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VALESSA VALENTIM VIANA

DISSIPATIVITY-BASED STATIC OUTPUT FEEDBACK STABILIZATION OF DYNAMICAL SYSTEMS

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Dissertação apresentada ao Curso de Mestrado Acadêmico em Engenharia Elétrica do Programa de Pós-Graduação em Engenharia Elétrica do Centro de Tecnologia da Universidade Federal do Ceará, como requisito parcial à obtenção do título de mestre em Engenharia Elétrica. Área de Concentração: Engenharia Elétrica

Orientador: Prof. Dr.-Ing. Diego de Sousa Madeira

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BANCA EXAMINADORA

Prof. Dr.-Ing. Diego de Sousa Madeira (Orientador)
Universidade Federal do Ceará (UFC)

Prof. Dr. Fabrício Gonzalez Nogueira
Universidade Federal do Ceará (UFC)

Prof. Dr. Ricardo Coração de Leão F. de Oliveira
Universidade de Campinas (UNICAMP)

Prof. Dr. Wilkley Bezerra Correia
Universidade Federal do Ceará (UFC)

Ao meu pai e à minha mãe;

Aos meus irmãos;

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(Marcel Proust)

RESUMO

Esta dissertação aborda o problema de estabilização de sistemas dinâmicos em tempo contínuo por realimentação linear estática de saída. São propostas três estratégias diferentes para estabilização de três categorias de sistemas, sendo eles sistemas lineares com incertezas, lineares a parâmetros variantes e não lineares com saturação na entrada e incertezas. Na formulação das condições de cada estratégia proposta foi utilizada uma noção de dissipatividade conhecida como condição de QSR-dissipatividade estrita, que se apresenta como uma condição necessária e suficiente para estabilização por realimentação de saída, sob certas suposições. Primeiramente, no caso de sistemas lineares, tendo em vista a consideração de um modelo mais realista do sistema, a estratégia proposta considera incertezas nas matrizes do sistema. Além disso, como um requisito de desempenho de malha-fechada, é considerada uma restrição na taxa de decaimento mínima do sistema. Já no caso de sistemas a parâmetros variantes, foi considerado que o sistema além de ser afetado por incertezas no modelo, também é afetado por uma entrada externa, que pode ser vista como um distúrbio no sistema. Então, a estratégia propõe a estabilização considerando a minimização do ganho $\mathcal{L}_2$. Nesse caso, é utilizada uma representação algébrico diferencial do sistema LPV que permite lidar com uma classe de sistemas onde as matrizes podem apresentar dependência polinomial ou racional no parâmetro. Por fim, considerou-se o caso de sistemas não lineares com saturação na entrada, que representam modelos mais fiéis à realidade, tanto por considerar as não-linearidades do sistema, como por considerar o problema de saturação na entrada. Nesse caso, o sistema também é transformado em uma representação algébrico diferencial tal que as matrizes do sistema podem apresentar dependência polinomial ou racional nos seus argumentos. Além disso, uma condição do setor é utilizada para lidar com a saturação na entrada do sistema e funções de Lyapunov mais genéricas são consideradas em busca de se obter resultados menos conservadores. Nos três casos, um algoritmo iterativo, baseado em desigualdades matriciais lineares, recentemente desenvolvido é aplicado visando computar os ganhos de realimentação enquanto minimiza uma função objetivo, que no caso de sistemas a parâmetros variantes consiste na minimização do ganho $\mathcal{L}_2$ e no caso dos sistemas não lineares consiste na maximização da região de atração. Por fim, são apresentados exemplos numéricos para mostrar a eficácia das estratégias propostas.

Palavras-chave: Realimentação de saída. Controle robusto. Dissipatividade. Desigualdades matriciais lineares.

ABSTRACT

This dissertation deals with the static output feedback control problem for continuous-time dynamical systems. Different strategies are proposed to stabilize three distinct classes of systems, namely the classes of linear time-invariant systems, linear parameter varying systems, and input saturated nonlinear systems, all considering uncertainties in the system model. The notion of strict QSR-dissipativity, also known as a necessary and sufficient condition for static output feedback stabilizability under certain circumstances, is applied to formulate new sufficient conditions in the form of linear matrix inequalities. In the case of uncertain linear systems, the proposed strategy considers more realistic models by including uncertainty in the system matrices. Moreover, a minimum bound for the decay rate of the system is a closed-loop performance constraint. In the linear parameter varying case, both uncertainties and external inputs are considered. Thus, the strategy suggests stabilization with the $\mathcal{L}_2$-gain performance criterion. In this case, it is considered a differential-algebraic representation that allows dealing with the broad class of systems whose matrices present rational or polynomial dependence on the parameters. Finally, uncertain saturated nonlinear systems are contemplated, which characterizes more realistic models by considering at the same time nonlinearities in the system model and the saturating actuator condition. In this case, the proposed strategy also transforms the system into a differential-algebraic representation allowing the system to present rational or polynomial dependence on the state and uncertain parameters. A generalized sector condition is used to deal with the saturation on the input, and rational Lyapunov functions (which are more generic than quadratic ones) are considered to obtain less conservative results compared to the recent literature. Furthermore, a recently developed iterative algorithm based on linear matrix inequalities is applied to compute the feedback gain matrices simultaneously to the minimization of an objective function. While the objective function is to minimize the $\mathcal{L}_2$ gain in the strategy for linear time-varying systems, in the approach for nonlinear ones, the objective function is the maximization of the region of attraction. For all cases, numerical examples are provided to highlight the effectiveness of the proposed strategies.

Keywords: Static output feedback. Robust control. Dissipativity. Linear matrix inequalities.

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LIST OF ABBREVIATIONS AND ACRONYMS

DAR	Differential algebraic representation
LMI	Linear matrix inequality
LPV	Linear parameter varying
LTI	Linear time-invariant
MIMO	Multiple-input multiple-output
RA	Region of attraction
SDP	Semidefinite programming
SOF	Static output feedback
SOS	Sum of Squares
SSF	Static state feedback

NOTATION

$A^\top$	Transpose of matrix A.
$A \succeq 0$	Matrix A is positive semi-definite.
$A \preceq 0$	Matrix A is negative semi-definite.
$\mathbb{S}_n$	The set of $n \times n$ symmetric matrices.
$\mathbb{S}_n^+$	The set of $n \times n$ symmetric positive definite matrices.
I_n	Identity matrices $n \times n$.
J_n	Exchange matrices $n \times n$ (i.e., anti-diagonal matrix with ones).
$\star$	Denotes blocks induced by symmetry.
$diag(A,B)$	Block-diagonal matrix of matrices A and B.
$\mathrm{He}\{A\}$	Operator $A + A^\top$.
∇	Gradient operator.
$tr(A)$	Trace of a matrix A.
$\mathcal{V}(\mathcal{X})$	Vertices of a polytope $\mathcal{X}$.
$f : \mathcal{X} \to \mathcal{Y}$	A (vector) function with domain $\mathcal{X}$ and codomain $\mathcal{Y}$.
$sign(x)$	The signum function, i.e, $sign(x) = -1$ if $x < 0$, $sign(x) = 0$ if $x = 0$, and $sign(x) = 1$ if $x > 0$.
$\mathcal{C}^1$	The set of functions whose partial derivatives exist and are continuous, i.e., $f \in \mathcal{C}^1$ means that f is continuously differentiable .
$\|f\|_2$	The l_2 norm of $f(t) : \mathbb{R}^+ \to \mathbb{R}^n$, given by $\sqrt{\int_0^t f^\top(\tau) f(\tau) d\tau}$.
$\|x\|$	The Euclidean norm of a vector x, given by $\sqrt{x^\top x}$.
$\cap$	Operator for intersection of sets.
$row_i(A)$	The i-th row of a matrix A.
$col_i(A)$	The i-th column of a matrix A.
$[A_i]_{row}^{i \in \mathbb{I}_n}$	Means that matrices $[A_1, \ldots, A_n]$ are arranged as a row.
$[A_i]_{col}^{i \in \mathbb{I}_n}$	Means that matrices $[A_1, \ldots, A_n]$ are arranged as a column.
$[A_i]_{diag}^{i \in \mathbb{I}_n}$	Means that matrices $[A_1, \ldots, A_n]$ are arranged as diagonal arrays.
$[A]_{diag}^{\mathbb{I}_n}$	Means that matrix A is arranged as diagonal arrays n times.

CONTENTS

1 INTRODUCTION

Stability is, in general, the most crucial expected characteristic of a dynamical system. Instability can be a characteristic of the original process. Moreover, the addition of external effects, such as disturbances, can lead a stable system to instability. Thus, to achieve stability of an unstable system, it must be affected by a control input computed by a controller. Feedback controllers using plant state information for feedback are commonly applied to stabilize dynamical systems. This type of controller assumes that plant states are available for measurement. However, measuring all plant states can be expensive or impossible due to physical constraints. In this case, applying state feedback controllers using all states for feedback becomes impossible. However, a state observer can be designed to estimate the states of the system that are unavailable for measurement to perform observer-based state feedback control. Moreover, controllers using only one state or a set of the process states for feedback appear as an alternative in this case. These are called output feedback controllers. (SYRMOS *et al.*, 1997).

Output feedback controllers are described as two classes of controllers, the dynamic and the static, the focus is the static one in this work. A Static output feedback (SOF) controller is the most simple controller compared with other control methods. It consists of a simple static gain multiplying the output vector of the system. Designing a SOF is an important and challenging problem in control theory. It results in non-convex conditions that are not allowed to be solved by Semidefinite programming (SDP) (GEROMEL *et al.*, 1996). Even though many strategies provide solutions to this problem, there is no exact solution that can guarantee the SOF stabilizability or determine that such a SOF does not exist, even for simple linear models (SADABADI; PEAUCELLE, 2016).

The first strategies proposing solutions to the SOF control problem have considered Linear time-invariant (LTI) models, where system dynamics are known to be linear, and the matrices of the system state-space model are constant and known. Trofino and Kucêra (1993), Kucêra and De Souza (1995), and Cao *et al.* (1998) have proposed necessary and sufficient conditions to stabilize this type of system via static output feedback. However, these conditions are nonlinear and difficult to test. With the aim of obtaining conditions that are easier to solve, such as the linear ones, many papers have been proposing iterative and non-iterative Linear matrix inequality (LMI)-based sufficient conditions to deal with this problem (VESELÝ, 2001; CRUSIUS; TROFINO, 1999; APKARIAN; NOLL, 2006; GAHINET; APKARIAN, 2011; GEROMEL *et al.*, 1996; GEROMEL *et al.*, 1998).

