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The Lambert-Kaniadakis W_κ function

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ABSTRACT

In the present work, we introduce the Lambert-Kaniadakis W_κ function. It is a generalization of the Lambert W function that solves the equation $W_\kappa(z)\exp_\kappa(W_\kappa(z)) = z$, where $\exp_\kappa(z)$ is the κ-exponential, a generalization of the exponential function proposed by Kaniadakis. Following, the W_κ function is used in the definition of the κ-disentropy. Analytical results and numerical calculations of W_κ are shown, as well some applications of the κ-disentropy are discussed.

1. Introduction

The Lambert W function is the solution of the equation

$$W(z)e^{W(z)} = z. \tag{1}$$

Several analytical solutions of problems in mathematics, physics and computer science uses the Lambert W function [1–6]. A generalization of (1) is the recently proposed Lambert-Tsallis W_q function [7]. It is the solution of $W_q(z)\exp_q(W_q(z)) = z$, where $\exp_q(z)$ is the q-exponential proposed by Tsallis [8]. Another possible generalization of (1) is

$$W_\kappa(z)\exp_\kappa\left(W_\kappa(z)\right) = z \tag{2}$$

where $\exp_\kappa(z)$ is the κ-exponential proposed by Kaniadakis [9]

$$\exp_\kappa(z) = \left[\sqrt{1+\kappa^2 z^2} + kz\right]^{\frac{1}{\kappa}}. \tag{3}$$

Furthermore, $\exp_{\kappa=0}(z) = e^z$ and, obviously, $W_{\kappa=0}(z) = W(z)$. Using (3) in (2) one gets

$$W_\kappa(z)\left[\sqrt{1+\kappa^2 W_\kappa^2(z)} + \kappa W_\kappa(z)\right]^{\frac{1}{\kappa}} = z. \tag{4}$$

Now, introducing $r = 1/\kappa$ and $x = W_{\frac{1}{r}}(z)$ in (4) one obtains

$$x\left[\sqrt{r^2+x^2} + x\right]^r = r^r z. \tag{5}$$

From (5) one can see that $W_\kappa(0) = 0$. The solutions of (5) are roots of polynomials. However, before to find out explicit formulas for $W_\kappa(z)$, it is important to find the branch point $(z_b, W_\kappa(z_b))$, that is, the point where two different real solutions meet. For example, for the Lambert function the branch point is $(-1/e, -1)$. The solution in the interval $-1/e \le z \le 0$ is named W_{-1} while the solution in the interval $-1/e \le z \le \infty$ is named W_0. The value of $W_\kappa(z_b)$ can be found doing $dW_\kappa/dz|_{z_b} = \infty$. Using (2) and (3), one has

$$\frac{dW_\kappa}{dz} = \left(\frac{dW_\kappa e_\kappa^{W_\kappa}}{dW_\kappa}\right)^{-1} = \left[\left(\frac{W_\kappa}{\sqrt{1+\kappa^2 W_\kappa^2}} + 1\right)e_\kappa^{W_\kappa}\right]^{-1}, \tag{6}$$

hence, the first derivative of W_κ is infinite for $W_\kappa = -\infty$ and $W_\kappa = -(1-\kappa^2)^{-1/2}$, that is valid in the interval $0 \le \kappa^2 < 1$. The value of z_b is found using (2) and (3) again

$$z_b = W_\kappa(z_b)e_\kappa^{W_\kappa(z_b)} = -\frac{1}{\sqrt{1-\kappa^2}}e_\kappa^{-1/\sqrt{1-\kappa^2}} = -\frac{(1-\kappa)^{\frac{1-\kappa}{2\kappa}}}{(1+\kappa)^{\frac{1+\kappa}{2\kappa}}}. \tag{7}$$

Thus the solution in the interval $z_b \le z < 0$ is $W_\kappa^-(z)$ while the solution in the interval $z_b \le z < \infty$ is $W_\kappa^+(z)$. Moreover, $\frac{d^2W_\kappa^+(z)}{dz^2} < 0$. On the other hand, initially $\frac{d^2W_\kappa^-(z)}{dz^2} > 0$ and it changes into $\frac{d^2W_\kappa^-(z)}{dz^2} < 0$. In order to see this clearly, let us initially consider $\kappa = 1/3$. Substituting $r = 3$ in (5) one has

$$\left(x^{2/3}\right)^2 + \frac{3}{2z^{1/3}}x^{2/3} - \frac{3}{2}z^{1/3} = 0. \tag{8}$$

