.6 requires the following lemma:

Lemma 4.4 *Let $G=(V(G),E(G))$ be a planar graph without C_4, (G,F) be a minimal pair and P be a connected component of F, then, the following inequality holds:*

$$\sum_{v\in V(P)} d_G(v) \geq 5|V(P)|+1. \tag{4.3}$$

The proof of Lemma 4.4 is done by considering an ordering over the strong subpaths in P. Let P_1 and P_2 be strong subpaths of P. Name the vertices of P in order, that is, $P=(v_1,\cdots,v_j)$. We say P_1 is *located to the left* of P_2 if there are vertices $v_{i_1}\in V(P_1)$ and $v_{i_2}\in V(P_2)$ such that $i_1<i_2$. Otherwise, P_2 is *located to the right* of P_1. Moreover, for an integer $m\in\{i_1,i_1+1,\cdots,i_2\}$, we say that v_m is a vertex *between* P_1 and P_2 if $v_m\notin V(P_1)\cup V(P_2)$. We say P_1 and P_2 are *consecutive* if there is no vertex v between P_1 and P_2 such that $d_G(v)\leq 5$.

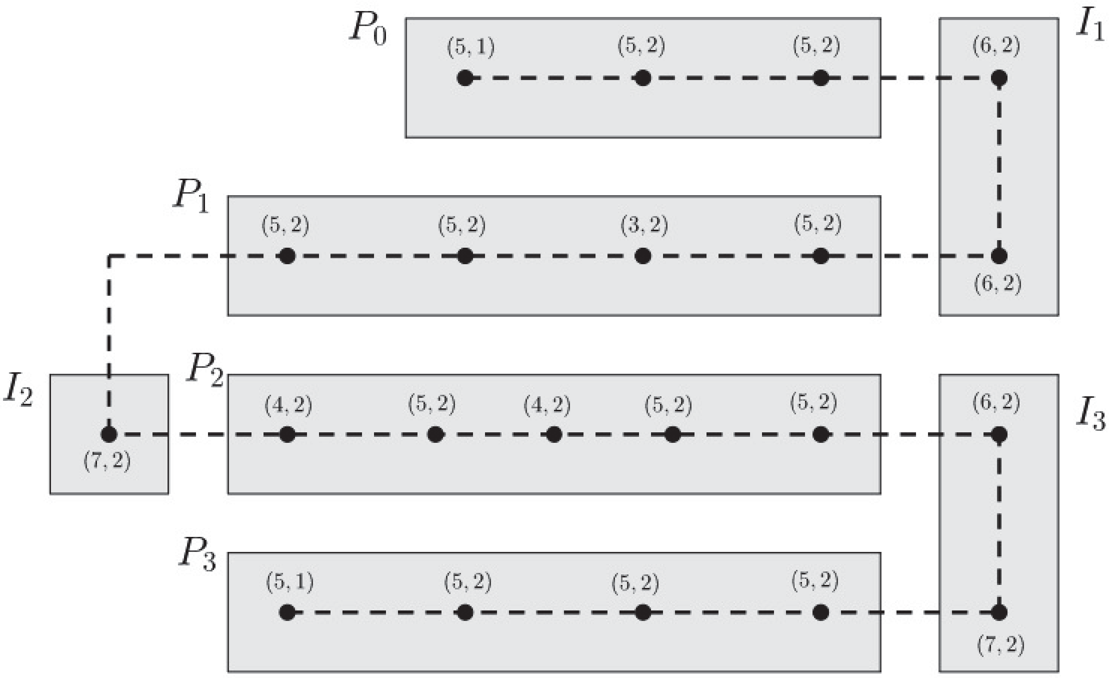
Let $P_0,\cdots,P_j$ be its strong subpaths such that for every $i\in\{1,\cdots,j\}$, the paths P_{i-1} and P_i are consecutive with P_{i-1} located to the left of P_i. Name $I_1,\cdots,I_j$ the maximal subpaths of P such that the vertices of I_i are located between P_{i-1} and P_i. It is worth noting that only the paths P_0 and P_j can be empty. We name the sequence $(P_0,I_1,P_1,\cdots,I_j,P_j)$ the *strong partition* of P. The *weight* of P, denoted by $\omega(P)$, is the integer that satisfy $\omega(P)=\sum_{v\in V(P)}(d_G(v)-5)$.

