Conditions for the existence of a generalization of Rényi divergence

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ABSTRACT

We give necessary and sufficient conditions for the existence of a generalization of Rényi divergence, which is defined in terms of a deformed exponential function. If the underlying measure μ is non-atomic, we found that not all deformed exponential functions can be used in the generalization of Rényi divergence; a condition involving the deformed exponential function is provided. In the case μ is purely atomic (the counting measure on the set of natural numbers), we show that any deformed exponential function can be used in the generalization.

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1. Introduction

Entropy has been widely employed as a key measure of information in dynamical systems. Information theory, the field that investigates the characterization and limits of information, allows a number of applications that span from areas such as communications, neurobiology, natural language processing, econometric and other physical systems [1].

Shannon [2] was the first to interpret that information was linked to probability and to propose the quantity as an information or uncertainty measure, which can be written as

$$H(\mathbf{p}) = -\sum_{i=1}^{n} p_i \ln p_i,$$

where $\mathbf{p}$ is the probability mass function of the source of information. The quantity was named as entropy by its similarity with Boltzmann entropy (see, for instance, [3]). Another well know measure of information was proposed by Tsallis [4], who defined the expression

$$H_q(\mathbf{p}) = \frac{1}{q-1}\left(1 - \sum_{i=1}^{n} p_i^q\right),$$

as a generalized entropy dependent on the parameter $q \geq 0$, since when $q \to 1$ it reduces to the Shannon entropy. Also, in [5] Tsallis defined the function $\ln_q(x) = \frac{x^{1-q}-1}{1-q}$ for any non negative q, as a generalized logarithm function, which was

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termed as q-logarithm, since $\ln_q(x) \to \ln(x)$, as $q \to 1$. As a consequence, Tsallis entropy generalizes Shannon entropy [5]. The uniqueness theorem for Tsallis entropy was presented in [6] by introducing a generalization of Shannon–Khinchin axiom. Furthermore, this theorem was generalized and simplified in [7]. Tsallis entropy plays a crucial role in nonextensive statistics also called Tsallis statistics [8].

On another way of visualizing the uncertainty of events and how to measure them, Rényi proposed a family if entropies that can be written as [9]:

$$H_\alpha(\mathbf{p}) = \frac{1}{1 - \alpha} \ln \left(\sum_{i=1}^n p_i^\alpha \right),$$

where α is the entropy order. Rényi entropy is then flexible in the sense it can, as in the Tsallis case, to provide several different expressions by choosing different entropy orders. Due to its properties for the case $\alpha = 2$, some researchers have been working in the field termed information theoretic learning (ITL) where several interesting properties arise for this entropy definition [10]. This flexibility of the model proposed by Rényi is one of our key interests on the investigation of work.

While entropy is an uncertainty measure, relative entropy can be interpreted as a measure of statistical distance between two probability distributions [1]. Relative entropy, or statistical divergence, plays an essential role in information geometry [11]. A well-known example is the relative (Shannon) entropy, or Kullback–Leibler (KL) divergence, given by

$$\mathcal{D}(\mathbf{p} \parallel \mathbf{q}) = \sum_{i=1}^n p_i \ln \left(\frac{p_i}{q_i} \right),$$

which was defined in [12]. It can also be interpreted as an analogous (non symmetric) of the squared of the Euclidean distance [13]. One possible generalization of this divergence is the Tsallis relative entropy [14] which is obtained when we replace the ordinary logarithm by the q-logarithm in the KL divergence which yields

$$\mathcal{D}_q(\boldsymbol{p} \parallel \boldsymbol{q}) = \sum_{i=1}^n p_i \ln_q \left(\frac{p_i}{q_i} \right).$$

Both KL divergence and Tsallis relative entropy satisfy important properties, such as non negativity, monotonicity and joint convexity, among others [15]. One must note that the expressions for the definitions of entropy and divergence have considered discrete distributions, as in the original works but it is straightforward to provide those expressions considering continuous distributions by replacing the summation by integrals and the probability mass function by the probability density function [1].

The investigation of more general divergences and study their properties has been the object of interest of many researchers in the last decades. The interest on a different statistical divergence metric is motivated, among others, in applications related to optimization and statistical learning since more flexible functions and expressions may be suitable to larger classes of data and signals and lead to more efficient information recovery methods [16–18]. To cite a few, the usage of divergence metric has been considered in several domains such as statistics (including statistical physics) and learning [10,19–21], econometrics [22–26], digital communications [27–30], signal and image processing [31–33], biomedical processing [34]. Also, quantum versions of generalized divergences are of interest in the literature [35,36].

The general rationale on the consideration of divergence in such optimization problems is usually to derive more robust (or suitable) metrics to statistically differentiate two distributions stating how close (or how different) they are from each other. Csiszár introduced yet another concept of divergence, the f-divergence defined as $\sum_i^n q_i f \left(\frac{p_i}{q_i} \right)$ [13] for any convex function $f(t)$ for $t > 0$ such that $f(1) = 0$. KL divergence and Tsallis relative entropy are also obtained as a particular case of the f-divergence. *Amari α-divergence* [37–39] is yet another divergence that can be seen as a special case of the f-divergence since such divergence reduces to the KL one, when $\alpha = \pm 1$. Bregman introduced in [40] a divergence which is induced by a convex differentiable function. In [11] a more general expression of a divergence function was introduced, the (ρ, τ)-divergence, that has as a especial case the *Zhang's α-divergence* which is based on the quasiarithmetic mean [41] and includes the Bregman divergence, the Amari α-divergence and the f-divergence as special cases. Furthermore, more recently, the (ρ, τ)-embedding was studied in [42,43] and Jain and Chhabra [44] introduced a new generalized divergence measure for increasing functions.

Some of the proposed generalized divergences rely on a more flexible function in order to exploit other statistical characteristics. The deformed exponentials proposed by Naudts [45] and further investigated in the context of statistical physics in [46] are one of such more flexible models. The idea of those deformed functions is that they relax some conditions of the classical exponential function and expand the number of degrees of freedom one can play so aspects of heavier tails, for example, can be more easily incorporated within the same framework. This is particularly of interest in some problem in econophysics when the distribution of the risk changes due to some external aspects of the economy (unforeseen events such as pandemics and crash of stock market).