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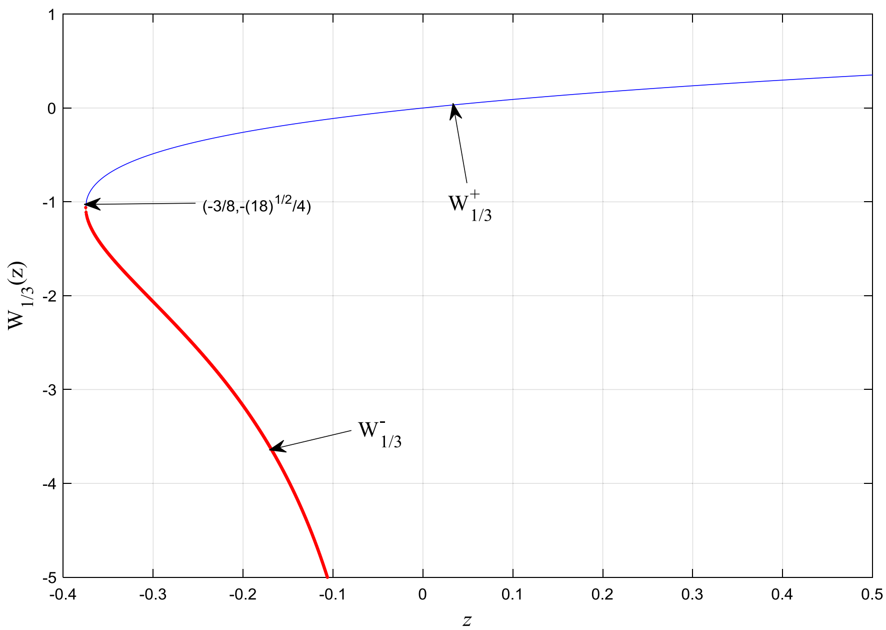

Fig. 1. $W^+_\kappa(z)$ and $W^-_\kappa(z)$ versus z for $\kappa = 1/3$.

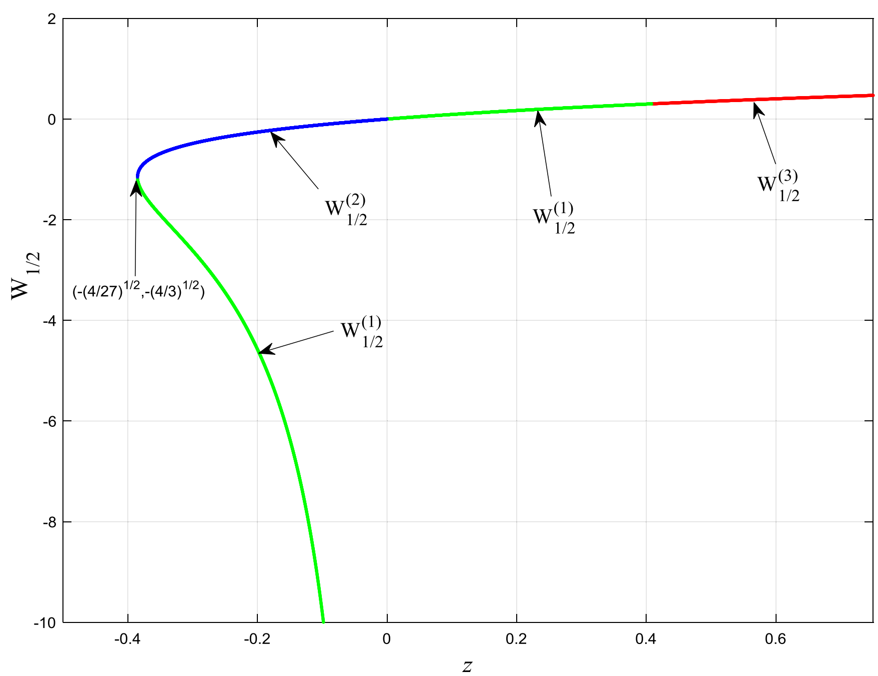

Fig. 2. $W_{1/2}$ versus z.