To prove Lemma 4.4, it is sufficient to prove that $\omega(P)\geq 1$, or in other words, that:

$$\omega(P_0)+\sum_{k=1}^{j}(\omega(I_k)+\omega(P_k))>0. \tag{4.4}$$

for any connected component P of F. We use the next lemma to prove Equation 4.4.

Figure 11 – Strong Partition of a path



Fonte: elaborado pelo autor.

Lemma 4.5 *Let P be a connected component of F and $P_0, I_1, P_1, ..., I_j, P_j$ be the strong partition of P. Let $k_1 < k_2 < ... < k_\lambda$ be the positive integers such that $\omega(P_{k_m}) = -2$. Furthermore, consider $k_0 = 0$. If $m \in \{0, 1, 2, 3, ..., \lambda - 1\}$, then the inequality below is true:*

$$S_m = \sum_{i=k_m+1}^{k_{(m+1)}} (\omega(I_i) + \omega(P_i)) \geq 0. \tag{4.5}$$

Demonstration. By contradiction, suppose that $S_m < 0$, for some $m \in \{0, 1, ..., \lambda - 1\}$. Note that $\omega(I_i) \geq 1$, $\omega(P_i) \geq -1$, for every $i \in \{k_m + 1, \cdots, k_{m+1} - 1\}$, and $\omega(P_{k_{m+1}}) = -2$, by the definition of the integers k_m. Consequently, we have:

$$S_m = \sum_{i=k_m+1}^{k_{m+1}} (\omega(I_i) + \omega(P_i)) = -1. \tag{4.6}$$

As it is an equality, we have that $\omega(I_i) = 1$ and $\omega(P_i) = -1$, for every $i \in \{k_m + 1, \cdots, k_{m+1} - 1\}$. In other words, for every $i \in \{k_m + 1, \cdots, k_{m+1} - 1\}$, the subpath I_i is a single vertex of degree 6 in G and P_i has exactly one vertex of type $(4,2)$ and every other vertex is a type $(5,2)$. Additionally, if $m = 0$, either P_0 is not empty or I_1 is a type $(6,1)$ vertex.

If $m = 0$, let P' be the smallest subpath of P containing both P_0 and I_1 and also containing every vertex of P_{k_1} of degree smaller than five. Similarly, if $m > 0$, let P' be the smallest subpath of P containing every vertex of both P_{k_m} and $P_{k_{m+1}}$ of degree smaller than five. Consider a circular 2-backbone 7-coloring φ of $(G-V(P'),F-V(P'))$. We want to prove that (P',P') is a_φ-colorable.

By Propositions 4.8 and 4.9, we can assume that each P_{k_m}, for $m \geq 1$, has a type $(3,2)$ vertex. By Proposition 4.10, we can also assume that P_{k_0} is not empty, therefore, using Proposition 4.8 we may assume that it has a type $(3,2)$ vertex as well. Therefore, we consider the end vertices of P to be type $(3,2)$ vertices.

Notice that in P' there are exactly $k_{m+1}-k_m-1$ vertices of type $(4,2)$. In terms of list coloring, that means that $a_\varphi(u) \geq 3$, for every u of type $(3,2)$ in P', $a_\varphi(u) \geq 4$, for every u of type $(5,2)$ in P', $a_\varphi(u) \geq 5$, for every u of type $(4,2)$ in P', and $a_\varphi(u) \geq 3$, for every u of type $(6,2)$ in P'. Without loss of generality, assume that all these inequalities are equalities.

