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RAFAEL ROCHA DE FARIAS

HYPERSURFACES WITH PRESCRIBED SCALAR CURVATURE IN GEOMETRIC FLOWS

FORTALEZA

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Thesis submitted to the Graduate Program in Mathematics of the Department of Mathematics of Federal University of Ceará in partial fulfillment o f t he n ecessary r equirements for the degree of Doctor in Mathematics. Area of expertise: Geometry.

Advisor: Prof. Dr. Jorge Herbert Soares de Lira.

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“There is a time for everything, and a season for every activity under the heavens.” (BIBLE, 1978, p. 366)

ABSTRACT

We consider translators to the extrinsic flow defined by S^α in $\mathbb{R} \times_h \mathbb{P}^n$ or in $\mathbb{P}^n \times_\chi \mathbb{R}$, where S is the extrinsic scalar curvature, $\alpha \in \{1/2, 1\}$, and $n \geq 3$. We show that there exist rotational bowl-type and translating catenoid-type translators in $\mathbb{P} \times_\chi \mathbb{R}$. In our main existence results we exhibit a one-parameter family of explicit solutions when $\alpha = 1/2$ in $\mathbb{P} \times_\chi \mathbb{R}$ when $\mathbb{P}$ is Hadamard complete manifold with a rotationally symmetric metric. We discuss the variational nature of solitons, we find a one-parameter family of null scalar curvature hypersurfaces when $\mathbb{P} \times_\chi \mathbb{R}$ is Einstein, we use maximum principle to show that if a translating soliton is contained in a slab in $I \times_h \mathbb{P}$ and it is parabolic with respect to the L map then it is contained in a leaf $\mathbb{P}_s = \{s\} \times \mathbb{P}$ and that if $\mathbb{P} \times_\chi \mathbb{R}$ has constant sectional curvature and a translating soliton is parabolic with respect to L_ζ map then it is not bounded from above.

Keywords: scalar curvature flow; translator; maximum principle.

RESUMO

Consideramos translators ao fluxo extrínseco definido por S^α em $\mathbb{R}\times_h \mathbb{P}^n$ ou em $\mathbb{P}^n \times_\chi \mathbb{R}$, onde S é a curvatura escalar extrínseca, $\alpha \in \{1/2, 1\}$ e $n \geq 3$. Mostramos que existem bowl-solitons rotacionais e translators tipo catenoide em $\mathbb{P}\times_\chi \mathbb{R}$. Em nosso principal resultado de existência exibimos uma família a um parâmetro de soluções explícitas quando $\alpha = 1/2$ em $\mathbb{P}\times_\chi \mathbb{R}$ quando $\mathbb{P}$ é uma variedade de Hadamard com uma métrica rotacionalmente simétrica. Discutimos a natureza variacional dos solitons, encontramos uma família a um parâmetro de hipersuperfícies de curvatura escalar nula quando $\mathbb{P}\times_\chi \mathbb{R}$ é Einstein, usamos o princípio do máximo para mostrar que se um soliton da curvatura escalar está contido em um slab em $I\times_h \mathbb{P}$ e é parabólico com respeito ao operador L então ele está contido em uma folha $\mathbb{P}_s = \{s\}\times\mathbb{P}$ e que se $\mathbb{P}\times_\chi \mathbb{R}$ possui curvatura seccional constante e um soliton pela curvatura escalar é parabólico com respeito ao operador L_ζ então ele não é limitado por cima.

Palavras-chave: fluxo pela curvatura escalar; translator; princípio do máximo.

CONTENTS

1 INTRODUCTION

We consider product manifolds $\bar{M}^{n+1} = I \times \mathbb{P}$ where I is an open interval in $\mathbb{R}$ and $(\mathbb{P}, \sigma)$ is a n-dimensional complete Riemannian manifold. In most cases, $I = \mathbb{R}$ or $I = (0, +\infty)$. We suppose that the Riemannian metric $\bar{g}$ in $\bar{M}$ is a warped metric with one of the following forms: either

$$\bar{g} = ds^2 + h^2(s)\sigma \tag{1.1}$$

or

$$\bar{g} = \chi^2 ds^2 + \sigma, \tag{1.2}$$

for some smooth positive functions $h : I \to \mathbb{R}$ and $\chi : \mathbb{P} \to \mathbb{R}$. Here, s is the natural coordinate in the factor I of the product $\bar{M}$. The vector field $X = h(s)\partial_s$ is a (closed) conformal vector field in the case of the warped metric (1.1) with $|X| = h$. In the particular case when h is constant, X is a parallel vector field. In the case of the warped structure defined in (1.2) the vector field $X = \partial_s$ is a Killing vector field with norm given by $|X| = \chi$. If χ is constant, X is a parallel vector field. We indicate the cases (1.1) and (1.2) by $\bar{M} = I \times_h \mathbb{P}$ and $\bar{M} = \mathbb{P} \times_\chi I$, respectively. This geometric setting encompasses space forms as $\mathbb{R}^{n+1}$ and $\mathbb{H}^{n+1}(\kappa)$ as well as Riemannian products as $\mathbb{H}^n(\kappa) \times \mathbb{R}$ among other examples.

The main results in this paper concern existence and uniqueness of hypersurfaces M immersed into $\bar{M}$ which are initial conditions to self-similar solutions of the extrinsic scalar curvature flow. This geometric flow is defined as a one-parameter family of hypersurfaces $\Psi : [0, t^*) \times M \to \bar{M}$, for some $t^* > 0$, satisfying

$$(\partial_t \Psi)^\perp = S^\alpha N. \tag{1.3}$$

The speed of this flow is a power ($\alpha = 1$ or $\alpha = 1/2$) of the second elementary symmetric function of the Weingarten map A of $\Psi_t(M) \doteq \{\Psi(t,x) : x \in M\}$, that is, the function

$$S = \sum_{i<j} \lambda_i(A)\lambda_j(A). \tag{1.4}$$

The Weingarten map A is computed with respect to the unit normal vector field $N = N|_{\Psi_t(M)}$ in (1.3) which defines an orientation for $\Psi_t(M)$, for each $t \in [0, t^*)$. The principal curvatures $\lambda_i(A)$ in (1.4) are, by definition, the eigenvalues of A. Note that the left-hand side in (1.3) involves only the orthogonal projection of the variational vector field $\partial_t \Psi$ onto the normal bundle. This means that the flow is defined up to local tangential diffeomorphisms in M. Fixed local coordinates in M, the flow (1.3) is described in terms of a fully nonlinear parabolic equation as described in Section 2.3. In Section 2.4, one proves that the condition of self-similarity has the following infinitesimal expression:

$$II_{-S^\alpha N}+\frac{c}{2}\pounds_{X^T}g=c\varphi g. \tag{1.5}$$

The coefficient φ in the right-hand side is given by the divergence of the conformal vector field $X=h(s)\partial_s$; more precisely, $(n+1)\varphi=\mathrm{div}_{\bar g}X$. In the Killing case, $\varphi=0$. Taking traces in (1.5) one obtains the scalar curvature soliton equation

$$S^\alpha=c\langle X,N\rangle, \tag{1.6}$$

where c is a constant related to the ratio between the parameters s of the flow of X and the time parameter in (1.3). We take (1.6) as the scalar curvature soliton equation in both *warped* structures, that is, either if X is a closed conformal vector field or a Killing vector field. If $\alpha=1/2$ this soliton equation is invariant by a re-scale of the metric. In Section 2.2, we establish the variational nature of the equation (1.6) when $\bar M$ has a warped metric of the form (1.1).

Our main existence results are obtained under the assumption that $\mathbb{P}$ is a Hadamard manifold with a rotationally invariant metric

$$\sigma=dr^2+\xi^2(r)g_{\mathbb{S}^{n-1}} \tag{1.7}$$

and that $\chi=\chi(r)$, where r is the Riemannian distance in $\mathbb{P}$ from some pole $o\in\mathbb{P}$. Some structural assumptions concerning the behavior of the warping functions χ and ξ are required to establish the existence of complete examples, namely

- **Main assumption 1.** either χ is a positive constant (that could be fixed as 1 up to some rescaling in I) or $\chi(r)\to+\infty$ as $r\to\infty$;
- **Main assumption 2.** The scalar curvature $r\mapsto S(r)$ of geodesic cylinders with radii $r\in(0,+\infty)$ is a decreasing function that converges to some finite value S_∞ as $r\to\infty$.

Besides this, by assuming that the metrics in $\mathbb{P}$ and $\bar M$ we are supposing tacitly that the warping functions satisfy

$$\xi(0)=0,\quad \xi'(0)=1\quad\text{and}\quad \xi^{(2k)}(0)=0$$

and

$$\chi(0)=1,\quad \chi'(0)=0\quad\text{and}\quad \xi^{(2k+1)}(0)=0.$$

In this setting, we have proved the existence of one-parameter families of scalar curvature flow solitons as described in the following statement.

Theorem 1. *Let $\mathbb{P}^n$, $n \geq 3$, be a Hadamard manifold with a rotationally invariant metric σ and suppose that χ depends only on the distance in $\mathbb{P}$ from a fixed pole $o \in \mathbb{P}$. We also suppose that the main assumptions on χ and ξ are in force.*

Fixed c and $\alpha \in \{1/2, 1\}$, there exists a one-parameter family of rotationally invariant scalar curvature flow solitons $\mathscr{C}_{n,\alpha,C_0}$ in $\bar{M} = \mathbb{P} \times_\chi I$ described as follows:

- *if $C_0 = 0$ the soliton $\mathscr{C}_{n,\alpha,C_0}$ is a rotationally invariant graph over $\mathbb{P}$. We refer to it a bowl soliton;*
- *if $C_0 < 0$ the soliton $\mathscr{C}_{n,\alpha,C_0}$ is defined for $r \geq r(C_0)$ where the radius $r(C_0)$ depends on C_0. This soliton contains a geodesic sphere of $\mathbb{P}$ along which its normal is perpendicular to X. We refer to them as translating catenoids;*
- *if $C_0 > 0$ the soliton $\mathscr{C}_{n,\alpha,C_0}$ is defined for $r \geq r(C_0)$ where the radius $r(C_0)$ depends on C_0. This soliton contains a geodesic sphere of $\mathbb{P}$ along which is singular in the sense that its normal is parallel to X along this sphere.*

The proof of these results is presented in sections 3 and 4. In Section 2 we establish the notion of self-similar solution for the scalar curvature flow. This motivates the definition of the scalar curvature flow solitons whose precise definition and fundamental equations are discussed in sections 2.3 and 2.4. Some basic geometric and analytical facts on extrinsic scalar curvature are established in Section 2.1 as well as some comments about the variational nature of solitons as critical points of a second-order geometric functional. In order to obtain the existence results above we model scalar curvature flow solitons as graphs of rotationally invariant functions that satisfy a set of differential equations which are deduced and discussed in Section 3. Finally, in Section 5 we apply suitable variants of the maximum principle to the linearized equations of solitons obtaining some uniqueness as well as non-existence results.

Mean curvature flow solitons have being a major topic of research with a massive body of papers devoted to it. We refer the reder to the encyclopedic approach to the subject in (ANDREWS *et al.*, 2022). The study of solitons for extrinsic non-linear geometric flows is relatively more recent and some references which were fundamental for our contributions here are (SANTAELLA, 2022), (RENGASWAMI, 2021. Diponível em https://arxiv.org/abs/2109.10456. Acesso em 12 dez. 2022), (LIMA; PIPOLI, 2022. Disponível em https://arxiv.org/abs/2211.03918. Acesso em 12 dez. 2022). The main analytical tools in this paper are based on the foundational works (REILLY, 1973), (ROSENBERG, 1993), (HOUNIE; LEITE, 1999b), (HOUNIE; LEITE, 1995), (HOUNIE; LEITE, 1999a), (PIGOLA *et al.*, 2005) and (ALÍAS *et al.*, 2016).

2 PRELIMINARIES

2.1 Variational formulae

Let M be a n-dimensional Riemannian manifold whose Riemannian metric we denote by g. In what follows we consider an isometric immersion $\psi : M \to \bar{M}$ of M into a Riemannian manifold $\bar{M}$ with metric $\bar{g}$. Given $\varepsilon > 0$, let $\Psi : (-\varepsilon, \varepsilon) \times M \to \bar{M}$ be a variation of ψ with $\Psi(0, \cdot) = \psi$ and variational vector field

$$\frac{\partial \Psi}{\partial t}\Big|_{t=0} = fN + T,$$

for some function $f \in C^\infty(M)$ and a tangent vector field T along ψ. From now on, $\cdot$ indicates Lie derivatives with respect to the parameter t. Hence, denoting the local components of the metric induced by $\Psi_t \doteq \Psi(t, \cdot)$ in M by g_{ij} one computes

$$\dot{g}_{ij} = \pounds_{\partial_t} g_{ij} = -2fh_{ij} + \underbrace{\nabla_i T_j + \nabla_j T_i}_{=\pounds_T g_{ij}},$$

where ∇ and h_{ij} are the Riemannian connection and the local components of the second fundamental form II in $\Psi_t(M)$, respectively. We also have

$$\dot{g}^{ij} = 2fh^{ij} \underbrace{-g^{ik} g^{j\ell}(\nabla_k T_\ell + \nabla_\ell T_k)}_{=\pounds_T g^{ij}}$$

The expressions above are consistent with the parallelism of ∇. Indeed,

$$\begin{aligned}
\nabla_{\partial_t} g(\partial_i, \partial_j) &= \pounds_{\partial_t} g_{ij} - \langle \bar{\nabla}_{\partial_t} \partial_i, \partial_j \rangle - \langle \partial_i, \bar{\nabla}_{\partial_t} \partial_j \rangle \\
&= -2fh_{ij} + \langle \nabla_{\partial_i} T, \partial_j \rangle + \langle \partial_i, \nabla_{\partial_j} T \rangle - \langle \bar{\nabla}_{\partial_i} fN, \partial_j \rangle - \langle \partial_i, \bar{\nabla}_{\partial_i} fN \rangle - \langle \nabla_{\partial_i} T, \partial_j \rangle - \langle \partial_i, \nabla_{\partial_j} T \rangle \\
&= 0,
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the action of $\bar{g}$ on a pair of tangent vectors and $\bar{\nabla}$ is the Riemannian connection in $(\bar{M}, \bar{g})$. Note that our convention to the second fundamental form is that

$$h_{ij} = -\langle \bar{\nabla}_{\partial_i} N, \partial_j \rangle.$$

Now, the variation of the unit normal vector N along $\Psi_t(M)$ is given at time $t = 0$ by

$$\langle \bar{\nabla}_{\partial_t} N, \partial_i \rangle = -\langle \bar{\nabla}_{\partial_t} \partial_i, N \rangle = -\langle \bar{\nabla}_{\partial_i} \partial_t, N \rangle = -\langle \bar{\nabla}_{\partial_i} (fN + T), N \rangle = -f_i - \langle \bar{\nabla}_{\partial_i} T, N \rangle.$$

Hence,

$$\bar{\nabla}_{\partial_s} N = -\nabla f - AT.$$

We also compute

$$
\begin{aligned}
\frac{d}{dt}h_{ij} &= -\frac{\partial}{\partial t}\langle\bar{\nabla}_{\partial_i}N,\partial_j\rangle = -\langle\bar{\nabla}_{\partial_t}\bar{\nabla}_{\partial_i}N,\partial_j\rangle - \langle\bar{\nabla}_{\partial_i}N,\bar{\nabla}_{\partial_t}\partial_j\rangle\\
&= -\langle\bar{\nabla}_{\partial_i}\bar{\nabla}_{\partial_t}N,\partial_j\rangle - \langle\bar{R}(\partial_t,\partial_i)N,\partial_j\rangle - \langle\bar{\nabla}_{\partial_i}N,\bar{\nabla}_{\partial_j}\partial_t\rangle\\
&= \langle\bar{\nabla}_{\partial_i}(\nabla f + AT),\partial_j\rangle + \langle\bar{R}(\partial_i,\partial_t)N,\partial_j\rangle - \langle\bar{\nabla}_{\partial_i}N,\bar{\nabla}_{\partial_j}(fN+T)\rangle\\
&= \langle\nabla_{\partial_i}\nabla f,\partial_i\rangle + \langle\nabla_{\partial_i}AT,\partial_j\rangle + \langle\bar{R}(\partial_i,\partial_t)N,\partial_j\rangle - f\langle\bar{\nabla}_{\partial_i}N,\bar{\nabla}_{\partial_j}N\rangle - \langle\bar{\nabla}_{\partial_i}N,\nabla_{\partial_j}T\rangle\\
&= f_{i;j} - f\langle A\partial_i,A\partial_j\rangle + \langle\bar{R}(\partial_i,\partial_t)N,\partial_j\rangle + \langle\nabla_{\partial_i}AT,\partial_j\rangle + \langle A\partial_i,\nabla_{\partial_j}T\rangle,
\end{aligned}
$$

where $f_{i;j}$ are the local components of the Hessian of f in M. Therefore the covariant derivative of the second fundamental form of $\Psi_t(M)$ with respect to the variational vector field ∂_t is given by

$$
\begin{aligned}
(\nabla_{\partial_t}h)_{ij} &= \frac{d}{dt}h_{ij} - \langle A(\nabla_{\partial_t}\partial_i),\partial_j\rangle - \langle A\partial_i,\nabla_{\partial_t}\partial_j\rangle\\
&= \frac{d}{dt}h_{ij} - \langle A(\nabla_{\partial_i}\partial_t),\partial_j\rangle - \langle A\partial_i,\nabla_{\partial_j}\partial_t\rangle\\
&= \frac{d}{dt}h_{ij} - \langle\bar{\nabla}_{\partial_i}(fN),A\partial_j\rangle - \langle A\partial_i,\bar{\nabla}_{\partial_j}(fN)\rangle - \langle A(\nabla_{\partial_i}T),\partial_j\rangle - \langle A\partial_i,\nabla_{\partial_j}T\rangle\\
&= \frac{d}{dt}h_{ij} + f\langle A(A\partial_i),\partial_j\rangle + f\langle A\partial_i,A\partial_j\rangle - \langle A(\nabla_{\partial_i}T),\partial_j\rangle - \langle A\partial_i,\nabla_{\partial_j}T\rangle\\
&= \frac{d}{dt}h_{ij} + 2f\langle A\partial_i,A\partial_j\rangle - \langle A(\nabla_{\partial_i}T),\partial_j\rangle - \langle A\partial_i,\nabla_{\partial_j}T\rangle.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
\dot{h}_{ij} &= f_{i;j} - f\langle A\partial_i,A\partial_j\rangle + \langle\bar{R}(\partial_i,\partial_t)N,\partial_j\rangle + \langle\nabla_{\partial_i}AT,\partial_j\rangle + \langle A\partial_i,\nabla_{\partial_j}T\rangle\\
&\quad + 2f\langle A\partial_i,A\partial_j\rangle - \langle A(\nabla_{\partial_i}T),\partial_j\rangle - \langle A\partial_i,\nabla_{\partial_j}T\rangle\\
&= f_{i;j} + f\langle A\partial_i,A\partial_j\rangle + f\langle\bar{R}(\partial_i,N)N,\partial_j\rangle + \langle(\nabla_{\partial_i}A)T,\partial_j\rangle + \langle\bar{R}(\partial_i,T)N,\partial_j\rangle.
\end{aligned}
$$

Therefore

$$
\dot{h}_{ij} = f_{i;j} + f\langle A\partial_i,A\partial_j\rangle + f\langle\bar{R}(\partial_i,N)N,\partial_j\rangle + \langle(\nabla_{\partial_i}A)T,\partial_j\rangle + \langle\bar{R}(\partial_i,T)N,\partial_j\rangle \tag{2.1}
$$

Since covariant derivative and contractions commute, we conclude that

$$
n\dot{H} = \mathrm{trace}(\dot{h}) = \Delta f + |A|^2 f + \mathrm{Ric}_{\bar{M}}(N,N)f + \langle \mathrm{div}A, T\rangle + \mathrm{Ric}_{\bar{M}}(T,N).
$$

At this point, we use Codazzi equation

$$
(\nabla_{\partial_i}A)\partial_j - (\nabla_{\partial_j}A)\partial_i = \bar{R}(\partial_j,\partial_i)N
$$

in its contracted version

$$-nH_j + \operatorname{div} A_j = (\bar{R}(\partial_j, \partial_i)N)^i = -\operatorname{Ric}_{\bar{M}}(\partial_j, N),$$

to conclude that

$$n\dot{H} = \Delta f + |A|^2 f + \operatorname{Ric}_{\bar{M}}(N,N) f + \langle \nabla(nH), T\rangle. \tag{2.2}$$

Moreover, denoting $\det g$ as the determinant of the metric, one has

$$\begin{aligned}
&\frac{d}{dt}\sqrt{\det g} = -\frac{1}{2}\sqrt{\det g}\,\dot{g}^{ij} g_{ij} = -\frac{1}{2}\sqrt{\det g}\big(2f g_{ij} h^{ij} - g^{ik} g_{ij} g^{j\ell}(\nabla_k T_\ell + \nabla_\ell T_k)\big) \\
&= -\frac{1}{2}\big(2nHf - 2g^{ij}\nabla_i T_j\big)\sqrt{\det g} = \big(\operatorname{div}_M T - nHf\big)\sqrt{\det g}.
\end{aligned}$$

The extrinsic scalar curvature S of $\Psi_t(M)$ is the second elementary symmetric function of the principal curvatures $\lambda_i = \lambda_i(II)$ of Ψ_t, that is,

$$S = \sum_{i<j} \lambda_i \lambda_j. \tag{2.3}$$

Note that

$$2S = n^2 H^2 - |A|^2, \tag{2.4}$$

where A is the Weingarten map of $\Psi_t(M)$ whose components are defined by

$$h^i_j = g^{ik} h_{kj}.$$

We use (2.4) in order to compute the first variation of the extrinsic scalar curvature. It is convenient to rewrite this expression in terms of the tensor

$$P^{ij} = nHg^{ij} - h^{ij}, \tag{2.5}$$

whose $(1,1)$ metrically equivalent form is

$$P = nHI - A, \tag{2.6}$$

as

$$2S = (nHg^{ij} - h^{ij})h_{ij} = P^{ij} h_{ij} = \operatorname{trace}(PA). \tag{2.7}$$

In order to obtain the first variation of S we start computing the first derivative of the squared norm of the Weingarten map as follows:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|A|^2 = \frac{1}{2}\nabla_{\partial_t}|A|^2 = \frac{1}{2}\nabla_{\partial_t}\operatorname{trace}(A^2) = h^{ij}\dot{h}_{ij} \\
&= h^{ij}\big(f_{i;j} + f\langle A\partial_i, A\partial_j\rangle + f\langle \bar{R}(\partial_i, N)N, \partial_j\rangle + \langle (\nabla_{\partial_i} A)T, \partial_j\rangle + \langle \bar{R}(\partial_i, T)N, \partial_j\rangle\big).
\end{aligned}$$

On the other hand,

$$\frac{1}{2}\frac{d}{dt}n^2H^2 = nH\big(\Delta f + |A|^2 f + \mathrm{Ric}_{\bar{M}}(N,N)f + n\pounds_T H\big)$$
$$= nHg^{ij}\big(f_{i;j} + g^{k\ell}h_{ik}h_{j\ell}f + \langle \bar{R}(\partial_i,N)N,\partial_j\rangle f + nH_iT_j\big).$$

Therefore

$$\frac{1}{2}\frac{d}{dt}S = (nHg^{ij} - h^{ij})f_{i;j} + (nHg^{ij} - h^{ij})g^{k\ell}h_{ik}h_{j\ell}f + (nHg^{ij} - h^{ij})\langle \bar{R}(\partial_i,N)N,\partial_j\rangle f$$
$$+ \frac{1}{2}\langle \nabla(n^2H^2 - |A|^2), T\rangle - h^{ij}\langle \bar{R}(\partial_i,T)N,\partial_j\rangle.$$

In terms of the tensor P, one has

$$\frac{1}{2}\frac{d}{dt}S = P^{ij}f_{i;j} + P^{ij}g^{k\ell}h_{ik}h_{j\ell}f + P^{ij}\langle \bar{R}(\partial_i,N)N,\partial_j\rangle f$$
$$+ \frac{1}{2}\pounds_T(n^2H^2 - |A|^2) - h^{ij}\langle \bar{R}(\partial_i,T)N,\partial_j\rangle.$$

Note that

$$g^{k\ell}h_{ik}P^{ij}h_{j\ell} = h_i^\ell P_j^i h_\ell^j = \mathrm{trace}(APA) = \mathrm{trace}(nHA^2 - A^3) = nH|A|^2 - \mathrm{trace}A^3. \tag{2.8}$$

We have

$$nH|A|^2 = nH(k_1^2 + \ldots + k_n^2) = k_1^2(k_1 + \ldots + k_n) + \ldots + k_n^2(k_1 + \ldots + k_n)$$
$$= k_1^3 + \ldots + k_n^3 + k_1^2k_2 + \ldots + k_1^2k_n + \ldots + k_n^2k_1 + \ldots + k_n^2k_{n-1}$$
$$= \mathrm{trace}A^3 + k_1(k_1k_2 + \ldots + k_1k_n) + \ldots + k_n(k_nk_1 + \ldots + k_nk_{n-1})$$
$$= \mathrm{trace}A^3 + k_1k_2(k_1 + k_2) + \ldots + k_nk_{n-1}(k_{n-1} + k_n)$$
$$= \mathrm{trace}A^3 + k_1k_2(nH - \sum_{\ell\neq 1,2} k_\ell) + \ldots + k_{n-1}k_n(nH - \sum_{\ell\neq n-1,n} k_\ell)$$
$$= \mathrm{trace}A^3 + nHS - 3\sum_{i<j<\ell} k_ik_jk_\ell$$

Therefore

$$g^{k\ell}h_{ik}P^{ij}h_{j\ell} = \mathrm{trace}(APA) = \mathrm{trace}A^2P = nHS - 3S_3.$$

Now we compute

$$P^{ij}f_{i;j} = \mathrm{trace}(P\nabla\nabla f) = (P^{ij}f_i)_{;j} - P^{ij}_{;j}f_i = \mathrm{div}(P\nabla f) - \langle \mathrm{div}P, \nabla f\rangle. \tag{2.9}$$

The divergence of the operator P is computed as follows:

$$\begin{aligned}\operatorname{div} P_j &= P^k_{j;k} = nH_k\delta^k_j + nH\delta^k_{j;k} - h^k_{j;k} = nH_j - (h^k_{k;j} + (\bar{R}(\partial_j,\partial_k)N)^k)\\ &= nH_j - nH_j + (\bar{R}(\partial_k,\partial_j)N)^k = \operatorname{Ric}_{\bar{M}}(\partial_j, N),\end{aligned}$$

where we used again Codazzi equation

$$h^k_{j;k} - h^k_{k;j} = \big((\nabla_{\partial_k}A)\partial_j - (\nabla_{\partial_j}A)\partial_k\big)^k = \big(\bar{R}(\partial_j,\partial_k)N\big)^k.$$