However, sometimes, LTI models do not faithfully represent the original system. As a consequence, it is important to consider deviations in the system models that are known as uncertainties. Uncertainties may result from unmodeled dynamics, parametric uncertainties, noise, and linearization (ZHOU; DOYLE, 1998). These deviations are considered in the control design to assure the *robustness* of the controllers. It means stability is guaranteed even if the model, for which the controller was designed, presents bounded deviations from the original system. Then, in the case of LTI models affected by uncertainties, the SOF design becomes even harder. Dong and Yang (2007), Dong and Yang (2013), Agulhari *et al.* (2010), Agulhari *et al.* (2012), Sereni *et al.* (2018), Felipe and Oliveira (2021) proposed sufficient conditions for the robust SOF design for this type of system.

Another interesting consideration is that the uncertain parameters are time-varying. A class of systems considering this is called Linear parameter varying (LPV) systems (CAIGNY *et al.*, 2009). The time-varying parameter can be assumed to be available or unavailable for measurement. An interesting problem called the gain scheduling control comes into question when the parameter is available for measurement (WEI *et al.*, 2014; APKARIAN *et al.*, 1995). The gain scheduling approach is based on the measurement of the time-varying parameter that adjusts the controller gains for the complete range of parameter variation (SHAMMA; ATHANS, 1991; RUGH; SHAMMA, 2000). It is well-known that the gain scheduling approach provides less conservative results for LPV systems when compared with other control methods (APKARIAN *et al.*, 1995; SHAMMA; ATHANS, 1991; MONTAGNER; PERES, 2005).

The static output feedback gain scheduling controller consists of a scheduled gain multiplying the output vector of the system. As excepted, static output feedback gain-scheduling design is also a challenging problem. In the literature, some works propose sufficient conditions to stabilize LPV systems via static output feedback gain scheduling control (AL-JIBOORY; ZHU, 2018; NGUYEN *et al.*, 2018; SERENI *et al.*, 2019; BEHROUZ *et al.*, 2021; SERENI *et al.*, 2022).

However, in a more realistic view, systems present more complex nonlinearities than those confined in LPV systems. Then, it is essential to develop control strategies for a model considering these complex nonlinearities in the system (KHALIL, 2002). The control design for this type of system is challenging, even when we have information on all states of the system. Therefore, the SOF design also becomes arduous in the case of nonlinear systems.

Recently, a concept known as dissipativity proved to be important for the SOF control problem, aiming at the stabilization of linear and nonlinear systems. Dissipativity theory was introduced some decades ago, and is a generalization of the notion of passivity. Numerous works in the literature have proposed passivity- and dissipativity-based stabilization strategies (HILL; MOYLAN, 1976; ASTOLFI *et al.*, 2002; FENG *et al.*, 2013; ORTEGA; GARCIA-CANSECO, 2004; SHISHKIN; HILL, 1995). Madeira (2022) proved, under mild assumptions, that a specific case of dissipativity called strict QSR-dissipativity is a necessary and sufficient condition for SOF exponential stabilizability of a class of linear and nonlinear systems. It is an important result, showing that dissipativity is a concept that can be used even in the most challenging control design problems.

Despite the effort to find solutions for the SOF control problem, at that moment, the search for a less conservative LMI-based method for designing static output feedback controllers is still being performed, even in the case of LTI systems. Then, this work addresses the problem of static output feedback robust stabilization of some classes of continuous-time dynamical systems. Since dissipativity proved to be important for the SOF control problem, we take advantage of this fact by using dissipativity in the development of the strategies.

First, a framework for the robust SOF stabilization of uncertain linear systems is proposed. The proposed strategy provides new sufficient LMI conditions for feedback stabilization with a lower bound on the decay rate that guarantees transient performance. The main contribution consists that our approach can provide less conservative results than those available in the literature.

Later, sufficient conditions for designing a gain scheduling SOF controller are provided to stabilize continuous-time LPV systems. The influence of external signals on the system are also considered for the stabilization with $\mathcal{L}_2$-gain performance. One of the main contributions of this strategy is the consideration that the system matrices can present other types of dependencies on the parameter, such as rational ones, since most papers only consider affine dependence. Moreover, another advantage is to consider that the system can be affected by measured and unmeasured time-varying parameters.

Finally, sufficient conditions for the stabilization of polynomial or rational nonlinear systems with an estimation of the region of attraction are proposed. The plant can be affected by input saturation and parametric uncertainties, thus increasing the level of complexity of the control design problem. Aiming at the stabilization of more complex systems, the proposed strategy deals with both linear and nonlinear control laws, and more generic Lyapunov functions are considered, leading to less conservative conditions.

In all proposed strategies in this work, to solve the resulting nonlinear inequalities, a recently developed iterative algorithm is used, from Alves Lima *et al.* (2022), where LMIs are solved at each iteration. Furthermore, numerical examples borrowed from the literature are presented to illustrate the effectiveness of all proposed approaches.

This dissertation is divided into 6 chapters, the main chapters where the proposed strategies are presented are chapters 3, 4, and 5. An overview of the following chapters is presented in the sequence.

- **Chapter 2**: Some theoretical preliminaries that are important for a better comprehension of the work are presented.
- **Chapter 3**: The SOF stabilization of uncertain LTI systems is addressed. We present the system definition, the problem formulation, and the development of the proposed strategy. Some numerical examples are provided to illustrate the effectiveness of the strategy. This chapter is based on the following article:
  – **VIANA, V. V.**; MADEIRA, D. de S. *Robust Static Output Feedback Stabilization of Linear Systems Using Dissipativity Theory*, Simpósio Brasileiro de Automação Inteligente (SBAI), Virtual, 2021.
- **Chapter 4**: The proposed strategy for $\mathcal{L}_2$-gain scheduling SOF stabilization of LPV systems is presented. First, we present the system definition, the problem formulation, and the development of the proposed strategy for the case without $\mathcal{L}_2$-gain performance. Later, the same is presented for the case considering the $\mathcal{L}_2$-gain performance. Numerical examples are provided to illustrate the efficiency of the strategy. This chapter is based on the following article:
  – **VIANA, V. V.**; MADEIRA, D. de S.; ALVES LIMA, T. *Dissipativity-based $\mathcal{L}_2$ gain-scheduled static output feedback design for rational LPV systems*, IEEE American Control Conference (ACC), Atlanta, Georgia, USA, 2022.
- **Chapter 5**: The SOF stabilization of uncertain input saturated nonlinear systems is proposed. The class of nonlinear systems considered, the problem formulation, and the proposed strategy are presented. Numerical examples from the literature are also investigated. This chapter is an extension of the following work that treats the same stabilization problem using quadratic Lyapunov functions instead of more generic ones as used in this dissertation.

  - ALVES LIMA, T.; MADEIRA, D. de S.; **VIANA, V. V.**; OLIVEIRA, R. C. L. F. *Static output feedback stabilization of uncertain rational nonlinear systems with input saturation*. Systems & Control Letters, Amsterdam, v. 168, p. 105359, 2022.
- **Chapter 6**: Finally, this chapter presents a summary of conclusions and perspectives of future works.

2 THEORETICAL PRELIMINARIES

In this chapter, we present some important preliminary contents from the literature. These contents provide a theoretical foundation for the main results developed in the following chapters. First, a general notion of dynamical systems introduces the types of systems used in this work. After that, both definitions of stability and the Lyapunov theory for stability analysis are presented. The dissipativity theory and its motivation for the SOF control problem are also shown. Later, the differential-algebraic representation is presented. Finally, in the last section, some mathematical tools from the literature are presented.

2.1 Dynamical Systems

A dynamical system is a description of a real process such that dynamics evolve in time. In mathematics, a function describing the variables of a system, called states, represents the dynamics at any instant in time. One of the classical representations of a dynamical system is given by the following linear time-invariant model,

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input. $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ are constant matrices representing the system (CHEN, 1999).

It is well-known that (2.1) is an elementary representation of a system since it only represents linear dynamics. A more complex model than (2.1) is given by an LPV system, where the dynamics depend on a parameter that evolves in a bounded set. The mathematical model of an LPV system is given by

$$\dot{x}(t) = A(\rho)x(t) + B(\rho)u(t), \tag{2.2}$$

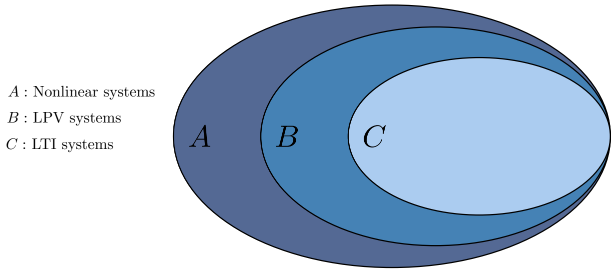
where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input (BRIAT, 2014). Moreover, $\rho(t) \in \Omega \subset \mathbb{R}^r$ is a vector of time-varying parameters available for measurement, where Ω is a bounded set. $A(\rho) \in \mathbb{R}^{n\times n}$ and $B(\rho) \in \mathbb{R}^{n\times m}$ are matrices depending on $\rho(t)$.

However, in general, real systems present more complex nonlinear dynamics, thus a more general mathematical model of a system is given by

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \tag{2.3}$$

where $t \geq 0$, $x(t) \in \mathbb{R}^n$ is the state vector with initial conditions $x(0) \in \mathcal{X} \subseteq \mathbb{R}^n$, with $0 \in \mathcal{X}$, $u(t)$ is the control input of the system such that $u(t) \in \mathcal{U} \subseteq \mathbb{R}^m$, $0 \in \mathcal{U}$. $f(x) : \mathcal{X} \to \mathbb{R}^n$ and $g(x) : \mathcal{X} \to \mathbb{R}^{n\times m}$ are continuously differentiable functions, i.e., $(f,g) \in \mathcal{C}^1$, and $f(0) = 0$, and the origin $(x(t),u(t)) = (0,0)$ is an equilibrium point of (2.3) (KHALIL, 2002). The set $\mathcal{X}$ is defined in section 2.1.1. Figure 1 shows the relationship between the three classes of systems, with nonlinear systems being the most general.

