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RENAN DE CASTRO SILVA CORDEIRO

TEMPORAL ABSTRACT ARGUMENTATION FRAMEWORKS

FORTALEZA

2023

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Undergraduate Thesis presented to the Graduation Course in Computer Science of the Science Center of the Federal University of Ceará, as a partial requirement for obtaining the degree of Bachelor of Computer Science.

Supervisor: Prof. Dr. João Fernando Lima Alcântara.

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To my family, thank you for your unwavering support during this academic endeavor.

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RESUMO

Para que os processos de tomada de decisão sejam automatizados em sistemas críticos do mundo real, agentes autônomos não apenas precisam ser poderosos e eficientes, mas também transparentes, responsáveis, imparciais, explicáveis, éticos e capazes de lidar com informações inconsistentes, não confiáveis e incompletas. Um formalismo possivelmente adequado para modelar o raciocínio com essas características é o Framework de Argumentação Abstrata, no qual argumentos sem estruturas detalhadas são avaliados unicamente com base em como interagem entre si. Justificativas a favor ou contra afirmações podem ser obtidas de maneira natural, seguindo um formato semelhante ao discurso humano. A expressividade desses frameworks foi expandida para incorporar várias noções, como suporte, preferências e pesos. O aspecto temporal inerente ao raciocínio humano levou ao desenvolvimento de frameworks temporais nos quais a aceitabilidade e as interações dos argumentos são relativas ao tempo. As abordagens atuais para incluir o tempo na argumentação permitem modelar a disponibilidade de argumentos ou de ataques, mas interações temporais mais gerais entre os argumentos não foram consideradas nos frameworks tradicionais de argumentação abstrata. Este trabalho revisa as principais abordagens de incorporação de tempo à argumentação na literatura e discute como uma abordagem de tradução simples é capaz de representar interações temporais variadas.

Palavras-chave: argumentação; temporalidade; explicabilidade; representação do conhecimento; inteligência artificial.

ABSTRACT

For decision-making processes to be automated in critical real-world systems, autonomous agents must not only be powerful and efficient, but transparent, accountable, unbiased, explainable, ethical, and capable of handling inconsistent, unreliable and incomplete information. A possibly suitable formalism for modeling reasoning with such features is the Abstract Argumentation Framework, in which arguments without intricate structures are evaluated solely based on their interaction with each other. Justifications for or against claims can be naturally obtained in a format similar to human discourse. The expressiveness of these frameworks has been expanded to incorporate various notions such as support, preferences and weights. The inherent temporal aspect in human reasoning led to the development of temporal frameworks in which arguments' acceptability and interactions are relative to time. Current approaches of including time in argumentation allow for modeling argument or attack availability, but general temporal interactions between arguments have not been considered in traditional abstract argumentation frameworks. This work reviews the main approaches of featuring time in argumentation from the literature and discusses how a simple translation approach is able to represent variegated temporal interactions.

Keywords: argumentation; temporality; explainability; knowledge representation; artificial intelligence.

LIST OF FIGURES

LIST OF SYMBOLS

$\mathfrak{A}$	Abstract Argumentation Framework
$\mathscr{A}$	Set of arguments
Att	Attack relation
S^+	Set of arguments defeated by S
$\emptyset$	Empty set
in	Accepted
out	Rejected
undec	Undecided
$\mathscr{L}$	Labelling
$\mathbb{R}^+$	Set of nonnegative real numbers
Δ	Timed Argumentation Framework
δ	Availability function
$\mathscr{T}_A$	Time interval of A
$\mathscr{T}^B_{(A\|S)}$	Interval of defense of A from B by S
$\subseteq_t$	Set-inclusion relation for t-profiles
Φ	Collection of propositional formulas
t	True
f	False
u	Undecided
$\mathscr{V}_2^D$	Set of two-valued interpretations for D
$\mathscr{V}_3^D$	Set of three-valued interpretations for D
$\leq_i$	Information order
$[v]_2^D$	Set of two-valued completions of v
$\wedge_i$	Consensus operator
Γ_D	Characteristic Function for ADF semantics
2^S	Power set of S

$\mathfrak{D}$	Timed Abstract Dialectical Framework
$\mathfrak{T}$	Temporal Abstract Argumentation Framework
$\mathscr{T}$	Set of time points
$\mathbb{N}$	Set of natural numbers
$\mathfrak{F}_{\mathfrak{T}}$	Characteristic Function for TeAF semantics
Sup	Support relation
β	Beta
$\mathfrak{S}$	Reflexive and transitive closure of Sup
$\blacktriangleright$	Attack relation for SETAFs

CONTENTS

1 INTRODUCTION

Argumentation allows reasoning about incomplete or unreliable knowledge in which arguments for and against claims interact. It has been found useful for a plethora of areas, such as Ambient Intelligence (Oguego *et al.* (2018)), robotics, law, medicine and more generally Explanable Artificial Intelligence (XAI) (Vassiliades *et al.* (2021)).

Abstract Argumentation Frameworks (AFs) proposed by Dung (1995) are a major formalism in formal argumentation due to its simplicity in representing arguments without structure and evaluating them solely based on their attack relation with each other. However, the task of instantiating abstract frameworks is not a trivial one, as multiple kinds of interactions between arguments are present in real world discourses. For this reason many proposals extend AFs' expressiveness by including notions such as support (Cayrol and Lagasquie-Schiex (2013)), preferences (Amgoud and Cayrol (2013)), probability (Li *et al.* (2011)), argument or attack weights (Amgoud *et al.* (2017), Coste-Marquis *et al.* (2012)), labels (Budán *et al.* (2015)), joint attacks (Flouris and Bikakis (2019a)), higher-order (Barringer *et al.* (2005)) or recursive attacks (Baroni *et al.* (2011)), and time availability (Cobo *et al.* (2010)).

The latter notion led to the definition of Timed Argumentation Frameworks (TAFs) (Cobo *et al.* (2010), Cobo *et al.* (2011), Budán *et al.* (2015)), in which some arguments may not be available for reasoning at some time intervals; and Temporal Probabilistic Abstract Argumentation Frameworks (TPAFs) (Bistarelli *et al.* (2023b), Bistarelli *et al.* (2023a)), in which this availability is expressed by probability distributions.

Time is a natural factor of argumentation in many orthogonal perspectives. Arguments' strength may vary with respect to the passage of time, or time may be treated as an object of discourse, susceptible to be reasoned about. Some of these aspects may be better understood with Example 1, illustrated by Figure 1 in which circles represent arguments and solid arrows represent attacks.

Example 1. *Bob must follow a medical prescription that requires him to take medication every other day. Due to its drowsiness-inducing effects, he is unable to drive on the days he takes it. Consider time in daily granularity starting from* 0*. The following arguments are presented:*

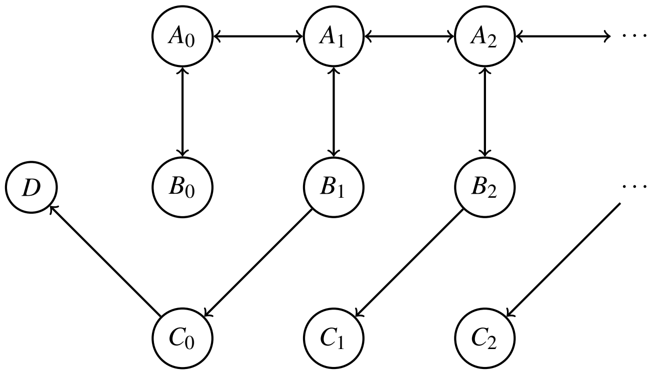
a) Bob took a medication that induces drowsiness today (argument A);

b) Bob doesn't feel sleepy today (argument B);

c) Bob won't be able to drive tomorrow (argument C);

d) *Bob is able to attend some specific party occuring at day* 1 *by driving (argument D).*

Figure 1 – Argumentation framework from Example 1



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The index subscripting each argument indicates the time point to which the corresponding argument is linked. For instance, A_3 represents argument A at time point 3. Each attack encodes a particular information. For example, the attack from A_0 to A_1 encodes the fact that if A_0 is accepted (Bob took medication at time point 0), then he will not take medication at time point 1 and therefore A_1 is rejected. Arguably, other attacks could be included, such as attacks from B to itself in consecutive points in time, mirroring A's behavior. Nevertheless, the attacks in Example 1 are sufficient to highlight some temporal aspects:

a) arguments can be linked to a relative point in time, such as today and tomorrow, or to an absolute point in time, such as the date of a specific party;
b) arguments linked to relative points in time can be accepted in some time instances, but rejected in others;
c) arguments accepted in some point in time can be used to justify or deny the acceptance of arguments (even itself) at a different time.

An important remark is that when arguments are accepted at a specific time point, the relative time reference "today" refers to such time point. For instance, accepting C_1 means that "Bob won't be able to drive tomorrow" is accepted at time point 1, which in turn means that "Bob won't be able to drive" at time point 2. Also, argument D is not linked to a relative time point, as it does not refer to relative dates, such as "today" or "tomorrow".

The acceptance or rejection of arguments in an argumentation framework naturally comes with justification. For example, accepting C_1 is justified by rejecting B_2, which in turn can be explained by the acceptance of A_2 and so on.

Four kinds of interactions were shown in Figure 1:

a) the static and mutual attack between A_0 and B_0 is the traditional notion of attack at a fixed time point, encoding the information that the medication makes Bob feel sleepy;

b) the past relative attack of A_0 over A_1 is a kind of temporal interaction, because if Bob took medication at time point 0, then he will not take medication at the next time point, assuming the medical prescription is being correctly followed;

c) the future relative attack of B_1 over C_0 is explained by C_0's use of a future time reference ("tomorrow") and encodes the information that Bob cannot drive (at a time point) if feeling sleepy (at that same time point);

d) the absolute attack of C_0 over D encodes the information that if Bob cannot drive at time 1 (acceptance of C_0), then Bob will not be able to attend the party (rejection of D) scheduled for that same time point.

Note that arguments B and C in Example 1 use negation in their text. If B were "Bob feels sleepy today", then the acceptance of A would lead to the acceptance of B. That is a positive interaction between arguments and can be modeled by using supports. The following example uses a support relation between arguments.

Example 2. *Continuing from Example 1, add argument E: "Bob took a medication that induces drowsiness this week" and assume day* 0 *is the first day of the week. This is illustrated by Figure 2, in which dashed edges represent support. Obviously, if Bob took medication on any day of the current week, then he took medication on that same week. Hence, arguments $A_0, \cdots, A_6$ support $E_0, \cdots, E_6$; arguments $A_7, \cdots, A_{13}$ support $E_7, \cdots, E_{13}$; and so on. Note that $E_0, \cdots, E_6$ possess the same meaning and, therefore, only E_0 is shown. Although the precise interpretation for support is not yet given, two of them will be presented later. Without considering a specific interpretation for support, it is reasonable to include the support from E_0 to $A_0, \cdots, A_6$, as accepting that Bob took medication this week contributes to accepting (despite not being a sufficient reason) that Bob took medication at a specific day of this week.*

Instead of using positive interactions, a similar framework can also be properly encoded by using joint attacks, as shown by the next example. A joint attack from a set of arguments S to an argument A indicates that A should be rejected if every argument in S is accepted.

Figure 2 – Argumentation framework from Example 2 by using support

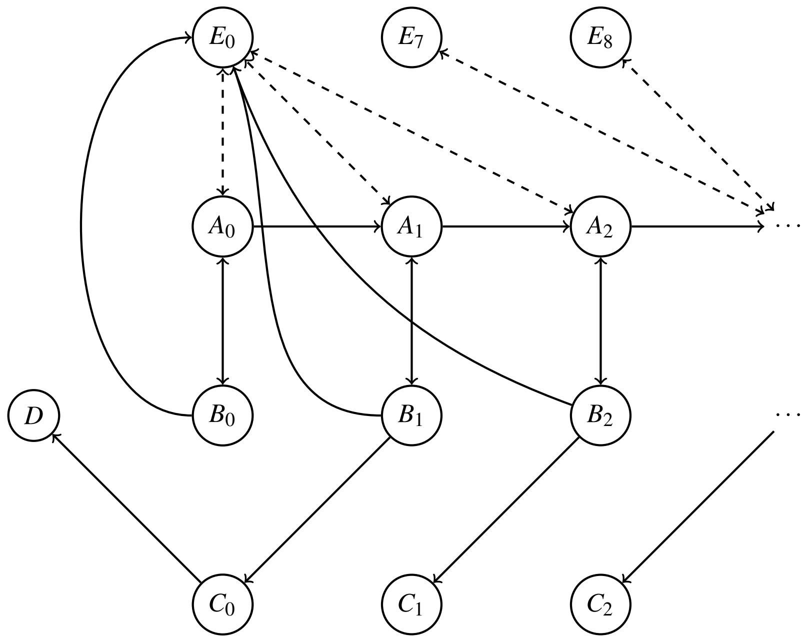

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Figure 3 – Argumentation framework from Example 2 by using joint attacks

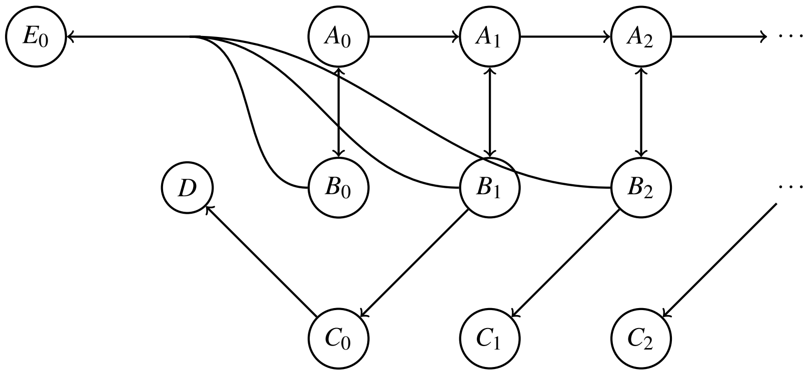

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Figure 3 illustrates this alternative. The joint attack from $S = \{B_0, \cdots, B_6\}$ to E_0 encodes the information that E_0 must be rejected if every argument in S is accepted, i.e., E_0 must be rejected if every argument $\{A_0, \cdots, A_6\}$ is rejected. That way, "Bob did not take medication this week" (rejection of E_0) if "Bob did not take medication today" (rejection of A_i) for every day $0 \leq i \leq 6$ of this week. The opposite direction is encoded by the attacks from E_0 to each $B_0, \cdots, B_6$, i.e., if E_0 is accepted, then $B_0, \cdots, B_6$ are rejected. Section 4 discusses expressiveness and limitations from the approaches briefly introduced above.

Although TAFs and TPAFs can represent time availability, argument acceptability in a particular time point t is determined exclusively by the available arguments at t, and more general temporal interactions cannot be directly expressed, such as the last three of those discussed above. For instance, the attack from A_0 to A_1 in Example 1 is an attack from an argument to itself in consecutive points in time. It is reasonable to accept A_0 and reject A_1, as these acceptability statuses refer to two distinct time instants. In a TAF, an argument attacking itself is not accepted in any time point, as every attack from A to B indicates that if A is accepted at a time point t, then B is rejected at this same time point.

The paper Brewka and Woltran (2010) introduces Abstract Dialetical Frameworks (ADFs), a powerful generalization of AFs in which complex interactions can be expressed. Whereas arguments may interact only by an attack relation in AFs, ADFs associate each argument A with a propositional formula expressing an acceptance condition for A that can depend on the acceptabilities of other arguments in the framework.

Recently, ADFs were extended by a new formalism, called Timed Abstract Dialetical Frameworks (tADFs) Prakken *et al.* (2020), capable of handling acceptance conditions changing over time, such as those expressed in Figures 1, 2 and 3, through the use of temporal propositional formulas. However, it is also reasonable to include this notion of time in less generic frameworks and see the limitations of expressiveness of different formalisms with respect to encoding distinct notions of temporal interactions. Temporal approaches for AFs, Bipolar Abstract Argumentation Frameworks (BAFs) and Frameworks with Sets of Attacking Arguments (SETAFs) (respectively from Cobo *et al.* (2011), Budán *et al.* (2017), Zhu (2020)) focuses strictly in time availability, without considering the more general temporal interactions discussed so far.

Hence, the objectives of this work are twofold:

a) to apply a simple translation approach for encoding temporal interactions in AFs, BAFs and SETAFs and investigate their limitations;
b) to show semantic relationships between the simple translation approach and other timed formalisms, such as TAFs and tADFs;

In the next section, some fundamental background definitions and related formalisms are introduced. In Section 3, AFs and its semantics are used to encode time. Next, time is included in Section 4 by using frameworks with support (BAFs) and frameworks with joint attacks (SETAFs). Timed formalisms related to the simple translation approach used in this work are presented and compared in Section 5. At last, the conclusion in Section 6 discusses future investigations and reviews the main contributions of this work.

2 BACKGROUND

In this chapter, the fundamental and simpler formalism upon which many others were built is introduced. Then, an approach for including temporality as availability is presented through Timed Abstract Argumentation Frameworks (TAFs). Later, Abstract Dialetical Frameworks (ADFs) and its timed version (tADFs) are described.

2.1 Abstract Argumentation Frameworks

In his seminal paper Dung (1995), Dung takes argumentation to its simplest form. Arguments have no structure and attacks are the only way arguments can interact with each other.

Definition 1. *An Abstract Argumentation Framework (*AF*) is a tuple* $\mathfrak{A} = (\mathscr{A}, Att)$*, where* $\mathscr{A}$ *is an enumerable set of arguments and* $Att \subseteq \mathscr{A} \times \mathscr{A}$ *is an attack relation. The attackers of an argument* $A \in \mathscr{A}$ *are denoted by the set* $Att(A) = \{B \mid (B, A) \in Att\}$.

Despite its simplicity, many interesting results follow. Reasoning about which arguments should be collectively accepted under some specific criteria can be determined exclusively by the relationship between arguments. Each criteria is called a *semantics* and a set of accepted arguments under a semantics is called an *extension*.

For a set of arguments to be collectively accepted, it must satisfy a property of internal coherence, called *conflict-freeness*, which ensures there is no attack between accepted arguments.

Definition 2 (Conflict-free sets)**.** *Let* $\mathfrak{A} = (\mathscr{A}, Att)$ *be an* AF*. A set* $S \subseteq \mathscr{A}$ *is conflict-free in* $\mathfrak{A}$ *iff there is no* $A, B \in S$ *such that* $(A, B) \in Att$.

Additionally, as arguments are collectively (and not individually) accepted, it is natural to extend the notion of attack to sets of arguments.

Definition 3 (Defeat in $\mathfrak{A}$)**.** *Let* $\mathfrak{A} = (\mathscr{A}, Att)$ *be an* AF*. A set* $S \subseteq \mathscr{A}$ *defeats* $A \in \mathscr{A}$ *iff there is* $B \in S$ *such that* $(B, A) \in Att$*. The set of all defeated arguments by* S *is denoted by* S^+.

The notion of defense stems from that of defeat. An argument must not be rejected due to attackers if there is a counterargument for every attacker.

Definition 4 (Defense in $\mathfrak{A}$)**.** *Let* $\mathfrak{A} = (\mathscr{A}, Att)$ *be an* AF*. A set* $S \subseteq \mathscr{A}$ *defends* $A \in \mathscr{A}$ *iff for every* $B \in Att(A)$ *it holds* S *defeats* B.

Extensions are derived from the concept of defense. At minimum, it is required that an extension does not defeat itself (i.e., it is conflict-free) and that it defends each of its arguments.

Definition 5 (Extension-based semantics of $\mathfrak{A}$)**.** *Let* $\mathfrak{A} = (\mathscr{A}, Att)$ *be an* AF. *A set* $S \subseteq \mathscr{A}$ *is:*

a) admissible if S is conflict-free and defends every argument in S;

b) complete if S is admissible and contains every defended argument;

c) grounded if S is $\subseteq$*-minimal complete;*

d) preferred if S is $\subseteq$*-maximal complete;*

e) semi-stable if S is complete with $\subseteq$*-maximal* $S \cup S^+$*;*

f) stable if S is conflict-free and $S \cup S^+ = \mathscr{A}$*.*

Although the definitions above are not the ones initially used by Dung, they have been shown to coincide (see Baroni *et al.* (2018)). Each semantics provides a reasonable criteria for determining sets of accepted arguments. The following example should bring clarity to their differences.