The proof here follows from induction on the number of type $(4,2)$ vertices in P'. If P' has no type $(4,2)$ vertex, that is, $k_{m+1}-k_m-1=0$, then it can be written as $P' = (w,v_1,\cdots,v_r,...v_j)$ such that $a_\varphi(w)=a_\varphi(v_r)=a_\varphi(v_j)=3$ and, for every other vertex v_p, we have $a_\varphi(v_p)=4$. Let $\ell_{wv}:\{w,v_1,\cdots,v_r\}\to\{1,\cdots,7\}$ be such that $\ell_{wv}(v_r)=2=a_\varphi(v_r)-1$ and $\ell_{wv}(u)=a_\varphi(u)$ for the remaining vertices. By Proposition 4.6, if $(w,v_1,\cdots,v_r)$ is ℓ_{wv}-colorable, then $P'=(w,v_1,\cdots,v_{r-1}\circ(v_r)\circ(v_{r+1},...,v_j)$ is a_φ-colorable. Additionally, let $\ell_w:\{w\}\to\{1,\cdots,7\}$ be such that $\ell_w(w)=1=\ell_{wv}(w)-2$. By Proposition 4.7, $(w,v_1,\cdots,v_r)$ is ℓ_{wv}-colorable as (w) is clearly ℓ_w-colorable. Therefore, if P' has no type $(4,2)$ vertex, then it is a_φ-colorable.

Now suppose otherwise and let $w \in V(P_{k_{m+1}-1})$ be a vertex of type $(4,2)$. We write P' as $P' = Q'\circ(w)\circ(v_1,\cdots,v_r,...,v_j)$ such that $a_\varphi(w)=a_\varphi(v_r)=a_\varphi(v_j)=3$ and, for every other vertex v_p, we have $a_\varphi(v_p)=4$. Notice that if $\ell_w:V(Q')\cup\{w\}\to\{1,\cdots,7\}$ is such that $\ell_w(w)=a_\varphi(w)-2$ and $\ell_w(u)=a_\varphi(u)$, for the remaining vertices, then $(Q'\circ(w),Q'\circ(w))$, with list assignment ℓ_w, is equivalent to $(Q'\circ(w^*),Q'\circ(w^*))$, with list assignment ϕ^*, where w^* is a type $(3,2)$ vertex and $\phi^*(v)=\ell_w(v)$, for every $v\in V(Q')$. Therefore, we just have to prove that $Q'\circ(w)$ being ℓ_w- colorable implies that P' is a_φ-colorable.

Observe that $a_\varphi(w)=5$, $a_\varphi(v_r)=a_\varphi(v_j)=3$ and for every other vertex v_p we have $a_\varphi(v_p)=4$. Let $\ell_{wv}:V(Q')\cup\{w,v_1,\cdots,v_r\}\to\{1,\cdots,7\}$ be such that $\ell_{wv}(v_r)=2=a_\varphi(v_r)-1$ and $\ell_{wv}(u)=a_\varphi(u)$ for the remaining vertices. By Proposition 4.6, if $Q'\circ(w,v_1,\cdots,v_r)$ is ℓ_{wv}-colorable, then $P'=Q'\circ(w,v_1,\cdots,v_{r-1}\circ(v_r)\circ(v_{r+1},...,v_j)$ is a_φ-colorable. Additionally, by Proposition 4.7, $Q'\circ(w,v_1,\cdots,v_r)$ is ℓ_{wv}-colorable, since $Q'\circ(w)$ is ℓ_w-colorable. Therefore,

P' is a_φ-colorable.

Thus, by assuming $\sum_{i=k_m+1}^{k_{m+1}} (\omega(I_i)+\omega(P_i)) = -1$, we get a contradiction. Therefore $\sum_{i=k_m+1}^{k_{m+1}} (\omega(I_i)+\omega(P_i)) \geq 0$. ■

Demonstration of Lemma 4.4. Equation 4.4, that is, $\omega(P) = \omega(P_0) + \sum_{k=1}^{j} (\omega(I_k)+\omega(P_k))$ can be rewritten as:

