Technical Note

The Bell-Shaped Unit Hydrograph for Overland Planes

José N. B. Campos[1]; Udinart P. Rabelo[2]; and Iran E. Lima Neto[3]

Abstract: This study presents a conceptual model to obtain the shape of the unit hydrograph in a small rectangular basin with a collecting channel on one side of the flow plane. In classical hydrology, flows are classified as linear, convergent, and divergent. In the proposed model, a rainfall of constant intensity is assumed, with the duration equal to the time of concentration of the basin, as in the rational method. The shape of the plane is simplified in order to obtain an analytical solution. It is observed that in the plane of diffuse flow, the flow begins as a convergent of repletion, then passes to divergent of repletion, and finishes as a convergent of depletion. The applied theory allows the development of a classic bell-shaped unit hydrograph, a very common form of the theoretical hydrographs found in the literature. The proposed methodology was also applied to a practical flood damping problem in an urban watershed. **DOI: 10.1061/(ASCE)IR.1943-4774.0001465.** *© 2020 American Society of Civil Engineers.*

Author keywords: Overland plane; Rational method; Unit hydrograph shape.

Introduction

Originally described by Sherman (1932), the concept of a unit hydrograph consists in a model to represent the streamflow hydrograph obtained from a rainfall hyetograph at the outlet of a basin. Traditionally, the derivation of the unit hydrograph was done based on observed data, encompassing practically all the premises of the method. Dooge (1959) presented the general equations of the unit hydrographs by the assumption that the reservoir action in a catchment can be separated from translation. Chow (1962) evaluated three methods of determining a unit hydrograph: (1) a direct derivation from observed hydrographs; (2) hydrograph syntheses for a large number of observed hydrographs; and (3) a construction of hydrographs based on theory.

The theoretical method of construction of Chow's hydrograph (1962) is to divide the watershed into a number of segments and calculate the contribution of each segment. Chow applied this method to a hypothetical basin of circular shape. The fictitious basin had a length L, along the stream, and was exposed to a uniform unit rain of duration t_d. The plane had a constant gradient S toward the basin outlet. The hydrographs obtained are shown in Fig. 1.

There are several other models, conceptual or empirical, that seek to establish the shape of a unit hydrograph. Classical studies on unit hydrographs are revised for instance by Maidment (1993) and Cleveland et al. (2008). More recent studies have developed other forms and methods for the construction of unit hydrographs. Bhunya et al. (2005) developed a hybrid model to obtain the unit hydrograph from the splitting of the Nash (1959) linear reservoir into two reservoirs with different storage coefficients. Nadarajah (2007) explored eleven methods of probability distribution to obtain unit hydrographs and presented nine programs written in Maple to obtain parameters that describe methods, such as time to peak, the time base, and peak discharge. Singh (2015) proposed a simple theory for instantaneous unit hydrographs based on two parameters, with conceptual and physical justification. This theory simplifies the computations involved in obtaining the streamflow from a complex rainfall event. Petroselli and Grimaldi (2018) developed a method to replace the rational formula in small watersheds with the scarcity of hydrological data. The method uses terrain digital elevation models and applies the instantaneous geomorphological unit hydrograph concept. Ghorbani et al. (2017) developed a nonlinear model to transmute a unit hydrograph into a probability distribution function with parameters optimization by two ways: programming in Mathematica and applying a genetic algorithm. Khaleghi et al. (2018) proposed a model to determine the shape of instantaneous unit hydrographs, which consists of a series of linear reservoirs that are connected to each other, and is referred to as the interconnected linear reservoir model.

The objective of this paper is to analytically develop a unit hydrograph (UH) based on hydraulics and hydrology fundamentals for application in small urban basins. This UH can be applied in urban basins where the hydrograph shape is important for the sizing of reservoirs and stormwater ponds. In the classic models of hydrological engineering practice, unit hydrographs (UHs) are bell-shaped. In this paper, the theoretical construction model (Chow 1962) is applied to a simplified overland flow regime. Theoretical flow types are defined to support the flow conditions that determine the occurrence of the inflection point of the UH. To the authors' knowledge, no previous study has been developed in this sense.