Figure 1 – Systems diagram.



Fonte: The author.

In the following chapters, different strategies for the stabilization of the three representations (2.1)-(2.3) of dynamical systems are developed. Aiming to deal with more realistic models, we assume that each model is affected by uncertainties, saturation on the control input, and external inputs. Details of each system representation are given in the following chapters.

2.1.1 Polytopic region

The set $\mathcal{X} \subseteq \mathbb{R}^n$ is a given polytope (with n_x vertices) of initial conditions $x(0)$ containing the origin. Such a polytope can be represented as the intersection of n_{xe} hyperplanes (COUTINHO; GOMES DA SILVA JR., 2010)

$$\mathcal{X} := \{x \mid a_k^\top x \leq 1, k = 1,\ldots,n_{xe}\}, \tag{2.4}$$

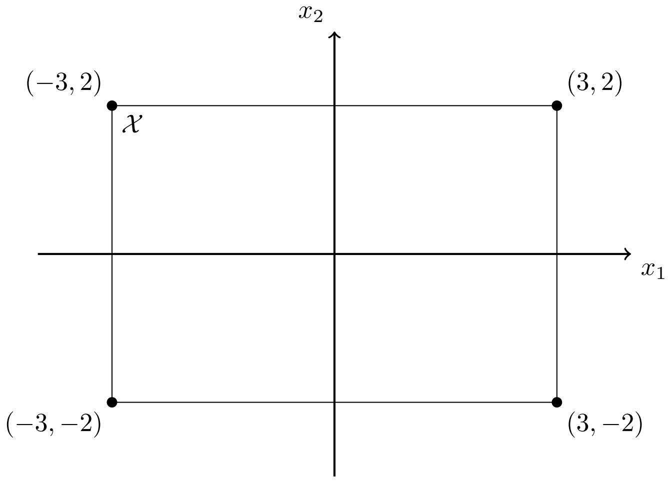
where the constant vectors $a_k \in \mathbb{R}^n$ can be determined by fulfilling $a_k^\top x = 1$ at all groups of adjacent vertices of $\mathcal{X}$. The set $\mathcal{X}$ does need to be positively invariant.

2.1.1.1 Example

Consider a polytopic set $\mathcal{X} \in \mathbb{R}^2$, as shown in Figure 2, with vertices $\mathcal{V} := \{v_1, v_2, v_3, v_4\}$, where

$$v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -3 \\ -2 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Figure 2 – Polytope $\mathcal{X}$ in $\mathbb{R}^2$.



Fonte: The author.

For each adjacent pair of vertices there exists an hyperplane defined by $\{x : a_k^\top x \leq 1\}$. The equations of each hyperplane of the polytope $\mathcal{X} \in \mathbb{R}^2$ are:

$$v_1, v_2 \in \{x : a_1^\top x = 1\}, \tag{2.5}$$

$$v_2, v_3 \in \{x : a_2^\top x = 1\}, \tag{2.6}$$

$$v_3, v_4 \in \{x : a_3^\top x = 1\}, \tag{2.7}$$

$$v_4, v_1 \in \{x : a_4^\top x = 1\}, \tag{2.8}$$

with that, we can define a set of linear equations to determine the vectors a_k's,

$$\begin{cases} a_1^\top v_1 = 1 \\ a_1^\top v_2 = 1 \end{cases} \quad \begin{cases} a_2^\top v_2 = 1 \\ a_2^\top v_3 = 1 \end{cases} \quad \begin{cases} a_3^\top v_3 = 1 \\ a_3^\top v_4 = 1 \end{cases} \quad \begin{cases} a_4^\top v_4 = 1 \\ a_4^\top v_1 = 1. \end{cases} \tag{2.9}$$

Therefore, by solving (2.9), we have

$$a_1 = \begin{bmatrix} 0.33 \\ 0 \end{bmatrix}, \ a_2 = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix}, \ a_3 = \begin{bmatrix} -0.33 \\ 0 \end{bmatrix}, \ a_4 = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix}.$$

2.2 Lyapunov Stability

In this section, some fundamental concepts regarding different forms of stability of equilibrium points are presented. First, to illustrate classical stability definitions, we consider an autonomous system

$$\dot{x}(t) = f(x(t)), \tag{2.10}$$

where $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state vector of the system and $f : \mathcal{X} \rightarrow \mathbb{R}^n$ is a continuous function on $\mathcal{X}$. Any point $\bar{x} \in \mathcal{X}$ such that $f(\bar{x}) = 0$ is an equilibrium point of the system. A formal definition of an equilibrium point is presented below (HADDAD; CHELLABOINA, 2011).

Definition 2.1 *A point $\bar{x} \in \mathcal{X}$ is said to be an equilibrium point of* (2.3) *at time $t^* \in [t_0, \infty)$, i.e. $x(t) = \bar{x}$ for all $t \geq t^*$, if $f(t, \bar{x}) = 0$ for all $t \geq t^*$.*

Since any equilibrium point can be translated to the origin of the system, without loss of generality, the following stability definition considers that the equilibrium point of the system is the origin, i.e., $\bar{x} = 0$ (KHALIL, 2002).

Definition 2.2 *The equilibrium point $\bar{x} = 0$ of* (2.10) *is*

1. *Stable if, for $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$||x(0)|| < \delta \Rightarrow ||x(t)|| < \varepsilon, \forall t \geq 0.$$

2. *Asymptotically stable if it is stable and δ can be chosen such that*

$$||x(0)|| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0.$$

3. *Unstable if it not stable.*

From Definition 2.2, an equilibrium point is stable if all state trajectories with initial conditions in a neighborhood nearby an equilibrium point stay nearby. It is unstable otherwise. Furthermore, it is asymptotically stable if all state trajectories with initial conditions in a region nearby an equilibrium point stay nearby and tend to the equilibrium point as the time approaches infinity (KHALIL, 2002). The stability of a dynamical system can also be analyzed based on the choice of a specific function, called the Lyapunov function. The stability analysis procedure is stated by the Lyapunov's theorem presented in the sequence (KHALIL, 2002).

Theorem 2.1 *Let $\bar{x} = 0$ be an equilibrium point of* (2.10) *and $\mathcal{X} \subseteq \mathbb{R}^n$ a domain containing $\bar{x} = 0$. Let $V : \mathcal{X} \rightarrow R$ be a continuous function such that*

$$V(0) = 0 \;\; and \;\; V(x) > 0 \;\; in \;\; \mathcal{X} - \{0\}, \tag{2.11}$$

$$\dot{V}(x) \leq 0 \;\; in \;\; \mathcal{X}. \tag{2.12}$$

Then, $\bar{x} = 0$ is stable. Furthermore, if

$$\dot{V}(x) < 0 \;\; in \;\; \mathcal{X} - \{0\} \tag{2.13}$$

then, $\bar{x} = 0$ is asymptotically stable.

Proof. (KHALIL, 2002). □

A problem arising with Lyapunov's theorem for stability analysis is how to choose the function $V(x)$. Lyapunov functions can be arbitrarily selected following the dynamics of the system. One of the widely used functions is the quadratic function $V(x) = x^\top Px,\ P = P^\top \in \mathbb{R}^{n\times n}$ due to its simplicity in formulating stability conditions. However, this function can be the reason for conservative results. (TROFINO; DEZUO, 2014). Then, polynomial or rational Lyapunov functions arise as an alternative.

Now, from Haddad and Chellaboina (2011) and Madeira (2022), some concepts of stability are formalized for a controlled system, such as (2.3). First, consider the following feedback control law $u(t) = \phi(x(t))$, where $\phi : \mathcal{X} \to \mathcal{U},\ \phi(0) = 0$, and $x(t)$ is the state vector satisfying (2.3) for all $t \geq 0$. Then, suppose that the mapping $\phi(\cdot)$ satisfies sufficient regularity conditions such that (2.14) has a unique solution.

$$\dot{x}(t) = f(x(t)) + g(x(t))\phi(x(t)) = F(x(t)). \tag{2.14}$$

From Haddad and Chellaboina (2011, p. 163), a lemma formalizing stability results for the controlled system is shown below.

Lemma 2.1 *Assume that the equilibrium point* $\bar{x} = 0$ *of* $F(x(t))$ *from* (2.14) *is asymptotically stable,* $F : \mathcal{X} \to \mathbb{R}^n$ *is continuously differentiable, and let* $\delta > 0$ *be such that the set* $\{x \in \mathbb{R}^n : \|x\| < \delta\} \subset \mathcal{X}$ *is contained in a set such that the state trajectories of* (2.14) *remain in this set and converge to the equilibrium point. Then there exists a continuously differentiable function* $V : \{x \in \mathbb{R}^n : \|x\| < \delta\} \to \mathbb{R}$ *such that* $V(0) = 0,\ V(x) > 0$*, and* $\dot{V}(x) = \nabla V^\top(x)F(x) < 0$ *for* $x \neq 0$ *in the following set* $\{x \in \mathbb{R}^n : \|x\| < \delta\}$.

2.2.1 Exponential stability

Exponential stability is a more specific definition of stability. It means that the trajectories of an asymptotically stable system converge faster than or as fast as a specifically known rate. As presented in Madeira (2022), $\bar{x} = 0$ of (2.3) is exponentially stable if and only if for all $x(t)$ contained in the set $\{x \in \mathbb{R}^n : \|x\| < \delta\}$,

$$\begin{gathered} \nabla V(x)^\top [f(x) + g(x)u(x)] \leq -\varepsilon V(x), \\ c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2, \end{gathered} \tag{2.15}$$

for scalars $(\varepsilon, c_1, c_2) > 0$ (HADDAD; CHELLABOINA, 2011, pg. 166), (FRADKOV; HILL, 1998). Furthermore,

$$\|\nabla V(x)\| \leq c_3 \|x\|, \tag{2.16}$$

for a scalar $c_3 > 0$.

2.2.2 *Estimated region of attraction*

The region (or domain) of attraction of a system is a set of admissible initial conditions such that the state trajectories converge to the equilibrium point without leaving this region. From the Lyapunov Theorem presented in section 2.2, it is possible to define an estimation for the domain of attraction of a system (KHALIL, 2002). Consider the following region

$$\mathcal{E} := \{x \in \mathbb{R}^n; V(x) \leq 1\} \subset \mathcal{X}, \tag{2.17}$$

where $V(x)$ and $\mathcal{X}$ are, respectively, the Lyapunov function and the domain considered in Theorem 2.1. If $\dot{V}(x) < 0$, $\forall x \in \mathcal{X} - \{0\}$, the region $\mathcal{E}$ is an estimated domain of attraction of system (2.10), i.e., every trajectory starting inside $\mathcal{E}$ converges to the origin without leaving $\mathcal{E}$.

