UNIVERSIDADE FEDERAL DO CEARÁ

CENTRO DE CIÊNCIAS

DEPARTAMENTO DE MATEMÁTICA

PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

ANDRÉ LUIZ ARAÚJO DA COSTA

CHARACTERIZATION OF LIPSCHITZ NORMALLY EMBEDDED COMPLEX CURVES AND LIPSCHITZ TRIVIAL VALUES OF POLYNOMIAL MAPPINGS

FORTALEZA

2023

ANDRÉ LUIZ ARAÚJO DA COSTA

CHARACTERIZATION OF LIPSCHITZ NORMALLY EMBEDDED COMPLEX CURVES AND LIPSCHITZ TRIVIAL VALUES OF POLYNOMIAL MAPPINGS

Thesis submitted to the Doctoral Program of the Departament of Mathematics of Universidade Federal do Ceará in partial fulfillment of the necessary requirements for the degree of Ph.D in Mathematics. Area of expertise: Theory of Singularities.

Advisor: Prof. Dr. Vincent Grandjean.

Co-advisor: Prof. Dr. Maria Michalska.

FORTALEZA

2023

Dados Internacionais de Catalogação na Publicação
Universidade Federal do Ceará
Sistema de Bibliotecas
Gerada automaticamente pelo módulo Catalog, mediante os dados fornecidos pelo(a) autor(a)

C87c Costa, André Luiz Araújo da.
Caracterização de curvas complexas Lipschitz normalmente mergulhadas e valores Lipschitz triviais de aplicações polinomiais / André Luiz Araújo da Costa. – 2023.
67 f. : il. color.

Tese (doutorado) – Universidade Federal do Ceará, Centro de Ciências, Programa de Pós-Graduação em Matemática , Fortaleza, 2023.
Orientação: Prof. Dr. Vincent Grandjean.
Coorientação: Prof. Dr. Maria Michalska.

1. Geometria Lipschitz. 2. Lipschitz normalmente mergulhado. 3. Valores Lipschitz triviais. I. Título.
CDD 510

ANDRÉ LUIZ ARAÚJO DA COSTA

CHARACTERIZATION OF LIPSCHITZ NORMALLY EMBEDDED COMPLEX CURVES AND LIPSCHITZ TRIVIAL VALUES OF POLYNOMIAL MAPPINGS

Thesis submitted to the Doctoral Program of the Departament of Mathematics of Universidade Federal do Ceará in partial fulfillment of the necessary requirements for the degree of Ph.D in Mathematics. Area of expertise: Theory of Singularities.

Approved in: 17/01/2023

EXAMINATION BOARD

Prof. Dr. Vincent Grandjean (Advisor)
Universidade Federal do Ceará (UFC)

Prof. Dr. Maria Michalska (Co-advisor)
Uniwersytet Łodzki

Prof. Dr. Alexandre Fernandes
Universidade Federal do Ceará (UFC)

Prof. Dr. Helge Pedersen
Universidade Federal do Ceará (UFC)

Prof. Dr. Lorenzo Fantini
Ecole Polytechnique

Prof. Dr. Zbigniew Jelonek
Uniwersytet Jagielloński

I dedicate this work to the memory of my aunt Francisca Lúcia Guerra Evangelista, who kept up her fight and her faith until death.

ACKNOWLEDGEMENTS

First, I thank God for the gift of loving Mathematics. The conclusion of this work would not be possible without all his support and care.

I dearly thank my wife, Priscila Costa, for her beauty, love and support in every moment, understanding all the long hours spent on this project and pushing me to be the best I can be.

I also thank my mother Francisca Isabel Guerra for always encouraging me and believing in me in any situation. I pray for God to honor her work and dedication put on growing her son.

I thank my advisors, Prof. Dr Vincent Grandjean and Prof. Maria Michalska, for accepting me as their student and guiding me on this work. I thank both of them for their patience and understandings while helping me to improve my mathematical skills. I also pray for God to honor their work and dedication put on growing me as mathematician.

I thank my pastors William and Samara Marques (the author of the epigraph) for their love and care for me. I also thank my friends Gerardo Mesquita, Carlos Sanches and Katy Sanches for their true cheer for me along this walk. I thank to my godchildren for the joy they bring to me.

I thank all the Professors of PGMAT whose taught me important mathematical topics through courses, specially, Maria Michalska, Vincent Grandjean, Alexandre Fernandes, Helge Pedersen, Lev Birbrair, Edson Sampaio, Kevin Langlois, Rodrigo Rodrigues, Fernanda Camargo, Francisco Pimentel and Alberto Maia.

I thank my colleagues and friends from UFC, for the memorable conversations without any sense and funny moments, specially Emanoel Souza, Tiago Gadelha, Thialita Moura, Bruno Rocha, José Erivamberto, Patrícia Regis, Sérgio Alvarez, Roger Oliveira, Danuso Rocha, Diego Silva, José Edilson, Felipe Fernandes, Rosa Tayane, Davi Lopes, Davi Ribeiro, André Gadelha.

I thank the members of this board for the corrections and suggestions to improve this text.

I also thank the secretary at the Department of Mathematics, Andrea Dantas, for all the support with the bureaucratic stuff along my academic years in the Master and Doctored at UFC.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

“There is still violence.” (MARQUES, 2019)

RESUMO

Estudamos a geometria Lipschitz das fibras de aplicações polinomiais complexas de dois pontos de vista: a equivalência entre as métricas induzida e intrínseca e a existência de estrutura local de feixe bi-Lipschitz de fibras sobre um conjunto de valores de uma aplicação polinomial. Provamos que a parte afim de uma curva algébrica projetiva conexa é Lipschitz normalmente mergulhada se, e somente se, as seguintes três condições são satisfeitas: sua parte afim é conexa; sua parte afim é localmente Lipschitz normalmente mergulhada em cada um dos seus pontos singulares; e seu grau é igual seu número de pontos no infinito. Além disso, mostramos que todo valor Lipschitz trivial de uma aplicação polinomial real ou complexa é a suspensão de um valor regular próprio de uma aplicação polinomial em menos variáveis. Por último, mostramos que esse resultado não é estendido para funções racionais.

Palavras-chave: geometria Lipschitz; Lipschitz normalmente mergulhado; valores Lipschitz triviais.

ABSTRACT

We study Lipschitz geometry of fibers of complex polynomial mappings from two points of view: the equivalence of inner and outer metrics of an algebraic curve and the existence of a locally bi-Lipschitz trivial fibre bundle structure over a subset of values of polynomial mappings. We prove that the affine part of a connected projective algebraic curve is Lipschitz normally embedded if and only if the following three conditions are satisfied: its affine part is connected, its affine part is locally Lipschitz normally embedded at each of its singular points; and its degree equals to the number of its points at infinity. Moreover, we show that any Lipschitz trivial value of a real or complex polynomial mapping is a suspension of a regular value of properness of a polynomial mapping in fewer variables. Last, we show that this result cannot extend to rational mappings.

Keywords: Lipschitz geometry; Lipschitz normally embedded; Lipschitz trivial values.

TABLE OF CONTENTS

1 INTRODUCTION

This PhD thesis explores the Lipschitz geometry of fibers of complex polynomial mappings from two points of view: the equivalence of inner and outer metrics of an algebraic curve and the existence of a locally bi-Lipschitz trivial fibre bundle structure over a subset of values of polynomial mappings.

There are two natural metrics on any arc-rectifiable subset X in $\mathbb{R}^n$: the outer metric, i.e., the distance induced by the Euclidean metric on $\mathbb{R}^n$ and the inner metric, given by the infimum of the lengths of rectifiable curves in X connecting the given pair of points. The set X is Lipschitz normally embedded (LNE) when these two metrics are equivalent in the sense that their ratio is uniformly bounded. This definition was introduced by Birbrair and Mostowski in [2].

The last decade has shown interest in investigating analytic set germs $(X, \mathbf{0})$ admitting LNE representatives. For example, Birbrair and Mendes obtained in [1] a criterion for a closed semialgebraic set to be LNE via the contact between real arcs on the set. Mendes and Sampaio proved in their recent paper [24] that the germ of a closed semialgebraic set at a point is LNE if and only if the family of links of this set at this point is uniformly LNE for any sufficiently small radius. For a rather exhaustive list of what is known on this topic one can consult [24] and also the recent short survey [7]. Outside the case of curve germs and isolated complex surface singularity germs, little is known in the higher dimensional complex setting.

Similarly, only a single paper has adressed when an algebraic set is LNE nearby infinity in the complex case, and under very restrictive hypotheses [10]. Last, there are very few results about gobally LNE algebraic subsets of $\mathbb{K}^n$. The only non-trivial example we know of is from [18]. It has non-isolated singularities, but is a $\mathbb{K}$-cone over a compact LNE set, thus essentially a projective result.

The first part of the thesis aims at explaining the problem of being globally LNE. It is devoted to study the case of algebraic curves being LNE as subsets of $\mathbb{C}^n$. We fully characterize which complex algebraic curves of $\mathbb{C}^n$ are LNE and which complex analytic curves of a compact complex manifold are LNE.

A stepping stone of the bundle-like properties of $\mathcal{C}^\infty$ mappings is Ehresmann's Theorem, stating that a proper $\mathcal{C}^\infty$ surjective submersion $f : X \to N$ between $\mathcal{C}^\infty$ manifolds is a locally $\mathcal{C}^\infty$ trivial fibration. In the case of polynomial functions, where singular points and non-compact fibers are allowed, Thom in the seminal paper [28] from 1969 proved that a polynomial function $f:\mathbb{K}^n\to\mathbb{K}$ is locally trivial at any value outside $\mathrm{Bif}(f)$, a finite set of values, called the set of bifurcation values.

After Thom's result, a significant literature about the $\mathcal{C}^0$ and the $\mathcal{C}^\infty$ local triviality has been developed. For example, Verdier studied the existence of $\mathcal{C}^0$ triviality of analytic mappings in [29]. In 1980, Hardt handled the semialgebraic case in [14]. Rabier generalized Ehresmann's Theorem to the cases of infinite dimension and non-proper setting about 25 years ago in [26]. In 2005, Jelonek and Kurdyka in [17] obtained constructive results about the so called set of generalized critical values which always contains the set of bifurcation values.

More recently, in 2019, Fernandes, Grandjean and Soares started the investigation of an intermediate case between $\mathcal{C}^0$ and $\mathcal{C}^\infty$ triviality in the paper [9]. They proved that a non-constant polynomial mapping $f:\mathbb{C}^n\to\mathbb{C}$ is locally Lipschitz trivial, i.e., admitting bi-Lipschitz trivialization, at a value if and only if it is a polynomial in a single variable.

In the second part of this thesis we fully characterize the polynomial mappings $f:\mathbb{K}^n\to\mathbb{K}^p$ taking Lipschitz trivial values in terms of proper and non-bifurcation values, generalizing the main result of [9].

Chapter 2 reviews some standard (and some less standard) definitions and technical tools. In Section 2.1 we present some notations that are used throughout the text. Section 2.2 recalls some definitions and properties on sub-manifolds, especially the k-slice criterion for a subset to be an embedded sub-manifold of a prescribed manifold. In Section 2.3 we recall some results on critical values including Sard's Theorem. In Section 2.4 we present the Conical Structure Theorem and the definition of tangent cone. In Sections 2.5 and 2.6 we recall some important concepts and properties of multiplicity of analytic sets and degree of algebraic sets, following [4]. In Section 2.7 we introduce the notion of Lipschitz normally embedded subset in a Riemannian manifold and provide some important tools to investigate the property of being LNE for curves. In Section 2.8 we present the definitions of $\mathcal{C}^\infty$ and $\mathcal{C}^0$ local triviality and the Theorems of Ehresmann and Thom. In Section 2.9 we present the definition of values of properness and some results of the papers [15], [16], [17] on the set of non-proper values of polynomial mappings. In Section 2.10 we present the definition of Lipschitz trivial values, the main object of study in Chapter 4. Sections 2.11 and 2.12 together present three well-known results that are used in next Chapters: Rademacher's Theorem, Hadamard's Inequality and Puiseux parameterization for curves.

In Chapter 3 we investigate the LNE property for curves. Section 3.1 is devoted to obtain a criterion for an isolated singularity germ to be LNE at the singularity (see Lemma 3.1.1). Section 3.2 presents a complete proof of a criterion for curve germs to be locally LNE at a prescribed point (see Definition 2.7.4). Moreover, we prove that a complex curve in a compact complex manifold is LNE if and only if it is connected and locally LNE at each of its singular points. In Section 3.3 we start the investigation of LNE property at infinity. We show that an analytic curve germ is LNE at infinity if and only if it is transverse to the hyperplane at infinity and its multiplicity at the prescribed point at infinity is 1. Section 3.4 works with algebraic curves. We prove the following key result: Let X^a be an affine curve and let X be its projective closure. Let $X^\infty = X \cap \mathbf{H}_\infty$, where $\mathbf{H}_\infty$ is the hyperplane at infinity. Assume that $\deg(X) = \operatorname{card}(X^\infty)$. Then, whenever the radius R is large enough, each connected component of $X^a \setminus B_R^{2n}$ is LNE (see Proposition 3.4.1). Section 3.5 considers the bounded part of affine algebraic curves. We prove that the bounded part $X \cap B_R^{2n}$ (for large enough radius R) of the affine part of a projective curve is LNE if and only if it is locally LNE at each of its singular points. Finally, in Section 3.6 we prove our main result on LNE curves, Theorem 3.6.1: Let X be a connected projective curve of $\mathbb{P}^n$ of degree $\deg(X)$ such that X^∞ is finite. The affine curve $X^a = X \setminus X^\infty$ is LNE in $\mathbb{C}^n$ if and only if the following conditions are satisfied: (1) X^a is connected; (2) X^a is locally LNE at each of its singular points; (3) $\operatorname{card}(X^\infty) = \deg(X)$.

Our goal in Chapter 4 is to decide when a real or complex polynomial mapping takes a Lipschitz trivial value (see Definition 2.10.1). In Section 4.2 we prove some general properties of mappings with Lipschitz trivial values. In Section 4.3 we characterize the polynomial mappings with Lipschitz trivial values, our main result on this topic is Theorem 4.3.1: Let $f : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial mapping with $\dim f^{-1}(\mathbf{c})^\infty = n-1-m$ for a value $\mathbf{c} \in \mathbb{K}^p$. The mapping f attains $\mathbf{c}$ as a Lipschitz trivial value if and only if there exist a polynomial mapping $g : \mathbb{K}^m \to \mathbb{K}^p$ which is proper at $\mathbf{c}$ and a linear surjective projection $\pi : \mathbb{K}^n \to \mathbb{K}^m$ such that $f = g \circ \pi$.

In Section 4.4 we present some consequences of Theorem 4.3.1 in the real and complex cases emphasizing the difference between them. In Section 4.5 we show that we cannot extend the category of mappings that satisfy the claim of our main result. We provide an example of rational function with empty indeterminacy locus that admits Lipschitz trivial values which are not values of properness of a function of the form described in Theorem 4.3.1.

2 PRELIMINARIES

2.1 Notations

Throughout the text we will use following notations.

$\mathbb{K}$: a field which might be $\mathbb{R}$ or $\mathbb{C}$.

$B_r^{n_{\mathbb{K}}}(x)$: open ball of $\mathbb{K}^n$ centred at x with radius r for $n_{\mathbb{K}} := dim_{\mathbb{R}}\mathbb{K}^n$.

$\mathbf{B}_r^{n_{\mathbb{K}}}(x)$: closed ball of $\mathbb{K}^n$ centred at x with radius r for $n_{\mathbb{K}} := dim_{\mathbb{R}}\mathbb{K}^n$.

$\mathbf{S}_r^{n_{\mathbb{K}}-1}(x)$: Euclidean sphere of $\mathbb{K}^n$ centred at x and of radius r.

If the center is the origin we simplify the three last notations to $B_r^{n_{\mathbb{K}}}$, $\mathbf{B}_r^{n_{\mathbb{K}}}$ and $\mathbf{S}_r^{n_{\mathbb{K}}-1}$.

eucl: Euclidean/Hermitian metric tensor over $\mathbb{K}^n$.

$S^{\perp}$: orthogonal complement of a linear subspace S of $\mathbb{K}^n$.

$K_0(\varphi) := \varphi(\mathrm{crit}(\varphi))$: Set of critical values of φ.

$\widehat{S^{\mathbb{K}}(\mathbf{x})}$: The real half-cone over S with vertex $\mathbf{x}$.

$C_{\mathbf{x}}(S)$: tangent cone to S at $\mathbf{x}$.

$T_{\mathbf{x}}(S)$: tangent space to S at $\mathbf{x}$.

$m(S,\mathbf{x})$: multiplicity of S at $\mathbf{x}$.

$i_{\mathbf{x}}(S,L)$: intersection index of S and L at $\mathbf{x}$.

$\mathbf{H}_\infty$: hyperplane at infinity of $\mathbb{KP}^n$.

$S^\infty := S \cap \mathbf{H}_\infty$: set of points at infinity of S.

$\mathrm{clos}(S)$: Euclidean closure of S.

$\overline{S}^{\mathbb{KP}^n}$: projective closure of S.

d_{inn}^S: inner distance in S.

d_{out}^S: outer distance in S.

$G_{\mathbb{C}}(n-p,n)$: Grassmanian of $(n-p)$-dimensional subspaces of $\mathbb{C}^n$.

∂M: boundary of a manifold M.

g_M: Riemannian metric tensor of a Riemannian manifold M.

d_M: distance provided by the Riemannian metric tensor of M.

$l_M(\gamma)$: length of the path γ w.r.t. the Riemannian metric g_M.

$J(f)$: Jelonek set of non-proper values of f.

$\mathrm{Bif}(f)$: set of bifurcation values of f.

$D_{\mathbf{p}}F$: derivative of F at $\mathbf{p}$.

$S^*:= S \setminus \{\mathbf{0}\}$: the set of elements of a subset S of $\mathbb{R}^n$ which are different from the origin.

$f(\mathbf{x}) = o(g(\mathbf{x}))$ means that $\lim\limits_{\mathbf{x}\to\mathbf{0}} \dfrac{|f(\mathbf{x})|}{|g(\mathbf{x})|} = +\infty$.
$f(\mathbf{x}) = o_\infty(g(\mathbf{x}))$ means that $\lim\limits_{\mathbf{x}\to\infty} \dfrac{|f(\mathbf{x})|}{|g(\mathbf{x})|} = 0$.
$f(\mathbf{x}) = O(g(\mathbf{x}))$ means that there exists $K > 0$ such that $|f(\mathbf{x})| \leq K \cdot |g(\mathbf{x})|$ whenever $|\mathbf{x}|$ is small enough.
$f(\mathbf{x}) = O_\infty(g(\mathbf{x}))$ means that there exists $K > 0$ such that $|f(\mathbf{x})| \leq K \cdot |g(\mathbf{x})|$ whenever $|\mathbf{x}|$ is large enough.

2.2 Sub-manifolds

Definition 2.2.1. *Suppose M is a $\mathcal{C}^\infty$ manifold with or without boundary. An embedded sub-manifold of M is a subset $S \subset M$ that is a manifold in the subspace topology, endowed with a $\mathcal{C}^\infty$ structure with respect to which the inclusion map $\iota : S \hookrightarrow M$ is a smooth embedding, i.e., an immersion that is a homeomorphism onto its image in the subspace topology.*

Remark 2.2.1. *An embedded sub-manifold $S \subset M$ is said to be properly embedded if the inclusion $\iota : S \hookrightarrow M$ is a proper map.*

Proposition 2.2.1. *Suppose M is a $\mathcal{C}^\infty$ manifold with or without boundary and $S \subset M$ is an embedded sub-manifold. Then S is properly embedded if and only if it is closed.*

Proof. See page 100 in [23], Proposition 5.5. □

Definition 2.2.2. *Let U be an open subset of $\mathbb{R}^n$ and $k \in \{0, \ldots, n\}$. A k-dimensional slice of U (or simply a k-slice) is any subset of the form*

$$\{(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) \in U : x_{k+1} = c_{k+1}, \ldots, x_n = c_k\}$$

for some constants $c_{k+1}, \ldots, c_n \in \mathbb{R}$.

Let M be a $\mathcal{C}^\infty$ n-dimensional manifold, and let (U, φ) be a $\mathcal{C}^\infty$ chart on M . If S is a subset of U such that $\varphi(S)$ is a k-slice of $\varphi(U)$, then we say that S is a k-slice of U.

Given a subset S of M and a nonnegative integer k, we say that S satisfies the local k-slice condition if each point of S is contained in the domain of a $\mathcal{C}^\infty$ chart (U, φ) for M such that $S \cap U$ is a single k-slice in U. Any such chart is called a slice chart for S in M, and the corresponding coordinates $(x_1, \ldots, x_n)$ are called slice coordinates.

Proposition 2.2.2. *Let M be a $\mathcal{C}^\infty$ n-dimensional manifold. If a subset S of M is an embedded k-dimensional sub-manifold, then S satisfies the local k-slice condition.*

Proof. See Theorem 5.8 in [23]. □

The k-slice condition can be adapted to the case of sub-manifolds with boundary.

Definition 2.2.3. *Let M be a $\mathcal{C}^\infty$ manifold and let (U,φ) be a $\mathcal{C}^\infty$ chart. A k-dimensional half-slice of $\varphi(U)$ is a subset of the form*

$$\{(x_1,\ldots,x_n) \in \varphi(U) : x_k \geq 0,\ x_{k+1} = c_{k+1},\ldots,\ x_n = c_n\}.$$

A subset S of M satisfies the local k-slice condition for sub-manifolds with boundary if each point of S is contained in the domain of a $\mathcal{C}^\infty$ chart (U,φ) for M such that $S\cap U$ is either an ordinary k-dimensional slice or a k-dimensional half-slice.

Proposition 2.2.3. *Let M be a $\mathcal{C}^\infty$ manifold. If $S \subset M$ is an embedded k-dimensional sub-manifold with boundary, then S satisfies the local k-slice condition for sub-manifolds with boundary.*

Proof. See Theorem 5.51 in [23]. □

Proposition 2.2.4. *Let S be a m-dimensional embedded $\mathcal{C}^\infty$ sub-manifold in $\mathbb{R}^n$ and let $p \in S$. There is a neighborhood U of p in S and $\varepsilon > 0$ such that $S\cap U$ is the graph of a $\mathcal{C}^\infty$ mapping*

$$T_pS \cap B^n_\varepsilon(p) \to (T_pS)^\perp,$$

where T_pS is the tangent space to S at p and $(T_pS)^\perp$ its orthogonal complement.

Proof. Without loss of generality, let us assume that $p = \mathbf{0}$ and $T_{\mathbf{0}}S = \mathbb{R}^m \times 0$. By the slice criterion and the Local Immersion Theorem, there is a neighborhood U of $\mathbf{0}$ in S and a $\varepsilon > 0$ such that the projection $\pi : S\cap U \to T_pS \cap B_\varepsilon(p)$ given by

$$\pi(x_1,\ldots,x_n) = (x_1,\ldots,x_m)$$

is a $\mathcal{C}^\infty$ diffeomorphism.

