Random Element Quantization for Finite Rate Feedback Systems

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***Abstract*—Finite rate feedback model based on Random Vector Quantization (RVQ) is efficient in terms of number of feedback bits. However, the required scaling of feedback bits (codebook size) to achieve full multiplexing gain becomes extremely large with high Signal-to-Noise Ratio (SNR) and *might* not be practical. This paper considers the use of a Random Element Quantization (REQ) approach, which quantizes the channel vector element-wise. A simple statistics is calculated and the presented simulation results validate our calculations. The main primary result is that, the required codebook size per receiver to achieve full multiplexing gain is kept very small, which would avoid high computational complexity and memory usage.**

***Index Terms*—limited feedback, random vector quantization, random element quantization, zero forcing.**

I. Introduction

Multiple Input Multiple Output (MIMO) transmission is considered one of the key features of current and future wireless networks to increase multiplexing gain (Capacity) and/or diversity. However, to increase multiplexing gain, the system needs to simultaneously transmit data to multiple receivers using, for instance, beamforming techniques.

The main problem in the practical implementation of beamforming techniques is that the transmitter needs to know the full Channel State Information (CSI) to be able to design the precoders. Several approaches have been proposed to obtain CSI at transmitter [1]. The simplest way is to introduce a finite feedback channel between transmitter and receivers [2]. In such a way, each receiver quantizes its channel vector to a finite number of bits and then feeds back these bits to the transmitter. It was shown that, in contrast to point-to-point systems, the level of CSI available to the transmitter affects the multiplexing gain of the MIMO interference/downlink channel [3]. Therefore, the required rate of feedback is clearly an important quantity in interference/downlink channel system models.

RVQ is the most considered quantization approach for the finite rate feedback model, see [1] and references therein. In this paper, on the other hand, we focus on a Random Element Quantization (REQ) approach, which quantizes the channel vector element-wise. Mathematical expressions of quantization error and required scaling of feedback bits are derived for REQ and validated through simulation. Furthermore it is shown that it achieves full multiplexing gain even for small codebook sizes.

The rest of this work is organized as follows. In section II we describe the system model. In Section III we review the main results of the RVQ approach from [3]. The REQ approach is presented in Section IV. Numerical results are shown in Section V. Finally, conclusions are drawn in Section VI.

Notation: Upper/lower boldface letters are used for matrices/vectors and $A^\dagger$ refers to the Hermitian of $\mathbf{A}$. The notation $\|\mathbf{x}\|$ refers to the Euclidean norm of vector $\mathbf{x}$, and $\angle(\mathbf{x},\mathbf{y})$ refers to the angle between vectors $\mathbf{x}$ and $\mathbf{y}$ with the standard convention $|\cos(\angle(\mathbf{x},\mathbf{y}))| = |\mathbf{x}^\dagger\mathbf{y}|/(\|\mathbf{x}\|\cdot\|\mathbf{y}\|)$.

II. System Model

Consider the downlink channel model shown in Fig. 1, in which there is a single transmitter with M transmit antennas and K single-antenna receivers.

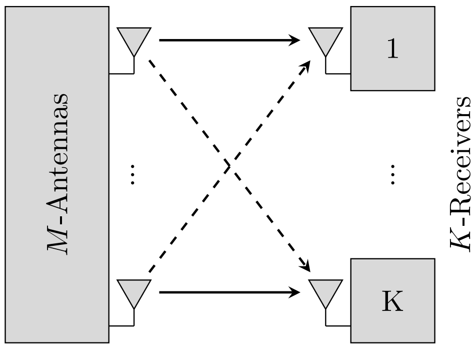

Fig. 1. M-K Downlink Channel Model.