2.3 Linear Matrix Inequalities

Linear matrix inequality is an essential tool for the formulation of control problems. It proved useful for stability analysis and synthesis using Lyapunov's theory. First, LMIs were solved analytically, limiting their application to small systems. In the 1990s, LMIs became more popular due to the development of efficient algorithms to solve these inequalities, which allowed the application of LMIs in more complex problems (BOYD *et al.*, 2004). Since then, LMIs have been extensively used to solve many control problems as a problem of convex optimization that can be easily solved. From Boyd *et al.* (2004), the representation of a linear matrix inequality is given by the following expression

$$X(x) \triangleq X_0 + \sum_{i=1}^{m} x_i X_i \succ 0 \tag{2.18}$$

where $x \in \mathbb{R}^m$ is the decision variable vector and $X_i \in \mathbb{R}^{n \times n}$, $i = 0, \ldots, m$, are given symmetric matrices. The problem in inequality (2.18) is to find x such that $X(x)$ is positive definite, i.e., $v^\top X(x) v > 0$ for all nonzero $v \in \mathbb{R}^n$. LMI (2.18) is a convex constraint on x, i.e., the set $\{x | F(x) > 0\}$ is convex.

2.4 Static Output Feedback Control Problem

In this section, an introduction to the static output feedback control problem will be proceeding. First, we intend to show the reason for the complexity of SOF design by comparison with static state feedback design using the concepts of Lyapunov theory. We consider the stabilization of simple LTI systems with quadratic Lyapunov functions. After that, some classical solutions for the SOF control problem for LTI systems will be presented. Consider the time-invariant linear system (2.1). First, suppose that all states of (2.1) are measurable. Then, it is possible to apply a static state feedback (SSF) control,

$$u(t) = Kx(t) \tag{2.19}$$

where $K \in \mathbb{R}^{m\times n}$ is a gain matrix to be determined. Using concepts of Lyapunov theory considering a quadratic Lyapunov function ($V(x) = x^\top Px > 0$), we obtain the following condition for the asymptotic stability of system (2.1) in closed-loop (BOYD *et al.*, 2004),

$$P > 0,\ A^\top P + K^\top B^\top P + PA + PBK < 0 \tag{2.20}$$

where K and P are decision variables. Clearly, $K^\top B^\top P$ and PBK are nonlinear terms, and this inequality can not be solved as a convex problem. To avoid these nonlinear terms, we can apply a change of variables, as follows. Consider a symmetric matrix $Q = Q^\top > 0 : PQ = I$, pre and post multiplying (2.20) by Q, we obtain

$$QA^\top + QK^\top B^\top + AQ + BKQ < 0. \tag{2.21}$$

Consider also that there exist a matrix $Y = KQ$, then the new condition for stability of system (2.1) is given by

$$Q > 0,\ QA^\top + Y^\top B^\top + AQ + BY < 0, \tag{2.22}$$

which is an LMI and can be solved as a convex problem. Therefore, in the static state feedback design, even if the original condition is nonlinear, we can apply simple manipulations to obtain a necessary and sufficient LMI condition that can be easily solved with SDP tools.

On the other hand, assume that is not possible to measure all states of system (2.1). In this case, we are not able to apply a SSF for system stabilization, then the static output feedback arises as an alternative, where

$$u(t) = Ky(t) = KCx(t). \tag{2.23}$$

Considering the same quadratic Lyapunov function ($V(x) = x^\top Px > 0$), we obtain the following condition for the stability of system (2.1) in closed-loop (SYRMOS *et al.*, 1997),

$$P > 0,\ A^\top P + C^\top K^\top B^\top P + PA + PBKC < 0. \tag{2.24}$$

where K and P are decision variables. Clearly, $C^\top K^\top B^\top P$ and $PBKC$ are nonlinear terms and this inequality can not be solved as a convex problem. In this case, even pre and post multiplying the condition by a symmetric matrix Q, it is not possible to use a change of variables to obtain a convex condition. It can be seen in the following condition that the decision variables Q and K are not side by side,

$$Q > 0,\ QA^\top + QC^\top K^\top B^\top + AQ + BKCQ < 0. \tag{2.25}$$

Well-known bilinear matrix inequality (BMI) solvers can be applied to find a solution to inequality (2.24) (HENRION *et al.*, 2005; KOČVARA; STINGL, 2003). However, the choice of a good initial guess is essential. Moreover, finding a solution for the SOF control problems is often a failed attempt in these solvers (SADABADI; PEAUCELLE, 2016). As summarized in the survey by Sadabadi and Peaucelle (2016), there exist a few classical convex solutions for the SOF control problem for LTI systems in the literature. One of them involves a change of variables with an auxiliary matrix, such that an equality constraint with this matrix and the Lyapunov matrix has to be satisfied. However, this constraint is difficult to test, being not possible in general. Another solution in terms of LMI conditions is proposed in Prempain and Postlethwaite (2001), where a series of assumptions in the system are necessary, such as the need for the plant to be square, in addition to other constraints involving the plant matrices. Then, the satisfaction of all assumptions is a disadvantage of this method. Another convex approach involving three steps was proposed by (BENTON; SMITH, 1998). The first step involves the design of a state feedback stabilizing gain, the second step consists of using the previously found state feedback gain to solve an LMI that has the Lyapunov function as a decision variable, and the third and last step is to use the Lyapunov function found in another LMI condition to find the SOF stabilizing gain. However, if any stage fails, no conclusions can be drawn. Having clarified the complexity of the SOF stabilizability for the simplest case of an LTI system, we can conclude that the SOF control problem for more complex systems such as uncertain systems, LPV systems, and general nonlinear systems is an even more complicated problem. A detailed review of the SOF control literature for these systems will be carried out in the following chapters.

2.5 Dissipativity Theory

The dissipativity theory investigates some input-output properties of a dynamical system, for example, the properties of dissipation, conservation, and mass transport of energy. Some examples of dissipative systems are electrical and mechanical systems. In control theory, dissipativity theory has been extensively used for stability analysis and control system design (BROGLIATO *et al.*, 2020). Brogliato *et al.* (2020) presented some mathematical definitions for this concept of dissipativity. Then, consider a dynamical system such as (2.3), a general definition of dissipativity is formally presented in Definition 2.3.

Definition 2.3 *System* (2.3) *is said to be dissipative along all possible trajectories of the system starting at* $x(0)$*, for all* $t \geq 0$*, if there exists a continuously differentiable storage function* $V(x) > 0$ *such that*

$$\dot{V}(x) \leq r(u,y). \tag{2.26}$$

From a practical point of view, a dissipative system stores only a fraction of the energy supplied to it through $r(u,y)$ and only a fraction of its stored energy $V(x)$ can be delivered to its surroundings. From Haddad and Chellaboina (2011), others definitions of dissipativity involving a specific supply rate are presented below.

Definition 2.4 *System* (2.3) *is said to be QSR-dissipative if it is dissipative with the following supply rate*

$$r(u,y) = y^\top Qy + 2y^\top Su + u^\top Ru, \tag{2.27}$$

where $Q \in \mathbb{S}_p$ *and* $R \in \mathbb{S}_m$ *are symmetric, and* $S \in \mathbb{R}^{p\times m}$.

Definition 2.5 *System* (2.3) *is said to be strictly QSR-dissipative if it is QSR-dissipative and there exists* $T(x) > 0$ *such that*

$$\dot{V} + T \leq y^\top Qy + 2y^\top Su + u^\top Ru, \tag{2.28}$$

where $Q \in \mathbb{S}_p$ *and* $R \in \mathbb{S}_m$ *are symmetric, and* $S \in \mathbb{R}^{p\times m}$.

Definition 2.6 *A dynamical system* (2.3) *is called exponentially QSR-dissipative if it is strictly QSR-dissipative with* $T(x) = \varepsilon V(x)$*, for some scalar* $\varepsilon > 0$.

After having defined the notions of dissipativity, some related results can be presented, such as a particular case of dissipativity called passivity. Then, from Definition 2.4, restricting the supply rate matrices to be $Q=0,\ S=\frac{1}{2}I$, and $R=0$, we obtain a condition for a system to be passive. Following this, we formalize the definition of a passive system (BROGLIATO *et al.*, 2020).

Definition 2.7 *System* (2.3) *is said to be passive if it is dissipative with the following supply rate*

$$r(u,y) = y^\top u. \tag{2.29}$$

Moreover, dissipativity theory can be related to the Lyapunov theory presented in section 2.2. For this, consider the storage function ($V(x)>0$) as a Lyapunov function of the system. Thus, if a system is QSR-dissipative with $Q\leq 0$, then the uncontrolled system is Lyapunov stable. On the other hand, in the case of strictly QSR-dissipative with $Q\leq 0$, the uncontrolled system is asymptotically stable (HADDAD; CHELLABOINA, 2011). Since a relation of dissipativity to stability was presented, consider that the system is dissipative with matrices $Q=-\frac{1}{\gamma}I,\ S=0$ and $R=\gamma I$, which is equivalent to

$$\dot{V}(x) \leq -\frac{1}{\gamma}\,y^\top y + \gamma\,u^\top u, \tag{2.30}$$

where γ is a positive scalar. Considering that $V(x)$ is a Lyapunov function, condition (2.30) represents that a system is input to output stable and has finite $\mathcal{L}_2$-gain from u to y. Then, dissipativity can also be related to the $\mathcal{L}_2$-gain stability (HILL; MOYLAN, 1980).

2.5.1 Applying dissipativity for SOF stabilization

As previous introduced, the concepts of dissipativity and stability are related. Thus, dissipativity has been extensively applied in stability analysis and control design. Furthermore, this concept of dissipativity has allowed the development of crucial results to the well-known SOF control problem. Recently, Madeira (2022) proved, under mild assumptions, that strict QSR-dissipativity is a necessary and sufficient condition for static output feedback stabilizability of LTI and nonlinear systems. The following lemma shows the result from Madeira (2022).

