

# A New Look at Nonlinear Time Series Prediction with NARX Recurrent Neural Network

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## Abstract

*The NARX network is a recurrent neural architecture commonly used for input-output modeling of nonlinear systems. The input of the NARX network is formed by two tapped-delay lines, one sliding over the input signal and the other one over the output signal. Currently, when applied to chaotic time series prediction, the NARX architecture is designed as a plain Focused Time Delay Neural Network (FTDNN); thus, limiting its predictive abilities. In this paper, we propose a strategy that allows the original architecture of the NARX network to fully explore its computational power to improve prediction performance. We use the well-known chaotic laser time series to evaluate the proposed approach in multi-step-ahead prediction tasks. The results show that the proposed approach consistently outperforms standard neural network based predictors, such as the FTDNN and Elman architectures.*

## 1. Introduction

Artificial neural networks (ANNs) have been successfully used as a tool for time series prediction and modeling in a variety of application domains, including financial time series prediction [8], river flow forecasting [2], biomedical time series modeling [7] and network traffic prediction [9, 1], just to mention a few. Usually, ANN models outperform traditional linear techniques, such as the well-known Box-Jenkins models [4], when the time series are noisy and nonlinear. In such cases, the universal approximation and generalization abilities of ANN models seems to justify their better prediction performance.

In nonlinear (e.g. chaotic) time series prediction, ANN models are commonly used as one-step-ahead predictors, estimating only the next value of a time series without feeding the predicted value back to the model's input regres-

sor. In other words, the input regressor contains only actual sample points of the time series. If the user is interested in a wider prediction horizon, a procedure known as multi-step-ahead prediction, the model's output should be fed back to the input regressor for a fixed but finite number of time steps. In this case, the input regressor's components, previously composed of actual sample points of the time series, are gradually replaced by predicted values.

If the prediction horizon tends to infinity, from some moment in time on, the input regressor will start to be composed only of previous estimated values of the time series. In this case, the multi-step-ahead prediction task becomes a *dynamic modeling* task, in which the ANN model acts as an autonomous system, trying to recursively emulate the dynamic behavior of the system that generated the nonlinear time series [12]. Multi-step ahead prediction and dynamic modelling are much more complex to deal with than one-step-ahead prediction, and it is believed that these are complex tasks in which ANN models play an important role, in particular recurrent neural architectures [21].

Recurrent ANNs have local and/or global feedback loops in their structure. Even though feedforward MLP-like networks can be easily adapted to process time series through an input tapped delay line, giving rise to the well-known Focused Time Delay Neural Network (FTDNN) [21], they can also be easily converted to simple recurrent architectures by feeding back the neuronal outputs of the hidden or output layers, giving rise to Elman and Jordan networks, respectively [14]. Recurrent neural networks (RNNs) are capable to represent arbitrary nonlinear dynamical mappings [19], such as those commonly found in nonlinear time series prediction tasks.

The previously described neural architectures are usually trained through the standard backpropagation algorithm. However, learning to perform tasks in which the temporal dependencies present in the input/output signals span long time intervals can be quite difficult using gra-

gradient descent [3]. In [16], the authors reported that learning such long-term temporal dependencies with gradient-descent techniques is more effective in a class of recurrent ANN architecture called *Nonlinear Autoregressive with exogenous input* (NARX) [17] than in simple MLP-based recurrent models. This occurs in part because the NARX model's input vector is cleverly built by means of a tapped-delay line sliding over the input signal together with another tapped-delay line over the network's output.

Despite the aforementioned advantages of the NARX network, its application to univariate time series prediction has been misdirected. In this type of application, the tapped-delay line over the output signal is eliminated, thus reducing the NARX network to a plain FTDNN architecture. Considering this under-utilization of the NARX network, we propose a simple strategy based on Takens' embedding theorem to allow the computational abilities of the original NARX network to be fully exploited in nonlinear time series prediction tasks.