Let $\pi^\perp : \mathbb{R}^n \to (T_pS)^\perp$ be the projection over $(T_pS)^\perp$ given by

$$(x_1,\ldots,x_n) \mapsto (x_{m+1},\ldots,x_n),$$

and consider the inverse $\pi^{-1} : T_pS \cap B^n_\varepsilon(p) \to S\cap U$. Notice that $S\cap U$ is the graph of $\pi^\perp \circ \pi^{-1} : S\cap U \to (T_pS)^\perp$. □

2.3 Critical points and critical values

Definition 2.3.1. *Let $\varphi : M \to N$ be a $\mathcal{C}^k$ mapping between $\mathcal{C}^k$ manifolds. A point $p \in M$ is critical if the pushfoward mapping $F^* : T_pM \to T_{\varphi(p)}N$ between the tangent spaces is not surjective.*

Definition 2.3.2. *Let $\varphi : M \to N$ be a $\mathcal{C}^k$ mapping between $\mathcal{C}^k$ manifolds and let $crit(\varphi)$ be the set of critical points of φ. The set*

$$K_0(\varphi) := \varphi(crit(\varphi))$$

is the set of critical values of φ. If $\boldsymbol{c} \notin K_0(\varphi)$, we say that $\boldsymbol{c}$ is a regular value.

The two Lemmas below are found on pages 16 and 17 of [25] and will play important role in the study of Lipschitz normally embeddedingness of affine part of unbounded sets.

Lemma 2.3.1. *Let $X \subset \mathbb{R}^n$ be an algebraic set and let $\Sigma(X)$ be the set of its singular points. A polynomial function $f : \mathbb{R}^n \to \mathbb{R}$ restricted to $X \setminus \Sigma(X)$ can have at most a finite number of critical values.*

Proof. The set of critical points of $f|_{X\setminus\Sigma(X)}$ has finitely many connected components. Since f is constant on each of those components, the number of critical values is at most the number of these connected components.

□

Lemma 2.3.2. *Let $X \subset \mathbb{C}^n$ be an algebraic set such that $\Sigma(X)$ is at most finite and let $x_0 \in X$. For a sufficiently small $\varepsilon > 0$ and a sufficiently large $R > 0$, the spheres $\mathbf{S}_\varepsilon^{2n-1}(x_0)$ and $\mathbf{S}_R^{2n-1}(x_0)$ intersect X transversally. Moreover, these intersections are $\mathcal{C}^\infty$ sub-manifolds.*

Proof. Consider the function $f : X \setminus \Sigma(X) \to \mathbb{R}$ given by $f(x) = |x_0 - x|^2$, which has finitely many critical values by Lemma 2.3.1. Given $\varepsilon > 0$ such that $\varepsilon^2 \in Im(f)$ and ε^2 is smaller then any positive critical value, the set

$$f^{-1}(\varepsilon^2) \cap (X \setminus \Sigma(X)) = \mathbf{S}_\varepsilon^{2n-1}(x_0) \cap (X \setminus \Sigma(X))$$

is a $\mathcal{C}^\infty$ sub-manifold since ε^2 is a regular value of $f|_{X\setminus\Sigma(X)}$. The same reasoning works for $\mathbf{S}_\varepsilon^{2n-1}(x_0) \cap (X \setminus \Sigma(X))$ by taking R^2 larger than any positive critical value. □

Theorem 2.3.1. *(Sard's Theorem) Let $\varphi : M \to N$ be a $\mathcal{C}^k$ mapping between $\mathcal{C}^\infty$ manifolds of dimension m, n, respectively. Then $K_0(\varphi)$ has null measure if $k \geq \max\{m - n + 1, 1\}$.*

Proof. See [27]. □

2.4 The Conical Structure Theorem and tangent cones

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.4.1. *Let Y be a subset of $\mathbb{K}^n$. Given a point $\boldsymbol{p} \in \mathbb{K}^n$, we define the real half-cone over Y with vertex $\boldsymbol{p}$ by*

$$\widehat{Y(\boldsymbol{p})} := \{tq + (1-t)\boldsymbol{p} \ : \ q \in Y,\ t \in [0, +\infty)\}.$$

Theorem 2.4.1. *(Conical Structure Theorem) Let $X \subset \mathbb{K}^n$ be an algebraic set and let $\boldsymbol{p}$ be a point of X. Given $\varepsilon > 0$, let $K_\varepsilon := X \cap \boldsymbol{S}^{2n-1}_\varepsilon(\boldsymbol{p})$ and let $X_\varepsilon := \widehat{K_\varepsilon(\boldsymbol{p})} \cap \boldsymbol{B}^{2n}_\varepsilon(\boldsymbol{p})$. There exists $\varepsilon > 0$ and a homeomorphism $\varphi : \boldsymbol{B}^{2n}_\varepsilon(\boldsymbol{p}) \to \boldsymbol{B}^{2n}_\varepsilon(\boldsymbol{p})$ such that*

1. $|\varphi(x) - \boldsymbol{p}| = |x - \boldsymbol{p}|$,
2. $\varphi|_{\boldsymbol{S}^{2n-1}_\varepsilon(\boldsymbol{p})}$ *is the identity mapping,*
3. $\varphi^{-1}(X \cap \boldsymbol{B}^{2n}_\varepsilon(\boldsymbol{p})) = X_\varepsilon$.

Moreover, if X has at most an isolated singularity at $\boldsymbol{p}$, the following property holds true:

4. $\varphi|_{X\setminus\{\boldsymbol{p}\}}$ *and* $\varphi^{-1}|_{X_\varepsilon\setminus\{\boldsymbol{p}\}}$ *are* $\mathcal{C}^\infty$.

Proof. See Theorem 9.3.6 in [3]. See also Theorem 2.10 in [25]. □

Let $X \subset \mathbb{K}^n$ be an analytic set.

Definition 2.4.2. *The geometric tangent cone $C_{g,\boldsymbol{0}}(X)$ of X at $\boldsymbol{0}$ is defined as the set of all $v \in \mathbb{K}^n$ such that there exist a sequence $\{x_k\} \subset X \setminus \{\boldsymbol{0}\}$ which converges to $\boldsymbol{0}$ and a sequence of numbers $\{t_k\} \subset \mathbb{K}$ for which the sequence $t_k x_k$ converges to v.*

Denote by $\mathcal{I}(X, \boldsymbol{0})$ the ideal of germs of analytic functions in $\mathbb{K}^n$ at $\boldsymbol{0}$ which vanish on $(X, 0)$.

Definition 2.4.3. *The algebraic tangent cone $C_{a,\boldsymbol{0}}(X)$ of X at $\boldsymbol{0}$ is the algebraic set defined by all polynomials in the ideal generated by the initial forms of all analytic functions f whose germs at $\boldsymbol{0}$ are in $\mathcal{I}(X, \boldsymbol{0})$.*

Remark 2.4.1. *For complex analytic (and algebraic) sets, the algebraic and geometric tangent cones are unique and coincide (see [30]). Then we use the simpler notation $C_{\boldsymbol{0}}(X)$.*

2.5 Multiplicity of analytic sets

Definition 2.5.1. *Let A be a locally closed set on a complex manifold X, i.e., A is the intersection of an open and a closed subset of X. Let $f : A \to Y$ be a continuous proper finite mapping into another complex manifold Y. The mapping f is an analytic ramified cover if:*

1. *there exists an analytic subset $\sigma \in Y$ with positive codimension and a natural number k such that $A \setminus f^{-1}(\sigma)$ is a complex sub-manifold in X and $f : A \setminus f^{-1}(\sigma) \to Y \setminus \sigma$ is a locally biholomorphic k-sheeted cover;*
2. *the set $f^{-1}(\sigma)$ is nowhere dense in A.*

The next result establishes that every pure p-dimensional analytic set can be locally represented as an analytic ramified cover of an open subset of $\mathbb{C}^n$.

Theorem 2.5.1. *Let $A \subset \mathbb{C}^n$ be a pure p-dimensional analytic set and let $\boldsymbol{x} \in A$. Then there exist a neighborhood U of $\boldsymbol{x}$, an open set $V \subset \mathbb{C}^p$ and a projection $\pi : A \cap U \to V$ which is a k-sheeted analytic ramified cover for some $k \in \mathbb{N}$.*

Proof. See Theorem 3.7 on page 38 of [4]. □

Let $A \subset \mathbb{C}^n$ be a pure p-dimensional analytic set and let $\mathbf{x} \in A$. Let $L \subset \mathbb{C}^n$ be a $(n-p)$-dimensional linear subspace such that $\mathbf{x}$ is an isolated point of $A \cap (\mathbf{x} + L)$. There exist an open neighborhood U of A and an open subset V of $L^{\perp} \in G_{\mathbb{C}}(p,n)$ such that the projection $\pi_L : A \cap U \to V$ is a ramified analytic cover, where $G_{\mathbb{C}}(p,n)$ is the Grassmanian of p-dimensional subspaces of $\mathbb{C}^n$.

Definition 2.5.2. *We define the intersection index of A and the plane $\boldsymbol{x} + L$ at $\boldsymbol{x}$ as the number of sheets of π_L. It is denoted by $i_{\boldsymbol{x}}(A, \boldsymbol{x} + L)$.*

Definition 2.5.3. *We define the multiplicity of A at $\boldsymbol{x}$ as*

$$m(A, \boldsymbol{x}) := \min\{i_{\boldsymbol{x}}(A, \boldsymbol{x} + L) : L \in G_{\mathbb{C}}(n-p, n)\}.$$

The minimum described in previous definition occurs for L in a open dense subset of $G_{\mathbb{C}}(n-p,n)$. In such a case we say that $\mathbf{x} + L$ and A are in general position at $\mathbf{x}$.

Proposition 2.5.2. *Let A be a pure p-dimensional analytic set in a neighborhood of $\boldsymbol{0} \in \mathbb{C}^n$ and let $L \in G_{\mathbb{C}}(n-p,n)$. The equality $m(A, \boldsymbol{0}) = i_{\boldsymbol{0}}(A, L)$ holds if and only if the plane L is transversal to A at $\boldsymbol{0}$, i.e., $C_{\boldsymbol{0}}(A) \cap L = \{\boldsymbol{0}\}$.*

Proof. See Proposition 2 on page 122 of [4]. □

The geometric definition of multiplicity given above is equivalent to the standard algebraic one in the case of hypersurfaces.

Proposition 2.5.3. *If f is the minimal defining function for the set $A = \{f = 0\}$, then*

$$m(A, \mathbf{0}) = ord_0(f).$$

Proof. See Corollary on page 122 of [4]. □

Remark 2.5.1. *Let $m(A, \mathbf{x}) \in \mathbb{N}_{\geq 1}$ be the multiplicity of A at $\mathbf{x}$. Observe that A is non-singular at $\mathbf{x}$ if and only if $m(A, \mathbf{x}) = 1$.*

2.6 Degree of algebraic sets

Let $\mathbb{CP}^n$ be the complex projective space with coordinates $[x_1 : \ldots : x_{n+1}]$. Let $\iota\colon \mathbb{C}^n \hookrightarrow \mathbb{CP}^n$ be the embedding given by $\iota(x_1, ..., x_n) = [x_1 : ... : x_n : 1]$ and let $p\colon \mathbb{C}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{CP}^n$ be the projection mapping given by $p(x_1, \ldots, x_{n+1}) = [x_1 : ... : x_{n+1}]$. Let $\mathbf{H}_\infty$ be the projective hyperplane defined by $\{x_{n+1} = 0\}$.

Let X be an algebraic set in $\mathbb{CP}^n$. Denote $\widetilde{X} := p^{-1}(X) \cup \{\mathbf{0}\}$. Let A be an algebraic set in $\mathbb{C}^n$ and let $\overline{\iota(A)}^{\mathbb{CP}^n}$ be the closure of $\iota(A)$ in $\mathbb{CP}^n$.

Remark 2.6.1. *Then set $\widetilde{X}$ is a homogeneous algebraic set in $\mathbb{C}^{n+1}$ and $\overline{\iota(A)}^{\mathbb{CP}^n}$ is an algebraic set in $\mathbb{CP}^n$.*

Definition 2.6.1. *Let X be an algebraic set in $\mathbb{CP}^n$. We define the degree of X by*

$$\deg(X) := m(\widetilde{X}, \mathbf{0}),$$

where $m(\widetilde{X}, \mathbf{0})$ is the multiplicity of $\widetilde{X}$ at $\mathbf{0} \in \mathbb{C}^{n+1}$.

Definition 2.6.2. *Let A be a complex algebraic set in $\mathbb{C}^n$. We define the degree of A by*

$$\deg(A) := \deg(\overline{\iota(A)}^{\mathbb{CP}^n}).$$

Proposition 2.6.1. *Let X be a pure d-dimensional algebraic set in $\mathbb{CP}^n$ and let Λ be an arbitrary $(n-d)$-dimensional plane in $\mathbb{CP}^n$ such that the set $X \cap \Lambda$ is 0-dimensional. Then*

$$\sum_{\mathbf{x} \in X \cap \Lambda} i_{\mathbf{x}}(X, \Lambda) = \deg(X).$$

Proof. See Proposition 2 on page 126 of [4]. □

Proposition 2.6.2. *Let A be a pure d-dimensional algebraic set in $\mathbb{C}^n$ and let L be an arbitrary $(n-d)$-dimensional plane in $\mathbb{C}^n$ such that the closures of A and L do not have points at infinity in common, i.e., $\overline{\iota(A)} \cap \overline{\iota(L)} \cap \boldsymbol{H}_\infty = \emptyset$. Then $\pi_L : A \to L^\perp$ is an analytic ramified cover with number of sheets equal to* $\deg(A)$.

Proof. See Corollary 1 on page 126 of [4]. □

2.7 Lipschitz normally embedded sets

Let M be $\mathcal{C}^k$ or $\mathbb{K}$-analytic manifold for $k \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ of positive $\mathbb{K}$-dimension m, and let g_M be a continuous Riemannian structure on M. The Riemannian metric g_M induces the distance function d_M on M: the distance between any pair of points of M is the infimum of the lengths $l_g(\gamma) = \int_a^b |\gamma'(t)|_g dt$ of rectifiable curves $\gamma\colon [a,b] \to M$ connecting the given pair of points. We say that a subset S of M is arc-rectifiable if any pair of points $\mathbf{s}, \mathbf{s}' \in S$ can be joined by a rectifiable path $\gamma\colon [a,b] \to S$. Obviously, any arc-rectifiable set is arcwise connected.

Definition 2.7.1. *Let S be a arc-rectifiable subset of M. We define the following two metric structures on S inherited from (M, g_M):*

1. *The outer distance on S is the distance function d^S_{out} on $S \times S$ obtained by restricting d_M to the subset $S \times S$ of $M \times M$.*
2. *The inner distance on S is the function*

$$d^S_{inn} : S \times S \to [0, +\infty)$$

such that, for any $\boldsymbol{s}, \boldsymbol{s}' \in S$, $d^S_{inn}(\boldsymbol{s}, \boldsymbol{s}')$ is given by the infimum of the lengths of the rectifiable paths lying in S joining $\mathbf{s}$ and $\mathbf{s}'$.

Remark 2.7.1. *Notice that we always have $d^S_{out} \leq d^S_{inn}$ since any rectifiable path joining $\boldsymbol{s}, \boldsymbol{s}'$ in S is also a rectifiable path joining these points in M.*

Definition 2.7.2. *Let d_1 and d_2 be two metrics on a set X. The metric spaces $E_1 = (X, d_1)$ and $E_2 = (X, d_2)$ are said to be equivalent if the identity mapping $id_X : E_1 \to E_2$ is bi-Lipschitz, i.e, there exists a constant $L > 1$ such that*

$$\frac{1}{L} \cdot d_1(\boldsymbol{x}, \boldsymbol{x}') \leq d_2(\boldsymbol{x}, \boldsymbol{x}') \leq L \cdot d_1(\boldsymbol{x}, \boldsymbol{x}')$$

for any $\boldsymbol{x}, \boldsymbol{x}' \in X$.

The definitions above yield immediately the question: when are the metric spaces (S, d^S_{out}) and (S, d^S_{inn}) equivalent?

Definition 2.7.3. *An arc-rectifiable subset S of the Riemannian manifold (M, g_M) is Lipschitz normally embedded (later shortened to LNE) in M if the metric spaces (S, d^S_{inn}) and (S, d^S_{out}) are equivalent, i.e., there exists a positive constant L_S such that*

$$d^S_{inn}(\boldsymbol{s}, \boldsymbol{s}') \leq L_S \cdot d^S_{out}(\boldsymbol{s}, \boldsymbol{s}')$$

for any $\boldsymbol{s}, \boldsymbol{s}' \in S$. Any constant L_S satisfying the inequality above is called a LNE constant of S.

Although we use the abbreviation LNE, it must be never forgotten that it is with respect to the distance d_M obtained from the Riemmanian metric g_M of M since the length of a rectifiable path is defined w.r.t. the outer metric d_M.

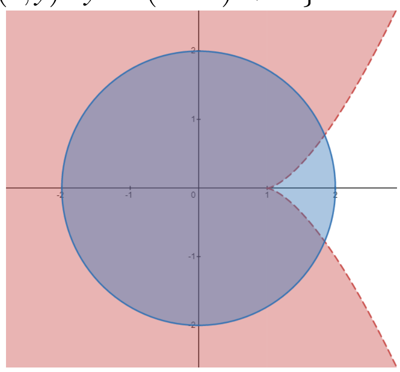
We can obtain some partial answer to the question above studying whether each point of S has a neighborhood in S which is LNE. In order to do so, we introduce a new notion.

Definition 2.7.4. *A subset S of (M, g_M) is locally LNE at the point $\boldsymbol{s} \in clos(S)$, the closure of S in M, if there exists an open neighbourhood U of $\boldsymbol{s}$ in M such that $S \cap U$ is LNE. The subset S is said to be locally LNE if it is locally LNE at each point of $clos(S)$.*

This notion is the best we can do to work nearby a point since we might have LNE and non-LNE representatives to the same germ.

Example 1. *The open ball B^2_1 centred at $\boldsymbol{0}$ of radius r is a LNE representative of the germ $(\mathbb{R}^2, \boldsymbol{0})$ since it is convex, while $\{(x,y) : y^2 - (x-r)^3 > 0\} \cap B^2_{2r}$ is a non-LNE one (see Lemma 3.2 in [12]) (see Figure 1).*

Figure 1 - The intersection of $\{(x,y) : y^2 - (x-1)^3 > 0\}$ with B^2_2.



Source: Created by the author.

The next Lemma slightly generalizes the statement of Proposition 2.4 from [18] although it has a similar proof. It presents a simple criterion for a compact subset of (M, g_M) to be LNE.

Lemma 2.7.1. *Let S be a compact connected subset of (M, g_M). It is LNE if and only if it is locally LNE at every point.*

Proof. We follow the proof of [18]. For each $\mathbf{s} \in S$, consider a LNE neighborhood $U_\mathbf{s}$ of $\mathbf{s}$ with Lipschitz constant $K_\mathbf{s}$. Let $\{U_i\}_{i=1,\cdots,l}$ be a finite subcover of $\{U_\mathbf{s}\}_{\mathbf{s}\in S}$ and let $K = \max\{K_i\}$. Consider the function $f : (S \times S) \setminus \Delta \to \mathbb{R}$ given by

$$(\mathbf{s}, \mathbf{s}') \mapsto \frac{d^S_{inn}(\mathbf{s}, \mathbf{s}')}{d^S_{out}(\mathbf{s}, \mathbf{s}')}.$$

Let $T \subset \bigcup_{i=1}^{l}(U_i \times U_i)$ be a neighborhood of the diagonal $\Delta = \{(\mathbf{s}, \mathbf{s}) : \mathbf{s} \in S\}$. Since the U_i are LNE, f is continuous on the compact set $(S \times S) \setminus T$ and hence bounded in $(M \times M) \setminus T$ by some $K' > 0$. Moreover, since $T \subset \bigcup_{i=1}^{l}(U_i \times U_i)$, f is bounded by K on $T \setminus \Delta$. Therefore, f is bounded by $\widetilde{K} = \max\{K, K'\}$ on $(S \times S) \setminus \Delta$. □

Now we see that bi-Lipschitz mappings preserve the property if being LNE.

Proposition 2.7.1. *Let $E_1 := (X, d_X)$ and $E_2 := (Y, d_Y)$ be two metric spaces and let S be a LNE subset of X. If $\varphi : E_1 \to E_2$ is a bi-Lipschitz mapping, then $\varphi(S)$ is LNE in E_2.*

Proof. Let $L > 1$ be a LNE constant for S, then we have

$$d^S_{inn}(\mathbf{x}, \mathbf{x}') \leq L \cdot d_X(\mathbf{x}, \mathbf{x}') \tag{2.1}$$

for any $\mathbf{x}, \mathbf{x}' \in S$. Let $K > 1$ be a bi-Lipschitz constant for φ, i.e.,

$$\frac{1}{K} \cdot d_Y(\varphi(\mathbf{x}), \varphi(\mathbf{x}')) \leq d_X(\mathbf{x}, \mathbf{x}') \leq K \cdot d_Y(\varphi(\mathbf{x}), \varphi(\mathbf{x}')), \tag{2.2}$$

for any $\mathbf{x}, \mathbf{x}' \in X$.

Given $\mathbf{y}, \mathbf{y}' \in \varphi(S)$, write $\mathbf{y} = \varphi(\mathbf{x})$ and $\mathbf{y}' = \varphi(\mathbf{x}')$. By the definition of length of a path and Estimates 2.2, for any rectifiable path $\gamma : [0,1] \to S$ joining $\mathbf{x}$ and $\mathbf{x}'$, we have

$$\frac{1}{K} \cdot l_Y(\varphi \circ \gamma) \leq l_X(\gamma) \leq K \cdot l_Y(\varphi \circ \gamma) \tag{2.3}$$

where l_X and l_Y are the lengths taken w.r.t. the metrics d_X and d_Y, respectively.

Estimates 2.1, 2.2 and 2.3 altogether yield

$$d^{\varphi(S)}_{inn}(\mathbf{y}, \mathbf{y}') \leq L \cdot d^S_{inn}(\mathbf{x}, \mathbf{x}') \leq K \cdot L \cdot d_X(\mathbf{x}, \mathbf{x}') \leq K^2 \cdot L \cdot d_Y(\mathbf{y}, \mathbf{y}').$$

□

In some sense, the corollary below tells that problems for the occurrence of the LNE property might happen at singularities or outside compact subsets.

Corollary 2.7.1. *Let N be a compact connected $\mathcal{C}^\infty$ embedded k-dimensional sub-manifold (with or without boundary) of a Riemannian manifold (M, g_M). Then N is LNE.*

Proof. By Proposition 2.2.2, each point $\mathbf{x} \in N$ is contained in the domain of a $\mathcal{C}^\infty$ chart $\varphi_\mathbf{x} : U_\mathbf{x} \to B_1^n$ for M such that $N \cap U_\mathbf{x}$ is a single k-slice in $U_\mathbf{x}$. If $\mathbf{x} \notin \partial N$, we set

$$\varphi_\mathbf{x}(N \cap U_\mathbf{x}) = \{(x_1, \ldots, x_n) \in B_1^n : x_{k+1} = 0, \ldots, \, x_n = 0\}.$$

If $\mathbf{x} \in \partial N$, we may assume that

$$\varphi_\mathbf{x}(N \cap U_\mathbf{x}) = \{(x_1, \ldots, x_n) \in B_1^n : x_1 \geq 0, \, x_{k+1} = 0, \ldots, \, x_n = 0\}.$$

In both cases we set $\varphi_\mathbf{x}(\mathbf{x}) = \mathbf{0}$ and denote $U'_\mathbf{x} := N \cap U_\mathbf{x}$.