Lemma 2.2 *A dynamical system* (2.3) *is exponentially stabilizable by linear SOF if and only if it is exponentially QSR-dissipative with* $R>0$ *and* $\Delta=0$*, where*

$$\Delta = SR^{-1}S^\top - Q. \tag{2.31}$$

A stabilizing SOF is given by

$$u = Ky,\ K = -R^{-1}S^{\top}. \tag{2.32}$$

It is important to highlight that the linear part of the dissipativity-based approach of Madeira (2022) is equivalent to the results of Peaucelle and Arzelier (2005), which applies a topological separation for establishing new necessary and sufficient conditions for linear SOF stabilization of LTI systems, involving the same nonlinear inequality $SR^{-1}S^{\top} - Q \geq 0$. However, Madeira (2022) provided necessary and sufficient stabilizability conditions for a broad class of nonlinear systems, which has not been done by any previous paper.

To summarize, there are some advantages of using dissipativity for SOF control design. First, dissipative-based techniques allow the manipulation of matrices (Q,S,R) to achieve stability. Moreover, from the results of Madeira (2022), dissipativity is a necessary and sufficient condition for SOF exponential stabilization. It has motivated the use of dissipative properties to address the stabilization problem via static output feedback control. It is important to state that the strategy from Madeira (2022) does not demand any fixed closed-loop dynamics such as a port-controlled Hamiltonian structure.

2.6 Differential Algebraic Representation - DAR

Nonlinear systems can present polynomial or rational dependence on the states of the system. To deal with polynomial or rational vector fields, proceeding with a transformation of the original model facilitates stability analysis and control synthesis. Then, Ghaoui and Scorletti (1996) proposed an alternative and exact representation of rational systems, namely the linear-fractional transformation (LFT). Later, a generalization of the LFT, called differential-algebraic representation (DAR), was proposed (COUTINHO *et al.*, 2002; COUTINHO *et al.*, 2008). While the matrices of the DAR can be affine functions of the states, matrices of the LFT are only constant. Thus, the DARs are more general than LFRs (AZIZI *et al.*, 2018). Furthermore, the DAR stores the nonlinear terms of the system in a vector of nonlinear terms. It can help in using some mathematical tools for linear systems to obtain LMI conditions. To illustrate this representation, consider the nonlinear system (2.3). A Differential Algebraic Representation of this system is given by

$$\begin{cases} \dot{x} = A_1(x)x + A_2(x)\pi + A_3(x)u, \\ y = C_1x + C_2\pi, \\ 0 = \Upsilon_1(x)x + \Upsilon_2(x)\pi + \Upsilon_3(x)u, \end{cases} \tag{2.33}$$

where $\pi(x,u) \in \mathbb{R}^{n_\pi}$ is an auxiliary vector that contains all nonlinear terms of (2.3) depending on x. $A_1(x) \in \mathbb{R}^{n\times n}, A_2(x) \in \mathbb{R}^{n\times n_\pi}, A_3(x) \in \mathbb{R}^{n\times m}$, $\Upsilon_1(x) \in \mathbb{R}^{n_\pi\times n}$, $\Upsilon_2(x) \in \mathbb{R}^{n_\pi\times n_\pi}$, $\Upsilon_3(x) \in \mathbb{R}^{n_\pi\times m}$ are affine matrices of x and $C_1 \in \mathbb{R}^{p\times n}, C_2 \in \mathbb{R}^{p\times n_\pi}$ are constant matrices. The DAR (2.33) is an alternative and exact representation of system (2.3).

The DAR of a system is not unique and a state-space representation (2.3) is well-posed in its DAR form if $\Upsilon_2(x)$ is a square full-rank matrix since from (2.33) we have

$$\pi(x,u) = -(\Upsilon_2^\top\Upsilon_2)^{-1}(\Upsilon_2^\top\Upsilon_1x + \Upsilon_2^\top\Upsilon_3u), \tag{2.34}$$

$$\dot{x} = (A_1 - A_2(\Upsilon_2^\top\Upsilon_2)^{-1}\Upsilon_2^\top\Upsilon_1)x + (A_3 - A_2(\Upsilon_2^\top\Upsilon_2)^{-1}\Upsilon_2^\top\Upsilon_3)u. \tag{2.35}$$

Applying DAR representation can be advantageous due to the possibility of using some tools that facilitate manipulations to obtain LMI conditions for evaluating some properties of the nonlinear system. Moreover, since trigonometric functions can be represented in rational forms, the DAR can also deal with nonlinear systems presenting trigonometric functions. A change of variables can be applied to transform trigonometric functions into rational ones without adding conservativeness. This procedure was used for robotic and inverted pendulum systems in Coutinho and Danes (2006), Danes and Bellot (2006), Rohr *et al.* (2009), and Azizi *et al.* (2018). It is important to highlight that the DAR can model the whole class of rational nonlinear systems without singularities at the origin as presented in Lemma 2.3 from Coutinho *et al.* (2008). Furthermore, a general procedure to find the DAR of autonomous systems presenting rational dependence can be found in the Trofino and Dezuo (2014). In the next subsection, a demonstration of how to obtain the DAR for a rational system is presented.

Lemma 2.3 *Let $x \in \sum \subset \mathbb{R}^{n_\rho}$ be a generic parameter, where $\sum$ is a compact set. For any rational matrix function $M : \sum \to \mathbb{R}^{n_1\times n_2}$ with no singularities at $\sum$, there exist constant matrices A_1, A_2 and affine matrix functions $\Upsilon_1(x), \Upsilon_2(x)$ with appropriate dimensions such that*

$$M(\rho) = A_1 - A_2(\Upsilon_2^\top\Upsilon_2)^{-1}\Upsilon_2^\top\Upsilon_1.$$

2.6.1 Example

Consider the following rational nonlinear system

$$\dot{x}(t) = a_1 x(t) + a_2 x^2(t) + a_3 x^3(t) + \frac{a_4 x^2(t) u(t)}{1 + x(t)}, \tag{2.36}$$

where $x(t)$ is the state and $u(t)$ is the control input. Note that, in this case, $-1 \notin \mathcal{X}$ has to be satisfied to find the DAR of this system. Then, we have to choose a vector π of nonlinear functions. The terms a_2x^2, a_3x^3 and $\frac{a_4x^2(t)u(t)}{1+x(t)}$ are the nonlinear ones of the system. Since matrices A_i's can be affine on the state $x(t)$, we can select a vector π, such as

$$\pi(x,u) = \begin{bmatrix} x^2 & \frac{xu}{1+x} \end{bmatrix}^T,$$

to obtain the following DAR matrices of system (2.36),

$$A_1 = a_1 + a_2 x,\;\; A_2 = \begin{bmatrix} a_3 x & a_4 x \end{bmatrix},\;\; A_3 = 0,$$

$$\Upsilon_1 = \begin{bmatrix} x \\ 0 \end{bmatrix},\;\; \Upsilon_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\; \Upsilon_3 = \begin{bmatrix} 0 \\ x \end{bmatrix}.$$

2.7 Auxiliary Lemmas

This section consists of presenting some mathematical tools from the literature that will be helpful later in this work. First, from Oliveira and Skelton (2001), we introduce a version of Finsler's Lemma below. This celebrated lemma is commonly used to formulate LMI conditions, which is our purpose in this work. It will be used in chapters 4 and 5, which treat the SOF control problem for LPV and nonlinear systems, respectively.

Lemma 2.4 *Consider $\mathcal{W} \subseteq \mathbb{R}^{n_s}$ a given polytopic set, and let $Q_d : \mathcal{W} \to \mathbb{R}^{n_q \times n_q}$ and $C_d : \mathcal{W} \to \mathbb{R}^{n_r \times n_q}$ be given matrix functions, with Q_d symmetric. Then, the following statements are equivalent*

i. $\forall w \in \mathcal{W}$ *the condition that* $z^\top Q_d(w) z > 0$ *is satisfied* $\forall z \in \mathbb{R}^{n_q} : C_d(w) z = 0$.

ii. $\forall w \in \mathcal{W}$ *there exists a certain matrix function* $L : \mathcal{W} \to \mathbb{R}^{n_q \times n_r}$ *such that* $Q_d(w) + L(w) C_d(w) + C_d(w)^\top L(w)^\top \succ 0$.

If C_d and Q_d are affine functions of w, and L is a constant matrix to be determined, then *ii*) becomes a parameter-dependent LMI condition which is sufficient for *i*). Clearly, C_d is an annihilator of the vector z, which is not unique. Further details and a systematic procedure for determining linear annihilators are presented in Trofino and Dezuo (2014) and Coutinho *et al.* (2008).

Next, we present a relaxation to check the negativity of a matrix polynomial depending quadratically on scalar parameters α_i such that $\alpha_i \geq 0$ and $\alpha_1 + \alpha_2 + \cdots \alpha_N = 1$. This relaxation will be used in chapter 3, which treats the SOF control problem for uncertain LTI systems.

Property 2.1 *If the following conditions hold*

$$Y_{ii} \prec 0, \textit{ for } i = 1, 2, ..., N, \tag{2.37}$$

$$Y_{ij} + Y_{ji} \prec 0, \textit{ for } 1 \leq i < j \leq N, \tag{2.38}$$

then it is true that

$$\sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N} \alpha_j Y_{ij} \prec 0, \tag{2.39}$$

where Y_{ij} is a symmetric matrix.

Proof. See (TANAKA *et al.*, 1998). □

3 STATIC OUTPUT FEEDBACK CONTROL OF UNCERTAIN LTI SYSTEMS

The challenging control problem of stabilizing linear systems with uncertainties by static output feedback has attracted considerable attention over the last decades. In the latest two decades, some works have proposed new solutions to this problem. Arzelier *et al.* (2003) proposed new sufficient conditions for the H_2 robust stabilization of uncertain LTI systems via SOF, while Dong and Yang (2013) proposed the stabilization via SOF considering the H_∞ performance. Dong and Yang (2007) also proposed a strategy to design robust SOF controllers for both cases of uncertain linear discrete and continuous-time systems. Furthermore, new strategies that use a two-stage procedure for SOF stabilization were developed (AGULHARI *et al.*, 2010; AGULHARI *et al.*, 2012), where the design of a state feedback gain is necessary in the first stage. Sereni *et al.* (2018) proposed the SOF control design using the two-stage procedure, considering a minimum decay rate restriction. More recently, Felipe and Oliveira (2021) proposed a new solution for the robust SOF stabilization problem by employing an iterative procedure in terms of LMI conditions.