Hence, the Lambert-Kaniadakis functions are

$$W^{\pm}_{1/3}(z) = \left(-\frac{3}{4z^{1/3}} \pm \frac{3}{4z^{1/3}}\sqrt{1+\frac{8}{3}z}\right)^{3/2}. \tag{9}$$

The branch point is $(-3/8, -\sqrt{18}/4)$. The plot of $W_{1/3}(z)$ versus z can be seen in Fig. 1.

For $\kappa = 1/2$ ($r = 2$), the Lambert-Kaniadakis function $W_{1/2}(z)$ is one of the roots of the polynomial $P(X) = x^3 - x^2/z + 2x - z$ and the brunch points is $(-(4/27)^{1/2}, -(4/3)^{1/2})$. In the interval $z \in (-(4/27)^{1/2}, (4/27)^{1/2})$, $P(X)$ has three real roots and two of them are useful, let us call it by $W^{(1)}_{1/2}$ and $W^{(2)}_{1/2}$. For $z^2 > (4/27)$, $P(X)$ has only one real root, $W^{(3)}_{1/2}$. The plot of $W_{1/2}$ versus z can be seen in Fig. 2.

For $\kappa = 1$ ($r = 1$) one has

$$W_{\kappa=1}(z) = \frac{z}{\sqrt{2z+1}}, \quad z > -1/2. \tag{10}$$

In $z = -1/2$, $W_1(z)$ presents a vertical asymptote with $W_1(-1/2^+) = -\infty$. The function is monotonically increasing with $W_1(+\infty) = +\infty$ and its concavity is $\frac{d^2 W^+_1(z)}{dz^2} < 0$.

For $\kappa > 1$, there is no branch points and $W_k(z)$ is defined in the entire real line. For example, Fig. 3 shows the plot of $W_\kappa(z)$ versus z for $\kappa = 0$, $\kappa = 1$ and $\kappa = 2$, in the interval $z \in [0, 10]$.

The curves in Fig. 3 were calculated numerically using the Halley method. According to the Halley method [1,2,5], the equation $f(x) = 0$ can be numerically solved using

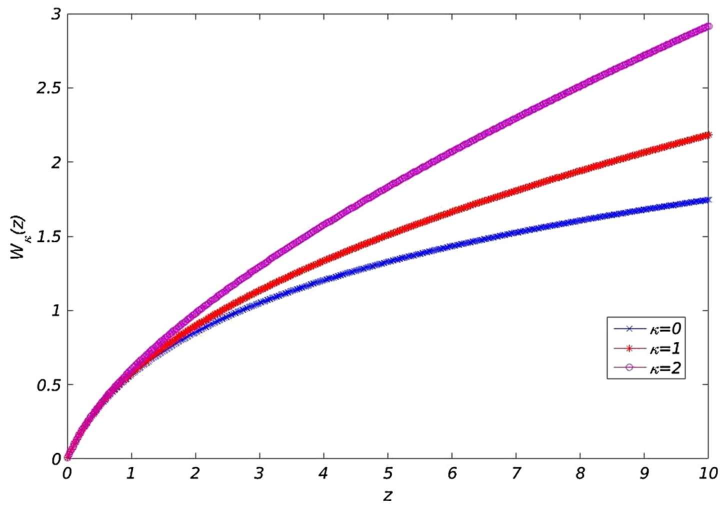

Fig. 3. $W_\kappa(z)$ versus z for $\kappa=0$, $\kappa=1$ and $\kappa=2$.

$$x(j+1)=x(j)-\frac{2f(x(j))f'(x(j))}{2[f'(x(j))]^2-f(x(j))f''(x(j))}. \tag{11}$$

Now, making $f(W_\kappa)=W_\kappa \exp_\kappa(W_\kappa)-z=0$ and using $d\exp_\kappa(y)/dy=\exp_\kappa(y)/(1+\kappa^2y^2)^{1/2}$ in (11), the function W_κ can be obtained numerically using

$$w_\kappa(j+1)=w_\kappa(j)-\frac{w_\kappa(j)e_\kappa^{w_\kappa(j)}-z}{\left[\frac{w_\kappa(j)}{\sqrt{1+\kappa^2w_\kappa^2(j)}}+1\right]e_\kappa^{w_\kappa(j)}-\frac{\left[\frac{w_\kappa(j)}{\sqrt{1+\kappa^2w_\kappa^2(j)}}+2\right]\left[w_\kappa(j)e_\kappa^{w_\kappa(j)}-z\right]}{2w_\kappa(j)+2\sqrt{1+\kappa^2w_\kappa^2(j)}}}. \tag{12}$$