One generalization of the exponential families of probability distributions was introduced in [47], with the so-called φ-families of probability distributions. This generalization was possible by replacing of the exponential function by a deformed exponential φ, with some appropriate conditions. In that work, the φ-divergence was defined between two

probability distributions in the same φ-family. The φ-divergence can be interpreted as the Bregman divergence associated to the normalizing function ψ, that is a convex differentiable function. Actually, the φ-divergence between two probability distributions **p** and **q**, that are in the same φ-family, is the normalizing function ψ, that appears when we write the probability distribution **q** as a function of **p**. In others words, φ-divergence appears naturally in the theory of information geometry [48]. Furthermore, the φ-divergence has an inherent relationship with Zhang's α-divergence [11].

Rényi divergence [9] is one of the most successful measures of dissimilarity between probability distributions, having found many applications [49]. It is given by

$$\mathcal{D}^{(\alpha)}(\mathbf{p} \parallel \mathbf{q}) = \frac{1}{\alpha - 1} \ln \left(\sum_{i=1}^{n} \frac{p_i^\alpha}{q_i^{\alpha-1}} \right),$$

where α is the order of the entropy (a free parameter). In [50] the authors proposed a generalization of Rényi divergence in terms of a deformed exponential function. In order that this generalization be well-defined, the deformed exponential function have to satisfy some suitable conditions, which we investigate in the present paper. We considered the cases in which the underlying measure μ is non-atomic or purely atomic (the counting measure on the set of natural numbers $\mathbb{N}$). Each case required distinctive techniques, and provided different results. If the measure μ is non-atomic, we found that not all deformed exponential functions can be used in the generalization of Rényi divergence; a condition involving the deformed exponential function is provided. In the case μ is the counting measure on $\mathbb{N}$, we prove that any deformed exponential function can be used to define the generalization of Rényi divergence. These results are found in Section 2. In what follows, we show how the Rényi divergence can be generalized in terms of a deformed exponential function; the limit cases are also discussed.

Let (T, Σ, μ) be a σ-finite measure space. All probability distributions (or probability measures) are assumed to have positive density w.r.t. the underlying measure μ. In other words, they belong to the collection

$$\mathcal{P}_\mu = \left\{ p \in L^0 : \int_T p \, d\mu = 1 \text{ and } p > 0 \right\},$$

where L^0 is the space of all real-valued, measurable functions on T, with equality μ-a.e.

The Rényi divergence of order $\alpha \in (0, 1)$ between probability distributions p and q in $\mathcal{P}_\mu$ is defined as

$$\mathcal{D}^{(\alpha)}(p \parallel q) = \frac{\kappa(\alpha)}{\alpha(\alpha - 1)}, \tag{1.1}$$

where

$$\kappa(\alpha) = -\log\left(\int_T p^\alpha q^{1-\alpha} \, d\mu \right). \tag{1.2}$$

For $\alpha \in \{0, 1\}$, the Rényi divergence is defined by taking a limit:

$$\mathcal{D}^{(0)}(p \parallel q) = \lim_{\alpha \downarrow 0} \mathcal{D}^{(\alpha)}(p \parallel q),$$

$$\mathcal{D}^{(1)}(p \parallel q) = \lim_{\alpha \uparrow 1} \mathcal{D}^{(\alpha)}(p \parallel q).$$

Expression (1.1) can be used to define the Rényi divergence $\mathcal{D}^{(\alpha)}(\cdot \parallel \cdot)$ for every $\alpha \in \mathbb{R}$. However, for $\alpha \notin (-1, 1)$ this expression may not be finite-valued for all p and q in $\mathcal{P}_\mu$. To avoid some technicalities, we assume that $\alpha \in [-1, 1]$. The standard form of the Rényi divergence found in the literature differs from (1.1) by a factor of $1/\alpha$. We chose to define $\mathcal{D}^{(\alpha)}(\cdot \parallel \cdot)$ as in (1.1) so that some symmetry could be preserved when the limits $\alpha \downarrow 0$ and $\alpha \uparrow 1$ are taken.

The generalization of Rényi divergence is based on an alternate interpretation of $\kappa(\alpha)$. Fixed $\alpha \in (0, 1)$, and given any p and q in $\mathcal{P}_\mu$, the function $\kappa(\alpha) := \kappa(\alpha; p, q)$ is the unique non-negative real number such that

$$\int_T \exp(\alpha \ln(p) + (1 - \alpha) \ln(q) + \kappa(\alpha)) \, d\mu = 1.$$

To generalize the Rényi divergence, we consider a deformed exponential $\varphi(\cdot)$ in the place of the exponential function. A *deformed exponential* $\varphi : \mathbb{R} \to [0, \infty)$ is a convex function such that $\lim_{u \to -\infty} \varphi(u) = 0$ and $\lim_{u \to \infty} \varphi(u) = \infty$. Given any p and q in $\mathcal{P}_\mu$, we take $\kappa(\alpha) = \kappa(\alpha; p, q) \geq 0$ so that

$$\int_T \varphi(\alpha \varphi^{-1}(p) + (1 - \alpha)\varphi^{-1}(q) + \kappa(\alpha)u_0) \, d\mu = 1, \tag{1.3}$$

where $u_0 : T \to (0, \infty)$ is a positive, measurable function satisfying a suitable condition. The existence and uniqueness of $\kappa(\alpha)$ as defined in (1.2) is guaranteed by the condition in (1.3), which will be investigated in the next section. We will show that the existence of u_0 depends on $\varphi(\cdot)$ and the underlying measure μ.