Example 3. *Let* $\mathfrak{A} = (\mathscr{A}, Att)$ *be the* AF *depicted in Figure 4 such that* $\mathscr{A} = \{A, B, C, D, E\}$ *and* $Att = \{(A,B), (A,D), (B,A), (D,C), (D,D), (E,E)\}$. $\mathfrak{A}$*'s extensions are computed as follows:*

a) $\emptyset, \{A\}, \{B\}, \{A,C\}$ *are admissible;*

b) $\emptyset, \{B\}, \{A,C\}$ *are complete;*

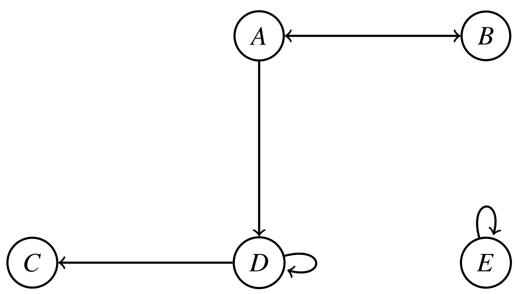
c) $\emptyset$ *is grounded;*

d) $\{B\}, \{A,C\}$ *are preferred;*

e) $\{A,C\}$ *is semi-stable;*

f) no set of arguments is stable.

Figure 4 – Argumentation framework from Example 3



Source: This author.

An equivalent and popular characterization from Caminada (2006) for defining these semantics uses three-valued labellings, in which arguments are explicitly labelled as **in** (accepted), **out** (rejected) or **undec** (undecided).

Definition 6 (Labelling of $\mathfrak{A}$)**.** *Let* $\mathfrak{A} = (\mathscr{A}, Att)$ *be an* AF. *A labelling of* $\mathfrak{A}$ *is a function* $\mathscr{L} : \mathscr{A} \to \{\mathbf{in}, \mathbf{out}, \mathbf{undec}\}$.

A labelling $\mathscr{L}$ is said to be conflict-free if there are no arguments $A, B \in \mathscr{A}$ such that $\mathscr{L}(A) = \mathscr{L}(B) = \mathbf{in}$ and $(A, B) \in Att$.

Definition 7 (Admissible labelling of $\mathfrak{A}$)**.** *A labelling* $\mathscr{L}$ *of an* AF $\mathfrak{A} = (\mathscr{A}, Att)$ *is admissible if for any* $A \in \mathscr{A}$*:*

a) if $\mathscr{L}(A) = \mathbf{in}$*, then for every* $B \in Att(A)$ *it holds* $\mathscr{L}(B) = \mathbf{out}$*;*

b) if $\mathscr{L}(A) = \mathbf{out}$*, then there exists* $B \in Att(A)$ *such that* $\mathscr{L}(B) = \mathbf{in}$*.*

Definition 8 (Complete labelling of $\mathfrak{A}$)**.** *A labelling* $\mathscr{L}$ *of an* AF $\mathfrak{A} = (\mathscr{A}, Att)$ *is complete if for any* $A \in \mathscr{A}$*:*

a) $\mathscr{L}(A) = \mathbf{in}$ *iff for every* $B \in Att(A)$ *it holds* $\mathscr{L}(B) = \mathbf{out}$*;*

b) $\mathscr{L}(A) = \mathbf{out}$ *iff there exists* $B \in Att(A)$ *such that* $\mathscr{L}(B) = \mathbf{in}$*.*

Each labelling $\mathscr{L}$ partitions $\mathscr{A}$ into three sets of respectively accepted, rejected and undecided arguments, denoted as $\mathbf{in}(\mathscr{L}), \mathbf{out}(\mathscr{L})$ and $\mathbf{undec}(\mathscr{L})$. It is convenient to denote a labelling $\mathscr{L}$ by $(\mathbf{in}(\mathscr{L}), \mathbf{out}(\mathscr{L}), \mathbf{undec}(\mathscr{L}))$. The other semantics can be defined by minimality or maximality, similarly to extensions:

Definition 9 (Labelling-based semantics of $\mathfrak{A}$)**.** *A labelling* $\mathscr{L}$ *of an* AF $\mathfrak{A} = (\mathscr{A}, Att)$ *is:*

a) grounded if $\mathscr{L}$ *is complete with* $\subseteq$*-minimal* $\mathbf{in}(\mathscr{L})$*;*

b) preferred if $\mathscr{L}$ *is complete with* $\subseteq$*-maximal* $\mathbf{in}(\mathscr{L})$*;*

c) semi-stable if $\mathscr{L}$ *is complete with* $\subseteq$*-minimal* $\mathbf{undec}(\mathscr{L})$*;*

d) stable if $\mathscr{L}$ *is complete with* $\mathbf{undec}(\mathscr{L}) = \emptyset$*.*

As proved in Caminada (2006), for each of the semantics above, there is a bijection between the set of conflict-free extensions and the set of conflict-free labellings of any AF $\mathfrak{A}$. For example, each extension S is mapped to the labelling $(S, S^+, \mathscr{A} - (S \cup S^+))$. This one-to-one correspondence is made clear after comparing the previously computed extensions of Example 3 with the labellings below of the same framework:

a) $(\emptyset,\emptyset,\mathscr{A}),(\{A\},\{B,D\},\{C,E\}),(\{B\},\{A\},\{C,D,E\})$ and $(\{A,C\},\{B,D\},$ $\{E\})$ are admissible;
b) $(\emptyset,\emptyset,\mathscr{A}),(\{B\},\{A\},\{C,D,E\}),(\{A,C\},\{B,D\},\{E\})$ are complete;
c) $(\emptyset,\emptyset,\mathscr{A})$ is grounded;
d) $(\{B\},\{A\},\{C,D,E\}),(\{A,C\},\{B,D\},\{E\})$ are preferred;
e) $(\{A,C\},\{B,D\},\{E\})$ is semi-stable;
f) no labelling is stable.

2.2 Timed Abstract Argumentation Frameworks

Many concerns arise when dealing with temporal reasoning and representation, such as how to structure time (linear or branching) and define granularity of properties and events (instant points or intervals). See Fisher (2008), Pani and Bhattacharjee (2001) for a more detailed discussion.

In Budán *et al.* (2015)'s approach for Timed Abstract Argumentation Frameworks (TAFs), time is incorporated to abstract frameworks by allowing arguments to be valid only in certain intervals of time called availability intervals. All concepts of defeat and defense are then adapted to ignore unavailable arguments. The time structure adopted corresponds to the set of nonnegative real numbers $\mathbb{R}^+$ and time intervals are periods of time without interruptions, e.g., $(1,3) = \{x \in \mathbb{R}^+ \mid 1 < x < 3\}$ is a time interval. Time intervals $(1,3),(1,3],[1,3),[1,3]$ are all distinct and follow the usual conventions for closed and open intervals: $[$ and $]$ denote the extreme points are included, whereas $($ and $)$ indicate otherwise.

Definition 10 (Timed Argumentation Framework (Budán *et al.* (2015))). *A Timed Argumentation Framework (*TAF*) is a tuple $\Delta = (\mathscr{A}, Att, \delta)$, where $\mathscr{A}$ is a set of arguments, $Att \subseteq \mathscr{A} \times \mathscr{A}$ is an attack relation and $\delta : \mathscr{A} \to 2^{\mathbb{R}^+}$ is an availability function associating each argument to a set of time intervals. When convenient, a set of time intervals might be flattened as only one set of real numbers.*

When deciding argument acceptability, each argument is bound to a set of time intervals. The resulting pair $(A, \mathscr{T}_A)$, where $A \in \mathscr{A}$ and $\mathscr{T}_A \subseteq \delta(A)$, is called a t-profile of A. Intuitively, $\mathscr{T}_A$ is the set of time intervals where A is available. In particular, the t-profile $(A, \delta(A))$ is called the basic t-profile of A. Similarly as with AFs, acceptance is determined collectively. It is then natural to consider collections of t-profiles:

Definition 11 (Collection of t-Profiles (Budán *et al.* (2015)))**.** *Let* $\Delta = (\mathscr{A}, Att, \delta)$ *be a* TAF. *A collection of t-profiles is a set* $S = \{(A_1, \mathscr{T}_{A_1}), \cdots, (A_n, \mathscr{T}_{A_n})\}$ *such that for every* $1 \leq i \leq n$ *it holds:*

a) $(A_i, \mathscr{T}_{A_i})$ *is a t-profile of* A_i*;*

b) $A_i \neq A_j$ *for every* $1 \leq j \leq n$ *with* $j \neq i$*;*

c) $\mathscr{T}_{A_i} \neq \emptyset$.

The notion of internal coherence is adapted for collections of t-profiles as follows.

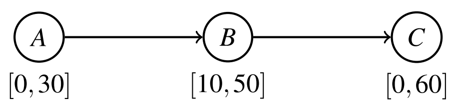
Definition 12 (Conflict-free in Δ (Budán *et al.* (2015)))**.** *A collection S of t-profiles is conflict-free in* Δ *if there are no t-profiles* $(A, \mathscr{T}_A), (B, \mathscr{T}_B) \in S$ *such that* $(A, B) \in Att$ *and* $\mathscr{T}_A \cap \mathscr{T}_B \neq \emptyset$.

For a t-profile $(A, \mathscr{T}_A)$ to be defended from t-profile $(B, \mathscr{T}_B)$ by a collection S of t-profiles at some time instant t, it must hold that if A and B are available at t, then there is a t-profile $(C, \mathscr{T}_C) \in S$ with C available at t such that $(C, B) \in Att$. Formally, the interval of when A is defended from B by S is computed as

$$\mathscr{T}^B_{(A|S)} = \mathscr{T}_A \cap \mathscr{T}_B \cap \bigcup_{(C,\mathscr{T}_C) \in S, (C,B) \in Att} \mathscr{T}_C.$$

Example 4. *Consider the* TAF *represented by Figure 5, in which nodes represent arguments, arrows represent attacks and the availability interval of each argument is near its corresponding node. C is defended from B by* $\{A\}$ *when A, B and C are available. B and C are required to be available because the interval being computed is that of when C is defended from B. Argument A is required to be available since it counterattacks B. The resulting interval of defense is* $\mathscr{T}^B_{(C|\{A\})} = \mathscr{T}_C \cap \mathscr{T}_B \cap \mathscr{T}_A = [10, 30]$.

Figure 5 – Defense of C from B by $\{A\}$ in a TAF



Source: TAF from Budán *et al.* (2015).

In traditional AFs, when a set S defends an argument A, it is usual to say that A is acceptable with respect to S. The concept of defense for TAFs is given in terms of acceptable t-profiles and it is considered that A is trivially defended from B whenever B is not available.

Definition 13 (Acceptable t-profile w.r.t. S)**.** *Let* $\Delta = (\mathscr{A}, Att, \delta)$ *be a* TAF. *The acceptable t-profile of A w.r.t. to a collection of t-profiles S is* $(A, \mathscr{T}_{(A|S)})$ *such that*

$$\mathscr{T}_{(A|S)} = \bigcap_{(B,A) \in Att} (\delta(A) - \delta(B)) \cup \mathscr{T}^{B}_{(A|S)}.$$

It is also reasonable to accept C when its only attacker is not available, and that is captured by taking the union of $\delta(A) - \delta(B)$ in Definition 13. Many semantics are expressed by minimizing or maximizing the set of accepted arguments (in this case, t-profiles). A relation $\subseteq_t$ is defined over collections of t-profiles such that $S \subseteq_t S'$ iff for any $(X, \mathscr{T}_X) \in S$ there exists $(X, \mathscr{T}'_X) \in S'$ such that $\mathscr{T}_X \subseteq \mathscr{T}'_X$.

Definition 14 (Semantics of Δ)**.** *Let* $\Delta = (\mathscr{A}, Att, \delta)$ *be a* TAF. *A collection of t-profiles S is:*

a) t-admissible if for every $(A, \mathscr{T}_A) \in S$ *it holds* $\mathscr{T}_A = \mathscr{T}_{(A|S)}$*;*

b) t-complete if S contains every t-profile acceptable w.r.t. S;

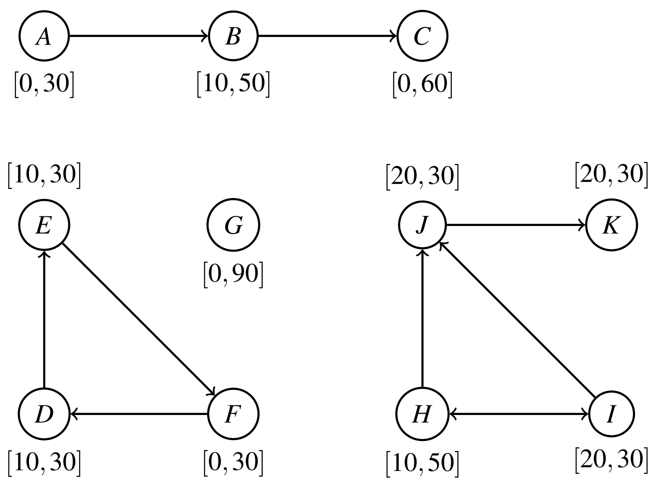
c) t-grounded if S is a $\subseteq_t$*-minimal among t-complete t-profiles;*

d) t-preferred if S is a $\subseteq_t$*-maximal among t-complete t-profiles;*

e) t-stable if S is conflict-free and for all $X \in \mathscr{A} - \bigcup_{(Y,\mathscr{T}_Y) \in S} Y$ *it holds*

$$\delta(X) - \bigcup_{(Y,\mathscr{T}_Y) \in S} \mathscr{T}_{(Y|S)} = \emptyset.$$

Figure 6 – Timed Argumentation Framework



Source: TAF from Budán *et al.* (2015).

In the TAF of Figure 6, the following two collections of t-profiles are the only t-preferred collections:

a)

$$S_1 = \{(A,\{[0,30]\}),(B,\{(30,50]\}),(C,\{[0,30],(50,60]\}),(F,\{[0,10)\})$$
$$(G,\{[0,90]\}),(H,\{[10,50]\},(K,\{[20,30]\})\};$$

b)

$$S_2 = \{(A,\{[0,30]\}),(B,\{(30,50]\}),(C,\{[0,30],(50,60]\}),(F,\{[0,10)\})$$
$$(G,\{[0,90]\}),(H,\{[10,20),(30,50]\},(I,\{[20,30]\}),(K,\{[20,30]\})\}.$$

Intuitively, B can be accepted over $(30,50]$ in S_1 and S_2, as the attack from the unavailable argument A can be ignored in such time frame. Although B is available at $[10,30]$, it is rejected in this time interval due to A's attack. Another example is the acceptance of H over $[10,50]$ in S_1. Even though I attacks H over $[20,30]$, argument H can defend itself from I. It is also reasonable to accept I over $[20,30]$ and reject H in this time interval and that is the decision encoded by S_2. Note that C is accepted over $[0,30]\cup(50,60]$ in S_1 and S_2. This interval is the union of $\mathscr{T}^{B}_{C|\{A\}} = [10,30]$ (previously computed in Example 4) and $\delta(C)-\delta(B) = [0,10)\cup(50,60]$, which means C is accepted when it is defended from its attackers or when it is not attacked by an available argument.

There are no t-stable collections, and the unique t-grounded collection is

$$S_3 = \{(A,\{[0,30]\}),(B,\{(30,50]\}),(C,\{[0,30],(50,60]\}),(F,\{[0,10)\}),(G,\{[0,90]\}),$$
$$(H,\{[10,20),(30,50]\}\}.$$

Note that neither H nor I are accepted over $[20,30]$ in the t-grounded collection (in the context of labellings, H and I would be labeled as undecided), which is expected given the t-grounded collection minimizes the acceptance of arguments among all t-complete collections S_1, S_2 and S_3.

Temporal interactions in TAFs are restricted to availability. The acceptance of A over $[0,30]$ in Figure 6 leads to the rejection of B over that same time interval, which explains why B is only accepted over $(30,50]$. The interaction represented by the attack (A,B) in a TAF indicates that A and B cannot be accepted at the same time. That differs from the approach of Prakken *et al.* (2020) applied to ADFs, which allows the acceptance of arguments at some time points to influence the acceptance of arguments at other time points, and that is the main inspiration for this work.

2.3 Abstract Dialectical Frameworks

Abstract Dialectical Frameworks (ADFs) extend Dung's frameworks by allowing advanced forms of interaction between arguments, and not only attacks. All arguments (which in ADFs are commonly called *statements*) are given an acceptance condition expressed as a propositional formula, which is sufficient to represent many kinds of interactions (referred to as *links*), such as support and joint-attacks.

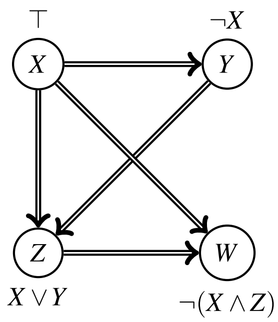
The definitions and theorems that follow are from Brewka and Woltran (2010), Brewka *et al.* (2013), but some adaptations were made to use labellings instead of extensions, and to not explicitly specify the set of links between statements.

Definition 15. *An abstract dialectical framework (*ADF*) is a tuple* (S,Φ) *where* S *is a set of statements and* $\Phi = \{\Phi_s\}_{s\in S}$ *is a collection of propositional formulas, one for each statement.*

A propositional formula expresses how an argument is accepted given the acceptability of each related argument. Three-valued functions will be used, similarly to the labelling approach for AFs.

Example 5. *Let* $D = (S,\Phi)$ *be an* ADF *such that* $S = \{X,Y,Z,W\}, \Phi_X = \top, \Phi_Y = \neg X, \Phi_Z = X \vee Y, \Phi_W = \neg(X \wedge Z)$. *It is illustrated by Figure 7, in which nodes represent statements, arrows represent links and the acceptance condition of each statement is near its corresponding node. A statement* X *is linked to* Y *if* X *appears in* Y*'s acceptance condition* Φ_Y*. The links can be obtained from* Φ*.*

Figure 7 – ADF from Example 5



Source: This author.

Many interactions may appear at the same time in an ADF. *The acceptance condition for Y is $\neg X$, which means Y is rejected if X is accepted, otherwise Y is accepted. This link from X to Y encodes an attack as in traditional* AF*s. The acceptance condition for Z is $X \vee Y$, which means Z is accepted if X or Y are accepted, otherwise Z is rejected. This positive interaction from X and Y to Z represents a notion of support. Finally, W's acceptance condition is $\neg(X \wedge Z)$, meaning W is rejected if X and Z are both accepted, otherwise W is accepted. That encodes a joint attack from $\{X, Z\}$ to W.*

Definition 16. *Let $D = (S, \Phi)$ be an* ADF. *A two-valued (resp. three-valued) interpretation v for D is a total function $v : S \to \{\mathbf{t}, \mathbf{f}\}$ (resp. $v : S \to \{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$). We denote $\mathcal{V}_2^D$ (resp. $\mathcal{V}_3^D$) for the set of all two-valued (resp. three-valued) interpretations for D.*

The acceptability $\mathbf{u}$ (undecided) has the least information. Valuations may be compared with each other with respect to an information order.

Definition 17. *Let $D = (S, \Phi)$ be an* ADF. *The information order $\leq_i$ over $\{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$ is the reflexive closure of $<_i$, where $\mathbf{u} <_i \mathbf{t}$ and $\mathbf{u} <_i \mathbf{f}$. This is generalised for three-valued interpretations for D in a point-wise fashion:*

$v_1 \leq_i v_2$ *if and only if* $\forall s \in S : v_1(s) \in \{\mathbf{t}, \mathbf{f}\} \to v_1(s) = v_2(s)$

Let $\mathrm{u} \in \mathcal{V}_3^D$ such that $\mathrm{u}(s) = \mathbf{u}$ for any $s \in S$. Note that u is the least information interpretation, i.e., for every $v \in \mathcal{V}_3^D$, it holds $\mathrm{u} \leq_i v$.

Definition 18. *Given $v \in \mathcal{V}_3^D$, we define $[v]_2^D = \{w \in \mathcal{V}_2^D \mid v \leq_i w\}$ as the set of all two-valued completions of v.*

It can be checked whether an argument acceptability is consensual with respect to distinct valuations.