Let $d_\mathbf{x}$ be the distance on $\varphi_\mathbf{x}(U'_\mathbf{x})$ obtained from the Riemannian metric $h_\mathbf{x} := (\varphi_\mathbf{x}^{-1})^* g_M|_{U'_\mathbf{x}}$. Since N is a $\mathcal{C}^\infty$ sub-manifold, we may assume that $\varphi_\mathbf{x}$ yields a bi-Lipschitz mapping $(U'_\mathbf{x}, g_M|_{U_\mathbf{x}}) \to (\varphi_\mathbf{x}(U'_\mathbf{x}), \mathrm{eucl}|_{\varphi_\mathbf{x}(U'_\mathbf{x})})$ with respect to the outer metrics. Therefore, the metric spaces $(\varphi_\mathbf{x}(U'_\mathbf{x}), |-|)$ and $(\varphi_\mathbf{x}(U'_\mathbf{x}), d_\mathbf{x})$ are equivalent. By Proposition 2.7.1, since φ is bi-Lipschitz and $\varphi_\mathbf{x}(U'_\mathbf{x})$ is LNE, so is $U'_\mathbf{x}$.

Therefore N is locally LNE at $\mathbf{x}$ and we conclude by Lemma 2.7.1. □

Suppose that (P, g_P) is another $\mathcal{C}^k$ manifold equipped with continuous Riemannian metric g_P, yielding the distance function d_P . The manifold $M \times P$ is naturally equipped with the Riemannian product metric $g_{M\times P} := g_M \otimes g_P$, whose associated distance function is $d_{M\times P} = d_M + d_P$. Let S be a subset of (M, g_M) naturally equipped with the metric d_{out}^S and consider a Lipschitz mapping $F : (S, d_{out}^S) \to (P, g_P)$ with Lipschitz constant K_F, i.e.,

$$d_P(F(\mathbf{s}), F(\mathbf{s}')) \leq K_F \cdot d_{out}^S(\mathbf{s}, \mathbf{s}') = K_F \cdot d_M(\mathbf{s}, \mathbf{s}'),$$

for any $\mathbf{s}, \mathbf{s}' \in S$. The metric space $S \times P$ comes naturally equipped with the product distance $d_{S\times P} := d_{out}^S + d_P$. We conclude this section with the following very useful and simple result.

Proposition 2.7.2. *Let S be a LNE (thus arc-rectifiable) subset of (M, g_M). If $F : (S, d_{out}^S) \to (P, g_P)$ is a Lipschitz mapping, then its graph is LNE in $(M \times P, d_{M\times P})$.*

Proof. Let λ be a LNE constant for S and let K_F be a Lipschitz constant for F: for any $\mathbf{x},\mathbf{x}' \in S$, we have

$$d_{inn}^S(\mathbf{x},\mathbf{x}') \leq \lambda \cdot d_{out}^S(\mathbf{x},\mathbf{x}') \text{ and } d_P(F(\mathbf{x}),F(\mathbf{x}')) \leq K_F \cdot d_{out}^S(\mathbf{x},\mathbf{x}').$$

Let $Q = M, M \times P$. Given a path $\gamma : [0,1] \to Q$, denote by $l_Q(\gamma)$ the length taken w.r.t. the Riemannian metric g_Q. Given $\varepsilon > 0$, there exists a rectifiable path $\gamma_\varepsilon : [0,1] \to S$ joining $\mathbf{x}$ and $\mathbf{x}'$ such that

$$l_M(\gamma_\varepsilon) < d_{inn}^S(\mathbf{x},\mathbf{x}') + \frac{\varepsilon}{(1+K_F)}.$$

Consider also the curve $\gamma_\varepsilon^F := (\gamma_\varepsilon, F \circ \gamma_\varepsilon) : [0,1] \to S \times P$. Then we have

$$\begin{aligned}
d_{inn}^{\mathrm{graph}(F)}((\mathbf{x},F(\mathbf{x})),(\mathbf{x}',F(\mathbf{x}'))) &\leq l_{M\times P}(\gamma_\varepsilon^F) \\
&\leq l_M(\gamma_\varepsilon) + K_F \cdot l_M(\gamma_\varepsilon) \\
&= (1+K_F) \cdot l_M(\gamma_\varepsilon) \\
&< (1+K_F) \cdot \left[d_{inn}^S(\mathbf{x},\mathbf{x}') + \frac{\varepsilon}{(1+K_F)} \right] \\
&= (1+K_F) \cdot d_{inn}^S(\mathbf{x},\mathbf{x}') + \varepsilon \\
&\leq (1+K_F) \cdot \lambda \cdot d_{out}^S(\mathbf{x},\mathbf{x}') + \varepsilon \\
&\leq (1+K_F) \cdot \lambda \cdot d_M(\mathbf{x},\mathbf{x}') + \varepsilon \\
&\leq (1+K_F) \cdot \lambda \cdot d_{M\times P}((\mathbf{x},F(\mathbf{x})),(\mathbf{x}',F(\mathbf{x}'))) + \varepsilon.
\end{aligned}$$

Therefore,

$$d_{inn}^{\mathrm{graph}(F)}((\mathbf{x},F(\mathbf{x})),(\mathbf{x}',F(\mathbf{x}'))) \leq (1+K_F) \cdot \lambda \cdot d_{M\times P}((\mathbf{x},F(\mathbf{x})),(\mathbf{x}',F(\mathbf{x}'))).$$

□

When $M = \mathbb{R}^m$ and $P = \mathbb{R}^p$, their product distance $d_M + d_P$ is not the Euclidean distance of $\mathbb{R}^{m+p}$, but it is equivalent to the Euclidean distance of $\mathbb{R}^{m+p}$, and this is how we will use Proposition 2.7.2 in the complex affine context.

2.8 Locally trivial fibrations and bifurcation values

The following notion lies in the heart of the second part of this thesis.

Definition 2.8.1. *A mapping $\varphi : \mathbb{K}^n \to \mathbb{K}^p$ is topologically trivial at the value $\boldsymbol{c} \in \mathbb{K}^p$, if there exist a neighbourhood $\mathcal{V}$ of $\boldsymbol{c}$ in $\mathbb{K}^p$ and a trivializing homeomorphism*

$$H : \varphi^{-1}(\boldsymbol{c}) \times \mathcal{V} \to \varphi^{-1}(\mathcal{V}) \tag{2.4}$$

which satisfies $(\varphi \circ H)(\boldsymbol{x}, \boldsymbol{t}) = \boldsymbol{t}$. In other words, the diagram below commutes.

$$\begin{array}{ccc} \varphi^{-1}(\boldsymbol{c}) \times \mathcal{V} & \xrightarrow{H} & \varphi^{-1}(\mathcal{V}) \\ & \searrow^{proj} & \downarrow \varphi \\ & & \mathcal{V} \end{array}$$

When H is a $\mathcal{C}^\infty$ diffeomorphism, the mapping φ is called $\mathcal{C}^\infty$ trivial at the value $\boldsymbol{c}$.

Remark 2.8.1. *In particular, the mapping φ is locally trivial at any value of the open subset $\mathbb{K}^p \setminus clos(Im(\varphi))$, the complement of the closure of the image of φ.*

Definition 2.8.2. *The values at which φ is not $\mathbb{C}^\infty$ trivial are called bifurcation values. Denote by $Bif_\infty(\varphi)$ be the set of bifurcation values.*

Theorem 2.8.1 ([28])**.** *Let $f : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial mapping, then $Bif_\infty(\varphi) \neq \mathbb{K}^p$. If $p = 1$, then $Bif_\infty(\varphi)$ is at most finite.*

The notion of $\mathcal{C}^\infty$ triviality is generally finer than that of $\mathcal{C}^0$ triviality and over the last fifty years, following the seminal paper [28], a significant literature about the $\mathcal{C}^0$ and the $\mathcal{C}^\infty$ local triviality has been developed (see for instance [29]; [14]; [26]; [20]; [17]).

A critical value $c \in K_0(\varphi)$ is a bifurcation value, since one cannot have a $\mathbb{C}^\infty$ trivial fibration at c. However, if we ask for topologically trivial fibration, we have examples of critical values which are topologically trivial as Examples 2 and 3.

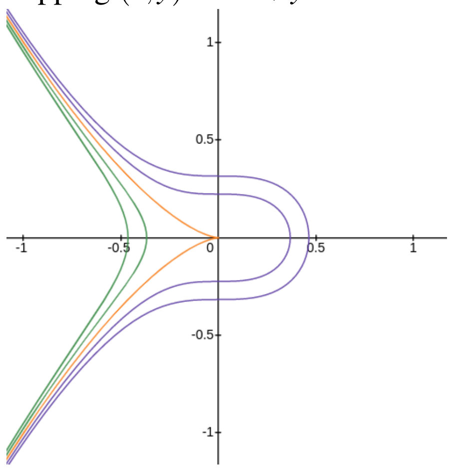
Example 2. *Note that the real polynomial function $x \mapsto x^{2023}$ is topologically trivial at each $c \in \mathbb{R}$, but not $\mathcal{C}^\infty$ trivial at 0 since $x \mapsto x^{\frac{1}{2023}}$ is not differentiable at 0.*

Example 3. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^3 + y^2$. The value $0 \in Im(f)$ is a topological trivial value although it is a critical value (see Figure 2).*

Moreover, we also have examples of bifurcation values which are regular.

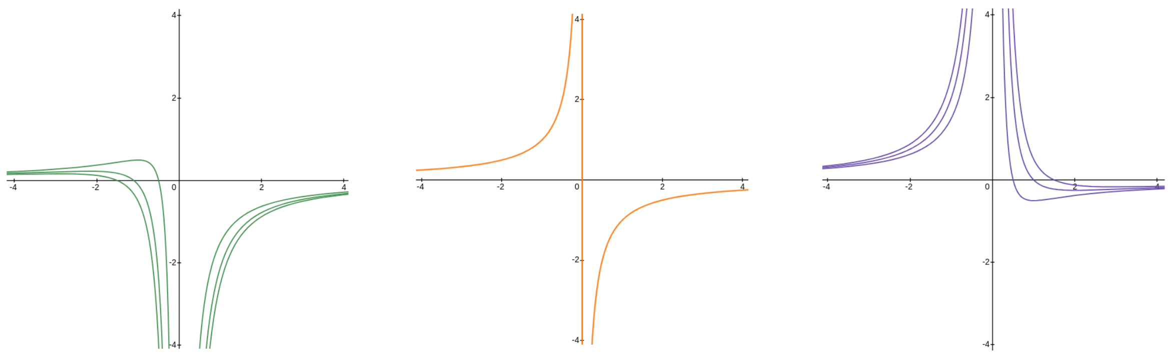
Example 4. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x + x^2 y$. The value $0 \in Im(f)$ is a regular bifurcation value of f since it has three connected components while any other value near 0 has only two connected components (see Figure 3).*

Figure 2 - Some levels of the mapping $(x,y) \mapsto x^3 + y^2$.



Source: Created by the author.
Note: The value 0 (orange), negative values near 0 (green) and positive values near 0 (purple) of the mapping $(x,y) \mapsto x^3 + y^2$.

Figure 3 - Some levels of the mapping $(x,y) \mapsto x + x^2 y$.



Source: Created by the author.
Note: The bifurcation value 0 (orange), three negative values near 0 (green) and three positive values near 0 (purple) of the mapping $(x,y) \mapsto x + x^2 y$.

We finish this section with the following result that provides a case when we obtain a $\mathcal{C}^\infty$ locally trivial fibration over a neighborhood of any value.

Theorem 2.8.2. *(Ehresmann's Theorem) Let $\varphi : M \to N$ be a proper submersion between $\mathcal{C}^\infty$ manifolds M and N. Then it is a $\mathcal{C}^\infty$ locally trivial fibration, i.e., for any $\boldsymbol{c} \in N$ there exists a neighbourhood $\mathcal{V}$ of $\boldsymbol{c}$ in N and a trivializing $\mathcal{C}^\infty$ diffeomorphism $H : \varphi^{-1}(\boldsymbol{c}) \times \mathcal{V} \to \varphi^{-1}(\boldsymbol{c})$ such that $(\varphi \circ H)(\boldsymbol{x}, \boldsymbol{t}) = \boldsymbol{t}$ for any $\boldsymbol{t} \in \mathcal{V}$.*

Proof. See page 154 in [6]. □

The phenomenon of being $\mathcal{C}^\infty$ locally trivial fibration for proper submersions presented in Theorem 2.8.2 motivates our next sub-section.

2.9 Values of properness and the Jelonek set

Definition 2.9.1. *A polynomial mapping $p : \mathbb{K}^n \to \mathbb{K}^p$ is dominant if its image $p(\mathbb{K}^n)$ is Zariski dense in $\mathbb{K}^p$, i.e., the unique algebraic subset of $\mathbb{K}^p$ containing $p(\mathbb{K}^n)$ is $\mathbb{K}^p$.*

Remark 2.9.1. *If $Y \subset \mathbb{K}^p$ contains an open subset of $\mathbb{K}^p$, then it is Zariski dense.*

Definition 2.9.2. *A continuous mapping $\varphi : X \to Y$ between topological spaces is proper if $\varphi^{-1}(K)$ is compact whenever K is compact.*

Definition 2.9.3. *A mapping $\varphi : \mathbb{K}^n \to \mathbb{K}^p$ is proper at the value $\mathbf{c} \in \mathbb{K}^p$, if there exists a neighbourhood $\mathcal{V}$ of $\mathbf{c}$ in $\mathbb{K}^p$ such that the restriction mapping of φ to $\varphi^{-1}(\mathcal{V})$ is proper. In this case we say that $\mathbf{c}$ is a value of properness of the mapping φ. The set $J(\varphi)$ of non-proper values of φ is called the Jelonek set.*

The Jelonek set of a real or complex polynomial mappings is always contained in an algebraic set of dimension at most $n-1$ as we can see in the next results.

Theorem 2.9.1. *Let $f = (f_1, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a dominant polynomial map. Then the set $J(f)$ is either empty or it is a hypersurface. Moreover, $Bif(f) = K_0(f) \cup J(f)$.*

Proof. See Theorem 15 in [15] and Proposition 4.1 in [17] . □

Theorem 2.9.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be a non-constant polynomial mapping. Then the set $J(f)$ is closed, semi-algebraic and for every non-empty connected component S of $J(f)$ we have $1 \leq \dim(S) \leq n-1$. Moreover, if $n = p$, then $Bif(f) = K_0(f) \cup J(f)$.*

Proof. See [16] and Proposition 4.2 in [17]. □

Remark 2.9.2. *In both real and complex cases in Theorems 2.9.1 and 2.9.2 when $n = p$, the set $Bif(f)$ is a closed semialgebraic set of Lebesgue measure zero (see also Theorem 3.3 in [17]), hence of positive codimension.*

When a polynomial mapping f is proper and regular at a value $\mathbf{c}$, the mapping f is $\mathcal{C}^\infty$ trivial at $\mathbf{c}$ by Remark 2.9.2 and Theorem 2.8.2 (Ehresmann's Theorem). Thus the set $\mathrm{Bif}(f) \setminus K_0(f)$ contains only non-proper values when it is non-empty.

2.10 Lipschitz trivial values

In this section we start an investigation on an intermediate case between $\mathcal{C}^0$ and $\mathcal{C}^\infty$ triviality: when and over which subset of values the mapping induces a locally bi-Lipschitz trivial fibre bundle structure. Our goal is to characterize polynomial mappings admitting Lipschitz trivial values, that is over a neighbourhood of which there is a bi-Lipschitz trivialization, problem recently raised in [9].

On a product $X \times Y$ of metric spaces we will consider the product metric.

Definition 2.10.1. *A Lipschitz trivial value $\boldsymbol{c} \in \mathbb{K}^p$ of the mapping $\varphi : \mathbb{K}^n \to \mathbb{K}^p$ is a value such that there exists a bi-Lipschitz trivializing mapping*

$$H : \varphi^{-1}(\boldsymbol{c}) \times \mathcal{V} \to \varphi^{-1}(\mathcal{V}) \tag{2.5}$$

over a neighborhood $\mathcal{V}$ of $\boldsymbol{c}$ which satisfies $(\varphi \circ H)(\boldsymbol{x},\boldsymbol{t}) = \boldsymbol{t}$. Let $L(\varphi)$ be the set of Lipschitz trivial values of φ.

Remark 2.10.1. *In light of Definition 2.10.1 and Remark 2.8.1, the subset $L(\varphi)$ is open and contains $\mathbb{K}^p \setminus clos(Im(\varphi))$. In particular, the mapping φ attains a Lipschitz trivial value only if its image is Zariski-dense in $\mathbb{K}^p$.*

Example 5. *Let $g : \mathbb{K}^2 \to \mathbb{K}^2$ be the dominant polynomial mapping given by $(x,y) \mapsto (x,xy)$. Given $(a,b) \in \mathbb{K}^2$, we have $g^{-1}(a,b) = \{(a,\frac{b}{a})\}$ if $a \neq 0$, $g^{-1}(0,0) = \{(0,y) : y \in \mathbb{K}\}$ and $f^{-1}(0,b) = \emptyset$ whenever $b \neq 0$. Let $\mathcal{V} = \{(a,b) \in \mathbb{K}^2 : a \neq 0\}$. For points $(x,y) \in \mathcal{U} := g^{-1}(\mathcal{V})$, the Jacobian matrix of f is $J(g) = \left(\begin{smallmatrix} 1 & 0 \\ y & x \end{smallmatrix}\right)$. Therefore, g is a proper submersion over $\mathcal{U}$ and by Theorem 2.8.2 (Ehresmann's Theorem), $g|_\mathcal{U}$ is a $\mathcal{C}^\infty$ locally trivial fibration. Since its fibers are compact and g is $\mathcal{C}^\infty$, by shrinking neighborhoods of the values in $\mathcal{V}$ to precompact ones, we conclude that $g|_\mathcal{U}$ is a bi-Lipschitz local trivial fibration, i.e., any value in $\mathcal{V}$ is a Lipschitz trivial value of g.*

Example 6. *Let $\varphi : \mathbb{K}^3 \to \mathbb{K}^2$ be the polynomial mapping defined as $(x,y,z) \mapsto (x,xy+xz)$. Notice that $\varphi = g \circ \pi$ where g is the mapping of Example 5 and $\pi : \mathbb{K}^3 \to \mathbb{K}^2$ is the linear surjective projection $(x,y,z) \mapsto (x,y+z)$. The set of Lipschitz trivial values of φ is the same as for g: $\mathbb{K}^2 \setminus (\{0\} \times \mathbb{K})$. Moreover, there is a single critical value $(0,0)$ (the same for g) whose level set is $\{0\} \times \mathbb{K}^2$, and none of the values $(0,b)$ with $b \neq 0$ are taken. Each level (a,b) with $a \neq 0$ is an affine line.*

If $\mathcal{W}$ is an open neighborhood of (a,b) in $\mathcal{V} := \mathbb{K}^2 \setminus (\{0\} \times \mathbb{K})$, the mapping H : $\varphi^{-1}(\mathcal{W}) \to \varphi^{-1}(a,b) \times \mathcal{W}$ defined as

$$(x,y,z) \mapsto \left(\left(a, \frac{y-z}{2} + \frac{b}{2a}, \frac{z-y}{2} + \frac{b}{2a} \right), (x, xy+xz) \right)$$

is a bi-Lipschitz trivializing homeomorphism. The inverse H^{-1} is given by

$$\big((a,y',z'),(c,d)\big) \mapsto \left(c, \frac{y'-z'}{2} + \frac{d}{2c}, \frac{z'-y'}{2} + \frac{d}{2c} \right).$$

For the case of polynomial mappings $\mathbb{C}^n \to \mathbb{C}$, we have the following result:

Theorem 2.10.1. *A non-constant complex polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}$ admits a Lipschitz trivial value if and only if it is a polynomial in a single variable.*

Proof. See [9]. □

2.11 Puiseux parameterization for curves

We present a version of Puiseux Theorem for analytic curves of any codimension.

Theorem 2.11.1. *Let $(Y, \boldsymbol{0})$ be an analytic curve germ of $\mathbb{C}^n$ at $\boldsymbol{0} \in \mathbb{C}^n$. Assume that Y is irreducible at $\boldsymbol{0}$ and let m be the multiplicity of Y at $\boldsymbol{0}$. Let L be the tangent cone of Y at $\boldsymbol{0}$, which is a complex line through $\boldsymbol{0}$. After an orthonormal change of complex coordinates we can assume that*

$$L := \{x = (x_1, \ldots, x_n) \in \mathbb{C}^n : x_2 = \ldots = x_n = 0\} = \mathbb{C} \times \boldsymbol{0}.$$

Then there exists a holomorphic map germ $F = (f_2, \ldots, f_n) : (\mathbb{C}, 0) \to (\mathbb{C}^{n-1}, \boldsymbol{0})$ such that

$$(Y, \boldsymbol{0}) = \{(s^m, f_2(s), \ldots, f_n(s)) : s \in (\mathbb{C}, 0)\},$$

where $f_j(s) = s^{a_j}\Phi_j(s)$, Φ_j is a holomorphic function with $\Phi_j(0) \neq 0$ and $a_j \geq m+1$ for each $j = 2, \ldots, n$.

Proof. See Proposition 1 of page 98 in [4]. □

2.12 Miscellaneous

The two next results will be used in the sequel to prove an interesting fact about the rank of a mappings admitting Lipschitz trivial values.

Theorem 2.12.1. *(Rademacher's Theorem) If $f : \mathbb{R}^m \to \mathbb{R}^n$ is a Lipschitz mapping, then it is differentiable almost everywhere.*

Proof. See Theorem 3.1.6 in [8]. □

Theorem 2.12.2. *(Hadamard's Inequality) Let D be a $n \times n$ matrix with complex entries and columns $d_1, \ldots, d_n$. Then*

$$|\det D| \leq \prod_{i=1}^{n} |d_i|.$$

Proof. See [13] or [22]. □

3 LIPSCHITZ NORMALLY EMBEDDED CURVES

In this chapter we investigate the LNE property for curves and obtain a characterization of the algebraic complex curves which are LNE in Theorem 3.6.1.

3.1 A criterion for isolated singularity germs to be LNE

Our first step is to obtain a criterion for isolated singularity germs to be LNE.