The remainder of the paper is organized as follows. In Section 2, we briefly describe the NARX recurrent network model and its main characteristics. In Section 3 we describe the basics of the nonlinear time series prediction problem and introduce our approach. The simulations and discussion of results are presented in Section 4. The paper is concluded in Section 5

## 2. The NARX Network

The Nonlinear Autoregressive model with Exogenous inputs (NARX) model is an important class of discrete-time nonlinear systems that can be mathematically represented as follows [15, 20]:

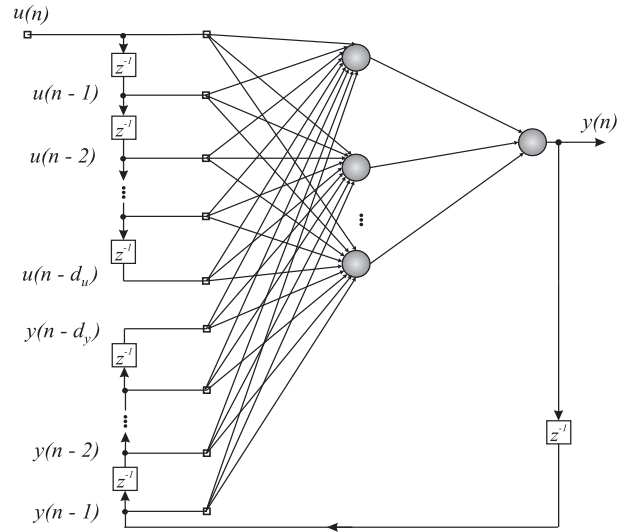
$$y(n+1) = f[y(n), \dots, y(n-d_y+1); u(n), u(n-1), \dots, u(n-d_u+1)] \quad (1)$$

or, in a compact form:

$$y(n+1) = f[\mathbf{y}(n); \mathbf{u}(n)] \quad (2)$$

where  $u(n) \in \mathbb{R}$  and  $y(n) \in \mathbb{R}$  denote, respectively, the input and output of the model at discrete time  $n$ , while  $d_u \geq 1$  and  $d_y \geq 1$ ,  $d_u \leq d_y$ , are the input-memory and output-memory orders. The vectors  $\mathbf{y}(n)$  and  $\mathbf{u}(n)$  denote the output and input regressors, respectively. Figure 1 shows the topology of an one-hidden-layer NARX network.

The function  $f(\cdot)$  is a (generally unknown) nonlinear function which should be approximated. When this is done by a multilayer Perceptron (MLP), the resulting topology is called a *NARX recurrent neural network* [6, 19]. This is a powerful class of dynamical models which has been shown to be computationally equivalent to Turing machines [22].



**Figure 1. A NARX network with  $d_u$  input and  $d_y$  output delays.**

The NARX network is trained basically under one out of two modes:

- **Series-Parallel (SP) Mode** - In this case, the output's regressor is formed only by actual values of the system's output:

$$\begin{aligned} \hat{y}(n+1) &= \hat{f}[\mathbf{y}_{sp}(n); \mathbf{u}(n)] \\ &= \hat{f}[y(n), \dots, y(n-d_y+1); \\ &\quad u(n), u(n-1), \dots, u(n-d_u+1)] \end{aligned} \quad (3)$$

- **Parallel (P) Mode** - In this case, estimated outputs are fed back and included in the output's regressor:

$$\begin{aligned} \hat{y}(n+1) &= \hat{f}[\mathbf{y}_p(n); \mathbf{u}(n)] \\ &= \hat{f}[\hat{y}(n), \dots, \hat{y}(n-d_y+1); \\ &\quad u(n), u(n-1), \dots, u(n-d_u+1)] \end{aligned} \quad (4)$$

It is worth noting that the feedback pathway shown in Figure 1 is present only in the Parallel Identification Mode. As a tool for nonlinear system identification, the NARX network has been successfully applied to a number of real-world input-output modelling problems, such as heat exchangers, waste water treatment plants, catalytic reforming systems in a petroleum refinery and nonlinear time series prediction.