At last, calling by W_κ^+ the upper branch and W_κ^- the lower branch, after some algebra the following limits can be obtained

$$\lim_{\substack{z\to\infty\\ \kappa>0}} W_\kappa^+(z)\approx(2\kappa)^{-1/(1+\kappa)}z^{\kappa/(1+\kappa)} \tag{13}$$

$$\lim_{\substack{z\to-\infty\\ \kappa>1}} W_\kappa^+(z)\approx-(2\kappa)^{1/(\kappa-1)}z^{\kappa/(\kappa-1)} \tag{14}$$

$$\lim_{\substack{z\to0^-\\ 0<\kappa<1}} W_\kappa^-(z)\approx-(2\kappa)^{-1/(1-\kappa)}z^{-\kappa/(1-\kappa)}. \tag{15}$$

2. The κ-disentropy

The κ-logarithm function and the Kaniadakis entropy (κ-entropy) are given, respectively, by [9,10]

$$\ln_\kappa(x)=\frac{x^\kappa-x^{-\kappa}}{2\kappa}, \tag{16}$$

$$S_\kappa=-\sum_n p_n\frac{p_n^\kappa-p_n^{-\kappa}}{2\kappa}=-\sum_n p_n\ln_\kappa(p_n). \tag{17}$$

In a recent work the disentropy was introduced as a measure of order or certainty [7]. The disentropies related to Shannon and Tsallis' entropies have been used in quantum and classical information theory [7,11], in radial basis function network [12] and as a quantumness measure [13]. The κ-disentropy related to κ-entropy is here defined as

$$D_\kappa=\sum_n p_nW_\kappa(p_n). \tag{18}$$

Differently of the disentropy proposed in [7], the κ-disentropy is not obtained from the relation between the functions $\ln_\kappa(z)$ and $W_\kappa(z)$. It is introduced ad-hoc. However, as one can see, D_κ still keeps the important properties. For example, it is maximal for a delta distribution and minimal for a uniform distribution. Considering the distribution $\{p, 1-p\}$, Fig. 4 shows the plot of κ-entropy and κ-disentropy versus p for $\kappa=0$ and $\kappa=1/2$.

The quantum κ-entropy is given by [14]:

$$S_\kappa(\rho)=\frac{\mathrm{Tr}(\rho^{1-\kappa}-\rho^{1+\kappa})}{2\kappa}=\sum_n\frac{\lambda_n^{1-\kappa}-\lambda_n^{1+\kappa}}{2\kappa}. \tag{19}$$

The corresponding quantum κ-disentropy is here defined as

$$D_\kappa=\sum_n\lambda_nW_\kappa(\lambda_n), \tag{20}$$

where λ_n is the nth eigenvalue of the density matrix ρ. One can note that Fig. 4 also describes the entanglement measured by S_κ and disentanglement [7] measured by D_κ of the pure two-qubit state $|\varphi\rangle=p^{1/2}|00\rangle+(1-p)^{1/2}|11\rangle$, versus p (the eigenvalues of the partial states $\rho_{A(B)}=\mathrm{Tr}_{B(A)}(|\varphi\rangle\langle\varphi|)$ are p and $(1-p)$).

3. Applications of W_κ and D_κ

In [10], considering a spherical surface as being the holographic screen, with a particle of mass M positioned in its centre, the Kaniadakis statistics was used to determine the relation between the number of bits of the holographic screen, N, and the holographic screen area A:

$$N=\frac{\ln\left(\frac{\kappa A}{4l_p^2}+\sqrt{\frac{\kappa^2A^2}{16l_p^4}+1}\right)}{\kappa\ln(c_1)}, \tag{21}$$

where l_p is the Planck's length. The total number of microstates is c_1^N, where c_1 stands for the number of internal degrees of freedom on the holographic screen. Thus, using the microcanonical ensemble definition where all states have the same probability, the κ-entropy S_κ is given by [10]