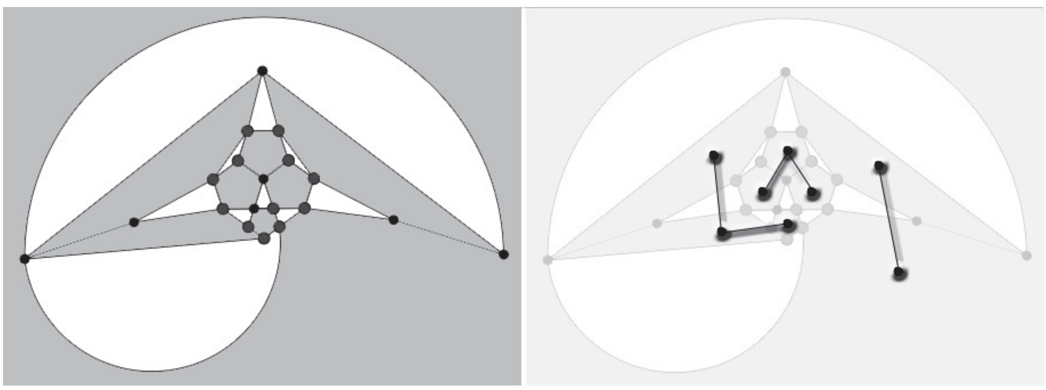
$$\omega(P_0) + \left(\sum_{m=0}^{\lambda-1} \sum_{i=k_m+1}^{k_{m+1}} (\omega(I_i)+\omega(P_i)) \right) + \sum_{i=k_\lambda+1}^{j-1} (\omega(I_i)+\omega(P_i)) + \omega(I_j) + \omega(P_j).$$

By Lemma 4.5, $\left(\sum_{m=0}^{\lambda-1} \sum_{i=k_m+1}^{k_{m+1}} (\omega(I_i)+\omega(P_i)) \right) \geq 0$. Also, by the definition of the numbers $k_0, k_1, \ldots, k_\lambda$, it holds that $\sum_{k=k_\lambda+1}^{j-1} (\omega(I_k)+\omega(P_k)) \geq 0$, as there is no P_k such that $\omega(P_k) = -2$, for $k > k_\lambda$. Consequently, we have that $\omega(I_j)+\omega(P_j) > 0$, as $\omega(P_j) = 0$. Therefore, Lemma 4.4 follows. ■

4.2.2 Islands

In Araujo et al. (ARAUJO *et al.*, 2018), a theorem similar to Theorem 1.1 is proven for planar graphs without C_4 and C_5. Here we drop the "no C_5" condition, we assume the existence of a counterexample for our theorem, then we prove two inequalities that contradict each other, denying the existence of a counterexample. In order to do that, we define islands as shown in the sequel, but first notice that since G contains no cycle of size four, then no pair of triangles can share an edge.

Figure 12 – Islands highlighted on the graph on the right



Fonte: elaborado pelo autor.

Let G be a plane graph and G^* be its dual. Let $G^*_{5+} \subseteq G^*$ be the subgraph of G^* induced by the vertices f of G^* with degree at least five. For some component J^* of G^*_{5+}, let

C be the set of vertices of G contained in some face corresponding to a vertex in J^*. We call the subgraph $J = G[C]$ an *island*. In Figure 12a, we see an example of a graph and its islands. We call J a *bad island* if J^* is a tree and every face of J corresponding to a vertex in J^* is a C_5. Otherwise, we call it a *good island*. Observe that if J is a good island, then at least one of the following conditions apply:

1. There is a cycle in J^*; or
2. There is a face $f \in F(J)$ such that $d_G(f) \geq 6$.

Let us prove a couple of propositions concerning the number of bad islands that a planar graph with no C_4 may have.

Proposition 4.12 *Let $G = (V(G), E(G))$ be a plane graph without C_4 and γ be the number of bad islands in G, then:*

$$3F_3 + F_5 \leq |E(G)| + \gamma. \tag{4.7}$$

Demonstration. We prove this by double counting edges in G. The main idea is to count the edges that lie in a face of size three, then, for each face of size five, count one edge that have not been counted on the triangles.

As G is a graph without C_4, then there are no two faces of size three sharing an edge. Therefore, we may count one edge for each face of size three without counting the same edge twice, so there are $3F_3$ edges on the triangles of G.