Methodology

Theoretical Flow Types in Planes with Different Geometries

Initially, three types of flows in fictitious planes were analyzed: (1) linear flow, which occurs in a rectangular plane in which the

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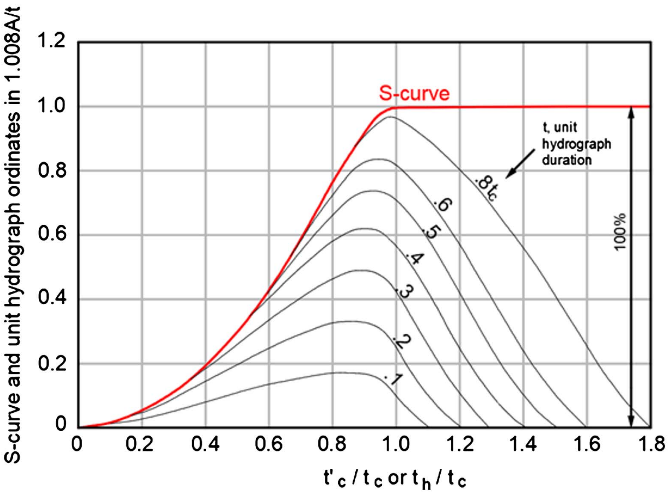

Fig. 1. *S*-curve and unit hydrograph for hypothetical circular basin adapted from Chow.

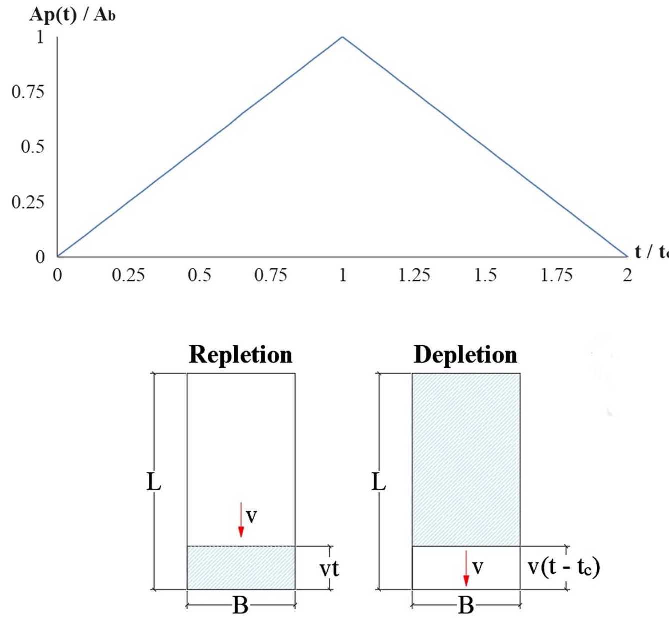

Fig. 2. Unit hydrograph for rectangular plane—linear flow for repletion and depletion.

derivative of the contributing area with time $dA(t)/dt$ is constant; (2) convergent flow, which occurs in a circular convergent sector in which the derivative $dA(t)/dt$ is positive; and (3) divergent flow, which occurs in a circular divergent sector in which the derivative $dA(t)/dt$ is negative.

The rational method formula for the fictitious planes can be written as

$$Q = C \cdot i \cdot A \tag{1}$$

where Q = discharge; C = runoff coefficient; i = rainfall intensity; and A = contributing area of the plane.

Thus, considering the $C \cdot i$ constant, the evolution of the contributing area at the plane outlet is analyzed in the following sections. Note that for the partial contributing areas in the construction of the hydrograph, the notation $Ap(t)$ is used, and the discharge at the outlet is $Qp(t) = CiAp(t)$, not considering that the area or the discharge in the rational method is time dependent, but that the area of effective contribution to the flow at the outlet depends on the intensity, which is dependent on the time of concentration (t_c). Hence, when $t = t_c$, Eq. (1) is applied.