$$S_\kappa=\frac{c_1^{N\kappa}-c_1^{-N\kappa}}{2\kappa} \tag{22}$$

while the κ-disentropy D_κ is

$$D_\kappa=W_\kappa\left(c_1^{-N}\right). \tag{23}$$

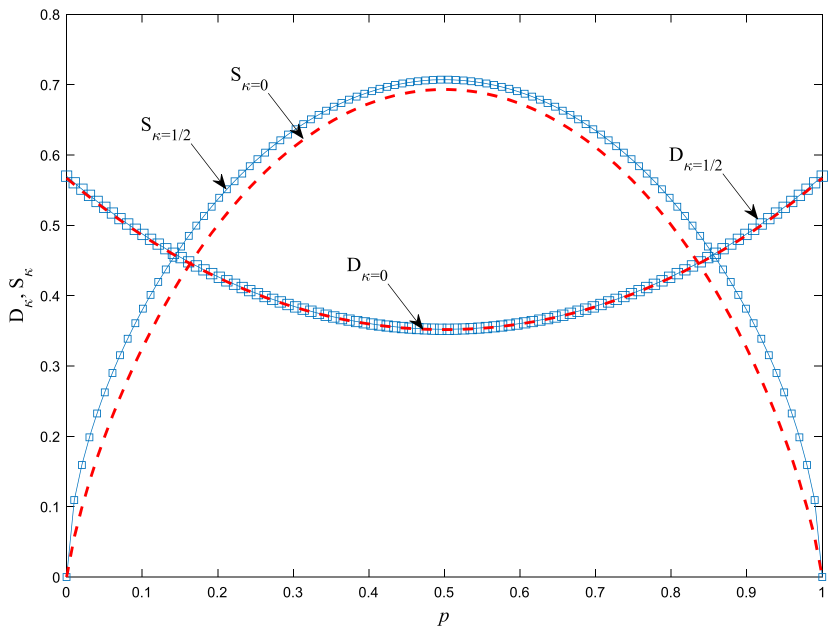

Fig. 4. S_κ and D_κ versus p for $\kappa = 0$ and $\kappa = 1/2$.

However, in order to recover the Boltzmann-Gibbs statistics when $\kappa = 0$, the condition $4\ln(c_1) = 1$ is required. Therefore, the disentropy as a function of the holographic screen area is given by

$$D_\kappa = W_\kappa\left[\left(\frac{\kappa A}{4l_p^2} + \sqrt{\frac{\kappa^2 A^2}{16 l_p^4} + 1}\right)^{-\frac{1}{\kappa}}\right]. \tag{24}$$

One can note in (24) that the larger the area A the smaller is the disentropy, as expected. Furthermore, when κ tends to zero, one recovers the disentropy of the black hole in Boltzmann-Gibbs statistics, $D = W(\exp(-A/4l_p^2))$.

In [15] it has been shown that for a normal (complex or real) matrix $B = UD_BU^\dagger$, $\exp_\kappa(B) = U\exp_\kappa(D_B)U^\dagger$ as long as $\|B\| < 1/\kappa$ (the square root of the largest eigenvalue of B is lower than $1/\kappa$). Thus, given a Hermitean matrix $H_A = UD_AU^\dagger$, one can define the family of κ-Hermitean matrices $H_\kappa = UD_\kappa U^\dagger$ where the nth eigenvalue of $H_\kappa(\xi_n)$ and the nth eigenvalue of $H_A(\lambda_n)$ are related by

$$\lambda_n e_\kappa^{\lambda_n} = \xi_n \quad \Leftrightarrow \quad \lambda_n = W_\kappa(\xi_n). \tag{25}$$

Using the κ-Hermitean matrices, one can construct the family of κ-quantum gates $U_\kappa = \exp(iH_\kappa)$. For example, if one starts with CNOT gate (that is also Hermitean) U_{CNOT}, one can construct the following family of unitary gates

$$U_k = \exp\left(iU_{cnot}e_\kappa^{U_{cnot}}\right) = V\begin{bmatrix}\exp(-ie_\kappa^{-1}) & & & \\ & \exp(e_\kappa^1) & & \\ & & \exp(e_\kappa^1) & \\ & & & \exp(e_\kappa^1)\end{bmatrix}V^\dagger, \tag{26}$$

where V is an unitary matrix whose columns are the eigenvectors of U_{CNOT}. One can also work in the opposite direction:

$$H_A e_\kappa^{H_A} = U_{cnot} \Rightarrow H_A = V\begin{bmatrix}W_k(-1) & & & \\ & W_k(1) & & \\ & & W_k(1) & \\ & & & W_k(1)\end{bmatrix}V^\dagger \tag{27}$$

and the family of κ-quantum gates constructed is

$$U = \exp(iH_A) = V\begin{bmatrix}\exp[iW_k(-1)] & & & \\ & \exp[iW_k(1)] & & \\ & & \exp[iW_k(1)] & \\ & & & \exp[iW_k(1)]\end{bmatrix}V^\dagger. \tag{28}$$