Definition 19. *The consensus operator $\sqcap_i$ assigns $\mathbf{t} \sqcap_i \mathbf{t} = \mathbf{t}$, $\mathbf{f} \sqcap_i \mathbf{f} = \mathbf{f}$ and $x \sqcap_i y = \mathbf{u}$ otherwise.*

Semantics are then defined by operators from approximation fixpoint theory from Denecker *et al.* (2004). The operator below updates the acceptance condition of those statements for which there is a consensus among all two-valued interpretations with at least as much information.

Definition 20. *Let $D = (S, \Phi)$ be an* ADF*. We define $\Gamma_D : \mathscr{V}_3^D \to \mathscr{V}_3^D$ as*

$\Gamma_D(v) : S \to \{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$ *such that* $\Gamma_D(v)(s) = \sqcap_i \{w(\Phi_s) \mid w \in [v]_2^D\}$

where $w(\Phi_s)$ is the application of valuation w over the acceptance condition Φ_s for statement s.

Example 6. *Consider the* ADF *from Example 5 illustrated by Figure 7. Let u be the valuation that assigns $\mathbf{u}$ to every statement. Note that $[u]_2^D$ is the set of all two-valued valuations with at least more information than u. As any valuation is more informative than u, $[u]_2^D$ is simply the set of all two-valued valuations. For any $w \in [u]_2^D$, it holds $w(\Phi_X) = w(\top) = \mathbf{t}$. Therefore, $\Gamma_D(u)(X) = \sqcap_i \{w(\Phi_X) \mid w \in [u]_2^D\} = \mathbf{t}$. Similarly, it holds $w(\Phi_Y) = w(\neg X) = \mathbf{f}$ for any $w \in [u]_2^D$ and thus $\Gamma_D(u)(Y) = \sqcap_i \{w(\Phi_Y) \mid w \in [u]_2^D\} = \mathbf{f}$. At last, $\Gamma_D(u)$ is a valuation v such that $v(X) = \mathbf{t}, v(Y) = \mathbf{f}, v(Z) = \mathbf{t}$ and $v(W) = \mathbf{f}$.*

Definition 21. *Let $D = (S, \Phi)$ be an* ADF *and $v \in \mathscr{V}_3^D$. It holds:*

a) v is admissible iff $v \leq_i \Gamma_D(v)$.

b) v is complete iff $v = \Gamma_D(v)$.

c) v is grounded iff v is $\leq_i$-minimal among complete interpretations in $\mathscr{V}_3^D$.

d) v is preferred iff v is $\leq_i$-maximal among complete interpretations in $\mathscr{V}_3^D$.

e) v is stable iff v is complete and $v \in \mathscr{V}_2^D$.

Example 7. *The valuation v from Example 6 is a two-valued valuation, as no statement is labelled $\mathbf{u}$. No other valuation can be more informative, i.e., $[v]_2^D = \{v\}$. This implies v is admissible, complete, grounded, preferred and stable.*

Figure 7 has shown how to encode an attack (X, Y) from a traditional Dung AF. More generally, any AF can be represented by an ADF when restricting the acceptance condition of each statement to formulae as $\neg A_1 \wedge \cdots \wedge \neg A_n$, where each $A_i \in \{A_1, \cdots, A_n\}$ is an atomic formula.

Definition 22. *For an* AF $\mathfrak{A} = (\mathscr{A}, Att)$*, define the* ADF *associated to $\mathfrak{A}$ as $D_{\mathfrak{A}} = (\mathscr{A}, \Phi)$ with $\Phi = \{\Phi_A\}_{A \in \mathscr{A}}$ and $\Phi_A = \bigwedge_{B \in Att(A)} \neg B$ for every $A \in \mathscr{A}$.*

The connection between labellings and interpretations is trivial.

Definition 23. *For any three-valued interpretation $v : S \to \{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$, we define an associated labelling $\mathscr{L}_v : S \to \{\mathbf{in}, \mathbf{out}, \mathbf{undec}\}$ such that:*

a) $\mathscr{L}_v(s) = \mathbf{in}$ *iff* $v(s) = \mathbf{t}$

b) $\mathscr{L}_v(s) = \mathbf{out}$ *iff* $v(s) = \mathbf{f}$

c) $\mathscr{L}_v(s) = \mathbf{undec}$ *iff* $v(s) = \mathbf{u}$

Finally, ADFs are not only strictly more expressive, but also generalizations of AFs.

Theorem 1. *Let* $\mathfrak{A} = (\mathscr{A}, Att)$ *be an* AF *and* $D_{\mathfrak{A}}$ *its associated* ADF. *A three-valued interpretation* v *of* $D_{\mathfrak{A}}$ *is admissible, complete, grounded, preferred, stable iff its associated labelling* $\mathscr{L}_v$ *is respectively an admissible, complete, grounded, preferred, stable in* $\mathfrak{A}$.

2.4 Timed Abstract Dialectical Frameworks

Prakken *et al.* (2020) add temporality to ADFs by a simple translation approach. Prakken's paper is the main influence for this work, in which the same approach is investigated in less generic formalisms, such as BAFs and SETAFs.

Definition 24. *A timed abstract dialectical framework (*tADF*) is a tuple* $\mathfrak{D} = (S, T, \Phi)$ *where* S *is a set of statements,* T *is a total ordered set of time states and* $\Phi = \{\Phi_{s_t} \mid s \in S, t \in T\}$ *is a set of propositional formulas, one for each statement* $s \in S$ *and time state* $t \in T$.

Any tADF (S, T, Φ) can be translated into an ADF (S', Φ') where $S' = \{(s,t) \mid s \in S, t \in T\}$ and $\Phi'_{(s,t)} = \Phi_{s_t}$ for every $s \in S$ and $t \in T$. Evaluating semantics of tADFs correspond to evaluating semantics of the associated ADF.

Definition 25. *For any* tADF $\mathfrak{D} = (S, T, \Phi)$*, the* ADF *associated to* $\mathfrak{D}$ *is* $(S \times T, \Phi)$.

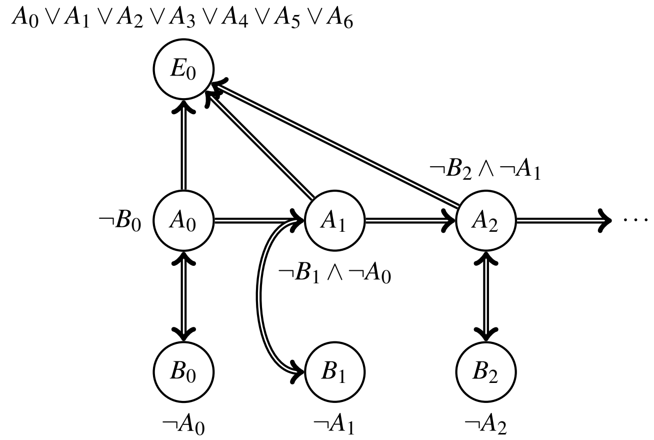
Example 8. *Recall the framework from Example 2. Figure 8 illustrates a fragment of this framework containing only arguments* A, B *and* E.

Shorthands facilitate the use of tADFs by compactly expressing many kinds of temporal interactions. In Example 8, argument E is accepted at time point 0 precisely if argument A is accepted at least once in $[0,6]$. The shorthands proposed by Prakken *et al.* (2020) are described below:

a) $\Phi_{c_t} = a_{\geq 1}^{[i,j]} := \bigvee_{i \leq k \leq j} a_k$ (c should be accepted at time state t if a is accepted at least once in $[i,j]$);

b) $\Phi_{c_t} = a_{\geq n}^{[i,j]} := \bigvee_{K \subseteq [i,j], |K| = n} \bigwedge_{k \in K} a_k$ (c should be accepted at time state t if a is accepted at least n times in $[i,j]$);

Figure 8 – tADF fragment from Figure 2



Source: This author.

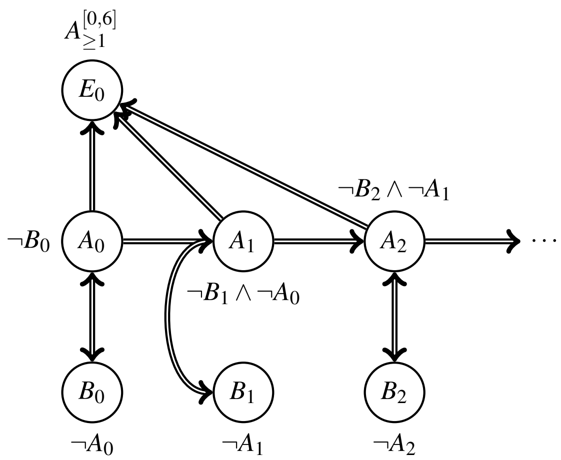
c) $\Phi_{c_t} = a^{[i,j]}_{\leq n} := \neg(a^{[i,j]}_{\geq n+1})$ (c should be accepted at time state t if a is accepted at most n times in $[i, j]$);

d) $\Phi_{c_t} = a^{[i,j]}_{\leq 1} := \neg(a^{[i,j]}_{\geq 2})$ (c should be accepted at time state t if a is accepted at most once in $[i, j]$);

e) $\Phi_{c_t} = a^{[i,j]}_{=n} := a^{[i,j]}_{\leq n} \wedge a^{[i,j]}_{\geq n}$ (c should be accepted at time state t if a is accepted exactly n times in $[i, j]$).

By using shorthands, Φ_{E_0} can be specified as $A^{[0,6]}_{\geq 1}$. More generally, the relation between A and E is explained by the intuitive notion that "if Bob took medication any day on this week, he took medication this week", encoded by $\Phi_{E_{7i}} = A^{[7i,7(i+1)-1]}_{\geq 1}$ and $\Phi_{E_{7i}} = \Phi_{E_{7i+j}}$ for every $0 \leq j < 7$ and $i \geq 0$. The equality $\Phi_{E_{7i}} = \Phi_{E_{7i+j}}$ for $i = 0 \leq j < 7$ simply means that $E_0, E_1, \cdots, E_6$ all interact in the same way and share the same acceptability degree. Application of shorthands on the tADF of Example 8 is shown in Figure 9.

Figure 9 – tADF fragment from Example 8 using shorthands



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3 ENCODING TIME IN AFS

The main contributions of this work start in this section, in which traditional AFs are extended for dealing with time. When including temporal interactions, arguments or attacks must be explicitly associated with temporal marks in order for past and future attacks be represented. Following the translation approach from tADFs to ADFs described in Prakken *et al.* (2020), arguments are treated as pairs (A,a), meaning "argument A" at "time instant a".

Definition 26 (TeAF)**.** *A Temporal Argumentation Framework* (TeAF) *is a tuple* $\mathfrak{T} = (\mathscr{A}, Att, \mathscr{T})$ *such that* $\mathscr{A}$ *is an enumerable set of arguments,* $Att \subseteq (\mathscr{A} \times \mathscr{T})^2$ *is an attack relation, and* $\mathscr{T}$ *is a total ordered enumerable set of time points. For* $(A,a) \in \mathscr{A} \times \mathscr{T}$*, we define* $Att(A,a) = \{(B,b) \in \mathscr{A} \times \mathscr{T} \mid ((B,b),(A,a)) \in Att\}$.

The attack (B_1, C_0) in Figure 1 will be specified as $((B,0),(C,1))$ in which $0 \leq 1$ are consecutive points in the timeline $\mathscr{T}$.

Example 9. *The framework in Figure 1 can be represented by the* TeAF $\mathfrak{T} = (\mathscr{A}, Att, \mathbb{N})$ *such that:*

a) $\mathscr{A} = \{A,B,C,D\}$*;*
b) $Att = X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5$*;*
c) $X_1 = \{((A,i),(A,i+1)) \mid i \in \mathbb{N}\}$*;*
d) $X_2 = \{((A,i),(B,i)) \mid i \in \mathbb{N}\}$*;*
e) $X_3 = \{((B,i),(A,i)) \mid i \in \mathbb{N}\}$*;*
f) $X_4 = \{((B,i+1),(C,i)) \mid i \in \mathbb{N}\}$*;*
g) $X_5 = \{((C,0),(D,i)) \mid i \in \mathbb{N}\}$.

The set of attacks will generally be infinite when the timeline is infinite, but most attack patterns can be expressed with a simpler notation. Some patterns can be recognized in Example 9:

a) X_1 contains every attack from A to itself in consecutive time points;
b) X_2 contains every attack from A (in a specific time point i) to B (in the same time point i);
c) X_4 contains attacks from the future, as accepting B at time $i+1$ implies rejecting C at time i;

d) X_5 contains attacks from C at time 0 to the argument D, which is linked to no relative time reference.

The following shorthands allow for a compact representation of many Temporal Abstract Argumentation Frameworks (TeAFs) containing the patterns discussed above.

Definition 27. *Let* $\mathfrak{T} = (\mathscr{A}, Att, \mathscr{T})$ *be a* TeAF, $i, j \in \mathscr{T}$ *be time points such that* $i \leq j$, *and* $A, B \in \mathscr{A}$ *be arguments. Attacks between arguments in the same time point can be expressed by the shorthands:*

a) $(A,B)_i^j = \{((A,t),(B,t)) \mid t \in \mathscr{T}, i \leq t \leq j\}$;

b) $(A,B)_i = \{((A,t),(B,t)) \mid t \in \mathscr{T}, i \leq t\}$;

c) $(A,B)^j = \{((A,t),(B,t)) \mid t \in \mathscr{T}, t \leq j\}$;

d) $(A,X) = \{((A,t),(X,t)) \mid t \in \mathscr{T}\}$;

e) Attacks may target or stem from an argument at each time point:

– $((A,i),X) = \{((A,i),(X,t)) \mid t \in \mathscr{T}\}$;

– $(X,(A,i)) = \{((X,t),(A,i)) \mid t \in \mathscr{T}\}$;

f) If the timeline can be enumerated by integers ($\mathscr{T} = \{\cdots, t_{-1}, t_0, t_1, \cdots\}$) *or naturals* ($\mathscr{T} = \{t_0, t_1, \cdots\}$), *the following relative attacks may also be abbreviated. For any* $s \in \mathbb{Z}$, *representing the difference of time points from the attacked argument to its attacker, the following shorthands abbreviate relative attacks:*

– $[A,B]_s = \{((A,t_i),(B,t_{i+s})) \mid i \in \mathbb{Z}\}$ *when the timeline is enumerated by integers;*

– $[A,B]_s = \{((A,t_i),(B,t_{i+s})) \mid i, i+s \in \mathbb{N}\}$ *when the timeline is enumerated by natural numbers.*

The attack relation of Example 9 can be abbreviated by the shorthands above: X_1, X_2, X_3, X_4 and X_5 are respectively $[A,A]_1, (A,B), (B,A), [B,C]_{-1}$ and $((C,0),D)$. Calculating semantics in frameworks with an attack relation described by a finite set of shorthands might be simpler than in frameworks with arbitrary attacks lacking discernible patterns.

Definition 28. *A* TeAF $\mathfrak{T} = (\mathscr{A}, Att, \mathscr{T})$ *is compact iff* $Att = \bigcup_{i=1}^{n} S_i$ *for* $n \in \mathbb{N}$ *finite and such that each* S_i *is a shorthand as in Definition 27.*

The TeAF in Example 9 is compact, because its attack relation is the set $[A,A]_1 \cup (A,B) \cup (B,A) \cup [B,C]_{-1} \cup ((C,0),D)$.

3.1 Extension-based semantics

For TeAFs, argument acceptance is time-dependent. Most definitions and propositions from this section are directly derived from those for AFs. Although some of the original definitions and properties for AFs are ommited, they can be easily obtained by the temporal version. In the following, notions of defeat and defense are refined by considering temporal interactions.

Definition 29 (Defeat in $\mathfrak{T}$)**.** *A set $S \subseteq \mathscr{A} \times \mathscr{T}$ defeats $(A,a) \in \mathscr{A} \times \mathscr{T}$ iff there exists $(B,b) \in Att(A,a)$ such that $(B,b) \in S$.*

Just as in AFs, the set of arguments (with associated time points) defeated by S is denoted as $S^+ = \{(A,a) \in \mathscr{A} \times \mathscr{T} \mid S \text{ defeats } (A,a)\}$.

Definition 30 (Defense in $\mathfrak{T}$)**.** *A set $S \subseteq \mathscr{A} \times \mathscr{T}$ defends $(A,a) \in \mathscr{A} \times \mathscr{T}$ iff for every $(B,b) \in Att(A,a)$ it holds that S defeats (B,b).*

For $(A,a) \in \mathscr{A} \times \mathscr{T}$, saying (A,a) is acceptable with respect to S is equivalent to saying that S defends (A,a).

Arguments can only be collectively accepted if they are internally consistent (i.e., there are no conflicts between accepted arguments).

Definition 31 (Conflict-free in $\mathfrak{T}$)**.** *A set $S \subseteq \mathscr{A} \times \mathscr{T}$ is conflict-free iff there is no $(A,a) \in S$ such that S defeats (A,a).*

Following the usual methodology in the literature, extensions are defined as conflict-free fixpoints of a defense operator.

Definition 32 (Characteristic Function)**.** *The Characteristic Function associated with a* TeAF *$\mathfrak{T} = (\mathscr{A}, Att, \mathscr{T})$ is a function $\mathscr{F}_{\mathfrak{T}} : 2^{\mathscr{A}\times\mathscr{T}} \to 2^{\mathscr{A}\times\mathscr{T}}$ such that*

$$\mathscr{F}_{\mathfrak{T}}(S) = \{(A,a) \in \mathscr{A} \times \mathscr{T} \mid S \textit{ defends } (A,a)\}$$

Definition 33 (Extension-based semantics of $\mathfrak{T}$)**.** *Let $\mathfrak{T} = (\mathscr{A}, Att, \mathscr{T})$ be a* TeAF*. A set $S \subseteq \mathscr{A} \times \mathscr{T}$ is:*

a) admissible if S is conflict-free and $S \subseteq \mathscr{F}_{\mathfrak{T}}(S)$;

b) complete if S is conflict-free and $S = \mathscr{F}_{\mathfrak{T}}(S)$;

c) grounded if S is $\subseteq$-minimal among all complete extensions;

d) preferred if S is $\subseteq$-maximal among all complete extensions;

e) semi-stable if $S \cup S^+$ is $\subseteq$-maximal among all complete extensions;

f) stable if S is complete and $S \cup S^+ = \mathscr{A} \times \mathscr{T}$.

Extensions for compact TeAFs generally follow certain patterns and can be expressed by simplified notations similar to attack shorthands. The set $\{(A, f(t)) \mid A \in S \subseteq \mathscr{A}, t \in \mathscr{T}\}$ is denoted as $S_{f(t)}$ for some function $f : \mathscr{T} \to \mathscr{T}$. When $S = \{X\}$, it can abbreviated even further as $X_{f(t)}$. For example, when $\mathscr{T} = \mathbb{Z}$, $\{A,B\}_{3i} = \{(X,3i) \mid X \in \{A,B\}, i \in \mathscr{T}\}$ and $Y_{i^2} = \{(Y,i^2) \mid i \in \mathscr{T}\}$.

Example 10. *For the framework in Example 9, we compute the following semantics:*

a) Complete: $\emptyset, A_{2i} \cup \{B,C\}_{2i+1} \cup D_i$ and $A_{2i+1} \cup \{B,C\}_{2i}$;

b) Grounded: $\emptyset$;

c) Semi-stable, stable and preferred: $A_{2i} \cup \{B,C\}_{2i+1} \cup D_i$ and $A_{2i+1} \cup \{B,C\}_{2i}$.

The next example shows that for some frameworks, every semantics differ.

Example 11. *Let $\mathfrak{T} = (\{A,B,C,D\}, Att, \mathbb{N})$, where $Att = [A,A]_1 \cup (A,B)_0^0 \cup (B,A)_0^0 \cup [B,B]_1 \cup ((B,2),D) \cup (C,C) \cup (C,D)$. We compute the following semantics:*

a) Complete: $\emptyset, A_{2i} \cup B_{2i+1}$ and $A_{2i+1} \cup B_{2i}$;

b) Grounded: $\emptyset$;

c) Preferred: $A_{2i} \cup B_{2i+1}$ and $A_{2i+1} \cup B_{2i}$;

d) Semi-stable: $A_{2i+1} \cup B_{2i}$;

e) Stable: none.

As expected, TeAFs are shown to generalize many properties of AFs, thus being a sound formalism for representing time in argumentation. In particular, the Fundamental Lemma is essential for demonstrating many AFs' properties. As TeAFs will also satisfy a corresponding version of the Fundamental Lemma, these properties are naturally preserved in TeAFs.