Lemma 3.1.1. *Let $(N_1,\mathbf{0}),\ldots,(N_s,\mathbf{0})$ be germs at $\mathbf{0}$ of $\mathcal{C}^\infty$ embedded sub-manifolds of $\mathbb{R}^n$ of positive dimensions and co-dimensions. Assume that $(Y,\mathbf{0}) := \bigcup_{j=1}^{s}(N_j,\mathbf{0})$ has an isolated singularity at $\mathbf{0}$. The germ $(Y,\mathbf{0})$ is locally LNE at $\mathbf{0}$ if and only if $T_{\mathbf{0}}N_j \cap T_{\mathbf{0}}N_k = \{\mathbf{0}\}$ for each $1 \le j < k \le s$.*

Proof. For each $j = 1,\ldots,s$, let $T_j := T_{\mathbf{0}}N_j$ and let $T_j^\perp$ be the orthogonal complement of T_j. By Proposition 2.2.4, there exists a radius $r_Y > 0$ and a $\mathcal{C}^\infty$ mapping

$$G_j : T_j \cap B^n_{2r_Y} \to (T_j)^\perp$$

whose graph Γ_j is a representative of $(N_j,\mathbf{0})$. Notice that the derivative $D_{\mathbf{0}}G_j$ is zero since $\mathrm{Ker}(D_{\mathbf{0}}G_j) = T_j$. Since G_j is $\mathcal{C}^\infty$, up to shrinking r_Y the mapping G_j is Lipschitz over B^n_r for any radius $0 < r \le r_Y$, with Lipschitz constant tending to 0 as r goes to 0. Denote $Y_{\le r} := Y \cap \mathbf{B}^n_r$. For $j = 1,\ldots,s$, we consider the representative of $(N_j,\mathbf{0})$ given by

$$N_j^{\le r} := \Gamma_j \cap \mathbf{B}^n_r.$$

For r_Y small enough, the $N_j^{\le r}$ are connected compact $\mathcal{C}^\infty$ embedded sub-manifolds with boundaries

$$\partial N_j^{\le r} := \Gamma_j \cap \mathbf{S}^{n-1}_r$$

which are diffeomorphic to $\mathbf{S}^{\dim T_j - 1}$. By Corollary 2.7.1, the representatives $N_j^{\le r}$ are LNE.

Given $j,k \in \{1,\ldots,s\}$ such that $j < k$ and a radius $r < r_Y$, let $\mathbf{x}_j \in (N_j^{\le r})^*$ and $\mathbf{x}_k \in (N_k^{\le r})^*$. Denote $r_j := |\mathbf{x}_j|$ and $r_k := |\mathbf{x}_k|$. Let 2α be the (non-oriented) angle between $\mathbf{x}_j$ and $\mathbf{x}_k$. By the Law of Sines, we have

$$\left|\frac{\mathbf{x}_j}{r_j} - \frac{\mathbf{x}_k}{r_k}\right| = 2\sin\alpha \tag{3.1}$$

and the Law of Cosines yields

$$|\mathbf{x}_j - \mathbf{x}_k|^2 = (r_j - r_k)^2 \cos^2 \alpha + (r_j + r_k)^2 \sin^2 \alpha. \tag{3.2}$$

By Estimate (3.2) we have

$$\sin \alpha \cdot (r_j + r_k) \leq |\mathbf{x}_j - \mathbf{x}_k| \leq r_j + r_k. \tag{3.3}$$

Assume that $T_j \cap T_k = \{\mathbf{0}\}$, for each pair $j \neq k$. For $j = 1, \ldots, s$, we consider the link of T_j at $\mathbf{0}$

$$S_j := T_j \cap \mathbf{S}_1^{n-1}.$$

Since the tangent spaces are transverse to each other we have

$$\delta := \min_{1 \leq j < k \leq s} \operatorname{dist}(S_j, S_k) > 0.$$

We can require that r_Y is small enough such that for each $j = 1, \ldots, s$, and each $\mathbf{x}_j \in (N_j^{\leq r_Y})^*$ the following estimates holds true

$$\operatorname{dist}\left(S_j, \frac{\mathbf{x}_j}{r_j}\right) \leq \frac{\delta}{4}.$$

For $1 \leq j < k \leq s$ and each $\mathbf{x}_j \in (N_j^{\leq r})^*$, $\mathbf{x}_k \in (N_k^{\leq r})^*$, we also find the following estimates

$$\left|\frac{\mathbf{x}_j}{r_j} - \frac{\mathbf{x}_k}{r_k}\right| \geq \frac{\delta}{2}. \tag{3.4}$$

By the choice of r and j,k, Equations (3.1) and Estimates (3.4), we have

$$\left|\frac{\mathbf{x}_j}{r_j} - \frac{\mathbf{x}_k}{r_k}\right| = 2 \sin \alpha \geq \frac{\delta}{2}. \tag{3.5}$$

By Equation (3.3) and Estimates (3.5) we have

$$\frac{\delta}{4}(r_k + r_j) \leq |\mathbf{x}_j - \mathbf{x}_k|. \tag{3.6}$$

Thus, for $l = j,k$, Estimates (3.6) yields

$$|\mathbf{x}_l| = r_l \leq \frac{4}{\delta}|\mathbf{x}_j - \mathbf{x}_k|. \tag{3.7}$$

Let A be a LNE constant for both $N_j^{\leq r}$ and $N_k^{\leq r}$. Since $\mathbf{0}$ is a isolated singularity of Y, we obtain

$$d_{inn}^{Y_{\leq r}}(\mathbf{x}_j, \mathbf{x}_k) \leq d_{inn}^{N_j^{\leq r}}(\mathbf{x}_j, \mathbf{0}) + d_{inn}^{N_k^{\leq r}}(\mathbf{x}_k, \mathbf{0}). \tag{3.8}$$

Estimates (3.8) and (3.7) yields

$$d_{inn}^{Y_{\leq r}}(\mathbf{x}_j,\mathbf{x}_k) \leq \frac{8A}{\delta}|\mathbf{x}_j-\mathbf{x}_k|.$$

Since each $N_j^{\leq r}$ is LNE, we conclude that $Y^{\leq r}$ is LNE.

Assume that $(Y,\mathbf{0})$ **is locally LNE at 0.** Since $(Y,\mathbf{0})$ has an isolated singularity at $\mathbf{0}$, we can assume that r_Y is small enough so that the following identity holds true whenever $j<k$:

$$(N_j^{\leq r_Y})^* \cap (N_k^{\leq r_Y})^* = \emptyset.$$

Given any pair $1 \leq j < k \leq s$, observe that to connect $\mathbf{x}_j$ to $\mathbf{x}_k$ within $Y^{\leq r_Y}$ it is necessary to go through $\mathbf{0}$. Therefore

$$d_{inn}^{Y}(\mathbf{x}_j,\mathbf{x}_k) \geq r_j + r_k. \tag{3.9}$$

Let $E := \{\{i_1,\ldots,i_t\} : 1 \leq i_1 < ... < i_t \leq s \ , \ \dim \bigcap_{j=1}^{t} T_{i_j} \geq 1\}$. Assume that E is not empty. Let J be an element of E and let

$$P_J := \bigcap_{j\in J} T_j.$$

For any $\mathbf{p} \in P_J$ and each $j \in J$, let $\mathbf{x}_j$ be the point $(\mathbf{p}, G_j(\mathbf{p}))$ of N_j. Since P_J is tangent at $\mathbf{0}$ to each N_l and $D_{\mathbf{0}}G_l$ is null, for any $l \in J$, as $\mathbf{p}$ goes to $\mathbf{0}$ in P_J we find that

$$|\mathbf{x}_j|,|\mathbf{x}_k| = |\mathbf{p}| + o(|\mathbf{p}|) \ \text{ and } \ |\mathbf{x}_j - \mathbf{x}_k| = o(|\mathbf{p}|)$$

for any pair j,k of J with $j \neq k$. Combining these last equations with Estimate (3.9) we have

$$\frac{d_{inn}^{Y}(\mathbf{x}_j,\mathbf{x}_k)}{|\mathbf{x}_j-\mathbf{x}_k|} \geq \frac{2\cdot[|\mathbf{p}|+o(|\mathbf{p}|)]}{o(|\mathbf{p}|)} \to +\infty$$

as $\mathbf{p}$ goes to $\mathbf{0}$. Therefore, the germ $(Y,\mathbf{0})$ cannot admit a LNE representative in any neighbourhood of $\mathbf{0}$. Necessarily E is empty. □

As a immediate consequence of the first part of proof of Lemma 3.1.1, we obtain the following result about LNE representatives.

Corollary 3.1.1. *Let Y be a representative of the germ $(Y,\mathbf{0})$ of Lemma 3.1.1. There exists a positive radius r_Y such that for each radius $r \in (0,r_Y]$, the subsets $Y \cap \mathbf{B}_r^n$ and $Y \cap B_r^n$ are LNE.*

3.2 The local case for curve germs

In this section we work in the local complex analytic category. We present in Proposition 3.2.1 a complete proof for a known local criterion for a complex analytic curve germ to be LNE (see also [5]). This result together with Proposition 3.2.1 fully characterize complex curves which are LNE in a given compact complex manifold. Let us set the following convention:

Convention 1. *A curve germ is a complex analytic curve germ at a prescribed point.*

Let $(Y,\mathbf{0})$ be a curve germ of $\mathbb{C}^n$ at $\mathbf{0} \in \mathbb{C}^n$. Assume that Y is irreducible at $\mathbf{0}$. Let m be the multiplicity of Y at $\mathbf{0}$. Let L be the tangent cone of Y at $\mathbf{0}$, which is a complex line through $\mathbf{0}$. The restriction to $(Y,\mathbf{0})$ of the (orthogonal) projection $\mathbb{C}^n \to L$ is a complex analytic finite mapping $p_L : (Y,\mathbf{0}) \to (L,\mathbf{0})$ inducing a holomorphic m-sheeted covering $(Y^*,\mathbf{0}) \to (L^*,\mathbf{0})$. After an orthonormal change of complex coordinates we can assume that

$$L := \{x = (x_1,\ldots,x_n) \in \mathbb{C}^n : x_2 = ... = x_n = 0\} = \mathbb{C} \times \mathbf{0}.$$

By Theorem 2.11.1 (Puiseux Theorem for curves), there exists a holomorphic map germ $F : (\mathbb{C},0) \to (\mathbb{C}^{n-1},\mathbf{0})$ such that

$$(Y,\mathbf{0}) = \{(s^m, F(s)) : s \in (\mathbb{C},0)\}. \tag{3.10}$$

More precisely, writing $F = (f_2,...,f_n)$, for each $j = 2,\ldots,n$, we know that

$$f_j(s) = s^{a_j}\Phi_j(s), \text{ with } \Phi_j(0) \neq 0 \text{ and } a_j \geq m+1$$

with the convention $a_j = \infty$ if and only if $f_j \equiv 0$. Moreover we require that $\gcd(m,a_2,\ldots,a_n) = 1$ in order to have a reduced representation.

Proposition 3.2.1. *Let $(X,\mathbf{0})$ be a curve germ of $(\mathbb{C}^n,\mathbf{0})$ and let $(X_1,\mathbf{0}),\ldots,(X_s,\mathbf{0})$ be its local irreducible components and let $m_1,\ldots,m_s$ be their respective multiplicities at $\mathbf{0}$. The germ $(X,\mathbf{0})$ admits a representative which is locally LNE at $\mathbf{0}$ if and only if the following conditions are satisfied:*

(i) $m_j = 1$ for each $j = 1,\ldots,s$,

(ii) $L_j \cap L_k = \{\mathbf{0}\}$ for each pair (j,k) such that $1 \leq j < k \leq s$ where L_j is the tangent cone to X_j at $\mathbf{0}$.

Proof. For each $j = 1,\dots,s$, let L_j be the tangent cone of X_j at $\mathbf{0}$, thus a complex line through $\mathbf{0}$. For each $j = 1,\dots,s$, the (orthogonal) projection $\pi_j : (X_j,\mathbf{0}) \to (L_j,\mathbf{0})$ is a complex finite mapping and it induces the germ of a holomorphic cover $(X_j^*,\mathbf{0}) \to (L_j^*,\mathbf{0})$ with $m_j \geq 1$ sheets.

Let $\lambda_1,\dots,\lambda_t$ be complex lines of $\mathbb{C}^n$ passing through the origin. We define the following 'angle'

$$\delta((\lambda_j)_{j=1,\dots,t}) := \min_{1\leq j<k\leq t}\{|\mathbf{u}_j-\mathbf{u}_k| : \mathbf{u}_j \in \lambda_j \cap \mathbf{S}_1^{2n-1} \text{ and } \mathbf{u}_k \in \lambda_k \cap \mathbf{S}_1^{2n-1}\}.$$

Observe that condition (ii) holds true if and only if $\delta((L_j)_{j=1,\dots,s}) > 0$.

Assume that (i) and (ii) are satisfied. By condition (i), each $(X_j,\mathbf{0})$ is a non-singular germ. Condition (ii) means that the lines L_j are pairwise transverse. By Lemma 3.1.1, the germ $(X,\mathbf{0})$ is locally LNE at $\mathbf{0}$.

Assume that (i) or (ii) is not satisfied. We prove that $(X,\mathbf{0})$ is not locally LNE at $\mathbf{0}$. First, we deal with the case where at least two components of $(X,\mathbf{0})$ are pairwise tangent.

Claim 3.2.1. *Assume $(X,\mathbf{0})$ is not irreducible and that $L := L_1 = \ldots = L_p$ for some $p \geq 1$ and $L_l \neq L_1$ for any $l \geq p+1$. Assume that $m_j = 1$, for $j = 1,\dots,p$. If $p \geq 2$, then X does not admit any representative locally LNE at $\mathbf{0}$.*

Proof of Claim 3.2.1. Assume $p \geq 2$. Up to a complex linear change of variables we can assume that L is not orthogonal to any of the L_j for $j \geq p+1$ and also that $L = \mathbb{C}\times\mathbf{0}$. Using parameterization (3.10), for each $j = 1,\dots,s$ we obtain a holomorphic map germ $F^j : (\mathbb{C},0) \to (\mathbb{C}^{n-1},\mathbf{0})$ such that

$$(X_j,\mathbf{0}) := \{(t^{m_j}, f_2^j(t),\dots,f_n^j(t)) : \ t \in (\mathbb{C},0)\}$$

and each f_k^j has multiplicity a_k^j at $\mathbf{0}$ with $a_k^j \geq m_j$. More precisely, if $j \leq p$ (case where $L_j = \mathbb{C}\times\mathbf{0}$), we have $a_k^j \geq m_j + 1$ for $k = 2,\dots,n$, while for $j \geq p+1$ (case where $L_j \neq \mathbb{C}\times\mathbf{0}$), there exists $k \geq 2$ such that $a_k^j = m_j$.

We can assume that there exists a positive radius r_X such that for each $1 \leq j < k \leq s$ the following holds true

$$(X_j \cap \mathbf{B}_{r_X}^{2n}) \cap (X_k \cap \mathbf{B}_{r_X}^{2n}) = \{\mathbf{0}\}.$$

Therefore to connect a point $\mathbf{x} \in X_j^* \cap \mathbf{B}_{rX}^{2n}$ to the point $\mathbf{x}' \in X_k^* \cap \mathbf{B}_{rX}^{2n}$ it is necessary to go through $\mathbf{0}$, and thus

$$d_{inn}^X(\mathbf{x},\mathbf{x}') \geq |\mathbf{x}| + |\mathbf{x}'|. \tag{3.11}$$

Let $(Y,\mathbf{0}) := \bigcup_{j=1}^{p}(X_j,\mathbf{0})$. We have $p = \sum_{j=1}^{p} m_j$. The orthogonal projection $\pi : \mathbb{C}^n \to L$ induces a germ of a holomorphic p-sheeted cover $Y^* \to L^*$ at $\mathbf{0}$.

Let $\mathbf{y} = (y,0,\ldots,0)$ be any point of L^* and let $\mathbf{x}_1,\ldots,\mathbf{x}_p$ be the points of $\pi^{-1}(y) \cap Y$. For any $1 \le j \le p$, we may assume that

$$\mathbf{x}_j = (y, f_2^j(y), \ldots, f_n^j(y))$$

where the multiplicity a_k^j of f_k^j is at least 2. Thus

$$|\mathbf{x}_j - \mathbf{y}| = |(0, f_2^j(y), \ldots, f_n^j(y)| \le \text{const} \cdot |y|^2 = \text{const} \cdot |\mathbf{y}|^2 \tag{3.12}$$

For $1 \le j < k \le p$, we have $\frac{p+1}{p} \le 2$. Then Estimate (3.12) yields

$$|\mathbf{x}_j - \mathbf{x}_k| \le |\mathbf{x}_j - \mathbf{y}| + |\mathbf{x}_k - \mathbf{y}| \le \text{const} \cdot |\mathbf{y}|^{\frac{p+1}{p}}. \tag{3.13}$$

Finally, by Estimates (3.11), for $1 \le j < k \le p$, we obtain

$$\frac{d_{inn}^X(\mathbf{x}_j,\mathbf{x}_k)}{|\mathbf{x}_j - \mathbf{x}_k|} \ge \frac{2|\mathbf{y}|}{\text{const} \cdot |\mathbf{y}|^{\frac{p+1}{p}}} = \frac{2}{\text{const} \cdot |\mathbf{y}|^{\frac{1}{p}}} \to \infty$$

as $|\mathbf{y}|$ goes to $\mathbf{0}$. Thus $(X,\mathbf{0})$ is not locally LNE at $\mathbf{0}$. □

Finally, we deal with the case when the multiplicity of an irreducible germ at $\mathbf{0}$ is higher than 1.

Claim 3.2.2. *If $m_1 \ge 2$, then the irreducible germ $(X_1,\mathbf{0})$ does not admit any representative locally LNE at $\mathbf{0}$.*

Proof of Claim 3.2.2. Assume that $m := m_1 \ge 2$ and that $L_1 = \mathbb{C} \times \mathbf{0}$. Let $\mathbf{y}$ be any point of L_1^*. Let $\mathbf{x}_1,\ldots,\mathbf{x}_m$ be the points of $\pi_1^{-1}(\mathbf{y})$. By definition of L_1 and using Puiseux Parametrization (3.10), whenever $|\mathbf{y}|$ is small enough we obtain the following estimate for each $j = 1,\ldots,m$

$$|\mathbf{x}_j - \mathbf{y}| \le \text{const} \cdot |\mathbf{y}|^{\frac{m+1}{m}}$$

after the same computations done in previous proof. Furthermore, for any $1 \le j < k \le m$ we get

$$|\mathbf{x}_j - \mathbf{x}_k| \le \text{const} \cdot |\mathbf{y}|^{\frac{m+1}{m}}.$$

Let γ be any path on X_1 connecting $\mathbf{x}_j$ to $\mathbf{x}_k$. Since X_1^* is a m-sheeted covering over L_1^* and the points $\mathbf{x}_j$ and $\mathbf{x}_k$ are in different sheets, the projection γ_1 of γ in L_1^* is not contractible. Thus we obtain the following estimates

$$l(\gamma) \ge l(\gamma_1) \ge 2|\mathbf{y}|.$$

Therefore,

$$\frac{d_{inn}^{X_1}(\mathbf{x}_j,\mathbf{x}_k)}{|\mathbf{x}_j-\mathbf{x}_k|} \geq \frac{2|\mathbf{y}|}{\text{const}\cdot|\mathbf{y}|^{\frac{m+1}{m}}} = \frac{2}{\text{const}\cdot|\mathbf{y}|^{\frac{1}{m}}} \to \infty$$

as $|\mathbf{y}|$ goes to $\mathbf{0}$ and thus $(X_1,\mathbf{0})$ cannot admit a representative locally LNE at $\mathbf{0}$. □

The case where $(X,\mathbf{0})$ is irreducible is covered by Claim 3.2.2. Therefore, Claims 3.2.1 and 3.2.2 combined show that both conditions (i) and (ii) are necessary for X to be LNE. □

Definition 3.2.1. *Let M be a complex manifold. A complex curve of M is a complex analytic subset of M, thus closed, and which is of local (complex) dimension* 1 *at each of its points.*

When the complex manifold M is compact, any complex curve X of M is compact. Therefore the set X_{sing} of singular points of X is either empty or consists of finitely many points. It is important to remember that Proposition 3.2.1 describes when the germ of X at a singular point is locally LNE.

Proposition 3.2.2. *A complex curve X of a compact complex Riemannian manifold M is LNE if and only if X is connected and is locally LNE at each point of its singular locus.*

Proof. If X is LNE, it is connected by definition and it is locally LNE at each of its points by Lemma 2.7.1.

Let $X_{\text{sing}} := \{\mathbf{x}_1,\ldots,\mathbf{x}_s\}$. For each $j=1,\ldots,s$, let $\mathcal{U}_j$ be an open neighborhood of $\mathbf{x}_j$ such that $X\cap\mathcal{U}_j$ is LNE in M. If $\bigcup_{j=1}^s X\cap\mathcal{U}_j = X$, then we conclude that X is LNE by Lemma 2.7.1 since it is compact. Otherwise, any point $\mathbf{x}$ of $X\setminus\bigcup_{j=1}^s X\cap\mathcal{U}_j$ is a non-singular point of X and by the slice criterion (Proposition 2.2.2) there exists an open neighborhood $\mathcal{U}$ of $\mathbf{x}$ in M and a chart $\Psi:\mathcal{U}\to B_2^{2m}$ of M such that $\Psi(\mathbf{x})=\mathbf{0}$ and

$$\Psi(X\cap\mathcal{U}) = \{(x_1,\ldots,x_{2m})\in B_2^{2m} : x_3=0,\ldots,\ x_n=0\}.$$

Observe that $\mathcal{V}_1 := \{(x_1,\ldots,x_{2m})\in B_1^{2m} : x_3=0,\ldots,\ x_n=0\}$ is LNE w.r.t. $eucl|_{\mathbf{B}_1^{2m}}$ since it is convex. Let $\mathbf{e}_1 := \Psi^*(eucl|_{\mathbf{B}_1^{2m}})$ be the pull-back of the Euclidean metric restricted to $\mathbf{B}_1^{2m}$. Let $\mathcal{W}_1 := \Psi^{-1}(\mathbf{B}_1^{2m})$. Since the restriction $\Psi|_{\mathcal{W}_1} : (\mathcal{W}_1, d_{\mathcal{W}_1}) \to (\mathbf{B}_1^m, |-|)$ is bi-Lipschitz, the distances provided by the Riemannian metrics $h_1 := g_M|_{\mathcal{W}_1}$ and $\mathbf{e}_1$ are equivalent. Thus $\Psi^{-1}(\mathcal{V}_1) = X\cap\Psi^{-1}(B_1^{2m})$ is LNE in (M,g_M). We conclude by Lemma 2.7.1. □

3.3 Preliminaries at infinity

In this section we start the investigation for the LNE property at infinity. We determine when an irreducible curve germ at a point at infinity is locally LNE at infinity. This notion was introduced in [10].

Definition 3.3.1. *A subset S of $\mathbb{C}^n$ is locally LNE at infinity (w.r.t. the euclidean metric of $\mathbb{C}^n$) if there exists a compact subset K of $\mathbb{C}^n$ such that $S \setminus K$ is LNE (w.r.t. the euclidean metric of $\mathbb{C}^n$).*

Convention 2. *Unless mentioned otherwise, being LNE is to be understood as being LNE as a subset of $\mathbb{C}^n$ w.r.t. the Euclidean metric.*

In light of Definition 3.3.1, we start this section with the following necessary result.