Despite the existence of several works proposing LMI-based solutions for the stabilization of uncertain LTI systems via SOF, at that moment, the search for a less conservative method is still being performed. Then, in this chapter, a new LMI-based strategy for the static output feedback stabilization of uncertain continuous-time linear systems is proposed. A minimum decay rate constraint is also considered. Moreover, the definition of strict QSR-dissipativity is used to formulate the conditions. An iterative algorithm allowing obtaining LMI conditions to solve the SOF control problem with SDP tools is applied.

3.1 System Description and Problem Formulation

Consider an uncertain LTI system such as

$$\begin{cases} \dot{x}(t) = A(\delta)x(t) + B(\delta)u(t), \\ y(t) = C(\delta)x(t), \end{cases} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the measured output. Moreover, $\delta \in \mathcal{D} \subset \mathbb{R}^q$ is a vector of constant uncertainties which accounts for deviations of the model description around its nominal part, and $A(\delta) \in \mathbb{R}^{n\times n}, B(\delta) \in \mathbb{R}^{n\times m}, C(\delta) \in \mathbb{R}^{p\times n}$ are uncertain matrices affine on δ. The uncertainties are bounded and the vector δ is assumed to lie $\mathcal{D}$ of $N = 2^q$ vertices, where q is the number of elements of δ. In addition, δ can be related with a set of constants α_i by

$$\delta = \sum_{i=1}^{N} \alpha_i \mathcal{V}(\delta)_i,$$

where the set containing the terms of $\alpha = \{\alpha_1, \ldots, \alpha_N\}$ can be defined as an unitary simplex (OLIVEIRA; PERES, 2007),

$$\Omega = \{\alpha \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_i = 1; \alpha_i \geq 0; i = 1, \ldots, N\}. \tag{3.2}$$

Since matrices $A(\delta), B(\delta)$, and $C(\delta)$ present an affine dependence on δ, they can be represented in a polytopic domain as follows,

$$\Theta = \{(A,B,C)(\delta) = \sum_{i=1}^{N} \alpha_i (A,B,C)_i,\ \alpha \in \Omega\}. \tag{3.3}$$

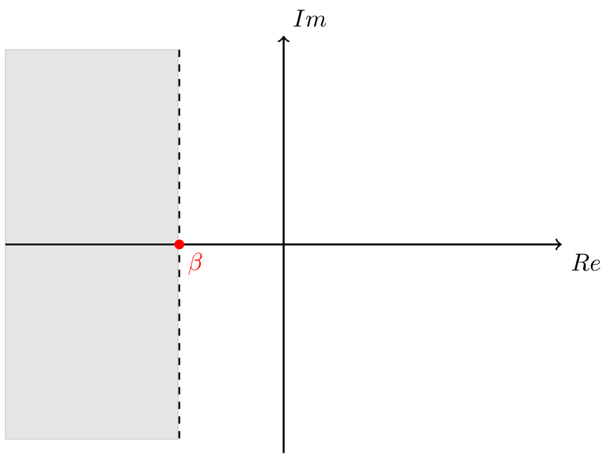
3.1.1 Decay rate

A performance index associated with the transient duration of system (3.1) is the decay rate. It consists on the largest β such that

$$\lim_{t \to \infty} e^{\beta t} ||x(t)|| = 0, \tag{3.4}$$

holds for all trajectories of the vector $x(t)$ (BOYD *et al.*, 2004). A geometric interpretation of the minimum decay rate involving the eigenvalues of the system can be given. As shown in Figure 3, it means that the eigenvalues of the system lie in the left half of the complex plane offset β units, at least, on the real axis, given by the gray area.

Figure 3 – Geometric interpretation of decay rate.



Fonte: The author.

Considering a Lyapunov function $V(x)$, a sufficient condition to compute a lower bound on the decay rate β at the same time that system stability is ensured is given by

$$\dot{V}(x) \leq -2\beta V(x), \tag{3.5}$$

holding for all trajectories of $x(t)$, with $\beta > 0$. If (3.5) is satisfied, then $V(x(t)) \leq V(x(0))e^{-2\beta t}$, so that $\|x(t)\| \leq e^{-\beta t}\kappa(P)^{\frac{1}{2}}\|x(0)\|$ for all trajectories, and therefore the decay rate of (3.1) is at least β (BOYD *et al.*, 2004).

Remark 3.1 *Considering the minimum decay rate of the closed-loop system as a performance criterion in the control design is important because the speed of the convergence of the system response can be increased, which is an essential criterion in many practical applications.*

3.1.2 Problem statement

The central problem, concerning system (3.1), that we propose a solution in this chapter, can be summarized as follows.

Problem 3.1 *Find a static output feedback gain K, i.e., a control law $u(t) = Ky(t)$, such that the closed-loop system given by*

$$\dot{x}(t) = (A(\delta) + B(\delta)KC(\delta))x(t) \tag{3.6}$$

is asymptotically stable for all $\delta \in \mathcal{D}$ with a lower bound on the decay rate given by β.

3.2 Static Output Feedback Design

In this section, Theorem 3.2 presents the proposed strategy that uses strict QSR-dissipativity and Property 2.1 to solve Problem 3.1. For the uncertain LTI system (3.1), strict QSR-dissipativity condition (2.5) with $T(x) = 2\beta V(x)$ can be rewritten as

$$t(x,u,\delta) = \nabla V^\top [A(\delta)x + B(\delta)u] + 2\beta V - y^\top Qy - 2y^\top Su - u^\top Ru \leq 0. \tag{3.7}$$

Moreover, in this case, a parameter dependent Lyapunov function is considered,

$$V(x,\delta) = x^\top P(\delta)x, \tag{3.8}$$

where $P(\delta)$ presents an affine dependence on the uncertainty δ. Then, this Lyapunov function can also be represented in a polytopic domain as given below

$$V(x,\delta) = x^\top \sum_{i=1}^{N} \alpha_i P_i x,\ \alpha \in \Omega. \tag{3.9}$$

Theorem 3.2 *Let $\mathcal{D}$ be a polytope of δ. Given some $\beta > 0$, suppose that there exist matrices $P_i \in \mathbb{S}_n^+$, $R \in \mathbb{S}_m^+$, $Q \in \mathbb{S}_p$ and $S \in \mathbb{R}^{p\times m}$ such that*

$$Y_{ii} \prec 0, \text{ for } i = 1,\ldots,N, \tag{3.10}$$

$$Y_{ij} + Y_{ji} \prec 0, \text{ for } 1 \le i < j \le N, \tag{3.11}$$

and

$$\Delta = SR^{-1}S^\top - Q \succeq 0, \tag{3.12}$$

where Y_{ij} is given by

$$Y_{ij} = \begin{bmatrix} P_iA_j + A_i^\top P_j + 2\beta P_i - C_i^\top QC_j & * \\ B_i^\top P_j - S^\top C_i & -R \end{bmatrix}, \tag{3.13}$$

then

(i) System (3.1) *is strictly QSR-dissipative for all $\delta \in \mathcal{D}$.*

(ii) The SOF given by

$$K = -R^{-1}S^\top, \tag{3.14}$$

asymptotically stabilizes (3.1) *for all $\delta \in \mathcal{D}$ with a lower bound on the decay rate given by β.*

Proof. First, if conditions (3.10) and (3.11) are satisfied, then by Lemma 2.1, the following holds

$$\sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j Y_{ij} = \begin{bmatrix} \sum\limits_{i=1}^N \alpha_i \sum\limits_{j=1}^N \alpha_j \Pi_{ij} & * \\ \sum\limits_{i=1}^N \alpha_i \sum\limits_{j=1}^N \alpha_j \Gamma_{ij} & \sum\limits_{i=1}^N \alpha_i \sum\limits_{j=1}^N \alpha_j(-R) \end{bmatrix} \prec 0, \tag{3.15}$$

where $\Pi_{ij} = P_iA_j + A_i^\top P_j + 2\beta P_i - C_i^\top QC_j$ and $\Gamma_{ij} = B_i^\top P_j - S^\top C_i$. Since $\sum_{i=1}^N \alpha_i = \sum_{j=1}^N \alpha_j = 1$, (3.15) can be rewritten as

$$\begin{bmatrix} \Psi_i & * \\ \sum\limits_{i=1}^N \alpha_i B_i^\top \sum\limits_{i=1}^N \alpha_i P_i - S^\top \sum\limits_{i=1}^N \alpha_i C_i & -R \end{bmatrix} \prec 0, \tag{3.16}$$

where Ψ_i is given by

$$He\{\sum_{i=1}^N \alpha_i P_i \sum_{i=1}^N \alpha_i A_i\} + 2\beta \sum_{i=1}^N \alpha_i P_i - \sum_{i=1}^N \alpha_i C_i^\top Q \sum_{i=1}^N \alpha_i C_i.$$

The summation of matrices A_i, B_i, C_i, P_i are defined in (3.3) and (3.9), then (3.16) can be expressed as

$$\begin{bmatrix} He\{P(\delta)A(\delta)\} + 2\beta P(\delta) - C(\delta)^\top QC(\delta) & * \\ B(\delta)^\top P(\delta) - S^\top C(\delta) & -R \end{bmatrix} \prec 0. \tag{3.17}$$

Multiplying (3.17) by $[x^\top \;\; u^\top]$ on the left and by $[x^\top \;\; u^\top]^\top$ on the right, we obtain

$$\begin{aligned} &x^\top P(\delta)A(\delta)x + x^\top A^\top(\delta)P(\delta)x + x^\top 2\beta P(\delta)x + x^\top P(\delta)B(\delta)u + u^\top B(\delta)^\top P(\delta)x - \\ &x^\top C(\delta)^\top QC(\delta)x - x^\top C(\delta)^\top Su - u^\top S^\top C(\delta)x - u^\top Ru < 0, \end{aligned} \tag{3.18}$$

as $y = C(\delta)x$ and $V = x^\top P(\delta)x$, (3.18) can be rewritten as

$$\nabla V^\top [A(\delta)x + B(\delta)u] + 2\beta V - y^\top Qy - 2y^\top Su - u^\top Ru < 0. \tag{3.19}$$