Linear Flow in a Rectangular Plane

The assumptions to determine the linear flow are those of the rational method (Maidment 1993). The plane has width B and length L, as shown schematically in Fig. 2. The rainfall is uniform over the basin with intensity i and duration t_c (time of concentration). The flow velocity v in the plane is equal to L/t_c. The effective contribution is equal to the rainfall intensity i times the runoff coefficient C. The total area of the plane is $A_b = L \times B$.

The construction of the hydrograph is done in two stages: (1) ascending phase (repletion), until t_c is reached; and (2) descending phase (depletion), after the rain ceases, from t_c to $2t_c$.

- Stage 1: $0 \leq t < t_c$ (linear repletion): the contributing area increases linearly with precipitation at t

$$Ap(t) = v \cdot t \cdot B \tag{2}$$

- Stage 2: $t_c \leq t < 2t_c$ (linear depletion): the contributing area decreases linearly with precipitation at t

$$Ap(t) = A_b - v(t - t_c)B \tag{3}$$

Fig. 2 shows the hydrograph resulting from Eqs. (2) and (3).

At this point, the construction of the subsequent hydrographs needs to be clarified: the inflection points refer to the instant of change in the flow type. In the ascending stages (repletion), an inflection point occurs when the contribution of the triangle above the diagonal of the flow plane begins. In the primary forms, when the convergent sector contribution is completed and the divergent sector starts, in the descending stage (depletion), the inflection point occurs when the depletion of the lower rectangle is completed, and the depletion of the upper triangle begins.

Convergent Flow in a Circular Sector

The assumptions to determine the convergent flow are that the circular sector has radius R and angle ϕ and that the rainfall is uniform over the basin with intensity i and duration t_c. The flow velocity in the plane is $v = R/t_c$. Fig. 3 shows schematically the convergent flow in the circular sector.

Similar to the previous case (rectangular plane—linear flow), the construction of the hydrograph is done in two stages: (1) ascending phase (repletion), until t_c is reached; and (2) descending phase (depletion), after the rain ceases, from t_c to $2t_c$.

- Stage 1: $0 \leq t < t_c$ (repletion, convergent flow): the contributing area increases from downstream to upstream. It begins with the sector near the vertex until t_c, when the entire area of the plane (A_b) will be contributing. Note that the second derivative $d^2Ap(t)/dt^2$ is positive. Thus, the variation of the contributing area over time is given by

$$Ap(t) = \left(\frac{\text{Ø}}{360}\right)\pi R^2 \tag{4}$$

or, defining $K = \pi x(\phi/360)$

$$Ap(t) = K(vt)^2 \tag{5}$$

- Stage 2: $t_c \leq t < 2t_c$ (depletion, divergent flow): the contributing area decreases over time. Initially, the area near the outlet ceases its contribution. The process proceeds to time $2t_c$ until the entire flow ceases. Note that the second derivative $d^2Ap(t)/dt^2$ is negative.