We define the generalization of Rényi divergence of order $\alpha \in (0, 1)$ by

$$\mathcal{D}_\varphi^{(\alpha)}(p \parallel q) = \frac{\kappa(\alpha)}{\alpha(1 - \alpha)}, \tag{1.4}$$

where $\kappa(\alpha)$ is given as in (1.3). For $\alpha \in \{0, 1\}$, the generalization is defined by taking a limit:

$$\mathcal{D}_\varphi^{(0)}(p \parallel q) = \lim_{\alpha \downarrow 0} \mathcal{D}_\varphi^{(\alpha)}(p \parallel q), \tag{1.5}$$

$$\mathcal{D}_\varphi^{(1)}(p \parallel q) = \lim_{\alpha \uparrow 1} \mathcal{D}_\varphi^{(\alpha)}(p \parallel q). \tag{1.6}$$

These limits are related to a generalization of the Kullback–Leibler divergence [51], the so-called φ-divergence, which was introduced by the authors in [47]. The φ-divergence is given by

$$\mathcal{D}_\varphi(p \parallel q) = \frac{\int_T \frac{\varphi^{-1}(p) - \varphi^{-1}(q)}{(\varphi^{-1})'(p)} d\mu}{\int_T \frac{u_0}{(\varphi^{-1})'(p)} d\mu}. \tag{1.7}$$

In the case $\varphi(\cdot)$ is the exponential function and $u_0 = 1$, the φ-divergence reduces to the Kullback–Leibler divergence. Under some conditions, the limits (1.5) and (1.6) are finite-valued and converges to the φ-divergence:

$$\mathcal{D}_\varphi^{(0)}(q \parallel p) = \mathcal{D}_\varphi^{(1)}(p \parallel q) = \mathcal{D}_\varphi(p \parallel q) < \infty. \tag{1.8}$$

These conditions are stated in Proposition 1 for the case involving the generalized Rényi divergence.

Proposition 1. *Assume that $\varphi(\cdot)$ is continuously differentiable. Consider the condition*

$$\int_T \varphi(\alpha \varphi^{-1}(p) + (1 - \alpha)\varphi^{-1}(q)) d\mu < \infty. \tag{1.9}$$

If expression (1.9) is satisfied for all $\alpha \in [\alpha_0, 0)$ and some $\alpha_0 < 0$, then

$$\mathcal{D}_\varphi^{(0)}(p \parallel q) = \frac{\partial \kappa}{\partial \alpha}(0) = \mathcal{D}_\varphi(q \parallel p) < \infty.$$

If expression (1.9) is satisfied for all $\alpha \in (1, \alpha_0]$ and some $\alpha_0 > 1$, then

$$\mathcal{D}_\varphi^{(1)}(p \parallel q) = -\frac{\partial \kappa}{\partial \alpha}(1) = \mathcal{D}_\varphi(p \parallel q) < \infty.$$

Notice that expression (1.9) always holds for $\alpha \in [0, 1]$, since $\varphi(\cdot)$ is convex. For a proof of Proposition 1 we refer to Lemma 4 and Proposition 5 in [50].

2. Existence conditions

The generalization of Rényi divergence requires that $\kappa(\alpha)$ be well-defined. To guarantee the existence and uniqueness of $\kappa(\alpha)$ as defined by (1.3), we assume that there exists a measurable function $u_0 : T \to (0, \infty)$ such that

$$\int_T \varphi(c + \lambda u_0) d\mu < \infty, \qquad \text{for all } \lambda > 0, \tag{2.1}$$

for each measurable function $c : T \to \mathbb{R}$ satisfying $\int_T \varphi(c) d\mu < \infty$. The existence of u_0 depends on the deformed exponential $\varphi(\cdot)$ and the underlying measure μ. In the case μ is non-atomic, not all deformed exponential functions admit the existence of a function u_0 satisfying (2.1). (A measure μ is said to be *non-atomic* if for any measurable set A with $\mu(A) > 0$ there exists a measurable subset $B \subset A$ such that $\mu(A) > \mu(B) > 0$.) We shall find a condition involving solely $\varphi(\cdot)$ which is equivalent to the existence of u_0. If μ is the counting measure on the set of natural numbers $T = \mathbb{N}$, we will show that, for any deformed exponential function $\varphi(\cdot)$, always there exists a function u_0 (to be more precise, a sequence) satisfying (2.1).

Many deformed exponential functions $\varphi(\cdot)$ can be used in the generalization of Rényi divergence. A standard example is the exponential function, which satisfies condition (2.1) for $u_0 = 1$. Another example is the Kaniadakis' κ-exponential [47,52]. For the deformed exponential function given below, we cannot find a function u_0 for which condition (2.1) holds.

Example 2. Let us consider the deformed exponential function

$$\varphi(u) = \begin{cases} e^{(u+1)^2/2}, & u \geq 0, \\ e^{(u+\frac{1}{2})}, & u \leq 0. \end{cases}$$

Assume that the underlying measure μ is non-atomic. Given any measurable function $u_0 : T \to (0, \infty)$, we can find a measurable function $c : T \to \mathbb{R}$ with $\int_T \varphi(c) d\mu < \infty$, for which condition (2.1) does not hold. This claim was proved by the authors in [50, Example2]. An alternate proof of this result follows from a proposition (which will be shown in this section) involving the existence of u_0.

The next result shows that condition (2.1) is appropriate for the existence of $\kappa(\alpha)$, since they are equivalent.

Proposition 3. *Assume that the measure μ is non-atomic. Fix any $\alpha \in (0, 1)$. A deformed exponential $\varphi \colon \mathbb{R} \to [0, \infty)$ and a measurable function $u_0 \colon T \to (0, \infty)$ satisfy condition* (2.1) *if, and only if, for each probability distributions p and q in $\mathcal{P}_\mu$, there exists a constant $\kappa(\alpha) := \kappa(\alpha; p, q)$ such that*

$$\int_T \varphi(\alpha\varphi^{-1}(p) + (1-\alpha)\varphi^{-1}(q) + \kappa(\alpha)u_0)d\mu = 1. \tag{2.2}$$

Proof. If condition (2.1) is satisfied, the existence and uniqueness of $\kappa(\alpha)$ follows from the Monotone Convergence Theorem and the continuity of $\varphi(\cdot)$.

Suppose that condition (2.1) does not hold. In this case, for some measurable function $c \colon T \to \mathbb{R}$ with $\int_T \varphi(c)d\mu < \infty$, and some $\lambda_0 \geq 0$, we have

$$\begin{cases} \int_T \varphi(c + \lambda u_0)d\mu < \infty, & \text{for } 0 \leq \lambda \leq \lambda_0, \\ \int_T \varphi(c + \lambda u_0)d\mu = \infty, & \text{for } \lambda_0 < \lambda, \end{cases} \tag{2.3}$$

or

$$\begin{cases} \int_T \varphi(c + \lambda u_0)d\mu < \infty, & \text{for } 0 \leq \lambda < \lambda_0, \\ \int_T \varphi(c + \lambda u_0)d\mu = \infty, & \text{for } \lambda_0 \leq \lambda. \end{cases} \tag{2.4}$$

Notice that (2.4) cannot be satisfied for $\lambda_0 = 0$. Let $\{T_n\}$ be a sequence of non-decreasing, measurable sets with $\mu(T_n) < \infty$ and $\mu(T \setminus \bigcup_{n=1}^{\infty} T_n) = 0$. Define $A_n = T_n \cap \{c \leq n\} \cap \{u_0 \leq n\}$, for each $n \geq 1$. Clearly, the sequence $\{A_n\}$ is non-decreasing and satisfies $\mu(A_n) < \infty$ and $\mu(T \setminus \bigcup_{n=1}^{\infty} A_n) = 0$. Moreover,

$$\int_{A_n} \varphi(c + \lambda u_0)d\mu \leq \varphi(n + \lambda n)\mu(A_n) < \infty,$$

for all $\lambda > 0$, and each $n \geq 1$.