Let $J \subseteq G$ be any island and let $F_5(J)$ be the number of faces of degree five in J. Also, let J^* be the component of G^*_{5+} corresponding to J. Notice that the number of edges in J that are not in a triangle is equal to the number of edges in J^*. As J^* is connected, then it has at least $|V(J^*)| - 1 = F_{5+}(J) - 1$ edges.

If J is a good island, we split into two cases:

1. If J^* has a cycle, then $|E(J^*)| \geq |V(J^*)| = F_{5+}(J) \geq F_5(J)$;
2. if J has a face f such that $d_G(f) \geq 6$, then $F_{5+}(J) > F_5(J)$ (or in other words $F_{5+}(J) - 1 \geq F_5(J)$). We conclude that $|E(J^*)| \geq |V(J^*)| - 1 = F_{5+}(J) - 1 \geq F_5(J)$.

If J is a bad island, we only know that $|E(J^*)| \geq F_{5+}(J) - 1 \geq F_5(J) - 1$.

Let $\mathfrak{B}$ be the set of bad islands and $\mathfrak{G}$ be the set of good islands of G. Since, by definition of island, each face of degree 5 is in exactly one island, then:

$$\begin{aligned}F_5 &= \sum_{J\in\mathfrak{G}} F_5(J) + \sum_{J\in\mathfrak{B}} F_5(J)\\ &\leq \sum_{J\in\mathfrak{G}} |E(J^*)| + \sum_{J\in\mathfrak{B}} \left(|E(J^*)|+1\right)\\ &= \gamma + \sum_{J\in\mathfrak{G}\cup\mathfrak{B}} |E(J^*)|.\end{aligned}$$

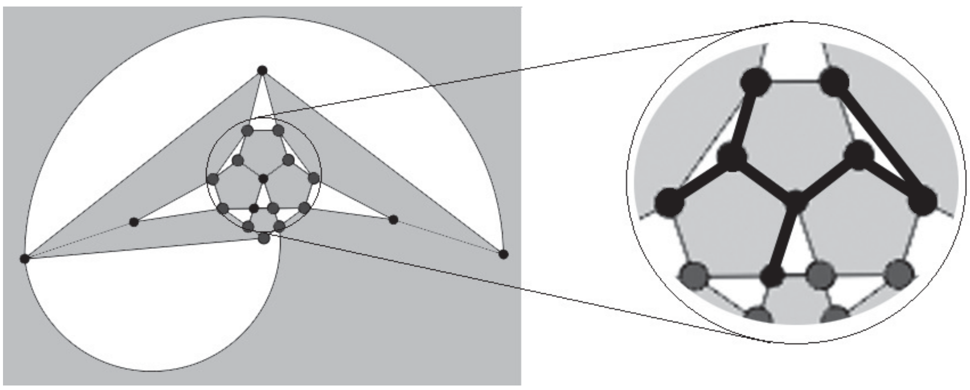
In $|E(J^*)|$ we are counting only edges between faces of length greater than or equal to five in an island, therefore no edge from any triangle is counted $|E(J^*)|$. Thus $3F_3 + \sum_{J\in\mathfrak{G}\cup\mathfrak{B}} |E(J^*)| \leq |E(G)|$. As a result we have:

$$3F_3 + F_5 \leq 3F_3 + \gamma + \sum_{J\in\mathfrak{G}\cup\mathfrak{B}} |E(J^*)| \leq |E(G)| + \gamma.$$

■

Let G be a plane graph without C_4 with γ bad islands. For each $v \in V(G)$, we name $\gamma(v)$ the number of bad islands which contain the vertex v. Similarly, if $H \subseteq G$, we name $\gamma(H)$ the number of bad islands which contain some vertex of H.

Figure 13 – Visualization of the inequalities



Fonte: elaborado pelo autor.