Replacing these expressions above one obtains

$$\begin{aligned}\frac{d}{dt}S &= \operatorname{div}(P\nabla f) + (nHS - 3S_3)f + \langle P, \operatorname{Ric}_{\bar{M}}\rangle f\\ &- \operatorname{Ric}_{\bar{M}}(\nabla f, N) + \frac{1}{2}\pounds_T(n^2H^2 - |A|^2) - h^{ij}\langle \bar{R}(\partial_i, T)N, \partial_j\rangle.\end{aligned}$$

If $\bar{M}$ is a space form with sectional curvatures κ one has

$$\langle \bar{R}(\partial_i,N)N,\partial_j\rangle = \kappa\big(\langle\partial_i,\partial_j\rangle\langle N,N\rangle - \langle\partial_i,N\rangle\langle\partial_j,N\rangle\big) = \kappa g_{ij}$$

and

$$P^{ij}\langle \bar{R}(\partial_i,N)N,\partial_j\rangle = \kappa \operatorname{tr} P = \kappa(n^2-n)H.$$

Moreover

$$\operatorname{Ric}_{\bar{M}}(\nabla f, N) = 0.$$

If $\bar{M} = I \times \mathbb{P}_\kappa$ is a Riemannian product (here, κ indicates the constant sectional curvature of $\mathbb{P}$ then

$$\begin{aligned}\langle \bar{R}(\partial_i,N)N,\partial_j\rangle &= \kappa\big(\langle\partial_i^h,\partial_j^h\rangle\langle N^h,N^h\rangle - \langle\partial_i^h,N^h\rangle\langle\partial_j^h,N^h\rangle\big)\\ &= \kappa\big((g_{ij} - \langle\partial_i,X\rangle\langle\partial_j,X\rangle)(1-\langle N,X\rangle^2) - \langle\partial_i,X\rangle\langle\partial_j,X\rangle\langle N,X\rangle^2\big)\\ &= \kappa(1-\langle N,X\rangle^2)g_{ij} - \kappa\langle\partial_i,X\rangle\langle\partial_j,X\rangle,\end{aligned}$$

where h indicates the orthogonal projection onto the tangent space of $\mathbb{P}$. Therefore

$$\begin{aligned}P^{ij}\langle \bar{R}(\partial_i,N)N,\partial_j\rangle &= \kappa(1-\langle N,X\rangle^2)\operatorname{trace} P - \kappa P(X^\top, X^\top)\\ &= (n^2-n)H\kappa(1-\langle N,X\rangle^2) - \kappa\big(nH|X^\top|^2 - A(X^\top,X^\top)\big),\end{aligned}$$

where the superscript $^{\top}$ means the tangent component of the vector in $\Psi_t(M)$. Moreover, if U is a tangent vector field,

$$
\begin{aligned}
\langle \bar{R}(\partial_i, U)N, \partial_j\rangle &= \kappa\big(\langle \partial_i^h, \partial_j^h\rangle\langle U^h, N^h\rangle - \langle \partial_i^h, N^h\rangle\langle \partial_j^h, U^h\rangle\big)\\
&= \kappa\big(-(g_{ij} - \langle \partial_i, X\rangle\langle \partial_j, X\rangle)\langle U, X\rangle\langle N, X\rangle - \langle \partial_i, X\rangle\langle \partial_j, X\rangle\langle X, N\rangle\langle U, X\rangle\\
&\quad + \langle \partial_i, X\rangle\langle \partial_j, U\rangle\langle X, N\rangle\big)\\
&= -\kappa g_{ij}\langle U, X\rangle\langle X, N\rangle + \kappa\langle \partial_i, X\rangle\langle \partial_j, U\rangle\langle X, N\rangle,
\end{aligned}
$$

what implies that

$$\mathrm{Ric}_{\bar{M}}(U, N) = -(n-1)\kappa\langle X, U\rangle\langle X, N\rangle.$$

2.2 Variational formulation in the case of conformal vector fields

Throughout this section, we restrict ourselves to warped products of the form $\bar{M} = I \times_h \mathbb{P}$, that is, to the geometric setting for which $X = h(s)\partial_s$ is a closed conformal vector field. In this case, one defines the function

$$\hat{\eta}(s) = \int_{s_0}^{s} h(\tau)d\tau \tag{2.10}$$

for some $s_0 \in I$ and denote its restriction to $\psi(M)$ by

$$\eta = \hat{\eta} \circ \psi. \tag{2.11}$$

Now we define geometric functionals whose critical points are scalar curvature flow solitons for $\alpha = 1$ accordingly to Definition (1.6).

Proposition 1. *Let dM be the Riemannian element volume in M induced by the isometric immersion $\psi : M \to \bar{M}$. Given the mean curvature H of ψ we define the functionals*

$$\mathscr{A}[\psi] = \int_M nH\, dM$$

and

$$\mathscr{A}_\eta[\psi] = \int_M nHe^{\hat{c}\eta}\, dM,$$

for some arbitrarily chosen constant $\hat{c}$. Given a variation Ψ of ψ with variational vector field

$$\partial_t\Psi|_{t=0} = fN,$$

for some smooth function f with $f|_{\partial M}=0$, one has the following expressions for the first variations of $\mathscr{A}$ and $\mathscr{A}_\eta$:

$$\delta\mathscr{A}[\psi]\cdot f=\int_M(-2S+\mathrm{Ric}_{\bar M}(N,N))f\,dM \tag{2.12}$$

and

$$\delta\mathscr{A}_\eta[\psi]\cdot f=\int_M(-2S+\hat c\langle X,N\rangle+\mathrm{Ric}_{\bar M}(N,N))fe^{\hat c\eta}\,dM \tag{2.13}$$

respectively.

Proof. We have

$$\begin{aligned}
&\frac{1}{\sqrt{\det g}}\frac{d}{dt}\big(nH\sqrt{\det g}\big)=\big(\Delta f+|A|^2f+\mathrm{Ric}_{\bar M}(N,N)f+n\pounds_T H\big)+nH\big(\mathrm{div}_M T-nHf\big)\\
&=\big(\Delta f+\mathrm{div}_M(nHT)\big)+\big(|A|^2-n^2H^2+\mathrm{Ric}_{\bar M}(N,N)\big)f\\
&=\big(\Delta f+\mathrm{div}_M(nHT)\big)+\big(-2S+\mathrm{Ric}_{\bar M}(N,N)\big)f
\end{aligned}$$

Therefore

$$\frac{d}{dt}\int_M nH\,dM=\int_M(-2S+\mathrm{Ric}_{\bar M}(N,N))f\,dM+\int_{\partial M}\langle\nabla f+nHT,\nu\rangle$$

and similarly (for normal variations, for simplicity)

$$\frac{d}{dt}\int_M nHe^{\hat c\eta}\,dM=\int_M(-2S+\hat c\langle\bar\nabla\eta,N\rangle+\mathrm{Ric}_{\bar M}(N,N))fe^{\hat c\eta}\,dM+\int_{\partial M}\langle\nabla f,\nu\rangle e^{\hat c\eta}$$

Since $\bar\nabla\eta=X$, this finishes the proof. □

Now we compute the second variation formula for normal variations (i.e, with $T=0$ on M) with $f|_{\partial M}=0$. Given a critical immersion of $\mathscr{A}_\eta$ at $t=0$ we have

$$\begin{aligned}
\frac{d^2}{dt^2}\Big|_{t=0}\mathscr{A}_\eta[\Psi_t]&=\int_M\frac{d}{dt}\Big|_{t=0}\big(-2S+\hat c\langle\bar\nabla\eta,N\rangle+\mathrm{Ric}_{\bar M}(N,N)\big)\big)\langle\partial_t,N\rangle e^{\hat c\eta}\,\mathrm{d}M\\
&+\int_M\underbrace{\big(-2S+\hat c\langle\bar\nabla\eta,N\rangle+\mathrm{Ric}_{\bar M}(N,N)\big)}_{=0}\frac{d}{dt}\Big|_{t=0}\langle\partial_t,N\rangle e^{\hat c\eta}\,\mathrm{d}M.
\end{aligned}$$

However, the expressions deduced above yield (for $T=0$)

$$\frac{d}{dt}\Big|_{t=0}S=\mathrm{div}(P\nabla f)+(nHS-3S_3)f+\langle P,\mathrm{Ric}_{\bar M}\rangle f-\mathrm{Ric}_{\bar M}(\nabla f,N).$$

Therefore, bearing in mind that $\bar\nabla\eta=X$, one gets

$$\frac{d}{dt}\Big|_{t=0}\langle\bar\nabla\eta,N\rangle=\frac{d}{dt}\Big|_{t=0}\langle X,N\rangle=\langle\bar\nabla_{\partial_t}X,N\rangle+\langle X,\bar\nabla_{\partial_t}N\rangle=\varphi\langle X,N\rangle-\langle X,\nabla f\rangle$$

where $\varphi = \operatorname{div}_{\bar{M}} X$. In the particular case of space forms we have

$\operatorname{Ric}_{\bar{M}}(N,N) = n\kappa$

and

$\operatorname{Ric}_{\bar{M}}(\nabla f, N) = 0.$

Moreover,

$\langle P, \operatorname{Ric}_{\bar{M}} \rangle = \kappa(n^2 - n)H.$

In this case we conclude that

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathscr{A}[\Psi_t] = -2\int_M f\big(\operatorname{div}(P\nabla f) + (nHS - 3S_3)f + \kappa(n^2-n)Hf\big)\, dM \tag{2.14}$$

and

$$\begin{aligned}\frac{d^2}{dt^2}\Big|_{t=0} \mathscr{A}_\psi[\Psi_t] = \int_M \Big(&-2\big(\operatorname{div}(P\nabla f) - \hat{c}\langle X, \nabla f\rangle + (nHS - 3S_3)f + \kappa(n^2-n)Hf\big) \\ &+ \hat{c}\phi\langle X, N\rangle\Big) f e^{\hat{c}\eta}\, dM.\end{aligned}$$

In Section (5.1) we deduce a *flux formula* that can be regarded as a conservation law due to the invariance of the functional $\mathscr{A}$ with respect to the flow of X in the case when it is a parallel vector field.

2.3 Scalar curvature flow solitons

Now we consider the variation of a given isometric immersion $\psi : M^n \to \bar{M}^{n+1}$ defined by the geometric flow whose speed is a function F of the Weingarten map (more precisely, of its principal values) given by

$$F(h_{ij}) = F(\boldsymbol{\kappa}) \doteq S^\alpha, \tag{2.15}$$

for $\alpha \in \{1/2, 1\}$ and $\boldsymbol{\kappa}$ the principal curvatures of the Weingarten map whose components are h_{ij}. This means that

$$\frac{\partial \Psi}{\partial t} = S^\alpha N. \tag{2.16}$$

Fixed a closed conformal or Killing vector field X on $\bar{M}$, let $\Phi : (\Omega_*, \Omega^*) \times \bar{M} \to \bar{M}$ denotes the flow generated by X defined in the maximal interval (Ω_*, Ω^*). Let s be the flow parameter in Φ and define

$$\widetilde{\Psi}_t(x) = \widetilde{\Psi}(t,x) = \Phi^{-1}(\sigma(t), \Psi_\tau(x)), \quad x \in M, \tag{2.17}$$

where $\sigma : (\omega_*, \omega^*) \to (\Omega_*, \Omega^*)$ is a reparametrization of the flow lines of X of the form

$$s = \sigma(t).$$

Equivalently we can write

$$\Psi(t,x) = \Phi(\sigma(t), \widetilde{\Psi}(t,x)), \quad (t,x) \in (\omega_*, \omega^*) \times M. \tag{2.18}$$

Definition 1. *Let $\bar{M}^{n+1}$ be a Riemannian manifold endowed with a closed conformal or Killing vector field $X \in \Gamma(T\bar{M})$. Given an m-dimensional Riemannian manifold M^m, we say that a scalar curvature flow $\Psi : (\omega_*, \omega^*) \times M \to \bar{M}$ is self-similar if there exists an isometric immersion $\psi : M \to \bar{M}$ and a reparametrization $\sigma : (\omega_*, \omega^*) \to (\Omega_*, \Omega^*)$ of the flow lines of X such that*

$$\Psi_t(M) = \Phi_{\sigma(t)}(\psi(M)), \tag{2.19}$$

for all $t \in (\omega_*, \omega^*)$, where $\Phi : (\Omega_*, \Omega^*) \times \bar{M} \to \bar{M}$ is the flow generated by X. In other terms, $\widetilde{\Psi}_t(M) = \psi(M)$, for all $t \in (\omega_*, \omega^*)$.

Remark 1. *Although Definition 1 does not require in principle any special properties of X, we will restrict ourselves to closed conformal vector and Killing fields.*

Recall that X is said to be conformal if the conformal Killing equation

$$\pounds_X \bar{g} = 2\varphi\, \bar{g} \tag{2.20}$$

holds, where

$$\varphi = \frac{1}{n+1} \operatorname{div}_{\bar{M}} X. \tag{2.21}$$

It turns out that each map $\Phi_s = \Phi(s, \cdot) : \bar{M} \to \bar{M}$, $s \in (\Omega_*, \Omega^*)$, is conformal in the sense that there exists a smooth positive function $\lambda : (\Omega_*, \Omega^*) \times \bar{M} \to \mathbb{R}$ such that

$$\Phi_s^* \bar{g}|_x = \lambda^2(s,x)\, \bar{g}|_x \tag{2.22}$$

for all $x \in \bar{M}$. It follows from (2.20) that

$$\bar{\varphi}(\Phi(s,x)) = \lambda(s,x)\, \partial_s \lambda(s,x), \tag{2.23}$$

for all $(s,x) \in (\Omega_*, \Omega^*) \times \bar{M}$. We suppose that there are no singular points of X in $\bar{M}$ by replacing $\bar{M}$ with a proper open subset of it, if necessary. Let $\mathbb{P}$ be a fixed integral leaf of the distribution orthogonal to X. It is convenient to parameterize the flow Φ by fixing initial conditions on $\mathbb{P}$, that is, we consider $\Phi : (\Omega_*, \Omega^*) \times \mathbb{P} \to \bar{M}$ as a global chart of $\bar{M}$. Having fixed this map, the integral leaves $\mathbb{P}_s := \Phi_s(\mathbb{P})$ are identified with the slices $\{s\} \times \mathbb{P}$, $s \in (\Omega_*, \Omega^*)$.

We then consider the particular case when the conformal factor depends only on the flow parameter, that is, $\lambda = \lambda(s)$. In this case, each leaf $\mathbb{P}_s$ is homothetic to $\mathbb{P}$. This particular case corresponds to warped product spaces. More precisely, given the change of variables

$$\varsigma = \int |X(s)|\, ds,$$

we can describe $\bar{M}$ as a *warped* product $I \times_h \mathbb{P}$ with warped Riemannian metric given by

$$\mathrm{d}\varsigma^2 + h^2(\varsigma)\,\sigma,$$

where σ is the metric in $\mathbb{P}$ and $h(\varsigma) = |X|(s(\varsigma))$. In this case we have

$$X = h(\varsigma)\partial_\varsigma \tag{2.24}$$

and

$$\bar{\nabla}_U X = \bar{\varphi}\, U, \text{ for all } U \in \Gamma(T\bar{M}), \tag{2.25}$$

with $\varphi = h'$ in this case. This means that X is *closed* in the sense that the 1-form metrically equivalent to X is closed.

For further reference, we mention that the principal curvatures of an integral leaf $\mathbb{P}_s$ with respect to $-X/|X|$ are given by $\varphi/|X|$ and its mean curvature is

$$\mathscr{H}(\Phi(s,x)) = -n \frac{\varphi}{|X|}\frac{X}{|X|}\Big|_{\Phi(s,x)}. \tag{2.26}$$

Proposition 2. *Let $\Psi : (\omega_*, \omega^*) \times M \to \bar{M}$ be a self-similar scalar curvature flow with respect to some vector field $X \in \Gamma(T\bar{M})$. Then for all $t \in (\omega_*, \omega^*)$ there exists a constant c_t such that*

$$c_t X = c_t \Psi_{t*} T + S^\alpha N, \tag{2.27}$$

where S is the extrinsic scalar curvature of $\Psi_t = \Psi(t,\cdot)$ and $T \in \Gamma(TM)$ is the pull-back by Ψ_t of the tangential component of X. Moreover, if X is closed conformal or Killing then

$$II_{-S^\alpha N} + \frac{c_t}{2} \pounds_T g = c_t \varphi g, \tag{2.28}$$

where g is the metric induced in M by Ψ_t, $II_{-S^\alpha N}$ is its second fundamental form in the direction of $-S^\alpha N$ and φ is the divergence of X.

Proof. Differentiating both sides in (2.18) with respect to t we obtain

$$\begin{aligned}\frac{d\Psi}{dt}\Big|_{(t,x)} &= \frac{\partial\Phi}{\partial s}\Big|_{(\sigma(t),\widetilde{\Psi}(t,x))}\frac{d\sigma}{dt}\Big|_t + \Phi_{\sigma(t)*}(\widetilde{\Psi}(t,x))\frac{d\widetilde{\Psi}}{dt}\Big|_{(t,x)}\\ &= X(\Psi(t,x))\frac{d\sigma}{dt}\Big|_t + \Phi_{\sigma(t)*}(\widetilde{\Psi}(t,x))\frac{d\widetilde{\Psi}}{dt}\Big|_{(t,x)},\end{aligned}\tag{2.29}$$

where $\Phi_\sigma = \Phi(\sigma,\cdot)$. Since Ψ is a self-similar scalar curvature flow with respect to X, there exists an isometric immersion $\psi : M \to \bar{M}$ such that $\Psi(0,\cdot) = \psi$ and $\widetilde{\Psi}_t(M) = \psi(M)$ for all $t \in (\omega_*, \omega^*)$. This implies that

$$\Phi_{\sigma(t)*}(\widetilde{\Psi}(t,x))\frac{d\widetilde{\Psi}}{dt}\Big|_{(t,x)} \in T_{\Psi(t,x)}\Psi_t(M).$$

We conclude that for all $t \in (\omega_*, \omega^*)$ the tangential component of $\frac{d\sigma}{dt}X$ onto $\Psi_t(M)$ is given by

$$X^\top(\Psi(t,x))\frac{d\sigma}{dt}\Big|_t = -\Phi_{\sigma(t)*}(\widetilde{\Psi}(t,x))\frac{d\widetilde{\Psi}}{dt}\Big|_{(t,x)}$$

where the superscript $\top$ denotes tangential projection. We note that the expression

$$c_t\Psi_{t*}T(t,x) = -\Phi_{\sigma(t)*}(\widetilde{\Psi}(t,x))\frac{d\widetilde{\Psi}}{dt}\Big|_{(t,x)}.\tag{2.30}$$

defines a vector field $T(t,\cdot) \in \Gamma(TM)$, for each $t \in (\omega_*, \omega^*)$, where $c_t = \frac{d\sigma}{dt}\big|_t$. We conclude from (2.29) that

$$S^\alpha N|_{\Psi(t,x)} = c_tX(\Psi(t,x)) - c_t\Psi_{t*}T(t,x).\tag{2.31}$$

Then, we rewrite (2.31) in the form

$$c_tX|_{\Psi_t} = c_t\Psi_{t*}T_t + S^\alpha N|_{\Psi_\tau}\tag{2.32}$$

where $T_t(x) = T(t,x)$. Note that (2.32) holds in both cases, namely of conformal and Killing vector fields.

Next, for a fixed t, one denotes by g_t and ∇, respectively, the induced metric and connection of the immersion Ψ_t. Hence, it follows from (2.32) that

$$c_t\bar{\nabla}_{\Psi_*U}X = c_t\Psi_*\nabla_UT + c_t(\bar{\nabla}_{\Psi_*U}\Psi_*T)^\perp + \bar{\nabla}_{\Psi_*U}S^\alpha N.\tag{2.33}$$

Taking the normal projection in both sides one has

$$c_t(\bar{\nabla}_{\Psi_*U}X)^\perp = c_t(\bar{\nabla}_{\Psi_*U}\Psi_*T)^\perp + \bar{\nabla}^\perp_{\Psi_*U}S^\alpha N.$$

If X is closed and conformal we have from (2.25) that

$$\bar{\nabla}_{\Psi_* U} X = \varphi \Psi_* U, \ \text{ for all } \ U \in \Gamma(TM), \tag{2.34}$$

which yields

$$c_t (\bar{\nabla}_{\Psi_* U} \Psi_* T)^\perp + \bar{\nabla}^\perp_{\Psi_* U} S^\alpha N = 0,$$

that is,

$$c_t \, II(T,U) + \bar{\nabla}^\perp_{\Psi_* U} S^\alpha N = 0. \tag{2.35}$$

Next, denoting by $II_{-S^\alpha N}$ the second fundamental form of Ψ_t in the opposite direction of the vector field $S^\alpha N$, one deduces from (2.20), that is, from the fact that X is conformal,

$$\begin{aligned} 2c_t \varphi \langle \Psi_* U, \Psi_* V \rangle &= c_t \langle \bar{\nabla}_{\Psi_* U} X, \Psi_* V \rangle + c_t \langle \Psi_* U, \bar{\nabla}_{\Psi_* V} X \rangle \\ &= c_t \langle \nabla_U T, V \rangle + c_t \langle U, \nabla_V T \rangle + \langle \bar{\nabla}_{\Psi_* U} S^\alpha N, \Psi_* V \rangle + \langle \Psi_* U, \bar{\nabla}_{\Psi_* V} S^\alpha N \rangle \\ &= c_t \, \pounds_T g(U,V) + 2II_{-S^\alpha N}(U,V), \end{aligned}$$

where we have omitted the subscript t for the sake of brevity. We then have proved that in the case when X is closed conformal Ψ_t satisfies the soliton equation

$$II_{-S^\alpha N} + \frac{c_t}{2} \pounds_T g = c_t \varphi \, g. \tag{2.36}$$

In the case when X is a Killing vector field expression (2.34) is replaced by

$$\bar{\nabla}_{\Psi_* U} X = -\langle \Psi_* U, X \rangle \bar{\nabla} \log \chi + \langle \bar{\nabla} \log \chi, \Psi_* U \rangle X, \tag{2.37}$$

where $\chi = |X|$. Hence, it follows from (2.33) that

$$\begin{aligned} c_t (\bar{\nabla}_{\Psi_* U} \Psi_* T)^\perp + \bar{\nabla}^\perp_{\Psi_* U} S^\alpha N &= -c_t \langle \Psi_* U, X \rangle \bar{\nabla}^\perp \log \chi + c_t \langle \bar{\nabla} \log \chi, \Psi_* U \rangle X^\perp \\ &= -c_t \langle \Psi_* U, X \rangle \bar{\nabla}^\perp \log \chi + \langle \bar{\nabla} \log \chi, \Psi_* U \rangle S^\alpha N. \end{aligned}$$

Moreover, combining (2.33) and (2.37) one has

$$\begin{aligned} 2II_{-S^\alpha N}(U,V) + c_t \pounds_T g(U,V) &= c_t \langle \bar{\nabla}_{\Psi_* U} X, \Psi_* V \rangle + c_t \langle \bar{\nabla}_{\Psi_* V} X, \Psi_* U \rangle \\ &= -\langle \Psi_* U, X \rangle \langle \bar{\nabla} \log \chi, \Psi_* V \rangle + \langle \bar{\nabla} \log \chi, \Psi_* U \rangle \langle X, \Psi_* V \rangle \\ &\quad - \langle \Psi_* V, X \rangle \langle \bar{\nabla} \log \chi, \Psi_* U \rangle + \langle \bar{\nabla} \log \chi, \Psi_* V \rangle \langle X, \Psi_* U \rangle = 0. \end{aligned}$$

Since the Killing equation implies that X is divergence-free when it is a Killing vector field, this completes the proof of Proposition 2. $\square$

2.4 Scalar curvature flow solitons in warped products

Motivated by the above geometric setting, we define a general notion of scalar curvature flow soliton with respect to a given vector field $X \in \Gamma(T\bar{M})$ as follows.

Definition 2. *An isometric immersion $\psi : M^n \to \bar{M}^{n+1}$ is a scalar curvature flow soliton with respect to $X \in \Gamma(T\bar{M})$ if*

$$c\,\langle X,N\rangle = S^{\alpha} \tag{2.38}$$

along ψ for $\alpha = 1$ or $\alpha = 1/2$ and some constant $c \in \mathbb{R}$.

With a slight abuse of notation, we also say that the hypersurface $\psi(M)$ itself is the scalar curvature flow soliton (with respect to the vector field X).

We observe that in case X is a either a closed conformal or a Killing vector field on $\bar{M}$, equation (2.38) is enough to deduce the following important consequences that we have considered in Proposition 2 in the context of extrinsic geometric flows.