Lemma 1 (Fundamental Lemma). *Let $\mathfrak{T} = (\mathscr{A}, Att, \mathscr{T})$ be a* TeAF, *S be an admissible extension of $\mathfrak{T}$, and $(A,a), (A',a') \in \mathscr{A} \times \mathscr{T}$ be acceptable with respect to S. Then*

1. $S' = S \cup \{(A,a)\}$ is admissible, and

2. (A',a') is acceptable with respect to S'.

As in Dung (1995), the next theorem results directly from the Fundamental Lemma:

Theorem 2. *Let $\mathfrak{T}$ be a* TeAF.

(1) The set of all admissible extensions of $\mathfrak{T}$ forms a complete partial order with respect to set inclusion;

(2) For each admissible extension S of $\mathfrak{T}$, there exists a maximal admissible extension $\mathscr{E}$ of $\mathfrak{T}$ such that $S \subseteq \mathscr{E}$.

From Theorem 2 and the admissibility of $\emptyset$, the following corollary holds:

Corollary 1. *Every* TeAF *possesses at least one maximal admissible extension with respect to set inclusion.*

Dung (1995) shows that a set of arguments S is stable if it is the set of arguments not defeated by S. Furthermore, he proves that every stable extension is a preferred extension, but not vice-versa. These properties are preserved in TeAFs with stable and preferred extensions:

Proposition 3. *Let $\mathfrak{T} = (\mathscr{A}, Att, \mathscr{T})$ be a* TeAF. *Then S is a stable extension of $\mathfrak{T}$ iff $S = \{(A,a) \mid (A,a)$ is not defeated by $S\}$.*

Proposition 4. *For any* TeAF $\mathfrak{T}$*, any stable extension of $\mathfrak{T}$ is a preferred extension of $\mathfrak{T}$. However, there is some* TeAF $\mathfrak{T}$ *such that not every preferred extension of $\mathfrak{T}$ is a stable extension of $\mathfrak{T}$.*

Lemmas 2 and 3 are related to the $\mathscr{F}_{\mathfrak{T}}$ operator: Lemma 2 guarantees $\mathscr{F}_{\mathfrak{T}}$ preserves the conflict-freeness property, while Lemma 3 shows $\mathscr{F}_{\mathfrak{T}}$ is monotonic.

Lemma 2. *Let $\mathfrak{T} = (\mathscr{A}, Att, \mathscr{T})$ be a* TeAF. *If $S \subseteq \mathscr{A} \times \mathscr{T}$ is conflict-free, then $\mathscr{F}_{\mathfrak{T}}(S)$ is also conflict-free.*

Lemma 3. *Let $\mathfrak{T} = (\mathscr{A}, Att, \mathscr{T})$ be a* TeAF. *Then $\mathscr{F}_{\mathfrak{T}}$ is monotonic with respect to set inclusion.*

Initially, preferred extensions were defined as maximal admissible extensions. Next, it is shown the equivalence to the alternative characterization of preferred extensions as maximal complete extensions (Lemma 4) as well as the existence of preferred/complete extensions (Theorem 5):

Lemma 4. *Let $\mathfrak{T}$ be a* TeAF. *It holds S is a preferred extension of $\mathfrak{T}$ iff S is a $\subseteq$-maximal admissible extension of $\mathfrak{T}$.*

The next result is an immediate consequence of Corollary 1, Lemma 4 and Definition 33:

Theorem 5. *Every* TeAF *has at least one preferred/complete extension.*

The grounded extension is uniquely defined for every TeAF $\mathfrak{T}$ and it coincides with the $\subseteq$-least fixpoint of $\mathscr{F}_{\mathfrak{T}}$.

Theorem 6. *Every* TeAF $\mathfrak{T}$ *possesses a unique grounded extension and it is the* $\subseteq$*-least fixpoint of* $\mathscr{F}_{\mathfrak{T}}$.

Just as for traditional AFs, complete extensions for a TeAF constitute a complete semilattice under set inclusion.

Theorem 7. *Let* $\mathfrak{T} = (\mathscr{A}, Att, \mathscr{T})$ *be a* TeAF*. The complete extensions of* $\mathfrak{T}$ *form a complete semilattice with respect to set inclusion.*

The results from this section are expected, as the complete semantics for TeAFs coincide with the complete semantics for AFs when each pair of argument and time instant from an TeAF is treated as an argument in an AF.

3.2 Labelling-based semantics

Semantics can also be defined by argument labellings as in Caminada (2006), in which the 3-valued model of acceptability degrees $(\mathbf{in}, \mathbf{out}, \mathbf{undec})$ are explicit.

Definition 34 (Labelling)**.** *Given a* TeAF $(\mathscr{A}, Att, \mathscr{T})$*, a labelling is a (total) function* $\mathscr{L} : \mathscr{A} \times \mathscr{T} \to \{\mathbf{in}, \mathbf{out}, \mathbf{undec}\}$. *We define* $\mathbf{in}(\mathscr{L}, t)$ *as* $\{(A,t) \in \mathscr{A} \times \mathscr{T} \mid \mathscr{L}(A,t) = \mathbf{in}\}$*,* $\mathbf{out}(\mathscr{L}, t)$ *as* $\{(A,t) \in \mathscr{A} \times \mathscr{T} \mid \mathscr{L}(A,t) = \mathbf{out}\}$ *and* $\mathbf{undec}(\mathscr{L}, t)$ *as* $\{(A,t) \in \mathscr{A} \times \mathscr{T} \mid \mathscr{L}(A,t) = \mathbf{undec}\}$.

A labelling $\mathscr{L}$ may be denoted as (I, O, U) where $I = \{(A,t) \in \mathscr{A} \times \mathscr{T} \mid \mathscr{L}(A,t) = \mathbf{in}\}$, $O = \{(A,t) \in \mathscr{A} \times \mathscr{T} \mid \mathscr{L}(A,t) = \mathbf{out}\}$ and $U = \{(A,t) \in \mathscr{A} \times \mathscr{T} \mid \mathscr{L}(A,t) = \mathbf{undec}\}$.

Given $t \in \mathscr{T}$, $\mathscr{L}_t$ denotes the function $\mathscr{L}_t(A) = \mathscr{L}(A,t)$, which returns the label of A at time t. Also, $\mathscr{L}_t$ can be described by (I, O, U) where $I = \mathbf{in}(\mathscr{L}_t) = \{A \in \mathscr{A} \mid \mathscr{L}_t(A) = \mathbf{in}\}, O = \mathbf{out}(\mathscr{L}_t) = \{A \in \mathscr{A} \mid \mathscr{L}_t(A) = \mathbf{out}\}$ and $U = \mathbf{undec}(\mathscr{L}_t) = \{A \in \mathscr{A} \mid \mathscr{L}_t(A) = \mathbf{undec}\}$ when convenient.

Definition 35 (Admissible labelling)**.** *A labelling* $\mathscr{L}$ *of* TeAF $(\mathscr{A}, Att, \mathscr{T})$ *is admissible if for every* $A \in \mathscr{A}$ *and* $a \in \mathscr{T}$ *it holds:*

a) $\mathscr{L}(A,a) = \mathbf{in}$ *if* $\mathscr{L}(B,b) = \mathbf{out}$ *for all* $B \in \mathscr{A}$ *and* $b \in \mathscr{T}$ *such that* $((B,b),(A,a)) \in Att$*;*

b) $\mathscr{L}(A,a) = \mathbf{out}$ *if* $\mathscr{L}(B,b) = \mathbf{in}$ *for some* $B \in \mathscr{A}$ *and* $b \in \mathscr{T}$ *such that* $((B,b),(A,a)) \in Att$.

Definition 36 (Complete labelling)**.** *A labelling* $\mathscr{L}$ *of* TeAF $(\mathscr{A}, Att, \mathscr{T})$ *is complete if for every* $A \in \mathscr{A}$ *and* $a \in \mathscr{T}$ *it holds:*

a) $\mathscr{L}(A,a) = \mathbf{in}$ *iff* $\mathscr{L}(B,b) = \mathbf{out}$ *for all* $(B,b) \in Att(A,a)$*;*

b) $\mathscr{L}(A,a) = \mathbf{out}$ *iff* $\mathscr{L}(B,b) = \mathbf{in}$ *for some* $(B,b) \in Att(A,a)$.

Refinements of the complete semantics are defined as usual by minimality/maximality with respect to the set-inclusion relation over the set of accepted/undecided arguments.

Definition 37 (Labelling-based semantics)**.** *A labelling* $\mathscr{L}$ *of* TeAF $(\mathscr{A}, Att, \mathscr{T})$ *is:*

a) grounded if $\mathbf{in}(\mathscr{L})$ *is* $\subseteq$*-minimal among all complete labellings;*

b) preferred if $\mathbf{in}(\mathscr{L})$ *is* $\subseteq$*-maximal among all complete labellings;*

c) semi-stable if $\mathbf{undec}(\mathscr{L})$ *is* $\subseteq$*-minimal among all complete labellings;*

d) stable if $\mathscr{L}$ *is complete and* $\mathbf{undec}(\mathscr{L}) = \emptyset$.

Example 12. *The labelling semantics for the* TeAF *defined in Example 11 are:*

a) Complete: $(\emptyset, \emptyset, \mathscr{A}), (A_{2i} \cup B_{2i+1}, A_{2i+1} \cup B_{2i}, C_i \cup D_i)$ *and* $(A_{2i+1} \cup B_{2i} \cup D_i, A_{2i} \cup B_{2i+1}, C_i)$*;*

b) Grounded: $(\emptyset, \emptyset, \mathscr{A})$*;*

c) Preferred: $(A_{2i} \cup B_{2i+1}, A_{2i+1} \cup B_{2i}, C_i \cup D_i)$ *and* $(A_{2i+1} \cup B_{2i} \cup D_i, A_{2i} \cup B_{2i+1}, C_i)$*;*

d) Semi-stable: $(A_{2i+1} \cup B_{2i} \cup D_i, A_{2i} \cup B_{2i+1}, C_i)$*;*

e) Stable: none.

The link between the extension-based and the labelling-based approach is that arguments labelled **in** (resp. **out**) are those defended (resp. defeated) by the set of accepted arguments.

Proposition 8. *Let* $\mathfrak{T} = (\mathscr{A}, Att, \mathscr{T})$ *be a* TeAF *and* $\mathscr{L} : \mathscr{A} \times \mathscr{T} \to \{\mathbf{in}, \mathbf{out}, \mathbf{undec}\}$ *a labelling of* $\mathfrak{T}$*. Then,* $\mathscr{L}$ *is complete iff for any* $A \in \mathscr{A}$ *and* $a \in \mathscr{T}$*:*

a) $\mathscr{L}(A,a) = \mathbf{in}$ *iff* $\mathbf{in}(\mathscr{L})$ *defends* (A,a)*;*

b) $\mathscr{L}(A,a) = \mathbf{out}$ *iff* $\mathbf{in}(\mathscr{L})$ *defeats* (A,a).

The following propositions can be used to provide alternative characterizations for the semantics, as done in Baroni *et al.* (2018), based on the minimization or maximization of arguments with a specific label.

Proposition 9. *Let $\mathscr{L}, \mathscr{L}'$ be complete labellings of a* TeAF $\mathfrak{T}$. *It holds:*

a) $\mathbf{in}(\mathscr{L}) \subseteq \mathbf{in}(\mathscr{L}')$ *iff* $\mathbf{out}(\mathscr{L}) \subseteq \mathbf{out}(\mathscr{L}')$*;*

b) $\mathbf{in}(\mathscr{L}) \subset \mathbf{in}(\mathscr{L}')$ *iff* $\mathbf{out}(\mathscr{L}) \subset \mathbf{out}(\mathscr{L}')$.

Proposition 10. *Let $\mathscr{L}, \mathscr{L}'$ be complete labellings of a* TeAF $\mathfrak{T}$. *It holds*

1. *If* $\mathbf{in}(\mathscr{L}) \subseteq \mathbf{in}(\mathscr{L}')$, *then* $\mathbf{undec}(\mathscr{L}') \subseteq \mathbf{undec}(\mathscr{L})$*;*
2. *If* $\mathbf{in}(\mathscr{L}) \subset \mathbf{in}(\mathscr{L}')$, *then* $\mathbf{undec}(\mathscr{L}') \subset \mathbf{undec}(\mathscr{L})$*;*
3. *If* $\mathbf{out}(\mathscr{L}) \subseteq \mathbf{out}(\mathscr{L}')$, *then* $\mathbf{undec}(\mathscr{L}') \subseteq \mathbf{undec}(\mathscr{L})$*;*
4. *If* $\mathbf{out}(\mathscr{L}) \subset \mathbf{out}(\mathscr{L}')$, *then* $\mathbf{undec}(\mathscr{L}') \subset \mathbf{undec}(\mathscr{L})$.

Proposition 11. *Let $\mathscr{L}, \mathscr{L}'$ be complete labellings of a* TeAF $\mathfrak{T}$. *It holds:*

a) If $\mathbf{in}(\mathscr{L}) = \mathbf{in}(\mathscr{L}')$, *then* $\mathscr{L} = \mathscr{L}'$*;*

b) If $\mathbf{out}(\mathscr{L}) = \mathbf{out}(\mathscr{L}')$, *then* $\mathscr{L} = \mathscr{L}'$.

3.3 Connection with TAFs

TeAFs are as expressive as TAFs. An attack (A,B) in a TAF $\Delta = (\mathscr{A}, Att, \delta)$ is represented in a TeAF $\mathfrak{T} = (\mathscr{A}', Att', \mathscr{T})$ by the set of attacks $\{((A,t),(B,t)) \mid t \in \mathscr{T}\}$ conveniently described by the shorthand (A,B) from Definition 27. Argument availability can be represented in TeAF $\mathfrak{T}$ by including a meta-argument X that attacks $(A,a) \in \mathscr{A} \times \mathscr{T}$ iff $a \in \delta(A)$. Thus, argument A is rejected at time point a in TeAF $\mathfrak{T}$ if A is not available at a according to the TAF Δ. That is formalized below.

Definition 38. *Let $\Delta = (\mathscr{A}, Att, \delta)$ be a* TAF *with an enumerable timeline. The corresponding* TeAF *is $\mathfrak{T}_\Delta = (\mathscr{A}', Att', \mathscr{T})$ such that:*

a) $\mathscr{A}' = \mathscr{A} \cup \{X\}$, *where* $X \notin \mathscr{A}$*;*

b) $Att' = \{((X,t),(A,t)) \mid A \in \mathscr{A}, t \in \mathscr{T} - \delta(A)\} \cup \{((A,t),(B,t)) \mid (A,B) \in Att, t \in \mathscr{T}\}$, *taking $\delta(A)$ as a set of time points, instead of as a set of intervals;*

c) $\mathscr{T}$ *is the timeline of* Δ.

In the previous definition, time points in TeAFs correspond exactly to those in TAFs. Sometimes, time points in TeAFs can be associated to entire time intervals in TAFs, allowing for a more compact representation. The following example displays this strategy.

Example 13. *In the* TAF *from Figure 5, A, B and C are respectively available in* $[0,30]$*,* $[10,50]$ *and* $[0,60]$*. This timeline can be divided as follows:*

a) In $[0,10)$*, only A and C are available.*

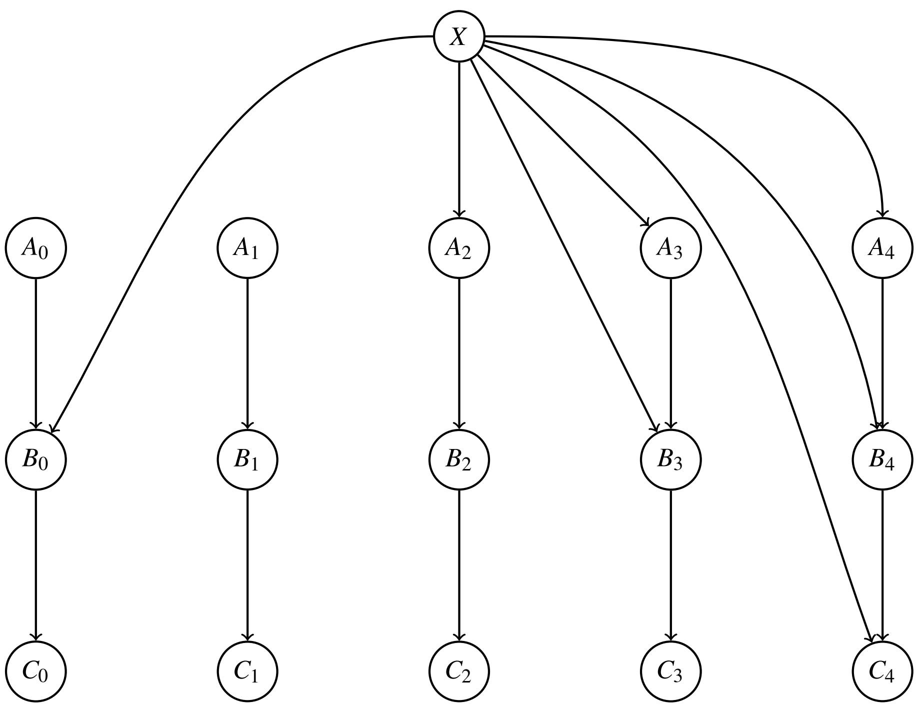
b) In $[10,30]$*, A, B and C are available.*

c) In $(30,50]$*, only B and C are available.*

d) In $(50,60]$*, only C is available.*

e) In any time point not included in some interval above, every argument is unavailable.

Figure 10 – Corresponding TeAF of a TAF, with time points associated to intervals



Source: This author.

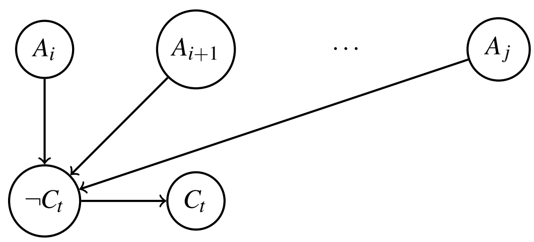
Acceptability only changes over time by arguments becoming available or unavailable. Thus, for every interval shown above, acceptability does not vary within the same interval. Let $\mathscr{T} = \{0,1,2,3,4\}$ *be the set of time points of the corresponding* TeAF*, where* $0,1,2,3$ *and* 4 *represent the intervals in the order they were listed. The* TAF *in Figure 5 can be represented by the* TeAF *in Figure 10. As meta-argument X interacts equally at every time point,* X_0, X_1, X_2, X_3

and X_5 are all represented by the sole meta-argument X. Semantics for the TAF *can be obtained by applying the semantics defined for* TeAF*s. For instance, B_2's acceptance in the* TeAF *indicates that B is accepted at interval* $(30,50]$ *in the* TAF.

3.4 Expressiveness limitations

Certain interactions cannot be easily expressed in TeAFs without including additional meta-arguments whose sole purpose is that of encoding such interactions. These are the same limitations encountered when using AFs to encode acceptance conditions of ADFs.

Figure 11 – TeAF fragment encoding $A^{[i,j]}_{\geq 1}$



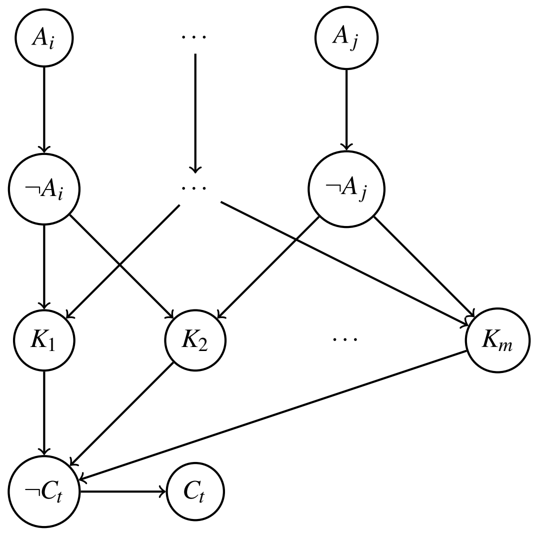
Source: This author.