If the function u_0 satisfies (2.3), we select a sufficiently large $n_0 \geq 1$ such that $\int_{T \setminus A_{n_0}} \varphi(c + \lambda_0 u_0)d\mu < 1$. Denote $B := T \setminus A_{n_0}$. Let $b_1, b_2 \colon T \to \mathbb{R}$ be measurable functions for which $p = \varphi(c_1)$ and $q = \varphi(c_2)$ are in $\mathcal{P}_\mu$, where $c_1 = b_1\chi_{T \setminus B} + (c + \lambda_0 u_0)\chi_B$ and $c_2 = b_2\chi_{T \setminus B} + (c + \lambda_0 u_0)\chi_B$. Moreover, we assume $b_1\chi_{T \setminus B} \neq b_2\chi_{T \setminus B}$. For any $\lambda > 0$, we can write

$$\int_T \varphi(\alpha\varphi^{-1}(p) + (1-\alpha)\varphi^{-1}(q) + \lambda u_0) \geq \int_B \varphi(c + (\lambda_0 + \lambda)u_0)d\mu$$

$$= \int_T \varphi(c + (\lambda_0 + \lambda)u_0)d\mu - \int_{A_{n_0}} \varphi(c + (\lambda_0 + \lambda)u_0)d\mu = \infty.$$

Thus, the constant $\kappa(\alpha)$, as defined by (2.2), cannot be found.

Now suppose that (2.4) is satisfied. Let $\{\lambda_n\}$ be a sequence in $(0, \lambda_0)$ such that $\lambda_n \uparrow \lambda_0$. We define inductively an increasing sequence $\{k_n\} \subseteq \mathbb{N}$ as follows. Choose $k_1 \geq 1$ such that $\int_{A_{k_1}} \varphi(c + \lambda_0 u_0)d\mu \geq 1$ and $\int_{A_{k_1}} \varphi(c + \lambda_1 u_0)d\mu \leq 2^{-(1+1)}$. Given k_{n-1} we select some $k_n > k_{n-1}$ such that

$$\int_{A_{k_n} \setminus A_{k_{n-1}}} \varphi(c + \lambda_0 u_0)d\mu \geq 1$$

and

$$\int_{A_{k_n} \setminus A_{k_{n-1}}} \varphi(c + \lambda_n u_0)d\mu \leq 2^{-(n+1)}.$$

Let us denote $B_1 = A_{k_1}$ and $B_n = A_{k_n} \setminus A_{k_{n-1}}$ for $n > 1$. Notice that the sets B_n are pairwise disjoint. Define $u = \sum_{n=1}^{\infty} \lambda_n u_0 \chi_{B_n}$ and $B = \bigcup_{n=1}^{\infty} B_n$. As a result of this construction, it follows that

$$\int_B \varphi(c + u)d\mu \leq \frac{1}{2}.$$

Let $b_1, b_2 \colon T \to \mathbb{R}$ be measurable functions for which $p = \varphi(c_1)$ and $q = \varphi(c_2)$ are in $\mathcal{P}_\mu$, where $c_1 = b_1\chi_{T \setminus B} + (c + u)\chi_B$ and $c_2 = b_2\chi_{T \setminus B} + (c + u)\chi_B$. In addition, we assume $b_1\chi_{T \setminus B} \neq b_2\chi_{T \setminus B}$. Fixed arbitrary $\lambda > 0$, we take $n_1 \geq 1$ such that $\lambda_n + \lambda \geq \lambda_0$ for all $n \geq n_1$. Observing that $\int_{B_n} \varphi(c + \lambda_0 u_0)d\mu \geq 1$, we can write

$$\int_T \varphi(\alpha\varphi^{-1}(p) + (1-\alpha)\varphi^{-1}(q) + \lambda u_0)d\mu \geq \int_B \varphi(c + u + \lambda u_0)d\mu$$

$$\geq \sum_{n=n_1}^{\infty} \int_{B_n} \varphi(c + (\lambda_n + \lambda)u_0)d\mu \geq \sum_{n=n_1}^{\infty} 1 = \infty,$$

which shows that $\kappa(\alpha)$ cannot be found. $\quad\square$

The analysis concerning the existence of u_0 implicates the use of different techniques, which depend on the measure μ be non-atomic or purely atomic (the counting measure on the set of natural numbers $T = \mathbb{N}$).

2.1. Non-atomic case

As shown in Example 2, where the measure μ was assumed to be non-atomic, not all deformed exponential functions accept the existence of a function u_0 satisfying (2.1). Supposing that μ is non-atomic, we will present an equivalent criterion for a deformed exponential function and a function u_0 to satisfy condition (2.1). Using this result, we will find a condition involving solely $\varphi(\cdot)$ which is equivalent to the existence of u_0. Throughout this subsection, we assume that the measure μ is non-atomic.

Proposition 4. *A deformed exponential $\varphi \colon \mathbb{R} \to [0, \infty)$ and a measurable function $u_0 \colon T \to (0, \infty)$ satisfy condition (2.1) if, and only if, for some constant $\alpha \in (0, 1)$, we can find a measurable function $c \colon T \to \mathbb{R} \cup \{-\infty\}$ such that $\int_T \varphi(c)d\mu < \infty$ and*

$$\alpha\varphi(u) \leq \varphi(u - u_0(t)), \qquad \text{for all } u \geq c(t), \tag{2.5}$$

for μ-a.e. $t \in T$.