Of particular interest for this paper is the issue of nonlinear time series prediction with the NARX network. In this type of application, the output-memory order is set  $d_y = 0$ , thus reducing the NARX network to a plain FTDNN archi-

ecture [18]:

$$\begin{aligned} y(n+1) &= f[\mathbf{u}(n)] \\ &= f[u(n), u(n-1), \dots, u(n-d_u+1)] \end{aligned} \quad (5)$$

where  $\mathbf{u}(n) \in \mathbb{R}^{d_u}$  is the input regressor. This simplified formulation of the NARX network eliminates a considerable portion of its representational capabilities as a recurrent network; that is, all the dynamic information that could be learned from the past memories of the output (feedback) path is discarded. For many practical applications, such as self-similar traffic modelling [11], the network must be able to robustly store information for a long period of time in the presence of noise.

It is worth emphasizing that the original formulation of the NARX network does not circumvent the problem of long-term dependencies, but it has only been demonstrated that it often performs much better than standard recurrent ANNs in such a class of problems, achieving much faster convergence and better generalization performance [17]. However, if the output memory is fully discarded as in Equation (5) these properties may no longer be observed. Considering this limited use of the potentialities of the NARX network, we propose a simple strategy to allow the computational abilities of the NARX network to be fully exploited in nonlinear time series prediction tasks.

### 3. Chaotic Time Series Prediction with NARX

The state of a deterministic dynamical system is the information necessary to determine the entire future evolution of the system. In discrete time, this evolution can be described by the following system of difference equations:

$$\mathbf{x}(n+1) = \mathbf{F}[\mathbf{x}(n)] \quad (6)$$

where  $\mathbf{x}(n) \in \mathbb{R}^d$  is the state of the system at time  $n$ , and  $\mathbf{F}[\cdot]$  is a nonlinear vector valued function. A time series is a set of measures  $\{x(n)\}$ ,  $n = 1, \dots, N$ , of a scalar quantity observed at the output of the system over time. This observable quantity is defined in terms of the state  $\mathbf{x}(n)$  of the underlying system as follows:

$$x(n) = h[\mathbf{x}(n)] + \varepsilon(t) \quad (7)$$

where  $h(\cdot)$  is a nonlinear scalar-valued function,  $\varepsilon$  is a random variable which accounts for modelling uncertainties and/or measurement noise. It is commonly assumed that  $\varepsilon(t)$  is drawn from a Gaussian white noise process. It can be inferred immediately from Equation (7) that the observations  $\{x(n)\}$  are seen as a projection of the multivariate state space of the system onto the one-dimensional space. Equations (6) and (7) describe together the state-space behavior of the dynamical system.

In order to perform prediction, one needs to reconstruct (estimate) as well as possible the state space of the system using the information provided by  $\{x(n)\}$ . Takens [23] has shown that, under very general conditions, the state of a deterministic dynamic system can be accurately reconstructed by a time window of finite length sliding over the observed time series as follows:

$$\mathbf{x}_1(n) \triangleq [x(n), x(n-\tau), \dots, x(n-(d_E-1)\tau)] \quad (8)$$

where  $x(n)$  is the value of the time series at time  $n$ ,  $d_E$  is the embedding dimension and  $\tau$  is the embedding delay. Equation (8) implements the delay embedding theorem. This theorem motivates the technique of using time-delay coordinate reconstruction in reproducing the phase space of an observed dynamical system; that is, a collection of time-lagged values in a  $d_E$ -dimensional vector space will provide sufficient information to reconstruct the states of the dynamical system. Thus, the purpose of time-delay embedding is to unfold the projection back to a multivariate state space that is representative of the original system.

The embedding theorem provides a sufficient condition for choosing the embedding dimension  $d_E$  large enough so that the projection is theoretically able to reconstruct the original state space. This theorem also provides a theoretical framework for nonlinear time series prediction, where the predictive relationship between the current state  $\mathbf{x}_1(t)$  and the next value of the time series is given by the following equation:

$$x(n+1) = g[\mathbf{x}_1(n)] \quad (9)$$

Once the embedding dimension  $d_E$  and delay  $\tau$  are chosen, one remaining task is to approximate the mapping function  $g(\cdot)$ . It has been shown that a feedforward neural network with enough neurons is capable of approximating any nonlinear function to an arbitrary degree of accuracy. Thus, it can provide a good approximation to the function  $g(\cdot)$  by implementing the following mapping:

$$\hat{x}(n+1) = \hat{g}[\mathbf{x}_1(n)] \quad (10)$$

where  $\hat{x}(n+1)$  is an estimate of  $x(n+1)$  and  $\hat{g}(\cdot)$  is the corresponding approximation of  $g(\cdot)$ . The estimation error,  $e(n+1) = x(n+1) - \hat{x}(n+1)$ , is commonly used to evaluate the quality of the approximation.