Take for instance the following interaction: argument C should be accepted at time t if A is accepted at least once in $[i,j]$. In tADFs, it can be encoded directly by defining C's propositional formula at time t as $\varphi_{c_t} = A^{[i,j]}_{\geq 1} = \bigvee_{i \leq k \leq j} A_k$. In TeAFs, a possible encoding for this interaction is illustrated by Figure 11. If any A_k for $i \leq k \leq j$ is accepted, then the meta-argument $\neg C_t$ is rejected and C_t is accepted. Argument C can only be rejected at time t if every A_k for $i \leq k \leq j$ is rejected.

For some interactions, the size of the meta-framework is unfeasible, as for expressing that C should be accepted at time t if A is accepted at least n times in $[i,j]$. In tADFs, this is encoded by the propositional formula $\varphi_{C_t} = A^{[i,j]}_{\geq n} = \bigvee_{K \subseteq [i,j], |K|=n} \bigwedge_{k \in K} a_k$. As for TeAFs, Figure 12 shows a possible encoding.

Let $\mathscr{K} = \{K \subseteq \{A_i, \cdots, A_j\} \mid |K| = n\}$ with $|\mathscr{K}| = m$ be the set of all sets of n arguments among $A_i, \cdots, A_j$. A meta-argument K_s $(1 \leq s \leq m)$ is added for each $K'_s \in \mathscr{K}$. Note that K_s is accepted if meta-argument $\neg A_k$ is rejected for every $A_k \in K'_s$. That means K_s is accepted if every argument in K'_s is accepted, i.e., A is accepted n times in $[i,j]$. The unviability of such approach comes from the enormous amount of meta-arguments introduced: for an interval $[i,j]$,

Figure 12 – TeAF fragment encoding $A^{[i,j]}_{\geq n}$



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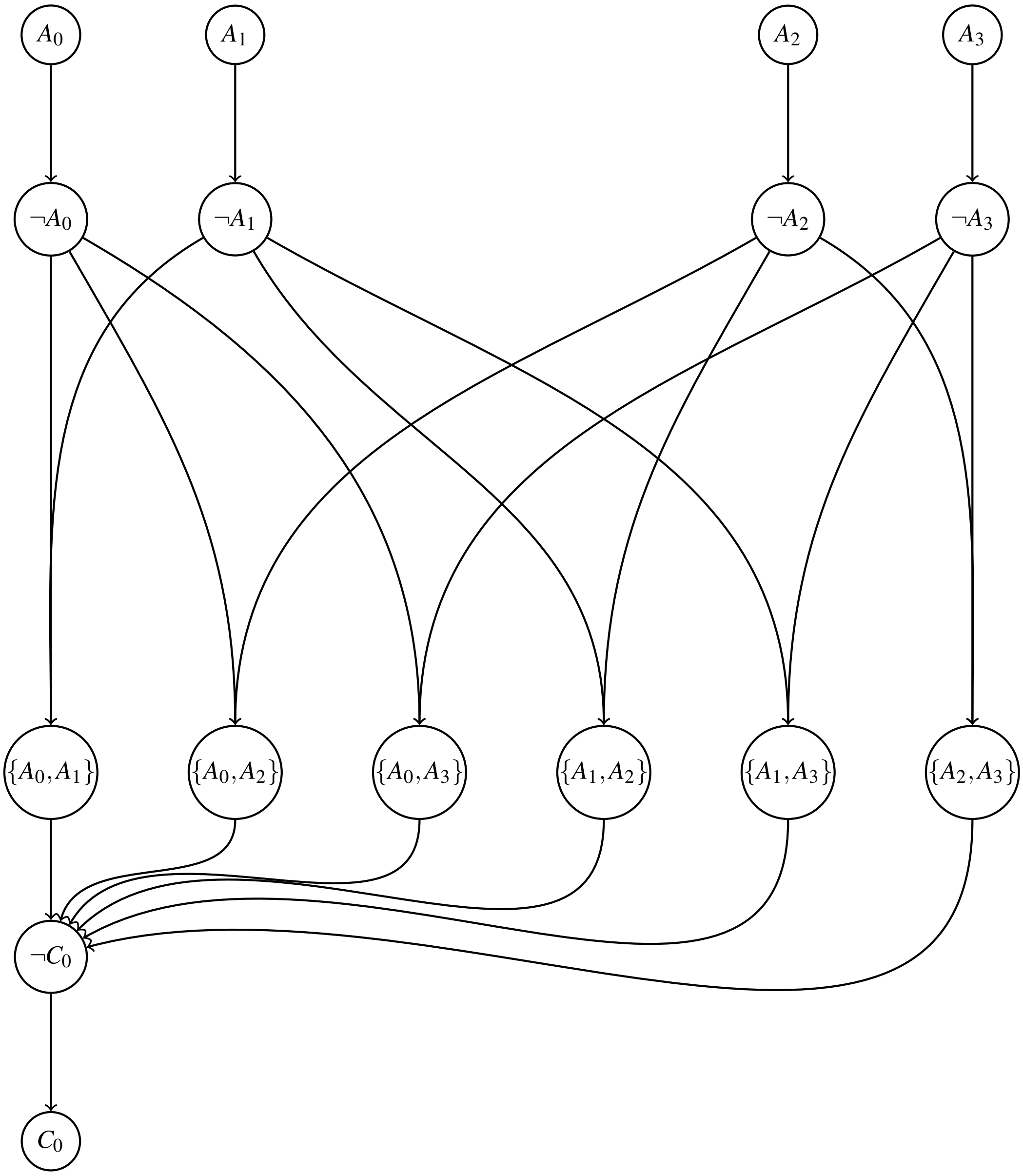
$\binom{j-i+1}{n}$ meta-arguments K_s are introduced.

Example 14. *The shorthand $A^{[0,3]}_{\geq 2}$ is encoded by the* TeAF *fragment in Figure 13.*

In the two examples above, a meta-argument $\neg C_t$ was used for indirectly expressing a positive interaction from an argument A_k (or K_s) to the argument C_t. Since only attacks are allowed in traditional AFs, support for C_t is expressed through an attack towards $\neg C_t$. In the next section, time is encoded in a framework with two independent relations between arguments: an attack and a support relation. It will facilitate the encoding of positive interactions such as those previously discussed and eliminate the need for using meta-argument $\neg C_t$.

Figure 13 – TeAF fragment encoding $C_0 = A^{[0,3]}_{\geq 2}$



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4 ENCODING TIME IN BAFS AND SETAFS

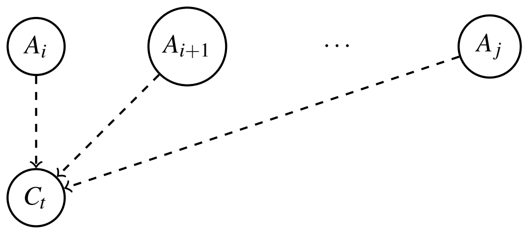
4.1 Representing time with support

The notion of bipolarity appears in many domains such as in knowledge and preference representation (Benferhat *et al.* (2002), Dubois and Prade (2005)), and has also been studied in the context of argumentation as in Cayrol and Lagasquie-Schiex (2013). In traditional AFs, positive interactions are encoded by the notion of defense, which in turn is defined exclusively from attacks and counterattacks. In BAFs, support is explicitly given as a relation between arguments, just like attacks.

Definition 39 (BAF)**.** *A Bipolar Abstract Argumentation Framework* (BAF) *is a tuple* $(\mathscr{A}, Att,$ $Sup)$ *where* $\mathscr{A}$ *is a set of arguments,* $Att \subseteq \mathscr{A} \times \mathscr{A}$ *is an attack relation and* $Sup \subseteq \mathscr{A} \times \mathscr{A}$ *is a support relation. The attackers of* $A \in \mathscr{A}$ *are denoted by* $Att(A) = \{B \in \mathscr{A} \mid (B,A) \in Att\}$*. Similarly, the direct supporters of* $A \in \mathscr{A}$ *are denoted by* $Sup(A) = \{B \in \mathscr{A} \mid (B,A) \in Sup\}$*.*

How attack and support interact is not a consensus in the literature, and many approaches give distinct interpretations for the support relation, such as those from Cayrol and Lagasquie-Schiex (2005), Boella *et al.* (2010), Nouioua and Risch (2011), Potyka (2021) (see Cohen *et al.* (2014) for a survey). In fact, the interpretation chosen should suit the application being modeled, as no interpretation will suffice to appropriately model every problem. A common interpretation for support is the deductive interpretation: if A is accepted and supports B, then B is accepted.

Figure 14 – BAF fragment with deductive support encoding $A^{[i,j]}_{\geq 1}$



Source: This author.

This interpretation can be used for modeling the temporal shorthands introduced in Prakken *et al.* (2020), as for instance the shorthand $\varphi_{C_t} = A^{[i,j]}_{\geq 1}$, which expresses that C should be accepted at time t if A is accepted at least once in $[i, j]$. One could try to encode this by the BAF of Figure 14, in which nodes represent arguments at some time point and dashed arrows represent support. Note that the translation approach was used, i.e., each argument in the framework actually represents a pair (argument, time instant).

Conversely, the shorthand $A^{[i,j]}_{\geq n}$ may be better encoded by an interpretation of support that takes quantity into account. Next, one such semantics is presented, namely the bi-complete semantics. After, the β-semantics with deductive interpretation is used to encode $A^{[i,j]}_{\geq 1}$.

4.1.1 Bi-complete semantics

In the bi-complete semantics proposed by Potyka (2021), acceptance is obtained from majority voting.

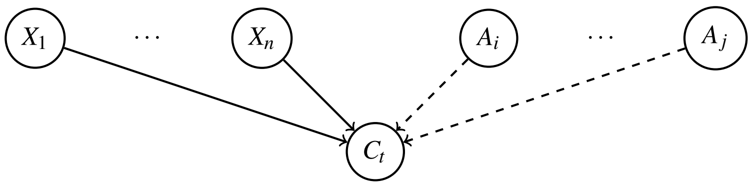
Definition 40 (Bi-complete)**.** *A labelling $\mathscr{L}$ of* BAF $\mathfrak{B} = (\mathscr{A}, Att, Sup)$ *is bi-complete iff for every $A \in \mathscr{A}$:*

a) $\mathscr{L}(A) = \textbf{in}$ *iff* $Att(A) \subseteq \textbf{out}(\mathscr{L})$ *or* $|Sup(A) \cap \textbf{in}(\mathscr{L})| > |Att(A) - \textbf{out}(\mathscr{L})|$*;*

b) $\mathscr{L}(A) = \textbf{out}$ *iff* $|Att(A) \cap \textbf{in}(\mathscr{L})| > |Sup(A) - \textbf{out}(\mathscr{L})|$.

It allows for compactly encoding $A^{[i,j]}_{\geq n+1}$, as shown in Figure 15. Note that meta-arguments $X_1, \cdots, X_n$ are always accepted, but arguments $A_i, \cdots, A_j$ are accepted according to the rest of the framework, which is not explicitly shown.

Figure 15 – BAF fragment encoding $A^{[i,j]}_{\geq n+1}$ in bi-complete semantics



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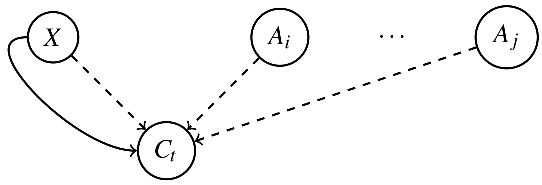
When there are at least $n+1$ accepted supporters, C_t is accepted (labelled **in**). Otherwise, C_t is labelled **out** (when there are less than n accepted supporters) or **undec** (when there are exactly n accepted supporters). This behavior is not exactly the one described by the shorthand $A^{[i,j]}_{\geq n+1}$ (according to which C_t should be rejected when A is accepted only n times in $[i, j]$), as argument C_t is labelled **undec** instead of **out** when it has the same amount n of accepted supporters and attackers. The main advantage of this strategy is that it is a very efficient approximation, as only n meta-arguments were added, and they can be shared among other shorthands whenever an always accepted argument is necessary.

Note however that this encoding does not make distinction between C_t supporters. The previous strategy is not capable of correctly encoding the shorthand $\varphi_{C_t} = A^{[i,j]}_{\geq n+1} \wedge B^{[i,j]}_{\geq n+1}$, as there are two independent counters to be tracked. For instance, for the shorthand $\varphi_{C_t} = A^{[i,j]}_{\geq 2} \wedge B^{[i,j]}_{\geq 2}$, argument C_t should not be accepted when A is accepted 4 times in $[i, j]$ and B is accepted only 1 time in $[i, j]$. However, the bi-complete encoding would detect 5 accepted supporters, which is more than the 4 meta-arguments attacking C_t. As a consequence, C_t would be accepted even though B is not accepted twice in $[i, j]$.

A problem arises when encoding $A^{[i,j]}_{\geq 1}$ with the strategy of $A^{[i,j]}_{\geq n+1}$ for $n = 0$, as C_t would be always accepted due to the lack of attackers. A solution is shown in Figure 16. By including both a support and an attack from meta-argument X, C_t is labeled **undec** when none of $A_i, \cdots, A_j$ are accepted. Otherwise, C_t has more accepted supporters than non-rejected attackers and is labeled **in**. It still is an approximate solution, as C_t is not rejected when all of $A_i, \cdots, A_j$ are rejected.

Figure 16 – BAF fragment encoding $A^{[i,j]}_{\geq 1}$ in bi-complete semantics



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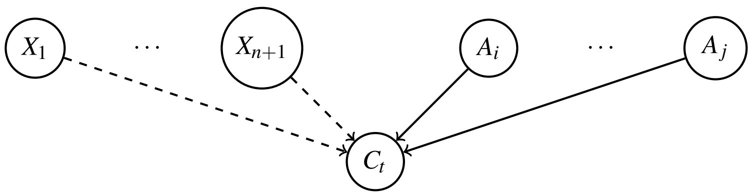
The solution for encoding $A^{[i,j]}_{\leq n}$ is almost symmetric to the encoding of $A^{[i,j]}_{\geq n+1}$. The resulting BAF is shown in Figure 17.

Figure 17 – BAF fragment encoding $A^{[i,j]}_{\leq n}$ in bi-complete semantics



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Again, the solution does not exactly represent the encoding $A^{[i,j]}_{\leq n}$ when A is accepted $n+1$ times in $[i,j]$, as C_t will be labelled **undec** instead of **out**.

4.1.2 β-complete semantics

The semantics of Alcântara and Cordeiro (2023) is based on the deductive interpretation of support, and is invariant to the inclusion of transitive supporters. For a BAF $(\mathscr{A}, Att, Sup)$, the reflexive and transitive closure of Sup is denoted by $\mathfrak{S}$. The supporters of an argument A are denoted by $\mathfrak{S}(A) = \{B \in \mathscr{A} \mid (B,A) \in \mathfrak{S}\}$. Defeat and defense are adapted to consider all supporters.

Definition 41 (Defeat/Defense in $\mathfrak{B}$)**.** *Let $\mathfrak{B} = (\mathscr{A}, Att, Sup)$ be a* BAF *and $S \subseteq \mathscr{A}$. For any $A \in \mathscr{A}$:*

a) S defeats A iff for every $A' \in \mathfrak{S}(A)$ there exists $B \in S$ such that $(B,A) \in Att$;

b) S defends A iff there exists $A' \in \mathfrak{S}(A)$ such that S defeats B for every $B \in Att(A')$.

Conflict-freeness follow as usual: a set $S \subseteq \mathscr{A}$ is conflict-free in $\mathfrak{B} = (\mathscr{A}, Att, Sup)$ iff there is no $A \in S$ such that S defeats A. Admissibility and its refinements are also naturally introduced. For the purposes of this discussion, β-complete extensions are enough and other β-semantics are omitted.

Definition 42 (β-semantics)**.** *Let $\mathfrak{B} = (\mathscr{A}, Att, Sup)$ be a* BAF*. A set $S \subseteq \mathscr{A}$ is:*

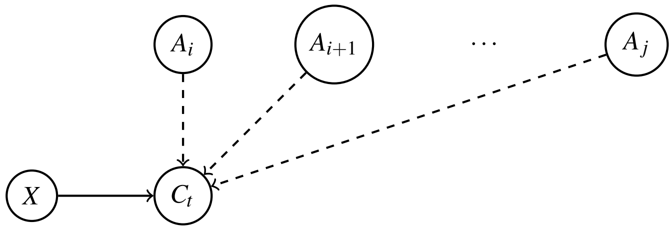
a) β-admissible iff S is conflict-free and defends every argument in S;

b) β-complete iff S is β-admissible and contains every argument defended by S.

Support is very strong in β-semantics, as only one accepted supporter leads to the acceptance of the supported argument. That suits the modeling of the temporal shorthand $A^{[i,j]}_{\geq 1}$: C_t is accepted when some of its supporters are accepted, and adding only one meta-argument X attacking C_t is enough for guaranteeing the rejection of C_t when A is never accepted in $[i,j]$. In this interpretation, C_t is rejected by default, i.e., C_t is rejected if every supporter is rejected.

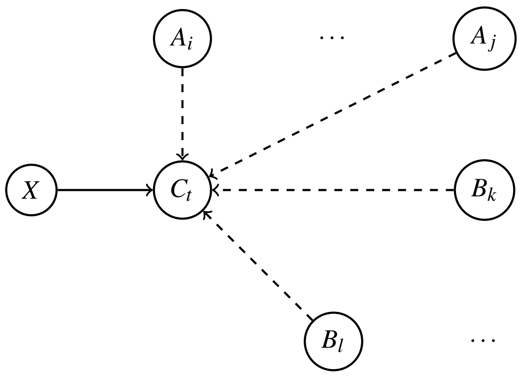
The resulting BAF is shown in Figure 18. Although a meta-argument X is used, it can be shared among multiple shorthands whenever an argument must be rejected by default. Hence, this approach is slightly more efficient than when TeAFs were used. Multiple shorthands can be encoded at once. For instance, the same strategy is applied for $\varphi_{C_t} = A^{[i,j]}_{\geq 1} \vee B^{[k,l]}_{\geq 1}$ in Figure 19.

Figure 18 – BAF fragment encoding $A^{[i,j]}_{\geq 1}$ in β-complete semantics



Source: This author.

Figure 19 – BAF fragment encoding $A^{[i,j]}_{\geq 1} \vee B^{[k,l]}_{\geq 1}$ in β-complete semantics



Source: This author.

It is still not trivial to encode the shorthand $\varphi_{C_t} = A^{[i,j]}_{\geq n}$, as many meta-arguments have to be added for each set of n time instants over $[i, j]$, just as it was done when encoding in AFs. Besides, arbitrarily combining shorthands is a complex task, since supporters are indistinguishable from each other. For example, the strategy in Figure 19 does not work for representing $\varphi_{C_t} = A^{[i,j]}_{\geq 1} \wedge B^{[k,l]}_{\geq 1}$. However, the strategy for representing temporal interactions in this section can be applied to any semantics for BAFs. In the next section, frameworks with high-order attacks facilitate the task of mixing shorthands by the connective $\wedge$. Recall that these are the same limitations encountered when using AFs to encode acceptance conditions of ADFs.

4.2 Representing time with joint attacks

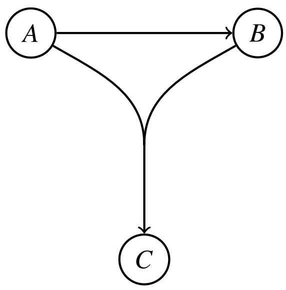
It is argued in Nielsen and Parsons (2006) that argumentation systems must allow for representing joint attacks on arguments. Similar to when using support, the increase in expressiveness will also allow for concisely representing some temporal interactions. SETAFs extend AFs by defining attacks as an interaction from a non-empty set of arguments to an argument.

Definition 43 (SETAF (Nielsen and Parsons (2006)))**.** *A Framework with Sets of Attacking Arguments (*SETAF*) is a pair* $(\mathscr{A}, Att)$ *where* $\mathscr{A}$ *is a set of arguments and* $Att \subseteq (2^{\mathscr{A}} - \{\emptyset\}) \times \mathscr{A}$ *is an attack relation. The set of attackers of* A *is denoted by* $Att(A) = \{\mathscr{B} \subseteq \mathscr{A} \mid (\mathscr{B}, A) \in Att\}$.

A set $S \subseteq \mathscr{A}$ is said to attack $A \in \mathscr{A}$ (denoted $S \blacktriangleright A$) iff there exists $S' \subseteq S$ such that $(S', A) \in Att$. Additionally, a set $S \subseteq \mathscr{A}$ is said to attack a set $\mathscr{B} \subseteq \mathscr{A}$ (denoted $S \blacktriangleright \mathscr{B}$) iff there exists $A \in \mathscr{B}$ such that S attacks A.

