Power Allocation Schemes for Multichannel Two-hop Relaying Systems

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***Abstract*—The usage of relays and OFDM can improve the performance of wireless systems in terms of data rates, coverage and reliability. In this paper we study joint subcarrier matching and power allocation for two-hop relay systems with the purpose of maximizing the total spectral efficiency. The problem is formulated as a mixed integer (binary) problem, but due to its complexity, the problem is separated into subcarrier matching and power allocation subproblems. Methods to solve both problems have been presented in previous works, but the solution to the power allocation problem still presented a high complexity. In this paper we demonstrate that the power allocation problem can be reformulated into a more tractable form, allowing us to develop suboptimal solutions based on water-filling with low computational cost. Numerical results show that the proposed suboptimal solutions are near-optimal and offer a good trade-off between performance and complexity.**

I. Introduction

One of the main drivers for the 4$^{\text{th}}$ Generation (4G) of wireless systems is the flexible and reliable provision of high data rates, which motivated the introduction of Multiple Input Multiple Output (MIMO) and Orthogonal Frequency Division Multiplexing (OFDM) as key transmission technologies of 4G system candidates, such as 3$^{\text{rd}}$ Generation Partnership Project (3GPP) Long Term Evolution Advanced (LTE-A) [1].

In spite of the rate improvements brought by MIMO-OFDM transmission techniques, these systems still suffer from reduced data rates at the cell edge where signal quality becomes lower due to attenuation and/or interference. In this context, the use of relays to reinforce the signal of a source and forward it to the receiver appears as promising solution to improve signal quality, especially at the cell edge [2].

In fact, the potential of relays to improve signal levels at the cell edge has already been focus of prior studies [3], and different protocols for this type of communication have been developped and evaluated, such as the Amplify-and-Forward (AF) and Decode-and-Forward (DF) protocols [2].

Considering the utilization of relays in a 4G OFDM-based system in which a Base Station (BS) sends data through a Resource Scheduler (RS) that forwards them to a Subscriber Station (SS), one important issue is how to ideally match the subcarriers of the BS-RS link, or hop, to those of the RS-SS link [4]. Indeed, in such two-hop scenarios the capacity (and the spectral efficiency, as well) of a given link binding a subcarrier of the BS-RS hop to another of the RS-SS hop is limited by the channel of worst hop. For example, suppose that a subcarrier experiences deep fading over the BS-RS channel. Then, even if the subcarrier in the RS-SS link has very good channel condition, the overall capacity of the BS-SS link will be limited by that of the BS-RS channel.

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Furthermore, the total rate achieved in this type of multichannel two-hop scenario also depends on how the power available at the BS and at the RS is distributed among subcarriers on each hop. Therefore, in order to maximize the overall capacity of BS-SS, it is important to optimally match the subcarriers across the two hops, as well as to optimize the power allocation at each and across the hops.

The joint subcarrier matching and power allocation in two-hop relay systems is the problem studied herein. This problem is a mixed integer (binary) problem, which is NP-hard and difficult to solve analytically and computationally [5], [6].

The problem of subcarrier matching in an AF relaying system was presented in [7, Hottinen et al.] with its optimality proved in [4]. One of the first works involving subcarrier matching and power allocation in an OFDM relay is [8], where suboptimal subcarrier matching is performed by applying the Hungarian algorithm [9] while the optimal power allocation is obtained by water-filling [10]. A joint subcarrier matching and power allocation problem was studied in [5], in which the authors provided the optimal subcarrier matching, but have made the unrealistic assumption of having a single total power constraint for all hops when performing the optimal power allocation via water-filling. Another joint subcarrier matching and power allocation problem was proposed in [11], which demonstrated an optimal subcarrier matching, as in [5], and used a subgradient method to find the optimum solution for the power allocation. The algorithm in [11] has the disadvantage of relying on an iterative numerical optimization method involving complex operations and depending on finding two suitable step sizes in order to ensure convergence. In [12], the problem is similar but it uses an algorithm called cap-limited water-filling, which has separated constraints for all hops and individual constraints for the power of each subchannel.