From (3.7), condition (3.19) implies that the system (3.1) is strictly QSR-dissipative for all $\delta \in \mathcal{D}$, completing the proof of item (i). In addition, the control input u is a static output feedback given by the following equation

$$u = -R^{-1}S^\top y, \tag{3.20}$$

by substitution of (3.20) into (3.19), we obtain

$$\dot{V} + 2\beta V < -y^\top \Delta y, \tag{3.21}$$

where $\Delta = SR^{-1}S^\top - Q$. Then, $\Delta \succeq 0$ is a sufficient condition for (3.5) to be satisfied, as follows

$$\dot{V} < -2\beta V, \tag{3.22}$$

and system (3.1) is asymptotically stabilizable for all $\delta \in \mathcal{D}$ by the SOF (3.12) with a lower bound on the decay rate given by β, completing the proof of all items. □

Remark 3.3 *Property 2.1 is used as a relaxation to check a matrix polynomial depending quadratically on the scalars α_i. It is important to highlight that there exist others relaxation techniques in the literature. Then, if a less conservative relaxation is used, it could lead to less conservative conditions.*

3.3 Iterative algorithm

Note that conditions (3.10) and (3.11) are LMI conditions and can be efficiently solved as a convex problem. However, condition (3.12) presents a nonlinearity on the term

$SR^{-1}S^\top$ and cannot be easily solved as a convex problem. Since $u = -R^{-1}S^\top y$, (3.12) is an alternative way to check the following condition

$$\begin{bmatrix} y \\ u \end{bmatrix} \begin{bmatrix} Q & \star \\ S^\top & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = y^\top Q y + 2y^\top S u + u^\top R u \leq 0, \tag{3.23}$$

which guarantees asymptotic stability of the closed-loop system. From the control law previous presented, $[S^\top \;\; R]$ is an annihilator of the extended vector $[y^\top \;\; u^\top]^\top$, then applying Finsler's Lemma on (3.23), this relation is equivalent to the following condition

$$\begin{bmatrix} Q & \star \\ S^\top & R \end{bmatrix} + \mathrm{He}\{L_s[S^\top \; R]\} \preceq 0, \tag{3.24}$$

which is a bilinear condition due to the term $\mathrm{He}\{L_s[S^\top \; R]\}$. Then, to deal with this bilinearity, Alves Lima *et al.* (2022) proposed an iterative algorithm. The main idea was to consider a relaxed version of (3.24) given by

$$\begin{bmatrix} Q & \star \\ S^\top & R \end{bmatrix} + \mathrm{He}\{L_s[S^\top \; R]\} + \lambda \begin{bmatrix} -\mathrm{I}_p & \star \\ 0 & 0 \end{bmatrix} \preceq 0, \tag{3.25}$$

where λ is an auxiliary scalar.

Moreover, the multiplier L_s is restrained to be of the form $L_s = \begin{bmatrix} -R^{-1}S^\top & -\mathrm{I}_m \end{bmatrix}^\top$, without any conservatism. To see this, note that (3.24) with this particular multiplier leads to

$$\begin{bmatrix} Q - \mathrm{He}\{SR^{-1}S^\top\} & \star \\ -S^\top & -R \end{bmatrix} \prec 0, \tag{3.26}$$

applying Schur complement, which is equivalent to $\Delta \succeq 0$. By this, matrices S and R obtained at each iteration can be mapped to L_s at the next iteration. However, the problem of how to initialize L_s at the first iteration arises. Alves Lima *et al.* (2022) proved that for any initialization considered for S_0 and R_0, condition (3.25) will always be feasible at the first iteration. Moreover, at each following iteration, the objective λ is nonincreasing. From Alves Lima *et al.* (2022), the theorem that formalizes this result is following presented.

Theorem 3.4 *The inequality* (3.25) *is always satisfied in the first iteration whatever S_0 and R_0 are used for initialization. Moreover, at each subsequent iteration in the while loop, the objective λ is nonincreasing.*

Proof. Assume that at the first iteration, LMIs (3.10) and (3.11) are satisfied for a solution set of matrices $\mathcal{Y}_1 = \{P_1(\delta), R_1, Q_1, S_1\}$. Thus, inequality (3.25) is also satisfied since $L_s = \begin{bmatrix} -R_0^{-1}S_0^\top & -I_m \end{bmatrix}^\top$ leads to the following condition

$$\begin{bmatrix} Q - \lambda_1 I_p - \mathrm{He}\{SR_0^{-1}S_0^\top\} & \star \\ -RR_0^{-1}S_0^\top & -R \end{bmatrix} \prec 0,$$

which can always hold with large enough λ_1 since R is a positive definite matrix. Then, due to the structure of inequality (3.25) derived from Finsler's lemma, at the next iteration there exists large enough λ_2 and a set of matrices $\mathcal{Y}_2$ satisfying the problem since this can be achieved at least with the trivial solution $\mathcal{Y}_2 = \mathcal{Y}_1$, $\lambda_2 = \lambda_1$. For each following iteration, the same logic applies, where there exists at least the trivial solution $\lambda_{i+1} = \lambda_i$, $\mathcal{Y}_{i+1} = \mathcal{Y}_i$, meaning that the new λ is at least as good as the one from the previous iteration. □

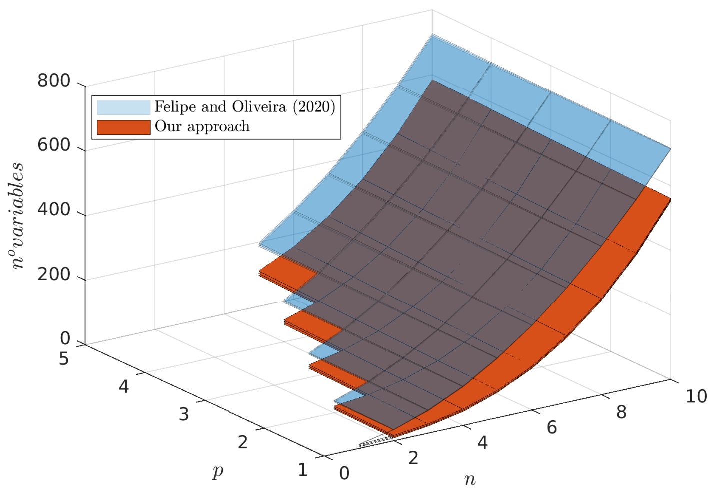
In this Algorithm, the multiplier L_s is updated in a while loop that searches for a solution to $\Delta \succeq 0$. The relaxed inequality (3.25) is the condition that aids the search for this solution. Moreover, matrices S_0 and R_0 are responsible for the update of L_s. The first term of the multiplier L_s is the gain $-R_0^{-1}S_0^\top$. Since condition (3.25) is feasible due to the relaxation variable λ, $-R_0^{-1}S_0^\top$ does not need to be a stabilizing gain. Then, for simplicity, the multiplier L_s is initialized with matrices $S_0 = 0$ and $R_0 = I$. The gain $K = -R^{-1}S^\top$ is a stabilizing one if $\lambda \leq 0$, since it ensures the fulfillment of (3.24). Note that the verification of $\Delta \succeq 0$ as a stop criterion is very important because this condition may be satisfied even with positive values of λ. Therefore, being helpful to decrease the number of iterations in Algorithm 1. The complete iterative algorithm adapted to solve the conditions of Theorem 3.2 is summarized in Algorithm 1.

Algoritmo 1: Control design algorithm.

input : β, k_{max}
output : K, R, S, Q, and P_i for $i = 1, \ldots, N$
$k \leftarrow 1$, $S_0 \leftarrow 0$, and $R_0 \leftarrow I$;
while $k \leq k_{max}$ **do**
  $L_s \leftarrow \begin{bmatrix} -R_0^{-1}S_0^\top & -I_m \end{bmatrix}^\top$;
  minimize λ s.t. (3.10), (3.11), (3.25);
  if $\lambda \leq 0$ *or* $\Delta \succeq 0$ **then**
    return $K = -R^{-1}S^\top$, R, S, and P_i;
  end
  $k \leftarrow k+1$, $S_0 \leftarrow S$, and $R_0 \leftarrow R$;
end

Another LMI-based iterative algorithm that also uses a slack variable through the Finsler's Lemma to solve a bilinear problem iteratively with the aid of a relaxation parameter as λ was proposed by Felipe and Oliveira (2021). This paper proposed a design procedure employing this iterative algorithm in the context of the SOF control design for continuous- and discrete-time uncertain LTI systems. Therefore, it is important to provide some comparisons between both algorithms. First, we note that the algorithm from Felipe and Oliveira (2021) presents more decision variables than Algorithm 1, considering that the parameter-dependent matrices from Felipe and Oliveira (2021) are polynomials of one degree. Figure 4 presents a surface representing the number of decision variables in function of the state vector dimension (n), the output vector dimension (p), and the input vector dimension (m). A variation of n from 1 to 10 and p from 1 to 5 for $m = 1,2,3$ is considered. Note that the surface for $p > n$ is not valid since $y = Cx$, and then $p \leq n$. Moreover, the dimension m was considered constant for all variation of n and p, then Figure 4 presents three planes in red and three in blue, which are very similar. As can be verified, for all combinations of n and p for the three values of m, our approach with Algorithm 1 has fewer decision variables. It shows an advantage of our strategy in terms of computational effort. Later, a numerical comparison will be proceeding in the examples section.

Figure 4 – Number of decision variables.



Fonte: The author.

3.4 Numerical Examples

In this section, three numerical examples to illustrate the effectiveness of our strategy are presented. We intend to focus on comparisons with some papers from the literature that also proposed LMI-based methods for the stabilization of uncertain LTI systems. The comparisons are in terms of the ability to find a stabilizing solution, maximum bounds for uncertainties, and maximum decay rate obtained. For the implementation we use Matlab R2018b and conventional tools as YALMIP parser (LOFBERG, 2004) and the SDP solver MOSEK (APS, 2019) release 9.3.10, in a PC equipped with: AMD Ryzen 5-3500u (2.10 GHz, 64 bits), 12 GB of RAM, Linux Satux. It is important to highlight that when using the previous tools for the implementation of the iterative algorithm, a factor that helps in the convergence of the algorithm is to multiply the term to be minimized by a small scalar.