Proposition 3. *Let $\psi : M^n \to \bar{M}^{n+1}$ be a scalar curvature flow soliton with respect to a closed conformal or Killing vector field $X \in \Gamma(T\bar{M})$. Then along ψ we have*

$$II_{-S^{\alpha}N} + \frac{c}{2}\pounds_T g = c\varphi g, \tag{2.39}$$

where g is the metric induced in M by ψ and $II_{-S^{\alpha}N}$ is its second fundamental form of ψ in the direction of $-S^{\alpha}N$. Here the vector field T is defined by $\psi_* T = X^{\top}$ and

$$\varphi = \frac{1}{n+1}\operatorname{div}_{\bar{M}} X \circ \psi. \tag{2.40}$$

Furthermore, if X is a closed conformal vector field then

$$\langle \nabla S^{\alpha}, \cdot\rangle N + c\,II(T,\cdot) = 0, \tag{2.41}$$

where II is the second fundamental tensor of ψ and ∇ is the Riemannian connection in M induced by ψ. If X is a Killing vector field then

$$\langle \nabla S^{\alpha}, \cdot\rangle N + c\,II(T,\cdot) = -c\langle T,\cdot\rangle \bar{\nabla}^{\perp}\log\chi + \langle \nabla \log\chi, \cdot\rangle S^{\alpha}N. \tag{2.42}$$

Proof. Using (2.38) by a direct computation we have for any tangent vector fields $U,V \in \Gamma(TM)$

$$\begin{aligned}
&c\langle \bar{\nabla}_{\psi_* U} X, \psi_* V\rangle + c\langle \bar{\nabla}_{\psi_* V} X, \psi_* U\rangle \\
&= c\langle \bar{\nabla}_{\psi_* U}\psi_* T, \psi_* V\rangle + c\langle \bar{\nabla}_{\psi_* V}\psi_* T, \psi_* U\rangle + \langle \bar{\nabla}_{\psi_* U} S^{\alpha}N, \psi_* V\rangle + \langle \bar{\nabla}_{\psi_* V} S^{\alpha}N, \psi_* U\rangle \\
&= c\langle \nabla_U T, V\rangle + c\langle \nabla_V T, U\rangle + 2II_{-S^{\alpha}N}(U,V).
\end{aligned}$$

On the other hand, if X is a closed conformal vector field it holds that

$$c\langle\bar{\nabla}_{\psi_* U}X,\psi_* V\rangle + c\langle\bar{\nabla}_{\psi_* V}X,\psi_* U\rangle = 2c\varphi\, g(U,V).$$

If X is a Killing field it follows from (2.37) that

$$c\langle\bar{\nabla}_{\psi_* U}X,\psi_* V\rangle + c\langle\bar{\nabla}_{\psi_* V}X,\psi_* U\rangle = 0.$$

In any case one concludes that

$$II_{-S^\alpha N} + \frac{c}{2}\pounds_T g = c\varphi g,$$

where φ is (up to some multiplicative constant) the divergence of X. Now, if X is closed conformal, using (2.25) one has

$$\begin{aligned}0 &= c(\varphi\,\psi_* U)^\perp = c(\bar{\nabla}_{\psi_* U}X)^\perp = c(\bar{\nabla}_{\psi_* U}\psi_* T)^\perp + c(\bar{\nabla}_{\psi_* U}X^\perp)^\perp\\ &= c\,II(T,U) + (\bar{\nabla}_{\psi_* U}(S^\alpha N)^\perp = c\,II(T,U) + \langle\nabla S^\alpha, U\rangle N.\end{aligned}$$

If X is a Killing vector field, one gets

$$\begin{aligned}-c\langle\Psi_* U,X\rangle\bar{\nabla}^\perp\log\chi + \langle\bar{\nabla}\log\chi,\Psi_* U\rangle S^\alpha N &= c(\bar{\nabla}_{\psi_* U}X)^\perp = c(\bar{\nabla}_{\psi_* U}\psi_* T)^\perp + c(\bar{\nabla}_{\psi_* U}X^\perp)^\perp\\ &= c\,II(T,U) + (\bar{\nabla}_{\psi_* U}(S^\alpha N)^\perp = c\,II(T,U) + \langle\nabla S^\alpha, U\rangle N.\end{aligned}$$

what concludes the proof of Proposition 3. □

Example 1. *In warped products of the form $\bar{M} = I\times_h\mathbb{P}$ a leaf $\mathbb{P}_s = \{s\}\times\mathbb{P}$ is an example of scalar curvature flow soliton. Indeed its extrinsic scalar curvature is given by*

$$S = \frac{n(n-1)}{2}\frac{h'^2(s)}{h^2(s)}$$

On the other hand $N = \partial_s$ and

$$\langle X,N\rangle = h(s).$$

Therefore $\mathbb{P}_s$ satisfies (1.6) *for $\alpha\in\{1/2,1\}$ if and only if $s = s_c$, where s_c is implicitly given by*

$$\frac{h'^2(s_c)}{h^{2+\frac{1}{\alpha}}(s_c)} = \frac{2c^{1/\alpha}}{n(n-1)} \doteq \beta_{n,c,\alpha}.$$

For some particular but relevant examples of warped products $\bar{M} = I\times_h\mathbb{P}$ we have totally umbilical leaves which are scalar curvature flow solitons as follows:

- *if* $\mathbb{H}^{n+1} = \mathbb{R} \times_h \mathbb{R}^n$ *for* $h(t) = e^s$*, then the leaves* $\mathbb{P}_s$ *are horospheres and* $\mathbb{P}_{s_c}$ *is a curvature flow soliton if* $s_c = -\alpha \log \beta_{n,c,\alpha}$*;*
- *if* $\mathbb{H}^{n+1} = (0,+\infty) \times_h \mathbb{S}^n$ *for* $h(t) = \sinh s$*, then the leaves* $\mathbb{P}_s$ *are geodesic spheres and* $\mathbb{P}_{s_c}$ *is a curvature flow soliton for* $\alpha = 1$ *if* $s_c = \frac{1}{2} + \sqrt{\frac{1}{4} + \beta_{n,c,1/2}}$*;*
- *if* $\mathbb{R}^{n+1} = (0,+\infty) \times_h \mathbb{S}^n$ *for* $h(t) = s$*, then the leaves* $\mathbb{P}_s$ *are geodesic spheres and* $\mathbb{P}_{s_c}$ *is a curvature flow soliton if* $s_c = (1/\beta_{n,c,\alpha})^{1/(2+1/\alpha)}$.

For further use, we define the soliton function by

$$\pi(s) \doteq \frac{n(n-1)}{2} h'^2(s) - c^{\frac{1}{\alpha}} h^{2+\frac{1}{\alpha}}.$$

3 SCALAR CURVATURE EQUATION IN WARPED PRODUCTS

In this section, we only consider the warped metric structure of the form (1.2), that is, a product manifold $\bar{M} = \mathbb{P} \times_\chi I$, where $(\mathbb{P}, \sigma)$ is a n-dimensional complete Riemannian manifold, endowed with a Riemannian metric of the form

$$\bar{g} = \chi^2 ds^2 + \sigma, \tag{3.1}$$

where $\chi : \mathbb{P} \to \mathbb{R}$ is a smooth positive function and s is the natural coordinate in the factor $\mathbb{R}$ in the product $\bar{M}$. If χ is constant, $(\bar{M}, \bar{g})$ is the Riemannian product of $\mathbb{P}$ and $\mathbb{R}$ and $X = \partial_s$ is a parallel vector field. In general, X is a Killing vector field whose flow is given by a one-parameter family of translations along its flow lines. In any case, the norm of X is given by the function χ in the expression of the metric (3.1). The leaves $\mathbb{P}_s = \{s\} \times \mathbb{P}$, for $s \in \mathbb{R}$, are totally geodesic hypersurfaces in $\bar{M}$ as a consequence of the Killing equation

$$\pounds_X \bar{g}_{ij} = \langle \bar{\nabla}_{\partial_i} X, \partial_j \rangle + \langle \partial_i, \bar{\nabla}_{\partial_j} X \rangle = 0,$$

where $\bar{\nabla}$ is the Riemannian connection in $(\bar{M}, \bar{g})$. Here, we are adopting local coordinates $x^1, \ldots, x^n$ in $\mathbb{P}$. Hence, ∂_i are the coordinate vector fields and

$$\bar{g}_{ij} = \bar{g}(\partial_i, \partial_j) \doteq \langle \partial_i, \partial_j \rangle$$

denotes the local components of $\bar{g}$ in terms of those coordinates. Hence

$$\bar{\nabla}_{\partial_i} \partial_j = \nabla_{\partial_i} \partial_j, \tag{3.2}$$

where ∇ is the Riemannian connection induced in $\mathbb{P}_t$. We also have

$$\langle \bar{\nabla}_X X, \partial_i \rangle = -\langle X, \bar{\nabla}_{\partial_i} X \rangle = -\frac{1}{2} \partial_i \chi^2. \tag{3.3}$$

Therefore

$$\bar{\nabla}_X X = -\frac{1}{2} \bar{\nabla} \chi^2 = -\frac{1}{2} \nabla \chi^2. \tag{3.4}$$

This also implies that

$$\bar{\nabla}_{\partial_i} X = \frac{1}{2\chi^2} \partial_i \chi^2 X. \tag{3.5}$$

3.1 Scalar curvature flow of graphs

Now, we resume our discussion about the extrinsic scalar curvature flow

$$(\partial_t \Psi)^{\perp} = S^{\alpha} N. \tag{3.6}$$

In what follows, we rule out the effect to local diffeomorphisms in (3.6) by defining a non-parametric formulation of the flow Ψ, namely

$$\Psi(t,x) = (x, u(t,x)), \tag{3.7}$$

where $u : [0,t^*) \times \mathbb{P} \to \mathbb{R}$ is a one-parameter family of smooth functions $u(t,\cdot)$ defined in $\mathbb{P}$. In this formulation, the hypersurface $\Psi_t(M)$ is the graph Σ_t of the function $u(t,\cdot)$. Since

$$\partial_i \Psi = \partial_i + \partial_i u X,$$

the induced metric in Σ_t has local components given by

$$g_{ij} = \sigma_{ij} + \chi^2 \partial_i u \partial_j u, \tag{3.8}$$

and its contravariant version has components

$$g^{ij} = \sigma^{ij} - \frac{\chi^2}{W^2} u^i u^j,$$

where $u^i = \sigma^{ij}\partial_j u$ are the components of the gradient ∇u (for a fixed t) and

$$W = \sqrt{1 + \chi^2 |\nabla u|^2}. \tag{3.9}$$

An orientation of Σ_t could be fixed by the unit normal vector field

$$N = \frac{1}{\chi W}\big(X - \chi^2 \nabla u\big). \tag{3.10}$$

In order to determine the second fundamental form of Σ_t, we compute

$$\bar{\nabla}_{\partial_i \Psi} \partial_j \Psi = \bar{\nabla}_{\partial_i} \partial_j + u_{i,j} X + u_j \bar{\nabla}_{\partial_i} X + u_i \bar{\nabla}_X \partial_j + u_i u_j \bar{\nabla}_X X,$$

from what follows that

$$\langle \bar{\nabla}_{\partial_i \Psi} \partial_j \Psi, X\rangle = \chi^2 u_{i,j} + \frac{1}{2} u_j \partial_i \chi^2 + \frac{1}{2} u_i \partial_j \chi^2$$

and

$$\begin{aligned}\langle \bar{\nabla}_{\partial_i \Psi} \partial_j \Psi, \chi^2 \nabla u\rangle &= \chi^2 \langle \nabla_{\partial_i} \partial_j, \nabla u\rangle + u_i u_j \langle \bar{\nabla}_X X, \chi^2 \nabla u\rangle \\ &= \chi^2 \langle \nabla_{\partial_i} \partial_j, \nabla u\rangle - \frac{1}{2} u_i u_j \langle \nabla \chi^2, \chi^2 \nabla u\rangle,\end{aligned}$$

where we used the expressions above for the covariant derivatives of the coordinate vector fields in $\bar{M}$. Therefore the components of the second fundamental form of Σ_t are given by

$$
\begin{aligned}
h_{ij} &= \langle \bar{\nabla}_{\partial_i\Psi}\partial_j\Psi, N\rangle = \frac{1}{\chi W}\langle \bar{\nabla}_{\partial_i\Psi}\partial_j\Psi, X-\chi^2\nabla u\rangle \\
&= \frac{\chi^2}{\chi W}\big(u_{i,j}-\langle \nabla_{\partial_i}\partial_j,\nabla u\rangle\big)+\frac{1}{2\chi W}\big(u_j\partial_i\chi^2+u_i\partial_j\chi^2\big)+\frac{\chi^2}{2\chi W}u_iu_j\langle\nabla\chi^2,\nabla u\rangle \\
&= \frac{\chi}{W}u_{i;j}+\frac{1}{2\chi W}\big(u_j\partial_i\chi^2+u_i\partial_j\chi^2\big)+\frac{\chi}{2W}u_iu_j\langle\nabla\chi^2,\nabla u\rangle,
\end{aligned}
$$

where $u_{i;j}$ are the local components of the Hessian of u (for a fixed t) in $\mathbb{P}$. We conclude that the components of the Weingarten map are given by

$$
a^i_j = g^{ik}h_{kj} = \frac{\chi}{W}g^{ik}u_{k;j}+\frac{1}{2\chi W}\big(u_jg^{ik}\partial_k\chi^2+g^{ik}u_k\partial_j\chi^2\big)+\frac{\chi^2}{2\chi W}g^{ik}u_ku_j\langle\nabla\chi^2,\nabla u\rangle.
$$

Note that

$$
g^{ik}u_k = \sigma^{ik}u_k-\frac{\chi^2}{W^2}u^iu^ku_k = u^i\Big(1-\frac{\chi^2}{W^2}|\nabla u|^2\Big) = \frac{1}{W^2}u^i
$$

and

$$
g^{ik}\partial_k\chi^2 = (\chi^2)^i-\frac{\chi^2}{W^2}u^i\langle\nabla\chi^2,\nabla u\rangle.
$$

Therefore

$$
\begin{aligned}
a^i_j &= \frac{\chi}{W}g^{ik}u_{k;j}+\frac{1}{2\chi W}\Big(u_j(\chi^2)^i-u^iu_j\frac{\chi^2}{W^2}\langle\nabla\chi^2,\nabla u\rangle\Big) \\
&\quad+\frac{1}{2\chi W}\frac{1}{W^2}u^i\partial_j\chi^2+\frac{\chi^2}{2\chi W}\frac{1}{W^2}u^iu_j\langle\nabla\chi^2,\nabla u\rangle.
\end{aligned}
$$

Eliminating some terms one obtains the following expression for the components of the Weingarten map in local coordinates

$$
a^i_j = \frac{\chi}{W}g^{ik}u_{k;j}+\frac{1}{2\chi W}u_j(\chi^2)^i+\frac{1}{2\chi W^3}u^i.(\chi^2)_j, \tag{3.11}
$$

where the sacalar curvature is given by

$$
S = \sum_{i<j}\varepsilon^{ij}_{k\ell}a^k_ia^\ell_j = \sum_{i<j}\begin{vmatrix} a^i_i & a^i_j \\ a^j_i & a^j_j \end{vmatrix}.
$$

3.1.1 Scalar curvature equation in terms of intrinsic operators

From the intrinsic point of view, the function u is the restriction of the coordinate s to the graph Σ. Since

$$
\bar{\nabla}s = \chi^{-2}\partial_s = \chi^{-2}X,
$$

the (intrinsic) gradient of $u:\Sigma\to\mathbb{R}$ is given by

$$\nabla^\Sigma u=\chi^{-2}X^\top, \tag{3.12}$$

where ∇^Σ denotes the Riemannian connection in (Σ,g) and $\top$ denotes the orthogonal projection onto the tangent space of Σ at a given point. Hence,

$$\nabla^\Sigma u=\frac{1}{\chi^2}X-\frac{1}{\chi W}N. \tag{3.13}$$

The local components of $\nabla^\Sigma u$ are given by

$$(\nabla^\Sigma u)^i=g^{ij}u_j=\sigma^{ij}u_j-\frac{\chi^2}{W^2}u^iu^ju_j=u^i\Big(1-\frac{\chi^2}{W^2}|\nabla u|^2\Big)=\frac{1}{W^2}u^i.$$

Therefore

$$\nabla^\Sigma u=g^{ij}u_i\partial_j\Psi=\frac{u^i}{W^2}\partial_i\Psi.$$

On the other hand

$$\langle X,\partial_i\Psi\rangle=\chi^2u_i,$$

what implies that

$$X^\top=\chi^2g^{ij}u_i\partial_j\Psi,$$

that is,

$$X^\top=\chi^2\nabla^\Sigma u,$$

verifying the expression 3.12 above. Now, we compute the intrinsic Hessian of $u:\Sigma\to\mathbb{R}$. Using 3.12 one has

$$\begin{aligned}\langle\nabla^\Sigma_{\partial_i\Psi}\nabla^\Sigma u,\partial_j\Psi\rangle&=\chi^{-2}\langle\nabla^\Sigma_{\partial_i\Psi}X^\top,\partial_j\Psi\rangle+\partial_i\chi^{-2}\langle X,\partial_j\Psi\rangle\\&=\chi^{-2}\langle\bar\nabla_{\partial_i\Psi}X,\partial_j\Psi\rangle-\langle X,N\rangle\langle\bar\nabla_{\partial_i\Psi}N,\partial_j\Psi\rangle+\partial_i\chi^{-2}\langle X,\partial_j\Psi\rangle\\&=\chi^{-2}\big(u_i\langle\bar\nabla_XX,\partial_j\rangle+u_j\langle\bar\nabla_{\partial_i}X,X\rangle\big)+\langle X,N\rangle h_{ij}+u_j\chi^2\partial_i\chi^{-2}.\end{aligned}$$

Hence, one obtains

$$\begin{aligned}\langle\nabla^\Sigma_{\partial_i\Psi}\nabla^\Sigma u,\partial_j\Psi\rangle&=\frac12\chi^{-2}\big(-u_i\partial_j\chi^2+u_j\partial_i\chi^2\big)+\langle X,N\rangle h_{ij}+u_j\chi^2\partial_i\chi^{-2}\\&=-\frac12\chi^{-2}\big(u_i\partial_j\chi^2+u_j\partial_i\chi^2\big)+\langle X,N\rangle h_{ij}.\end{aligned}$$

In sum, the Hessian of $u = t|_\Sigma$ in (Σ, g) has components given by

$$\langle \nabla^\Sigma_{\partial_i \Psi} \nabla^\Sigma u, \partial_j \Psi \rangle = \frac{\chi}{W} h_{ij} - \frac{1}{2} \chi^{-2} \big(u_i \partial_j \chi^2 + u_j \partial_i \chi^2 \big) \tag{3.14}$$

$$= \frac{\chi}{W} h_{ij} - \big(u_i \partial_j \log \chi + u_j \partial_i \log \chi \big). \tag{3.15}$$

Therefore

$$\Delta_\Sigma u = g^{ij} \langle \nabla^\Sigma_{\partial_i \Psi} \nabla^\Sigma u, \partial_j \Psi \rangle = nH \frac{\chi}{W} - 2 \langle \nabla^\Sigma u, \nabla^\Sigma \log \chi \rangle.$$

Moreover

$$|A|^2 = \frac{\chi^2}{W^2} g^{ik} g^{j\ell} h_{ij} h_{k\ell}$$
$$= g^{ik} g^{j\ell} \Big(\nabla^\Sigma_{\partial_i} \nabla^\Sigma_{\partial_j} u + u_i \partial_j \log \chi + u_j \partial_i \log \chi \Big) \Big(\nabla^\Sigma_{\partial_k} \nabla^\Sigma_{\partial_\ell} u + u_k \partial_\ell \log \chi + u_\ell \partial_k \log \chi \Big)$$
$$= |\nabla^\Sigma \nabla^\Sigma u|^2 + 2 \langle \nabla^\Sigma_{\nabla^\Sigma u} \nabla^\Sigma u, \nabla^\Sigma \log \chi \rangle + 2 |\nabla^\Sigma u|^2 |\nabla^\Sigma \log \chi|^2 + 2 \langle \nabla^\Sigma u, \nabla^\Sigma \log \chi \rangle^2.$$

We conclude that

$$2S = n^2 H^2 - |A|^2 = \frac{W^2}{\chi^2} \big(\Delta_\Sigma u + 2 \langle \nabla^\Sigma u, \nabla^\Sigma \log \chi \rangle \big)^2$$
$$- |\nabla^\Sigma \nabla^\Sigma u|^2 - 2 \langle \nabla^\Sigma_{\nabla^\Sigma u} \nabla^\Sigma u, \nabla^\Sigma \log \chi \rangle - 2 |\nabla^\Sigma u|^2 |\nabla^\Sigma \log \chi|^2 - 2 \langle \nabla^\Sigma u, \nabla^\Sigma \log \chi \rangle^2. \tag{3.16}$$

This expression shows that the scalar curvature flow soliton is a solution of a fully nonlinear PDE of the form

$$F(u, \nabla^\Sigma u, \nabla^\Sigma \nabla^\Sigma u) = \psi(\nabla^\Sigma u),$$

where ψ encodes the expression involving $\langle X, N \rangle$.

3.2 Scalar curvature of rotationally invariant hypersurfaces

In this section, we consider the particular case when $\mathbb{P}$ is a Hadamard manifold and its Riemannian metric σ is rotationally invariant in the sense that there exists a pole $o \in \mathbb{P}$ and Gaussian global coordinates $(r, \theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$ centered at o in terms of which σ may be written as

$$\sigma = dr^2 + \xi^2(r) d\theta^2, \tag{3.17}$$

where $d\theta^2$ is the standard metric in $\mathbb{S}^{n-1}$ and ξ is a smooth positive function that extends smoothly as $r \to 0^+$ with

$$\xi(0) = 0, \quad \xi'(0) = 1, \quad \xi^{(2k)}(0) = 0$$

for $k \in \mathbb{N}$. We also suppose that $\chi = |X|$ depends (smoothly) only on $r = \mathrm{dist}_P(0,x)$, that is, $\chi(x) = \chi(r(x))$ with

$$\chi(0) = 1, \quad \chi'(0) = 0, \quad \chi^{(2k+1)}(0) = 0$$

for $k \in \mathbb{N}$. Recall that we are assuming the validity of the main sctructural assumptions about χ and ξ as stated above.

In sum, the metric $\bar{g}$ in $\bar{M} = \mathbb{R} \times \mathbb{P}$ has a doubly warped structure described in terms of the "cylindrical" coordinates (r, θ, τ) as

$$\bar{g} = dr^2 + \xi^2(r)d\theta^2 + \chi^2(r)ds^2.$$

Recall that $X = \partial_s$ is Killing vector field (in fact, a parallel vector field if χ is constant) with

$$\bar{\nabla}_{\partial_r} X = \frac{\partial_r \chi}{\chi} X = \frac{\chi'(r)}{\chi(r)} X$$

and

$$\bar{\nabla}_X X = -\chi(r)\chi'(r)\partial_r.$$

Moreover we have

$$\langle \bar{\nabla}_{\partial_{\theta^i}} X, X \rangle = \langle \bar{\nabla}_{\partial_{\theta^i}} X, \partial_r \rangle = \langle \bar{\nabla}_{\partial_{\theta^i}} X, \partial_{\theta^j} \rangle = 0,$$

$$\bar{\nabla}_{\partial_r} \partial_r = 0 \quad \text{and} \quad \bar{\nabla}_{\partial_{\theta^i}} \partial_r = \frac{\xi'(r)}{\xi(r)} \partial_{\theta^i}.$$

A rotationally invariant hypersurface $\Sigma \subset \bar{M}$ may be parameterized in clylindrical coordinates as

$$\Psi(\tau, \vartheta) = (r(\tau), \vartheta, s(\tau)).$$

Hence the tangent space at a given point at Σ is spanned by the coordinate vector field

$$\partial_\tau \Psi = \dot{r}\partial_r + \dot{s}\partial_s$$

and

$$\partial_{\theta^i} \Psi = \partial_{\theta^i}|_\Psi,$$

where the superscript $\dot{}$ denotes derivatives with respect to the parameter τ. The induced metric in Σ has local components in terms of the coordinates $(\tau, \theta^1, \ldots, \theta^{n-1})$ given by

$$g_{\tau\tau} = \dot{r}^2 + \chi^2(r)\dot{s}^2,$$

$$g_{\tau i} = 0,$$

$$g_{ij} = \xi^2(r)\theta_{ij}.$$

An orientation for Σ is given by

$$N = \frac{1}{\chi W}\big(-\chi^2(r)\dot{s}\partial_r + \dot{r}\partial_s\big),$$

where

$$W = \sqrt{\dot{r}^2 + \chi^2(r)\dot{s}^2}.$$

Now we compute

$$\bar{\nabla}_{\partial_\tau\Psi}(\chi W N) = -\frac{d}{d\tau}(\chi^2\dot{s})\partial_r + \frac{d}{d\tau}\dot{r}\partial_s - \chi^2\dot{s}\bar{\nabla}_{\partial_\tau\Psi}\partial_r + \dot{r}\bar{\nabla}_{\partial_\tau\Psi}\partial_\tau$$
$$= -\frac{d}{d\tau}(\chi^2\dot{s})\partial_r + \frac{d}{d\tau}\dot{r}\partial_s + \big(-\chi^2\dot{s}^2\bar{\nabla}_{\partial s}\partial_r + \dot{r}^2\bar{\nabla}_{\partial_r}\partial_s + \dot{r}\dot{s}\bar{\nabla}_{\partial_s}\partial_s\big)$$
$$= -\frac{d}{d\tau}(\chi^2\dot{s})\partial_r + \frac{d}{d\tau}\dot{r}\partial_s + \Big(\big(-\chi^2\dot{s}^2 + \dot{r}^2\big)\frac{\chi'(r)}{\chi(r)}\partial_s - \dot{r}\dot{s}\chi(r)\chi'(r)\partial_r\Big),$$

from what follows that

$$\bar{\nabla}_{\partial_\tau\Psi}(\chi W N) = -\frac{d}{d\tau}(\chi^2\dot{s})\partial_r + \frac{d}{d\tau}\dot{r}\partial_s + \big(\dot{r}^2 - \chi^2\dot{s}^2\big)\frac{\chi'(r)}{\chi(r)}\partial_s - \dot{r}\dot{s}\chi(r)\chi'(r)\partial_r.$$

Therefore the component of the second fundamental form of Σ in the direction of $\partial_\tau\Psi$ is given by

$$h_{\tau\tau} = -\langle\bar{\nabla}_{\partial_\tau\Psi}N, \partial_\tau\Psi\rangle = \frac{1}{\chi W}\Big(\dot{r}\frac{d}{d\tau}(\chi^2\dot{s}) - \chi^2\dot{s}\frac{d}{d\tau}\dot{r} + \dot{r}^2\dot{s}\chi(r)\chi'(r) - \dot{s}\big(\dot{r}^2 - \chi^2\dot{s}^2\big)\chi(r)\chi'(r)\Big).$$

Hence, we have

$$h_{\tau\tau} = \frac{1}{\chi W}\Big(\dot{r}\frac{d}{ds}(\chi^2\dot{s}) - \chi^2\dot{s}\frac{d}{ds}\dot{r} + \chi'\chi^3\dot{s}^3\Big). \tag{3.18}$$

Now we compute

$$\bar{\nabla}_{\partial_{\theta^i}\Psi}(\chi W N) = \bar{\nabla}_{\partial_{\theta^i}}\big(-\chi^2\dot{s}\partial_r + \dot{r}\partial_s\big) = -\chi^2\dot{s}\bar{\nabla}_{\partial_{\theta^i}}\partial_r = -\chi^2\dot{s}\frac{\xi'(r)}{\xi(r)}\partial_{\theta^i}.$$

Therefore

$$h_{\tau i} \doteq -\langle\bar{\nabla}_{\partial_{\theta^i}\Psi}N, \partial_\tau\rangle = 0 \tag{3.19}$$

and

$$h_{ij} \doteq -\langle\bar{\nabla}_{\partial_{\theta^i}\Psi}N, \partial_{\theta^j}\rangle = \frac{1}{\chi W}\chi^2\dot{s}\frac{\xi'(r)}{\xi(r)}g_{ij}. \tag{3.20}$$