A way to turn the joint subcarrier matching and power allocation for OFDM-based two-hop systems into an easier

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problem is to separate the subcarrier matching and power allocation, solve them optimally and then rejoin them afterwards. Following this approach, we simplified the problem and developed near optimal solutions based on water-filling methods which have low computational costs. The main contributions of this work are:

- We study the joint subcarrier matching and power allocation problem in a two-hop multichannel system considering different power constraints at each hop in contrast to previous works that considered a single power constraint at the BS-RS and RS-SS hops. Moreover, we also state a relation which simplifies this problem and allows us to devise an approach with better computational costs.
- We propose near-optimal power allocation algorithms with low computational cost that apply to the joint subcarrier matching and power allocation problem considered here.

The remainder of this paper is organized as follows. In Section II we present the system model and an overview of the problem, establishing the joint subcarrier matching and power allocation problem. In Section III we state a useful relation between the power of both sides, BS-RS and RS-SS, which will help us in restating the optimization problem with a set of variables of only one node. The two suboptimal algorithms are proposed in Section IV. Section V presents numerical results and discusses the obtained results. Lastly, conclusions are drawn in Section VI.

II. System Model and Problem Overview

A. System Model

A two-hop multichannel OFDM environment is considered, which consists of a BS, a fixed RS and one SS, as illustrated in Fig. 1. The RS operation consists in the allocation of subcarriers to the BS-RS and RS-SS links. It is assumed that every allocated subcarrier is received by the SS. In Fig. 1, π_{ij} are binary variables indicating whether the bits transmitted from subcarrier i at the BS-RS hop are mapped to subcarrier j at the RS-SS hop.

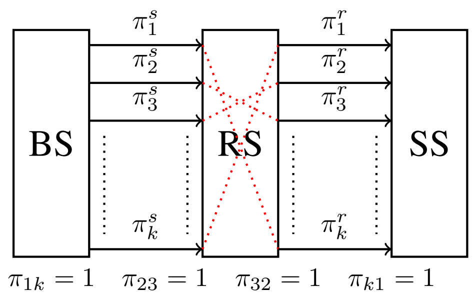

Figure 1. Schematic of a two-hop OFDM environment.

The relaying strategy used is the DF, in which the RS decodes the signal and simply forwards the signal to the SS. The OFDM cyclic prefix and the channel coherence time are considered sufficiently long and all nodes are assumed to have perfect frequency and time synchronization. Channel state information is assumed to be available at the RS node. Additive White Gaussian Noise (AWGN) is assumed, with a same noise variance σ^2 being perceived by all subcarriers and hops. Each subcarrier and each hop experiences independent fading. Indicating the subcarrier index by i and the hops by the superscript s, for the BS-RS hop, and r, for the RS-SS hop, the spectral efficiency of each subcarrier in the BS-RS hop is given by

$$R_i^s(P_i^s) = \frac{1}{2N}\log_2\left(1 + P_i^s g_i^s\right), \tag{1}$$

where P_i^s is the power allocated to subcarrier i, N is the number of subcarriers and g_i^s is the ratio between channel gain h_i^s and noise power σ^2. The same equation holds for the RS-SS hop, with the superscript s replaced by r, and all the involved elements related to the RS-SS link. Once the matching is known, we can define a spectral efficiency for the BS-SS link as the matched capacity of all hops

$$R_{i,j} = \min\{R_i^s(P_i^s), R_j^r(P_j^r)\}. \tag{2}$$

B. Problem Overview

Throughout this paper, one-to-one subcarrier communication is considered, which means bits of one subcarrier can not be spread to multiple subcarriers in the same hop. The joint subcarrier matching and power allocation optimization problem can be written as:

$$\min_{P_i^s, P_i^r, \pi_{ij}} -\sum_{i=1}^{N}\min\left\{R_i^s(P_i^s), \sum_{j=1}^{N}\pi_{ij}R_i^r(P_i^r)\right\}, \tag{3a}$$

$$\text{s.t.} \sum_{i=1}^{N} P_i^s \le P_t^s, \tag{3b}$$

$$\sum_{i=1}^{N} P_i^r \le P_t^r, \tag{3c}$$

$$-P_i^s \le 0, \forall i, \quad -P_j^r \le 0, \forall j, \tag{3d}$$

$$\sum_{i,j}^{N} \pi_{ij} = 1, \forall \quad 1 \le i, j \le N, \tag{3e}$$

$$\pi_{i,j} \in \{0, 1\}, \tag{3f}$$

where P_t^s, P_t^r are power constraints for the BS-RS and RS-SS links. Differently from [5], which assumes a total power constraint for both links, we consider a more realistic scenario where each hop has its own power constraint.