To allow simultaneous transmission to the K receivers, we use the well-known Zero-Forcing (ZF) technique to design the beamforming vectors at the transmitter. In ZF, the beamforming vector $\mathbf{v}_i$ of receiver i is simply chosen to be the i-th normalized column of the inverse of the concatenated channel matrix of all receivers $\mathbf{H}$. Thus, $\mathbf{v}_i$ is orthogonal to all other channel vectors, i.e, $\mathbf{h}_i^\dagger\mathbf{v}_j = 0, \forall i \neq j$. Using ZF, the received signal at receiver i is given by:

$$y_i = \sum_{j=1}^{K} \mathbf{h}_i^\dagger \mathbf{v}_j s_j + n_i, \tag{1}$$

where $\mathbf{h}_i \in \mathbb{C}^{M\times 1}$ is the channel vector of i-th receiver, s_j is the scalar symbol intended for the j-th receiver and n_i is

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the noise term, that is assumed to be independent complex Gaussian with unit variance. The Signal to Interference-plus-Noise Ratio (SINR) at the i-th receiver is given by

$$\gamma_i = \frac{\frac{P}{M}|\mathbf{h}_i^\dagger \mathbf{v}_i|^2}{1+\sum_{j\neq i}^{K}\frac{P}{M}|\mathbf{h}_i^\dagger \mathbf{v}_j|^2}, \qquad (2)$$

where P is total transmit power. The channel is assumed to be block fading, with independent fading from block to block. The entries of the channel vectors are distributed as i.i.d unit variance complex Gaussian. Using finite rate feedback model, each receiver quantizes its channel to B bits and feeds back the bits to the transmitter (assumed error-free and delay-zero). The quantization is performed using a codebook that is known at the transmitter and the receivers. Each receiver is assumed to use a different and independently generated codebook to avoid non-zero probability that multiple receivers return the same quantization vector.

With ZF, if the transmitter has perfect channel knowledge (denoted as CSI_p), all interference will be equal to 0, thus (1) is reduced to $y_i = \mathbf{h}_i^\dagger \mathbf{v}_i s_i + n_i$ and (2) is reduced to $\frac{P}{M}|\mathbf{h}_i^\dagger \mathbf{v}_i|^2$. On the other hand, if the transmitter has a quantized version of channel knowledge (denoted as CSI_q), some interference will still exist and its power mainly depends on the accuracy of quantization.

III. Review of RVQ Important Results

In [3], which uses the same system model as above, the authors studied the RVQ approach. Next, some of its main results are reviewed, which will be useful in later derivations.

In RVQ, the codebook $\mathcal{C}$ has 2^B codewords independently chosen from the isotropic distribution on the M-dimensional unit sphere. Each receiver quantizes its channel direction $\mathbf{h}_i^* = \frac{\mathbf{h}_i}{\|\mathbf{h}_i\|}$ to the quantization vector $\hat{\mathbf{h}}_j^*$ that is closest to its channel vector, where closeness is measured in terms of the angle between two vectors. Mathematically, receiver i computes quantization index j according to:

$$j = \arg\min(\sin^2(\angle(\mathbf{h}_i^*, \hat{\mathbf{h}}_j^*))), \quad j = 1, \ldots, 2^B, \qquad (3)$$

and feeds back index j to the transmitter.

A. Expectations of Quantization Error

From a statistical point of view, the quantization error $Z = \sin^2(\angle(\mathbf{h}_i^*, \hat{\mathbf{h}}_i^*))$ is the minimum of 2^B independent beta $(M-1, 1)$ random variables. The expectation of this quantity has been computed in closed form as [4]

$$\mathbb{E}[Z] = 2^B \cdot \beta\left(2^B, \frac{M}{M-1}\right),$$

where $\beta(\cdot)$ denotes the beta function. This can also be upper bounded as [3, lemma 1]

$$\mathbb{E}[Z] < 2^{\frac{-B}{M-1}}. \qquad (4)$$

B. Performance Offset

If a system with CSI_q maintains a bounded rate offset, we say that the full multiplexing gain is achieved. Define the rate Offset $\Delta R(P)$ to be the difference between the per receiver sum-rate achieved by CSI_p and CSI_q, assuming ZF beamforming, i.e,

$$\Delta R(P) = \frac{1}{K}\left[R_{\text{CSI}_p} - R_{\text{CSI}_q}\right]. \qquad (5)$$

where R_κ refers to the system sum-rate achieved using κ. It was shown by [3, Theorem 1] that $\Delta R(P)$ can be upper bounded by

$$\Delta R(P) < \log_2(1 + P\cdot\mathbb{E}[Z]) < \log_2\left(1 + P\cdot 2^{\frac{-B}{M-1}}\right). \qquad (6)$$