Therefore the principal values of the Weingarten map

$$h^a_b = g^{ac}h_{cb}$$

of $\Sigma \subset \bar{M}$ are given by

$$\kappa_r \doteq h^\tau_\tau = \frac{1}{\chi W^3}\Big(\dot r \frac{d}{d\tau}(\chi^2 \dot s) - \chi^2 \dot s \frac{d}{d\tau}\dot r + \chi' \chi^3 \dot s^3\Big), \tag{3.21}$$

in ∂_r direction, and

$$\kappa_\theta \doteq \frac{1}{W}\chi \dot s \frac{\xi'(r)}{\xi(r)}, \tag{3.22}$$

in any direction of $\mathbb{S}^{n-1}$. Therefore the extrinsic scalar curvature of Σ is given by

$$S = (n-1)\kappa_r \kappa_\theta + \frac{(n-1)(n-2)}{2}\kappa_\theta^2,$$

that is,

$$S = (n-1)\frac{1}{W^4}\frac{\xi'(r)}{\xi(r)}\dot s\Big(\dot r \frac{d}{d\tau}(\chi^2 \dot s) - \chi^2 \dot s \frac{d}{d\tau}\dot r + \chi' \chi^3 \dot s^3\Big) + \frac{(n-1)(n-2)}{2}\frac{1}{W^2}\chi^2 \dot s^2 \frac{\xi'^2(r)}{\xi^2(r)}. \tag{3.23}$$

Now, since

$$\langle X, N\rangle = \frac{\chi}{W}\dot r,$$

the soliton equation for $\alpha = 1/2$, that is,

$$S = c^2 \langle X, N\rangle^2,$$

may be written as

$$(n-1)\frac{1}{W^4}\frac{\xi'(r)}{\xi(r)}\dot s\Big(\dot r \frac{d}{ds}(\chi^2 \dot s) - \chi^2 \dot s \frac{d}{ds}\dot r + \chi' \chi^3 \dot s^3\Big) + \frac{(n-1)(n-2)}{2}\frac{1}{W^2}\chi^2 \dot s^2 \frac{\xi'^2(r)}{\xi^2(r)} = c^2 \frac{\chi^2}{W^2}\dot r^2$$

or

$$(n-1)\frac{1}{\chi^2 W^2}\frac{\xi'(r)}{\xi(r)}\dot s\Big(\dot r \frac{d}{ds}(\chi^2 \dot s) - \chi^2 \dot s \frac{d}{ds}\dot r + \chi' \chi^3 \dot s^3\Big) + \frac{(n-1)(n-2)}{2}\dot s^2 \frac{\xi'^2(r)}{\xi^2(r)} = c^2 \dot r^2 \tag{3.24}$$

For $\alpha = 1$ the scalar curvature flow soliton is given by

$$(n-1)\frac{1}{W^4}\frac{\xi'(r)}{\xi(r)}\dot s\Big(\dot r \frac{d}{ds}(\chi^2 \dot s) - \chi^2 \dot s \frac{d}{ds}\dot r + \chi' \chi^3 \dot s^3\Big) + \frac{(n-1)(n-2)}{2}\frac{1}{W^2}\chi^2 \dot s^2 \frac{\xi'^2(r)}{\xi^2(r)} = c\frac{\chi}{W}\dot r. \tag{3.25}$$

3.2.1 First-order ODEs of rotationally invariant solitons

If we assume in the previous calculations that τ is the arc-length parameter of the profile curve

$$\alpha(\tau) = (r(\tau), s(\tau))$$

of Σ, then

$$\dot{r}^2 + \chi^2(r)\dot{s}^2 = 1,$$

that is, $W \equiv 1$. Moreover, we can define a smooth determination of the angle ϕ between the tangent to the curve α at a given point and the coordinate direction ∂_r. One has

$$\cos\phi = \langle \dot{r}\partial_r + \dot{s}\partial_s, \partial_r \rangle = \dot{r}.$$

In fact, one defines ϕ such that

$$\dot{r} = \cos\phi, \tag{3.26}$$

$$\chi(r)\dot{s} = \sin\phi. \tag{3.27}$$

Hence one has

$$\dot{r}\frac{d}{d\tau}(\chi\dot{s}) - \chi\dot{s}\frac{d}{d\tau}\dot{r} = \cos^2\phi\dot{\phi} + \sin^2\phi\dot{\phi} = \dot{\phi}.$$

Therefore

$$\dot{r}\frac{d}{d\tau}(\chi^2\dot{s}) - \chi^2\dot{s}\frac{d}{d\tau}\dot{r} + \chi'\chi^3\dot{s}^3 = \chi(r)\dot{\phi} + \chi(r)\chi'(r)\dot{r}^2\dot{s} + \chi'(r)\chi^3(r)\dot{s}^3$$
$$= \chi(r)\dot{\phi} + \chi(r)\chi'(r)\dot{s}(\dot{r}^2 + \chi^2\dot{s}^2) = \chi(r)\dot{\phi} + \chi(r)\chi'(r)\dot{s}$$
$$= \chi(r)\dot{\phi} + \chi'(r)\sin\phi.$$

Therefore

$$S = (n-1)\frac{\xi'(r)}{\xi(r)}\sin\phi\dot{\phi} + \left((n-1)\frac{\xi'(r)}{\xi(r)}\frac{\chi'(r)}{\chi(r)} + \frac{(n-1)(n-2)}{2}\frac{\xi'^2(r)}{\xi^2(r)}\right)\sin^2\phi.$$

If it is the cylinder, $\phi \equiv \pi/2$, and by the above expression, the scalar curvature of the cylinder is the term between brackets. We refer

$$S(r) \doteq (n-1)\frac{\xi'(r)}{\xi(r)}\frac{\chi'(r)}{\chi(r)} + \frac{(n-1)(n-2)}{2}\frac{\xi'^2(r)}{\xi^2(r)} \tag{3.28}$$

as the scalar curvature of the cylinder and S alone as the scalar curvature of M. Rewriting the scalar curvature of M,

$$S = (n-1)\frac{\xi'(r)}{\xi(r)}\sin\phi\dot{\phi} + S(r)\sin^2\phi. \tag{3.29}$$

In general, the scalar curvature flow soliton for $\alpha \in \{1/2, 1\}$ is equivalent to the following first-order systems of ODEs:

$$\begin{cases} \dot{r} = \cos\phi \\ \chi(r)\dot{s} = \sin\phi \\ (n-1)\frac{\xi'(r)}{\xi(r)}\sin\phi\dot{\phi} = c^{1/\alpha}\chi^{1/\alpha}(r)\cos^{1/\alpha}\phi - S(r)\sin^2\phi \end{cases} \tag{3.30}$$

Theorem 2. *For each $r_0 > 0$ and $s_0 \in I$ there exists a rotationally invariant scalar curvature soliton Σ_{α,n,r_0} for $\alpha \in \{1/2, 1\}$ in $M \times_\chi I$ given by a graph of a function defined in $M \backslash B_o(r_0)$ that meets M orthogonally along its boundary $\partial B_o(r_0) \subset \mathbb{P}_{s_0}$.*

For $\alpha = 1/2$, the rotationally invariant hypersurface $\mathscr{C}_{\alpha,n,r_0} \doteq \Sigma_{\alpha,n,r_0} \cup R_{s_0}(\Sigma_{\alpha,n,r_0})$ where $R : \bar{M} \to \bar{M}$ is the reflection through $\mathbb{P}_{s_0}$ is a scalar curvature flow soliton

Proof. We denote by ϕ the solution of the system (3.30) for initial conditions

$$r(0) = r_0 > 0, \quad s(0) = 0 \quad \text{and} \quad \phi(0) = \frac{\pi}{2},$$

with $\phi(\tau) \in [-\pi/2, \pi/2]$ for τ in some interval of the form $[0, \tau_\infty)$ or $[0, +\infty)$. It follows from the third equation in the system that

$$\dot{\phi}(0) < 0$$

and therefore ϕ decreases for $\tau > 0$ close to 0. Note that ϕ does not attain the value 0 since otherwise we would have $\chi(r) = 0$ for some $r > 0$. Indeed, if $\phi(\tau) \to 0$ as $\tau \to \tau_*$ for some $\tau_* > 0$ one has $\dot{r} \to 1$ as $\tau \to \tau_*$. Therefore, we can write s as a function of r for r close to $\lim_{\tau \to \tau_*} r$. Since

$$\begin{aligned}
\frac{d^2 s}{dr^2} = \frac{\ddot{s}\dot{r} - \dot{s}\ddot{r}}{\dot{r}^3} &= \frac{1}{\cos^3\phi}\left(\frac{1}{\chi}\left(\dot{\phi} - \frac{\chi'}{\chi}\sin\phi\right)\cos^2\phi + \frac{1}{\chi}\sin^2\phi\dot{\phi}\right) \\
&= \frac{1}{\chi(r)\cos^3\phi}\left(\dot{\phi} - \frac{\chi'(r)}{\chi(r)}\sin\phi\cos^2\phi\right),
\end{aligned}$$

one concludes that

$$\frac{d^2 s}{dr^2} \to \frac{1}{\chi(r_*)} \lim_{r \to r_*} \dot{\phi}(r).$$

On the other hand the third equation in (3.30) implies that $\lim_{\tau \to \tau_*} \dot{\phi}$ is not finite. Hence, the graph is not C^2 at $r_* = \lim_{\tau \to \tau_*} r$. Hence, ϕ stays positive wherever it is defined. In particular, it follows that there does not exists $\tau_* > 0$ such that $\phi(\tau) \to -\frac{\pi}{2}$ as $\tau \to \tau_*$. Similarly, we also observe that if

$$\phi(\tau) \to \frac{\pi}{2} \quad \text{and} \quad r(\tau) \to r_*$$

as $\tau \to \tau_*$ for some $\tau_* > 0$ and $r_* > 0$ finite, then it follows from the third equation in (3.30) that

$$(n-1)\frac{\xi'(r_*)}{\xi(r_*)} \lim_{\tau \to \tau_*} \dot{\phi} = -S(r_*) < 0, \qquad \lim_{\tau \to \tau_*} \dot{r} = 0$$

and

$$\lim_{\tau\to\tau_*} \ddot{r} = -\lim_{\tau\to\tau_*} \dot{\phi} > 0.$$

Therefore $r = r_*$ is either a strict local minimum of the map $\tau \mapsto r(\tau)$ or a vertical asymptote of $\tau \mapsto (r(\tau), s(\tau))$ curve. In sum, if τ_* is finite, r_* is attained and the curve is in upward direction, that is, $\phi(\tau) \searrow \pi/2$ since $\dot{\phi} < 0$ as $\tau \to \tau_*$. By continuity of ϕ, it can happen only for $\tau = \tau_*$: in fact, given two distinct minimum points it would exists a local maximum point with $\dot{r} = \cos\phi = 0$ and $\ddot{r} = -\sin\phi\dot{\phi} \leq 0$ which contradicts the third equation in (3.30). Therefore, the tangent of the curve can be vertical at most once and $r_* > r_0$ is a global minimum, contradicting the choice of the initial condition. Now, suppose that $\tau_* = \infty$. It follows that $\dot{s} > 0$, that is, s is a function of r for $r \in [r_0, r_*)$ with

$$\frac{ds}{dr} = \frac{\dot{s}}{\dot{r}} = \chi(r)\frac{\sin\phi}{\cos\phi} \to +\infty$$

and

$$\dot{\phi}(\tau) \to -S(r_*) < 0$$

as $\tau \to +\infty$ and $r \to r_*$. Hence $\phi(\tau) > \frac{\pi}{2}$ for τ large enough which contradicts the fact that $\frac{ds}{dr} > 0$ for large values of τ.

From this point on we also follow the ideas in the proofs of propositions 1 to 3 in (LIMA; PIPOLI, 2022. Disponível em https://arxiv.org/abs/2211.03918. Acesso em 12 dez. 2022). We denote

$$m(\tau) = \sin^2\phi(\tau).$$

Using this function we rewrite the third equation in (3.30) as

$$(n-1)\frac{\xi'(r)}{\xi(r)}\frac{1}{2\cos\phi}\dot{m} = c^{1/\alpha}\chi^{1/\alpha}(r)\big(1-m\big)^{1/2\alpha} - S(r)m.$$

Since

$$\frac{d\tau}{dr} = \frac{1}{\cos\phi},$$

we conclude that the derivative of m with respect to r is given by

$$\frac{dm}{dr} = \frac{2}{n-1}c^{1/\alpha}\chi^{1/\alpha}(r)\frac{\xi(r)}{\xi'(r)}\big(1-m\big)^{\frac{1}{2\alpha}} - \frac{2}{n-1}\frac{\xi(r)}{\xi'(r)}S(r)m \tag{3.31}$$

Given $r_0 > 0$, let m_{r_0} be the solution of (3.31) with initial condition $m_{r_0}(r_0) = 1$. This corresponds to the solution of (3.30) with that $\phi(0) = \pi/2$ as we are considering above. Therefore, (3.31) implies that $m'_{r_0}(r_0) < 0$ what implies that m_{r_0} is decreasing for $r > r_0$ near r_0. Suppose that m_{r_0} vanishes for some $r_1 > r_0$. If this is the case, (3.31) implies that $m'_{r_0}(r_1) > 0$ which is a contradiction since $m_{r_0}(r)$ would be negative for $r < r_1$ near r_1. Hence $m_{r_0}(r) > 0$ for $r > 0$. Suppose that there exists a critical point $r_* > r_0$ of m_{r_0}. Hence,

$$
\begin{aligned}
\frac{d^2m}{dr^2}\Big|_{r=r_*} &= \frac{2}{n-1}c^{1/\alpha}\left(\chi^{1/\alpha}(r)\frac{\xi(r)}{\xi'(r)}\right)'(1-m)^{\frac{1}{2\alpha}} - \frac{2}{n-1}\left(\frac{\xi(r)}{\xi'(r)}S(r)\right)'m \\
&= \frac{2}{n-1}\frac{\frac{\xi(r)}{\xi'(r)}S(r)m}{\chi^{1/\alpha}(r)\frac{\xi(r)}{\xi'(r)}}\left(\chi^{1/\alpha}(r)\frac{\xi(r)}{\xi'(r)}\right)' - \frac{2}{n-1}\left(\frac{\xi(r)}{\xi'(r)}S(r)\right)'m \\
&= \frac{2}{n-1}\frac{\xi(r)}{\xi'(r)}S(r)m\left(\frac{\left(\chi^{1/\alpha}(r)\frac{\xi(r)}{\xi'(r)}\right)'}{\chi^{1/\alpha}(r)\frac{\xi(r)}{\xi'(r)}} - \frac{\left(\frac{\xi(r)}{\xi'(r)}S(r)\right)'}{\frac{\xi(r)}{\xi'(r)}S(r)}\right) \\
&= \frac{2}{n-1}\frac{\xi(r_*)}{\xi'(r_*)}S(r_*)m(r_*)\frac{d}{dr}\Big|_{r=r_*}\log\left(\frac{\chi^{1/\alpha}(r)}{S(r)}\right) > 0
\end{aligned}
$$

where we used the fact that χ is an increasing function and $S(r)$ decreases with increasing r. Hence, a critical point is also a *strict* minimum point. Therefore m'_{r_0} does not vanish at other points besides $r = r_*$. It follows that m_{r_0} is increasing for $r > r_*$ wherever it is defined. Now, suppose that there exists $r_2 > r_*$ such that $m_{r_0}(r_2) = 1$. This implies that $m'_{r_0}(r_2) < 0$ what is a contradiction since $m'_{r_0}(r_{**}) > 0$ for $r_{**} > r_*$ and close to r_*: indeed, as we have seen, there are no points in $r \in (r_{**}, r_2)$ with $m'_{r_0}(r) = 0$. Therefore, $m_{r_0}(r)$ is always strictly less than 1 for $r > r_*$ wherever it is defined. Hence, m_{r_0} can be defined for $r \in (r_*, +\infty)$ and therefore in the interval $(r_0, +\infty)$. On the other hand, if there are no critical points of m_{r_0} in its interval of definition, then m_{r_0} decreases with increasing r and does not attain the value 1. We conclude that in this case m_{r_0} is defined for any $r \in (r_0, +\infty)$.

We conclude that the solution of (3.30) with initial conditions $r(0) = r_0, s(0) = 0$ and $\phi(0) = \pi/2$ is given by a graph defined for $r \in [r_0, +\infty)$ whose slope ϕ does not attain the values 0 and $\pm\pi/2$ elsewhere. Moreover, ϕ decreases in $[r_0, +\infty)$ if there are no critical points for m_{r_0}. In the case when there is (a unique) critical point $r_* > r_0$ of m_{r_0}, the angle ϕ decreases in $[r_0, r_*)$ and is an increasing function in the interval $(r_*, +\infty)$.

Moreover, the graph has a non zero asymptotic angle ϕ_∞ implicitly given by $\dot{\phi} \to 0$ as $\tau \to +\infty$, that is, by

$$
\lim_{\tau \to +\infty}\left(c^{1/\alpha}\chi^{1/\alpha}(r)(1-m)^{1/2\alpha} - S(r)m\right) = 0.
$$

In the case of Riemannian products (where $\chi = 1$) we have

$$c^{1/\alpha} \lim_{\tau \to +\infty} (1-m)^{1/2\alpha} = S_\infty \lim_{\tau \to +\infty} m$$

where $S_\infty = \lim_{r\to+\infty} S(r)$. For instance, in $\mathbb{R}^{n+1}$ one has $S_\infty = 0$ and $\phi_\infty = \frac{\pi}{2}$ whereas in $\mathbb{H}^n \times \mathbb{R}$ one has $S_\infty = \frac{1}{2}(n-1)(n-2)$ and

$$\frac{\cos^{1/\alpha}\phi_\infty}{\sin^2\phi_\infty} = \frac{S_\infty}{c^{1/\alpha}}.$$

In the case of non-trivial warped products with $\chi(r) \to +\infty$ as $r \to +\infty$ and finite S_∞ one has

$$c^{1/\alpha} \lim_{\tau \to +\infty} \frac{(1-m)^{1/2\alpha}}{m} = \lim_{r \to +\infty} \frac{S_\infty}{\chi^{1/\alpha}(r)} = 0.$$

Hence, $m \to 1$ as $\tau \to +\infty$ in this case, that is, $\phi_\infty = \frac{\pi}{2}$.

It remains to prove that m_{r_0} is bounded away from 0. In order to get a lower bound to m_{r_0} we observe from (3.31) that

$$-\frac{2}{n-1}\frac{\xi(r)}{\xi'(r)} S(r) m_{r_0}(r) \le \frac{dm_{r_0}}{dr} \le \frac{2}{n-1} c^{1/\alpha} \chi^{1/\alpha} \frac{\xi(r)}{\xi'(r)} (1-m)^{1/2\alpha}$$

for $r \ge r_0$. In particular one deduces that

$$\log m_{r_0}(r) \ge -\frac{2}{n-1}\int_{r_0}^{r} \frac{\xi(\varsigma)}{\xi'(\varsigma)} S(\varsigma)\, d\varsigma = \log\left(\chi^{-2}(r)\xi^{2-n}(r)\right) - \log\left(\chi^{-2}(r_0)\xi^{2-n}(r_0)\right)$$

where we used the fact that

$$-\frac{2}{n-1}\frac{\xi(r)}{\xi'(r)} S(r) = -2\frac{\chi'(r)}{\chi(r)} - (n-2)\frac{\xi'(r)}{\xi(r)}.$$

Therefore

$$m_{r_0}(r) \ge \frac{\chi^{-2}(r)\xi^{2-n}(r)}{\chi^{-2}(r_0)\xi^{2-n}(r_0)}$$

We also have

$$m_{r_0}(r) \le \frac{2}{n-1} c^{1/\alpha} \int_{r_0}^{r} \chi^{1/\alpha}(\varsigma) \frac{\xi(\varsigma)}{\xi'(\varsigma)}\, d\varsigma$$

for $r \ge r_0$. By first and second equations in 3.30,

$$\chi = \frac{\sin\phi}{\dot{s}} = \frac{\sin\phi}{\cos\phi}\frac{\dot{r}}{\dot{s}} \Rightarrow \chi\cos\phi = \frac{\sin\phi}{s'}.$$

By the third equation in (3.30) and knowing that $\dot{\phi} \to 0$ and $\phi \to \pi/2$ as $r \to \infty$,

$$\lim_{r\to\infty}\left[c^{1/\alpha}\left(\frac{\sin\phi}{s'}\right)^{1/\alpha} - S(r)\sin^2\phi\right] = 0 \Rightarrow \left(\frac{c}{s'_\infty}\right)^{1/\alpha} = S_\infty \Rightarrow s'_\infty = \frac{c}{S_\infty^\alpha},$$

where $s' \to s'_\infty$ as $r \to \infty$.

For $\alpha = 1/2$, let R be the reflection over a horizontal hyperplane $M \times \{t_0\}$ in $M \times \mathbb{R}$. Suppose that Σ is a translator in $M \times \mathbb{R}$ with unit normal ν. Define $\overline{\Sigma} = R(\Sigma)$ with unit normal $\overline{\nu} = -R_* \nu$. Then $\overline{\Sigma}$ is a translator as well. In fact, S is invariant by change of orientation. This and the fact that R is an isometry give that S of Σ at p coincides with $\overline{S}$ of $\overline{\Sigma}$ at $R(p)$. Therefore,

$$\sqrt{\overline{S} \circ R} = \sqrt{S} = c\langle \nu, \partial_t \rangle = c\langle R_* \nu, R_* \partial_t \rangle = c\langle \overline{\nu}, \partial_t \rangle,$$

which give us that $\overline{\Sigma}$ is a translator as well. This finishes the proof. $\square$

Theorem 3. *For each $s_o \in I$, there exists a rotationally invariant scalar curvature soliton $\mathscr{C}_{\alpha,n,0}$ for $\alpha \in \{1/2, 1\}$ in $M \times_\chi I$ given by a graph of a function defined in $\mathbb{P}_{s_0}$. We refer to this entire graph as a bowl soliton.*

Moreover, for each $r_0 > 0$ there exists a rotationally invariant scalar curvature soliton Σ_{α,n,r_0} for $\alpha \in \{1/2, 1\}$ in $M \times I$ given by a graph of a function defined in $M \setminus B_o(r_0)$ that meets M tangencially along its boundary $\partial B_o(r_0) \subset \mathbb{P}$. The rotationally invariant hypersurface $\mathscr{C}_{\alpha,n,r_0} \doteq \Sigma_{\alpha,n,r_0} \cup R_{s_0}(\Sigma_{\alpha,n,r_0})$, where $R_{s_0} : \bar{M} \to \bar{M}$ is the reflection throwgh $\mathbb{P}_{s_0}$, is a scalar curvature flow soliton.

Proof. It is convenient to consider the angle $\psi = \frac{\pi}{2} - \phi$. Hence,

$$\chi \dot{s} = \cos\psi, \quad \dot{r} = \sin\psi \tag{3.32}$$

Hence one has

$$\dot{r}\frac{d}{d\tau}(\chi\dot{s}) - \chi\dot{s}\frac{d}{d\tau}\dot{r} = -\dot{\psi}.$$

Therefore

$$\begin{aligned}
&\dot{r}\frac{d}{d\tau}(\chi^2\dot{s}) - \chi^2\dot{s}\frac{d}{d\tau}\dot{r} + \chi'\chi^3\dot{s}^3 = -\chi(r)\dot{\psi} + \chi(r)\chi'(r)\dot{r}^2\dot{s} + \chi'(r)\chi^3(r)\dot{s}^3 \\
&= -\chi(r)\dot{\psi} + \chi(r)\chi'(r)\dot{s}(\dot{r}^2 + \chi^2\dot{s}^2) = -\chi(r)\dot{\psi} + \chi(r)\chi'(r)\dot{s} \\
&= -\chi(r)\dot{\psi} + \chi'(r)\cos\psi.
\end{aligned}$$

We conclude that

$$S = -(n-1)\frac{\xi'(r)}{\xi(r)}\cos\psi\dot{\psi} + \left((n-1)\frac{\xi'(r)}{\xi(r)}\frac{\chi'(r)}{\chi(r)} + \frac{(n-1)(n-2)}{2}\frac{\xi'^2(r)}{\xi^2(r)}\right)\cos^2\psi.$$

and

$$S=-(n-1)\frac{\xi'(r)}{\xi(r)}\cos\psi\dot{\psi}+S(r)\cos^2\psi. \tag{3.33}$$

In general, the scalar curvature flow soliton for $\alpha\in\{1/2,1\}$ is equivalent to the following first-order systems of ODEs:

$$\begin{cases} \chi(r)\dot{s}=\cos\psi \\ \dot{r}=\sin\psi \\ -(n-1)\frac{\xi'(r)}{\xi(r)}\cos\psi\dot{\psi}=c^{1/\alpha}\chi^{1/\alpha}(r)\sin^{1/\alpha}\psi-S(r)\cos^2\psi \end{cases} \tag{3.34}$$

Denoting

$$m(\tau)=\cos^2\psi(\tau)$$

one has from the third equation above that

$$\frac{dm}{dr}=\frac{2}{n-1}\frac{\xi(r)}{\xi'(r)}c^{1/\alpha}\chi^{1/\alpha}(r)(1-m)^{1/2\alpha}-\frac{2}{n-1}\frac{\xi(r)}{\xi'(r)}S(r)m \tag{3.35}$$

Consider the initial condition $m(0)=0$ for (3.35). This corresponds to the solution of (3.34) with that $\psi(0)=\pi/2$. Therefore, (3.35) implies that $m'(0)>0$ what implies that m is increasing for $r>0$ near 0. Suppose that $m(r_1)=1$ for some $r_1>0$. If this is the case, (3.35) implies that $m'(r_1)<0$ which is a contradiction since in this case we would have $m(r)>1$ for $r<r_1$ near r_1. Hence $m(r)<1$ for $r>0$. Suppose that there exists a critical point $r_*>0$ of m. Hence,

$$\begin{aligned}
\frac{d^2m}{dr^2}\Big|_{r=r_*} &= \frac{2}{n-1}c^{1/\alpha}\left(\chi^{1/\alpha}(r)\frac{\xi(r)}{\xi'(r)}\right)'(1-m)^{\frac{1}{2\alpha}}-\frac{2}{n-1}\left(\frac{\xi(r)}{\xi'(r)}S(r)\right)'m \\
&= \frac{2}{n-1}\frac{\frac{\xi(r)}{\xi'(r)}S(r)m}{\chi^{1/\alpha}(r)\frac{\xi(r)}{\xi'(r)}}\left(\chi^{1/\alpha}(r)\frac{\xi(r)}{\xi'(r)}\right)'-\frac{2}{n-1}\left(\frac{\xi(r)}{\xi'(r)}S(r)\right)'m \\
&= \frac{2}{n-1}\frac{\xi(r)}{\xi'(r)}S(r)m\left(\frac{\left(\chi^{1/\alpha}(r)\frac{\xi(r)}{\xi'(r)}\right)'}{\chi^{1/\alpha}(r)\frac{\xi(r)}{\xi'(r)}}-\frac{\left(\frac{\xi(r)}{\xi'(r)}S(r)\right)'}{\frac{\xi(r)}{\xi'(r)}S(r)}\right) \\
&= \frac{2}{n-1}\frac{\xi(r_*)}{\xi'(r_*)}S(r_*)m(r_*)\frac{d}{dr}\Big|_{r=r_*}\log\left(\frac{\chi^{1/\alpha}(r)}{S(r)}\right)>0,
\end{aligned}$$

where we used the fact that χ is an increasing function and $S(r)$ decreases with increasing r. Hence, a critical point is also a *strict* minimum point. This contradicts the fact that there should exists a local maximum point before r_*, since m is increasing near $r=0$. It follows that m is increasing for $r>0$ wherever it is defined. Now, suppose that there exists $r_2>0$ such that

$m(r_2) = 0$. This implies that $m'(r_2) > 0$ what is a contradiction since we would have $m(r) < 0$ for $r < r_2$ near r_2. Therefore, $m(r)$ is always strictly greater than 0 for $r > 0$ wherever it is defined. We conclude that m can be defined for $r \in (0, +\infty)$ and that m_0 increases with increasing r and does not attain the values 0 and 1 for $r > 0$.