The joint subcarrier matching and power allocation problem in (3) can be viewed as a mixed binary integer optimization problem as in [5], [13]. The optimization variables are the power allocated to all subcarriers and hops, P_i^s and P_i^r, and the $N \times N$ binary matching matrix, $\boldsymbol{\pi}$. By separating the subcarrier matching and power allocation problems we aim at the simplification of the power allocation solution since it becomes a convex problem, as it is shown in the next section.

III. Power allocation problem for two hops

The optimal subcarrier matching is proved in [11], which is to sort channel gains in descending order and match them. Given this subcarrier matching, the power allocation can be studied alone, and using its convexity, it can be attacked by Lagrangian methods [14], such as water-filling [10].

The model above can be simplified based on results that will be presented in this section. First, an explicit solution for the matching is used, which eliminates the binary variable from the problem. Second, a linear relation between P_i^s and P_i^r is established for each $i = 1, 2, \ldots, N$, that allows us to rewrite the problem as a function of variables from the BS-RS link only. Notice that, without loss of generality, the constant $1/2N$ was removed from each expression and the logarithm basis was changed.

With the subcarrier matching problem solved, the problem established in (3) can be simplified into another one without subcarrier matching and the binary variable π:

$$\min_{P_i^s, P_i^r} -\sum_{i=1}^{N} \min\left\{R_i^s(P_i^s), R_i^r(P_i^r)\right\}, \tag{4a}$$

$$\text{s.t.} \sum_{i=1}^{N} P_i^s \le P_t^s, \tag{4b}$$

$$\sum_{i=1}^{N} P_i^r \le P_t^r, \tag{4c}$$

$$-P_i^s \le 0, \forall i, \quad -P_j^r \le 0, \forall j, \tag{4d}$$

where the problem above is convex and can be solved using existing standard tools, such as the interior-point method [14]. In [11], [13], this same problem has been found and solved using the interior point and subgradient methods. Differently from these works, our objective here is to devise a simplified version of this problem. With the following Lemma, a linear relation between BS-RS and RS-SS power can be stated and can then be used to decrease the complexity of problem (4).

Lemma 1: There is one optimal solution (π^*, P^{s*}, P^{r*}) for (3) such that, for all $i = 1, 2, \ldots, N$:

i) $\pi_{ii}^* = 1$;
ii) $P_i^{r*} = \frac{g_i^s}{g_i^r} P_i^{s*}$ and, consequently, $R_{ii}^* = R_i^{s*} = R_i^{r*}$.

Proof: Let us assume that $(\bar{\pi}, \bar{P}^s, \bar{P}^r)$ is an optimal solution for (3) which satisfies (i). So, a new solution (π^*, P^{s*}, P^{r*}) has to be built, where $\pi^* = \bar{\pi}$, $P_i^{s*} = \frac{2^{\bar{R}_{ii}}-1}{g_i^s}$ e $P_i^{r*} = \frac{2^{\bar{R}_{ii}}-1}{g_i^r}$. It is straightforward to show that this solution satisfies (i) and (ii). The matching is feasible, since $\pi^* = \bar{\pi}$, which guarantees (i). Furthermore, the i-th subcarrier from BS-RS and RS-SS links receives the necessary power to assure $R_i^{s*} = R_i^{r*} = \bar{R}_{ii}$, and, therefore, $R_{ii}^* = \bar{R}_{ii}$, which does not change the value of the objective function. Finally, $\bar{R}_{ii} = \min\{\bar{R}_i^s, \bar{R}_i^r\}$ and the definition of powers proposed, leads to $P_i^{s*} \le \bar{P}_i^s$ and $P_i^{r*} \le \bar{P}_j^r, \forall 1 \le j \le N$. ■

The second item from Lemma 1 could be demonstrated using a Lagrangian approach as well, but it is not as straightforward as the one provided above. Using Lemma 1, problem (4) can be simplified by changing the RS-SS variables into BS-RS variables or the opposite. Therefore, a new optimization problem with regard to the BS-RS hop can be stated as shown in (5). Since there are less optimization variables and constraints in (5) than in (4), numerical solutions can be obtained with less computational power. Moreover, based on (5), suboptimal solutions with futher lower computational costs can be developed when compared to previous works such as [11], [13].