Therefore, in order to maintain a rate offset no larger than $\log_2(b)$ [1] (per user), i.e.,

$$\Delta R(P) < \log_2\left(1 + P\cdot 2^{\frac{-B}{M-1}}\right) = \log_2(b), \qquad (7)$$

it was shown by [3, Theorem 3] that it is sufficient to scale the number of feedback bits per receiver (by inverting (7) and solving for B as a function of b and P) according to:

$$B \approx \frac{(M-1)}{3}P_{dB} - (M-1)\log_2(b-1). \qquad (8)$$

With this scaling of feedback bits B, we have $\Delta R(P) \leq \log_2(b)$ for all P, as desired. Note that a 1-dB offset corresponds to $b = 10^{1/10} = 1.259$.

IV. Random Element Quantization Approach

A. Motivations

Obviously, there is a trade-off between number of feedback bits and system performance. Using the RVQ approach, the number of feedback bits B should be scaled linearly according to (8). For practical use, the codebook size should be as small as possible to avoid high computational complexity (minimize delay) and memory usage. Table I shows some calculations using (8) assuming M=5.

TABLE I
Feedback Bits B for varying SNR and offset b [M=5].

	SNR (dB)	B	Codebook Size
b = 1.259	5	$14.46 \approx 15$	$2^{15} = 32768$
	25	$41.13 \approx 42$	$2^{42} = 4.39 \times 10^{12}$

As we can see from Table I, for high SNR values (P_{dB}), the required codebook size becomes extremely large and *might* not be practical.

[1] The rate offset of $\log_2(b)$ (per user) can be translated into a power offset, which is a more useful metric from the design perspective [3].

B. Random Element Quantization (REQ) Scheme

Another methodology to quantize the channel vector is by quantizing it one element at a time, such as the scalar approach briefly mentioned in [5]. In this paper we investigate it further, referring to it as Random Element Quantization (REQ). In REQ, the receiver i quantizes its channel vector $\mathbf{h}_i = [h_1 \quad h_2 \quad \dots \quad h_M]$ *element-wise* using a codebook $\mathcal{C}$ that has 2^B 1-dimensional complex numbers (codewords). The channel coefficients $h_m \in \mathbb{C}$ and codewords $w_j \in \mathbb{C}$ are distributed as i.i.d unit variance complex Gaussian random variables (Rayleigh fading). Each element of $\mathbf{h}_i$ will be separately quantized from codebook $\mathcal{C}$ with codeword w_j that attains minimum euclidean distance. The euclidean distance between two complex numbers, h_1 and h_2, is given by

$$d_r(h_1, h_2) = \|h_1 - h_2\| \tag{9}$$

Therefore, in REQ, receiver i calculates the euclidean distance d_r as given by (9) for each channel element h_m with all 2^B codewords, then selects the index j that indicates the codeword w_j that gives the minimum value. Mathematically, this can be written as,

$$j = \arg\min\{d_r(h_{i,m}, w_j)\}, \quad j = 1, \dots, 2^B, \tag{10}$$

where $h_{i,m}$ refers to element m of channel vector i. The quantized channel element is then $\hat{h}_{i,m} = w_j$. Algorithm 1 summarizes the REQ quantization scheme.