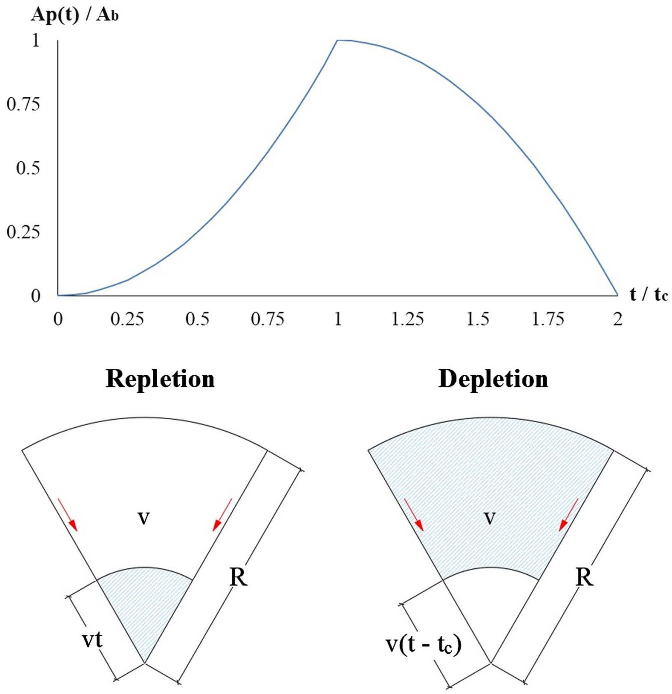

Fig. 3. Unit hydrograph convergent sector—flow type for repletion and depletion.

Hence, the variation of the contributing area over time can be described by the following equation

$$Ap(t) = A_b - K[v(t - t_c)]^2 \quad (6)$$

Fig. 3 shows the hydrograph resulting from Eqs. (5) and (6).

Divergent Flow in a Circular Sector

The assumptions to determine the divergent flow are the same already described for the convergent flow. The only different is the flow direction, which is the opposite of the previous case, as shown schematically in Fig. 4.

Again, the construction of the hydrograph is done in two stages: (1) ascending phase (repletion), until t_c is reached; and (2) descending phase (depletion), after the rain ceases, from t_c to $2t_c$.

- Stage 1: $0 \leq t < t_c$ (repletion, divergent flow): the contributing area increases from downstream to upstream, but the second derivative $d^2Ap(t)/dt^2$ is negative.

  The variation of the contributing area over time is given by

$$Ap(t) = KR^2 - K(R - vt)^2 \quad (7)$$

- Stage 2: $t_c \leq t < 2t_c$ (depletion, convergent flow): the contributing area decreases over time, but the second derivative $d^2Ap(t)/dt^2$ is positive.

The variation of the contributing area over time can be described by the following equation:

$$Ap(t) = KR^2 - K[R - v(t - t_c)]^2 \quad (8)$$

Fig. 4 shows the hydrograph resulting from Eqs. (7) and (8).

Theoretical Construction of the Unit Hydrograph for a Square Basin with a Side Collecting Channel

For the theoretical construction of the bell-shaped UH, a diffuse square flow plane was conceived with a collecting channel on one side, as shown schematically in Fig. 5. To provide an analytical

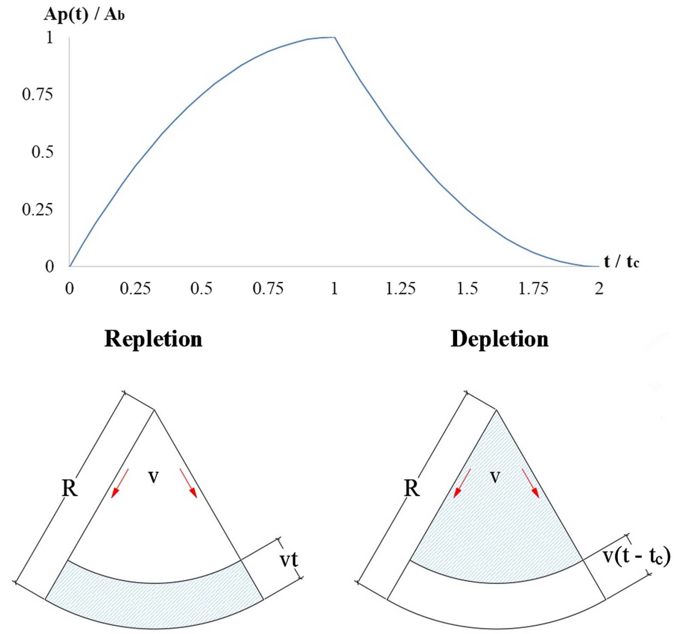

Fig. 4. Unit hydrograph divergent sector—flow type for repletion and depletion.