$$(Q): \ \max \sum_{i=1}^{N} \ln\left(1 + P_i^s \cdot g_i^s\right) \tag{5a}$$

$$\text{s.t.}: \sum_{i=1}^{N} P_i^s \le P_t^s, \tag{5b}$$

$$\sum_{j=1}^{N} \frac{g_i^s}{g_i^r} P_i^s \le P_t^r, \tag{5c}$$

$$P_i^s \ge 0, \forall i. \tag{5d}$$

IV. Suboptimal Algorithms for Power Allocation

In order to further simplify the determination of the solution of (5), we propose in this section two suboptimal algorithms. When restrictions (5b) or (5c) are not considered, the following relaxations are proposed:

$$(Q^s): \ \max \sum_{i=1}^{N} \ln\left(1 + P_i^s g_i^s\right) \qquad (Q^r): \ \max \sum_{i=1}^{N} \ln\left(1 + P_i^r g_i^r\right)$$

$$\text{s.t.}: \ \sum_{i=1}^{N} P_i^s \le P_t^s, \qquad \text{s.t.}: \ \sum_{i=1}^{N} P_i^r \le P_t^r,$$

$$P_i^s \ge 0, \ \forall i, \qquad P_i^r \ge 0, \ \forall i.$$

Each subproblem (Q^r) or (Q^s) can be solved in polynomial time by the water-filling method [10]. The solution of one subproblem can be an optimal solution for (5) if it generates a feasible solution for the other subproblem by the transformation expressed in Lemma 1. Next, the suboptimal solutions are further developed.

A. Water-filling with Power Scaling

The idea of this algorithm is to solve the problem for one hop (BS-RS or RS-SS) and then scale the power as to respect the constraint for the other hop. The steps are summarized as follows.

i) Sort the subcarriers at BS-RS and RS-SS links in descending order and match the subcarriers in pairs by the order of the channel power gains (e.g. $h_{\pi_1}^s \sim h_{\pi_1}^r$), which means that the bits transported on the subcarrier π_1^s will be retransmitted on the subcarrier π_1^r;
ii) The problem (Q^s) is solved using water-filling as follows:

$$P_i^s = \left(\frac{1}{\lambda} - \frac{1}{g_i^s}\right)^+, \tag{6}$$

where $(x)^+ = \max(x, 0)$ and λ can be found by the following equation:

$$\sum_{i=1}^{N} P_i^s = P_t^s; \tag{7}$$

iii) P_i^s, P_i^r can be found by means of Lemma 1. Check if the power constraint for RS-SS link is violated. If it is violated, scale both P_i^s and P_i^r by the factor $P_t^r / \sum_{i=1}^{N} P_i^r$;
iv) Now, the total system spectral efficiency can be calculated by equation:

$$R_i^s(P_i^s) = \frac{1}{2N} \log_2(1 + P_i^{s,r} g_i^{s,r}). \quad (8)$$

v) Repeat steps 2-4 for the RS-SS link, i.e. , solve the problem (Q^r). The scaling factor is now $P_t^s / \sum_{i=1}^{N} P_i^s$;
vi) Choose the maximum spectral efficiency between BS-RS and RS-SS links;

B. Minimum water-filling

In this case, the idea is to solve the problem with the minimum power constraint, and then scale the power of the other node, if it is necessary. The steps are summarized as follows:

i) Sort the subcarriers at BS-RS and RS-SS links in descending order and match the subcarriers in pairs by the order of the channel power gains (e.g. $h_{\pi_1}^s \sim h_{\pi_1}^r$), which means that the bits transported on the subcarrier π_1^s will be retransmitted on the subcarrier π_1^r;
ii) Take the minimum power constraint of (Q^s) and (Q^r) and solve the problem by water-filling. Then, evaluate the power of the other node using Lemma 1;
iii) Check if the power constraint for the other node is violated. If it is violated, scale both P_i^s and P_i^r by the respective factor, $P_t^s / \sum_{i=1}^{N} P_i^s$ if the RS-SS link is the one with minimum power constraint, or $P_t^r / \sum_{i=1}^{N} P_i^r$ if BS-RS has the minimum power constraint;
iv) Now, the total system spectral efficiency can be calculated by equation:

$$R_i^s(P_i^s) = \frac{1}{2N} \log_2(1 + P_i^{s,r} g_i^{s,r}). \quad (9)$$

V. Numerical Results and Discussion

Computer simulations are used to evaluate the system model. The performance metric used for comparison among different approaches is the total spectral efficiency involving both the BS-RS and RS-SS links.