Example 15. *Let* $(\mathscr{A}, Att)$ *be a* SETAF *such that* $\mathscr{A} = \{A, B, C\}$ *and* $Att = \{(\{A\}, B), (\{A, B\}, C)\}$*, depicted in Figure 20 in which nodes represent arguments and solid arrows represent joint attacks. Note that* $\{A, B\}$ *attacks* B*, as there exists* $\{A\} \subset \{A, B\}$ *such that* $\{A\} \in Att(B)$*. The joint attack from* $\{A, B\}$ *to* C *intuitively means that if* A *and* B *are accepted, then* C *must be rejected.*

Figure 20 – SETAF from Example 15



Source: This author.

This interpretation of joint attack is formalized next through the use of labellings.

Definition 44 (Complete labelling (Adapted from Flouris and Bikakis (2019b)))**.** *A labelling* $\mathscr{L}$ *of a* SETAF $(\mathscr{A}, Att)$ *is complete iff for any* $A \in \mathscr{A}$ *it holds:*

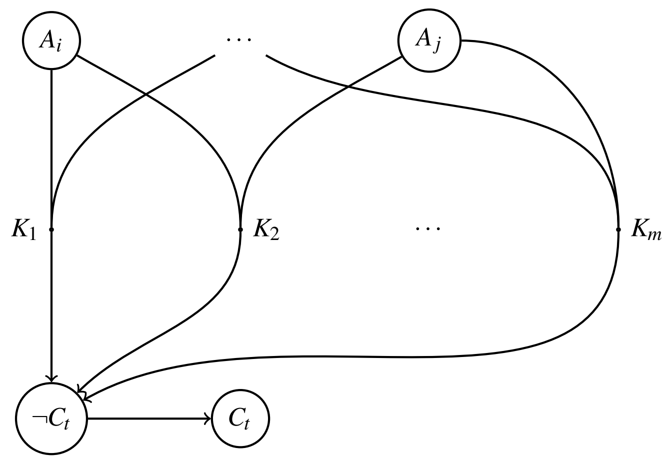
a) $\mathscr{L}(A) = \mathbf{in}$ *iff for every* $\mathscr{B} \subseteq \mathscr{A}$ *such that* $\mathscr{B} \blacktriangleright A$ *there exists* $B \in \mathscr{B}$ *such that* $\mathscr{L}(B) = \mathbf{out}$*;*

b) $\mathscr{L}(A) = \mathbf{out}$ *iff there exists* $\mathscr{B} \subseteq \mathscr{A}$ *such that* $\mathscr{B} \blacktriangleright A$ *and* $\mathscr{L}(B) = \mathbf{in}$ *for every* $B \in \mathscr{B}$.

The unique complete labelling of the SETAF from Example 15 labels A, C as **in** and B as **out**. The rejection of B comes from an attacker $\{A\}$ being accepted, whereas the acceptance of C comes from not having accepted attackers, as every attacker of C contains a non-accepted argument B.

Let $\mathscr{K} = \{K \subseteq \{A_i, \cdots, A_j\} \mid |K| = n\}$ with $|\mathscr{K}| = m$ be the set of all sets of n arguments among $A_i, \cdots, A_j$. Recall the strategy employed in Figure 12 where a meta-argument K_i $(1 \leq i \leq m)$ is included for every set $K_i' \in \mathscr{K}$, and K_i is accepted iff each argument in K_i' is accepted. Figures 21 and 23 use a different strategy by adding joint attacks instead of including meta-arguments. A label placed near each joint attack indicates the attacker and assists the comparison of this approach with the one using TeAFs. There is a joint attack from each set $K' \in \mathscr{K}$ to the meta-argument $\neg C_t$. That means meta-argument $\neg C_t$ is rejected iff there are n accepted arguments among $A_i, \cdots, A_j$. Hence, argument C_t is accepted iff A is accepted n times in $[i, j]$.

Figure 21 – SETAF fragment encoding $A_{\geq n}^{[i,j]}$



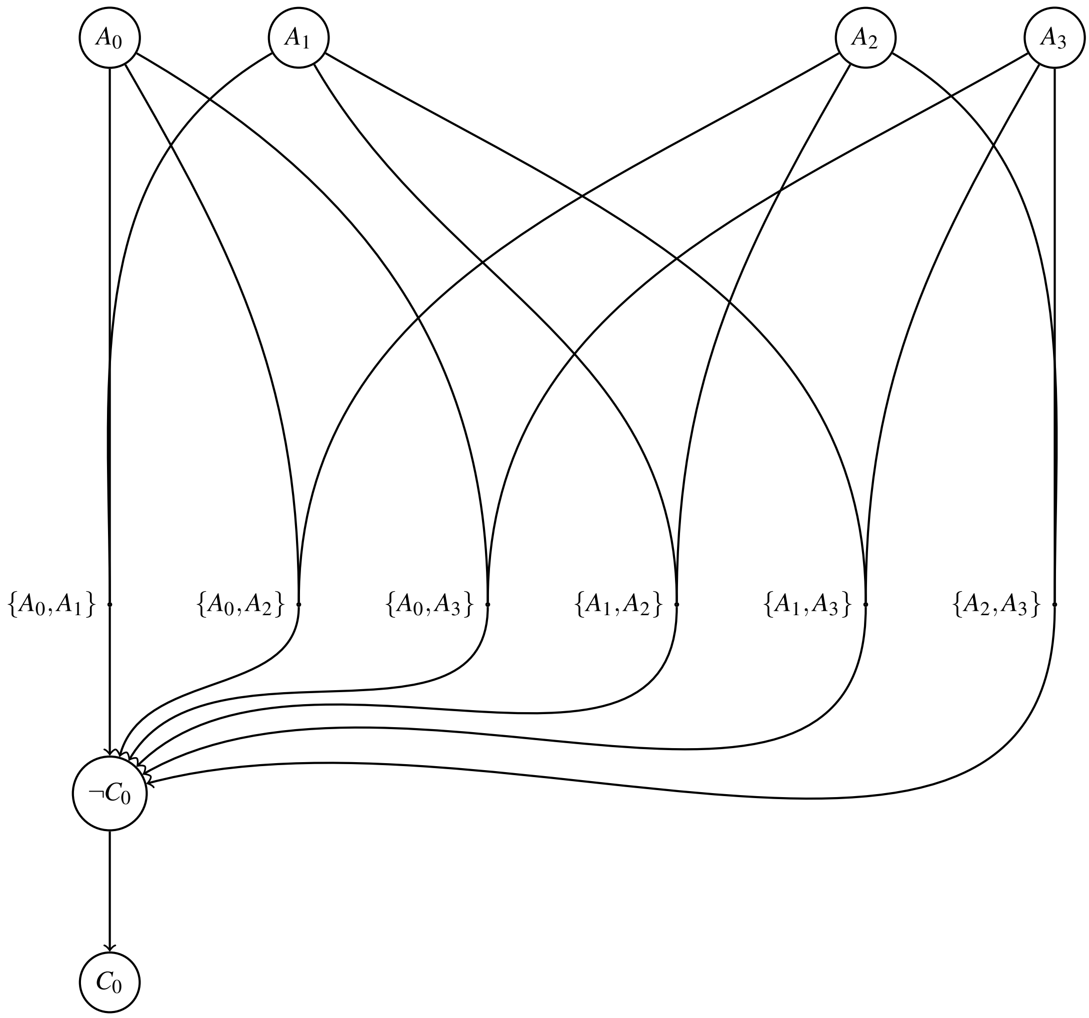
Source: This author.

Example 16. *The shorthand $A_{\geq 2}^{[0,3]}$ is encoded by the* SETAF *fragment in Figure 22. Compare it with the approach using* TeAF*s shown in Figure 13.*

Note that the shorthand $A_{\geq n}^{[i,j]} \wedge B_{\geq m}^{[k,l]}$ can be encoded by simply adding joints attacks towards $\neg C_t$ from every set K containing n elements among $A_i, \cdots, A_j$ or containing m elements among $B_k, \cdots, B_l$. Hence, compared to BAFs, SETAFs facilitate mixing temporal shorthands by the connective $\wedge$.

Although no meta-arguments were added in the solution of Figure 23, it contains an enormous amount of joint attacks. The use of support with a suitable interpretation in SETAFs

Figure 22 – SETAF fragment encoding $C_0 = A^{[0,3]}_{\geq 2}$



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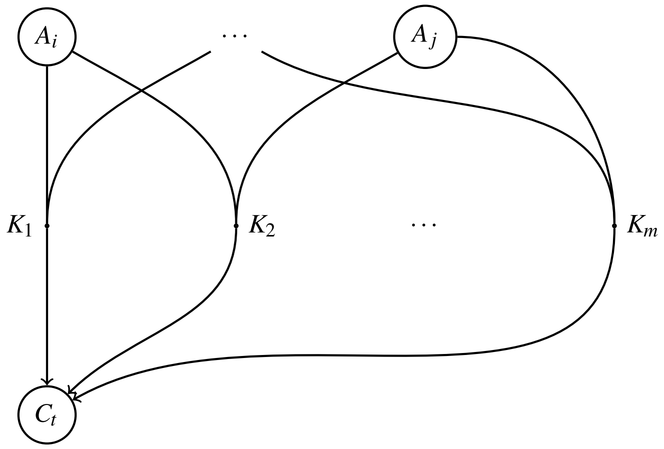
may allow for a more concise solution, and it is a topic worth investigating in future works.

Figure 23 – SETAF fragment encoding $A^{[i,j]}_{\leq n-1}$



Source: This author.

5 RELATED WORKS

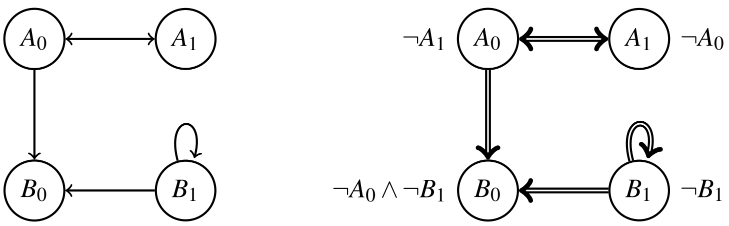
Time in abstract argumentation has been tackled in many ways. In the works of Cobo *et al.* (2010), Cobo *et al.* (2011), Budán *et al.* (2015), Budán *et al.* (2017), Zhu (2020), arguments are considered nonexistent in certain intervals of time, called availability intervals. Fundamental notions, such as defeat, defense and conflict-freeness, are adapted to disregard attacks from unavailable arguments. Argument acceptability becomes dependent on time and semantics inform *when* each argument is accepted, rather *which* arguments are accepted. Recalling the TAF from Figure 5, the attack from A to B signifies that B must be rejected whenever A is accepted. That notion of attack in TAFs (Cobo *et al.* (2011)) also applies to bipolar TAFs (Budán *et al.* (2017)) and TAFs with sets of attacking arguments (Zhu (2020)). In contrast, an attack from (A,a) to (B,b) in TeAFs indicates that accepting A at time point a implies rejecting B at time point b. Hence, both A and B can be accepted at the same time if $a \neq b$. Additionally, the connection between TeAFs and TAFs has been discussed in Section 3.3.

The probability approach used in TPAFs from Bistarelli *et al.* (2023a), Bistarelli *et al.* (2023b) provides other interpretations for attacks over time. An attack from A to B does not necessarily imply a conflict between A and B. A *partial conflict* from A to B occurs when A attacks B and both of them have positive probability of occurring at the same time. An *included conflict* from A to B occurs when A attacks B and there is no positive probability of B occurring without A occurring. A *total conflict* from A to B occurs when A attacks B and there is no positive probability of one occurring without the other. These notions still associate conflict with argument availability. They do not allow for A and B being accepted if A attacks B and they are available at the same time interval. That also differs from the expressiveness of TeAFs, where the attack from (A,a) to (B,b) encodes that B at time b must not be accepted if A is accepted at time a. Note how A and B might be accepted at the same time if $a \neq b$.

Timed Abstract Dialectical Frameworks (tADFs), introduced in Prakken *et al.* (2020), can be used to represent acceptance conditions changing over time. In a simple translation approach, each node in a tADF graph is annotated with a time point, just as was done in this work. TeAFs are very similar to tADFs, but the former is less general, as it is based on AFs instead of on ADFs. As discussed in Section 3.4, TeAFs may encode some shorthands of tADFs, at the cost of introducing many meta-arguments. Conversely, tADFs can easily represent TeAFs, just as ADFs can represent AFs. A TeAF and its corresponding tADF are shown in Figure 24.

Definition 45. *Let* $\mathfrak{T} = (\mathscr{A}, Att, \mathscr{T})$ *be an* TeAF. *The corresponding* tADF *is* $\mathfrak{D}_{\mathfrak{T}} = (\mathscr{A}, \mathscr{T}, \Phi)$ *with* $\Phi_{A_a} = \bigwedge_{(B,b) \in Att(A,a)} \neg B_b$ *for every* $A \in \mathscr{A}$ *and* $a \in \mathscr{T}$.

Figure 24 – Simple TeAF and its corresponding tADF



Source: This author.

In Budán *et al.* (2012), Extended Timed Abstract Argumentation Frameworks ($E-$TAFs) extend TAFs with the capability of representing availability of attacks. Intuitively, when an attack is unavailable, it is ignored, as if did not exist in the framework in the first place. There is a conflict between arguments A and B when A, B and the attack (A,B) are available at the same time. An attack (A,A) in an E-TAF indicates that A is never accepted when the attack (A,A) is available. This does not allow for representing that A is rejected in a future time point if it is accepted in the present, as encoded by the attack (A_0,A_1) in the TeAF of Figure 24.

Barringer *et al.* (2012) continues the investigation of Barringer *et al.* (2005) and presents an extensive study about time in argumentation. Many temporal interactions are considered, such as relative attacks, e.g. the attacks between A_0 and A_1 in Figure 24, and argument strength varying over time. However, it is applied to argumentation networks instead of to traditional Dung (1995) frameworks.

The topic of forgetting an argument is studied in Baumann *et al.* (2020), Baumann and Berthold (2021). It gives interpretations and limitations for what forgetting means in argumentation frameworks. It is a very different approach in which time is not the focus. The passage of time can only be noticed by forgetting arguments, i.e., they define operators for modifying a framework in order to represent forgetting, whereas in this work, an entire framework (TeAF) already encodes all temporal interactions between arguments.

6 CONCLUSION AND FUTURE WORKS

In this work, traditional Dung (1995) frameworks (AFs) were extended with the notion of time, allowing for the description of many kinds of temporal interactions between arguments. The resulting formalism is called Temporal Abstract Argumentation Frameworks (TeAFs) and its properties are thoroughly presented so that it becomes clear that it generalizes AFs and preserves its fundamental properties. The key insight is that arguments and their acceptability can be linked to time points. That is the approach taken by Prakken *et al.* (2020) with tADFs, a framework in which each argument is given an explicit acceptance condition. Instead, in TeAFs, this condition is given implicitly by an attack relation, but not all conditions can be compactly described by attacks. For instance, the TeAF fragment in Figure 13 contains many meta-arguments with the sole purpose of better representing an acceptance condition of a tADF. This lack of expressiveness due to allowing only attacks can be partially overcome by including time in frameworks with support (BAFs) and frameworks with joint attacks (SETAFs). This work discusses their capabilities and limitations for expressing the temporal shorthands used in tADFs.

In future works, algorithmic efficiency and complexity analysis are essential for applying these frameworks in practice. Besides, relating temporal argumentation with formalisms in other fields, such as Logic Programming, is also an interesting research direction, given the close relationship between Argumentation and Logic Programming established by Caminada *et al.* (2015). Additionally, many paths are open for further improving TeAFs' expressiveness, such as:

a) handling gradual argument acceptability, in which arguments are not labeled as **in**, **out** or **undec**, but instead are given acceptance values over an interval of real numbers;
b) defining semantics restricted to a particular time interval of interest, in which the accepted arguments are selected with respect to that interval, such as treating differently arguments that have the same acceptability over an entire interval;
c) proposing other shorthands for compactly encoding temporal interactions.

BIBLIOGRAPHY

ALCÂNTARA, J.; CORDEIRO, R. Bipolar argumentation frameworks with a dual relation between defeat and defence (under review). In: JOURNAL OF LOGIC AND COMPUTATION. [*S. l.*], 2023.

AMGOUD, L.; BEN-NAIM, J.; DODER, D.; VESIC, S. Acceptability semantics for weighted argumentation frameworks. In: INTERNATIONAL JOINT CONFERENCES ON ARTIFICAL INTELLIGENCE (IJCAI). **Twenty-Sixth International Joint Conference on Artificial Intelligence (IJCAI 2017)**. [*S. l.*], 2017.

AMGOUD, L.; CAYROL, C. On the acceptability of arguments in preference-based argumentation. **CoRR**, abs/1301.7358, 2013. Available at: http://arxiv.org/abs/1301.7358.

BARONI, P.; CERUTTI, F.; GIACOMIN, M.; GUIDA, G. Afra: Argumentation framework with recursive attacks. **International Journal of Approximate Reasoning**, Elsevier, v. 52, n. 1, p. 19–37, 2011.

BARONI, P.; GABBAY, D.; GIACOMIN, M.; TORRE, L. Van der. Handbook of formal argumentation. p. 159–201, 2018.

BARRINGER, H.; GABBAY, D.; WOODS, J. Temporal dynamics of support and attack networks: From argumentation to zoology. **Mechanizing Mathematical Reasoning: Essays in Honor of Jörg H. Siekmann on the Occasion of His 60th Birthday**, Springer, p. 59–98, 2005.

BARRINGER, H.; GABBAY, D. M.; WOODS, J. Modal and temporal argumentation networks. **Argument & Computation**, IOS Press, v. 3, n. 2-3, p. 203–227, 2012.

BAUMANN, R.; BERTHOLD, M. Limits and possibilities of forgetting in abstract argumentation. In: **19 th International Workshop on Non-Monotonic Reasoning**. [*S. l.*: *s. n.*], 2021. p. 61.

BAUMANN, R.; GABBAY, D.; RODRIGUES, O. Forgetting an argument. In: **Proceedings of the AAAI Conference on Artificial Intelligence**. [*S. l.*: *s. n.*], 2020. v. 34, n. 03, p. 2750–2757.

BENFERHAT, S.; DUBOIS, D.; KACI, S.; PRADE, H. Bipolar representation and fusion of preferences on the possibilistic logic framework. **KR**, v. 2, p. 421–432, 2002.

BISTARELLI, S.; DAVID, V.; SANTINI, F.; TATICCHI, C. Temporal probabilistic argumentation frameworks. **argument**, v. 12, p. 7l, 2023.

BISTARELLI, S.; DAVID, V.; SANTINI, F.; TATICCHI, C. Towards a temporal probabilistic argumentation framework. In: **The International FLAIRS Conference Proceedings**. [*S. l.*: *s. n.*], 2023. v. 36.

BOELLA, G.; GABBAY, D. M.; TORRE, L. van der; VILLATA, S. Support in abstract argumentation. In: FRONTIERS IN ARTIFICIAL INTELLIGENCE AND APPLICATIONS, IOS PRESS. **Proceedings of the Third International Conference on Computational Models of Argument (COMMA'10)**. [*S. l.*], 2010. p. 40–51.

BREWKA, G.; ELLMAUTHALER, S.; STRASS, H.; WALLNER, J. P.; WOLTRAN, S. Abstract dialectical frameworks revisited. In: **Proceedings of the Twenty-Third international joint conference on Artificial Intelligence**. [*S. l.*: *s. n.*], 2013. p. 803–809.

BREWKA, G.; WOLTRAN, S. Abstract dialectical frameworks. In: **Proceedings of the Twelfth International Conference on Principles of Knowledge Representation and Reasoning**. [*S. l.*: *s. n.*], 2010. p. 102–111.

BUDÁN, M. C.; LUCERO, M. G.; CHESÑEVAR, C.; SIMARI, G. R. Modeling time and valuation in structured argumentation frameworks. **Information Sciences**, v. 290, p. 22–44, 2015. ISSN 0020-0255. Available at: https://www.sciencedirect.com/science/article/pii/S0020025514008172.