If we assume  $\mathbf{u}(n) = \mathbf{x}_1(n)$  and  $y(n+1) = x(n+1)$  in Equation (5), then it leads to an intuitive interpretation of the nonlinear state-space reconstruction procedure as equivalent to the time series prediction problem whose the goal is to compute an estimate of  $x(n+1)$ . Thus, the only thing we have to do is to train a FTDNN model [21]. Once training is completed, the FTDNN can be used for predicting the next samples of the time series.

Despite the correctness of the FTDNN approach, recall that it is derived from a simplified version of the NARX network by eliminating the output memory. In order to use the

full computational abilities of the NARX network for non-linear time series prediction, we propose novel definitions for its input and output regressors. Firstly, the input signal regressor, denoted by  $\mathbf{u}(n)$ , is defined by the delay embedding coordinates of Equation (8):

$$\begin{aligned}\mathbf{u}(n) &= \mathbf{x}_1(n) \\ &= [x(n), x(n - \tau), \dots, x(n - (d_E - 1)\tau)]\end{aligned}\quad (11)$$

where we set  $d_u = d_E$ . In words, the input signal regressor  $\mathbf{u}(n)$  is composed of  $d_E$  actual values of the observed time series, separated from each other of  $\tau$  time steps.

Secondly, since the NARX network can be trained in two different modes, the output signal regressor  $\mathbf{y}(n)$  can be written as follows:

$$\mathbf{y}_{sp}(n) = [x(n), \dots, x(n - d_y + 1)] \quad (12)$$

$$\mathbf{y}_p(n) = [\hat{x}(n), \dots, \hat{x}(n - d_y + 1)] \quad (13)$$

where the output regressor for the SP mode in Equation (12) contains  $d_y$  past values of the actual time series, while the output regressor the P mode in Equation (13) contains  $d_y$  past values of the estimated time series. For a suitably trained network, these outputs are estimates of previous values of  $x(n + 1)$ , and should obey the following predictive relationships implemented by the NARX network:

$$\hat{x}(n + 1) = \hat{f}[\mathbf{y}_{sp}(n), \mathbf{u}(n)] \quad (14)$$

$$\hat{x}(n + 1) = \hat{f}[\mathbf{y}_p(n), \mathbf{u}(n)] \quad (15)$$

where the nonlinear function  $\hat{f}(\cdot)$  be readily implemented through a MLP trained with backpropagation. The NARX networks trained according to Equations (14) and (15) are denoted onwards by NARX-SP and NARX-P networks, respectively.

Note that, unlike the FTDNN-based approach for the nonlinear time series prediction problem, the proposed approach makes full use of the output signal regressor  $\mathbf{y}_{sp}(n)$  (or  $\mathbf{y}_p(n)$ ). Equations (11) and (12) are valid only for one-step-ahead prediction tasks. If one is interested in multi-step-ahead or recursive prediction tasks, the estimates  $\hat{x}$  should also be inserted into the regressors in a recursive fashion.

The proposed approach is summarized as follows. A recurrent NARX network is defined so that its input regressor  $\mathbf{u}(n)$  contains samples of the measured variable  $x(n)$  separated  $\tau > 0$  time steps from each other, while the output regressor  $\mathbf{y}(n)$  contains actual or estimated values of the same variable, but sampled at consecutive time steps. As training proceeds, these estimates should become more and more similar to the actual values of the time series, indicating convergence of the training process. Thus, it is interesting to note that the input signal regressor supplies medium- to long-term information about the dynamical behavior of

the time series, since the delay  $\tau$  is always much larger than unity, while the output regressor, once the network has converged, supplies short-term information about the same time series.