Algorithm 1 Random Element Quantization (REQ)

for $i = 1$ to K **do**
  Estimate $\mathbf{h}_i = [h_1 \quad h_2 \quad \dots \quad h_M]$ (perfectly).
  for $m = 1$ to M **do**
    1) Calculate $d_r(h_m, w_j), j = 1, .., 2^B$ using (9).
    2) Find codeword w_j that gives min d_r as (10).
    3) Receiver i feeds back index j to transmitter.
    4) Transmitter uses fed back index j to look-up codeword w_j from same codebook $\mathcal{C}$, then estimates $\hat{h}_m$ with receiver i as $\hat{h}_m = w_j$.
  end for
end for

C. REQ Important Statistics

Quantizing channel vector $\mathbf{h}_i$ using Algorithm 1, the quantized channel vector $\hat{\mathbf{h}}_i$ has an expected *equal* element-wise quantization error. An important quantity is the quantization error between two vectors direction, i.e.,

$$Z = \sin^2(\angle(\mathbf{h}_i^*, \hat{\mathbf{h}}_i^*)) \tag{11}$$

It can be easily understood that $\hat{\mathbf{h}}_i$ is inside the M-dimensional sphere centered at $\mathbf{h}_i$ with radius equal to the element-wise error.

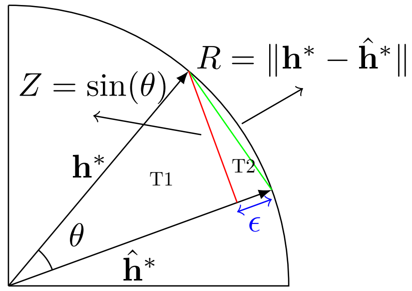

Fig. 2. Quantization Error: $\sin(\angle(\mathbf{h}_i^*, \hat{\mathbf{h}}_i^*))$ Geometry.

1) Expectation of REQ Quantization Error: Refer to Fig. 2, where we have:

$$\sin(\theta) = \sin(\angle(\mathbf{h}_i^*, \hat{\mathbf{h}}_i^*)) = \frac{Z}{\|\mathbf{h}_i^*\|} = Z, \tag{12}$$

since $\mathbf{h}_i^*$ and $\hat{\mathbf{h}}_i^*$ are normalized vectors and z is the projection of $\mathbf{h}_i^*$ on $\hat{\mathbf{h}}_i^*$ (same as projection of $\hat{\mathbf{h}}_i^*$ on $\mathbf{h}_i^*$). Using the well-known Pythagoras algorithm on triangle T_2 from the same figure, we can see that,

$$R^2 = Z^2 + \epsilon^2 \Rightarrow Z^2 = R^2 - \epsilon^2. \tag{13}$$

Therefore, the quantization error can be written as

$$Z^2 = \sin^2(\angle(\mathbf{h}_i^*, \hat{\mathbf{h}}_i^*)) = R^2 - \epsilon^2. \tag{14}$$

where $R = \|\mathbf{h}_i^* - \hat{\mathbf{h}}_i^*\|$ is equal to the sphere radius (error) which is the minimum of 2^B independent Rayleigh random variables. Without loss of generality, the Cumulative Distribution Function (CDF) of the minimum value of R is given by Theorem 1.

Theorem 1: Let $X_1, \dots, X_n$ be independent random variables, all with the same distribution. Let $R = \min\{X_1, \dots, X_n\}$.

Then, the CDF of R is given by

$$F_R(r) = \mathbb{P}(R \le r) = 1 - [1 - F_X(r)]^n. \tag{15}$$

Proof 1: Since all X_n have the same distribution, then

$$\mathbb{P}(X_n > r) = \mathbb{P}(X > r), \forall n = 1, 2, \dots, n.$$

Since

$$\begin{aligned}\{R \le r\} &= 1 - \{\min\{X_1, \dots, X_n\} > r\} \\ &= 1 - \left\{\prod_{i=1}^{n}\{X_i > r\}\right\},\end{aligned}$$

we have,

$$\mathbb{P}(R \le r) = 1 - \mathbb{P}\left(\left\{\prod_{i=1}^{n}\{X_i > r\}\right\}\right).$$