BUDÁN, M. C.; LUCERO, M. G.; CHESNEVAR, C. I.; SIMARI, G. R. Modelling time and reliability in structured argumentation frameworks. In: **Thirteenth International Conference on the Principles of Knowledge Representation and Reasoning**. [*S. l.*: *s. n.*], 2012.

BUDÁN, M. C.; LUCERO, M. G.; VIGLIZZO, I.; SIMARI, G. R. A labeled argumentation framework. **Journal of Applied Logic**, Elsevier, v. 13, n. 4, p. 534–553, 2015.

BUDáN, M. C.; COBO, M. L.; MARTINEZ, D. C.; SIMARI, G. R. Bipolarity in temporal argumentation frameworks. **International Journal of Approximate Reasoning**, v. 84, p. 1–22, 2017. ISSN 0888-613X. Available at: https://www.sciencedirect.com/science/article/pii/S0888613X17300713.

CAMINADA, M. On the issue of reinstatement in argumentation. In: SPRINGER. **European Workshop on Logics in Artificial Intelligence**. [*S. l.*], 2006. p. 111–123.

CAMINADA, M.; SÁ, S.; ALCÂNTARA, J.; DVOŘÁK, W. On the equivalence between logic programming semantics and argumentation semantics. **International Journal of Approximate Reasoning**, Elsevier, v. 58, p. 87–111, 2015.

CAYROL, C.; LAGASQUIE-SCHIEX, M.-C. On the acceptability of arguments in bipolar argumentation frameworks. In: SPRINGER. **European Conference on Symbolic and Quantitative Approaches to Reasoning and Uncertainty**. [*S. l.*], 2005. p. 378–389.

CAYROL, C.; LAGASQUIE-SCHIEX, M.-C. Bipolarity in argumentation graphs: Towards a better understanding. **International Journal of Approximate Reasoning**, v. 54, n. 7, p. 876–899, 2013. ISSN 0888-613X. Special issue: Uncertainty in Artificial Intelligence and Databases. Available at: https://www.sciencedirect.com/science/article/pii/S0888613X13000509.

COBO, M.; MARTINEZ, D.; SIMARI, G. On admissibility in timed abstract argumentation frameworks. In: . [*S. l.*: *s. n.*], 2010. v. 215, p. 1007–1008.

COBO, M. L.; MARTINEZ, D. C.; SIMARI, G. R. Acceptability in timed frameworks with intermittent arguments. In: ILIADIS, L.; MAGLOGIANNIS, I.; PAPADOPOULOS, H. (Ed.). **Artificial Intelligence Applications and Innovations**. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011. p. 202–211. ISBN 978-3-642-23960-1.

COHEN, A.; GOTTIFREDI, S.; GARCÍA, A. J.; SIMARI, G. R. A survey of different approaches to support in argumentation systems. **The Knowledge Engineering Review**, Cambridge University Press, v. 29, n. 5, p. 513–550, 2014.

COSTE-MARQUIS, S.; KONIECZNY, S.; MARQUIS, P.; OUALI, M. A. Weighted attacks in argumentation frameworks. In: **Thirteenth international conference on the principles of knowledge representation and reasoning**. [*S. l.*: *s. n.*], 2012.

DENECKER, M.; MAREK, V. W.; TRUSZCZYńSKI, M. Ultimate approximation and its application in nonmonotonic knowledge representation systems. **Information and Computation**, v. 192, n. 1, p. 84–121, 2004. ISSN 0890-5401. Available at: https://www.sciencedirect.com/science/article/pii/S0890540104000306.

DUBOIS, D.; PRADE, H. A bipolar possibilistic representation of knowledge and preferences and its applications. In: SPRINGER. **International Workshop on Fuzzy Logic and Applications**. [*S. l.*], 2005. p. 1–10.

DUNG, P. M. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. **Artificial intelligence**, Elsevier, v. 77, n. 2, p. 321–357, 1995.

FISHER, M. Temporal representation and reasoning. **Foundations of Artificial Intelligence**, Elsevier, v. 3, p. 513–550, 2008.

FLOURIS, G.; BIKAKIS, A. A comprehensive study of argumentation frameworks with sets of attacking arguments. **International Journal of Approximate Reasoning**, v. 109, p. 55–86, 2019. ISSN 0888-613X. Available at: https://www.sciencedirect.com/science/article/pii/S0888613X18304742.

FLOURIS, G.; BIKAKIS, A. A comprehensive study of argumentation frameworks with sets of attacking arguments. **International Journal of Approximate Reasoning**, Elsevier, v. 109, p. 55–86, 2019.

LI, H.; OREN, N.; NORMAN, T. J. Probabilistic argumentation frameworks. In: SPRINGER. **International workshop on theorie and applications of formal argumentation**. [*S. l.*], 2011. p. 1–16.

NIELSEN, S. H.; PARSONS, S. A generalization of dung's abstract framework for argumentation: Arguing with sets of attacking arguments. In: SPRINGER. **International Workshop on Argumentation in Multi-Agent Systems**. [*S. l.*], 2006. p. 54–73.

NOUIOUA, F.; RISCH, V. Argumentation frameworks with necessities. In: SPRINGER. **Scalable Uncertainty Management: 5th International Conference, SUM 2011, Dayton, OH, USA, October 10-13, 2011. Proceedings 5**. [*S. l.*], 2011. p. 163–176.

OGUEGO, C.; AUGUSTO, J.; MUÑOZ, A.; SPRINGETT, M. Using argumentation to manage users' preferences. **Future Generation Computer Systems**, v. 81, p. 235–243, 2018. ISSN 0167-739X. Available at: https://www.sciencedirect.com/science/article/pii/S0167739X1630379X.

PANI, A.; BHATTACHARJEE, G. Temporal representation and reasoning in artificial intelligence: A review. **Mathematical and Computer Modelling**, Elsevier, v. 34, n. 1-2, p. 55–80, 2001.

POTYKA, N. Generalizing complete semantics to bipolar argumentation frameworks. In: SPRINGER. **Symbolic and Quantitative Approaches to Reasoning with Uncertainty: 16th European Conference, ECSQARU 2021, Prague, Czech Republic, September 21–24, 2021, Proceedings 16**. [*S. l.*], 2021. p. 130–143.

PRAKKEN, H. *et al.* Timed abstract dialectical frameworks: A simple translation-based approach. **Computational Models of Argument: Proceedings of COMMA 2020**, IOS Press, v. 326, p. 103, 2020.

TARSKI, A. A lattice-theoretical fixpoint theorem and its applications. 1955.

VASSILIADES, A.; BASSILIADES, N.; PATKOS, T. Argumentation and explainable artificial intelligence: a survey. **The Knowledge Engineering Review**, Cambridge University Press, v. 36, p. e5, 2021.

ZHU, J. **A Temporal Argumentation Framework with Sets of Attacking Arguments**. Tese (Doctorate) – UNIVERSITY OF ABERDEEN, 2020.

7 PROOFS

7.1 Proofs from Chapter 2

Theorem 1. *Let* $\mathfrak{A} = (\mathscr{A}, Att)$ *be an* AF *and* $D_{\mathfrak{A}}$ *its associated* ADF. *A three-valued interpretation* v *of* $D_{\mathfrak{A}}$ *is admissible, complete, grounded, preferred, stable iff its associated labelling* $\mathscr{L}_v$ *is respectively an admissible, complete, grounded, preferred, stable in* $\mathfrak{A}$.

Proof. Is is trivial that for any $u, v \in \mathscr{V}_2^{D_{\mathfrak{A}}}$, it holds $u \leq_i v$ iff $\mathbf{in}(\mathscr{L}_u) \subseteq \mathbf{in}(\mathscr{L}_v)$ and $\mathbf{out}(\mathscr{L}_u) \subseteq \mathbf{out}(\mathscr{L}_v)$. Then, maximizing $\mathbf{in}(\mathscr{L}_v)$ and $\mathbf{out}(\mathscr{L}_v)$ coincides with maximizing v w.r.t. $\leq_i$. Therefore, a proof for the complete semantics suffices to show the equivalence for the grounded, preferred and stable semantics. Also, the proof for the admissible semantics is very similar to the proof shown below.

Let v be a three-valued interpretation of $D_{\mathfrak{A}}$. Then, v is complete iff $v = \Gamma_{D_{\mathfrak{A}}}(v)$ iff $v(A) = \sqcap_i \{w(\Phi_A) \mid w \in [v]_2^{D_{\mathfrak{A}}}\}$ for every $A \in \mathscr{A}$. Then:

a) ($\Rightarrow$) Assume v is complete and let $A \in \mathscr{A}$:
   - Assume $\mathscr{L}_v(A) = \mathbf{out}$, i.e., $v(A) = \mathbf{f}$. Then, $\sqcap_i \{w(\Phi_A) \mid w \in [v]_2^{D_{\mathfrak{A}}}\} = \mathbf{f}$, which means $w(\Phi_A) = \mathbf{f}$ for every $w \in [v]_2^{D_{\mathfrak{A}}}$. In particular, $v(\Phi_A) = \mathbf{f}$. As $\Phi_A = \bigwedge_{B \in Att(A)} \neg B$, we conclude $v(B) = \mathbf{t}$ for some $B \in Att(A)$. Hence, $\mathscr{L}_v(B) = \mathbf{in}$ for some $B \in Att(A)$;
   - Assume $\mathscr{L}_v(A) = \mathbf{in}$, i.e., $v(A) = \mathbf{t}$. Then, $\sqcap_i \{w(\Phi_A) \mid w \in [v]_2^{D_{\mathfrak{A}}}\} = \mathbf{t}$, which means $w(\Phi_A) = \mathbf{t}$ for every $w \in [v]_2^{D_{\mathfrak{A}}}$. In particular, $v(\Phi_A) = \mathbf{t}$. As $\Phi_A = \bigwedge_{B \in Att(A)} \neg B$, we conclude $v(B) = \mathbf{f}$ for every $B \in Att(A)$. From 1, $\mathscr{L}_v(B) = \mathbf{out}$ for every $B \in Att(A)$;
   - Assume $\mathscr{L}_v(A) = \mathbf{undec}$, i.e., $v(A) = \mathbf{u}$. Then, $\sqcap_i \{w(\Phi_A) \mid w \in [v]_2^{D_{\mathfrak{A}}}\} = \mathbf{u}$. If $v(\Phi_A) \in \{\mathbf{t}, \mathbf{f}\}$, then $w(\Phi_A) = v(\Phi_A) \neq \mathbf{u}$ for every $w \in [v]_2^{D_{\mathfrak{A}}}$, which contradicts $\sqcap_i \{w(\Phi_A) \mid w \in [v]_2^{D_{\mathfrak{A}}}\} = \mathbf{u}$. Hence, $v(\Phi_A) = \mathbf{u}$. As $\Phi_A = \bigwedge_{B \in Att(A)} \neg B$, there is some $B \in Att(A)$ such that $\mathscr{L}_v(B) \neq \mathbf{out}$ and it holds $\mathscr{L}_v(B) \neq \mathbf{in}$ for every $B \in Att(A)$.

b) ($\Leftarrow$) Assume $\mathscr{L}_v$ is complete and let $A \in \mathscr{A}$:
   - Assume $v(A) = \mathbf{f}$, i.e., $\mathscr{L}_v(A) = \mathbf{out}$. As v is complete, there is $B \in Att(A)$ such that $\mathscr{L}_v(B) = \mathbf{in}$, i.e., $v(B) = \mathbf{t}$. Thus, $v(\Phi_A) = \mathbf{f}$;
   - Assume $v(A) = \mathbf{t}$, i.e., $\mathscr{L}_v(A) = \mathbf{in}$. As v is complete, for every $B \in Att(A)$ it holds $\mathscr{L}_v(B) = \mathbf{out}$, i.e., $v(B) = \mathbf{f}$. Thus, $v(\Phi_A) = \mathbf{t}$;
   - Assume $v(A) = \mathbf{u}$, i.e., $\mathscr{L}_v(A) = \mathbf{undec}$. As v is complete, there is $B \in Att(A)$ such

that $\mathscr{L}_v(B) \neq$ **out** (i.e., $v(B) \neq$ **f**) and for every $B \in Att(A)$ it holds $\mathscr{L}_v(B) \neq$ **in** (i.e., $v(B) \neq$ **t**). Thus, $v(\Phi_A) = \mathbf{u}$.

Therefore, $v(A) = v(\Phi_A)$ for every $A \in \mathscr{A} \Rightarrow v = \Gamma_{D_{\mathfrak{A}}}(v) \Rightarrow v$ is complete.

□

7.2 Proofs from Chapter 3

Lemma 1 (Fundamental Lemma)**.** *Let $\mathfrak{T} = (\mathscr{A}, Att, \mathscr{T})$ be a* TeAF*, S be an admissible extension of $\mathfrak{T}$, and $(A,a),(A',a') \in \mathscr{A} \times \mathscr{T}$ be acceptable with respect to S. Then*

1. *$S' = S \cup \{(A,a)\}$ is admissible, and*
2. *(A',a') is acceptable with respect to S'.*

Proof.

a) It suffices to show S' is conflict-free. By absurd, assume S' is not conflict-free. This means there exists $(B,b) \in S'$ such that S' defeats (B,b). Then, there exists $(C,c) \in S'$ such that $(C,c) \in Att(B,b)$.

There are two possibilities:

– $(B,b) \in S$: in this case, S does not defeat (B,b), because S is conflict-free. Therefore, $(C,c) \notin S$ and $(C,c) = (A,a)$. Also, S defends (B,b) and hence S defeats (A,a), which is an attacker of (B,b). As S also defends (A,a), it is an absurd given that S is conflict-free.

– $(B,b) = (A,a)$: given that S defends (A,a) and that $(C,c) \in Att(A,a)$, we know $(C,c) \notin S$ as S is conflict-free. Thus, $(C,c) = (A,a)$ and S defeats (A,a). As S also defends (A,a), it is an absurd given that S is conflict-free.

b) As $(A',a') \in \mathscr{F}_M(S)$, it is clear $(A',a') \in \mathscr{F}_M(S \cup \{(A,a)\})$, i.e., (A',a') is acceptable with respect to S'.

□

Theorem 2. *Let $\mathfrak{T}$ be a* TeAF.

(1) The set of all admissible extensions of $\mathfrak{T}$ forms a complete partial order with respect to set inclusion;

(2) For each admissible extension S of $\mathfrak{T}$, there exists a maximal admissible extension $\mathscr{E}$ of $\mathfrak{T}$ such that $S \subseteq \mathscr{E}$.

Proof. Let $\text{ADM}_M = \{S \mid S \text{ is an admissible extension of } M\}$. We will show $(\text{ADM}_M, \subseteq)$ is a complete partially ordered set:

a) Let $(\mathscr{D}, \subseteq)$ be a directed set with $\mathscr{D} \subseteq \text{ADM}_M$. We have to prove

$\sup(\mathscr{D}) = \bigcup\{S \mid S \in \mathscr{D}\} \in \text{ADM}_M$

– $\sup(\mathscr{D})$ is conflict-free. By absurd, suppose $\sup(\mathscr{D})$ is not conflict-free. This means there exists $(A,a) \in \sup(\mathscr{D})$ such that $\sup(\mathscr{D})$ defeats (A,a), i.e., there exists $(B,b) \in \sup(\mathscr{D})$ such that $(B,b) \in Att(A,a)$. As $(A,a) \in \sup(\mathscr{D})$, there exists $S \in \mathscr{D}$ such that $(A,a) \in S$ and therefore S defends (A,a). Similarly, as $(B,b) \in \sup(\mathscr{D})$, there exists $S' \in \mathscr{D}$ such that S' defends (B,b). As $(B,b) \in Att(A,a)$ and S defends (A,a), we know S defeats (B,b). Given that $(\mathscr{D}, \subseteq)$ is a directed set, there exists $S'' \in \mathscr{D}$ such that $S \subseteq S''$ and $S' \subseteq S''$. It follows that S'' both defeats B and defends B. It is an absurd as S'' is conflict-free.

– $\sup(\mathscr{D}) \subseteq \mathscr{F}_M(\sup(\mathscr{D}))$: $(A,a) \in \sup(\mathscr{D}) \Rightarrow$ there exists $S \in \mathscr{D} \subseteq \text{ADM}_M$ such that $(A,a) \in S \Rightarrow$ there exists $S \in \mathscr{D}$ such that $(A,a) \in \mathscr{F}_M(S) \Rightarrow$ in consequence of Lemma 1, $(A,a) \in \mathscr{F}_M(\sup(\mathscr{D})) = \mathscr{F}_M(\bigcup\{S \mid S \in \mathscr{D}\})$.

b) Let $S \in \text{ADM}_M$ and $\mathscr{G} \subseteq \{S' \mid S' \text{ is an admissible extension of } M \text{ and } S \subseteq S'\}$ such that $(\mathscr{G}, \subseteq)$ is a directed set. According to the previous item, $\mathscr{E} = \sup(\mathscr{G})$ is an admissible extension of M. Indeed, by definition, $\mathscr{E}$ is a maximal admissible extension of M such that $S \subseteq \mathscr{E}$.

□

Proposition 12. *Let $\mathfrak{T} = (\mathscr{A}, Att, \mathscr{T})$ be a* TeAF*. Then S is a stable extension of $\mathfrak{T}$ iff $S = \{(A,a) \mid (A,a) \text{ is not defeated by } S\}$.*

Proof. $S = \{(A,a) \mid (A,a) \text{ is not defeated by } S\}$ iff $S = \{(A,a) \mid \text{for every } (B,b) \in Att(A,a) \text{ it holds } (B,b) \notin S\}$ and S is conflict-free and $S \cup S^+ = \mathscr{A} \times \mathscr{T}$ iff $S = \mathscr{F}_M(S)$ and S is conflict-free and $S \cup S^+ = \mathscr{A} \times \mathscr{T}$ iff S is a stable extension of M. □

Proposition 13. *For any* TeAF *$\mathfrak{T}$, any stable extension of $\mathfrak{T}$ is a preferred extension of $\mathfrak{T}$. However, there is some* TeAF *$\mathfrak{T}$ such that not every preferred extension of $\mathfrak{T}$ is a stable extension of $\mathfrak{T}$.*

Proof. Let S be a stable extension of M. By absurd, suppose S is not a preferred extension of M. This means there exists a complete extension S' of M such that $S \subset S'$. As any argument in $S' - S$ is defeated by S, it follows S' is not conflict-free. An absurd, as S' is a complete extension of M.

In order to show the reverse does not hold, we present the TeAF $M = (\mathscr{A}, Att, \{0\})$ with $\mathscr{A} = \{A\}$, $Att = \{(A,0),(A,0)\}$. It is clear that the empty set is a preferred extension of M which is not stable. □

Lemma 2. *Let $\mathfrak{T} = (\mathscr{A}, Att, \mathscr{T})$ be a* TeAF. *If $S \subseteq \mathscr{A} \times \mathscr{T}$ is conflict-free, then $\mathscr{F}_{\mathfrak{T}}(S)$ is also conflict-free.*

Proof. Let $S \subseteq \mathscr{A} \times \mathscr{T}$ be a conflict-free set. By absurd, assume $\mathscr{F}_M(S)$ is not conflict-free. This means there exists $(A,a) \in \mathscr{F}_M(S)$ such that $\mathscr{F}_M(S)$ defeats (A,a), i.e., there exists $(B,b) \in Att(A,a) \cap \mathscr{F}_M(S)$. Then, as S defends (A,a), we know that S defeats (B,b). However, S also defends (B,b). It is an absurd as S is conflict-free. □

Lemma 3. *Let $\mathfrak{T} = (\mathscr{A}, Att, \mathscr{T})$ be a* TeAF. *Then $\mathscr{F}_{\mathfrak{T}}$ is monotonic with respect to set inclusion.*

Proof. We have to prove that if $S \subseteq S'$, then $\mathscr{F}_M(S) \subseteq \mathscr{F}_M(S')$. This result is straightforward, because if (A,a) is defended by S, then (A,a) is also defended by any superset of S. □

Lemma 4. *Let $\mathfrak{T}$ be a* TeAF. *It holds S is a preferred extension of $\mathfrak{T}$ iff S is a $\subseteq$-maximal admissible extension of $\mathfrak{T}$.*

Proof.

a) Let S be a $\subseteq$-maximal admissible extension of M, i.e., S is conflict-free and $S \subseteq \mathscr{F}_M(S)$. From Lemma 2, it is clear $\mathscr{F}_M(S)$ is also conflict-free. From the monotonicity of $\mathscr{F}_M$ (Lemma 3), we obtain

$$\mathscr{F}_M(S) \subseteq \mathscr{F}_M(\mathscr{F}_M(S))$$

Hence, $\mathscr{F}_M(S)$ is also an admissible extension of M. As $S \subseteq \mathscr{F}_M(S)$ and S is a maximal admissible extension of M, it follows $S = \mathscr{F}_M(S)$ and consequently S is a preferred extension of M.

b) Let S be a preferred extension of M. By absurd, assume S is not a maximal admissible extension of M. Then there exists a maximal admissible extension S' of M such that $S \subset S'$. From the previous case, we obtain S' is also a preferred extension of M. It is an absurd as $S \subset S'$.