Inequalities similar to (2.5) will be assumed to hold for μ-a.e. $t \in T$. Accordingly, we will omit this assumption hereafter. The proof of Proposition 4 requires some preliminary results.

Lemma 5. *Let μ be a non-atomic, σ-finite measure. If $\{\alpha_m\}$ is a sequence of positive, real numbers, and $\{u_m\}$ is a sequence of finite-valued, non-negative, measurable functions, such that*

$$\int_T u_m d\mu \geq 2^m \alpha_m, \quad \text{for all } m \geq 1,$$

then there exist an increasing sequence $\{m_n\}$ of natural numbers and a sequence $\{A_n\}$ of pairwise disjoint, measurable sets such that

$$\int_{A_n} u_{m_n} d\mu = \alpha_{m_n}, \quad \text{for all } n \geq 1.$$

A proof of Lemma 5 is found in [53, Lemma 8.3]. We use Lemma 5 to prove the result stated below.

Lemma 6. *Suppose that we cannot find $\alpha \in (0, 1)$ and a measurable function $c \colon T \to \mathbb{R} \cup \{-\infty\}$ such that $\int_T \varphi(c)d\mu < \infty$ and*

$$\alpha\varphi(u) \leq \varphi(u - u_0(t)), \qquad \text{for all } u \geq c(t). \tag{2.6}$$

Then there exist sequences $\{c_n\}$ and $\{A_n\}$ of measurable functions, and pairwise disjoint, measurable sets, respectively, such that

$$\int_{A_n} \varphi(c_n)d\mu = 1 \quad \text{and} \quad \int_{A_n} \varphi(c_n - u_0)d\mu \leq 2^{-n}, \quad \text{for all } n \geq 1. \tag{2.7}$$

Proof. For each $m \geq 1$, we define the function

$$f_m(t) = \sup\{u \in \mathbb{R} : 2^{-m}\varphi(u) > \varphi(u - u_0(t))\},$$

where we use the convention $\sup \emptyset = -\infty$. We will verify that f_m is measurable. For each rational number r, define the measurable sets

$$E_{m,r} = \{t \in T : 2^{-m}\varphi(r) > \varphi(r - u_0(t))\}$$

and the simple functions $u_{m,r} = r\chi_{E_{m,r}}$. Let $\{r_i\}$ be an enumeration of the rational numbers. For each $m, k \geq 1$, consider the non-negative, simple functions $v_{m,k} = \max_{1 \leq i \leq k} u_{m,r_i}$. Moreover, denote $B_{m,k} = \bigcup_{i=1}^{k} E_{m,r_i}$. By the continuity of $\varphi(\cdot)$, it follows that $\varphi(v_{m,k})\chi_{B_{m,k}} \uparrow \varphi(f_m)$ as $k \to \infty$, which shows that f_m is measurable. Since (2.6) is not satisfied, we have that $\int_T \varphi(f_m)d\mu = \infty$ for all $m \geq 1$. In virtue of the Monotone Convergence Theorem, for each $m \geq 1$, we can find some $k_m \geq 1$ such that the function $v_m = v_{m,k_m}$ and the set $B_m = B_{m,k_m}$ satisfy $\int_{B_m} \varphi(v_m)d\mu \geq 2^m$. Clearly, we have that $\varphi(v_m)\chi_{B_m} < \infty$ and $2^{-m}\varphi(v_m)\chi_{B_m} \geq \varphi(v_m - u_0)\chi_{B_m}$. By Lemma 5, there exist an increasing sequence $\{m_n\}$ of indices and a sequence $\{A_n\}$ of pairwise disjoint, measurable sets such that $\int_{A_n} \varphi(v_{m_n})d\mu = 1$. Clearly, $\int_{A_n} \varphi(v_{m_n} - u_0)d\mu \leq 2^{-m_n}$. Denoting $c_n = v_{m_n}$, we obtain (2.7). $\quad\square$

Proof of Proposition 4. Assume that $\varphi(\cdot)$ and u_0 satisfy condition (2.1). Suppose that expression (2.5) does not hold. Let $\{c_n\}$ and $\{A_n\}$ be as stated in Lemma 6. Denote $A = \bigcup_{n=1}^{\infty} A_n$. Then we define $c = c_0 \chi_{T \setminus A} + \sum_{n=1}^{\infty} c_n \chi_{A_n}$, where $c_0 \colon T \to \mathbb{R}$ is any measurable function such that $\int_{T \setminus A} \varphi(c_0) d\mu < \infty$. Using (2.7), we can write

$$\int_T \varphi(c) d\mu = \int_{T \setminus A} \varphi(c_0) d\mu + \sum_{n=1}^{\infty} \int_{A_n} \varphi(c_n) d\mu$$

$$= \int_{T \setminus A} \varphi(c_0) d\mu + \sum_{n=1}^{\infty} 1 = \infty. \tag{2.8}$$

In addition, it follows that

$$\int_T \varphi(c - u_0) d\mu = \int_{T \setminus A} \varphi(c_0 - u_0) d\mu + \sum_{n=1}^{\infty} \int_{A_n} \varphi(c_n - u_0) d\mu$$

$$\leq \int_{T \setminus A} \varphi(c_0) d\mu + \sum_{n=1}^{\infty} 2^{-n} < \infty.$$

By condition (2.1), we get $\int_T \varphi(c) d\mu = \int_T \varphi(c - u_0 + u_0) d\mu < \infty$, which is a contradiction to (2.8).