Since all X_n are independent

$$\begin{aligned}\mathbb{P}(R \le r) &= 1 - \mathbb{P}\{X_1 > r\} \dots \mathbb{P}\{X_n > r\} \\ &= 1 - [\mathbb{P}\{X > r\}]^n \\ &= 1 - [1 - \mathbb{P}\{X \le r\}]^n \\ F_R(r) &= 1 - [1 - F_X(r)]^n\end{aligned}$$

□

The CDF of the Rayleigh distribution is given by [6]

$$F_X(x) = 1 - e^{-x^2/2\sigma^2}. \tag{16}$$

Using the result of Theorem 1, substituting (16) in (15), the CDF of the minimum value is given by

$$F_R(r) = 1 - e^{-r^2 2^B/2\sigma_r^2}. \tag{17}$$

The Probability Density Function (PDF) of R is just the derivative of (17) and it is given by

$$f_R(r) = \frac{2^B r}{\sigma_r^2} e^{-r^2 2^B/2\sigma_r^2}. \tag{18}$$

Therefore, the expectation of minimum value R is

$$\mathbb{E}\left[R\right] = \int_0^\infty r f_R(r) dr = \frac{1.25(\sigma_r^2)^{0.5}}{(2^B)^{0.5}} = 0.82 \cdot 2^{-B/2}, \tag{19}$$

where we used the fact that the variance of Rayleigh-distributed random variable is calculated as $\sigma_r^2 = \frac{4-\pi}{2}\sigma^2 = 0.43$, and $\sigma^2 = 1$ is the variance of codewords that are i.i.d Gaussian random variables with mean equal to 0 [6]. Therefore, by substituting R by the expectation of its minimum value in (14), the quantization error is given by:

$$\begin{aligned} Z^2 = \sin^2(\angle(\mathbf{h}_i^*, \hat{\mathbf{h}}_i^*)) &= \mathbb{E}\left[R\right]^2 - \epsilon^2 \\ &= (0.82 \cdot 2^{-B/2})^2 - \epsilon^2 \\ &= 0.67 \cdot 2^{-B} - \epsilon^2. \end{aligned} \tag{20}$$

Note that the ϵ value becomes very small with sufficiently large codebook. Neglecting ϵ, the quantization error can be upper bounded as

$$Z^2 = \sin^2(\angle(\mathbf{h}_i^*, \hat{\mathbf{h}}_i^*)) < 0.67 \cdot 2^{-B}. \tag{21}$$

The most important feature to notice from (21) is that the quantization error is not a function of the number of transmit antennas M.

2) Scaling Feedback Bits: Following the same approach given by (6), a constant rate offset $\Delta R(P)$ can be upper bounded by

$$\Delta R(P) < \log_2\left(1 + P \cdot 0.67 \cdot 2^{-B}\right). \tag{22}$$

Hence, in order to characterize a sufficient scaling of feedback bits, we set the rate gap upper bound in (22) to be equal to the maximum allowable gap of $\log_2 b$, i.e,

$$\Delta R(P) < \log_2\left(1 + P \cdot 0.67 \cdot 2^{-B}\right) = \log_2 b. \tag{23}$$

By inverting the expression in (23) and solving for B as a function of b and P we get

$$B \approx \frac{P_{dB}}{3} + \log_2(0.67) - \log_2(b-1). \tag{24}$$

where B here is the per element feedback bits required, thus total feedback bits per channel vector is B_{tot} = MB. With this scaling of feedback bits B, we have $\Delta R(P) \leq \log_2(b)$ for all P, as desired. A 1-dB offset corresponds to $b = 10^{1/10} = 1.259$.

V. Numerical Results

In this section we compare REQ and RVQ approaches in terms of quantization error, required scaling of feedback bits, sum-rate and Bit Error Rate (BER) performance.