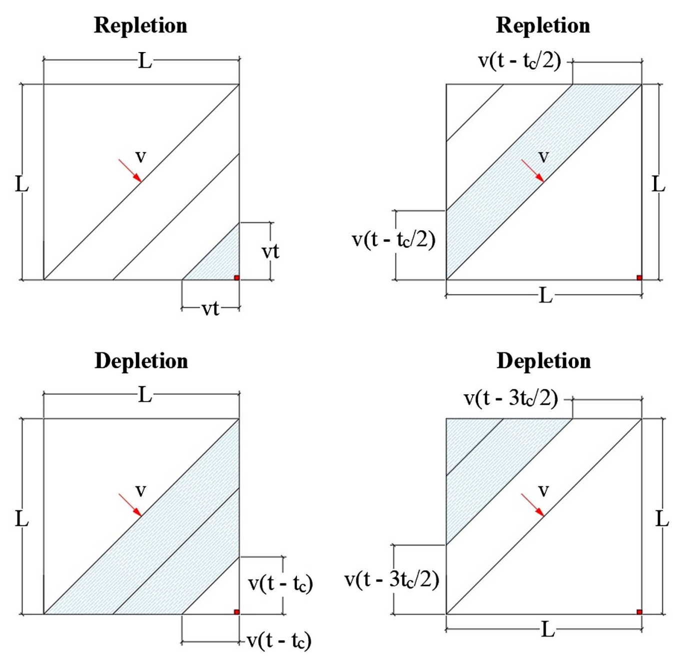

Fig. 5. Four steps of the function $Ap(t)$ for the square plane with side collector.

treatment, it is assumed that the velocity in the plane, toward the channel, is equal to the velocity in the channel relative to the outlet. With this assumption, the increase in the contributing area in the outlet occurs along a line parallel to the main diagonal. The theoretical construction of UH occurs in four stages, described as follows:

Stage 1: From $t = 0$ to $t_c/2$ (Convergent Repletion)

Due to the assumption of equal velocities in the plane and in the channel, the increase in the contributing area occurs in the convergent flow form (Fig. 3). Thus, the contributing area refers to an isosceles right triangle [Fig. 5(a)] and can be described by the following the formula:

$$Ap(t) = \frac{1}{2} \cdot v \cdot t \cdot v \cdot t = \frac{1}{2} v^2 t^2 \quad (9)$$

Observe that when $t = t_c/2$, the contributing area is $A_b/2$.

Stage 2: From $t = t_c/2$ to t_c (Divergent Repletion)
In this stage, the increase in the contributing area starts from the widest part of the diagonal of the trapezoid square and follows to the vertex of the upper rectangle triangle [Fig. 5(b)]. It represents the process of the initial phase of the flow in a divergent circular sector (Fig. 4). Hence, the equation for the contributing area is

$$Ap(t) = \frac{A_b}{2} + \left(L\sqrt{2} - \frac{v(t - \frac{t_c}{2})}{\sqrt{2}} \right) v \left(t - \frac{t_c}{2} \right) \bigg/ \sqrt{2} \quad (10)$$

Stage 3: From t_c to $1.5t_c$ (Divergent Depletion)
At this stage, depletion of the most upstream part (below the main diagonal) begins [Fig. 5(c)]. The flow is similar to the depletion of the convergent circular sector (Fig. 3). That is, it is a repletion with divergent flow. At time t_c, the entire basin is contributing $[A(t) = A_b]$. Thus, the following equation describes the contributing area:

$$Ap(t) = A_b - \frac{1}{2} v^2 (t - t_c)^2 \quad (11)$$

Stage 4: From $1.5t_c$ to $2t_c$ (Convergent Depletion)
At the final stage, the depletion of the most downstream part (above the main diagonal) begins [Fig. 5(d)]. In this case, the flow is similar to the depletion of the divergent circular sector (Fig. 4). At $t = 1.5t_c$, the contributing area is $A_b/2$, and the process continues until total depletion is reached. Thus, the contributing area follows the formula

$$Ap(t) = \frac{A_b}{2} - \left(L\sqrt{2} - \frac{v(t - 1.5t_c)}{\sqrt{2}} \right) v(t - 1.5t_c)/\sqrt{2} \quad (12)$$