Conversely, suppose that expression (2.5) holds. Let $\widetilde{c} \colon T \to \mathbb{R}$ be any measurable function satisfying $\int_T \varphi(\widetilde{c}) d\mu < \infty$. Denote $A = \{t : \widetilde{c}(t) + u_0(t) \geq c(t)\}$. We use inequality (2.5) to write

$$\alpha \int_T \varphi(\widetilde{c} + u_0) d\mu \leq \alpha \int_A \varphi(\widetilde{c} + u_0) d\mu + \alpha \int_{T \setminus A} \varphi(c) d\mu$$

$$\leq \int_A \varphi(\widetilde{c}) d\mu + \alpha \int_{T \setminus A} \varphi(c) d\mu < \infty.$$

As a result, we can conclude that $\int_T \varphi(\widetilde{c} + n u_0) d\mu < \infty$ for all $n \geq 1$. Consequently, $\int_T \varphi(\widetilde{c} + \lambda u_0) d\mu < \infty$ for all $\lambda > 0$. □

In Proposition 4, if we consider $u_0 = 1$ then the function $c(t)$ can be chosen to be constant. Clearly inequality (2.5) with $u_0 = 1$ holds for all $u \geq \text{ess inf} c(t)$. As a result, we can replace $c(t)$ with $\text{ess inf} c(t)$ if the measure μ is finite. On the other hand, assume $\mu(T) = \infty$. Then $\int_T \varphi(c) d\mu < \infty$ implies $\text{ess inf} c(t) = a_\varphi := \inf\{u \in \mathbb{R} : \varphi(u) > 0\}$. It cannot be the case $a_\varphi > -\infty$, since we would have $0 < \alpha \varphi(u) \leq \varphi(u - 1) = 0$ for $a_\varphi < u \leq a_\varphi + 1$. Consequently, the function $c(t)$ can be replaced by $\text{ess inf} c(t) = -\infty$; and inequality (2.5) holds for all $u \in \mathbb{R}$.

Next we present a class of deformed exponential functions which admit $u_0 = 1$.

Example 7. We will show that the Kaniadakis' κ-exponential $\exp_\kappa(\cdot)$ and $u_0 = 1$ satisfy condition (2.1). The κ-exponential $\exp_\kappa \colon \mathbb{R} \to (0, \infty)$ for $\kappa \in [-1, 1]$ is defined as

$$\exp_\kappa(u) = \begin{cases} (\kappa u + \sqrt{1 + \kappa^2 u^2})^{1/\kappa}, & \text{if } \kappa \neq 0, \\ \exp(u), & \text{if } \kappa = 0, \end{cases}$$

Its inverse, the so calledκ-logarithm $\log_k \colon (0, \infty) \to \mathbb{R}$, is given by

$$\log_\kappa(v) = \begin{cases} \dfrac{v^\kappa - v^{-\kappa}}{2\kappa}, & \text{if } \kappa \neq 0, \\ \ln(v), & \text{if } \kappa = 0. \end{cases}$$

We will verify that there exist $\alpha \in (0, 1)$ and $\lambda > 0$ for which

$$\lambda \leq \log_\kappa(v) - \log_\kappa(\alpha v), \qquad \text{for all } v > 0. \tag{2.9}$$

Some manipulations imply that the derivative of $\log_\kappa(v) - \log_\kappa(\alpha v)$ is negative for $0 < v \leq v_0$ and positive for $v \geq v_0$, where

$$v_0 = \left(\frac{\alpha^{-\kappa} - 1}{1 - \alpha^\kappa}\right)^{\frac{1}{2\kappa}} = \left(\frac{1}{\alpha}\right)^{\frac{1}{2}} > 0.$$

Consequently, the difference $\log_\kappa(v) - \log_\kappa(\alpha v)$ attains a minimum at v_0; given $\alpha \in (0, 1)$, inequality (2.9) is satisfied for some $\lambda > 0$. Inserting $v = \exp_\kappa(u)$ into (2.9), we can write

$$\alpha \exp_\kappa(u) \leq \exp_\kappa(u - \lambda), \qquad \text{for all } u \in \mathbb{R}. \tag{2.10}$$

If $n \in \mathbb{N}$ is such that $n\lambda \geq 1$, then a repeated application of (2.10) yields

$$\alpha^n \exp_\kappa(u) \leq \exp_\kappa(u - n\lambda) \leq \exp_\kappa(u - 1), \qquad \text{for all } u \in \mathbb{R}.$$

Proposition 4 implies that $u_0 = 1$ satisfies condition (2.1).

Now we show an equivalent criterion for the existence of u_0 satisfying (2.1).

Proposition 8. *Let $\varphi\colon \mathbb{R} \to [0, \infty)$ be a deformed exponential. Then we can find a measurable function $u_0\colon \mathbb{R} \to (0, \infty)$ for which condition (2.1) holds if, and only if,*

$$\limsup_{u \to \infty} \frac{\varphi(u)}{\varphi(u - \lambda_0)} < \infty, \tag{2.11}$$

for some $\lambda_0 > 0$.

Proof. By Proposition 4 we can conclude that the existence of u_0 implies (2.11). Conversely, assume that expression (2.11) holds for some $\lambda_0 > 0$. In this case, there exists $M \in (1, \infty)$ and $\overline{c} \in \mathbb{R}$ such that $\frac{\varphi(u)}{\varphi(u - \lambda_0)} \leq M$ for all $u \geq \overline{c}$. Let $\{\lambda_n\}$ be any sequence in $(0, \lambda_0]$ such that $\lambda_n \downarrow 0$. For each $n \geq 1$, define

$$c_n = \sup\{u \in \mathbb{R} : \alpha\varphi(u) > \varphi(u - \lambda_n)\}, \tag{2.12}$$

where $\alpha = 1/M$ and we adopt the convention $\sup \emptyset = -\infty$. From the choice of $\{\lambda_n\}$ and α, it follows that $-\infty \leq c_n \leq \overline{c}$. We claim that $\varphi(c_n) \downarrow 0$. If the sequence $\{c_n\}$ converges to some $c > -\infty$, the equality $\alpha\varphi(c_n) = \varphi(c_n - \lambda_n)$ implies $\alpha\varphi(c) = \varphi(c)$ and then $\varphi(c) = 0$. In the case $c_n \downarrow -\infty$, it is clear that $\varphi(c_n) \downarrow 0$. Let $\{T_k\}$ be a sequence of pairwise disjoint, measurable sets with $\mu(T_k) < \infty$ and $\mu(T \setminus \bigcup_{k=1}^{\infty} T_k) = 0$. Thus we can select a sub-sequence $\{c_{n_k}\}$ such that $\sum_{k=1}^{\infty} \varphi(c_{n_k})\mu(T_k) < \infty$. Let us define $c = \sum_{k=1}^{\infty} c_{n_k} \chi_{T_k}$ and $u_0 = \sum_{k=1}^{\infty} \lambda_{n_k} \chi_{T_k}$. From (2.12) it follows that

$$\alpha\varphi(u) \leq \varphi(u - u_0(t)), \qquad \text{for all } u \geq c(t).$$

Proposition 4 implies that $\varphi(\cdot)$ and u_0 satisfy condition (2.1). $\square$

For the deformed exponential function $\varphi(\cdot)$ given in Example 2, it follows that

$$\limsup_{u \to \infty} \frac{\varphi(u)}{\varphi(u - \lambda_0)} = \limsup_{u \to \infty} \frac{e^{(u+1)^2/2}}{e^{(u-\lambda_0+1)^2/2}}$$
$$= \limsup_{u \to \infty} e^{u\lambda_0 - (\lambda_0^2 - \lambda_0)/2} = \infty,$$

which shows that $\varphi(\cdot)$ cannot be used in the generalization of Rényi divergence.