A. Quantization Error

Fig. 3 shows the results for the quantization error comparing both approaches using two systems, M=K=5 and M=K=10.

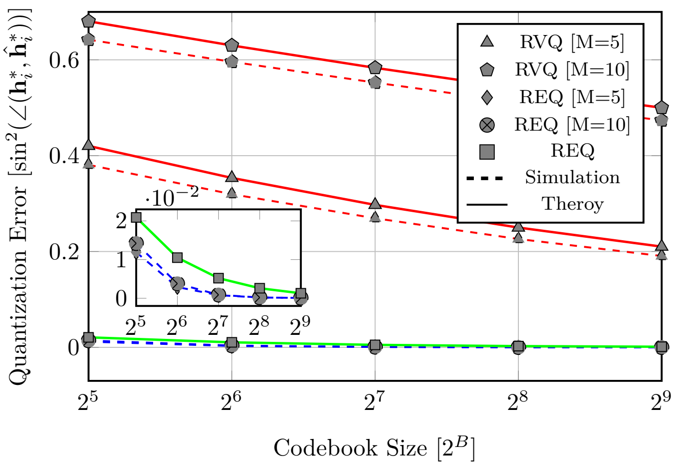

Fig. 3. Quantization Error: REQ vs RVQ.

From Fig. 3, we can see that REQ achieves much less quantization error. By quantizing the channel vector $\mathbf{h}_i$ element-wise, the resulting channel vector $\hat{\mathbf{h}}_i$ becomes more accurate, which reduces the quantization error. Note that, in contrast to RVQ, REQ is not a function of M.

B. Required Scaling of Feedback Bits

In this section we show some calculations using (8) and (24) to compare between required scaling of feedback bits of RVQ and REQ, respectively. We considered a scenario with M=K=5 and 1-dB offset. Table II show the results.

TABLE II
Scaling Feedback Bits for REQ and RVQ with Corresponding Quantization Error [M=5, 1-dB].

SNR(dB)	5	10	15	20
B [RVQ]	14.4642	21.1309	27.7976	34.4642
B [REQ/Element]	3.0383	4.7050	6.3716	8.0383
B [REQ-Total]	15.1915	23.5250	31.8580	40.1915
$\sin^2(\angle(\mathbf{h}_i^*, \hat{\mathbf{h}}_i^*))$ [RVQ]	0.0816	0.0257	0.0081	0.0025
$\sin^2(\angle(\mathbf{h}_i^*, \hat{\mathbf{h}}_i^*))$ [REQ]	0.0816	0.0257	0.0081	0.0025

As we can see in Table II, the total feedback bits required using REQ ($M \times B$/element), is a little larger than corresponding bits using RVQ (could be smaller for other parameter values, e.g, M=5, b=3, SNR<10). Notice that both schemes achieved exactly the same quantization error ($\sin^2(\angle(\mathbf{h}_i^*, \hat{\mathbf{h}}_i^*))$) required to maintain the full multiplexing gain ($M = 5$) and desirable constant rate offset (1-dB). Moreover, it can be easily verified that, even if the scenario parameters change (b and M), both approaches still maintain the exact quantization error.

C. Sum-Rate and BER Performance

For the sake of a fair comparison between the RVQ and REQ approaches in terms of number of feedback bits, we consider a scenario such that REQ has the same total number of feedback bits as RVQ. Fig. 4 shows sum-rate results where we assume M=K=3 and REQ uses B=5 bits per element, thus 15 bits in total.

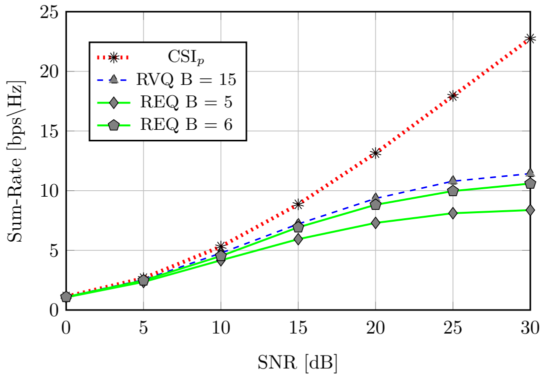

Fig. 4. Sum-Rate: REQ vs RVQ [M=K=3].