A special care must be taken when one chooses the value of κ, since in general W_κ is not defined in the whole real straight line. For example, in (29) one cannot use $\kappa = 1/2$, since $W_{1/2}(-1)$ is not real.

Entropy plays an important role in image processing algorithms. For example, the segmentation algorithm, one of the most basic tasks in image processing that aims to separate the main object of the background, can be 'tuned' by maximizing the entropy [16–19]. Basically, the segmented image is constructed setting all pixels with value smaller than t to the value '0' (black) and all pixels with value larger or equal to t to the value '255' (white). For an $N \times N$ image there are N^2 pixels. The value of the k-th pixel is $v(k)$. For a given threshold value t, the set of pixels $A = \{a_1, a_2, \ldots, a_s\}$ is composed only by pixels with values larger or equal than t while the set of pixels $B = \{b_1, b_2, \ldots, b_r\}$ is composed only by pixels with values lower than t $(r + s = N^2)$. Thus, the probability distributions for object $P(A) = \{p_a(1), \ldots, p_a(k), \ldots, p_a(s)\}$ and background $P(B) = \{p_b(1), \ldots, p_b(k), \ldots, p_b(r)\}$ can be constructed:

$$p_a(k) = v(a_k)/\sum_{i=1}^{s} v(a_i) \tag{29}$$

$$p_b(k) = v(b_k)/\sum_{i=1}^{r} v(b_i). \tag{30}$$

The κ-entropy of the distribution $P(A)$ $(P(B))$ is $S_\kappa(A)$ $(S_\kappa(B))$. The values of $S_\kappa(A)$ and $S_\kappa(B)$ depend on the value of t. The best value of t is the one that maximizes $SAB_\kappa = S_\kappa(A)[1 + (\kappa S_\kappa(B))^2]^{1/2} + S_\kappa(B)[1 + (\kappa S_\kappa(A))^2]^{1/2}$. The same method can be used employing the κ-disentropy instead of κ-entropy. The

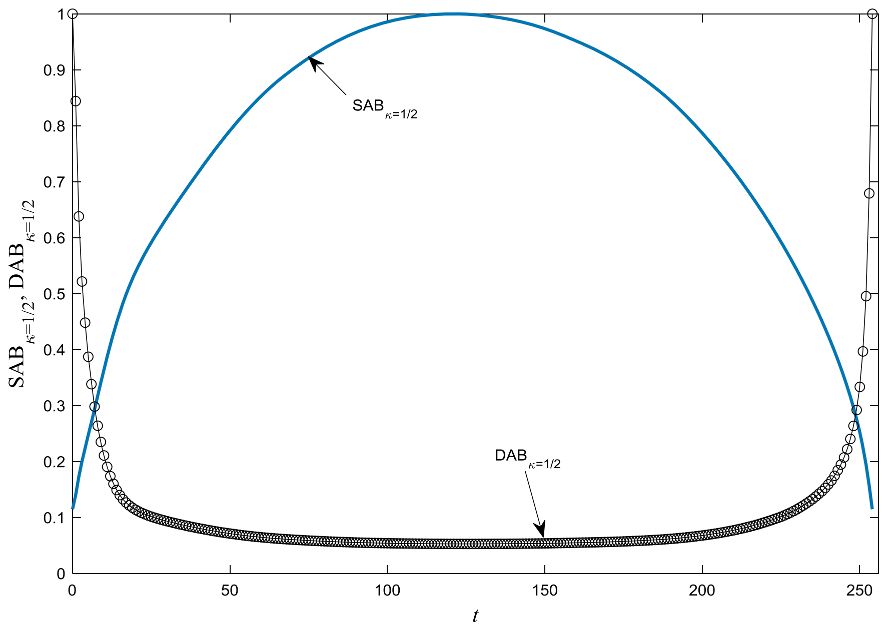

Fig. 5. Normalized κ-disentropy and κ-entropy of the image of the star V838 monocerotis versus the threshold value t, for $\kappa = 1/2$.