## 4. Simulations

In this paper, we evaluate the NARX-P and NARX-SP models using the chaotic laser time series, a highly non-linear data set that has been widely used for benchmark studies [24]. This time series comprises measurements of the intensity pulsations of a single-mode Far-Infrared-Laser  $\text{NH}_3$  in a chaotic state [13]. It was made available worldwide during a time series prediction competition organized by the Santa Fe Institute and, since then, has been used in benchmark studies. This time series has 1500 sample points which have been rescaled to the range  $[-1, 1]$ . The rescaled time series was further split into two sets for the purpose of performing 1-fold cross-validation, so that the first 1000 samples were used for training and the remaining 500 samples for testing.

All the networks evaluated in this paper have two-hidden layers and one output neuron. All neurons in both hidden layers and the output neuron use the hyperbolic tangent activation function. The standard backpropagation algorithm is used to train the networks for 3000 epochs, with learning rate equal to 0.01. No momentum term is used. In what concerns the Elman network, only the neuronal outputs of the first hidden layer are fed back to the input layer.

The number of neurons,  $N_{h,1}$  and  $N_{h,2}$ , in the first and second hidden layers, respectively, are the chosen according to the following heuristics:

$$N_{h,1} = 2d_E + 1 \quad \text{and} \quad N_{h,2} = \sqrt{N_{h,1}} \quad (16)$$

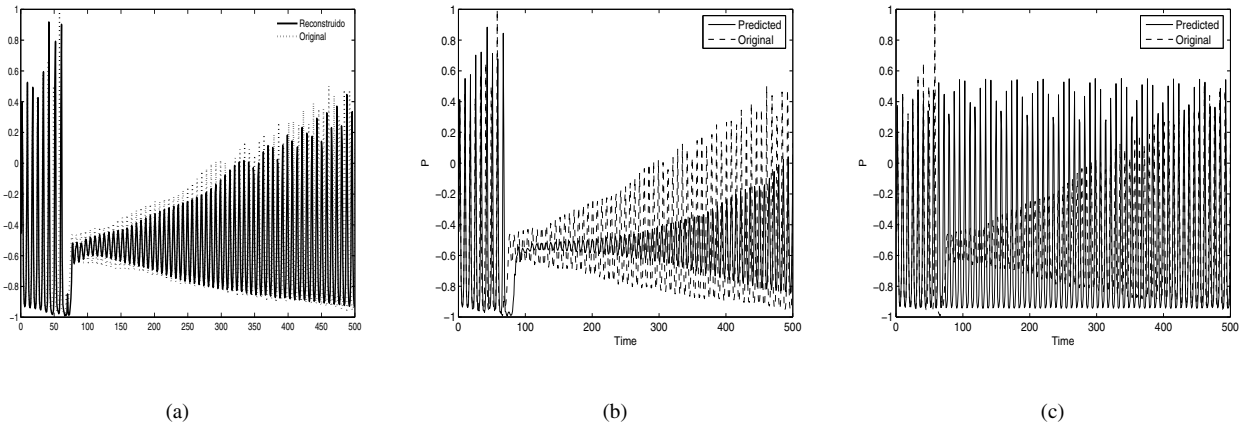
where  $N_{h,2}$  is rounded up towards the next integer number.

In this paper, we used Cao's method [5], which is a variant of the false neighbors' method, for estimating  $d_E = 7$ . A choice of  $\tau = 2$  that keeps the coordinates more uncorrelated in time is made based on the first minimum of the mutual information curve [10] of the time series. The number of neurons in each hidden layer is  $N_{h,1} = 15$  and  $N_{h,2} = 4$ , for all networks whose performances are being compared. The order of the output regressor in NARX-P and NARX-SP models is set to  $n_y = \tau d_E = 2 \times 7 = 14$ .

The networks are evaluated in terms of the *Normalized Mean Squared Error* (NMSE), defined as follows:

$$NMSE(N) = \frac{1}{N \cdot \sigma_x^2} \sum_{n=1}^N e^2(n) = \frac{\hat{\sigma}_e^2}{\hat{\sigma}_x^2} \quad (17)$$

where  $N$  is the horizon prediction (i.e., how many steps into the future a given network has to predict),  $\hat{\sigma}_x^2$  is the sample



**Figure 2. Recursive predictions for the laser time series: (a) NARX-SP, (b) Elman, and (c) FTDNN.**

variance of the actual time series, and  $\hat{\sigma}_e^2$  is the sample variance of the sequence of estimation errors. All the reported values of NMSE are mean values averaged over 10 training/testing runs.