Discussion

The application of Chow's method to construct hydrograph forms was done analytically and provided the formulation of a classic bell-shaped hydrograph with two inflection points, as depicted in Table 1 and Fig. 6. The hydrograph was constructed with four types of flow: (1) convergent flow in repletion; (2) divergent flow in repletion; (3) divergent flow in depletion; and (4) convergent flow in depletion. The hypothetical data used in simulation were as follows: area $(Ab) = 100{,}000$ unit area; time of concentration $(t_c) = 3{,}600$ unit time; length $(L) = \sqrt{A_b} = 316.228$ unit length; velocity $(v) = 2 \cdot L/t_c = 0.176$ unit velocity; and time step $(t) =$ 100 unit time.

Recent studies have presented many advances in obtaining unit hydrographs (UH), mainly in the classes of probabilistic models and geomorphological models, according to Bhunya et al. (2011) and Singh et al. (2014). Classic models, such as soil conservation service (SCS), have been widely used, although they present some inconsistencies. The conceptual models were responsible for defining the standard form of UH, with a minimum number of parameters, by several methodologies, and obtained satisfactory results in several areas of hydrology. The proposed UH belongs to this class of conceptual models, with the advantage that the hydrograph was derived in a totally analytical way, with the classic bell-shaped form found in most hydrological situations. Despite the simplifications

Table 1. Example of hypothetical unit hydrograph for a square plane flow

t/tc	$Ap(t)/A_b$
0.028	0.002
0.056	0.006
0.083	0.014
0.111	0.025
0.139	0.039
0.167	0.056
0.194	0.076
0.222	0.099
0.250	0.125
0.278	0.154
0.306	0.187
0.333	0.222
0.361	0.261
0.389	0.302
0.417	0.347
0.444	0.395
0.472	0.446
0.500	0.500
0.528	0.554
0.556	0.605
0.583	0.653
0.611	0.698
0.639	0.739
0.667	0.778
0.694	0.813
0.722	0.846
0.750	0.875
0.778	0.901
0.806	0.924
0.833	0.944
0.861	0.961
0.889	0.975
0.917	0.986
0.944	0.994
0.972	0.998
1.000	1.000
1.028	0.998
1.056	0.994
1.083	0.986
1.111	0.975
1.139	0.961
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1.667	0.222
1.694	0.187
1.722	0.154
1.750	0.125
1.778	0.099
1.806	0.076
1.833	0.056
1.861	0.039

Table 1. (*Continued.*)

t/tc	$Ap(t)/A_b$
1.889	0.025
1.917	0.014
1.944	0.006
1.972	0.002
2.000	0.000

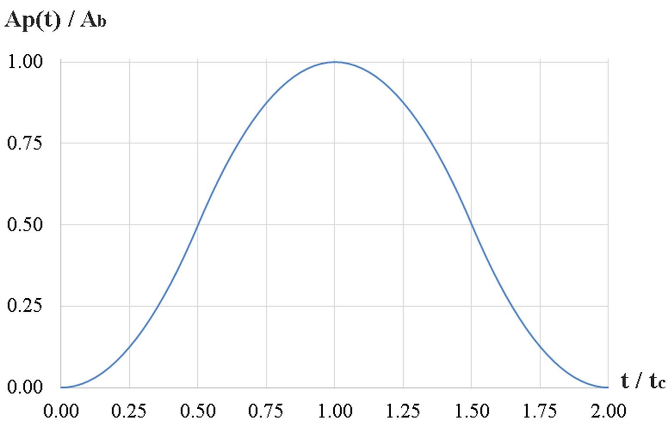

Fig. 6. Conceptual unit hydrograph for the square plane flow.

and the simple area considered (rectangular with diagonal output), the area repletion and depletion phases can be visualized quite clearly, and the applied method can be expanded and adapted to new situations.