In sum, the solution of (3.34) with initial conditions $r(0) = 0, s(0) = s_0$ and $\psi(0) = \pi/2$ is given by a graph defined for $r \in [0, +\infty)$ whose slope ψ does not attain the value $\pi/2$ elsewhere. Moreover, ψ is a increasing function in the interval $(0, +\infty)$.

One can also deduce that this graph has the asymptotic limit determined in the case of initial conditions $\phi(0) = \frac{\pi}{2}$. □

Now, we prove the existence of rotationally invariant hypersurfaces with zero scalar curvature (this means to fix $c = 0$ in the soliton equation) in warped $\bar{M} = \mathbb{P} \times_\chi I$ which is an Einstein manifold.

Theorem 4. *There exists a one-parameter family of rotationally invariant hypersurfaces with zero scalar curvature in $\bar{M} = \mathbb{P} \times_\chi I$ when this warped space is an Einstein manifold.*

Proof. We use the flux formula (5.9) to deduce a first integral to the zero curvature scalar equation. Indeed, let M_{s_-,s_+} be the open subset of M between the leaves $\mathbb{P}_{s_-}$ and $\mathbb{P}_{s_+}$ with $s_- < s_+$. It follows that

$$\int_{\partial D_{s_-}} \langle P\nabla\eta, \nu\rangle e^{-\zeta} d\partial D_{s_-} = \int_{\partial D_{s_+}} \langle P\nabla\eta, \nu\rangle e^{-\zeta} d\partial D_{s_+} = C$$

for some constant C. Here, $\partial D_{s_\pm} = M \cap P_{s_\pm}$ and $\partial M_{s_-,s_+} = \partial D_{s_-} \cup \partial D_{s_+}$. Note that

$$\nu = \partial_\tau \Psi = \dot{r}\partial_r + \dot{s}\partial_s$$

and, since

$$\nabla\eta = X^\top = \partial_s^\top = X - \langle X, N\rangle N = X - \chi(r)\dot{r},$$

one gets

$$\begin{aligned}\langle P\nabla\eta, \nu\rangle &= \langle X^\top, (nHI - A)\partial_\tau\Psi\rangle = nH\langle X, \dot{s}\partial_s + \dot{r}\partial_r\rangle - \langle X, A\partial_\tau\Psi\rangle \\ &= nH\dot{s}\langle X, X\rangle - \kappa_r\langle X, \partial_\tau\Psi\rangle = nH\chi^2(r)\dot{s} - \kappa_r\chi^2(r)\dot{s} \\ &= (n-1)\kappa_\theta\chi^2(r)\dot{s}.\end{aligned}$$

But the zero scalar curvature hypersurface is rotacionally symmetric by hypothesis and $D_s \subset P_s$, therefore

$$d\partial D = \xi^{n-1}(r)d\theta,$$

where $d\theta$ is the standard volume element in $\mathbb{S}^{n-1}$. We conclude that

$$(n-1)\kappa_\theta \chi^2(r)\dot{s}e^{-\chi^{-2}}\xi^{n-1}(r)\,\mathrm{vol}(\mathbb{S}^{n-1}) = C, \tag{3.36}$$

where

$$\kappa_\theta = \frac{\xi'(r)}{\xi(r)}\chi(r)\dot{s}.$$

Hence we obtain the following first integral for the zero scalar curvature equation:

$$e^{-\chi^{-2}(r)}\chi^3(r)\xi^{n-2}(r)\xi'(r)\dot{s} = \frac{C}{(n-1)\,\mathrm{vol}(\mathbb{S}^{n-1})}. \tag{3.37}$$

We conclude from this expression that the solution is defined for $r = 0$ if and only if $C = 0$, what corresponds to the trivial solution $s =$ constant. Now consider $C \neq 0$. On the hand, $e^{-\chi^{-2}(r)}\chi^2(r)\xi^{n-2}(r)\xi'(r)$ is an increasing function of r and it is zero when $r = 0$. On the other hand, $\chi(r)\dot{s} = \sin\phi \leq 1$. Therefore, by continuity, a smooth solution is defined for each C in $r \geq r^* > 0$, where r_* is defined implictly by

$$e^{-\chi^{-2}(r_*)}\chi^2(r_*)\xi^{n-2}(r_*)\xi'(r_*) = \frac{C}{(n-1)\,\mathrm{vol}(\mathbb{S}^{n-1})},$$

and at $r = r_*$ one has $\chi(r_*)\dot{s} = 1$. This means that $\phi = \pm\pi/2$ at $r = r_*$. $\square$

4 ROTATIONALLY INVARIANT SOLITONS FOR $\alpha = 1/2$ AS GRAPHS

In this section, we consider $n \geq 3$ and a non-parallel Killing vector field X. We fix $\tau = r$, that is, we suppose that the rotationally invariant scalar curvature flow soliton Σ can be described as a graph of a function $s = u(r)$ defined over some region in $\mathbb{P}$. In this case one has

$$\dot{r} = 1 \quad \text{and} \quad \dot{s} = u'(r).$$

Hence, 3.24 can be rewritten in terms of these parameters as

$$(n-1)\frac{1}{\chi^2 W^2}\frac{\xi'(r)}{\xi(r)}u'(r)\big((\chi^2(r)u'(r))' + \chi'(r)\chi^3(r)u'^3(r)\big) + \frac{(n-1)(n-2)}{2}u'^2(r)\frac{\xi'^2(r)}{\xi^2(r)} = c^2.$$

Therefore

$$(n-1)\frac{1}{W^2}\frac{\xi'(r)}{\xi(r)}u'(r)\Big(\chi(r)\big(\chi(r)u'(r)\big)' + \chi'(r)\chi(r)u'(r)\big(1+\chi^2(r)u'^2(r)\big)\Big)$$
$$+\frac{(n-1)(n-2)}{2}\chi^2(r)u'^2(r)\frac{\xi'^2(r)}{\xi^2(r)} = c^2\chi^2(r).$$

Hence, one has

$$(n-1)\frac{1}{W^2}\frac{\xi'(r)}{\xi(r)}\chi(r)u'(r)\Big(\big(\chi(r)u'(r)\big)' + \frac{\chi'(r)}{\chi(r)}\chi(r)u'(r)\big(1+\chi^2(r)u'^2(r)\big)\Big)$$
$$+\frac{(n-1)(n-2)}{2}\chi^2(r)u'^2(r)\frac{\xi'^2(r)}{\xi^2(r)} = c^2\chi^2(r).$$

Denoting

$$v(r) = \big(\chi(r)u'(r)\big)^2, \tag{4.1}$$

one obtains

$$(n-1)\frac{1}{W^2}\frac{\xi'(r)}{\xi(r)}\Big(\frac{1}{2}v'(r) + \frac{\chi'(r)}{\chi(r)}v(r)\big(1+v(r)\big)\Big) + \frac{(n-1)(n-2)}{2}v(r)\frac{\xi'^2(r)}{\xi^2(r)} = c^2\chi^2(r).$$

Note that $v = W^2 - 1$. Therefore, expressing the equation above in terms of W yields

$$(n-1)\frac{\xi'(r)}{\xi(r)}\Big(\frac{W'(r)}{W(r)} + \frac{\chi'(r)}{\chi(r)}\big(W^2(r)-1\big)\Big) + \frac{(n-1)(n-2)}{2}\big(W^2(r)-1\big)\frac{\xi'^2(r)}{\xi^2(r)} = c^2\chi^2(r).$$

Rearranging terms and multiplying both sides by $1/W^2$ one gets

$$(n-1)\frac{\xi'(r)}{\xi(r)}\frac{W'(r)}{W^3(r)} - \left((n-1)\frac{\xi'(r)}{\xi(r)}\frac{\chi'(r)}{\chi(r)} + \frac{(n-1)(n-2)}{2}\frac{\xi'^2(r)}{\xi^2(r)} + c^2\chi^2(r)\right)\frac{1}{W^2}$$
$$+\left((n-1)\frac{\xi'(r)}{\xi(r)}\frac{\chi'(r)}{\chi(r)} + \frac{(n-1)(n-2)}{2}\frac{\xi'^2(r)}{\xi^2(r)}\right) = 0. \tag{4.2}$$

In terms of the scalar curvature $S(r)$ of geodesic cylinders this equations can be rewritten as

$$(n-1)\frac{\xi'(r)}{\xi(r)}\frac{W'(r)}{W^3(r)} - \big(S(r)+c^2\chi^2(r)\big)\frac{1}{W^2} + S(r) = 0. \tag{4.3}$$

Define

$$g(r) \doteq \frac{1}{W^2}. \tag{4.4}$$

Multiplying both sides equation (4.2) by $\xi(r)/\xi'(r)$, it becomes

$$(n-1)g'(r) + 2\big(S(r)+c^2\chi^2(r)\big)\frac{\xi(r)}{\xi'(r)}g(r) = 2S(r)\frac{\xi(r)}{\xi'(r)}. \tag{4.5}$$

An integrating factor $\mu(r)$ for this first-order ODE equation satisfies

$$(n-1)\mu'(r) = 2\big(S(r)+c^2\chi^2(r)\big)\frac{\xi(r)}{\xi'(r)}\mu.$$

Hence, we have

$$\frac{\mu'(r)}{\mu(r)} - 2\frac{\chi'(r)}{\chi(r)} - (n-2)\frac{\xi'(r)}{\xi(r)} = \frac{2c^2}{n-1}\chi^2(r)\frac{\xi(r)}{\xi'(r)}.$$

Therefore, we may choose the integrating factor

$$\mu(r) = \chi^2(r)\xi^{n-2}(r)\exp\left(\frac{2c^2}{n-1}\int_0^r \chi^2(\rho)\frac{\xi(\rho)}{\xi'(\rho)}d\rho\right).$$

Denote

$$G(r) = \exp\left(\frac{2c^2}{n-1}\int_0^r \chi^2(\rho)\frac{\xi(\rho)}{\xi'(\rho)}d\rho\right). \tag{4.6}$$

Hence,

$$(n-1)\mu(r)g(r) = C_0 + \int_0^r 2S(\rho)\frac{\xi(\rho)}{\xi'(\rho)}\mu(\rho)\,d\rho. \tag{4.7}$$

Therefore

$$g(r) = \frac{1}{n-1}\left(\frac{C_0}{\mu(r)} + \frac{1}{\mu(r)}\int_0^r 2S(\rho)\frac{\xi(\rho)}{\xi'(\rho)}\mu(\rho)\,d\rho\right). \tag{4.8}$$

We conclude that

$$u'^2(r) = \frac{n-1}{\chi^2(r)}\left(\frac{C_0}{\mu(r)} + \frac{1}{\mu(r)}\int_0^r 2S(\rho)\frac{\xi(\rho)}{\xi'(\rho)}\mu(\rho)\,d\rho\right)^{-1} - \frac{1}{\chi^2(r)}. \tag{4.9}$$

Since

$$2S(\rho)\frac{\xi(\rho)}{\xi'(\rho)}\mu(\rho) = (n-1)\left(2\frac{\chi'(r)}{\chi(r)} + (n-2)\frac{\xi'(r)}{\xi(r)}\right)\chi^2(r)\xi^{n-2}(r)G(r)$$
$$= (n-1)\big(\chi^2\xi^{n-2}\big)'G(r),$$

we can rewrite (4.9) as

$$u'^2(r) = \xi^{n-2}(r)G(r)\left(\frac{C_0}{n-1} + \int_0^r (\chi^2(\rho)\xi^{n-2}(\rho))'G(\rho)\,d\rho\right)^{-1} - \chi^{-2}(r). \tag{4.10}$$

It follows that

$$u'(r) = \pm\sqrt{\frac{\xi^{n-2}(r)G(r)}{\hat{C}_0 + \int_0^r (\chi^2(\rho)\xi^{n-2}(\rho))'G(\rho)\,d\rho} - \frac{1}{\chi^2(r)}}, \tag{4.11}$$

where $\hat{C}_0 = C_0/(n-1)$ is a constant of integration. Integrating (4.11),

$$u(r) = \pm\int_{r_*}^{r}\sqrt{\frac{\xi^{n-2}G_n(\lambda)}{\hat{C}_0 + \int_0^\lambda (\chi^2(\rho)\xi^{n-2}(\rho))'G_n(\rho)\,d\rho} - \frac{1}{\chi^2(\lambda)}}\,d\lambda, \tag{4.12}$$

where $r \geq r_*$ and $r_* \geq 0$ depends on C_0.

In what follows, we consider the positive square root in the expression of u' and u. We could take negative sign in (4.11) and (4.12) as well. However, this is equivalent to reflect $\psi(M)$ with respect to $\mathbb{P}_0 = \mathbb{P} \times \{0\}$. We will see below that for $C_0 < 0$ the solution is defined for $r \geq r_*$ for given $r_* > 0$ depending on C_0. In this case we have

$$\lim_{r\to r_*} u'(r) = +\infty$$

and the reflection with respect to $\mathbb{P}_0$ produces (at least C^1) complete solitons to which we refer to as *translating catenoids*. For $C_0 > 0$, we will prove that the solution is defined for $r \geq r_*$ for some $r_* > 0$ that depends on C_0 and

$$\lim_{r\to r_*} u'(r) = 0.$$

In this case, the rotationally invariant hypersurface in singular along its intersection with $\mathbb{P}_0$, that is, along the geodesic sphere of $\mathbb{P}_0$ centered at o with radius r_*.

Observe that if we take $C_0 = 0$ in (4.12) we have (4.14). These hypersurfaces are referred as *bowl solitons*.

Now we summarize the discussion above and state some existence results for scalar curvature flow soliton given by rotationally invariant graphs in $\bar{M} = \mathbb{P} \times_\chi I$ for $n = \dim\mathbb{P} \geq 3$ under the assumptions that $\mathbb{P}$ is a Hadamard manifold with rotationally invariant metric and that $\chi = \chi(r)$ is a non-constant radial function under the structural assumptions stated in the Introduction.

Theorem 5. *The one-parameter family of rotationally invariant scalar curvature flow solitons $\mathscr{C}_{n,\alpha,C_0}$ in $\bar{M} = \mathbb{P} \times_\chi I$ are described as follows: given*

$$G(r) = \exp\left(\frac{2c^2}{n-1}\int^r \chi^2(\rho)\frac{\xi(\rho)}{\xi'(\rho)}\,d\rho\right), \tag{4.13}$$

the soliton $\mathscr{C}_{n,\alpha,C_0}$ is the graph of the rotationally invariant function

$$u(r)=\int_{r_*}^{r}\sqrt{\frac{\xi^{n-2}G(\lambda)}{C_0+\int_0^{\lambda}(\chi^2(\rho)\xi^{n-2}(\rho))'G(\rho)\,d\rho}-\frac{1}{\chi^2(\lambda)}}\,d\lambda, \tag{4.14}$$

for $n\geq 3$. The integration limit r_* depends on C_0. For instance $r_*=0$ if $C_0=0$.

In the subsequent sections, we provide further information on the behavior of these solitons near $r=r_*$ as well as on their asymptotic regimen.

4.1 General behavior near the origin

We expect that the behavior of the solitons near $r=0$ are similar for different metrics, since they are locally Euclidean, but the behavior at infinity may be different, as we have seen in the proof of the existence results. Note that

$$r\to\chi^2(r)\frac{\xi(r)}{\xi'(r)}$$

is integrable near $r=0$ since $\chi(0)=1$ and $\frac{\xi(r)}{\xi'(r)}\sim r\to 0$ as $r\to 0^+$. Hence, $G(r)\to 0$ and $\mu(r)\to 0$ as $r\to 0^+$. In the same way one has

$$G'(r)=\frac{2c^2}{n-1}G(r)\chi^2(r)\frac{\xi(r)}{\xi'(r)}\to 0$$

as $r\to 0^+$ and

$$\mu'(r)=(\chi^2(r)\xi^{n-2}(r))'G(r)+\chi^2(r)\xi^{n-2}(r)G'(r)\to 0$$

as $r\to 0^+$. On the other hand we have

$$\begin{aligned}2S(r)\frac{\xi(r)}{\xi'(r)}\mu(r)&=\left(2(n-1)\frac{\chi'(r)}{\chi(r)}+(n-1)(n-2)\frac{\xi'(r)}{\xi(r)}\right)\chi^2(r)\xi^{n-2}(r)G(r)\\&=(n-1)\big(\chi^{2'}(r)\xi^{n-2}(r)+\chi^2(r)\xi^{n-2\prime}(r)\big)G(r)=(n-1)\big(\chi^2(r)\xi^{n-2}(r)\big)'G(r)\to 0\end{aligned}$$

as $r\to 0^+$. Hence, taking $r_*=0$ in the integral term in (4.8), one has by L'Hôpital's rule

$$\begin{aligned}&\lim_{r\to 0^+}\frac{1}{\mu(r)}\int_0^r 2S(\rho)\frac{\xi(\rho)}{\xi'(\rho)}\mu(\rho)\,d\rho\\&=\lim_{r\to 0^+}\frac{\big(2(n-1)\frac{\chi'(r)}{\chi(r)}+(n-1)(n-2)\frac{\xi'(r)}{\xi(r)}\big)\chi^2(r)\xi^{n-2}(r)G(r)}{\chi^2(r)\xi^{n-2}(r)G'(r)+(\chi^2(r)\xi^{n-2}(r))'G(r)}\\&=\lim_{r\to 0^+}\frac{(n-1)(\chi^2(r)\xi^{n-2}(r))'G(r)}{\frac{2c^2}{n-1}\chi^2(r)\frac{\xi(r)}{\xi'(r)}\chi^2(r)\xi^{n-2}(r)G(r)+(\chi^2(r)\xi^{n-2}(r))'G(r)}\\&=\lim_{r\to 0^+}\frac{n-1}{\frac{2c^2}{n-1}\chi^2(r)\frac{\xi(r)}{\xi'(r)}\frac{\chi^2(r)\xi^{n-2}(r)}{(\chi^2(r)\xi^{n-2}(r))'}+1}=n-1,\end{aligned}$$

since $\chi'(r)/\chi(r) \to 0$ and $\xi'(r)/\xi(r) \to +\infty$ and therefore

$$\frac{\chi^2(r)\xi^{n-2}(r)}{(\chi^2(r)\xi^{n-2}(r))'} = \frac{1}{2\frac{\chi'(r)}{\chi(r)} + (n-2)\frac{\xi'(r)}{\xi(r)}} \to 0$$

as $r \to 0^+$. From these computations we conclude that $g(r)$ is well-defined near $r = 0$ if we fix $C_0 = 0$. Hence, for this particular case one has

$$\frac{1}{1 + (\chi^2(r)u'^2(r))} = \frac{1}{W^2} = \frac{1}{(n-1)\mu(r)} \int_0^r 2S(\rho)\frac{\xi(\rho)}{\xi'(\rho)}\mu(\rho)\,d\rho \tag{4.15}$$

Note that $W \to 1$ as $r \to 0^+$, that is, $u'(r) \to 0$ as $r \to 0^+$, what implies that the rotationally invariant hypersurface described by the graph of u hits the rotation axis $r = 0$ orthogonally. Therefore, this hypersurface is smooth.

We now consider the case $C_0 < 0$. It follows from the previous analysis that the solution u could not be smoothly extended to $r = 0$ in this case. Hence, we fix $r_* = r_*^- > 0$, where r_*^- is defined implicitly by

$$C_0 = -\int_0^{r_*^-} 2S(\rho)\frac{\xi(\rho)}{\xi'(\rho)}\mu(\rho)\,d\rho < 0.$$

Note that with this choice

$$\lim_{r \to r_*} g(r) = 0,$$

what implies that $u'(r) \to \pm\infty$ as $r \to r_*$. Hence, in this case the rotationally invariant hypersurface hits the leaf $\mathbb{P}_{s_0} = \mathbb{P} \times \{s_0\}$, with $s_0 = u(r_*)$, orthogonally. Later on, we will discuss how to obtain a complete (only C^1 in principle) hypersurface reflecting the graph of u with respect to $\mathbb{P}_{s_0}$.

Now, we discuss the case when $C_0 > 0$. In this case, we fix $r_* = r_*^+ > 0$, defined implicitly by

$$C_0 + \int_0^{r_*^+} 2S(\rho)\frac{\xi(\rho)}{\xi'(\rho)}\mu(\rho)d\rho = (n-1)\mu(r_*^+)$$

or

$$\begin{aligned} C_0 &= (n-1)\chi^2(r_*^+)\xi^{n-2}(r_*^+)G(r_*^+) - (n-1)\int_0^{r_*^+} \left(\chi^2(\rho)\xi^{n-2}(\rho)\right)'G(\rho)\,d\rho \\ &= (n-1)\chi^2(r_*^+)\xi^{n-2}(r_*^+)G(r_*^+) - (n-1)\chi^2(r_*^+)\xi^{n-2}(r)G(r)\Big|_0^{r_*^+} \\ &\quad + 2c^2\chi^2(r_*^+)\xi^{n-2}(r_*^+)\chi^2(r)\frac{\xi(r_*^+)}{\xi'(r_*^+)}G(r_*^+) \\ &= 2c^2\chi^2(r_*^+)\xi^{n-2}(r_*^+)\chi^2(r_*^+)\frac{\xi(r_*^+)}{\xi'(r_*^+)}G(r_*^+) > 0. \end{aligned}$$

For this choice of C_0 we have

$$g(r) = \frac{1}{(n-1)\mu(r)}\left((n-1)\mu(r_*) + \int_{r_*}^{r} 2S(\rho)\frac{\xi(\rho)}{\xi'(\rho)}\mu(\rho)\,d\rho\right). \tag{4.16}$$

Note that in this case $u'(r) \to 0$ as $r \to r_*$. Then it meets $\mathbb{P}_{s_0}$ tangencially. Hence, the reflection with respect to $\mathbb{P}_{s_0}$ produces a singular solution along the geodesic sphere in $\mathbb{P}_{s_0}$ with radius r_*.

In sum, we have obtained two examples of rotationally invariant scalar curvature flow solitons given up to integration by

$$g(r) = \frac{1}{(n-1)\mu(r)}\int_{r_*}^{r} 2S(\rho)\frac{\xi(\rho)}{\xi'(\rho)}\mu(\rho)\,d\rho \tag{4.17}$$

$$= \frac{1}{\chi^2\xi^{n-2}G(r)}\int_{r_*}^{r}(\chi^2(\rho)\xi^{n-2}(\rho))'G(\rho)\,d\rho$$

$$= 1 + \frac{\chi^2(r_*)\xi^{n-2}(r_*)G(r_*)}{\chi^2(r)\xi^{n-2}(r)G(r)} - \frac{2c^2/(n-1)}{\chi^2(r)\xi^{n-2}(r)G(r)}\int_{r_*}^{r}\chi^4(\rho)\frac{\xi^{n-1}(\rho)}{\xi'(\rho)}G(\rho)\,d\rho \tag{4.18}$$

where

$$r_* = \begin{cases} 0, \text{ if } C_0 = 0, \\ r_*^-, \text{ if } C_0 < 0, \end{cases} \tag{4.19}$$

and

$$g(r) = \frac{\mu(r_*)}{\mu(r)} + \frac{1}{(n-1)\mu(r)}\int_{r_*}^{r} 2S(\rho)\frac{\xi(\rho)}{\xi'(\rho)}\mu(\rho)\,d\rho. \tag{4.20}$$

where $r_* = r_*^+$, if $C_0 > 0$.