□

Theorem 6. *Every* TeAF $\mathfrak{T}$ *possesses a unique grounded extension and it is the $\subseteq$-least fixpoint of $\mathscr{F}_{\mathfrak{T}}$.*

Proof. From Theorem 5, we know M has at least one complete extension S'. From the monotonicity of $\mathscr{F}_M$ (Lemma 3) and the well-known Knaster-Tarski Theorem from Tarski (1955), we obtain the least fixpoint S of $\mathscr{F}_M$ exists. As $S \subseteq S'$ and S' is conflict-free, then S is also conflict-free. This means S is the unique grounded extension of M. $\square$

Theorem 7. *Let* $\mathfrak{T} = (\mathscr{A}, Att, \mathscr{T})$ *be a* TeAF. *The complete extensions of* $\mathfrak{T}$ *form a complete semilattice with respect to set inclusion.*

Proof. Let $\text{ADM}_M = \{S \mid S \text{ is an admissible extension of } M\}, \text{COMP}_M = \{S \mid S \text{ is a complete extension of } M\}$ and $\mathscr{G} \in \text{COMP}_M$ the grounded extension of M. We will show $(\text{COMP}_M, \subseteq)$ is a complete semilattice:

a) Each nonempty subset of COMP_M has a greatest lower bound:
Let $\mathfrak{S} \subseteq \text{COMP}_M$ and $\mathfrak{S} \neq \emptyset$. We define $LB = \{\mathscr{E} \in \text{ADM}_M \mid \forall C' \in \mathfrak{S} : \mathscr{E} \subseteq C'\}$.
LB is not empty, as clearly $\mathscr{G} \in LB$. Given that $\mathscr{E} \subseteq C'$ for any $\mathscr{E} \in LB$ and for any $C' \in \mathfrak{S}$, we obtain from Lemma 3, $\mathscr{F}_M(\mathscr{E}) \subseteq \mathscr{F}_M(C') = C'$, i.e., $\forall \mathscr{E} \in LB, \forall C' \in \mathfrak{S} : \mathscr{F}_M(\mathscr{E}) \subseteq C'$.
Thus,

$$\forall \mathscr{E} \in LB : \mathscr{F}_M(\mathscr{E}) \in LB. \tag{7.1}$$

Let $C = \bigcup\{\mathscr{E} \mid \mathscr{E} \in LB\}$. Now we have to prove $C \in \text{ADM}_M$:

– C is conflict-free: by absurd, suppose C is not conflict-free. This means there exists $(A,a) \in C$ such that C defeats (A,a), i.e., there exists $(B,b) \in Att(A,a) \cap C$. As $(A,a) \in C$, there exists $\mathscr{E} \in LB$ such that $\mathscr{E}$ defends (A,a) and therefore $\mathscr{E}$ defeats (B,b). As $(B,b) \in C$, there exists $\mathscr{E}' \in LB$ such that $\mathscr{E}'$ defends (B,b). It is clear $\mathscr{E} \cup \mathscr{E}' \in LB$ as both $\mathscr{E}$ and $\mathscr{E}'$ are in LB. It follows $\mathscr{E} \cup \mathscr{E}'$ both defeats B and defends B. It is an absurd as $\mathscr{E} \cup \mathscr{E}'$ is conflict-free;

– $C \subseteq \mathscr{F}_M(C)$: $A \in C \Rightarrow$ there exists $\mathscr{E} \in LB \subseteq \text{ADM}_M$ such that $A \in \mathscr{E} \Rightarrow$ there exists $\mathscr{E} \in LB$ such that $A \in \mathscr{F}_M(\mathscr{E}) \Rightarrow$ in consequence of Lemma 1, $A \in \mathscr{F}_M(C)$.

It follows from the definition of C that $\forall C' \in \mathfrak{S} : C \subseteq C'$. As consequence, $C \in LB$ and according to Equation 7.1, $\mathscr{F}_M(C) \in LB$.
As $C \in \text{ADM}_M$, we know $C \subseteq \mathscr{F}_M(C)$; as $\mathscr{F}_M(C) \in LB$, we know $\mathscr{F}_M(C) \subseteq C$ by C's definition. Then $\mathscr{F}_M(C) = C$, i.e., $C \in \text{COMP}_M$.

The complete extension C is a lower bound of $\mathfrak{S}$ because for every $C' \in \mathfrak{S}$ it holds $C \subseteq C'$. Furthermore, C is the greatest lower bound of $\mathfrak{S}$ as $C \in LB$ and for every $\mathscr{E} \in LB$ it holds $\mathscr{E} \subseteq C$.

b) For each chain $(\mathfrak{S}, \subseteq)$ of $(\mathrm{COMP}_M, \subseteq)$, the set $\mathfrak{S}$ has a least upper bound:
Firstly, note that $(\mathfrak{S}, \subseteq)$ is also a chain of $(\mathrm{ADM}_M, \subseteq)$. This means $(\mathfrak{S}, \subseteq)$ is a directed set of $(\mathrm{ADM}_M, \subseteq)$. Let $S = \bigcup\{S' \mid S' \in \mathfrak{S}\} \in \mathrm{ADM}_M$.
By Theorem 2, there exists a maximal admissible extension $\mathscr{E}$ of M such that $S \subseteq \mathscr{E}$. We know $\mathscr{E} \subseteq \mathscr{F}_M(\mathscr{E})$. From the monotonicity of $\mathscr{F}_M$ (Lemma 3), we obtain $\mathscr{F}_M(\mathscr{E}) \subseteq \mathscr{F}_M(\mathscr{F}_M(\mathscr{E}))$ and, from Lemma 2, we obtain $\mathscr{F}_M(\mathscr{E})$ is conflict-free. This means $\mathscr{F}_M(\mathscr{E}) \in \mathrm{ADM}_M$. As $\mathscr{E}$ is a maximal admissible extension of M, it follows $\mathscr{F}_M(\mathscr{E}) = \mathscr{E}$, i.e., $\mathscr{E} \in \mathrm{COMP}_M$; besides, $\mathscr{E}$ is an upper bound of $\mathfrak{S}$.
Let $\mathfrak{S}' = \{\mathscr{E} \mid \mathscr{E} \in \mathrm{COMP}_M$ and $\mathscr{E}$ is an upper bound of $\mathfrak{S}\}$. Obviously, $\mathfrak{S}' \neq \emptyset$. Note also that S is a lower bound of $\mathfrak{S}'$. From the previous item, we know $\mathfrak{S}'$ has a greatest lower bound C'' in $(\mathrm{COMP}_M, \subseteq)$.
As $S \subseteq C''$, we conclude C'' is the least upper bound of $\mathfrak{S}$.

□

Proposition 14. *Let $\mathfrak{T} = (\mathscr{A}, Att, \mathscr{T})$ be a* TeAF *and $\mathscr{L} : \mathscr{A} \times \mathscr{T} \to \{\mathbf{in}, \mathbf{out}, \mathbf{undec}\}$ a labelling of $\mathfrak{T}$. Then, $\mathscr{L}$ is complete iff for any $A \in \mathscr{A}$ and $a \in \mathscr{T}$:*

a) $\mathscr{L}(A,a) = \mathbf{in}$ iff $\mathbf{in}(\mathscr{L})$ defends (A,a);

b) $\mathscr{L}(A,a) = \mathbf{out}$ iff $\mathbf{in}(\mathscr{L})$ defeats (A,a).

Proof. ($\Longrightarrow$) Let $\mathscr{L}$ be a complete labelling of M. Note that

$$\begin{aligned}\mathscr{L}(A,a) = \mathbf{out} &\iff \exists (B,b) \in Att(A,a) : \mathscr{L}(B,b) = \mathbf{in} \\ &\iff \exists (B,b) \in \mathbf{in}(\mathscr{L}) : (B,b) \in Att(A,a) \\ &\iff \mathbf{in}(\mathscr{L}) \text{ defeats } (A,a)\end{aligned}$$

and,

$$\begin{aligned}\mathscr{L}(A,a) = \mathbf{in} &\iff \forall (B,b) \in Att(A,a) : \mathscr{L}(B,b) = \mathbf{out} \\ &\iff \forall (B,b) \in Att(A,a) : \mathbf{in}(\mathscr{L}) \text{ defeats } (B,b) \\ &\iff \mathbf{in}(\mathscr{L}) \text{ defends } (A,a).\end{aligned}$$

($\Longleftarrow$) Assume that for any $A \in \mathscr{A}$ and $a \in \mathscr{T}$ it holds (i) $\mathscr{L}(A,a) = \mathbf{in}$ iff $\mathbf{in}(\mathscr{L})$ defends (A,a), and (ii) $\mathscr{L}(A,a) = \mathbf{out}$ iff $\mathbf{in}(\mathscr{L})$ defeats (A,a).

Then, $\mathscr{L}$ is complete, because

$$\begin{aligned}\mathscr{L}(A,a) = \mathbf{out} &\Longleftrightarrow \mathbf{in}(\mathscr{L}) \text{ defeats } (A,a)\\ &\Longleftrightarrow \exists (B,b) \in \mathbf{in}(\mathscr{L}) : (B,b) \in Att(A,a)\\ &\Longleftrightarrow \exists (B,b) \in Att(A,a) : \mathscr{L}(B,b) = \mathbf{in}\end{aligned}$$

and,

$$\begin{aligned}\mathscr{L}(A,a) = \mathbf{in} &\Longleftrightarrow \mathbf{in}(\mathscr{L}) \text{ defends } (A,a)\\ &\Longleftrightarrow \forall (B,b) \in Att(A,a) : \mathbf{in}(\mathscr{L}) \text{ defeats } (B,b)\\ &\Longleftrightarrow \forall (B,b) \in Att(A,a) : \mathscr{L}(B,b) = \mathbf{out}\end{aligned}$$

□

Proposition 15. *Let $\mathscr{L}, \mathscr{L}'$ be complete labellings of a* TeAF $\mathfrak{T}$. *It holds:*

a) $\mathbf{in}(\mathscr{L}) \subseteq \mathbf{in}(\mathscr{L}')$ *iff* $\mathbf{out}(\mathscr{L}) \subseteq \mathbf{out}(\mathscr{L}')$*;*

b) $\mathbf{in}(\mathscr{L}) \subset \mathbf{in}(\mathscr{L}')$ *iff* $\mathbf{out}(\mathscr{L}) \subset \mathbf{out}(\mathscr{L}')$.

Proof.

a) ($\Rightarrow$) Assume $\mathbf{in}(\mathscr{L}) \subseteq \mathbf{in}(\mathscr{L}')$. It follows $(A,a) \in \mathbf{out}(\mathscr{L}) \Rightarrow$ there exists $(B,b) \in Att(A,a)$ such that $\mathscr{L}(B,b) = \mathbf{in} \Rightarrow$ (as $\mathbf{in}(\mathscr{L}) \subseteq \mathbf{in}(\mathscr{L}')$) there exists $(B,b) \in Att(A,a)$ such that $\mathscr{L}'(B,b) = \mathbf{in} \Rightarrow (A,a) \in \mathbf{out}(\mathscr{L}')$;

($\Leftarrow$) Assume $\mathbf{out}(\mathscr{L}) \subseteq \mathbf{out}(\mathscr{L}')$. It follows $(A,a) \in \mathbf{in}(\mathscr{L}) \Rightarrow$ for every $(B,b) \in Att(A,a)$ it holds $\mathscr{L}(B,b) = \mathbf{out} \Rightarrow$ (as $\mathbf{out}(\mathscr{L}) \subseteq \mathbf{out}(\mathscr{L}')$) for every $(B,b) \in Att(A,a)$ it holds $\mathscr{L}'(B,b) = \mathbf{out} \Rightarrow (A,a) \in \mathbf{in}(\mathscr{L}')$.

b) ($\Rightarrow$) It follows $\mathbf{in}(\mathscr{L}) \subset \mathbf{in}(\mathscr{L}') \Rightarrow \mathbf{in}(\mathscr{L}) \subseteq \mathbf{in}(\mathscr{L}')$ and $\mathbf{in}(\mathscr{L}') \nsubseteq \mathbf{in}(\mathscr{L}) \Rightarrow \mathbf{out}(\mathscr{L}) \subseteq \mathbf{out}(\mathscr{L}')$ and $\mathbf{out}(\mathscr{L}') \nsubseteq \mathbf{out}(\mathscr{L}) \Rightarrow \mathbf{out}(\mathscr{L}) \subset \mathbf{out}(\mathscr{L}')$;

($\Leftarrow$) It follows $\mathbf{out}(\mathscr{L}) \subset \mathbf{out}(\mathscr{L}') \Rightarrow \mathbf{out}(\mathscr{L}) \subseteq \mathbf{out}(\mathscr{L}')$ and $\mathbf{out}(\mathscr{L}') \nsubseteq \mathbf{out}(\mathscr{L})$ $\Rightarrow \mathbf{in}(\mathscr{L}) \subseteq \mathbf{in}(\mathscr{L}')$ and $\mathbf{in}(\mathscr{L}') \nsubseteq \mathbf{in}(\mathscr{L}) \Rightarrow \mathbf{in}(\mathscr{L}) \subset \mathbf{in}(\mathscr{L}')$.

□

Proposition 16. *Let $\mathscr{L}, \mathscr{L}'$ be complete labellings of a* TeAF $\mathfrak{T}$. *It holds*

1. *If* $\mathbf{in}(\mathscr{L}) \subseteq \mathbf{in}(\mathscr{L}')$*, then* $\mathbf{undec}(\mathscr{L}') \subseteq \mathbf{undec}(\mathscr{L})$*;*
2. *If* $\mathbf{in}(\mathscr{L}) \subset \mathbf{in}(\mathscr{L}')$*, then* $\mathbf{undec}(\mathscr{L}') \subset \mathbf{undec}(\mathscr{L})$*;*
3. *If* $\mathbf{out}(\mathscr{L}) \subseteq \mathbf{out}(\mathscr{L}')$*, then* $\mathbf{undec}(\mathscr{L}') \subseteq \mathbf{undec}(\mathscr{L})$*;*
4. *If* $\mathbf{out}(\mathscr{L}) \subset \mathbf{out}(\mathscr{L}')$*, then* $\mathbf{undec}(\mathscr{L}') \subset \mathbf{undec}(\mathscr{L})$.

Proof.

a) Assume $\mathbf{in}(\mathscr{L}) \subseteq \mathbf{in}(\mathscr{L}')$. From Lemma 9, it follows $\mathbf{out}(\mathscr{L}) \subseteq \mathbf{out}(\mathscr{L}')$. Then, $(A,a) \in \mathbf{undec}(\mathscr{L}') \Rightarrow (A,a) \notin \mathbf{in}(\mathscr{L}') \cup \mathbf{out}(\mathscr{L}') \Rightarrow (A,a) \notin \mathbf{in}(\mathscr{L}) \cup \mathbf{out}(\mathscr{L}) \Rightarrow (A,a) \in \mathbf{undec}(\mathscr{L})$;

b) Assume $\mathbf{in}(\mathscr{L}) \subset \mathbf{in}(\mathscr{L}')$. From Lemma 9, it follows $\mathbf{out}(\mathscr{L}) \subset \mathbf{out}(\mathscr{L}')$. It also follows $\mathbf{in}(\mathscr{L}) \subseteq \mathbf{in}(\mathscr{L}')$ and $\mathbf{in}(\mathscr{L}') \not\subseteq \mathbf{in}(\mathscr{L})$ and $\mathbf{out}(\mathscr{L}) \subseteq \mathbf{out}(\mathscr{L}')$ and $\mathbf{out}(\mathscr{L}') \not\subseteq \mathbf{out}(\mathscr{L})$;

   From the previous item, we obtain $\mathbf{undec}(\mathscr{L}') \subseteq \mathbf{undec}(\mathscr{L})$. As $\mathbf{in}(\mathscr{L}') \not\subseteq \mathbf{in}(\mathscr{L})$, there exists $(A,a) \in \mathbf{in}(\mathscr{L}')$ such that $(A,a) \notin \mathbf{in}(\mathscr{L})$. It is clear $(A,a) \notin \mathbf{out}(\mathscr{L})$ (otherwise (A,a) would be in $\mathbf{out}(\mathscr{L}')$). Thus, $A \in \mathbf{undec}(\mathscr{L})$. It implies $\mathbf{undec}(\mathscr{L}) \not\subseteq \mathbf{undec}(\mathscr{L}')$. Consequently, $\mathbf{undec}(\mathscr{L}') \subset \mathbf{undec}(\mathscr{L})$;

c) Similar to proof of item (a);

d) Similar to proof of item (b).

□

Proposition 17. *Let $\mathscr{L}, \mathscr{L}'$ be complete labellings of a* TeAF $\mathfrak{T}$. *It holds:*

a) If $\mathbf{in}(\mathscr{L}) = \mathbf{in}(\mathscr{L}')$, then $\mathscr{L} = \mathscr{L}'$;

b) If $\mathbf{out}(\mathscr{L}) = \mathbf{out}(\mathscr{L}')$, then $\mathscr{L} = \mathscr{L}'$.

Proof.

a) It follows $\mathbf{in}(\mathscr{L}) = \mathbf{in}(\mathscr{L}') \Rightarrow \mathbf{in}(\mathscr{L}) \subseteq \mathbf{in}(\mathscr{L}')$ and $\mathbf{in}(\mathscr{L}') \subseteq \mathbf{in}(\mathscr{L}') \Rightarrow$ from Lemma 9, $\mathbf{out}(\mathscr{L}) \subseteq \mathbf{out}(\mathscr{L}')$ and $\mathbf{out}(\mathscr{L}') \subseteq \mathbf{out}(\mathscr{L}') \Rightarrow \mathbf{out}(\mathscr{L}) = \mathbf{out}(\mathscr{L}') \Rightarrow$ as $\mathbf{in}(\mathscr{L}) = \mathbf{in}(\mathscr{L}')$, it holds $\mathbf{undec}(\mathscr{L}) = \mathbf{undec}(\mathscr{L}') \Rightarrow \mathscr{L} = \mathscr{L}'$ as $\mathbf{in}(\mathscr{L}) = \mathbf{in}(\mathscr{L}'), \mathbf{out}(\mathscr{L}) = \mathbf{out}(\mathscr{L}')$ and $\mathbf{undec}(\mathscr{L}) = \mathbf{undec}(\mathscr{L}')$.

b) It follows $\mathbf{out}(\mathscr{L}) = \mathbf{out}(\mathscr{L}') \Rightarrow \mathbf{out}(\mathscr{L}) \subseteq \mathbf{out}(\mathscr{L}')$ and $\mathbf{out}(\mathscr{L}') \subseteq \mathbf{out}(\mathscr{L}') \Rightarrow$ from Lemma 9, $\mathbf{in}(\mathscr{L}) \subseteq \mathbf{in}(\mathscr{L}')$ and $\mathbf{in}(\mathscr{L}') \subseteq \mathbf{in}(\mathscr{L}') \Rightarrow \mathbf{in}(\mathscr{L}) = \mathbf{in}(\mathscr{L}') \Rightarrow$ as $\mathbf{out}(\mathscr{L}) = \mathbf{out}(\mathscr{L}')$, it holds $\mathbf{undec}(\mathscr{L}) = \mathbf{undec}(\mathscr{L}') \Rightarrow \mathscr{L} = \mathscr{L}'$ as $\mathbf{in}(\mathscr{L}) = \mathbf{in}(\mathscr{L}'), \mathbf{out}(\mathscr{L}) = \mathbf{out}(\mathscr{L}')$ and $\mathbf{undec}(\mathscr{L}) = \mathbf{undec}(\mathscr{L}')$.

□