The Subcarrier Matching (SM) and Power Allocation (PA) schemes to be compared are the following:

i) *Random*: the bits transmitted on the subcarrier π_i^s (at BS-RS) will be retransmitted on the subcarrier π_i^r (at RS-SS) and no kind of sorting is done, i.e., the matching is random; power is equally allocated over all subcarriers, $P_i^s = P_i^r = P_t^s/N$;
ii) *Optimal SM*: the bits transmitted on the subcarrier π_i^s (at BS-RS) will be retransmitted on the subcarrier π_i^r (at RS-SS) and all subcarriers are sorted in descending order by their channel gains, i.e., $h_1^s > h_2^s > \cdots > h_N^s$ and $h_1^r > h_2^r > \ldots > h_N^r$; the power is equally allocated over all subcarriers, $P_i^s = P_i^r = P_t^s/N$;
iii) *Power Scaling PA*: the optimal subcarrier matching algorithm in strategy ii) is combined with the power allocation from section IV-A;
iv) *Minimum PA*: the optimal subcarrier matching algorithm in strategy ii) is combined with the power allocation from section IV-B;
v) *Optimal SM+PA*: the optimal subcarrier matching algorithm in strategy ii) is combined with optimal power allocation obtained by numerically solving the convex problem (5).

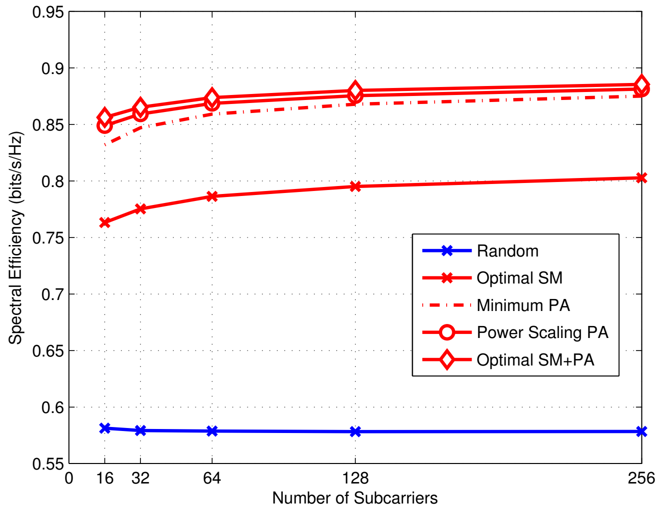

Figure 2. Spectral Efficiency versus Number of Subcarriers under a Low SNR Scenario

In the computer simulations, we assume that each subcarrier undergoes independent Rayleigh fading, with unitary variance and the mean channel gain ($E[h_i^s]$ and $E[h_i^r]$) may vary, depending on the case we are dealing with. Spectral efficiency is estimated using the Shannon capacity formula [15]. The total available power for all subcarriers is defined as $P_t^s = N \cdot SNR^s \cdot \sigma_N^2$ and $P_t^r = N \cdot SNR^r \cdot \sigma_N^2$. A total of 3,000 independent trials are run for calculating averages for each spectral efficiency point.

The relationship between the number of available subcarriers and spectral efficiency is shown in Figs. 2 and 3 for scenarios with average Signal-to-Noise Ratio (SNR) values of 5dB and 20dB, respectively. At both figures, the worst algorithm is Random, which shows a large gap of spectral efficiency to the others. With the Optimal SM, the gap to the other algorithms with non-uniform PA is decreased and it is a good choice when the SNR is high, because the gap is small. The Minimum PA algorithm proposed in Section IV-B has a good performance under low SNR scenarios, because it is close to the Optimal SM+PA solution. Under high SNR, its performance is below the Optimal SM, but increases with the number of subcarriers, and with 256 subcarriers approaches the Optimal SM+PA solution again.