Lemma 3.3.1. *For any positive radius R, the subsets $\mathbb{C}^n \setminus B_R^{2n}$ and $\mathbb{C}^n \setminus \boldsymbol{B}_R^{2n}$ are LNE.*

Proof. It is the same proof for both. We do only the first case.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n \setminus B_R^{2n}$. Assume that $|\mathbf{x}| \le |\mathbf{y}|$. If $|\mathbf{x}| \le |\mathbf{x} - \mathbf{y}|$, we have

$$\begin{aligned} d_{\mathbb{C}^n \setminus B_R^{2n}}(\mathbf{x}, \mathbf{y}) &\le |\mathbf{y}| - |\mathbf{x}| + \pi |\mathbf{x}| \\ &\le |\mathbf{x} - \mathbf{y}| + \pi |\mathbf{x} - \mathbf{y}| \\ &= (\pi + 1)|\mathbf{x} - \mathbf{y}|. \end{aligned}$$

Now assume $|\mathbf{x} - \mathbf{y}| < |\mathbf{x}| \le |\mathbf{y}|$. Therefore, the angle θ between the vectors $\mathbf{x}$ and $\mathbf{y}$ is less than $\pi/3$. The length of the circular arc joining $\mathbf{x}$ and $\frac{|\mathbf{x}|}{|\mathbf{y}|}\mathbf{y}$ is $\theta \cdot |\mathbf{x}|$. If $\theta = 0$, then $d_{\mathbb{C}^n \setminus B_R^{2n}}(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$. For $0 < \theta < \frac{1}{3}$, we have

$$\begin{aligned} d_{\mathbb{C}^n \setminus B_R^{2n}}(\mathbf{x}, \mathbf{y}) &\le |\mathbf{y}| - |\mathbf{x}| + \theta \cdot |\mathbf{x}| \\ &\le |\mathbf{x} - \mathbf{y}| + \theta \cdot |\mathbf{x}| \\ &= |\mathbf{x} - \mathbf{y}| + \frac{\theta}{\sqrt{2 - 2\cos(\theta)}} \cdot \left| \mathbf{x} - \frac{|\mathbf{x}|}{|\mathbf{y}|}\mathbf{y} \right| \\ &\le |\mathbf{x} - \mathbf{y}| + \frac{\pi}{3} \cdot \left| \mathbf{x} - \frac{|\mathbf{x}|}{|\mathbf{y}|}\mathbf{y} \right| \\ &\le |\mathbf{x} - \mathbf{y}| + \frac{\pi}{3} \cdot |\mathbf{x} - \mathbf{y}| \\ &= \left(1 + \frac{\pi}{3}\right) \cdot |\mathbf{x} - \mathbf{y}|. \end{aligned}$$

□

The affine space $\mathbb{C}^n$ embeds in $\mathbb{CP}^n$ as $\mathbf{x} \mapsto [\mathbf{x}:1]$, that is $\mathbb{CP}^n = \mathbb{C}^n \sqcup \mathbf{H}_\infty$. Any point of $\mathbf{H}_\infty$ is called a point at infinity. Let S be a subset of $\mathbb{CP}^n$, the affine part S^a of S is defined as $S^a := S \cap \mathbb{C}^n = S \setminus \mathbf{H}_\infty$. The subset S is said to be affine if $S = S^a$. When S is affine we can consider $(S, \mathbf{H}_\infty)$ as subset germ of S in $\mathbb{C}^n$ and just write (S, ∞) for $(S, \mathbf{H}_\infty)$. Similarly if λ is a point of $\mathbf{H}_\infty$, we might consider (S, λ) as a subset germ in $\mathbb{C}^n$.

Let $(Y, \mathbf{y}_0)$ be a curve germ at the point $\mathbf{y}_0$ of $\mathbb{CP}^n$. We will denote by Y any representative of the germ $(Y, \mathbf{y}_0)$ in $\mathbb{CP}^n$.

Let λ be a point of $\mathbf{H}_\infty$ and let (Y, λ) be an irreducible curve germ at λ which is not contained in $\mathbf{H}_\infty$. Thus the tangent cone to Y at λ is a line L of the vector space $T_\lambda \mathbb{CP}^n$. The line L is either transverse to $\mathbf{H}_\infty$ or contained in $T_\lambda \mathbf{H}_\infty$, we will investigate both cases.

After a linear change of coordinates in $\mathbb{C}^n$, we can assume that

$$\lambda := [1:0:\ldots.:0].$$

Let $\mathcal{A}_1$ be the affine chart of $\mathbb{CP}^n$ defined as $\{z_1 \neq 0\}$. Let $[1:\mathbf{w}:z]$ be affine coordinates in $\mathcal{A}_1$. We further write $\mathbf{w} = (v, \mathbf{w}') \in \mathbb{C} \times \mathbb{C}^{n-2}$. We identify $T_\lambda \mathbb{CP}^n$ with $\mathcal{A}_1$ as well. Since $\mathbf{x}$ are the affine coordinates of $\mathbb{C}^n$, in $\mathbb{C}^n \cap \mathcal{A}_1$ we write

$$[x_1 : \mathbf{y} : 1] = [\mathbf{x} : 1] = [1 : \mathbf{w} : z].$$

If L is contained in $\{z = 0\}$, after a linear transformation of $\{z = 0\}$ in $\mathcal{A}_1$ (corresponding to a linear transformation of $\{x_1 = 0\}$ in $\mathbb{C}^n$) we can further require that

$$L = \{[1 : w_1 : 0 : \ldots : 0] : w_1 \in \mathbb{C}\}.$$

The line L is transverse to $\mathbf{H}_\infty$ at λ if and only if there exists $\mathbf{a} \in \mathbb{C}^{n-1}$ such that

$$L = \{[1 : z\mathbf{a} : z] : z \in \mathbb{C}\}.$$

After the affine translation $\mathbf{x} \mapsto (x_1, \mathbf{y} - \mathbf{a})$ in $\mathbb{C}^n$, corresponding to the linear change of coordinates $(\mathbf{w}, z) \mapsto (\mathbf{w}', z) = (\mathbf{w} - z\mathbf{a}, z)$ in $\mathcal{A}_1$, we can assume that the line L has the following equation

$$L = \{[1 : 0 : \ldots : 0 : z] : z \in \mathbb{C}\}.$$

The multiplicity $m := m(Y, \lambda)$ is not affected by this linear change of coordinates. Let

$$Y^* := Y \setminus \{\lambda\} \text{ and } L^* := L \setminus \{\lambda\}.$$

The projection $p_L : (Y^*, \lambda) \to (L^*, \lambda)$ is a holomorphic m-sheeted covering.

First, we investigate the LNE property of (Y, λ) in the case of transversality of Y to $\mathbf{H}_\infty$ at λ.

Lemma 3.3.2. *Let $\lambda \in \mathbf{H}_\infty$ and let (Y,λ) be an irreducible curve germ with multiplicity m at λ. Assume that the tangent cone L to Y at λ is transverse to $\mathbf{H}_\infty$. The germ (Y^a,λ) is locally LNE at infinity if and only if $m = 1$.*

Proof. Since λ is transverse to $\mathbf{H}_\infty$, we may assume that the line L has the following equation

$$L = \{[1:0:\ldots:0:z] : z \in \mathbb{C}\}.$$

Using parameterization (3.10) we find

$$(Y,\lambda) = \{[1:F(z):z^m] = [1:O(z^p):z^m] : z \in (\mathbb{C},0)\}$$

for a positive integer $p \geq m+1$ maximal for this property. In this case, $F \equiv 0$ would mean $p = \infty$, which corresponds to the case $Y^a = L$. In the affine point of view, where $x_1 = \frac{1}{z}$, we obtain

$$(Y^a,\lambda) = \{\left(x_1^m, x_1^m F(x_1^{-1})\right) = \left(x_1^m, O_\infty\left(x_1^{-(p-m)}\right)\right) : x_1 \in (\mathbb{C},\infty)\}$$

in such a way that if $(x_1,\mathbf{y}) \in (Y^a,\infty)$, then $\mathbf{y} \to \mathbf{0}$ as $x_1 \to \infty$. Let us consider the holomorphic map germ $(\mathbb{C},\infty) \to (\mathbb{C}^{n-1},0)$ given by

$$x_1 \mapsto G(x_1) := x_1^m F\left(x_1^{-1}\right) = O_\infty\left(x_1^{-(p-m)}\right).$$

Assume that $m = 1$. In this case, the germ (Y,λ) is non-singular at λ and hence

$$\mathbf{w} = z^p \alpha(z)$$

where $p \geq 2$ and $\alpha(z) \in \mathbb{C}\{z\}^{n-1}$ with $\alpha(0) \neq 0$. In affine coordinates we have

$$\mathbf{y} = G(x_1).$$

Since G is a power series in x_1^{-1} converging for $|x_1| \geq R_1 > 0$, it goes to 0 as x_1 goes to infinity. Thus its derivative goes to 0 as x_1 goes to infinity, therefore G yields a Lipschitz mapping $\mathbb{C} \setminus B_R^2 \to \mathbb{C}^{n-1}$ for any radius $R \geq R_1$. Proposition 2.7.2 and Lemma 3.3.1 together imply that $\{(x_1, G(x_1)) : |x_1| \geq R\}$ is LNE since it is the graph of a Lipschitz mapping over a LNE subset of $\mathbb{C}$. Therefore the germ (Y^a,λ) is locally LNE at infinity.

Assume that $m \geq 2$. Let $z \in \mathbb{C}^*$ and let ζ be a m-th root of z. Let $\mathbf{x}_0 := [1:F(\zeta):z]$ and $\mathbf{x}_1 := [1:F(\omega\zeta):z]$ where $\omega = e^{\frac{2\pi i}{m}}$ and let $\mathbf{x}_0^a := \left(\frac{1}{z}, \frac{F(\zeta)}{z}\right)$ and $\mathbf{x}_1^a := \left(\frac{1}{z}, \frac{F(\omega\zeta)}{z}\right)$ their respective coordinates in $\mathbb{C}^n$. Recall that $F(z) = O(z^p)$ for $p \geq m+1$. Therefore, for $|z|$ small enough we have

$$|\mathbf{x}_1^a - \mathbf{x}_0^a| = \left|\frac{F(\zeta)}{z} - \frac{F(\omega\zeta)}{z}\right| \leq \text{const} \cdot |z|^{\frac{p}{m}-1} = \text{const} \cdot \frac{1}{|x_1|^{\frac{p-m}{m}}} \tag{3.14}$$

where $\frac{p-m}{m} > 0$. We can further consider a representative of (Y^a, λ) taken outside a large Euclidean closed ball $\mathbf{B}^{2n}$ centred at $\mathbf{0}$ in such a way that the projection mapping

$$\pi : Y^a \setminus \mathbf{B}^{2n} \to \mathbb{C} \setminus \mathbf{B}^2,\ [x_1 : \mathbf{y} : 1] \mapsto x_1$$

is a m-sheeted holomorphic covering, where $\mathbf{B}^2$ is the Euclidean ball of $\mathbb{C}$ centred at $\mathbf{0}$ which is the image of $\mathbf{B}^{2n}$ under the projection onto the x_1-axis.

Let $\gamma\colon [0,1] \to Y^a$ be a rectifiable path from $\mathbf{x}_0^a = \gamma(0)$ to $\mathbf{x}_1^a = \gamma(1)$. We write $\gamma(t) = (x_1(t), \mathbf{y}(t))$. The path $\pi \circ \gamma : t \mapsto x_1(t)$ is well defined over $[0,1]$ and is a loop since $\pi(\mathbf{x}_0^a) = \pi(\mathbf{x}_1^a) \in \mathbb{C}^*$. Therefore it is not contractible in $\mathbb{C}^*$ and

$$l(\pi \circ \gamma) = \int_0^1 |x_1'(t)| dt \leq l(\gamma).$$

Since $\mathbf{x}_0^a$ and $\mathbf{x}_1^a$ are in different sheets of the covering, the path $\pi \circ \gamma$ has to avoid B^2 in $\mathbb{C}$, then it is not contractible. Thus the following estimate holds true

$$2|x_1(0)| \leq l(\pi \circ \gamma) \leq l(\gamma)$$

whenever $|x_1(0)|$ is big enough, i.e., when $\mathbf{x}_0^a$ and $\mathbf{x}_1^a$ are taken sufficiently 'close to infinity'. Combining this last estimate with Estimate (3.14), we obtain

$$\frac{d_{inn}^{Y^a}(\mathbf{x}_0^a, \mathbf{x}_1^a)}{|\mathbf{x}_0^a - \mathbf{x}_1^a|} \geq \frac{2}{\text{const}} \cdot \frac{|x_1|}{|x_1|^{\frac{m-p}{m}}} = \frac{2}{\text{const}} \cdot |x_1|^{\frac{p}{m}}$$

which goes to ∞ as $|x_1|$ goes to ∞ since $p \geq m+1$. Therefore, no representative of the germ (Y^a, λ) can be locally LNE at infinity. □

The next result presents the case of tangency of Y to $\mathbf{H}_\infty$. Its proof will follow from arguments similar to those used for the transverse case.

Lemma 3.3.3. *Let $\lambda \in \boldsymbol{H}_\infty$ and let (Y, λ) be an irreducible curve germ at λ which is not contained in $\boldsymbol{H}_\infty$. Assume that the tangent cone L to Y at λ is contained in $T_\lambda \boldsymbol{H}_\infty$. Then the germ (Y^a, λ) is not locally LNE at infinity.*

Proof. Since L is contained in $T_\lambda \mathbf{H}_\infty$ we have

$$L = \{[1 : w_1 : 0 : \ldots : 0] : w_1 \in \mathbb{C}\}.$$

Using parameterization (3.10) again we find

$$(Y, \lambda) = \{[1 : w_1^m : F(w_1)] = [1 : w_1^m : O(w_1^{m+1})] : w_1 \in (\mathbb{C}, 0)\}.$$

Since the curve germ is not contained in $\mathbf{H}_\infty$, there exist a positive integer $p \geq m+1$, a function $\varphi \in \mathbb{C}\{w_1\}$ satisfying $\varphi(0) \neq 0$ such that

$$z(w_1) = w_1^p \cdot \varphi(w_1).$$

Writing $F = (\mathbf{w}', z) = (w_3, \ldots, w_n, z)$, for each $j = 3, \ldots, n$, there exists an exponent $q_j \in \mathbb{N}_{\geq m+1} \cup \{\infty\}$ and a function $\Psi_j \in \mathbb{C}\{w_1\}$ such that

$$w_j(w_1) = w_1^{q_j} \cdot \Psi_j(w_1)$$

where $\Psi_j(0) \neq 0$ whenever the function w_j is not the null function, equivalently if $q_j < \infty$. Recall that by parameterization (3.10) we know that $\gcd(m, q_3, \ldots, q_n, p) = 1$.

Since $\varphi(0) \neq 0$, we can reparameterize the affine part (Y^a, λ) as

$$(Y^a, \lambda) = \left\{ [s^p : O_\infty\left(|s|^{p-m}\right) : 1] : s \in (\mathbb{C}, \infty) \right\}.$$

Since $q_j > m$ for all j, for $(x_1, \mathbf{y}) \in (Y^a, \lambda)$ from the affine point of view where $x_1 = s^p$, we obtain

$$|\mathbf{y}| = |x_1|^{\frac{p-m}{p}} [1 + o(1)]$$

whenever $|x_1|$ is large enough. We can further consider a representative of (Y^a, λ) taken outside a large Euclidean closed ball $\mathbf{B}^{2n}$ centred at $\mathbf{0}$ in such a way that the projection mapping

$$\pi : Y^a \setminus \mathbf{B}^{2n} \to \mathbb{C} \setminus \mathbf{B}^2, [x_1 : \mathbf{y} : 1] \mapsto x_1$$

is a p-sheeted holomorphic covering whenever $|x_1|$ is large enough, where $\mathbf{B}^2$ is the Euclidean ball of $\mathbb{C}$ centred at $\mathbf{0}$ image of $\mathbf{B}^{2n}$ under the projection on the x_1-axis. Let $\mathbf{x}, \mathbf{x}'$ in Y^a such that

$$\pi(\mathbf{x}) = \pi(\mathbf{x}') = x_1.$$

Since $p \geq m+1 \geq 2$, we assume that $\mathbf{x} \neq \mathbf{x}'$. If $|x_1|$ is large enough, we find

$$|\mathbf{x}|, |\mathbf{x}'| = |x_1| (1 + o_\infty(1))$$

and then we obtain

$$|\mathbf{x} - \mathbf{x}'| = O_\infty\left(|x_1|^{\frac{p-m}{p}}\right).$$

The reasoning done in the proof of the case $m \geq 2$ of Lemma 3.3.2 (taking a path γ joining $\mathbf{x}$ and $\mathbf{x}'$ and bounding its length by the length of its projection $\pi \circ \gamma$) yields the same type of estimate in this case:

$$\frac{d_{inn}^{Y^a}(\mathbf{x}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \geq \frac{2}{\text{const}} \cdot \frac{|x_1|}{|x_1|^{\frac{m-p}{m}}} = \frac{2}{\text{const}} \cdot |x_1|^{\frac{p}{m}}$$

which goes to ∞ as $|x_1|$ goes to ∞ since $p \geq m+1$. Therefore, no representative of the germ (Y^a, λ) can be locally LNE at λ. $\square$

3.4 Unbounded part of affine curves

In this section we pass to the algebraic category. It is a continuation of Section 3.3, but working with explicit representatives of germs at infinity.

Convention 3. *A curve in $\mathbb{C}^n$ is an affine algebraic curve. A curve in $\mathbb{CP}^n$ is a projective algebraic curve.*

Let X be a curve of $\mathbb{CP}^n$. We assume that $X^\infty = X \cap \mathbf{H}_\infty$ consists of finitely many points $\lambda_1, \ldots, \lambda_p$. Then $p \leq \deg(X)$. The germ of the affine part X^a at infinity is

$$(X^a, \infty) = \sqcup_{j=1}^p (X^a, \lambda_j).$$

Given any positive radius R, let us consider

$$X_R := X \cap \mathbf{S}_R^{2n-1} = X^a \cap \mathbf{S}_R^{2n-1}.$$

Using the Euclidean inversion $\text{inv}_{2n} : \mathbf{x} \mapsto \frac{\mathbf{x}}{|\mathbf{x}|^2}$, the euclidean closure of the image $\text{inv}_{2n}(X^a)$ of X^a is a real algebraic set of $\mathbb{R}^{2n} \supset \text{inv}_{2n}(\mathbb{C}^n)$, with an isolated singularity at $\mathbf{0}$. The Local Conic Structure at $\mathbf{0}$ (see Theorem 2.4.1) when combined with the inversion yields a Conical Structure Theorem at infinity, which implies that there exists a positive radius R_X such that for $R, R' \geq R_X$ the links X_R and $X_{R'}$ are diffeomorphic. Moreover for $R \geq R_X$ the sub-manifold with boundary

$$X_{\geq R}^a := X^a \setminus B_R^{2n}$$

is $\mathcal{C}^\infty$ diffeomorphic to the cylinder $X_R \times [R, +\infty)$, mapping X_r onto $X_R \times r$. Let $\mathcal{C}$ be a connected component of $X_{\geq R}^a$. Therefore the closure of $\mathcal{C}$ in $\mathbb{CP}^n$ intersects the hyperplane at infinity $\mathbf{H}_\infty$ in a single point $\lambda_{\mathcal{C}}$ since X is a complex curve. Two such connected components $\mathcal{C}_1, \mathcal{C}_2$ may accumulate at the same point at infinity, i.e., $\lambda_{\mathcal{C}_1} = \lambda_{\mathcal{C}_2}$.

For $R \geq R_X$, let $\mathcal{C}_1^R, \ldots, \mathcal{C}_e^R$, be the connected components of $X_{\geq R}^a$. Necessarily we have $e \leq \deg(X)$. A first key piece to your main result is the following:

Proposition 3.4.1. *Let X^a be an affine curve and let X be its projective closure. Assume that $\deg(X) = card(X^\infty)$. Then, for large enough radius R, each connected component $\mathcal{C}_j^R$ is LNE, for $j = 1, \ldots, \deg(X)$.*

Proof. Let $d := \deg(X)$ and let $i_\lambda(X, \mathbf{H}_\infty)$ be the local intersection index of the germ (X, λ) with $(\mathbf{H}_\infty, \lambda)$. Proposition 2.6.1 yields

$$\sum_{\lambda \in X^\infty} i_\lambda(X, \mathbf{H}_\infty) = \deg(X).$$

By hypothesis, we obtain $\mathrm{card}(X^\infty) = \sum_{\lambda \in X^\infty} i_\lambda(X, \mathbf{H}_\infty)$. Since $i_\lambda(X, \mathbf{H}_\infty) \geq m(X, \lambda) \geq 1$, we conclude that $i_\lambda(X, \mathbf{H}_\infty) = 1$ for any $\lambda \in X^\infty$. The curve germ (X, λ) is non-singular and, by Proposition 2.5.2, it is in general position with $\mathbf{H}_\infty$ at λ for each $\lambda \in X^\infty$, i.e., the tangent cone to X at λ is transverse to $\mathbf{H}_\infty$. Thus we are in the hypotheses of Lemma 3.3.2.

Let $X^\infty = \{\lambda_1, \ldots, \lambda_d\}$ and assume that $(X^a, \lambda_j) = (\mathcal{C}_j^R, \lambda_j)$ for each $j = 1, \ldots, d$, and each $R \geq R_X$. Let L_j be the line tangent to X at λ_j. Since it is transverse to $\mathbf{H}_\infty$ we can consider L_j as an affine line of $\mathbb{C}^n$. Let $\mathbf{e}_j$ be a unit vector of $\mathbb{C}^n$ such that the point $\lambda_j \in \mathbf{H}_\infty$ corresponds to the complex line $\mathbb{C}\mathbf{e}_j$. Let H_j be the complex hyperplane orthogonal to $\mathbb{C}\mathbf{e}_j$. Since L_j is transverse to $\mathbf{H}_\infty$ there exists $\mathbf{a}_j \in H_j$ such that

$$L_j = \mathbb{C}\mathbf{e}_j + \mathbf{a}_j.$$

From the proof of Lemma 3.3.2, we deduce that (X^a, λ_j) is a Lipschitz and holomorphic graph over $\mathbb{C} \setminus B^2_{r_j} = \mathbb{C}\mathbf{e}_j \setminus B^2_{r_j}$ for each $j = 1, \ldots, d$.

Let

$$r_X := \max_{j=1,\ldots,d} r_j.$$

For each $R \geq r_X$, let

$$Y_j^R := \{(s, G_j(s)) = s\mathbf{e}_j + G_j(s) \in \mathbb{C}\mathbf{e}_j \times H_j : s \in \mathbb{C} \setminus B^2_R\}$$

where $G_j : \mathbb{C} \setminus B^2_{r_j} \to \mathbb{C}^{n-1}$ is the holomorphic mapping taking values in $\mathbb{C}^{n-1} = H_j$, as the one obtained in the proof of Lemma 3.3.2. We recall that

$$G_j(s) = g_j\left(\frac{1}{s}\right)$$

where $g_j(t) \in \mathbb{C}\{t\}^{n-1}$ and $g_j(0) = \mathbf{a}_j$. Therefore,

$$\lim_{|s| \to \infty} G_j'(s) = 0.$$

Since G_j goes to $\mathbf{a}_j$ and its derivative goes to $\mathbf{0}$ as s goes to ∞, for every $\varepsilon > 0$, there exists $r_\varepsilon \geq \max\{R_X, r_X\}$ such that

$$|G_j(x_1) - G_j(x_1')| < \varepsilon |x_1 - x_1'| \;\text{ and }\; |G_j(x_1)|^2 < |\mathbf{a}_j|^2 + \varepsilon^2$$

whenever $x_1, x_1' \in \mathbb{C}\mathbf{e}_j \setminus B^2_{r_\varepsilon}$. Thus the mapping $\Gamma_j : s \mapsto s\mathbf{e}_j + G_j(s)$ is holomorphic over $\mathbb{C} \setminus B^2_{r_\varepsilon}$ and is Lipschitz, with Lipschitz constant smaller than $(1+\varepsilon)$. Since Γ_j is bi-holomorphic, up to taking a smaller ε we find that $p_j := \Gamma_j^{-1}$, that is the projection onto $\mathbb{C}\mathbf{e}_j$, is Lipschitz with constant larger than $(1-\varepsilon)$.