A deformed exponential function $\varphi(\cdot)$ that satisfies (2.11) does not increase faster then $u \mapsto e^{\lambda u}$ for some $\lambda \geq 1$. Expression (2.11) is equivalent to the existence of constants $K \geq 1$ and $c \in \mathbb{R} \cup \{-\infty\}$ such that

$$\frac{\varphi(u)}{\varphi(u - \lambda_0)} \leq K, \qquad \text{for all } u \geq c.$$

Fixed any $v \geq 0$ we take an integer $n \geq 0$ such that $n\lambda_0 \leq v < (n+1)\lambda_0$. For $u \geq c$, we can write

$$\varphi(u + v) \leq \varphi(u + (n+1)\lambda_0)$$
$$\leq K^{n+1}\varphi(u)$$
$$\leq K^{v/\lambda_0 + 1}\varphi(u)$$
$$= K\varphi(u)e^{\lambda v},$$

where $\lambda = \log(K)/\lambda_0$. Therefore, the function $\varphi(\cdot)$ cannot increase faster then $u \mapsto e^{\lambda u}$.

2.2. Purely atomic case

In this subsection, we will assume that μ is the counting measure on the set of natural numbers $T = \mathbb{N}$. Due to this assumption, notation changes a little. Sequences and summations are considered in the place of functions and integrals. Condition (2.1) is rewritten as follows. We assume that there exists a sequence $\{u_{0,i}\} \subset (0, \infty)$ such that

$$\sum_{i=1}^{\infty} \varphi(c_i + \lambda u_{0,i}) < \infty, \qquad \text{for all } \lambda > 0, \tag{2.13}$$

for each sequence $\{c_i\} \subset \mathbb{R}$ such that $\sum_{i=1}^{\infty} \varphi(c_i) < \infty$. Beyond these changes, proofs of results involving condition (2.13) require distinct techniques. In this subsection, we shall find an equivalent criterion for a deformed exponential function and a sequence $\{u_{0,i}\}$ to satisfy condition (2.13). We will prove that, in the case μ is the counting measure, any deformed exponential function $\varphi(\cdot)$ admits a sequence $\{u_{0,i}\}$ for which condition (2.13) holds.

Proposition 9. *A deformed exponential $\varphi\colon \mathbb{R} \to [0, \infty)$ and a sequence $\{u_{0,i}\}$ satisfy condition (2.13) if, and only if, for some constants $\alpha \in (0, 1)$ and $\varepsilon > 0$, we can find a sequence $\{c_i\} \subseteq \mathbb{R} \cup \{-\infty\}$ such that $\sum_{i=1}^{\infty} \varphi(c_i) < \infty$ and*

$$\alpha\varphi(u) \leq \varphi(u - u_{0,i}), \qquad \text{for all } u > c_i \text{ with } \varphi(u - u_{0,i}) < \varepsilon. \tag{2.14}$$

To prove Proposition 9, we require a preliminary lemma.

Lemma 10. *Suppose that we cannot find $\alpha \in (0, 1)$, $\varepsilon > 0$ and a sequence $\{c_i\} \subset \mathbb{R} \cup \{-\infty\}$ such that $\sum_{i=1}^{\infty} \varphi(c_i) < \infty$ and*

$$\alpha \varphi(u) \leq \varphi(u - u_{0,i}), \qquad \text{for all } u > c_i \text{ with } \varphi(u - u_{0,i}) < \varepsilon. \tag{2.15}$$

Then there exist sequences $\{\{c_{n,i}\}\}$ and $\{A_n\}$ of finite-valued real numbers, and pairwise disjoint sets in $\mathbb{N}$, respectively, such that

$$\frac{1}{2} \leq \sum_{i \in A_n} \varphi(c_{n,i}) \quad \text{and} \quad \sum_{i \in A_n} \varphi(c_{n,i} - u_{0,i}) \leq 2^{-n}, \tag{2.16}$$

for each $n \geq 1$.

Proof. For each $m \geq 1$, we define the sequence $\{f_{m,i}\} \subset \mathbb{R} \cup \{-\infty\}$ by

$$f_{m,i} = \sup\{u \in \mathbb{R} : 2^{-m}\varphi(u) > \varphi(u - u_{0,i}) \text{ and } \varphi(u - u_{0,i}) \leq 2^{-m-1}\},$$

where we use the convention $\sup \emptyset = -\infty$. Since (2.15) is not satisfied, we have that $\sum_{i=1}^{\infty} \varphi(f_{m,i}) = \infty$ for each $m \geq 1$. We will consider the following cases.

Case 1. There exists a strictly increasing sequence $\{m_n\} \subseteq \mathbb{N}$ for which the set $B_n = \{i : \varphi(f_{m_n,i} - u_{0,i}) = 2^{-m_n-1}\}$ has an infinite number of elements. Then we can select a strictly increasing sequence $\{i_n\} \subseteq \mathbb{N}$ such that

$$2^{-m_n}\varphi(f_{m_n,i_n}) \geq \varphi(f_{m_n,i_n} - u_{0,i_n}) = 2^{-m_n-1},$$

which implies $\varphi(f_{m_n,i_n}) \geq 1/2$. Expression (2.16) follows with $c_{n,i} = f_{m_n,i}$ and $A_n = \{i_n\}$.