Proposition 4.13 *Let G be a plane graph without C_4, $w \in V(G)$ and $T \subseteq G$ a tree such that $|V(T)| = t$, then:*

$$\gamma(w) \leq \left\lfloor \frac{d_G(w)}{2} \right\rfloor \text{ and } \gamma(T) \leq 1 - t + \frac{1}{2} \sum_{v\in V(T)} d_G(v). \tag{4.8}$$

Demonstration. For the first inequality, $\gamma(w) \leq \left\lfloor \frac{d_G(w)}{2} \right\rfloor$, it is sufficient to notice that w is incident to $d_G(w)$ faces. As two adjacent faces cannot be both from different islands, then there are at most $\left\lfloor \frac{d_G(w)}{2} \right\rfloor$ islands containing w.

Thus, we can count, for each vertex $v \in T$, at most $\left\lfloor \frac{d_G(v)}{2} \right\rfloor \leq \frac{d_G(v)}{2}$ different islands. Hence, in T there are at most $\frac{1}{2}\sum_{v\in V(T)} d_G(v)$ islands incident to T.

However, every edge in T belongs to at most one island, as both sides of the edge cannot be part of different islands. So when we count $\frac{1}{2}\sum_{v\in V(T)} d_G(v)$ islands, we are counting every island incident to the edges of T twice. To take that into account, we remove $t-1$ from $\frac{1}{2}\sum_{v\in V(T)} d_G(v)$, resulting in $\gamma(T) \leq -(t-1) + \frac{1}{2}\sum_{v\in V(T)} d_G(v)$. ■

4.2.3 Tool Lemmas

Now that the reader is familiar with the notion of islands and Lemma 4.4, we are ready to present the lemmas that lead us to the contradiction we get when assuming the existence of a minimal pair.

Lemma 4.6 *Let (G,F) be a minimal pair such the plane representation of G has γ bad islands. The following inequality holds:*

$$|E(G)| \geq 2|V(G)| + \frac{\gamma}{3}. \tag{4.9}$$

Demonstration. For any P component of F, we have by Proposition 4.13 that:

$$\gamma(P) \leq -(|V(P)|-1) + \frac{1}{2}\sum_{v\in V(P)} d_G(v).$$

Then:

$$1 \geq \gamma(P) + |V(P)| - \frac{1}{2}\sum_{v\in V(P)} d_G(v).$$

On the other hand, Lemma 4.4 states that:

$$\sum_{v\in V(P)} d_G(v) \geq 5|V(P)| + 1$$

$$\sum_{v\in V(P)} d_G(v) \geq 5|V(P)| + \gamma(P) + |V(P)| - \frac{1}{2}\sum_{v\in V(P)} d_G(v)$$

$$2\sum_{v\in V(P)} d_G(v) \geq 10|V(P)| + 2\gamma(P) + 2|V(P)| - \sum_{v\in V(P)} d_G(v)$$

$$3\sum_{v\in V(P)} d_G(v) \geq 12|V(P)| + 2\gamma(P)$$

Because two components of the spanning subgraph F do not share any vertex, let $\mathscr{P}$ be the set of components of F. It follows that:

$$2|E(G)| = \sum_{v \in V(G)} d_G(v) = \sum_{P \in \mathscr{P}} \sum_{v \in V(P)} d_G(v).$$

Also, since F is a spanning forest, we get:

$$\sum_{P \in \mathscr{P}} \gamma(P) \geq \gamma \text{ and } \sum_{P \in \mathscr{P}} |V(P)| \geq |V(G)|.$$

Summing up $3\sum_{v \in V(P)} d_G(v) = 12|V(P)| + 2\gamma(P)$ for all components, we get $6|E(G)| \geq 12|V(G)| + 2\gamma$ and the lemma follows. ■