If ε and r_ε are given, for each $j = 1, \dots, d$, we obtain the inclusions

$$Y_j^{R_{j,\varepsilon}} \subset \mathcal{C}_j^{R_{j,\varepsilon}} \subset Y_j^R \text{ where } R_{j,\varepsilon} := \sqrt{R^2 + |\mathbf{a}_j|^2 + \varepsilon^2}$$

whenever $R \geq r_\varepsilon$.

By Lemma 2.3.2, $\partial \mathcal{C}_j^R$ is a $\mathcal{C}^\infty$ sub-manifold. Then we consider the following smooth Jordan curve of L_j:

$$Z_j^R := p_j(\partial \mathcal{C}_j^R),$$

which is diffeomorphic to $\mathbf{S}^1$. Let D_j^R be the open disk bounded by Z_j^R. In order to use Lemma 2.7.2 we prove the following:

Claim 3.4.1. *$L_j \setminus D_j^R$ is LNE.*

Proof of Claim 3.4.1. We know that D_j^R contains the closed ball $\mathbf{B}^2_{(1-\varepsilon)R}$ centred at $0 \in L_j$. Since Z_j^R is a compact connected sub-manifold of L_j it is LNE with LNE constant K_Z. Let z and z' be two points of E. The only case to deal with is when the segment $I = [z, z']$ intersects D_j^R. In this case we can assume that $z' \notin Z_j^R$.

Let $z_1, \dots, z_{k-1}$ be the points of Z_j^R which belong to I. If z is not in Z_j^R let $z_0 := z$ and $z_k := z'$. The indexation is done in such a way that the (real) vector $z_j - z_i$ is positively co-linear to $z' - z$ whenever $i < j$. Observe that for $i = 1, \dots, k-1$, we have

$$d_{inn}^{L_j - D_j^R}(z_i, z_{i+1}) \leq d_{inn}^{Z_j^R}(z_i, z_{i+1}) \leq K_Z \cdot |z_{i+1} - z_i|,$$

since $Z_j^R = \partial(L_j - D_j^R)$. Let a_0 be the minimum of the indices i such that $z_i \in I$, that is $a_0 \in \{0, 1\}$. Since $K_Z \geq 1$, we deduce

$$d_{inn}^{L_j - D_j^R}(z, z') \leq \sum_{i=a_0}^{k-1} d_{inn}^{L_j - D_j^R}(z_i, z_{i+1}) \leq \sum_{i=a_0}^{k-1} K_Z \cdot |z_{i+1} - z_i| = K_Z \cdot |z' - z|.$$

□

Since $\mathbf{B}^2_R \cap \mathbb{C}\mathbf{e}_j \subset D_j^{R_{j,\varepsilon}}$ and the mapping Γ_j is Lipschitz over $\mathbb{C}\mathbf{e}_j \setminus B^2_R$, it is also Lipschitz over $\mathbb{C}\mathbf{e}_j \setminus D_j^R$. The result follows from Proposition 2.7.2.

□

There are two cases when $\text{card}(X^\infty) < \deg(X)$. The first one has already been mentioned in Lemma 3.3.3, when the germ (X,λ) is irreducible and with tangent cone L at λ contained in the hyperplane at infinity. The second is when (X,λ) is not irreducible.

Lemma 3.4.1. *Let X be a curve of $\mathbb{CP}^n$. Let λ be an isolated point of X^∞. If the germ (X,λ) is not irreducible then X^a is not LNE.*

Proof. We can assume that $\lambda = [1:0:\ldots.:0]$. Let $(X_1,\lambda),\ldots,(X_N,\lambda)$ be the irreducible components of (X,λ). Fix $R \geq R_X$ so that $S := X^a_{\geq R}$ is not connected. Notice that $(X^a,\lambda) = (S,\lambda)$. For each connected component $\mathcal{C}$ of S accumulating at λ, there exists a unique index $j \in \{1,\ldots,N\}$ such that $(\mathcal{C},\infty) = (X^a_j,\lambda)$.

Let $\mathcal{C}_1,\mathcal{C}_2$ be two connected components of S whose germ at ∞ are $(X^a_1,\lambda),(X^a_2,\lambda)$, respectively. Let $(\mathbf{x}_k)_k \subset \mathcal{C}_1$ and $(\mathbf{x}'_k)_k \subset \mathcal{C}_2$ be two sequences converging to ∞ and such that

$$\mathbf{x}_k = (k,\mathbf{y}_k) \ \textit{and} \ \mathbf{x}'_k = (k,\mathbf{y}'_k).$$

Therefore we obtain

$$|\mathbf{x}_k - \mathbf{x}'_k| = o_\infty(k).$$

Let γ_k be a rectifiable path on X^a connecting $\mathbf{x}_k$ and $\mathbf{x}'_k$. Since the sequences are in different connected components of S, the length of γ_k satisfies the following inequality

$$l(\gamma_k) \geq \text{dist}(\mathbf{x}_k,\partial S) + \text{dist}(\mathbf{x}'_k,\partial S),$$

where ∂S is the boundary of S, which is compact. Moreover, for k large enough we have

$$\text{dist}(\mathbf{x}_k,\partial S) \geq \frac{k}{2} \ \text{ and } \ \text{dist}(\mathbf{x}'_k,\partial S) \geq \frac{k}{2}.$$

Therefore,

$$\frac{l(\gamma_k)}{|\mathbf{x}_k - \mathbf{x}'_k|} \geq \frac{k}{o_\infty(k)} \to \infty$$

as k goes to ∞. We conclude that X^a cannot be LNE.

□

3.5 Bounded part of affine curves

In this section we use the exact same objects, notations and hypotheses from Section 3.4. We investigate the LNE property to a 'bounded part' of X^a, the affine part of an algebraic projective curve X.

Let X_{sing} be the singular locus of X, consisting at most of finitely many points. Observe that the affine part $(X_{\text{sing}})^a$ of X_{sing} is exactly the set of singular points $(X^a)_{\text{sing}}$ of the affine part X^a of X. Let us denote it by X^a_{sing}.

Recall that we consider a positive radius R_X such that for $R, R' \geq R_X$ the links X_R and $X_{R'}$ are diffeomorphic. For $R \geq R_X$ the subset

$$X^a_{\leq R} := X^a \cap \mathbf{B}^{2n}_R$$

is a semi-algebraic subset of $\mathbb{C}^n$ with $\mathcal{C}^\infty$ boundary $X_R := X \cap \mathbf{S}^{2n-1}_R$. If the affine part X^a is connected, then for $R \geq R_X$ the subset $X^a_{\leq R}$ is also connected.

The next result is another key piece of our main result on LNE curves.

Proposition 3.5.1. *Let X be a projective curve such that its affine part X^a is connected and not empty. For $R \geq R_X$, the subset $X^a_{\leq R}$ is LNE if and only if X^a is locally LNE at each point of X^a_{sing}.*

Proof. Let $S := X^a_{\leq R}$. Since $S \times S$ is connected, compact and semi-algebraic (thus arc-rectifiable), the function d^S_{inn} admits a maximum l_S.

If S is LNE, than we can take S as a LNE neighborhood of each point. Therefore, S is locally LNE at each point of X^a_{sing}.

Assume that X^a is locally LNE at each point of X^a_{sing}. Let $\mathbf{x}$ be a point of $X^a \setminus X^a_{\text{sing}}$. If $|\mathbf{x}| < R$, the germ $(S, \mathbf{x})$ is bi-holomorphic to the germ $(\mathbb{C}, 0)$. In the other case, when $|\mathbf{x}| = R$, we recall that $X^a_{\leq R} \cap \mathbf{S}^{2n-1}_R$ is a $\mathcal{C}^\infty$ sub-manifold by Lemma 2.3.2 and then the germ $(S, \mathbf{x})$ is C^∞ diffeomorphic to the germ $(\mathbb{R} \times [0, \infty), \mathbf{0})$. In both cases, S is locally LNE at $\mathbf{x}$ by Proposition 2.7.1. Therefore S is locally LNE at each of its points. Since it is compact, we conclude that S is LNE by Lemma 2.7.1. □

3.6 Characterization of affine LNE curves

Now that we have all the ingredients, we prove our main result on LNE property :

Theorem 3.6.1. *Let X be a connected projective curve of $\mathbb{CP}^n$ of degree $\deg(X)$ such that X^∞ is finite. The affine curve X^a is LNE if and only if the following conditions are satisfied:*

1. *X^a is connected;*
2. *X^a is locally LNE at each of its singular points;*
3. *$card(X^\infty) = \deg(X)$.*

Proof. **Assume X^a is LNE.** It is connected and it is locally LNE at each of its point, thus we obtained (1) and (2). By Lemma 3.4.1, for each $\lambda \in X^\infty$, the germ (X,λ) is irreducible. Moreover, by Lemma 3.3.3 we deduce also that (X,λ) is transverse to $\mathbf{H}_\infty$ at each λ of X^∞. By Lemma 3.3.2, we conclude that, for each $\lambda \in X^\infty$, the germ (X,λ) is non-singular, thus $m(X,\lambda) = 1$. Since X is transverse to $\mathbf{H}_\infty$ at λ, by Proposition 2.5.2, we have

$$i_\lambda(X,\mathbf{H}_\infty) = m(X,\lambda) = 1$$

for each $\lambda \in$. By Proposition 2.6.1,

$$\deg(X) = \sum_{\lambda \in X^\infty} i_\lambda(X,\mathbf{H}_\infty) = \operatorname{card}(X^\infty).$$

Assume that conditions (1), (2), (3) are satisfied. Then there exists R_X such that $X^a_{\leq R}$ is a connected and X_R is a $\mathcal{C}^\infty$ compact embedded sub-manifold whenever $R \geq R_X$. In this case, hypothesis (2) and Proposition 3.5.1 guarantee that $X^a_{\leq R}$ is LNE.

Condition (3) together with Proposition 3.4.1 yields that each connected component of $X^a_{\geq R}$ is LNE whenever R is large enough. Denote $d := \deg(X)$. Let $X^\infty = \{\lambda_j\}_{j=1,\dots,d}$ and let $\mathcal{C}_1,\dots,\mathcal{C}_d$ be the connected components of $X^a_{\geq R}$, indexed in such a way that λ_j is the accumulation point at infinity of $\mathcal{C}_j$.

Let $\mathcal{C}_0 := X^a_{\leq R}$. Since each $\mathcal{C}_j$ is LNE, we may consider a positive constant A which is a LNE constant for each $\mathcal{C}_j$, i.e., for any $j = 0,\dots,d$ we have

$$d_{inn}^{\mathcal{C}_j}(\mathbf{x},\mathbf{x}') \leq A|\mathbf{x}-\mathbf{x}'|$$

whenever $\mathbf{x},\mathbf{x}' \in \mathcal{C}_j$.

Claim 3.6.1. *There exists a positive constant A' such that for any $0 \leq j < k \leq d$ we have*

$$d_{inn}^{X^a}(\boldsymbol{x}_j,\boldsymbol{x}_k) \leq A'|\boldsymbol{x}_j-\boldsymbol{x}_k|$$

for any $\boldsymbol{x}_j \in \mathcal{C}_j$ and $\boldsymbol{x}_k \in \mathcal{C}_k$.

Proof of Claim 3.6.1. For $j \geq 1$, let $\mathbb{C}_j$ be the complex line of $\mathbb{C}^n$ corresponding to the complex line direction λ_j. Let $S_j := \mathbb{C}_j \cap \mathbf{S}_1^{2n-1}$ be the unit circle of $\mathbb{C}_j$ centred at the origin. Whenever $j \neq k$, the complex lines $\mathbb{C}_j$ and $\mathbb{C}_k$ only meet at $\mathbf{0}$, therefore the intersection $S_j \cap S_k$ is empty. For any $1 \leq j < k \leq d$, let

$$\delta_{j,k} := \operatorname{dist}(S_j,S_k) > 0,$$

the Euclidean distance between S_j and S_k, and let

$$\delta := \min_{1\leq j<k\leq d} \delta_{j,k} > 0.$$

Since $\mathcal{C}_j$ accumulate at λ_j at infinity, we can assume that R is large enough so that

$$\sup\left\{\operatorname{dist}\left(S_j, \frac{\mathbf{x}}{|\mathbf{x}|}\right) : \mathbf{x} \in \mathcal{C}_j\right\} \leq \frac{\delta}{4}$$

for each $j = 0, \ldots, d$.

Let $\mathbf{x}_j \in \mathcal{C}_j$ and $\mathbf{x}_k \in \mathcal{C}_k$, for a given pair of indices $1 \leq j < k \leq d$. For $l = k, j$ we define

$$\mathbf{u}_l := \frac{\mathbf{x}_l}{|\mathbf{x}_l|} \;\text{ and }\; r_l := |\mathbf{x}_l|.$$

Let $2\alpha \in [0, \pi]$ be the non-oriented angle between $\mathbf{u}_j$ and $\mathbf{u}_k$. We split the proof in two cases.

Case 1: $1 \leq j < k \leq d$.

For $l = j, k$, let $\mathbf{y}_l$ be a point of $\partial\mathcal{C}_l \subset X_R^a$ which realizes the minimum of $d_{inn}^{\mathcal{C}_j}(\mathbf{x}_l, \partial\mathcal{C}_l)$,

$$d_{inn}^{\mathcal{C}_j}(\mathbf{x}_l, \partial\mathcal{C}_l) = d_{inn}^{\mathcal{C}_j}(\mathbf{x}_l, \mathbf{y}_l).$$

Observe that the set of points $\mathbf{y}_l$ satisfying this condition is closed, thus compact.

By choice of R and j, k and the Law of Sines we have

$$|\mathbf{u}_j - \mathbf{u}_k| = 2\sin(\alpha) \geq \frac{\delta}{2}.$$

Assume that $r_j \geq r_k$. We recall identity (3.2)

$$|\mathbf{x}_j - \mathbf{x}_k|^2 = (r_j + r_k)^2 \sin^2\alpha + (r_j - r_k)^2 \cos^2\alpha$$

from which we deduce again Estimate (3.6)

$$\frac{\delta}{4}(r_k + r_j) \leq |\mathbf{x}_j - \mathbf{x}_k|.$$

Since $r_j, r_k \geq R$, for $l = j, k$, Estimate (3.6) yields the following estimate

$$|\mathbf{x}_l - \mathbf{y}_l| \leq r_l + R \leq \frac{4}{\delta} \cdot |\mathbf{x}_j - \mathbf{x}_k|. \tag{3.15}$$

as well as

$$|\mathbf{y}_j - \mathbf{y}_k| \leq 2R \leq \frac{4}{\delta} \cdot |\mathbf{x}_j - \mathbf{x}_k|. \tag{3.16}$$

Since, we obviously have

$$d_{inn}^{X^a}(\mathbf{x}_j,\mathbf{x}_k) \le d_{inn}^{\mathcal{C}_j}(\mathbf{x}_j,\mathbf{y}_j) + d_{inn}^{\mathcal{C}_0}(\mathbf{y}_j,\mathbf{y}_k) + d_{inn}^{\mathcal{C}^k}(\mathbf{y}_k,\mathbf{x}_k),$$

combining Estimates (3.15) and (3.16) with $\mathcal{C}_0$, $\mathcal{C}_j$, $\mathcal{C}_k$ being LNE with LNE constant A yield the following expected inequality

$$d_{inn}^{X^a}(\mathbf{x}_j,\mathbf{x}_k) \le \frac{16A}{\delta} \cdot |\mathbf{x}_j - \mathbf{x}_k|.$$

Case 2: $j = 0$ and $1 \le k \le d$. We can assume $r_0 < R < r_k$. For $l = 0,k$, we define the following distances

$$\delta := d_{inn}^{X^a}(\mathbf{x}_0,\mathbf{x}_k) \;\text{ and }\; \delta_l := d_{inn}^{\mathcal{C}_l}(\mathbf{x}_l,\partial\mathcal{C}_k).$$

If $r_k \le 2R$, we obtain a LNE constant from $X^a_{\le 2R}$. Let us concentrate in the case when $r_k \ge 2R$. Then we find that

$$\frac{r_k}{2} \le |\mathbf{x}_k - \mathbf{x}_0| \le 2r_k.$$

Since $\mathcal{C}_k$ and $\mathcal{C}_0$ are LNE, we have the following estimates

$$\frac{r_k}{2} \le d_{inn}^{\mathcal{C}_k}(\mathbf{x}_k,\partial\mathcal{C}_k) \le 2Ar_k \;\text{ and }\; d_{inn}^{\mathcal{C}_0}(\mathbf{x}_0,\partial\mathcal{C}_k) \le 2AR.$$

Since $r_k \ge 2R$, we therefore deduce the following estimate

$$d_{inn}^{X^a}(\mathbf{x}_0,\mathbf{x}_k) \le d_{inn}^{\mathcal{C}_0}(\mathbf{x}_0,\partial\mathcal{C}_k) + d_{inn}^{\mathcal{C}_k}(\mathbf{x}_k,\partial\mathcal{C}_k) \le Ar_k + 2Ar_k \le 6A|\mathbf{x}_0 - \mathbf{x}_k|$$

proving the second case. □

Claim 3.6.1 establishes the desired LNE properties between any pair of points belonging to different subsets $\mathcal{C}_j$. Since each of these subset is LNE, the result is proved. □

Example 7. *Let $[x:y:z]$ be projective coordinates of $\mathbb{CP}^2$ so that $\mathbb{C}^2$ is the affine chart given by $z \ne 0$. Consider the non-singular quadrics*

$$P := \{[x:y:z] \in \mathbb{CP}^2 : yz - x^2 = 0\} \;\text{ and }\; H := \{[x:y:z] \in \mathbb{CP}^2 : xy - z = 0\}.$$

The parabola P and the hyperbola H are isomorphic via the linear change of coordinates $(x,z) \to (z,x)$. The affine parts are

$$P^a := \{(x,y) \in \mathbb{C}^2 : y - x^2 = 0\} \text{ and } H^a := \{(x,y) \in \mathbb{C}^2 : xy - 1 = 0\}.$$

Thus,

$$1 = card(P^\infty) < \deg(P) = \deg(H) = card(H^\infty) = 2.$$

By Theorem 3.6.1, the hyperbola H^a is LNE since $H^\infty = \{[1:0:0],[0:1:0]\}$ while the parabola P^a is not LNE because P^∞ has only the point $[0:1:0]$.

Given a polynomial function $f:\mathbb{C}^n \to \mathbb{C}$, we denote its leading term by f^*. As consequence of Theorem 3.6.1, we obtain next result.

Corollary 3.6.1. *Let $f = (f_1,...,f_n) : \mathbb{C}^{n+1} \to \mathbb{C}^n$ be dominant polynomial mapping and let $d := \prod \deg(f_j)$. The following are equivalent:*

1. *All non-singular fibers of f over $\mathbb{C}^n \setminus K_0(f)$ are LNE.*
2. *There exists a LNE fiber of f over $\mathbb{C}^n \setminus Bif(f)$.*
3. *The vanishing set $\{\boldsymbol{x} \in \mathbb{C}^n : f_1^*(\boldsymbol{x}) = \ldots = f_n^*(\boldsymbol{x}) = 0\}$ consists of d distinct lines, where f_j^* is the leading term of f_j.*

Proof. The implication $(1) \Rightarrow (2)$ is obvious.

Let $\mathbf{c} = (\mathbf{c}_1,\ldots,\mathbf{c}_n) \in \mathbb{C}^n \setminus \mathrm{Bif}(f)$ such that $f^{-1}(\mathbf{c})$ is LNE. Since $\mathbf{c}$ is generic and f is dominant, the fiber $f^{-1}(\mathbf{c})$ is an algebraic curve. Moreover, for each $t = (t_1,\ldots,t_n) \notin K_0(f)$, the polynomials $(f_j - t_j)$ are square free. Therefore, the degree of the fiber $f^{-1}(\mathbf{c})$ is equal to $\deg(f)$. By Theorem 3.6.1, $\{\mathbf{x} \in \mathbb{C}^n : f_1^*(\mathbf{x}) = \ldots = f_n^*(\mathbf{x}) = 0\}$ consists of d distinct lines.

Let $t \notin K_0(f)$. Since $\mathrm{card}(f^{-1}(t)^\infty)$ is maximal, by Theorem 3.6.1 we conclude that $f^{-1}(t)$ is LNE. □

Next we present an example of polynomial mapping which has an LNE singular fiber while its generic non-singular fibers are not LNE.

Example 8. *Let $f:\mathbb{C}^2 \to \mathbb{C}$ be the polynomial mapping defined as $f(x,y) = (xy)^2$. For each $t \in \mathbb{C}$, denote $X_t := f^{-1}(t)$. If $t \neq 0$, then $X_t = \{(x,y) \in \mathbb{C}^2 : xy - \sqrt{t} = 0\} \cup \{(x,y) \in \mathbb{C}^2 : xy + \sqrt{t} = 0\}$ is not connected, thus not LNE. Yet the germ $(X_0,\boldsymbol{0})$ is LNE. By Theorem 3.6.1, X_0 is LNE since $card(X_0^\infty) = \deg(\overline{X_0}) = 2$.*

4 LIPSCHITZ TRIVIAL VALUES OF POLYNOMIAL MAPPINGS

4.1 Statement of the main result

Our main result on Lipschitz trivial values, Theorem 4.3.1, when combined with Proposition 4.2.2 implies the following

Theorem 4.1.1. *Let $f : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial mapping and let $n-1-m$ be the dimension of the set of accumulation points at infinity of the fibre $f^{-1}(\boldsymbol{c})$. The mapping f attains the Lipschitz trivial value $\boldsymbol{c}$ if and only if*

$$f = g \circ \pi,$$

for a linear surjective projection $\pi : \mathbb{K}^n \to \mathbb{K}^m$ and a polynomial mapping $g : \mathbb{K}^m \to \mathbb{K}^p$ for which $\boldsymbol{c}$ is a regular value of properness.

In the complex case, the statement equivalently requires that $m = p$ and the polynomial mapping $g : \mathbb{C}^p \to \mathbb{C}^p$ be dominant and generically finite (see Corollary 4.4.1). Therefore, either almost all values of a complex polynomial mapping are Lipschitz trivial or there are none. In contrast with the complex case, there exist non-proper real polynomial mappings admitting values of properness.

Moreover, we show that the Theorem 4.3.1 cannot extend without further hypotheses to a wider class of rational mappings.