Let us give more precise estimates for the behavior of u near $r = r_*$ for solitons with $C_0 = 0$, $C_0 < 0$ and $C_0 > 0$. Fix $C_0 = 0$. By (4.17) we expand in Taylor, knowing that

$\xi^{(2k)}(0) = 0$ and $\chi^{(2k+1)}(0) = 0$,

$$
\begin{aligned}
a(r) &\doteq \int_{r_*}^{r} \chi^4(\rho)\frac{\xi^{n-1}(\rho)}{\xi'(\rho)}G(\rho)\,d\rho = \sum_{j=0}^{\infty} \frac{d^j a(\rho)}{d\rho^j}\bigg|_{\rho=0}\frac{r^j}{j!} \\
&= \frac{r^n}{n!}\frac{(n-1)!\xi'(0)^{n-1}\chi(0)^4}{\xi'(0)} + \frac{r^{n+1}}{(n+1)!}\frac{d^n}{d\rho^n}\left[\frac{\chi^4(\rho)\xi^{n-1}(\rho)}{\xi'(\rho)}G(\rho)\right]_{\rho=0} \\
&+ \frac{r^{n+2}}{(n+2)!}\frac{d^{n+1}}{d\rho^{n+1}}\left[\frac{\chi^4(\rho)\xi^{n-1}(\rho)}{\xi'(\rho)}G(\rho)\right]_{\rho=0} + O(r^{n+3}) \\
&= \frac{r^n}{n} + \frac{r^{n+1}}{(n+1)!}\sum_{j=0}^{n}\binom{n}{j}\left[\frac{d^{n-j}}{d\rho^{n-j}}\xi^{n-1}(\rho)\right]_{\rho=0}\frac{d^j}{d\rho^j}\left[\frac{\chi^4(\rho)}{\xi'(\rho)}G(\rho)\right]_{\rho=0} \\
&+ \frac{r^{n+2}}{(n+2)!}\sum_{j=0}^{n+1}\binom{n+1}{j}\left[\frac{d^{n+1-j}}{d\rho^{n+1-j}}\xi^{n-1}(\rho)\right]_{\rho=0}\frac{d^j}{d\rho^j}\left[\frac{\chi^4(\rho)}{\xi'(\rho)}G(\rho)\right]_{\rho=0} + O(r^{n+3}) \\
&= \frac{r^n}{n} + \frac{r^{n+1}}{(n+1)!}\left[\underbrace{\left(\frac{d^n}{d\rho^n}\xi^{n-1}(\rho)\right)_{\rho=0}}_{=0} + n\left(\frac{d^{n-1}}{d\rho^{n-1}}\xi^{n-1}(\rho)\right)_{\rho=0}\underbrace{\frac{d}{d\rho}\left(\frac{\chi^4(\rho)}{\xi'(\rho)}G(\rho)\right)_{\rho=0}}_{=0}\right] \\
&+ \frac{r^{n+2}}{(n+2)!}\Bigg[\left(\frac{d^{n+1}}{d\rho^{n+1}}\xi^{n-1}(\rho)\right)_{\rho=0} + (n+1)\underbrace{\left(\frac{d^n}{d\rho^n}\xi^{n-1}(\rho)\right)_{\rho=0}}_{=0}\frac{d}{d\rho}\left(\frac{\chi^4(\rho)}{\xi'(\rho)}G(\rho)\right)_{\rho=0} \\
&\qquad + \binom{n+1}{2}\left(\frac{d^{n-1}}{d\rho^{n-1}}\xi^{n-1}(\rho)\right)_{\rho=0}\frac{d^2}{d\rho^2}\left(\frac{\chi^4(\rho)}{\xi'(\rho)}G(\rho)\right)_{\rho=0}\Bigg] + O(r^{n+3}) \\
&= \frac{r^n}{n} + \frac{r^{n+2}}{(n+2)!}\Bigg[\frac{n-1}{6}(n+1)![\xi'(0)]^{n-1}\xi'''(0) \\
&\qquad + \frac{(n+1)!}{2}[\xi'(0)]^{n-1}\left(4\chi''(0) + \frac{2c^2}{n-1} - \xi'''(0)\right)\Bigg] + O(r^{n+3}) \\
&= \frac{r^n}{n} + \frac{r^{n+2}}{(n+2)}\left[\frac{n-4}{6}\xi'''(0) + 2\chi''(0) + \frac{c^2}{n-1}\right] + O(r^{n+3}),
\end{aligned}
$$

where

$$
\left(\frac{d^{n+1}}{d\rho^{n-1}}\xi^{n-1}\right)_{\rho=0} = \sum_{k=1}^{n-1} k(n-k)(n-1)![\xi'(0)]^{n-1}\xi'''(0) = \frac{n-1}{6}(n+1)!\xi'''(0).
$$

Now we expand in Taylor

$$
\begin{aligned}
b(r) &\doteq \chi^2(r)\xi^{n-2}(r)G(r) = \sum_{j=0}^{\infty} \frac{d^j b(\rho)}{d\rho^j}\bigg|_{\rho=0} \frac{r^j}{j!} \\
&= \frac{r^{n-2}}{(n-2)!}(n-2)![\xi'(0)]^{n-2}\chi^2(0)G(0) \\
&+ \frac{r^{n-1}}{(n-1)!}\sum_{j=0}^{n-1}\binom{n-1}{j}\left[\frac{d^{n-1-j}}{d\rho^{n-1-j}}\xi^{n-2}(\rho)\right]_{\rho=0}\left[\frac{d^j}{d\rho^j}(\chi^2(\rho)G(\rho))\right]_{\rho=0} \\
&+ \frac{r^n}{n!}\sum_{j=0}^{n}\binom{n}{j}\left[\frac{d^{n-j}}{d\rho^{n-j}}\xi^{n-2}(\rho)\right]_{\rho=0}\left[\frac{d^j}{d\rho^j}(\chi^2(\rho)G(\rho))\right]_{\rho=0} + O(r^{n+1}) \\
&= r^{n-2} + \frac{r^n}{n!}\left[\left(\frac{d^n}{d\rho^n}\xi^{n-2}(\rho)\right)_{\rho=0} + \binom{n}{2}\left(\frac{d^{n-2}}{d\rho^{n-2}}\xi^{n-2}(\rho)\right)_{\rho=0}\left(\frac{d^2}{d\rho^2}(\chi^2(\rho)G(\rho))\right)_{\rho=0}\right] + O(r^{n+1}) \\
&= r^{n-2} + r^n\left(\frac{n-2}{6}\xi'''(0) + \chi''(0) + \frac{c^2}{n-1}\right) + O(r^{n+1}).
\end{aligned}
$$

Dividing both expressions gives us

$$
\frac{a(r)}{b(r)} = \frac{r^2}{n} + \left[\frac{n^2-4n+2}{3n(n-2)}\xi'''(0) + \frac{3n-2}{n(n-2)}\chi''(0) + \frac{2c^2}{n(n-2)}\right]r^4 + O(r^5).
$$

Therefore,

$$
\begin{aligned}
g(r) &= 1 - \frac{2c^2}{n-1}\frac{a(r)}{b(r)} \\
&= 1 - \frac{2c^2}{n(n-1)}r^2 - \frac{2c^2}{n(n-1)(n-2)}\left[\frac{n^2-4n+2}{3}\xi'''(0) + (3n-2)\chi''(0) + 2c^2\right]r^4 + O(r^5).
\end{aligned}
$$

Now we write the equation in terms of $(\chi u')^2$, giving

$$
\begin{aligned}
(\chi u')^2 &= \frac{2c^2}{n(n-1)}\left[1 + \left(\frac{2c^2(n^2-2)}{n(n-1)(n-2)} + \frac{1}{n-2}\left(\frac{n^2-4n+2}{3}\xi'''(0) + (3n-2)\chi''(0)\right)\right)r^2\right]r^2 \\
&\quad + O(r^5). \Rightarrow \\
\chi(r)u'(r) &= \sqrt{\frac{2c^2}{n(n-1)}}r \\
&\quad \times \sqrt{1 + \left(\frac{2c^2(n^2-2)}{n(n-1)(n-2)} + \frac{1}{n-2}\left(\frac{n^2-4n+2}{3}\xi'''(0) + (3n-2)\chi''(0)\right)\right)r^2 + O(r^3)} \\
&= \sqrt{\frac{2c^2}{n(n-1)}}r\left[1 + \frac{1}{n-2}\left(\frac{c^2(n^2-2)}{n(n-1)} + \frac{n^2-4n+2}{6}\xi'''(0) + \frac{(3n-2)}{2}\chi''(0)\right)r^2 + O(r^3)\right] \\
&= \sqrt{\frac{2c^2}{n(n-1)}}r + \frac{1}{n-2}\left(\frac{c^2(n^2-2)}{n(n-1)} + \frac{n^2-4n+2}{6}\xi'''(0) + \frac{(3n-2)}{2}\chi''(0)\right)\sqrt{\frac{2c^2}{n(n-1)}}r^3 + O(r^4).
\end{aligned}
$$

By

$$
\chi(r) = 1 + \frac{\chi''(0)}{2}r^2 + O(r^4) \Rightarrow \frac{1}{\chi(r)} = 1 - \frac{\chi''(0)}{2}r^2 + O(r^4),
$$

we have

$$u'(r)=\sqrt{\frac{2c^2}{n(n-1)}}r+\frac{1}{n-2}\left(\frac{c^2(n^2-2)}{n(n-1)}+\frac{n^2-4n+2}{6}\xi'''(0)+\frac{3}{2}(n-1)\chi''(0)\right)\sqrt{\frac{2c^2}{n(n-1)}}r^3$$
$$+O(r^4).$$

Integrating,

$$u(r)=\sqrt{\frac{c^2}{2n(n-1)}}r^2+\frac{1}{4(n-2)}\left(\frac{c^2(n^2-2)}{n(n-1)}+\frac{n^2-4n+2}{6}\xi'''(0)+\frac{3}{2}(n-1)\chi''(0)\right)\sqrt{\frac{2c^2}{n(n-1)}}r^4$$
$$+O(r^5).$$

We begin applying (2.2) to a normal variation of geodesic cylinders in $\bar{M}$ for which $f=1$ and $T=0$. Hence the variation of the mean curvature of geodesic spheres in $\mathbb{P}$ obeys the Ricatti equation

$$-(n-1)\Big(\frac{\xi'(r)}{\xi(r)}\Big)'=(n-1)\frac{\xi'^2(r)}{\xi^2(r)}+\underbrace{\mathrm{Ric}_P(\partial_r,\partial_r)}_{=(n-1)K_P}$$

. Hence,

$$\Big(\frac{\xi(r)}{\xi'(r)}\Big)'=\frac{\xi^2(r)}{\xi'^2(r)}\Big(\frac{\xi'^2(r)}{\xi^2(r)}+\frac{1}{n-1}\mathrm{Ric}_P(\partial_r,\partial_r)\Big)=1+\frac{1}{n-1}\frac{\xi^2(r)}{\xi'^2(r)}\mathrm{Ric}_P(\partial_r,\partial_r).$$

Now the normal evolution of the mean curvature of geodesic cylinders is given by

$$-\Big(\frac{\chi'}{\chi}+(n-1)\frac{\xi'}{\xi}\Big)'=\Big(\frac{\chi'^2}{\chi^2}+(n-1)\frac{\xi'^2}{\xi^2}\Big)+\mathrm{Ric}_{\bar{M}}(\partial_r,\partial_r).$$

Therefore

$$-\Big(\frac{\chi'}{\chi}\Big)'=\frac{\chi'^2}{\chi^2}+\mathrm{Ric}_{\bar{M}}(\partial_r,\partial_r)-(n-1)\Big(\frac{\xi'^2(r)}{\xi^2(r)}+\frac{1}{n-1}\mathrm{Ric}_P(\partial_r,\partial_r)\Big)+(n-1)\frac{\xi'^2}{\xi^2}$$
$$=\frac{\chi'^2}{\chi^2}+\mathrm{Ric}_{\bar{M}}(\partial_r,\partial_r)-\mathrm{Ric}_P(\partial_r,\partial_r)=\frac{\chi'^2}{\chi^2}+\chi^{-2}\bar{R}(X,\partial_r)\partial_r,X\rangle\doteq\frac{\chi'^2}{\chi^2}+K^\perp,$$

where we use the fact that $P\subset\bar{M}$ is totally geodesic and We conclude establishing the Jacobi equation

$$\frac{\chi''(r)}{\chi(r)}=-K^\perp. \tag{4.21}$$

Using these expressions one obtains

$$\frac{(\chi^2(r)\xi^{n-2}(r))'}{\chi^2(r)\xi^{n-2}(r)}=2\frac{\chi'(r)}{\chi(r)}+(n-2)\frac{\xi'(r)}{\xi(r)}=2\left(\frac{\chi'(r_*)}{\chi(r_*)}-\Big(\frac{\chi'^2(r_*)}{\chi^2(r_*)}+K^\perp\Big)(r-r_*)\right)$$
$$+(n-2)\left(\frac{\xi'(r_*)}{\xi(r_*)}-\Big(\frac{\xi'^2(r_*)}{\xi^2(r_*)}-K_P\Big)(r-r_*)\right)+O\big((r-r_*)^2\big)$$
$$=2\frac{\chi'(r_*)}{\chi(r_*)}+(n-2)\frac{\xi'(r_*)}{\xi(r_*)}-\left(2\frac{\chi'^2(r_*)}{\chi^2(r_*)}+(n-2)\frac{\xi'^2(r_*)}{\xi^2(r_*)}+2K^\perp-(n-2)K_P\right)(r-r_*)$$
$$+O\big((r-r_*)^2\big).$$

Moreover for $r_* > 0$,

$$\frac{G'(r)}{G(r)} = \frac{2c^2}{n-1}\chi^2(r)\frac{\xi(r)}{\xi'(r)} = \frac{2c^2}{n-1}\chi^2(r_*)\frac{\xi(r_*)}{\xi'(r_*)}\left(1+2\frac{\chi'(r_*)}{\chi(r_*)}+\frac{\xi'(r_*)}{\xi(r_*)}\Big(1+\frac{\xi^2(r_*)}{\xi'^2(r_*)}K_P\Big)(r-r_*)\right) + O\big((r-r_*)^2\big)$$

Finally, one has for $r_* > 0$

$$\begin{aligned}\frac{\mu'(r)}{\mu(r)} &= 2\frac{\chi'(r)}{\chi(r)} + (n-2)\frac{\xi'(r)}{\xi(r)} + \frac{G'(r)}{G(r)}\\ &= 2\frac{\chi'(r_*)}{\chi(r_*)} + (n-2)\frac{\xi'(r_*)}{\xi(r_*)} - \left(2\frac{\chi'^2(r_*)}{\chi^2(r_*)} + (n-2)\frac{\xi'^2(r_*)}{\xi^2(r_*)} + 2K^\perp - (n-2)K_P\right)(r-r_*)\\ &+ \frac{2c^2}{n-1}\chi^2(r_*)\frac{\xi(r_*)}{\xi'(r_*)}\left(1+2\frac{\chi'(r_*)}{\chi(r_*)}+\frac{\xi'(r_*)}{\xi(r_*)}\Big(1+\frac{\xi^2(r_*)}{\xi'^2(r_*)}K_P\Big)(r-r_*)\right) + O\big((r-r_*)^2\big).\end{aligned}$$

Moreover

$$\begin{aligned}2S(r)\frac{\xi(r)}{\xi'(r)}\mu(r) &= (n-1)\mu(r_*)\left(2\frac{\chi'(r)}{\chi(r)} + (n-2)\frac{\xi'(r)}{\xi(r)}\right)\left(1+\Big(2\frac{\chi'(r_*)}{\chi(r_*)} + (n-2)\frac{\xi'(r_*)}{\xi(r_*)}\Big)(r-r_*)\right.\\ &-\Big(2\frac{\chi'^2(r_*)}{\chi^2(r_*)} + (n-2)\frac{\xi'^2(r_*)}{\xi^2(r_*)} + 2K^\perp - (n-2)K_P\Big)(r-r_*)^2\\ &\left.+\frac{2c^2}{n-1}\chi^2(r_*)\frac{\xi(r_*)}{\xi'(r_*)}\Big(\Big(1+2\frac{\chi'(r_*)}{\chi(r_*)}\Big)(r-r_*)+\frac{\xi'(r_*)}{\xi(r_*)}\Big(1+\frac{1}{n-1}\frac{\xi^2(r_*)}{\xi'^2(r_*)}K_P\Big)(r-r_*)^2\Big)+O\big((r-r_*)^3\big)\right)\end{aligned}$$

with

$$\begin{aligned}2\frac{\chi'(r)}{\chi(r)} + (n-2)\frac{\xi'(r)}{\xi(r)} =& 2\frac{\chi'(r_*)}{\chi(r_*)} + (n-2)\frac{\xi'(r_*)}{\xi(r_*)}\\ &-\left(2\frac{\chi'^2(r_*)}{\chi^2(r_*)} + (n-2)\frac{\xi'^2(r_*)}{\xi^2(r_*)} + 2K^\perp - (n-2)K_P\right)(r-r_*) + O\big((r-r_*)^2\big).\end{aligned}$$

Hence

$$\begin{aligned}2S(r)\frac{\xi(r)}{\xi'(r)}\mu(r) &= (n-1)\mu(r_*)\left(2\frac{\chi'(r_*)}{\chi(r_*)} + (n-2)\frac{\xi'(r_*)}{\xi(r_*)}\right)\left(1+\Big(2\frac{\chi'(r_*)}{\chi(r_*)} + (n-2)\frac{\xi'(r_*)}{\xi(r_*)}\Big)(r-r_*)\right.\\ &\left.+\frac{2c^2}{n-1}\chi^2(r_*)\frac{\xi(r_*)}{\xi'(r_*)}\Big(1+2\frac{\chi'(r_*)}{\chi(r_*)}\Big)(r-r_*)+O\big((r-r_*)^2\big)\right)\\ &-(n-1)\mu(r_*)\left(2\frac{\chi'^2(r_*)}{\chi^2(r_*)} + (n-2)\frac{\xi'^2(r_*)}{\xi^2(r_*)} + 2K^\perp - (n-2)K_P\right)\times\\ &\times\left((r-r_*)+\Big(2\frac{\chi'(r_*)}{\chi(r_*)} + (n-2)\frac{\xi'(r_*)}{\xi(r_*)}\Big)(r-r_*)^2+\frac{2c^2}{n-1}\chi^2(r_*)\frac{\xi(r_*)}{\xi'(r_*)}\Big(1+2\frac{\chi'(r_*)}{\chi(r_*)}\Big)(r-r_*)^2\right).\end{aligned}$$

and

$$\begin{aligned}\frac{\mu(r_*)}{\mu(r)} &= 1-\Big(2\frac{\chi'(r_*)}{\chi(r_*)} + (n-2)\frac{\xi'(r_*)}{\xi(r_*)}\Big)(r-r_*)-\frac{2c^2}{n-1}\chi^2(r_*)\frac{\xi(r_*)}{\xi'(r_*)}\Big(1+2\frac{\chi'(r_*)}{\chi(r_*)}\Big)(r-r_*)\\ &\quad+O\big((r-r_*)^2\big)\\ &= 1-\left[s(r_*)+\frac{2c^2}{n-1}\chi^2(r_*)\frac{\xi(r_*)}{\xi'(r_*)}\left(1+2\frac{\chi'(r_*)}{\chi(r_*)}\right)\right](r-r_*)+O(r-r_*)^2\end{aligned}$$

One concludes that

$$\frac{1}{(n-1)\mu(r)}\int_{r_*}^{r} 2S(\rho)\frac{\xi(\rho)}{\xi'(\rho)}\mu(\rho)\,d\rho = \left(2\frac{\chi'(r_*)}{\chi(r_*)}+(n-2)\frac{\xi'(r_*)}{\xi(r_*)}\right)\Bigg((r-r_*)$$
$$-\frac{1}{2}\Big(2\frac{\chi'(r_*)}{\chi(r_*)}+(n-2)\frac{\xi'(r_*)}{\xi(r_*)}\Big)(r-r_*)^2-\frac{c^2}{n-1}\chi^2(r_*)\frac{\xi(r_*)}{\xi'(r_*)}\Big(1+2\frac{\chi'(r_*)}{\chi(r_*)}\Big)(r-r_*)^2+O\big((r-r_*)^3\big)\Bigg)$$
$$-\left(2\frac{\chi'^2(r_*)}{\chi^2(r_*)}+(n-2)\frac{\xi'^2(r_*)}{\xi^2(r_*)}+2K^{\perp}-(n-2)K_P\right)\left(\frac{1}{2}(r-r_*)^2+O\big((r-r_*)^3\big)\right)$$

Therefore for $C_0<0$ and $r_*=r_->0$ we have

$$g(r)=s(r_-)(r-r_-)-\frac{1}{2}\Bigg(s^2(r_-)+\frac{2c^2}{n-1}s(r_-)\chi^2(r_-)\frac{\xi(r_-)}{\xi'(r_-)}\Big(1+2\frac{\chi'(r_-)}{\chi(r_-)}\Big)+ \tag{4.22}$$
$$s(r_-)\Big(2\frac{\chi'^2(r_-)}{\chi^2(r_-)}+(n-2)\frac{\xi'^2(r_-)}{\xi^2(r_-)}+2K^{\perp}-(n-2)K_P\Big)\Bigg)(r-r_-)^2+O\big((r-r_-)^3\big). \tag{4.23}$$

where

$$s(r)=2\frac{\chi'(r)}{\chi(r)}+(n-2)\frac{\xi'(r)}{\xi(r)}=\frac{(\chi^2(r)\xi^{n-2}(r))'}{\chi^2(r)\xi^{n-2}(r)}. \tag{4.24}$$

For $C_0>0$,

$$g(r)=\frac{\mu(r_*)}{\mu(r)}+\frac{1}{(n-1)\mu(r)}\int_{r_*}^{r} 2S(\rho)\frac{\xi(\rho)}{\xi'(\rho)}\mu(\rho)\,d\rho,$$

and

$$\left(\frac{\mu(r_*)}{\mu(r)}\right)''\Bigg|_{r=r_*}=2\left(\frac{\mu'(r_*)}{\mu(r_*)}\right)^2-\frac{\mu''(r_*)}{\mu(r_*)}=\left(\frac{\mu'(r_*)}{\mu(r_*)}\right)^2-\left(\frac{\mu'(r_*)}{\mu(r_*)}\right)'$$
$$=\left(s(r_*)+\frac{2c^2}{n-1}\chi^2(r_*)\frac{\xi(r_*)}{\xi'(r_*)}\right)^2-s'(r_*)-\frac{2c^2}{n-1}\left[2\chi(r_*)\chi'(r_*)\frac{\xi(r_*)}{\xi'(r_*)}+\chi^2(r_*)\left(\frac{\xi(r_*)}{\xi'(r_*)}\right)'\right]$$
$$=\left(s(r_*)+\frac{2c^2}{n-1}\chi^2(r_*)\frac{\xi(r_*)}{\xi'(r_*)}\right)^2+2\left(\frac{\chi'(r_*)}{\chi(r_*)}\right)^2+2K^{\perp}+(n-2)\left(\frac{\xi'(r_*)}{\xi(r_*)}\right)^2+(n-2)K_P$$
$$-\frac{2c^2}{n-1}\chi(r)\left[2\chi'(r_*)\frac{\xi(r_*)}{\xi'(r_*)}+\chi(r_*)\left(1+\left(\frac{\xi(r_*)}{\xi'(r_*)}\right)^2K^{\perp}\right)\right],$$

Now we have to add this term, divided by 2, into the quadratic coefficient in (4.22)

$$g(r)=1-\frac{2c^2}{n-1}\chi^2(r_*)\frac{\xi(r_*)}{\xi'(r_*)}\left(1+2\frac{\chi'(r_*)}{\chi(r_*)}\right)(r-r_*)$$
$$+\frac{1}{2}\Bigg[\left(s(r_*)+\frac{2c^2}{n-1}\chi^2(r_*)\frac{\xi(r_*)}{\xi'(r_*)}\right)^2+2\left(\frac{\chi'(r_*)}{\chi(r_*)}\right)^2+2K^{\perp}+(n-2)\left(\frac{\xi'(r_*)}{\xi(r_*)}\right)^2+(n-2)K_P$$
$$-\frac{2c^2}{n-1}\chi(r)\left[2\chi'(r_*)\frac{\xi(r_*)}{\xi'(r_*)}+\chi(r_*)\left(1+\left(\frac{\xi(r_*)}{\xi'(r_*)}\right)^2K^{\perp}\right)\right]-s^2(r_*)$$
$$-\frac{2c^2}{n-1}s(r_*)\chi^2(r_*)\frac{\xi(r_*)}{\xi'(r_*)}\Big(1+2\frac{\chi'(r_-)}{\chi(r_-)}\Big)-$$
$$-s(r_*)\Big(2\frac{\chi'^2(r_-)}{\chi^2(r_-)}+(n-2)\frac{\xi'^2(r_-)}{\xi^2(r_-)}+2K^{\perp}-(n-2)K_P\Big)\Bigg](r-r_*)^2+O(r-r_*)^3.$$

4.2 Asymptotic limit

Now we discuss the limit of the asymptotic limit of the solutions. We may rewrite (4.5) as

$$g'(r) = \frac{2}{n-1}\frac{\xi}{\xi'}\left[S - \left(S + c^2\chi^2\right)g\right] := F(r, g(r)). \tag{4.25}$$

This suggests us that $g(r) \approx S/(S+c^2\chi^2)$, or equivalently $u'^2 \approx c^2/S$, for r sufficiently large.

Theorem 6. *Suppose that*

$$\left(\frac{S}{S+c^2\chi^2}\right)' = o\left(\left[\ln\left(\xi^2\chi^{n-2}\right)\right]'\right) \tag{4.26}$$

$$\frac{1}{S}\left(\ln\frac{\chi^2}{S}\right)' = o\left\{\max\left(\frac{\xi}{\xi'}(S+c^2\chi^2), \frac{\chi'}{\chi}\right)\right\} \tag{4.27}$$

$$\frac{h}{S}\left(\ln\frac{\chi^2}{S}\right)' = o\left\{\max\left(\frac{\xi}{\xi'}(S+c^2\chi^2), \frac{\chi'}{\chi}\right)\right\} \tag{4.28}$$

$$\frac{h'}{h} = o\left\{\max\left(\frac{\xi}{\xi'}(S+c^2\chi^2), \frac{\chi'}{\chi}\right)\right\} \tag{4.29}$$

as $r \to +\infty$ for some smooth positive function h. Then the rotationally symmetric translating solitons $\mathscr{C}_{n,\alpha,C_0}$, for $\alpha = 1/2$, $n \geq 3$ and $C_0 \in \mathbb{R}$, are described, outside a cylinder over a geodesic ball $B_R(o) \subset \mathbb{P}$, as graphs or bi-graphs of functions with the following asymptotic behavior

$$u'^2(r) = \frac{c^2}{S(r)} + o\left(\frac{1}{h(r)}\right) \tag{4.30}$$

as $r \to +\infty$.