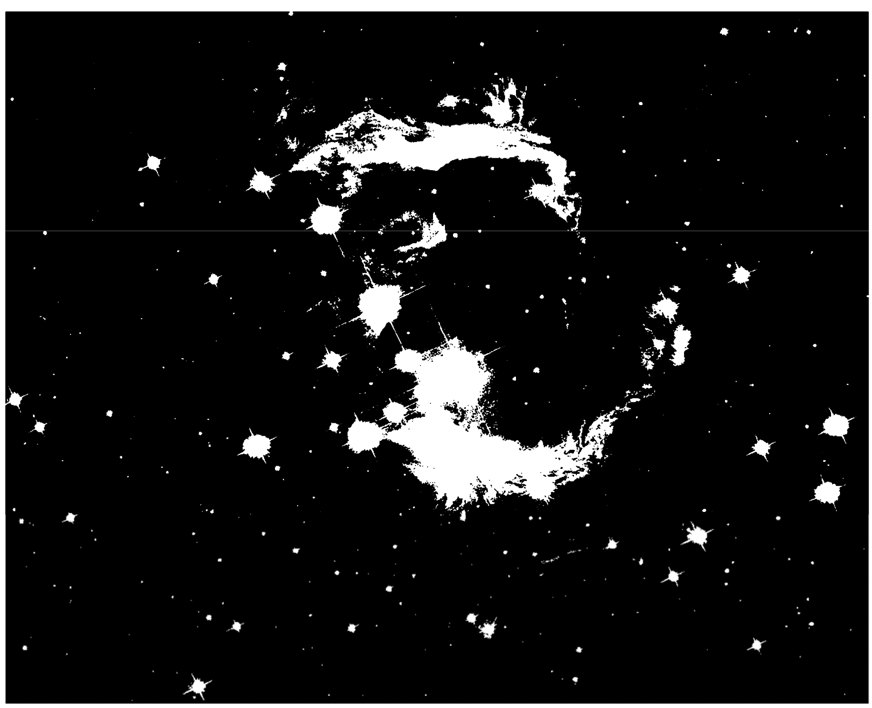

Fig. 6. Image of the star V838 monocerotis segmented by an algorithm based on the minimization of the disentropy $D_{\kappa=1/2}$. Image n° 21 found in https://www.spacetelescope.org/images/archive/top100/.

κ-disentropy of the distribution $P(A)$ ($P(B)$) is $D_\kappa(A)$ ($D_\kappa(B)$). The values of $D_\kappa(A)$ and $D_\kappa(B)$ depend on the value of t. The best value of t is the one that minimizes $DAB_\kappa = D_\kappa(A)[1 + (\kappa D_\kappa(B))^2]^{1/2} + D_\kappa(B)[1 + (\kappa D_\kappa(A))^2]^{1/2}$.

As an example, let us consider the image of the star V838 monocerotis (image n° 21 found in https://www.spacetelescope.org/images/archive/top100/). The κ-disentropy and κ-entropy versus threshold value t, for $\kappa = 1/2$, can be seen in Fig. 5.

The segmentation protocol described in [16] but using the minimization of the κ-disentropy $DAB_\kappa(\kappa = 1/2)$ produces the segmented image shown in Fig. 6 ($t = 124$).

4. Conclusions

The present work introduced the Lambert-Kaniadakis W_κ function. As an application of the W_κ function, the κ-disentropy was introduced. Then, we used the κ-disentropy to calculate the disentanglement of pure two-qubit states, we provided a formula for the κ-disentropy of a holographic screen as a function of its area and we described an image segmentation algorithm based on the minimization of the κ-disentropy. Hence, like the q-disentropy [7], the κ-disentropy can find applications in several different areas of physics and engineering. In particular, since $W_\kappa(z)$ accepts negative values in its arguments ($z \geq z_b$ for $0 < \kappa < 1$), like the q-disentropy, one can also calculate the κ-disentropy of the Wigner function of some highly quantum states, what cannot be done with κ-entropy.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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