Application

The proposed method is applied in this study for the prediction of the inflow and outflow hydrographs for flood damping in the Santo Anastácio Pond, which is located in an urban basin in Fortaleza, Brazil. The basin area $A = 143{,}400$ m^2, and the pond has a maximum depth $h = 4.0$ m and a capacity $V = 305{,}000$ m^3. Considering the intensity-duration-frequency equation of Silva et al. (2013) $\{i = [2{,}345.29T^{0.173}/(t + 28.31)^{0.904}]\}$, a flood with return period $T = 10$ years, and a time of concentration $t = 60$ min, the method described in the previous sections was applied to obtain the inflow hydrograph. The flood damping was calculated by using the Puls method, considering a volume-height relationship of the form $V = 4{,}765.625 \cdot h^3$ and an outflow equation represented by $Q = 91.9 \cdot (h - 4)^{3/2}$. The predicted hydrographs are shown in Fig. 7. This illustrates how the method proposed in the present paper can be easily applied to practical problems.

Summary and Conclusions

This research presented a theoretical construction of a bell-shaped unit hydrograph (UH). Overall, six types of flows were defined in order to compose the UH: (1) linear flow repletion, (2) linear flow depletion, (3) convergent flow repletion, (4) divergent flow depletion, (5) divergent flow repletion, and (6) convergent flow depletion.

Chow's method was used for the construction of the UH. The water catchment area used to present this conceptual model had the form of a square flowing to a side collecting channel that drained the waters to the outlet. To allow an analytical treatment of the problem, it was assumed that the velocity in the plane toward the outlet is equal to the velocity of the flow in the side collector. A bell-shaped hydrograph was constructed with four segments: convergent repletion flow, divergent repletion flow, divergent depletion flow, and convergent depletion flow. The inflection points in the hydrograph are in the ascending stages (repletion), which refer to the instant at which the convergent flow ends and the divergent flow begins, and in the descending stages (depletion), which refer to the instant that the divergent flow of depletion ends and the convergent flow of depletion initiates.

An application of the method presented in this study illustrated how the bell-shaped unit hydrograph together with the Puls method can be used to assess flood damping by ponds in urban watersheds. The proposed methodology can also be extended to more complex planes such as rectangles, triangles, and circles, but other geometrical conditions must be considered to obtain an analytical solution. In such cases, numerical methods using finite-differences or finite-elements, and specialized software such as the Hydrologic Modeling System (HEC/HMS), could also be employed.

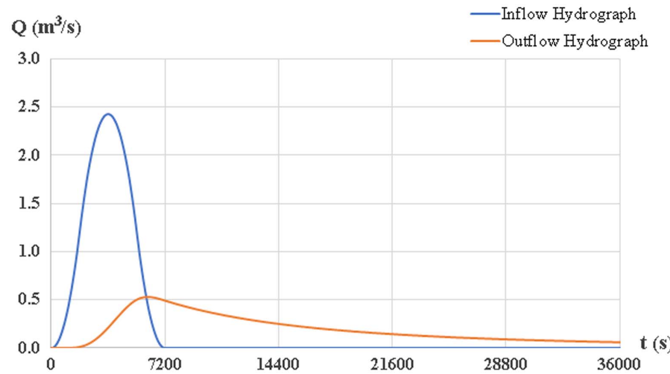

Fig. 7. Prediction of the inlet and outlet hydrographs for the Santo Anastácio Pond, in Fortaleza, Brazil.

Data Availability Statement

All data, models, and code generated or used during the study appear in the published article.

Acknowledgments

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