4.2 General properties for mappings with Lipschitz trivial values

In this section we present general properties of differentiable mappings with Lipschitz trivial values. Let $\varphi : \mathbb{K}^n \to \mathbb{K}^p$ be a mapping. The next result emphasizes the rigid asymptotic behaviour of levels near a Lipschitz trivial value.

Proposition 4.2.1. *Assume that $\boldsymbol{c}$ is a Lipschitz trivial value of $\varphi : \mathbb{K}^n \to \mathbb{K}^p$. There exists a neighbourhood $\mathcal{V}$ of $\boldsymbol{c}$ such that the following properties hold:*

(i) the mapping φ is Lipschitz on $\varphi^{-1}(\mathcal{V})$;

(ii) there exist $0 < \delta < \varepsilon$ such that

$$T_\delta(\varphi^{-1}(\boldsymbol{c})) \subset \varphi^{-1}(\mathcal{V}) \subset T_\varepsilon(\varphi^{-1}(\boldsymbol{c})),$$

where the open tube $T_r(S)$ of radius r around a subset S of $\mathbb{K}^n$ is defined as

$$T_r(S) := \{\boldsymbol{x} \in \mathbb{K}^n : dist(\boldsymbol{x}, S) < r\}.$$

Proof. Let $d_{\mathbf{c}} := d_{out}^{f^{-1}(\mathbf{c})}$ be the outer metric on $\varphi^{-1}(\mathbf{c})$ and let $\mathcal{U} := \varphi^{-1}(\mathcal{V})$. Since φ has a Lipschitz trivial value at $\mathbf{c}$, there exist an open ball $\mathcal{V} = B_r^p(\mathbf{c})$ of $\mathbb{K}^p$ and a bi-Lipschitz homeomorphism

$$G = (\varphi, \psi) : \mathcal{U} \mapsto \mathcal{V} \times \varphi^{-1}(\mathbf{c}).$$

Therefore, there exists $L > 1$ such that for any $\mathbf{x}, \mathbf{x}' \in \mathcal{U}$ we have

$$\frac{1}{L}\left\|\mathbf{x} - \mathbf{x}'\right\| \leq \left\|G(\mathbf{x}) - G(\mathbf{x}')\right\| \leq L\left\|\mathbf{x} - \mathbf{x}'\right\|. \tag{4.1}$$

Point (i) follows from inequalities (4.1) since for all $\mathbf{x}, \mathbf{x}' \in \mathcal{U}$ we find

$$\|\varphi(\mathbf{x}) - \varphi(\mathbf{x}')\| \leq \|\varphi(\mathbf{x}) - \varphi(\mathbf{x}')\| + d_{\mathbf{c}}(\psi(\mathbf{x}), \psi(\mathbf{x}')) \leq L\left\|\mathbf{x} - \mathbf{x}'\right\|.$$

To prove (ii), define the following radii

$$\delta := \frac{r}{L} \text{ and } \varepsilon := Lr.$$

Given $\mathbf{x}' \in T_\delta(\varphi^{-1}(\mathbf{c}))$ and $\mathbf{x} \in \varphi^{-1}(\mathbf{c})$, we have $\|\mathbf{x} - \mathbf{x}'\| < \delta$. Inequalities (4.1) yields

$$\|\varphi(\mathbf{x}) - \mathbf{c}\| = \|\varphi(\mathbf{x}) - \varphi(\mathbf{x}')\| \leq L\|\mathbf{x} - \mathbf{x}'\| < L\delta = r.$$

Therefore, $T_\delta(\varphi^{-1}(\mathbf{c})) \subset \mathcal{U}$. Note that for $\mathbf{t} \in \mathcal{V}$ and $\mathbf{x}' \in \varphi^{-1}(\mathbf{t})$, there exists $\mathbf{x} \in \varphi^{-1}(\mathbf{c})$ such that $\psi(\mathbf{x}) = \psi(\mathbf{x}')$. Therefore, estimates (4.1) provide

$$\frac{1}{L}\left\|\mathbf{x} - \mathbf{x}'\right\| \leq \left\|G(\mathbf{x}) - G(\mathbf{x}')\right\| = \|\mathbf{c} - \mathbf{t}\| + d_{\mathbf{c}}(\psi(\mathbf{x}), \psi(\mathbf{x}')) = \|\mathbf{c} - \mathbf{t}\|.$$

Thus we obtain $\|\mathbf{x} - \mathbf{x}'\| < rL = \varepsilon$ and hence $\mathcal{U} \subset T_\varepsilon(\varphi^{-1}(\mathbf{c}))$. □

Remark 4.2.1. *Point (i) of Proposition 4.2.1 implies that each first order partial derivative of each component of the mapping φ is bounded over $\varphi^{-1}(\mathcal{V})$.*

Property 4.2.1. *Let $\tau : \mathbb{K}^m \to \mathbb{K}^p$ be a mapping Lipschitz trivial at the value $\boldsymbol{c}$ and let $\pi : \mathbb{K}^n \to \mathbb{K}^m$ be a linear surjective projection. Then the mapping $\tau \circ \pi : \mathbb{K}^n \to \mathbb{K}^p$ is Lipschitz trivial at the value $\boldsymbol{c}$.*

Proof. Let $\varphi := \tau \circ \pi$. Up to a $\mathbb{K}$-linear change of coordinates in $\mathbb{K}^n$, we can assume that for any subset V of $\mathbb{K}^p$ the following holds true

$$\varphi^{-1}(V) = \tau^{-1}(V) \times \mathbb{K}^{n-m}.$$

Denote $(\mathbf{u},\mathbf{v}) \in \mathbb{K}^m \times \mathbb{K}^{n-m} = \mathbb{K}^n$. If the bi-Lipschitz homeomorphism $G : \tau^{-1}(\mathcal{V}) \mapsto \mathcal{V} \times \tau^{-1}(\mathbf{c})$ provides trivialisation of τ over a neighbourhood $\mathcal{V}$ of $\mathbf{c}$, then the mapping

$$H : \varphi^{-1}(\mathcal{V}) \to \mathcal{V} \times \varphi^{-1}(\mathbf{c}), \quad (\mathbf{u},\mathbf{v}) \to (G(\mathbf{u}),\mathbf{v})$$

is a bi-Lipschitz homeomorphism trivialising φ over $\mathcal{V}$. □

Property 4.2.2. *Let $\varphi : \mathbb{K}^n \to \mathbb{K}^p$ be a smooth mapping with a nowhere dense set of critical values. Any regular value of φ that is also a value of properness is a Lipschitz trivial value.*

Proof. Let $\mathcal{V}$ be a non-empty open subset of $\mathbb{K}^p$ such that φ is proper over $\mathcal{V}$. Since $K_0(\varphi)$ is closed and nowhere dense we can further assume that $\mathrm{clos}(\mathcal{V})$ does not intersect with $K_0(\varphi)$. Therefore, the mapping φ is $\mathcal{C}^\infty$ locally trivial over $\mathcal{V}$ by Ehresmann's Theorem (see Theorem 2.8.2). The restriction of any $\mathcal{C}^\infty$ trivialisation of φ over $\mathcal{V}$ to any open subset $\mathcal{U}$ relatively compact in $\mathcal{V}$ is necessarily bi-Lipschitz over $\mathcal{U}$. □

In Examples 2 and 3 we observed that a mapping might admit a topologically trivial value that is critical. In the next result we show that such a phenomenon cannot happen if we require bi-Lipschitz triviality.

Proposition 4.2.2. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}^p$ be a $\mathcal{C}^k$ mapping with $k \geq \max(n-p+1,1)$. Any Lipschitz trivial value of φ is a regular value.*

The proof will follow from the next result.

Lemma 4.2.1. *Let $f : \mathcal{U} \to \mathcal{V}$ be a $\mathcal{C}^1$ mapping, where $\mathcal{U}$ and $\mathcal{V}$ are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively. If there exists a bi-Lipschitz homeomorphism*

$$(f,\psi) : \mathcal{U} \to \mathcal{V} \times F$$

where F is a C^1 sub-manifold of some $\mathbb{R}^q$, then f has no critical points in $\mathcal{U}$.

Proof. Let $H : \mathcal{V} \times F \to \mathcal{U}$ be the inverse of the bi-Lipschitz homeomorphism (f,ψ). Since H is bi-Lipschitz, it is differentiable almost everywhere by Rademacher's Theorem (see Theorem 2.12.1).

Let $\mathbf{x}_0 \in \mathcal{V} \times F$ be a point at H is differentiable so that H^{-1} is differentiable at $\mathbf{y}_0 := H(\mathbf{x}_0)$. Let $\psi : \mathcal{W} \to B_1^n$ be a coordinate chart for $\mathcal{V} \times F$ defined in a neighborhood of $\mathcal{W}$ of $\mathbf{x}_0$.

Let $\widetilde{H} := H \circ \psi^{-1}$. Since H is bi-Lipschitz, $\widetilde{H}$ is also bi-Lipschitz and thus there exists $K > 1$ such that

$$0 < \frac{1}{K} \leq \left| \frac{\partial \widetilde{H}_i^{-1}}{\partial x_j}(\mathbf{y}_0) \right| \leq K,$$

where $\frac{\partial \widetilde{H}_i^{-1}}{\partial x_j}(\mathbf{y}_0)$ are the entries of the Jacobian matrix of $D_{\mathbf{y}_0}\widetilde{H}^{-1}$. By Theorem 2.12.2 (Hadamard's Inequality), we have

$$\frac{1}{|\det(D_{\psi(\mathbf{x}_0)}\widetilde{H})|} = |\det(D_{\mathbf{y}_0}\widetilde{H}^{-1})| \leq \left(\sqrt{nK^2}\right)^n.$$

Thus,

$$|\det(D_{\psi(\mathbf{x}_0)}\widetilde{H})| \geq \frac{1}{\left(\sqrt{nK^2}\right)^n} > 0.$$

Therefore, for a sequence $(\mathbf{x}_n)_n$ of points at which the computations above hold, any limit of the form $D = \lim_{\mathbf{x}_n \to \mathbf{x}} D_{\mathbf{y}_n} H$ at the given point $\mathbf{y}$ of $\mathcal{V} \times F$ has also rank n. Thus, the differential of f must have rank p at each point of $\mathcal{U}$. □

Proof of Proposition 4.2.2. Let $\mathbf{c}$ be a Lipschitz trivial value of φ. If $\mathbf{c}$ does not lie in $Im(\varphi)$, the image of φ, then it belongs to $\mathbb{R}^p \setminus \mathrm{clos}(Im(\varphi))$, thus is a regular value.

Assume $\mathbf{c}$ is a value taken by φ. Let $\mathcal{V}$ be an open neighbourhood of $\mathbf{c}$ over which φ is Lipschitz trivial. By hypothesis φ satisfies Sard's Theorem (see Theorem 2.3.1), then there exists a regular value $\mathbf{t}$ in $\mathcal{V}$. Let $F := \varphi^{-1}(\mathbf{t})$ and let $H : \varphi^{-1}(\mathcal{V}) \to \mathcal{V} \times F$ be the bi-Lipschitz mapping trivializing φ over $\mathcal{V}$. Thus, by Lemma 4.2.1, the mapping φ has no critical point in $\varphi^{-1}(\mathcal{V})$, therefore $\mathbf{c}$ is a regular value. □

Definition 4.2.1. *Given $S \subset \mathbb{K}^n$, we define its accumulation set at infinity as*

$$S^\infty := \overline{S}^{\mathbb{KP}^n} \cap \boldsymbol{H}_\infty.$$

The next Property shows the rigidity of the asymptotic behavior at infinity of the fibers taken over a Lipschitz trivial neighborhood.

Property 4.2.3. *Let $\varphi : \mathbb{K}^n \to \mathbb{K}^p$ be a mapping locally Lipschitz trivial over the connected open subset $\mathcal{V} \subset \mathbb{K}^p$. Then the family $(\varphi^{-1}(\boldsymbol{t})^\infty)_{t\in\mathcal{V}}$ of accumulation sets at infinity of the levels of φ is constant, i.e.*

$$\varphi^{-1}(\boldsymbol{t})^\infty = \varphi^{-1}(\mathcal{V})^\infty, \ \forall \boldsymbol{t} \in \mathcal{V}.$$

Proof. Consider two sequences $(\mathbf{x}_k)_k$ and $(\mathbf{x}'_k)_k$ of $\mathbb{K}^n$ satisfying the following property: there exists a positive constant A such that

$$|\mathbf{x}_k - \mathbf{x}'_k| \ \leq \ A \ \text{for} \ k \gg 1.$$

If furthermore $|\mathbf{x}_k|$ goes to ∞ and does so such that $[\mathbf{x}_k : 1] \to [\lambda : 0] \in \mathbf{H}_\infty$ as $k \to \infty$, then we deduce that $|\mathbf{x}'_k|$ goes to ∞ and $[\mathbf{x}'_k : 1] \to [\lambda : 0]$ as k goes to ∞.

Let $\mathbf{c} \in \mathcal{V}$ be a Lipschitz trivial value of φ. Up to taking a smaller $\mathcal{V}$ containing $\mathbf{c}$, point (ii) of Proposition 4.2.1 states that $\varphi^{-1}(\mathcal{V})$ is contained in the open tube $T_\varepsilon(\varphi^{-1}(\mathbf{c}))$ for some positive radius ε. The first part of the proof then gives the result. □

4.3 Characterization of polynomial mappings with Lipschitz trivial values

Let $f : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial mapping. The level $f^{-1}(\mathbf{t})$ is denoted by $F_\mathbf{t}$. We use the convention that $\dim \emptyset = -1$. In this section we prove our main theorem on Lipschitz trivial values.

Theorem 4.3.1. *Let $f : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial mapping with $\dim F_{\boldsymbol{c}}^\infty = n-1-m$ for a value $\boldsymbol{c} \in \mathbb{K}^p$. If the mapping f attains $\mathbf{c}$ as a Lipschitz trivial value, then there exist a polynomial mapping $g : \mathbb{K}^m \to \mathbb{K}^p$ proper at $\mathbf{c}$ and a linear surjective projection $\pi : \mathbb{K}^n \to \mathbb{K}^m$ such that*

$$f = g \circ \pi.$$

We start with the following key result.

Lemma 4.3.1. *Let $f : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial mapping and assume that the point $[1 : 0 : \cdots : 0]$ lies in $F_{\boldsymbol{c}}^\infty$. If there exists a neighbourhood $\mathcal{V}$ of $\boldsymbol{c}$ such that f satisfies points (i) and (ii) of Proposition 4.2.1, then the mapping f does not depend on the coordinate x_1.*

Proof. Since $[1 : 0 : \cdots : 0] \in F_\mathbf{c}^\infty$, there exists an arc $\gamma : I \to F_\mathbf{c}$ parametrized as

$$\gamma(t) = (t^d, p(t) + A_0(1/t)) \in \mathbb{K} \times \mathbb{K}^{n-1},$$

where I is a connected component of the complement of an Euclidean ball of $\mathbb{K}$, the mapping $p : \mathbb{K} \to \mathbb{K}^{n-1}$ is polynomial of degree $\leq d-1$, and A_0 is a $\mathbb{K}$-analytic map germ $(\mathbb{K},0) \to (\mathbb{K}^{n-1},\mathbf{0})$.

Consider the following dominant polynomial mapping

$$\Gamma : \mathbb{K} \times \mathbb{K}^{n-1} \to \mathbb{K}^n, \ \ (t,\varepsilon) \mapsto (t^d, p(t)+\varepsilon).$$

Let $\gamma_\varepsilon : \mathbb{K} \to \mathbb{K}^n$ be the polynomial arc

$$\gamma_\varepsilon : t \mapsto \gamma_\varepsilon(t) := \Gamma(t,\varepsilon).$$

Since $A_0(1/t) \to \mathbf{0}$ as $|t| \to \infty$, we conclude that

$$\|\gamma_\varepsilon(t) - \gamma(t)\| = \|\varepsilon - A_0(1/t)\| \to \|\varepsilon\|.$$

Take a neighbourhood $\mathcal{V}$ of $\mathbf{c}$ in $\mathbb{K}^p$ such that f satisfies points (i) and (ii) of Proposition 4.2.1. Point (ii) of Proposition 4.2.1 and the definition of Γ guarantee the existence of constants $\delta > 0$ and $R > 0$ such that

$$\gamma_\varepsilon(t) \in f^{-1}(\mathcal{V})$$

for any $\|\varepsilon\| < \delta$ and $t \in I$ such that $|t| > R$. Since f is Lipschitz on $f^{-1}(\mathcal{V})$, we get

$$\|f(\gamma_\varepsilon(t)) - \mathbf{c}\| \leq L \cdot \|\varepsilon - A_0(1/t)\| \to L \cdot ||\varepsilon\|$$

as $|t| \to \infty$. Therefore, whenever $\|\varepsilon\| < \delta$, the polynomial mapping $t \mapsto f \circ \gamma_\varepsilon(t)$ is bounded, thus constant. Writing $\mathbf{x} = (x_1, \mathbf{y})$, we deduce

$$\mathbf{0} \equiv \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma_\varepsilon)(t) = d \cdot t^{d-1} \cdot \partial_{x_1} f(\gamma_\varepsilon(t)) + \partial_{\mathbf{y}} f(\gamma_\varepsilon(t)) \cdot p'(t). \tag{4.2}$$

By Remark 4.2.1, the first order partial derivatives of f are bounded along γ_ε, thus $\partial_{\mathbf{y}} f(\gamma_\varepsilon(t))$ is bounded. Since p has degree at most $d-1$, the term $d \cdot t^{d-1} \cdot \partial_{x_1} f(\gamma_\varepsilon(t))$ would be the unique term of maximal degree $d-1$. By equation 4.2, we conclude that

$$(\partial_{x_1} f) \circ \gamma_\varepsilon \equiv 0.$$

Since the subset $\mathbb{K} \times \{\varepsilon : \|\varepsilon\| < \delta\} \subset \mathbb{K}^n$ is open and non-empty and the mapping Γ is dominant, we conclude that the mapping $\partial_{x_1} f$ is identically null. $\square$

Proof of Theorem 4.3.1. Note that f satisfies the claim of Proposition 4.2.1. If $\dim F_{\mathbf{c}}^{\infty} = -1$, then the fibre $F_{\mathbf{c}}$ is compact and from point (ii) of Proposition 4.2.1 the subset $f^{-1}(\mathcal{V})$ is compact for a small compact neighbourhood $\mathcal{V}$ of $\mathbf{c}$. Thus f is proper at $\mathbf{c}$ and taking π as the identity mapping of $\mathbb{K}^n$ yields the claim.

Assume $F_{\mathbf{c}}^{\infty}$ is of dimension $n-1-m \geq 0$. Thus there exist $n-m$ points $[\mathbf{v}_1 : 0], \ldots, [\mathbf{v}_{n-m} : 0]$ of $F_{\mathbf{c}}^{\infty}$ such that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{n-m}$ are $\mathbb{K}$-linearly independent in $\mathbb{K}^n$. Take a $\mathbb{K}$-linear change of coordinates $\ell : \mathbb{K}^n \to \mathbb{K}^n$ such that $\ell(\mathbf{v}_j) = \mathbf{e}_j$ for $j = 1, \ldots, n-m$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard orthonormal basis of $\mathbb{K}^n$. Applying Lemma 4.3.1 we conclude that the polynomial mapping $f \circ \ell$ depends only on $\mathbf{u} := (x_{n-m+1}, \ldots, x_n)$. Let g be the polynomial mapping restriction of $f \circ \ell$ to $\mathbb{K}^m$, the subspace of $\mathbb{K}^n$ generated by $\mathbf{e}_{n-m+1}, \ldots, \mathbf{e}_n$. Let $\pi_0 : \mathbb{K}^n \to \mathbb{K}^m$ be the orthogonal projection of $\mathbb{K}^n$ onto the subspace $\mathbb{K}^m$. Therefore, we find

$$f = g \circ \pi_0 \circ \ell^{-1}.$$

Note that $(f \circ \ell)^{-1}(\mathbf{t}) = \mathbb{K}^{n-m} \times g^{-1}(\mathbf{t})$. Since

$$n - 1 - m = \dim F_{\mathbf{c}}^{\infty} = n - m + \dim g^{-1}(\mathbf{c})^{\infty},$$

we deduce that $g^{-1}(\mathbf{c})$ is compact. From point (ii) of Proposition 4.2.1 applied to the levels of $f \circ \ell$ over $\mathcal{V}$ we get that $g^{-1}(\mathcal{V})$ is bounded. Thus $\mathbf{c}$ is a value of properness of g. $\square$

Corollary 4.3.1. *If polynomial mapping $f : \mathbb{K}^n \to \mathbb{K}^p$ attains a Lipschitz trivial value $\mathbf{c}$, then the connected components of its fibers over a neighbourhood $\mathcal{V}$ of $\mathbf{c}$ are LNE.*

Proof. By Theorem 4.3.1, there exist a polynomial mapping $g : \mathbb{K}^m \to \mathbb{K}^p$ which is proper at $\mathbf{c}$ and a linear surjective projection $\pi : \mathbb{K}^n \to \mathbb{K}^m$ such that

$$f = g \circ \pi.$$

By Proposition 4.2.2, all the values in $\mathcal{V}$ are regular. Therefore, for any $\mathbf{c}' \in \mathcal{V}$, the level set $g^{-1}(\mathbf{c}')$ is a compact sub-manifold of $\mathbb{K}^m$ (thus LNE by Corollary 2.7.1) and $f^{-1}(\mathbf{c}')$ is a cylinder over this set, thus LNE. $\square$

4.4 The set of Lipschitz trivial values of real and complex mappings

This section presents some consequences of Theorem 4.3.1. In particular, complex polynomial mappings admitting Lipschitz trivial values have a very rigid structure, while the real setting allows for more variety.

Corollary 4.4.1. *A complex polynomial mapping $f:\mathbb{C}^n\to\mathbb{C}^p$ attains a Lipschitz trivial value if and only if there exist a dominant polynomial mapping $g:\mathbb{C}^p\to\mathbb{C}^p$ and a linear surjective projection $\pi:\mathbb{C}^n\to\mathbb{C}^p$ such that*

$$f=g\circ\pi.$$

In such a case we get

$$L(f)=\mathbb{C}^p\setminus Bif(g).$$

In particular, the set of regular Lipschitz trivial values is either empty or the complement of an algebraic hypersurface.

Proof. Assume there exists a dominant polynomial mapping $g:\mathbb{C}^p\to\mathbb{C}^p$ such that $f=g\circ\pi$ for some linear surjective projection $\pi:\mathbb{C}^n\to\mathbb{C}^p$. Therefore g is generically finite and by Properties 4.2.1 and 4.2.2, the set $\mathrm{L}(f)$ of Lipschitz trivial values of f is not empty. For the converse statement, note that f is dominant since a Lipschitz trivial value is attained. Moreover, we have $n-p=\dim F_{\mathbf{c}}=1+\dim F_{\mathbf{c}}^{\infty}$ for a generic level $\mathbf{c}$ of f, so Theorem 4.3.1 gives the claim.