Proof. We define the functions

$$f_{\pm} = (1 \pm \varepsilon)\frac{S}{S+c^2\chi^2}. \tag{4.31}$$

We claim that for every fixed $\varepsilon > 0$ and $r_0 > 0$ there exists $r_1 > r_0$ such that $f_-(r_1) < g(r_1)$. If not, there exists $r_0 > 0$ such that for every $r > r_0$ we have $g(r) \leq f_-(r)$.Therefore, by (4.25),

$$g'(r) \geq \frac{2\varepsilon}{n-1}\frac{\xi}{\xi'}S \Rightarrow g(r) - g(r_*) \geq \varepsilon\int_{r_*}^{r}\frac{2}{n-1}\frac{\xi(\rho)}{\xi'(\rho)}S(\rho)\,d\rho = \varepsilon\ln\left(\frac{\chi^2(r)\xi^{n-2}(r)}{\chi^2(r_*)\xi^{n-2}(r_*)}\right) \underset{r\to+\infty}{\longrightarrow} +\infty \tag{4.32}$$

which contradicts $0 \leq g \leq 1$. Analogously, for every fixed $\varepsilon > 0$ and $r_0 > 0$ there exists $r_1 > r_0$ such that $g(r_1) < f_+(r_1)$. If not, there exists $r_0 > 0$ such that for every $r > r_0$ we have $g(r) \geq f_-(r)$. Therefore

$$g'(r) \leq -\frac{2\varepsilon}{n-1}\frac{\xi}{\xi'}S \Rightarrow g(r) - g(r_*) \leq -\frac{2\varepsilon}{n-1}\int_{r_*}^{r}\frac{\xi(\rho)}{\xi'(\rho)}S(\rho)\,d\rho \underset{r\to+\infty}{\longrightarrow} -\infty \tag{4.33}$$

a contradiction. Assuming that $S(r)$ is a decreasing function for a sufficient large r,

$$\left(\frac{S}{S+c^2\chi^2}\right)' = \frac{S'(S+c^2\chi^2)-S(S'+2c^2\chi\chi')}{(S+c^2\chi^2)^2} = -c^2\chi\frac{(-S')\chi+2S\chi'}{(S+c^2\chi^2)^2} < 0.$$

Since

$$F(r,f_-(r)) = \frac{2\varepsilon}{n-1}\frac{\xi}{\xi'}S > 0 > (f_-)'(r),$$

we conclude from a standard comparison argument that there exists some $r_1 > r_0$ such that for every $r > r_0$,

$$f_-(r) < g(r).$$

The inequality

$$-\frac{2\varepsilon}{n-1}\frac{\xi}{\xi'}S = F(r,f_+(r)) < (f_+)'(r)$$

is true if and only if

$$\varepsilon\frac{d}{dr}\ln(\chi^2\xi^{n-2}) = \frac{2\varepsilon}{n-1}\frac{\xi}{\xi'}S > -(1+\varepsilon)\left(\frac{S}{S+c^2\chi^2}\right)'.$$

By (4.26), it is satisfied for some sufficiently large r.

$$\frac{2}{n-1}\frac{\xi}{\xi'}S = \frac{d}{dr}\ln(\chi^2\xi^{n-2}),$$

Again by standard comparison argument, there exists $r_2 > r_1$ such that for every $r > r_2$ we have

$$g(r) < f_+(r).$$

We set

$$g(r) = \frac{1}{1+c^2\chi^2(r)\left(\frac{1}{S}+\psi(r)\right)}.$$

Note that, fixing $\varepsilon > 0$, there exists $r_2 > 0$ such that for every $r > r_2$ we have

$$c^2\chi^2\psi = \frac{1}{g} - \frac{S+c^2\chi^2}{S} \therefore c^2\chi^2|\psi| < \frac{\varepsilon}{1-\varepsilon}\left(1+\frac{c^2\chi^2}{S}\right) \therefore c^2\chi^2S|\psi| < \frac{\varepsilon}{1-\varepsilon}\left(S+c^2\chi^2\right),$$

since

$$\frac{(1-\varepsilon)S}{S+c^2\chi^2} < g < \frac{(1+\varepsilon)S}{S+c^2\chi^2} \therefore \frac{1}{1+\varepsilon}\frac{S+c^2\chi^2}{S} < \frac{1}{g} < \frac{1}{1-\varepsilon}\frac{S+c^2\chi^2}{S}$$
$$\therefore -\frac{\varepsilon}{1+\varepsilon}\frac{S+c^2\chi^2}{S} < \frac{1}{g} - \frac{S+c^2\chi^2}{S} < \frac{\varepsilon}{1-\varepsilon}\frac{S+c^2\chi^2}{S}.$$

Now we compute ψ':

$$
\begin{aligned}
c^2[(\chi^2)'\psi+\chi^2\psi'] &= -\left[\frac{g'}{g^2}+\left(1+\frac{c^2\chi^2}{S}\right)'\right]\\
&= -\left[\left(\frac{S+c^2\chi^2(1+S\psi)}{S}\right)^2\frac{2}{n-1}\frac{\xi}{\xi'}\left[S-\left(S+c^2\chi^2\right)\frac{S}{S+c^2\chi^2(1+S\psi)}\right]+c^2\frac{(\chi^2)'S-\chi^2S'}{S^2}\right]\\
&= -\left[\frac{2}{n-1}\frac{\xi}{\xi'}\frac{(S+c^2\chi^2(1+S\psi))^2}{S^2}\frac{S[S+c^2\chi^2(1+S\psi)]-S(S+c^2\chi^2)}{S+c^2\chi^2(1+S\psi)}+c^2\frac{(\chi^2)'S-\chi^2S'}{S^2}\right]\\
&= -\left[\frac{2}{n-1}\frac{\xi}{\xi'}\frac{(S+c^2\chi^2(1+S\psi))^2}{S^2}\frac{c^2\chi^2S^2\psi}{S+c^2\chi^2(1+S\psi)}+c^2\frac{(\chi^2)'S-\chi^2S'}{S^2}\right]\\
&= -\left[\frac{2}{n-1}\frac{\xi}{\xi'}(S+c^2\chi^2(1+S\psi))c^2\chi^2\psi+c^2\chi^2\frac{S(\ln\chi^2)'-S'}{S^2}\right].
\end{aligned}
$$

Finally,

$$
\psi' = -\left[\frac{2}{n-1}\frac{\xi}{\xi'}(S+c^2\chi^2(1+S\psi))+2\frac{\chi'}{\chi}\right]\psi-\frac{1}{S}\left(\ln\frac{\chi^2}{S}\right)'.
$$

We claim that $\lim_{r\to+\infty}\psi(r)=0$. If it is not the case, we may have $\liminf_{r\to+\infty}\psi(r)<0$ or $\limsup_{r\to+\infty}\psi(r)>0$. In first case, there exists $\delta>0$ such that $\liminf_{r\to+\infty}\psi(r)<-\delta$, therefore there are arbitrary large r_* such that

$$
\psi(r_*)<-\delta.
$$

for every such $r_*>r_3$ we have

$$
\begin{aligned}
\psi'(r_*) &> \delta\left[\frac{2}{n-1}\frac{\xi}{\xi'}(S+c^2\chi^2-c^2\chi^2S|\psi|)+2\frac{\chi'}{\chi}\right]\psi-\frac{1}{S}\left(\ln\frac{\chi^2}{S}\right)'\\
&> \delta\left[\frac{2}{n-1}\frac{1-2\varepsilon}{1-\varepsilon}\frac{\xi}{\xi'}(S+c^2\chi^2)+2\frac{\chi'}{\chi}\right]\psi-\frac{1}{S}\left(\ln\frac{\chi^2}{S}\right)'\\
&> \tilde{\delta}>0
\end{aligned}
$$

by (4.27), where $r_3>r_2$ is large enough. Now we define

$$
r^* := \sup\{r>r_3 :\ \psi(t)<-\delta\ \forall t\in(r_2,r)\},
$$

consequently $\psi(r^*)=-\delta$ and $\psi'(r^*)\geq\tilde{\delta}$. We may define

$$
r_4 := \sup\{r :\ \psi(t)>-\delta\ \forall\, t\in(r^*,r)\}.
$$

This is finite since $\liminf\psi(r)<-\delta$. Therefore $\psi(r_4)=-\delta$ and consequently $\psi'(r_4)\geq\tilde{\delta}$. But it would imply that $\psi(r_4)<-\delta$ for $r<r_4$ sufficiently closed, which is a contradiction. Now

suppose that $\limsup_{r\to+\infty} \psi(r) > 0$. If it is the case, there exists a $\delta > 0$ such that $\psi(r_*) > \delta$ for r_* arbitrarily large. For every such $r_* > r_3$ we have

$$\begin{aligned}
\psi'(r_*) &< -\delta\left[\frac{2}{n-1}\frac{\xi}{\xi'}(S+c^2\chi^2-c^2\chi^2S|\psi|)+(\ln\chi^2)'\right]\psi-\frac{1}{S}\left(\ln\frac{\chi^2}{S}\right)' \\
&< -\delta\left[\frac{2}{n-1}\frac{1-2\varepsilon}{1-\varepsilon}\frac{\xi}{\xi'}(S+c^2\chi^2)+(\ln\chi^2)'\right]\psi-\frac{1}{S}\left(\ln\frac{\chi^2}{S}\right)' \\
&< -\tilde{\delta} < 0
\end{aligned}$$

whenever r_3 is large enough. As above, this leads to a contradiction. Thus

$$\lim_{r\to+\infty}\psi(r) = 0.$$

Now we estimate the rate of convergence $\psi \to 0$. Let

$$\lambda(r) = h(r)\psi(r),$$

where $h(r)$ satisfies conditions (4.28) and (4.29). Therefore

$$\begin{aligned}
\lambda'(r) &= h'(r)\psi(r)+h(r)\psi'(r) \\
&= -\left[\frac{2}{n-1}\frac{\xi}{\xi'}(S+c^2\chi^2(1+S\psi))+(\ln\chi^2)'\right]\psi h-\frac{1}{S}\left(\ln\frac{\chi^2}{S}\right)' h+h'\psi \\
&= -\lambda\left[\frac{2}{n-1}\frac{\xi}{\xi'}(S+c^2\chi^2(1+S\psi))-\frac{h'}{h}+2\frac{\chi'}{\chi}\right]-\frac{h}{S}\left(\ln\frac{\chi^2}{S}\right)'.
\end{aligned}$$

Assuming that $\lambda(r) < -\delta$ for arbitrary large r and using (4.28) and (4.29), there exists $r_3 > r_2$ sufficiently large so that for every such $r > r_3$ we have

$$\lambda'(r) > \delta\left[\frac{2}{n-1}\frac{\xi}{\xi'}(S+c^2\chi^2)\frac{1-2\varepsilon}{1-\varepsilon}-\frac{h'}{h}+2\frac{\chi'}{\chi}\right]-\frac{h}{S}\left(\ln\frac{\chi^2}{S}\right)' > \tilde{\delta} > 0.$$

This leads to a contradiction as before. Now assuming that $\lambda(r) > \delta$ for arbitrary large r, again there exists $r_3 > r_2$ sufficiently large so that for every such $r > r_3$ we have

$$\lambda'(r) < -\delta\left[\frac{2}{n-1}\frac{\xi}{\xi'}(S+c^2\chi^2)\frac{1-2\varepsilon}{1-\varepsilon}-\frac{h'}{h}+2\frac{\chi'}{\chi}\right]-\frac{h}{S}\left(\ln\frac{\chi^2}{S}\right)' < -\tilde{\delta} < 0.$$

This leads to contradictions as before. Therefore $\lambda(r) \to 0$ and the proof is completed. $\square$

5 APPLICATIONS OF THE MAXIMUM PRINCIPLE FOR SOLITONS

5.1 Some fundamental elliptic equations for solitons

Proposition 4. *Suppose that $\bar{M} = \mathbb{P} \times_\chi I$ has the warped structure given by $\chi^2(s)ds^2 + \sigma$. Let X be the Killing vector field ∂_s and let M be a scalar curvature flow soliton in $\bar{M}$ for $\alpha \in \{1/2, 1\}$. Consider the function $\eta : M \to \mathbb{R}$ defined by*

$$\eta(x) = s(\psi(x)), \tag{5.1}$$

for $x \in M$. One has

$$L_\zeta \eta \doteq e^{-\zeta} \operatorname{div}_M(e^\zeta P \nabla u) = 2S\langle \chi^{-2} X, N\rangle + \operatorname{Ric}_{\bar{M}}(\chi^{-2} X^\top, N), \tag{5.2}$$

where $\zeta = |X|^{-2}$.

Proof. One has

$$\nabla^\Sigma \eta = (\bar{\nabla} s)^\top = \chi^{-2} X^\top.$$

Hence,

$$P\nabla\eta = nH\chi^{-2}X^\top - \chi^{-2}AX^\top.$$

Therefore

$$\begin{aligned}
\langle \nabla_{\partial_i} P\nabla\eta, \partial_j\rangle &= \langle \nabla_{\partial_i}(nH\chi^{-2}X^\top), \partial_j\rangle - \langle \nabla_{\partial_i}(\chi^{-2}AX^\top), \partial_j\rangle \\
&= \partial_i(nH)\langle \chi^{-2}X^\top, \partial_j\rangle + nH\langle \nabla_{\partial_i}(\chi^{-2}X^\top), \partial_j\rangle - \partial_i\chi^{-2}\langle AX^\top, \partial_j\rangle - \chi^{-2}\langle \nabla_{\partial_i} AX^\top, \partial_j\rangle.
\end{aligned}$$

Taking traces, one obtains

$$\operatorname{div}(P\nabla\eta) = \langle \nabla(nH), \chi^{-2}X^\top\rangle + nHg^{ij}\langle \nabla_{\partial_i}(\chi^{-2}X^\top), \partial_j\rangle - \langle AX^\top, \nabla\chi^{-2}\rangle - \chi^{-2}g^{ij}\langle \nabla_{\partial_i} AX^\top, \partial_j\rangle.$$

However

$$\begin{aligned}
g^{ij}\langle \nabla_{\partial_i} AX^\top, \partial_j\rangle &= g^{ij}\langle (\nabla_{\partial_i} A)X^\top, \partial_j\rangle + g^{ij}\langle \nabla_{\partial_i} X^\top, A\partial_j\rangle \\
&= (\operatorname{div} A)X^\top + g^{ij}\langle \bar{\nabla}_{\partial_i} X, A\partial_j\rangle - g^{ij}\langle X, N\rangle\langle \bar{\nabla}_{\partial_i} N, A\partial_j\rangle \\
&= (\operatorname{div} A)X^\top + g^{ij}\langle \bar{\nabla}_{\partial_i} X, A\partial_j\rangle + g^{ij}\langle X, N\rangle\langle A\partial_i, A\partial_j\rangle \\
&= (\operatorname{div} A)X^\top + g^{ij}\langle \bar{\nabla}_{\partial_i} X, A\partial_j\rangle + \langle X, N\rangle|A|^2.
\end{aligned}$$

Now we use the following expression for the covariant derivatives of X:

$$\langle \bar{\nabla}_{\mathrm{u}} X, \mathrm{v} \rangle = \langle \mathrm{v}, X \rangle \langle \bar{\nabla} \log \chi, \mathrm{u} \rangle - \langle \mathrm{u}, X \rangle \langle \bar{\nabla} \log \chi, \mathrm{v} \rangle. \tag{5.3}$$

This expression can be proved as follows: using the decomposition

$$\mathrm{u} = \chi^{-2} \langle \mathrm{u}, X \rangle X + \mathrm{u}^{\perp},$$

we have

$$\langle \bar{\nabla}_{\mathrm{u}} X, \mathrm{v} \rangle = \chi^{-4} \langle \mathrm{u}, X \rangle \langle \mathrm{v}, X \rangle \langle \bar{\nabla}_X X, X \rangle + \langle \bar{\nabla}_{\mathrm{u}^{\perp}} X, \mathrm{v}^{\perp} \rangle + \chi^{-2} \langle \mathrm{u}, X \rangle \langle \bar{\nabla}_X X, \mathrm{v}^{\perp} \rangle + \chi^{-2} \langle \mathrm{v}, X \rangle \langle \bar{\nabla}_{\mathrm{u}^{\perp}} X, X \rangle.$$

By the identities

$$\bar{\nabla}_X X = -\frac{1}{2} \bar{\nabla} \chi^2 \quad \text{and} \quad \langle \bar{\nabla}_{\mathrm{u}^{\perp}} X, X \rangle = \frac{1}{2} \langle \bar{\nabla} \chi^2, \mathrm{u}^{\perp} \rangle,$$

using the fact that the leaves perpendicular to X are totally geodesic and that $\chi^2 = |X|^2$ is constant along the flow lines of X, one concludes that

$$\langle \bar{\nabla}_{\mathrm{u}} X, \mathrm{v} \rangle = -\langle \mathrm{u}, X \rangle \langle \bar{\nabla} \log \chi, \mathrm{v} \rangle + \langle \mathrm{v}, X \rangle \langle \bar{\nabla} \log \chi, \mathrm{u} \rangle$$

as stated above. Using 5.3 and the fact that the expression above is skew-symmetric, one obtains

$$g^{ij} \langle \nabla_{\partial_i} A X^{\top}, \partial_j \rangle = (\mathrm{div} A) X^{\top} + \langle X, N \rangle |A|^2.$$

Moreover one has

$$\begin{aligned}
g^{ij} \langle \nabla_{\partial_i} (\chi^{-2} X^{\top}), \partial_j \rangle &= g^{ij} \partial_i \chi^{-2} \langle X, \partial_j \rangle + \chi^{-2} g^{ij} \langle \nabla_{\partial_i} X^{\top}, \partial_j \rangle \\
&= \langle \nabla \chi^{-2}, X^{\top} \rangle + \chi^{-2} g^{ij} \big(\langle \bar{\nabla}_{\partial_i} X, \partial_j \rangle - \langle X, N \rangle \langle \bar{\nabla}_{\partial_i} N, \partial_j \rangle \big) \\
&= \langle \nabla \chi^{-2}, X^{\top} \rangle + \chi^{-2} g^{ij} \langle \bar{\nabla}_{\partial_i} X, \partial_j \rangle + nH \chi^{-2} \langle X, N \rangle.
\end{aligned}$$

Now, using 5.3 we have

$$g^{ij} \langle \bar{\nabla}_{\partial_i} X, \partial_j \rangle = 0.$$

Hence,

$$g^{ij} \langle \nabla_{\partial_i} (\chi^{-2} X^{\top}), \partial_j \rangle = \langle \nabla \chi^{-2}, X^{\top} \rangle + nH \chi^{-2} \langle X, N \rangle.$$

Therefore

$$\begin{aligned}
L\eta \doteq \mathrm{div}(P \nabla \eta) &= \langle \nabla (nH), \chi^{-2} X^{\top} \rangle + nH \langle \nabla \chi^{-2}, X^{\top} \rangle + n^2 H^2 \chi^{-2} \langle X, N \rangle - \langle A X^{\top}, \nabla \chi^{-2} \rangle \\
&\quad - \chi^{-2} (\mathrm{div} A) X^{\top} - \chi^{-2} \langle X, N \rangle |A|^2.
\end{aligned}$$

Now, using the contracted version of Codazzi identity and the expression for S one obtains

$$L\eta \doteq \operatorname{div}(P\nabla\eta) = 2S\chi^{-2}\langle X,N\rangle + \chi^{-2}\operatorname{Ric}_{\bar{M}}(X^\top,N) + nH\langle\nabla\chi^{-2},X^\top\rangle - \langle AX^\top,\nabla\chi^{-2}\rangle.$$

Finally, note that

$$nH\langle\nabla\chi^{-2},X^\top\rangle - \langle AX^\top,\nabla\chi^{-2}\rangle = \langle PX^\top,\nabla\chi^{-2}\rangle.$$

Since $X^\top = \nabla\eta$ we conclude that

$$\operatorname{div}(P\nabla\eta) - \langle\nabla\chi^{-2},P\nabla\eta\rangle = 2S\chi^{-2}\langle X,N\rangle + \chi^{-2}\operatorname{Ric}_{\bar{M}}(X^\top,N). \tag{5.4}$$

Denoting

$$\zeta = \chi^{-2}, \tag{5.5}$$

one has

$$L_\zeta\eta = 2S\langle\chi^{-2}X,N\rangle + \operatorname{Ric}_{\bar{M}}(\chi^{-2}X^\top,N), \tag{5.6}$$

where

$$L_\zeta\eta = \operatorname{div}(P\nabla\eta) - \langle\nabla\chi^{-2},P\nabla\eta\rangle = e^{\zeta}\operatorname{div}(e^{-\zeta}P\nabla\eta),$$

and the proof is finished. $\square$

In Riemannian products, this expression becomes

$$L_\zeta\eta = 2S\langle\chi^{-2}X,N\rangle - (n-1)\kappa\langle X,N\rangle. \tag{5.7}$$

Theorem 7. *Let $\bar{M} = \mathbb{P}\times_\chi I$ a warped space which is an Einstein manifolds. Let M be a compact manifold with boundary and let $\psi : M \to \bar{M}$ be an isometric immersion with constant scalar curvature S. Hence,*

$$2S\int_M \zeta\langle X,N\rangle\, e^{-\zeta}dM = \int_{\partial M}\langle P\nabla\eta,\nu\rangle e^{-\zeta}d\partial M, \tag{5.8}$$

where $\zeta = \chi^{-2} = |X|^{-2}$ and ν is the exterior unit outwards vector field along $\partial M \subset M$. Moreover if D is a hypersurface in $\bar{M}$ with $\partial M = \partial D$ such that $M\cup D$ is an oriented cycle then

$$\int_M\langle\chi^{-2}X,N\rangle e^{-\zeta}dM = \int_D\langle\chi^{-2}X,N_D\rangle e^{-\zeta}dD, \tag{5.9}$$

where N and $-N_D$ determine an orientation to $M\cup D$.

Proof. Integrating both sides in

$$L_\zeta \eta = 2S\langle \chi^{-2}X, N\rangle$$

with respect to the measure $e^{-\zeta}dM$ and applying the divergence theorem, one obtains

$$\int_M 2S\langle \chi^{-2}X, N\rangle\, e^{-\zeta}dM = \int_M \operatorname{div}_M\left(e^{-\zeta}P\nabla\eta\right)dM = \int_{\partial M}\langle P\nabla\eta, \nu\rangle e^{-\zeta}d\partial M,$$

where ν is the outwards unit conormal vector field along $\partial M \subset M$. Since $\nabla\eta = X^\top$ we conclude that

$$\int_M 2S\langle \chi^{-2}X, N\rangle\, e^{-\zeta}dM = \int_M \operatorname{div}_M\left(e^{-\zeta}P\nabla\eta\right)dM = \int_{\partial M}\langle P\nabla\eta, \nu\rangle e^{-\zeta}d\partial M. \tag{5.10}$$

On the other hand since $\operatorname{div}_{\bar{M}} X = 0$ and χ is constant along the flow lines of X one gets

$$\int_\Omega \operatorname{div}_{\bar{M}}\left(e^{-\zeta}\chi^{-2}X\right)d\bar{M} = \int_\Omega \left(e^{-\zeta}\chi^{-2}\operatorname{div}_{\bar{M}} X + \langle\bar{\nabla}e^{-\zeta}\chi^{-2}, X\rangle\right)d\bar{M} = 0. \tag{5.11}$$

Therefore the divergence theorem applied to the oriented cycle $M\cup D = \partial\Omega$ yields

$$\int_M \langle\chi^{-2}X, N\rangle e^{-\zeta}dM = \int_D \langle\chi^{-2}X, N_D\rangle e^{-\zeta}dD,$$

where N and $-N_D$ determine an orientation to $M\cup D$. □

Proposition 5. *Suppose that $\bar{M} = I\times_h \mathbb{P}$ has the warped structure given by $ds^2 + h^2(s)\sigma$. Let X be the closed conformal vector field $h(s)\partial_s$ and let M be a scalar curvature flow soliton in $\bar{M}$ for $\alpha \in \{1/2, 1\}$. Consider the function $\eta : M \to \mathbb{R}$ defined by*

$$\eta(x) = \int_{s_0}^{s(x)} h(\varsigma)\,d\varsigma, \tag{5.12}$$

for $x \in M$ and some $s_0 \in I$. One has

$$L\eta \doteq \operatorname{div}_M(P\nabla\eta) = 2S\langle X, N\rangle + \operatorname{Ric}_{\bar{M}}(X^\top, N) + (n^2-n)H\varphi, \tag{5.13}$$

where $\varphi = h'\circ\psi$.

Proof. One has

$$\nabla\eta = h(s(x))\partial_s^\top = X^\top.$$

Hence,

$$P\nabla\eta = nHX^\top - AX^\top.$$

Therefore

$$\langle \nabla_{\partial_i} P \nabla \eta, \partial_j \rangle = \partial_i(nH)\langle X^\top, \partial_j\rangle + nH\langle \nabla_{\partial_i} X^\top, \partial_j\rangle - \langle \nabla_{\partial_i} AX^\top, \partial_j\rangle.$$

Taking traces, one obtains

$$\mathrm{div}_M(P\nabla\eta) = \langle \nabla(nH), X^\top\rangle + nHg^{ij}\langle \nabla_{\partial_i} X^\top, \partial_j\rangle - g^{ij}\langle \nabla_{\partial_i} AX^\top, \partial_j\rangle.$$

However

$$\begin{aligned} g^{ij}\langle \nabla_{\partial_i} AX^\top, \partial_j\rangle &= g^{ij}\langle (\nabla_{\partial_i} A)X^\top, \partial_j\rangle + g^{ij}\langle \nabla_{\partial_i} X^\top, A\partial_j\rangle \\ &= (\mathrm{div}_M A)X^\top + g^{ij}\langle \bar\nabla_{\partial_i} X, A\partial_j\rangle - g^{ij}\langle X, N\rangle\langle \bar\nabla_{\partial_i} N, A\partial_j\rangle \\ &= (\mathrm{div}_M A)X^\top + g^{ij}\langle \partial_i, A\partial_j\rangle\varphi + g^{ij}\langle X, N\rangle\langle A\partial_i, A\partial_j\rangle \\ &= (\mathrm{div}_M A)X^\top + nH\varphi + \langle X, N\rangle |A|^2. \end{aligned}$$

Moreover one has

$$g^{ij}\langle \nabla_{\partial_i} X^\top, \partial_j\rangle = g^{ij}\big(\langle \bar\nabla_{\partial_i} X, \partial_j\rangle - \langle X, N\rangle\langle \bar\nabla_{\partial_i} N, \partial_j\rangle\big) = n\varphi + nH\langle X, N\rangle.$$

Therefore

$$L\eta \doteq \mathrm{div}_M(P\nabla\eta) = \langle \nabla(nH), X^\top\rangle + n^2H\varphi + n^2H^2\langle X, N\rangle - (\mathrm{div}_M A)X^\top - nH\varphi - \langle X, N\rangle|A|^2.$$

Now, using the contracted version of Codazzi identity and the expression for S, one obtains

$$L\eta \doteq \mathrm{div}(P\nabla\eta) = 2S\langle X, N\rangle + \mathrm{Ric}_{\bar M}(X^\top, N) + (n^2-n)H\varphi. \tag{5.14}$$

Proposition 6. *Suppose that $\bar M = I \times_h \mathbb{P}$ is an Einstein manifold with the warped structure given by $ds^2 + h^2(s)\sigma$. Let X be the closed conformal vector field $h(s)\partial_s$ and let M be a scalar curvature flow soliton in $\bar M$ for $\alpha = 1$. Hence,*

$$LS + c\langle \nabla\eta, \nabla S\rangle = -\big(2nc\varphi + (nHS - 3S_3)\big)S,$$

where $\varphi = h' \circ \psi$.