The simulations aim to evaluate, in qualitative and quantitative terms, the predictive ability of all networks of interest. Once they have been trained, the networks are required to provide estimates of the future values of the laser time series for a certain prediction horizon  $N$ . The predictions are executed in a recursive fashion until desired prediction horizon is reached, i.e., during  $N$  time steps the predicted values are fed back in order to take part in the composition of the regressors. In this sense, the NMSE quantity in Equation (17) is better understood as a multi-step-ahead NMSE. For the NARX-SP network in particular, the predicted values, during multi-step ahead predictions, should be fed back to both the input regressor  $\mathbf{u}(n)$  and output regressor  $\mathbf{y}_{sp}(n)$ .

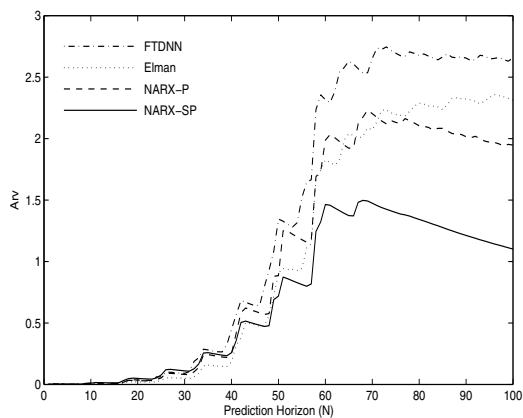
The results are shown in Figures 2(a), 2(b) and 2(c), for the NARX-SP, Elman and FTDNN networks, respectively. A visual inspection illustrates clearly that the NARX-SP model performed better than the other two neural architectures. It is important to point out that a critical situation occurs around time step 60, where the laser intensity collapses suddenly from its highest value to its lowest one; then, it starts recovering the intensity gradually. The NARX-SP model is able to emulate the laser dynamics very closely. The Elman's network was doing well until the critical point. From this point onwards, it was unable to emulate the laser dynamics faithfully, i.e., the predicted laser intensities have much lower amplitudes than the actual ones. The FTDNN network had a very poor predictive performance. From a dynamical point of view the output of the FTDNN seems to be stuck in a limit cycle, since it only oscillates endlessly.

It is worth mentioning that the previous results did not mean that the FTDNN and Elman networks cannot learn the dynamics of the chaotic laser. Indeed, it was shown to be possible in [12]. The results only shows that for the same small number of  $N_{h,1} + N_{h,2} + 1 = 20$  neurons, short length of the training time series, and number of training epochs, the NARX-P and NARX-SP networks perform better than the FTDNN and Elman networks; that is, the former architectures are computationally more powerful than the latter ones in capturing the nonlinear dynamics of the chaotic laser.

The multi-step-ahead predictive performances of all networks can be assessed in more quantitative terms by means of NMSE curves, that show the evolution of NMSE as a function of the prediction horizon  $N$ . Figure 3 shows the obtained results. It is worth emphasizing two types of behavior in this figure. Below the critical time step  $N = 60$ , the NMSE values reported are approximately the same, with a small advantage to the Elman network. This means that while the critical point is not reached, all networks predict well the time series. Above  $N = 60$ , the NARX-P and NARX-SP models reveal their superior performance.

## 5. Conclusions

In this paper, we proposed a strategy that allows the original architecture of the NARX network to fully explore its computational power to improve prediction performance. We used the well-known chaotic laser time series to evaluate the proposed approach in multi-step-ahead prediction tasks. The results have shown that the proposed approach consistently outperforms standard neural network based predictors, such as the FTDNN and Elman architectures. Currently, we are investigating the performances of



**Figure 3. Multi-step-ahead NMSE curves.**

the NARX-P and NARX-SP approaches in other time series prediction applications, such as self-similar traffic prediction and electric load prediction.

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