Case 2. There exists a strictly increasing sequence $\{m_n\} \subseteq \mathbb{N}$ for which the set B_n, as defined above, has a finite number of elements. Let us denote $C_n = \mathbb{N} \setminus B_n = \{i : \varphi(f_{m_n,i} - u_{0,i}) < 2^{-m_n-1}\}$. By the continuity of $\varphi(\cdot)$, we have that $2^{-m_n}\varphi(f_{m_n,i}) = \varphi(f_{m_n,i} - u_{0,i})$ for all $i \in C_n$. Because $\varphi(f_{m_n,i}) \leq 1/2$ for each $i \in C_n$, and $\sum_{i=1}^{\infty} \varphi(f_{m_n,i}) = \infty$ for all $n \geq 1$, we can find a strictly increasing sequence $\{k_n\} \subset \mathbb{N}$ for which the set $A_n = C_n \cap \{k_{n-1}, \ldots, k_n - 1\}$ satisfies

$$\frac{1}{2} \leq \sum_{i \in A_n} \varphi(f_{m_n,i}) \leq 1.$$

The second inequality above in conjunction with $2^{-m_n}\varphi(f_{n,i}) = \varphi(f_{n,i} - u_{0,i})$ implies

$$\sum_{i \in A_n} \varphi(f_{m_n,i} - u_{0,i}) \leq 2^{-m_n}.$$

Thus expression (2.16) follows with $c_{n,i} = f_{m_n,i}$. □

Proof of Proposition 9. To show that condition (2.13) implies inequality (2.14), one can proceed as in the proof of Proposition 4, using Lemma 10 in the place of Lemma 6.

Suppose that inequality (2.14) is satisfied. Let $\{\widetilde{c}_i\}$ be any sequence of real numbers such that $\sum_{i=1}^{\infty} \varphi(\widetilde{c}_i) < \infty$. Denote $A = \{i : \widetilde{c}_i + u_{0,i} \geq c_i\}$ and $B = \{i \in A : \varphi(\widetilde{c}_i) \leq \varepsilon\}$. We use inequality (2.5) to write

$$\alpha \sum_{i=1}^{\infty} \varphi(\widetilde{c}_i + u_{0,i}) \leq \alpha \sum_{i \in A \cap B} \varphi(\widetilde{c}_i + u_{0,i}) + \alpha \sum_{i \in A \setminus B} \varphi(\widetilde{c}_i + u_{0,i}) + \alpha \sum_{i \in T \setminus A} \varphi(c_i)$$

$$\leq \sum_{i \in A} \varphi(\widetilde{c}_i) + \alpha \sum_{i \in A \setminus B} \varphi(\widetilde{c}_i + u_{0,i}) + \alpha \sum_{i \in T \setminus A} \varphi(c_i) < \infty. \tag{2.17}$$

To conclude that the second summation in (2.17) is finite, we observed that the set $T \setminus B$ is finite. In consequence, it follows that $\sum_{i=1}^{\infty} \varphi(\widetilde{c}_i + nu_{0,i}) < \infty$ for all $n \geq 1$; and then $\sum_{i=1}^{\infty} \varphi(\widetilde{c}_i + \lambda u_{0,i}) < \infty$ for all $\lambda > 0$. □

The result stated below shows that any deformed exponential function $\varphi(\cdot)$ can be used in the generalization of Rényi divergence, in the case μ is the counting measure.

Proposition 11. *Let $\varphi \colon \mathbb{R} \to [0, \infty)$ be a deformed exponential. Then we can find a sequence $\{u_{0,i}\}$ for which condition (2.13) holds.*

Proof. Let $\{\lambda_n\} \subset (0, \infty)$ be any decreasing sequence converging to 0. Fix any $\alpha \in (0, 1)$ and $\eta \in \mathbb{R}$ such that $\alpha \varphi(\eta) < \varphi(\eta - \lambda_1)$. Denoting $\varepsilon = \varphi(\eta - \lambda_1)$, we define

$$\widetilde{c}_n = \sup\{u \in \mathbb{R} : \alpha \varphi(u) > \varphi(u - \lambda_n) \text{ and } \varphi(u - \lambda_n) \leq \varepsilon\}, \qquad \text{for each } n \geq 1,$$

where we adopt the convention $\sup \emptyset = -\infty$. Clearly, the sequence $\{\widetilde{c}_n\} \subset [-\infty, \eta)$ is decreasing. We claim that $\varphi(\widetilde{c}_n) \downarrow 0$. If the sequence $\{\widetilde{c}_n\}$ converges to some $c > -\infty$, inequality $\alpha\varphi(\widetilde{c}_n) \geq \varphi(\widetilde{c}_n - \lambda_n)$ implies $\alpha\varphi(c) \geq \varphi(c)$ and then $\varphi(c) = 0$. In the case $\widetilde{c}_n \downarrow -\infty$, it is clear that $\varphi(\widetilde{c}_n) \downarrow 0$. Thus we can select a sub-sequence $c_i = \widetilde{c}_{n_i}$ such that $\sum_{i=1}^{\infty} \varphi(c_i) < \infty$ and

$$\alpha\varphi(u) \leq \varphi(u - u_{0,i}), \qquad \text{for all } u > c_i \text{ with } \varphi(u) < \varepsilon,$$

where $u_{0,i} = \lambda_i$. From Proposition 9, it follows that $\{u_{0,i}\}$ satisfies condition (2.13). $\quad\square$

Such general models provide more robust methods to devise different distributions and improve the capability of inference of which the distribution better fits the available data. For example, in [33], the authors employ a φ-divergence to the problem of image segmentation achieving better results than classical image processing methods. In their case, the selected φ function complies the existence conditions discussed in this work. To fail meeting such existence conditions, in the problem of image classification (segmentation can be one step in the process) would lead to some classes of images (it would depend on the probability distribution of the images) being erroneously assumed as different ones since the divergence would not include all the statistical characteristics of the image data.

3. Conclusions

This paper provided the existence conditions of a generalized Rényi divergence so a deformed exponential function can be used to model the statistical distribution. Such conditions admit the design of a robust model by assuming any deformed exponential which provides the use of purely atomic measure. For the non-atomic case not all deformed exponentials can be used to generalize the Rényi divergence although there are a fair amount of functions that comply with the existence conditions and therefore be applied to problems based on statistical divergence optimization. The results presented in this paper allow to consider discrete distributions (such as the one we can find in digital applications) to devise the differentiation between two probability distributions, which brings a greater number of possibilities of applications in several areas such as signal and image processing and possible extensions to quantum cases.

CRediT authorship contribution statement

Rui F. Vigelis: Conceptualization, Investigation, Writing. **Luiza H.F. de Andrade:** Investigation, Writing. **Charles C. Cavalcante:** Investigation, Writing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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