Proof. Since

$$\nabla S^\alpha = -cAX^\top,$$

one has

$$\begin{aligned} P^{ij}\langle \nabla_{\partial_i}\nabla S^\alpha, \partial_j\rangle &= -c(nHg^{ij} - h^{ij})\langle \nabla_{\partial_i} AX^\top, \partial_j\rangle \\ &= -c\big(nHg^{ij} - h^{ij}\big)\big(\langle (\nabla_{\partial_i} A)X^\top, \partial_j\rangle + \langle \nabla_{\partial_i} X^\top, A\partial_j\rangle\big). \end{aligned}$$

Therefore

$$P^{ij}\langle\nabla_{\partial_i}\nabla S^\alpha,\partial_j\rangle=-nHc\,\mathrm{div}A(X^\top)+ch^{ij}\langle(\nabla_{\partial_i}A)X^\top,\partial_j\rangle$$
$$-c\big(nHg^{ij}-h^{ij}\big)\big(n\varphi h_{ij}+\langle X,N\rangle\langle A\partial_i,A\partial_j\rangle\big).$$

However using Codazzi equation one has

$$\langle(\nabla_{\partial_i}A)X^\top,\partial_j\rangle=\langle(\nabla_{X^\top}A)\partial_i,\partial_j\rangle+\langle\bar R(X^\top,\partial_i)N,\partial_j\rangle.$$

Hence,

$$P^{ij}\langle\nabla_{\partial_i}\nabla S^\alpha,\partial_j\rangle=-nHc\,\mathrm{div}A(X^\top)+ch^{ij}\big(\langle(\nabla_{X^\top}A)\partial_i,\partial_j\rangle-ch^{ij}\langle\bar R(\partial_i,X^\top)N,\partial_j\rangle\big)$$
$$-c\big(nHg^{ij}-h^{ij}\big)\big(n\varphi h_{ij}+\langle X,N\rangle\langle A\partial_i,A\partial_j\rangle\big).$$

We conclude that

$$P^{ij}\langle\nabla_{\partial_i}\nabla S^\alpha,\partial_j\rangle=-nHc\,\mathrm{div}A(X^\top)+\frac{c}{2}\langle\nabla|A|^2,X^\top\rangle-ch^{ij}\langle\bar R(\partial_i,X^\top)N,\partial_j\rangle$$
$$-c\big(n\varphi\,\mathrm{tr}(PA)+\langle X,N\rangle\,\mathrm{tr}(PA^2)\big).$$

Since

$$\mathrm{div}A(X^\top)=\langle\nabla(nH),X^\top\rangle-\mathrm{Ric}_{\bar M}(X^\top,N),$$

one obtains

$$P^{ij}\langle\nabla_{\partial_i}\nabla S^\alpha,\partial_j\rangle=\frac{c}{2}\langle\nabla(|A|^2-n^2H^2),X^\top\rangle+c\big(nHg^{ij}-h^{ij}\big)\langle\bar R(\partial_i,X^\top)N,\partial_j\rangle$$
$$-c\big(2Sn\varphi+\langle X,N\rangle(nHS-3S_3)\big).$$

Since

$$Lu=\mathrm{div}(P\nabla u)=P^{ij}u_{i;j}+\mathrm{Ric}_{\bar M}(\nabla u,N),$$

one concludes that

$$LS^\alpha=-c\langle\nabla S,X^\top\rangle+\mathrm{Ric}_{\bar M}(\nabla S^\alpha,N)+cP^{ij}\langle\bar R(\partial_i,X^\top)N,\partial_j\rangle$$
$$-c\big(2Sn\varphi+\langle X,N\rangle(nHS-3S_3)\big).$$

If $\bar M$ is an Einstein manifold, then the terms involving $\bar R$ vanish and we are left with

$$LS^\alpha=-c\langle\nabla S,X^\top\rangle-c\big(2Sn\varphi+\langle X,N\rangle(nHS-3S_3)\big).$$

Fixing $\alpha=1$ and using (1.6) one obtains

$$LS+c\langle\nabla\eta,\nabla S\rangle=-\big(2n\varphi c+(nHS-3S_3)\big)S.$$

Proposition 7. *Suppose that $\bar{M} = \mathbb{P} \times_\chi I$ is an Einstein manifold with the warped structure given by $\chi^2(s)ds^2 + \sigma$. Let X be the Killing vector field ∂_s and let M be a scalar curvature flow soliton in $\bar{M}$ for $\alpha = 1$. Hence,*

$$LS + c\langle \nabla\eta, \nabla S\rangle = -(nHS - 3S_3)S.$$

Proof. Since

$$\nabla S^\alpha = -cAX^\top,$$

one has

$$P^{ij}\langle \nabla_{\partial_i}\nabla S^\alpha, \partial_j\rangle = -c(nHg^{ij} - h^{ij})\langle \nabla_{\partial_i} AX^\top, \partial_j\rangle$$
$$= -c\big(nHg^{ij} - h^{ij}\big)\big(\langle(\nabla_{\partial_i}A)X^\top, \partial_j\rangle + \langle \nabla_{\partial_i}X^\top, A\partial_j\rangle\big).$$

Therefore

$$P^{ij}\langle \nabla_{\partial_i}\nabla S^\alpha, \partial_j\rangle = -nHc\,\mathrm{div}A(X^\top) + ch^{ij}\langle(\nabla_{\partial_i}A)X^\top, \partial_j\rangle$$
$$-c\big(nHg^{ij} - h^{ij}\big)\big((\log\chi)_i\langle A\partial_j, X\rangle - h^k_j(\log\chi)_k\langle X, \partial_i\rangle + \langle X, N\rangle\langle A\partial_i, A\partial_j\rangle\big).$$

However using Codazzi equation one has

$$\langle(\nabla_{\partial_i}A)X^\top, \partial_j\rangle = \langle(\nabla_{X^\top}A)\partial_i, \partial_j\rangle + \langle \bar{R}(X^\top, \partial_i)N, \partial_j\rangle.$$

Hence,

$$P^{ij}\langle \nabla_{\partial_i}\nabla S^\alpha, \partial_j\rangle = -nHc\,\mathrm{div}A(X^\top) + ch^{ij}\big(\langle(\nabla_{X^\top}A)\partial_i, \partial_j\rangle - \langle \bar{R}(\partial_i, X^\top)N, \partial_j\rangle\big)$$
$$-c\big(nHg^{ij} - h^{ij}\big)\big((\log\chi)_i\langle AX^\top, \partial_j\rangle - h^k_j(\log\chi)_k\langle X, \partial_i\rangle + \langle X, N\rangle\langle A\partial_i, A\partial_j\rangle\big).$$

We conclude that

$$P^{ij}\langle \nabla_{\partial_i}\nabla S^\alpha, \partial_j\rangle = -nHc\,\mathrm{div}A(X^\top) + \frac{c}{2}\langle \nabla|A|^2, X^\top\rangle - ch^{ij}\langle \bar{R}(\partial_i, X^\top)N, \partial_j\rangle$$
$$-c\big(\langle P\nabla\log\chi, AX^\top\rangle - \langle PX^\top, A\nabla\log\chi\rangle + \langle X, N\rangle\,\mathrm{tr}(PA^2)\big).$$

The symmetry of A and P implies that

$$\langle P\nabla\log\chi, AX^\top\rangle - \langle PX^\top, A\nabla\log\chi\rangle.$$

Hence,

$$\mathrm{div}A(X^\top) = \langle \nabla(nH), X^\top\rangle - \mathrm{Ric}_{\bar{M}}(X^\top, N)$$

and one obtains

$$P^{ij}\langle\nabla_{\partial_i}\nabla S^{\alpha},\partial_j\rangle=\frac{c}{2}\langle\nabla(|A|^2-n^2H^2),X^{\top}\rangle+c\big(nHg^{ij}-h^{ij}\big)\langle\bar{R}(\partial_i,X^{\top})N,\partial_j\rangle$$
$$-c\langle X,N\rangle(nHS-3S_3).$$

Since

$$Lu=\operatorname{div}(P\nabla u)=P^{ij}u_{i;j}+\operatorname{Ric}_{\bar{M}}(\nabla u,N),$$

one concludes that

$$LS^{\alpha}=-c\langle\nabla S,X^{\top}\rangle+\operatorname{Ric}_{\bar{M}}(\nabla S^{\alpha},N)+cP^{ij}\langle\bar{R}(\partial_i,X^{\top})N,\partial_j\rangle$$
$$-c\langle X,N\rangle(nHS-3S_3).$$

If $\bar{M}$ is an Einstein manifold, then the terms involving $\bar{R}$ vanish and we are left with

$$LS^{\alpha}=-c\langle\nabla S,X^{\top}\rangle-c\langle X,N\rangle(nHS-3S_3).$$

Fixing $\alpha=1$ and using (1.6) one obtains

$$LS+c\langle\nabla\eta,\nabla S\rangle=-(nHS-3S_3)S.$$

5.2 Variants of the maximum principle

The linearized equation of the scalar curvature flow soliton has in its principal part the divergence-type operator

$$Lu=\operatorname{div}(P\nabla u),\quad u\in C^2(\Sigma),$$

where $P=nHI-A$. Note that the principal values μ_i of P are given by

$$\mu_i=nH-\lambda_i,$$

for $i=1,\ldots,n$, where λ_i are the principal curvatures of $\Sigma=\psi(M)$.

One of the main analytical tools used in this paper is the weak maximum principle either for the operator L or for its weighted counterpart

$$L_{\zeta}=L-\langle\nabla\zeta,\nabla\cdot\rangle \tag{5.15}$$

for some $\zeta\in C^1(M)$. A general discussion on the weak maximum principle for a very large class of operators can be found in (ALÍAS *et al.*, 2016), (ALBANESE *et al.*, 2013) and (BESSA;

PESSOA, 2014). Indeed, in these references we could find a detailed analysis of *trace operators*, a class that includes L_ζ since

$$L_\zeta u = \operatorname{tr}(P \circ \nabla^2 u) + \langle \operatorname{div} P, \nabla u\rangle - \langle \nabla\zeta, \nabla u\rangle.$$

Definition 3. *Definition 4.1 in (ALÍAS et al., 2016). Let M be a Riemannian manifold and let $\zeta \in C^1(M)$. We say that the weak maximum principle holds for the operator L_ζ on M if for any $u \in C^2(M)$ with*

$$u^* = \sup_M u < \infty$$

and for each $\gamma < u^*$ we have

$$\inf_{\Omega_\gamma} L_\zeta u \le 0,$$

where

$$\Omega_\gamma = \{x \in M : u(x) > \gamma\}.$$

Theorem 3.1 in (ALÍAS *et al.*, 2016) gives a sufficient condition for the validity of the weak maximum principle which is rooted in the notion of stochastic completeness and in the approach to the maximum principle built by Pigola, Rigoli and Setti. The condition known as Khas'minski condition is the existence of a $v \in C^2(M)$ satisfying

$$\begin{cases} L_\zeta v \le Av & \text{on} \quad M\backslash K, \\ v(x) \to +\infty & \text{as} \quad x \to \infty \ \text{ in } \ M, \end{cases} \tag{5.16}$$

for some constant $A \in \mathbb{R}$ and some compact set $K \subset M$. We note that the first condition in (5.16) can be substituted, for instance, with $L_\zeta v \le A$ on $M\backslash K$. Occasionally, we shall also use an equivalent form of the weak maximum principle that which can be stated as follows.

Definition 4. *Definition 4.6 in (ALÍAS et al., 2016). We say that the open weak maximum principle holds for the operator L_ζ on M if for each $F \in C^0(\mathbb{R})$, for each open set $\Omega \subset M$ with $\partial\Omega \neq \emptyset$ and for each $v \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ satisfying*

$$\begin{cases} L_\zeta v \ge F(v) \ on \ \ \Omega \\ \sup_\Omega v < +\infty, \end{cases} \tag{5.17}$$

we have that either $\sup_\Omega v = \sup_{\partial\Omega} v$ *or* $F(\sup_\Omega v) \le 0$.

Parabolicity for L_ζ, in the sense of the validity of a Liouville type theorem for bounded above L_ζ-subharmonic functions, can be expressed as a stronger form of the weak maximum principle. Indeed,

Definition 5. *A Riemannian manifold M is strongly parabolic with respect to L_ζ if for any non-constant $u \in C^2(M)$ with $u^* = \sup_M u < \infty$ and for each $\gamma < u^*$ we have*

$$\inf_{\Omega_\gamma} L_\zeta u < 0,$$

with Ω_γ as above.

In the case of the operator L_ζ, the three forms of parabolicity, that is, Ahlfors, Liouville and strong parabolicity, are in fact equivalent. See Section 4.4 of (ALÍAS *et al.*, 2016). We recall that M is Liouville L_ζ-parabolic if a function $u \in C^2(M)$ with $u^* < +\infty$ and $L_\zeta u \geq 0$ is necessarily constant. We also recall that Ahlfors parabolicity expresses as follows: L_ζ is Ahlfors parabolic on M if for each open set $\Omega \subset M$ with $\partial\Omega \neq \emptyset$ and for each non-constant $v \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ satisfying

$$\begin{cases} L_\zeta v \geq 0 & \text{on} \quad \Omega \\ \sup_\Omega v < +\infty, \end{cases} \tag{5.18}$$

we have

$$\sup_\Omega v = \sup_{\partial\Omega} v.$$

See theorems 4.10 and 4.11 in (ALÍAS *et al.*, 2016) at this respect. Of course, for parabolicity, Khas'minskii type test still applies appropriately stated in the following mildly stronger form: Let M be complete and assume the existence of $v \in C^2(M)$ such that $v(x) \to +\infty$ as $x \to \infty$ in M, and

$$\begin{cases} L_\zeta v \leq 0 \ \text{ if } \ \nabla\zeta \equiv 0 \text{ and} \\ L_\zeta v < 0 \ \text{ if } \ \nabla\zeta \not\equiv 0 \text{ on } M \backslash K, \end{cases} \tag{5.19}$$

for some compact set $K \subset M$. Then the operator L_ζ is parabolic on M.

Remark 2. *Always in case M is complete a sufficient condition can be given for parabolicity with respect to L_ζ in terms of the growth of a weighted volume of the boundary of geodesic balls. More precisely, having fixed an origin $o \in M$ let*

$$\mathrm{vol}_\zeta(\partial B_r) = \int_{\partial B_r} e^{-\zeta}\, dM, \tag{5.20}$$

where ∂B_r is the boundary of the geodesic ball B_r centered at o and of radius r. Let μ_- and μ_+ be the minimum and maximum principal values of P. Suppose that $\mu_- > 0$ in M and

$$\inf_M \frac{\mu_-}{\mu_+} > 0.$$

If

$$\frac{1}{\mu_+ \mathrm{vol}_\zeta(\partial B_r)} \notin L^1(+\infty)$$

then M is L_ζ-parabolic. Note that the above request is also necessary on a Riemannian model manifold with $\zeta \equiv 1$. Other more elaborated results can be found in Chapter 4 of (ALÍAS et al., 2016) like Theorem 4.14.

5.3 Applications of the maximum principle

Next result is a direct consequence of Proposition 4 and Definition 5.

Theorem 8. *Let $\bar{M}^{n+1} = I \times_h \mathbb{P}$ be a warped product with constant sectional curvature and $h' > 0$. Let $\psi : M^n \to \bar{M}^{n+1}$ be a scalar curvature flow soliton for $c > 0$. Suppose that $H > 0$ in $\psi(M)$ and that η is bounded above on M. If M is parabolic with respect to the operator L then $\psi(M)$ is contained in the leaf $\mathbb{P}_{s_c} = \{s_c\} \times \mathbb{P}$, with s_c given implicitly by the equation $\pi(s_c) = 0$.*

Proof. It follows from Proposition (4) that

$$L\eta = 2S\langle X, N\rangle + (n^2 - n)Hh' > 0$$

Hence η is not bounded above unless it is constant. Therefore $s \circ \psi$ is constant. Since $\psi(M)$ is a leaf and a soliton we necessarily have $s \circ \psi \equiv s_c$. This finishes the proof. □

Now we state and prove a similar result for solitons in warped spaces of the form $\bar{M} = \mathbb{P} \times_\chi I$.

Theorem 9. *Let $\bar{M}^{n+1} = \mathbb{P} \times_\chi I$ be a warped product with constant sectional curvature. Let $\psi : M^n \to \bar{M}^{n+1}$ be a scalar curvature flow soliton for $c > 0$. If M is parabolic with respect to the operator L_ζ where $\zeta = \chi^{-2}$ then η is not bounded above.*

Proof. It follows from Proposition (5) that

$$L\eta = 2S\langle X, N\rangle > 0$$

Hence if η is bounded above then it is constant. Therefore $s \circ \psi$ is constant. Since a leaf $\mathbb{P}_s$ is not a soliton we reach to a contradiction that finishes the proof. □

Theorem 10. *Let $\psi : M^n \to \bar{M}^{n+1}$ be a scalar curvature curvature flow soliton with $S > 0$ in a warped product $\bar{M} = I \times_h \mathbb{P}$. Suppose that $\bar{M}$ is Einstein. Assume the validity of the weak maximum principle for the operator $L_{-c\eta}$ on M. Finally suppose that $\sup_M S_3 < +\infty$ and that $\psi(M)$ has an elliptic point. Then either*

$$\sup_M (nHS - 3S_3 + 2nc\varphi) \geq 0, \tag{5.21}$$

or

$$S \equiv 0 \ \text{ on } \ M. \tag{5.22}$$

Proof. If $\sup_M (nHS - 3S_3 + 2nc\varphi) \geq 0$ there is nothing to prove. Otherwise, suppose that

$$\sup_M (nHS - 3S_3 + 2nc\varphi) < 0.$$

Then Newton-Maclaurin inequalities imply that

$$\left(\frac{2n}{n-1}\right)^{1/2} S^{1/2} S \leq nHS < -2nc\varphi + 3S_3 \leq -2nc\varphi + 3\sup_M S_3.$$

Therefore

$$\sup_M S \leq \left(\frac{n-1}{2n}\right)^{1/3} \sup_M \left(3S_3 - 2nc\varphi\right)^{1/3}. \tag{5.23}$$

Furthermore,

$$\inf_M (-nHS + 3S_3 - 2nc\varphi) \doteq C > 0. \tag{5.24}$$

By the weak maximum principle for the operator $L_{-c\eta}$ on M, there exists a sequence $\{x_k\}_{k=1}^{\infty}$ in M such that

$$S(x_k) > \sup_M S - \frac{1}{k} \ \text{ and } \ L_{-c\eta} S(x_k) < \frac{1}{k}. \tag{5.25}$$

It follows that

$$(-nH(x_k)S(x_k) + 3S_3(x_k) - 2nc\varphi(x_k))S(x_k)) = L_{-c\eta} S(x_k) < \frac{1}{k}$$

from which we infer

$$C\left(\sup_M S - \frac{1}{k}\right) < \inf_M (-nHS + 3S_3 - 2nc\varphi)S < \frac{1}{k}.$$

Passing to the limit as $k \to +\infty$ we conclude that $\sup_M S = 0$, that is, $S \equiv 0$ on M. $\square$

Similarly, we get the following result.

Theorem 11. *Let $\psi : M^n \to \bar{M}^{n+1}$ be a scalar curvature flow soliton with respect in a warped product $\bar{M}^{n+1} = I \times_h \mathbb{P}$. Suppose that $\bar{M}$ is Einstein. Assume the validity of the weak maximum principle for the operator $L_{-c\eta}$ on M. Finally suppose that $\inf_M S > 0$. Then*

$$\inf_M \big(2n\varphi c + nHS - 3S_3\big) \leq 0. \tag{5.26}$$

Proof. Denote

$$\inf_M S \doteq C_1 > 0.$$

Setting $u = -S$ one has

$$L_{-c\eta} u = \big(2n\varphi c + nHS - 3S_3\big)S.$$

Thus assuming by contradiction that

$$\inf_M \big(2n\varphi c + nHS - 3S_3\big) = C_2 > 0$$

we deduce

$$L_{-c\eta} u \geq C_1 C_2 > 0.$$

An application of the weak maximum principle directly gives he desired contradiction. □

Theorem 12. *Let $\psi : M^n \to \bar{M}^{n+1} = I \times_h \mathbb{P}$ be a complete scalar curvature flow soliton with respect to $X = h\partial_s$. Assume that $\mathbb{P}$ has constant sectional curvature κ. If*

$$\Lambda = \frac{1}{n-1} \sup_M \left[ch' + |nhA| + |A|^2 + \left(\frac{n-1}{2}\right)^{\alpha} |A|^{1+2\alpha} - (n-1)\varkappa \right] < +\infty, \tag{5.27}$$

where

$$\varkappa = \min\left\{ -\frac{h''}{h}, \frac{\kappa}{h^2} - \frac{h'^2}{h^2} \right\}, \tag{5.28}$$

the maximum principle is valid for the operator $L_{-c\eta}$ on M.

Proof. We follow the proof of Theorem 5.1 of (ALIAS *et al.*, 2020). Let $\{E_i\}_{i=1}^n$ be a local orthonormal frame on M and V, W vector fields on M. Contracting Gauss equation,

$$\begin{aligned}
\mathrm{Ric}_M(V,W) &= \mathrm{R}_M(V,E_i,E_i,W) \\
&= \mathrm{R}_{\bar{M}}(V,E_i,E_i,W) + \langle AV,W\rangle\langle AE_i,E_i\rangle - \langle AV,E_i\rangle\langle AW,E_i\rangle \\
&= \mathrm{R}_{\bar{M}}(V,E_i,E_i,W) + nH\langle AV,W\rangle - \langle AV,AW\rangle
\end{aligned}$$

Using equation (5.8) of (ALIAS *et al.*, 2020),

$$
\begin{aligned}
&\mathrm{R}_{\bar{M}}(V,E_i,E_i,W) = \left(\frac{h'^2}{h^2}-\frac{\kappa}{h^2}\right)\left(\langle\hat{V},\hat{E}_i\rangle\langle\hat{E}_i,\hat{W}\rangle-\langle\hat{V},\hat{W}\rangle\langle\hat{E}_i,\hat{E}_i\rangle\right)\\
&+\frac{h''}{h}\Big(\langle V,\partial_s\rangle\langle E_i,\partial_s\rangle\langle\hat{E}_i,\hat{W}\rangle-\langle V,\partial_s\rangle\langle W,\partial_s\rangle\langle\hat{E}_i,\hat{E}_i\rangle-\langle E_i,\partial_s\rangle\langle E_i,\partial_s\rangle\langle\hat{V},\hat{W}\rangle+\langle E_i,\partial_s\rangle\langle W,\partial_s\rangle\langle\hat{V},\hat{E}_i\rangle\Big)\\
&=\left(\frac{h'^2}{h^2}-\frac{\kappa}{h^2}\right)[(n-2)\langle V,\partial_s\rangle\langle W,\partial_s\rangle-(n-1-|\partial_s^\top|^2)\langle V,W\rangle]\\
&-\frac{h''}{h}[(n-2)\langle V,\partial_s\rangle\langle W,\partial_s\rangle+|\partial_s^\top|^2\langle V,W\rangle]
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathrm{Ric}_M(V,W) = &-\left(\frac{h''}{h}|\partial_s^\top|^2+(n-1-|\partial_s^\top|^2)\right)\left(\frac{h'^2}{h^2}-\frac{\kappa}{h^2}\right)\langle V,W\rangle\\
&+(n-2)\left(\frac{h'^2}{h^2}-\frac{h''}{h}-\frac{\kappa}{h^2}\right)\langle V,\partial_s\rangle\langle W,\partial_s\rangle+nH\langle AV,W\rangle-\langle AV,AW\rangle. \qquad (5.29)
\end{aligned}
$$

Now we take $W=V$ so that $|V|=1$. Observe that

$$0 = S^\alpha\langle AV,V\rangle - S^\alpha\langle AV,V\rangle \geq c\nabla^2\eta(V,V)-ch'-|S|^\alpha|A|.$$

By arithmetic-geometric mean inequality,

$$|S| \leq \sum_{i<j}|k_ik_j| \leq \sum_{i<j}\frac{k_i^2+k_j^2}{2} = \frac{n-1}{2}|A|^2.$$

Therefore

$$0 \geq c\nabla^2\eta(V,V)-ch'-\left(\frac{n-1}{2}\right)^\alpha|A|^{1+2\alpha}.$$

Applying the above inequality into (5.29), we have

$$
\begin{aligned}
&\mathrm{Ric}_M(V,V) \geq c\nabla^2\eta(V,V)-ch'-\left(\frac{h''}{h}|\partial_s^\top|^2+(n-1-|\partial_s^\top|^2)\right)\left(\frac{h'^2}{h^2}-\frac{\kappa}{h^2}\right)\\
&\qquad+(n-2)\left(\frac{h'^2}{h^2}-\frac{h''}{h}-\frac{\kappa}{h^2}\right)\langle V,\partial_s\rangle\langle V,\partial_s\rangle-|nHA|-|A|^2-\left(\frac{n-1}{2}\right)^\alpha|A|^{1+2\alpha}\Rightarrow\\
&\mathrm{Ric}_M(V,V)-c\nabla^2\eta(V,V) \geq \left[-ch'-|nhA|-|A|^2-\left(\frac{n-1}{2}\right)^\alpha|A|^{1+2\alpha}+(n-1)\varkappa\right],
\end{aligned}
$$

where $\varkappa$ is defined in (5.28). Using (5.27) we have

$$\mathrm{Ric}_M - c\nabla^2\eta \geq -(n-1)\Lambda g. \qquad (5.30)$$

By Proposition 8.6 of (ALÍAS *et al.*, 2016) there exists a geodesic ball $B_{R_0}(o) \subset M$ centered at $o \in M$ with sufficiently small radius $R_0 > 0$ and a constant $C = C(B_{R_0}) > 0$ such that

$$\Delta r(x) + c\langle \nabla\eta, \nabla r(x)\rangle \leq C + (n-1)\Lambda r(x) \quad \text{on} \quad M \setminus B_{R_0},$$

where $r(x) = \operatorname{dist}(o,x)$. Using Proposition 8.11 of (ALÍAS *et al.*, 2016), we deduce

$$\int_{B_r} e^{c\eta}\, dM \leq \int_0^r e^{C\lambda + (n-1)\Lambda\frac{\lambda^2}{2}}\, d\lambda + D,$$

where $D > 0$ and C as above. Therefore

$$\liminf_{r\to+\infty} \frac{1}{r^2} \log \int_{B_r} e^{c\eta}\, dM < +\infty.$$

For $L_{-c\eta} = e^{-c\eta}\operatorname{div}(e^{c\eta}P\nabla\cdot)$, we apply Theorem 4.1 of (ALÍAS *et al.*, 2016) with $T = P$, $\varphi(x,t) = te^{c\eta(x)}$, $A(x) = e^{c\eta(x)}$, $b(x) = 1$ to prove the validity of the weak maximum principle for the operator $L_{-c\eta}$ on M.

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