Lemma 4.7 *Let (G,F) be a minimal pair, such that G has γ bad islands. The following inequality holds:*

$$|E(G)| \leq 2|V(G)| - 4 + \frac{\gamma}{3}. \tag{4.10}$$

Demonstration. Recall that $F(G)$ is the set of faces of the plane graph G. It holds that $2|E(G)| - 6|F(G)| = \sum_{f \in F(G)} d_G(f) - \sum_{f \in F(G)} 6$. As G does not contain a leaf, $G \neq K_{1,2}$, then a face f has degree four only if that face is a C_4. So $\sum_{f \in F(G)} (d_G(f) - 6) \geq -3F_3 - F_5 \geq -|E(G)| - \gamma$ by the Proposition 4.12. Therefore, by applying Euler's formula to the left-hand side, $2|E(G)| - 6(2 - |V(G)| + |E(G)|) \geq -|E(G)| - \gamma$, which implies that $|E(G)| \leq 2|V(G)| - 4 + \frac{\gamma}{3}$. ■

Demonstration of Theorem 1.1. By contradiction, assume the Theorem 1.1 is not true. Therefore, there exists pair a (G,F) such that $CBC_2(G,F) > 7$. Without loss of generality, consider (G,F) as a minimal pair. By Lemma 4.7, we have $|E(G)| \leq 2|V(G)| - 4 + \frac{\gamma}{3}$, whereas Lemma 4.6 states that $|E(G)| \geq 2|V(G)| + \frac{\gamma}{3}$. Therefore, we get the following contradiction:

$$2|V(G)| + \frac{\gamma}{3} \leq 2|V(G)| - 4 + \frac{\gamma}{3} \Rightarrow 0 \leq -4.$$

We conclude that there cannot exist such pair (G,F). ■

5 CONCLUSION

Inspired by previous works done by Broersma et al. (BROERSMA *et al.*, 2007) and Araujo et al. (ARAUJO *et al.*, 2018) we were able to present a proof for a particular case of an open problem they proposed, that we state in the following theorem:

Theorem 5.1 *Let G be a planar graph without cycles of size four and let F be an spanning induced path forest of G, then* $CBC_2(G,F) \leq 7$.

Even though the result we prove seems like a small improvement to the one presented by Araujo et al., our approach required a more detailed study on the properties of a minimal counterexample. This indicates the level of complexity needed to prove Conjecture 3.1.

In our proof, we have $q = 2$. But, as presented in Chapter 3, some authors work with other possible values of q. Therefore, we ask the following question:

Question 5.1 *For any planar graph G without cycles of length four or five, F a spanning induced path forest of G and* $q \geq 2$ *an integer, is it true that* $CBC_q(G,F) \leq 4q-1$*? What if we drop the "no* C_5*" condition?*

One thing we strongly need in our proof is that each component of the subgraph F was an induced subgraph of G. This was needed for the Propositions 4.3, 4.4 and 4.5, as we exclude some available colors for w in a way that we are still able to color the vertex v. But if v was adjacent to other vertices of the component, then the induction argument could not be used. The following question arises:

Question 5.2 *Is it possible to extend the results in Propositions 4.3, 4.4 and 4.5 to non-induced subgraphs of G?*

As for the problem itself, we notice that it is closely related to the problem studied by Araujo (ARAUJO *et al.*, 2018). In their work, they showed that if G is a planar graph without cycles of size 4 and 5, then $CBC_2(G,T) \leq 7$, for any spanning forest T of G. Aiming to generalize their results, and ours, it is natural to ask what would happen if T were a spanning forest of G:

Question 5.3 *Let G be a planar graph without* C_4 *and T be a spanning induced forest of G, does* $CBC_2(G,T) \leq 7$ *still hold?*

Finally, inspired by the works of Bu et al. (BU; BAO, 2015; BU; LI, 2011; BU *et al.*, 2013; BU; ZHANG, 2011; BU *et al.*, 2012), we were able to generalize the results they obtained, avoiding the restrictions imposed on the graph G, but still with the backbone restricted to a spanning tree or its subgraphs. Therefore, the next step is to research similar problems with cyclic graphs. The following question arises:

Question 5.4 *If G is a cyclic connected graph, q is an integer greater than or equal to 2 and $k \geq \max\{\chi(G), \lceil \frac{\chi(G)}{2} \rceil + q\}$, does there exist a connected spanning cyclic subgraph H of G such that* $\text{girth}(G) \leq \text{girth}(H)$ *and (G,H) is q-backbone k-colorable?*

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