For the second part of the assertion, $\mathrm{L}(g)\cap K_0(g)$ is empty by Proposition 4.2.2 and since g is generically finite $\mathrm{L}(g)\cap J(g)$ is empty as well. Thus $\mathrm{L}(g)\cap(J(g)\cup K_0(g))$ is empty. We recall that $J(g)\cup K_0(g)=\mathrm{Bif}(g)$ and if non-empty, it is an algebraic hypersurface by Theorem 2.9.1. Finally, by Property 4.2.2 we have $\mathrm{L}(g)=\mathbb{C}^p\setminus\mathrm{Bif}(g)$. Therefore, $\mathrm{L}(f)=\mathbb{C}^p\setminus\mathrm{Bif}(g)$. □

As a consequence, we recover the main result of [9], see Theorem 2.10.1. The real case is more nuanced than the complex one and Lipschitz trivial values admit a richer structure.

Corollary 4.4.2. *Let $f:\mathbb{R}^n\to\mathbb{R}^p$ be a polynomial mapping admitting a Lipschitz trivial value. There exists a polynomial mapping $g:\mathbb{R}^m\to\mathbb{R}^p$ and linear surjective projection $\pi:\mathbb{R}^n\to\mathbb{R}^m$ such that $f=g\circ\pi$ and*

$$L(f)=\mathbb{R}^p\setminus(J(g)\cup K_0(g)).$$

Moreover, the mapping g is unique up to linear changes of coordinates.

Proof. Let $\mathbf{c}$ be a Lipschitz trivial value of f. By Theorem 4.3.1 there exists a polynomial mapping $g:\mathbb{R}^m\to\mathbb{R}^p$ proper at $\mathbf{c}$ and linear surjective projection $\pi:\mathbb{R}^n\to\mathbb{R}^m$ such that $f=g\circ\pi$. Moreover, up to a linear change of coordinates in $\mathbb{R}^n$, we have $Df=Dg\oplus\mathbf{0}:\mathbb{R}^m\times\mathbb{R}^{n-m}\to\mathbb{R}^p$, thus $K_0(g)=K_0(f)$. By Property 4.2.1 and Theorem 4.3.1, the Lipschitz trivial values of g are values of properness and by Proposition 4.2.2 they are regular. Thus, $\mathrm{L}(g) \subset \mathbb{R}^p \setminus (J(g) \cup K_0(g))$. The other inclusion is given by Property 4.2.2. Therefore,

$$\mathrm{L}(f) = \mathrm{L}(g) = \mathbb{R}^p \setminus (J(g) \cup K_0(g)).$$

To show uniqueness take $g : \mathbb{R}^m \to \mathbb{R}^p$ proper at $\mathbf{c}$ such that $f = g \circ \pi$ for a linear surjective mapping $\pi : \mathbb{R}^n \to \mathbb{R}^m$. For any polynomial mapping $h : \mathbb{R}^k \to \mathbb{R}^p$ and a linear surjective projection $\sigma : \mathbb{R}^n \to \mathbb{R}^k$ such that $f = h \circ \sigma$, we get, up to a linear change of coordinates $\ell : \mathbb{R}^k \to \mathbb{R}^k$, that

$$(h \circ \ell)^{-1}(\mathbf{t}) = g^{-1}(\mathbf{t}) \times \mathbb{R}^{k-m}$$

for $\mathbf{t} \in \mathbb{R}^p$, since at least the level $\mathbf{c}$ of g is compact. Thus either $m < k$ and h does not attain a proper value, or $m = k$ and $h \circ \ell = g$. □

Example 9. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the suspension at infinity of the Motzkin polynomial given by*

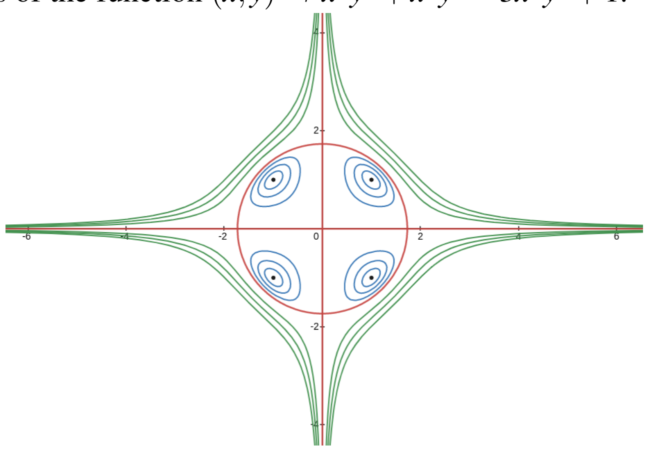
$$f(x,y,z) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1.$$

In notations of Theorem 4.3.1, we have $f = g \circ \pi$ where $\pi(x,y,z) = (x,y)$ and $g(x,y) = f(x,y,z)$. We have $J(g) = [1,\infty)$ and $K_0(g) = \{0,1\}$. Moreover,

$$L(f) = \mathbb{R} \setminus (J(g) \cup K_0(g)) = (-\infty, 0) \cup (0,1).$$

Indeed, the values of $[1,+\infty)$ are not Lipschitz trivial values of f, since f does not satisfy the necessary condition (ii) of Proposition 4.2.1.

Figure 4 - Some levels of the function $(x,y) \mapsto x^4y^2 + x^2y^4 - 3x^2y^2 + 1$.



Source: Created by the author.
Note: In green we have three values larger than 1 and, in blue we have three values smaller than 1. The value 0 consists of four points (black) while the value 1 is the union of the two axes with a circle (red).

Example 9 illustrates that the set of Lipschitz trivial values of a real polynomial mapping can be open and not dense in the image, whereas for complex mappings Lipschitz trivial values follow a local-global principle as stated in Corollary 4.4.1.

For polynomial mappings Property 4.2.3 is refined as the following necessary condition on the fibres.

Property 4.4.1. *Let $f:\mathbb{K}^n\to\mathbb{K}^p$ be a polynomial mapping locally Lipschitz trivial over the connected open subset $\mathcal{V}\subset\mathbb{K}^p$. There exists a $\mathbb{K}$-linear subspace A of $\mathbb{K}^n$ of positive codimension such that*

$$\widehat{F_{\boldsymbol{t}}^{\infty}}=A, \quad \textit{for all } \boldsymbol{t}\in L(f)\cap Im(f),$$

where $\widehat{F_{\boldsymbol{t}}^{\infty}}$ is the $\mathbb{K}$-cone of $\mathbb{K}^n$ over $F_{\boldsymbol{t}}^{\infty}$ with vertex at the origin, and where the cone over the empty set is defined as the null subspace.

Example 10. *The polynomial mapping $f:\mathbb{K}^3\to\mathbb{K}^2$, defined as $(x,y,z)\mapsto(x,xy+z)$, is surjective and $\mathcal{C}^\infty$ trivial at each $\boldsymbol{t}\in\mathbb{K}^2$. For each $\boldsymbol{t}=(a,b)$, we have the fibre*

$$F_{\boldsymbol{t}}=\{(a,y,b-ay)\in\mathbb{K}^3: y\in\mathbb{K}\}$$

which is an affine line. Taking $F_{\boldsymbol{0}}$ as model fibre, we have a global $\mathcal{C}^\infty$ trivialization $H:F_{\boldsymbol{0}}\times\mathbb{K}^2\to\mathbb{K}^3$ given by

$$H((0,y,0),(a,b))=(a,y,b-ay).$$

The accumulation set at infinity of the value $\boldsymbol{t}=(a,b)$ is $F_{\boldsymbol{t}}^{\infty}=\{[0:1:-a]\}$. Moreover,

$$\widehat{F_{\boldsymbol{t}}^{\infty}}=\{(0,\lambda,-\lambda a):\lambda\in\mathbb{K}\}.$$

Yet, the $\mathbb{K}$-linear subspace $(\widehat{F_{\boldsymbol{t}}^{\infty}})_{t\in\mathbb{K}^2}$ is nowhere locally constant. Therefore Property 4.4.1 implies that this mapping cannot admit any Lipschitz trivial value.

Given a real polynmial mapping $f:\mathbb{R}^n\to\mathbb{R}^p$ we consider its complexification $f_{\mathbb{C}}:\mathbb{C}^n\to\mathbb{C}^p$, the complex mapping defined by the same polynomial expressions. We will see that every real Lipschitz trivial value of $f_{\mathbb{C}}$ is a Lipschitz trivial value of f.

Proposition 4.4.1. *If $f_{\mathbb{C}}$ admits a Lipschitz trivial value, then $L(f)$ is a semi-algebraic dense open subset of $\mathbb{R}^p$. More precisely,*

$$L(f_{\mathbb{C}})\cap\mathbb{R}^p\ \subset\ L(f).$$

Proof. Denote $F_{\mathbb{C},\mathbf{t}} := f_{\mathbb{C}}^{-1}(\mathbf{t})$. Since $\mathrm{L}(f_{\mathbb{C}}) \cap Im(f_{\mathbb{C}})$ is not empty, Corollary 4.4.1 and Property 4.2.1 imply the existence of a $\mathbb{C}$-linear subspace $A_{\mathbb{C}}$ of $\mathbb{C}^n$ of dimension $n-p$ such that for each $\mathbf{t} \in \mathbb{C}^p \setminus \mathrm{Bif}(f_{\mathbb{C}})$, the level $F_{\mathbb{C},\mathbf{t}}$ is a disjoint union of finitely many affine subspaces of $\mathbb{C}$-dimension $n-p$, parallel to $A_{\mathbb{C}}$. Moreover, by Lemma 4.2.1, $f_{\mathbb{C}}$ has full rank on the preimage $f_{\mathbb{C}}^{-1}(\mathcal{V})$ of an open set $\mathcal{V}$ of Lipschitz trivial values. Therefore f has full rank on $f_{\mathbb{C}}^{-1}(\mathcal{V}) \cap \mathbb{R}^p$ and thus $f(\mathbb{R}^n)$ contains an open subset of $\mathbb{R}^p$. Therefore, there exists an open set $\mathcal{V}'$ in $\mathbb{R}^p$ such that the level $F_{\mathbf{t}}$ is of dimension $n-p$ for any $\mathbf{t} \in \mathcal{V}'$. As the intersection of the complex fibre $F_{\mathbb{C},\mathbf{t}}$ with $\mathbb{R}^n$, the fibre $F_{\mathbf{t}}$ is necessarily a disjoint union of parallel real affine subspaces. Therefore, $\dim f^{-1}(\mathcal{V}')^{\infty} = n-1-p$. Note that f, as the restriction of $f_{\mathbb{C}}$ to $\mathbb{R}^n$, satisfies assumptions (i) and (ii) of Lemma 4.3.1 since $f^{-1}(\mathbf{t}) = f_{\mathbb{C}}^{-1}(\mathbf{t}) \cap \mathbb{R}^n$. Using Theorem 4.3.1 we get that $f = g \circ \pi$ for some real linear surjective projection $\pi : \mathbb{R}^n \to \mathbb{R}^p$ and real polynomial mapping $g : \mathbb{R}^p \to \mathbb{R}^p$. Thus $f_{\mathbb{C}} = g_{\mathbb{C}} \circ \pi_{\mathbb{C}}$ and necessarily $g_{\mathbb{C}}$ is generically finite. Since Corollaries 4.4.1 and 4.4.2 yield

$$\mathrm{L}(f_{\mathbb{C}}) = \mathbb{C}^p \setminus \mathrm{Bif}(g_{\mathbb{C}}) \quad \text{and} \quad \mathrm{L}(f) = \mathbb{R}^p \setminus \mathrm{Bif}(g),$$

we get the claim since $\mathrm{Bif}(g) = J(g) \cup K_0(g)$ is semi-algebraic of positive codimension by Theorem 2.9.2. □

4.5 Rational functions and Lipschitz trivial values

It is natural to ask whether we can extend the category of mappings that satisfy the claim of our main result. We answer negatively, as Proposition 4.5.1 demonstrates that Theorem 4.3.1 is sharp in the sense that it does not hold for rational but non-polynomial mappings.

Let $f : \mathbb{K}^n \dashrightarrow \mathbb{K}$ be a rational function, $n \geq 2$. Its indeterminacy locus $I(f)$ is the subset of $\mathbb{K}^n$ where denominator and numerator vanish simultaneously (for all representations of f as a fraction). Let $\overline{\mathbb{K}}$ be the compactification of $\mathbb{K}$ defined as follows

$$\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \text{ and } \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}.$$

Property 4.5.1. *Assume that the rational function $f : \mathbb{K}^n \dashrightarrow \mathbb{K}$ does not extend continuously through the point $\boldsymbol{x}_0 \in \mathbb{K}^n$, i.e., the subset $J_{\boldsymbol{x}_0} := \{\lim_{\boldsymbol{x} \to \boldsymbol{x}_0} f(\boldsymbol{x})\} \subset \overline{\mathbb{K}}$ of accumulation values of f at $\boldsymbol{x}_0$ does not reduce to a single value in $\mathbb{K}$. Then $L(f) \cap J_{\boldsymbol{x}_0} = \emptyset$.*

Proof. If $\mathbb{K} = \mathbb{C}$ then $J_{\mathbf{x}_0} = \overline{\mathbb{C}}$. When $\mathbb{K} = \mathbb{R}$, we have $[a,b] \subset J_{\mathbf{x}_0}$ for any two values $a,b \in J_{\mathbf{x}_0}$, hence the set $J_{\mathbf{x}_0}$ is closed and has non-empty interior . Suppose $J_{\mathbf{x}_0} \cap \mathrm{L}(f)$ is non-empty, thus it is open. In such a case, there exists an open subset $\mathcal{V}$ of $J_{\mathbf{x}_0} \cap \mathrm{L}(f)$ such that f is Lipschitz trivial over $\mathcal{V}$. Then any trivializing bi-Lipschitz homeomorphism satisfies Estimates (4.1), contradicting the fact that $\mathbf{x}_0$ lies in the closure of any level $f^{-1}(t)$ when $t \in \mathcal{V}$. □

Corollary 4.5.1. *A complex rational function with Lipschitz trivial values has empty indeterminacy locus.*

Proof. Let $f : \mathbb{C}^n \dashrightarrow \mathbb{C}$ be a rational function with Lipschitz trivial values. Assume that f does not extend continuously through $\mathbf{x}_0 \in \mathbb{C}$. Then $J_{\mathbf{x}_0} = \overline{\mathbb{C}}$, thus $J_{\mathbf{x}_0} \cap \mathrm{L}(f)$ has non-empty interior. By Property 4.5.1, we have $\mathrm{L}(f) \cap J_{\mathbf{x}_0} = \emptyset$, a contradiction. □

On the other hand, the real setting is more flexible. Real rational functions may extend continuously (or even smoothly) through their indeterminacy locus onto $\mathbb{R}^n$, in such a case they are called regulous.

Proposition 4.5.1. *There exist rational functions $f : \mathbb{K}^n \dashrightarrow \mathbb{K}$ with empty indeterminacy locus that admit Lipschitz trivial values which are not values of properness, and are never of the form $g \circ \pi$ with $g : \mathbb{K}^m \dashrightarrow \mathbb{K}$ a rational function and $\pi : \mathbb{K}^n \to \mathbb{K}^m$ a linear surjective projection with $n > m$.*

Proof. Let $h : \mathbb{R}^{n-1} \to \mathbb{R}$ be the non-constant function $\mathbf{x} \mapsto h(\mathbf{x}) := 1 + \sum_{i=1}^{n-1} x_i^2$ and consider the rational $\mathcal{C}^\infty$ function $f : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(\mathbf{x}, y) = y - \frac{1}{h(\mathbf{x})}.$$

We have $I(f) = \emptyset$ and f has no critical point. Observe that the partial derivatives of f are uniformly bounded over $\mathbb{R}^n$, thus f is a Lipschitz function over $\mathbb{R}^n$.

For $c \in \mathbb{R}$ define the following mapping

$$G : \mathbb{R}^n \to \mathbb{R} \times f^{-1}(c), \ (\mathbf{x}, y) \mapsto \left(f(\mathbf{x}, y), \left(\mathbf{x}, c + \frac{1}{h(\mathbf{x})} \right) \right).$$

It is a Lipschitz homeomorphism with inverse

$$G^{-1}\left(t, \left(\mathbf{x}, c + \frac{1}{h(\mathbf{x})} \right) \right) = \left(\mathbf{x}, t + \frac{1}{h(\mathbf{x})} \right).$$

The inverse G^{-1} is also Lipschitz, thus each value c of $\mathbb{R}$ is Lipschitz trivial for f. Since any level of f is a graph over $\mathbb{R}^n$, the function f cannot be proper at c. Last, there exists no vector $\mathbf{v}$ of $\mathbb{R}^n \setminus \mathbf{0}$ such that $\partial_{\mathbf{v}} f \equiv 0$ since f is $\mathcal{C}^\infty$, $\partial_{x_j} f = \dfrac{2x_j}{[h(\mathbf{x})]^2}$ and $\partial_y f \equiv 1$. □

Remark 4.5.1. *The function f defined in the proof of Proposition 4.5.1 is regulous [21]; [19]; [11].*

5 CONCLUSION

We first proved that the affine part X^a of a connected projective algebraic curve X is Lipschitz normally embedded if and only if the following three conditions are satisfied: X^a is connected, X^a is locally Lipschitz normally embedded at each of its singular points; and $\deg(X) = \mathrm{card}(X^\infty)$. This result deals only with the case of complex algebraic sets with dimension 1. We shall find families of complex algebraic sets with dimension higher than 1 which are LNE. A natural step toward this target is to look for families of non-singular sets, because we only need to deal with the accumulation set at infinity. A nice first step would be to find properties at infinity which are sufficient to guarantee the LNE property.

We also proved that a polynomial mapping $f : \mathbb{K}^n \to \mathbb{K}^p$ attains a Lipschitz trivial value $\mathbf{c}$ if and only if there exist a polynomial mapping $g : \mathbb{K}^m \to \mathbb{K}^p$, for which the value $\mathbf{c}$ is a regular value of properness, and a linear surjective projection $\pi : \mathbb{K}^n \to \mathbb{K}^m$ such that $f = g \circ \pi$. We observed that this statement might not be true for non-polynomial mappings. The next step on this topic is to investigate the existence of some intermediate property for non-polynomial mappings.

REFERENCES

[1] BIRBRAIR, L.; MENDES, R. Arc criterion of normal embedding. **Springer Proceedings in Mathematics & Statistics. NBMS 2015, BMMS 2015**, v. 222, n. 2015, 2018.

[2] BIRBRAIR, L.; MOSTOWSKI, T. Normal embeddings of semialgebraic sets. **Michigan Mathematical Journal**, v. 47, n. 1, p. 125–132, 2000.

[3] BOCHNAK, J.; COSTE, M.; ROY, M.-F. **Real algebraic geometry**. Berlin: Springer-Verlag, 1998.

[4] CHIRKA, E. M. **Complex analytic sets**. Dordrecht: Kluwer Academic Publishers, 1989.

[5] DENKOWSKI, M.; TIBAR, M. Testing lipschitz non normally embedded complex spaces. **Bull. Math. Soc. Sci. Math. Roumanie**, v. 62, n. 1, p. 93–100, 2019.

[6] EHRESMANN, C. Les connexions infinitésimales dans un espace fibré différentiable. **Séminaire Bourbaki**, n. 1, exp. n. 24, p. 153-168, 1952.

[7] FANTINI, L.; PICHON, A. On lipschitz normally embedded singularities. **arXiv.org**, [Ithaca, N. Y.], 2022. Available in: https://arxiv.org/abs/2202.13725. Accessed on: 14 nov. 2022.

[8] FEDERER, H. **Geometric measure theory.** Berlin: Springer, 1996

[9] FERNANDES, A.; GRANDJEAN, V.; SOARES, H. A note on the local Lipschitz triviality of values of complex polynomial functions. **Mathematische Zeitschrift**, v. 296, p. 861–874, 2020.

[10] FERNANDES, A.; SAMPAIO, J. E. On lipschitz rigidity of complex analytic sets. **The Journal of Geometric Analysis**, v. 30, p. 706–718, 2020.

[11] FICHOU, G.; HUISMAN, J.; MANGOLTE, F.; MONNIER, J.-P. Fonctions régulues. **Journal für die reine und angewandte Mathematik**, v. 2016, n. 718, p. 103–151, 2016.

[12] FISCHER, A.; MARSHALL, M. Extending piecewise polynomial functions in two variables. **Annales de la Faculté des sciences de Toulouse** : Mathématiques, v. 6, n. 2, p. 253–268, 2013.

[13] HADAMARD, J. Resolution d'une question relative aux determinants. **Bull. des Sciences Math.**, v. 2, p. 240–246, 1893.

[14] HARDT, R. Semi-algebraic local-triviality in semi-algebraic mappings. **Model theory Mathematics**, v. 102, n. 2, p. 291–302, 1980.

[15] JELONEK, Z. The set of points at which a polynomial map is not proper. **Annales Polonici and Mathematici**, v. 58, n. 3, p. 259–266, 1993.

[16] JELONEK, Z. Geometry of real polynomial mappings. **Mathematische Zeitschrift**, v. 239, p. 321–333, 2002.

[17] JELONEK, Z.; KURDYKA, K. Quantitative generalized Bertini-Sard theorem for smooth affine varieties. **Discrete & Computational Geometry**, v. 34, n. 4, p. 659–678, 2005.

[18] KERNER, D.; PEDERSEN, H. M.; RUAS, M. A. S. Lipschitz normal embeddings in the space of matrices. **Mathematische Zeitschrift**, v. 290, p. 485–507, 2018.

[19] KUCHARZ, W. Rational maps in real algebraic geometry. **Advances in Geometry**, v. 9, n.4, p. 517–539, 2009.

[20] KURDYKA, K.; ORRO, P.; SIMON, S. Semialgebraic Sard theorem for generalized critical values. **Journal of Differential Geometry**, v. 56, n. 1, p. 67–92, 2000.

[21] KÓLLAR, J.; NOWAK, K. Continuous rational functions on real and p-adic varieties. **Mathematische Zeitschrift**, v. 279, n. 1, p. 85–97, 2015.

[22] LANGE, K. Hadamard's determinant inequality. **The American Mathematical Monthly**, Taylor & Francis, v. 121, n. 3, p. 258–259, 2014.

[23] LEE, J. M. **Introduction to smooth manifolds**. New York, 2012.

[24] MENDES, R.; SAMPAIO, J. E. On link of Lipschitz normally embedded sets. **arXiv.org**, 2018. Available in: https://arXiv:1801.05842. Accessed on: 14 nov. 2022.

[25] MILNOR, J. **Singular points of complex hypersurfaces**. Princeton, N.J.: Princeton University Press, 1968.

[26] RABIER, P. Ehresmann fibrations and Palais-Smale conditions for morphisms of Finsler manifolds. **Annals of Mathematics**, v. 146, n. 3, p. 647–691, 1997.

[27] SARD, A. The measure of the critical values of differentiable maps. **Bull. Amer. Math. Soc.**, v. 48, p. 883–890, 1942.

[28] THOM, R. Ensembles et morphismes stratifiés. **Bull. Amer. Math. Soc.**, v. 75, p. 240–284, 1969.

[29] VERDIER, J.-L. Stratifications de Whitney et théorème de Bertini-Sard. **Inventiones Mathematicae**, v. 36, n. 1, p. 295–312, 1976.

[30] WHITNEY, H. Tangents to an analytic variety. **Annals of Mathematics**, v. 81, n. 3, p. 496–549, 1965.