The Power Scaling PA algorithm, proposed in Section IV-A, has the best suboptimal performance, being very close (at least 0.5% of relative error) to the Optimal SM+PA. As it can be seen when analysing the figures, the gap between algorithms with PA and the one with uniform PA is smaller than the gap with regard to the algorithm without optimal SM, implying

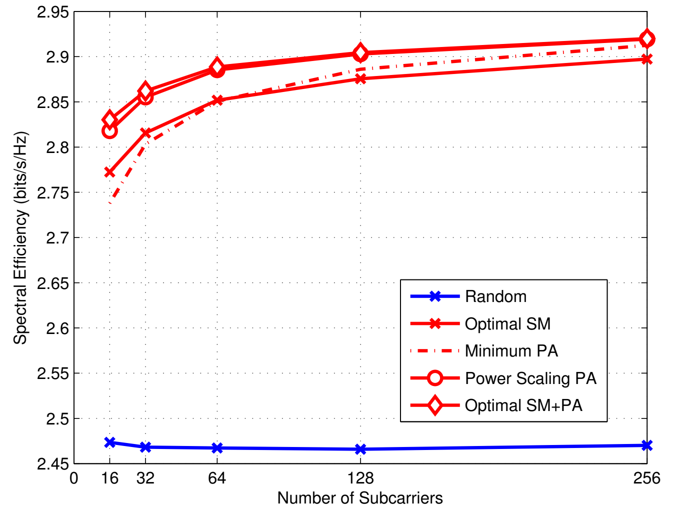

Figure 3. Spectral Efficiency versus Number of Subcarriers under a High SNR Scenario

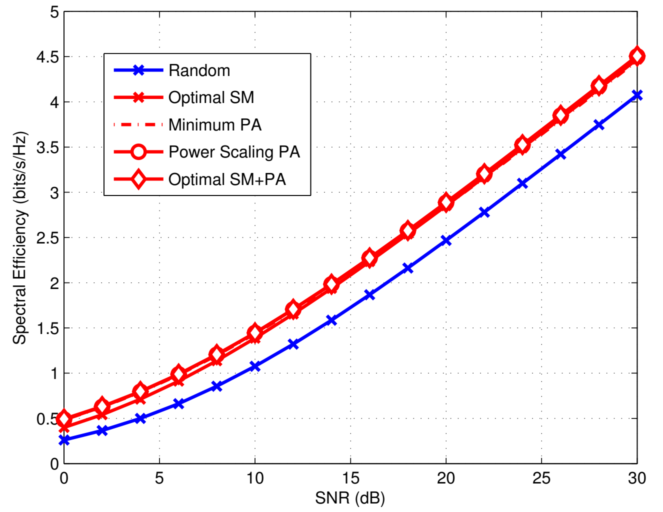

Figure 4. Spectral Efficiency versus SNR with 64 subcarriers

that using optimal SM provides more performance gain than using some form of PA. Therefore, it is justifiable to employ a suboptimal algorithm, such as the proposed Power Scaling PA, which presents low complexity and achieves good performance independent of the scenario (low and high SNR).

Fig. 4 shows the spectral efficiency versus the SNR for a 64-subcarrier scenario. As expected, with an increasing SNR the spectral efficiency increases as well. Again, the worst algorithm is Random. At low SNR, close to 10 dB, there is a gap between the Optimal SM algorithm and the other ones, but with increasing SNR this gap decreases and its curve can not be distinguished from the others. The Power Scaling PA algorithm, has the best performance among all the suboptimal algorithms and the smallest gap to the optimal solution.

VI. Conclusions

The joint subcarrier matching and power allocation problems were studied in order to maximize the total spectral efficiency in a multicarrier system which employs one relay and considering separate power constraints for each hop. The formulated problem is an NP-hard mixed-binary integer programming problem, similar to the one in [5]. The problem is separated in two, where the first is the optimal subcarrier matching and the second is the optimal power allocation given a certain subcarrier matching solution. The solution to the former is provided in [11], which is to match subcarriers in descending order of their instantaneous power gains.

In this paper, for the power allocation problem, an expression between the capacity of both links BS-RS and RS-SS is derived, so that the problem can be reformulated in a more tractable form and the complexity decreased, allowing us to propose two different suboptimal solutions to the problem. It is important to remark that the complexity of the proposed power allocation algorithms is considerably low. The simulation results illustrate that the proposed algorithms, Minimum PA and Power Scaling PA, have a close-to-optimal performance. In comparison to other schemes with or without subcarrier matching and any suboptimal power allocation, the proposed Power Scaling PA algorithm presented the best performance.

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