We can see from Fig. 4 that the REQ approach with 5 bits achieves lower sum-rates compared to RVQ at high SNR values. This was expected since REQ scaling, see Table II, requires a little more feedback bits. At the same figure, we included results with REQ using B=6 bits (total of 18 bits), where we can see that the sum-rates are improved and got closer to RVQ sum-rates. The corresponding BER calculations are shown in Fig. 5, where we can see that RVQ achieves better BER performance than REQ, for the considered feedback bits.

It should be noticed that, even though REQ requires a slightly higher amount of total feedback bits to achieve the same performance as RVQ, it requires significantly smaller codebook sizes (2^6 versus 2^{15}).

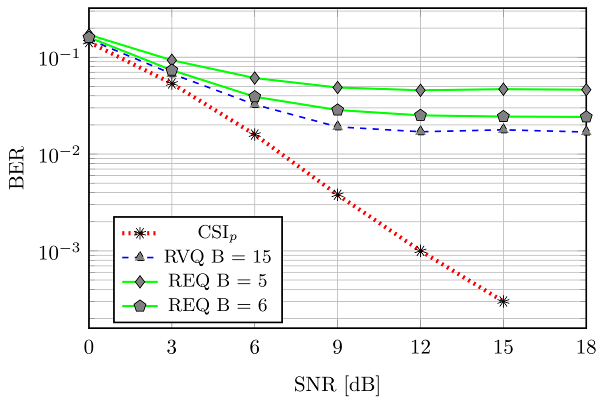

Fig. 5. Bit Error Rate [M=K=3].

The number of feedback bits considered in Fig. 4 are assumed to be fair, at least with the case of REQ B=5. However, to achieve full multiplexing gain, the feedback bits should be scaled linearly with the SNR according to (24) for REQ and (8) for RVQ. The scaling of (8) for RVQ has been shown at [3, Fig. 5]. In the following, Fig. 6 shows scaling of (24) for REQ assuming the same scenario as above, i.e., M=K=3.

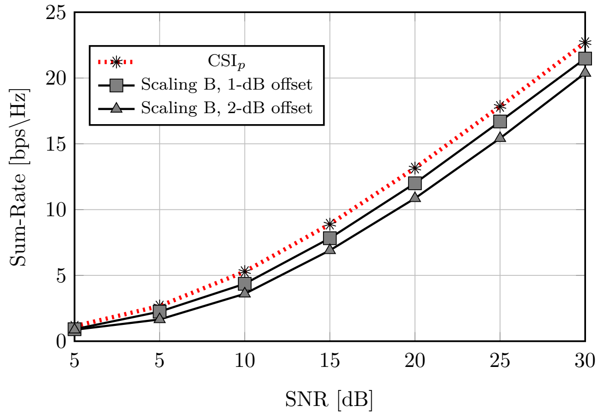

Fig. 6. Scaling Feedback Bits for REQ [M=K=3].

As we can see from Fig. 6, the projected error offset is in fact achieved and the scaling of REQ feedback bits according to (24) achieved full multiplexing gain with the required rate offsets.

VI. Conclusions

In this work, the Random Element Quantization (REQ) is analyzed, for which mathematical expressions of quantization error and required scaling of feedback bits are derived and validated through simulation. The REQ approach is compared to RVQ, whose codebook size becomes extremely large with high SNR values. A detailed analysis is presented in terms of quantization error, sum-rates, BER and required scaling of feedback bits. The main primary conclusion is that, even though the total feedback bits with the REQ approach are a little larger than with RVQ, the codebook size per receiver is kept very small, which would avoid the high computational complexity and memory usage.

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