3.4.1 Examples 1 and 2

Consider the linearized model of a VTOL helicopter considered in Keel *et al.* (1988),

$$\dot{x}(t) = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.010 & 0.0024 & -4.0208 \\ 0.1002 & p_1 & -0.707 & p_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0.4422 & 0.1761 \\ p_3 & -7.5922 \\ -5.52 & 4.49 \\ 0 & 0 \end{bmatrix} u(t). \quad (3.27)$$

where x_1 is the horizontal velocity, x_2 is the vertical velocity, x_3 is the pitch rate, x_4 is the pitch angle, u_1 is the collective pitch control and u_2 is the longitudinal cyclic pitch control. In this example, the same parameters considered in Felipe and Oliveira (2021) are used, that are $p_1 = 0.3681,\ p_3 = 3.5446$, an uncertain $p_2 = 1.42 \pm \delta$, and the output of the system as $y(t) = Cx(t)$, where $C = [I_2\ \ 0]$.

Consider also the model of a mechanical system with two masses and two springs from Peaucelle and Arzelier (2001)

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & \frac{1}{m_1} & 0 \\ 0 & 0 & 0 & \frac{1}{m_2} \\ -(k_1+k_2) & k_2 & -\frac{c_0}{m_1} & 0 \\ k_2 & -k_2 & 0 & -\frac{c_0}{m_2} \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -\frac{1}{m_1} & 0 \\ 0 & -\frac{1}{m_2} \end{bmatrix} u(t) \quad (3.28)$$

where the states x_1 and x_2 are the positions of masses m_1, m_2, respectively, and $x_3 = m_1\dot{x}_1$, $x_4 = m_2\dot{x}_2$. The parameters k_1 and k_2 are the stiffness of the springs and c is the viscous friction coefficient. In this example, the same parameters values of Felipe and Oliveira (2021) are considered, that are $m_1 = 1,\ m_2 = 0.5,\ k_1 = 1,\ k_2 = 1$, an uncertain $c = 2 \pm \delta$, and an output $y(t) = Cx(t)$, where $C = [0\ \ I_2\ \ 0]$.

The purpose of both examples is to find the SOF control gain that guarantees system stability to a maximum bound for the uncertainty δ. Applying Algorithm 1 to both systems, the SOF gain

$$K = \begin{bmatrix} -287.6788 & 53.6541 \\ 335.2621 & -50.9685 \end{bmatrix}$$

for the mechanical system (3.28) after 158 iterations in 33.3186s, and

$$K = \begin{bmatrix} 27.3674 & -115.8107 \\ 9.2324 & -39.1456 \end{bmatrix}$$

is designed for the VTOL helicopter system (3.27). In the latter case, the iterative algorithm was applied several times until finding the best uncertainty. First, we run Algorithm 1 with $\delta = 0$, while on the next run, we add δ of 0.2 and make $S = -K'$ for initialization instead of $S = 0$, where K is the stabilizing gain found in the previous run. Thus, the average of iterations of 23.4925 and the average time of 4.9649s are obtained for each execution of a value of δ.

Table 1 presents comparisons in terms of the maximum bound for the uncertainty δ obtained. A comparison of the obtained result with two important papers from the literature is presented. One of them is Agulhari *et al.* (2012) which deals with SOF control design through a two-stage procedure based on LMI conditions. A stabilizing state feedback controller with polynomial or rational dependence on the parameters is designed in the first stage. After that, this state feedback controller is used in the second stage to find the SOF gain. The other one is Felipe and Oliveira (2021) which proposes a design procedure employing an iterative algorithm based on LMI conditions. It is important to highlight this iterative algorithm presents more decision variables than the one used in our work. As can be seen, our strategy achieved the best result in terms of maximum value of δ for both cases.

Remark 3.5 *Results presented in Table 1 for Agulhari et al. (2012) were computed and presented by Felipe and Oliveira (2021).*

Table 1 – Maximum value of δ such that the system is stabilizable.

	Maximum δ	
	VTOL (3.27)	Mass-spring (3.28)
Our approach	13.20	10.60
(FELIPE; OLIVEIRA, 2021)	8.496	10.101
(AGULHARI *et al.*, 2012)	4.845	2.026

Fonte: The author.

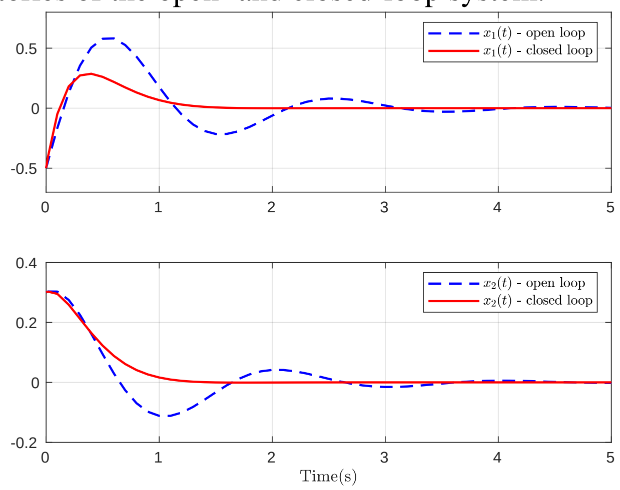
3.4.2 Example 3

Consider the illustrative system analyzed in Sereni *et al.* (2018), where the two vertices of the uncertain system are as follows

$$A_1 = \begin{bmatrix} -1 & 10 \\ -1 & -1 \end{bmatrix}, \; A_2 = \begin{bmatrix} a & -4 \\ -2 & -3 \end{bmatrix}, \; B_2 = \begin{bmatrix} b \\ 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} -9 \\ 0 \end{bmatrix}, \; C_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^\top, \; C_2 = C_1. \tag{3.29}$$

Applying Algorithm 1 with $a = 80$, $b = -180$, and a decay rate $\beta = 2.2$, that are values characterizing a system that Sereni *et al.* (2018) failed to stabilize, the SOF gain $K = 0.5689$ which stabilizes this system is obtained after 15 iterations in 8.199s. Figure 5 presents simulation results of the open- and closed-loop system for initial conditions $x(0) = [-0.5 \;\; 0.3]^\top$. Note that the state trajectories of the open-loop system converge asymptotically to the equilibrium point with an oscillatory behavior, while the state trajectories of the closed-loop system converge faster than the open-loop and without oscillation. It happens due to the constraint on the decay rate of the closed-loop system considered in the design conditions.

Figure 5 – State trajectories of the open- and closed-loop system.



Fonte: The author.

4 GAIN SCHEDULING CONTROL OF RATIONAL LPV SYSTEMS

Time-varying parameter systems model a broad class of dynamical systems, such as those with linear or some nonlinear behavior. The time-varying parameters can be variables internal to the system or result from an approximation of a nonlinear system, being the last one called quasi-LPV systems (BRIAT, 2014). Examples of practical applications involving LPV systems are car active suspension (SERENI *et al.*, 2019), power systems (NOGUEIRA *et al.*, 2018), missile autopilots (WHITE *et al.*, 2007), and wind turbines (SHIRAZI *et al.*, 2012). Due to a wide range of applications that can be modeled as LPV systems, the control of LPV systems has attracted lots of attention in the literature. As explained in the introduction of this work, the gain scheduling control arises as an interesting alternative in the LPV framework. Then, the gain scheduling SOF control problem is also an interesting issue that has been investigated over the last decades.

Gain scheduling SOF design for continuous-time LPV systems have been proposed in Al-Jiboory and Zhu (2018), Nguyen *et al.* (2018), Sereni *et al.* (2019), Sereni *et al.* (2022). Recently, a gain scheduling SOF design procedure with $\mathcal{H}_2/\mathcal{H}_\infty$ performance has been developed (BEHROUZ *et al.*, 2021). However, as it is common in the field, these strategies consider the polytopic approach, then the LPV system can only be affine on the parameter. Few works consider a polynomial or rational dependence on the parameter. A gain scheduling state feedback design was developed for rational LPV systems in Bouali *et al.* (2006), Bouali *et al.* (2007). In Bouali *et al.* (2008), gain scheduling dynamic output feedback design with $\mathcal{H}_2$ performance has been proposed for rational LPV systems. In Masubuchi and Suzuki (2008), a procedure for designing dynamic gain scheduling controllers for rational LPV systems in the descriptor form was developed. It is important to state that Bouali *et al.* (2006), Bouali *et al.* (2007), Bouali *et al.* (2008), Masubuchi and Suzuki (2008) do not apply static output feedback. Recently, Polcz *et al.* (2020) proposes a novel method to compute the $\mathcal{L}_2$-gain for rational LPV systems, however no control law is designed. Many solutions to the gain scheduling static output feedback design for discrete-time LPV systems have also been recently developed (SADEGHZADEH, 2017; CAIGNY *et al.*, 2010; PEIXOTO *et al.*, 2020; PEIXOTO *et al.*, 2021). However, none of them consider rational dependence on the parameter.

Since no work has proposed the gain scheduling SOF design for LPV systems with rational dependence on the parameter, in this chapter, we suggest a first solution for this control problem. We consider LPV systems presenting rational, or polynomial, dependence on the parameter. Moreover, we assume that the system can be affected by two classes of time-varying parameters, the available and unavailable for measurement. First, in section 4.1, we present an approach for the gain scheduling SOF design. Later, in section 4.2, we consider that the system is affected by external disturbances, then we propose the gain scheduling SOF design considering the $\mathcal{L}_2$-gain performance. Here, we also use the definition of strict QSR-dissipativity, in this case, together with Finsler's Lemma. Lastly, we also apply an iterative algorithm that allows solving the conditions as an SDP problem and an optimization algorithm to minimize the upper bound of the $\mathcal{L}_2$-gain.

4.1 Gain Scheduling SOF Design

In this section, we present an approach for the gain scheduling static output feedback stabilization of rational LPV systems that are affected by two classes of time-varying parameters.

4.1.1 System description and problem formulation

Consider an LPV system of the form

$$\begin{cases} \dot{x}(t) = A(\rho,\delta)x(t) + B(\rho,\delta)u(t), \\ y(t) = C(\rho)x(t), \end{cases} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the output that is available for measurement. $\rho(t) \in \Omega \subset \mathbb{R}^r$ is a vector of time-varying parameters available for measurement and $\delta(t) \in \Pi \subset \mathbb{R}^v$ is a vector of time-varying uncertainties, unavailable for measurement, which accounts for deviations of the model description. Moreover, $\rho(t)$ and $\delta(t)$ can vary arbitrarily in time. Matrices $A(\rho,\delta(t)) \in \mathbb{R}^{n\times n}, B(\rho,\delta(t)) \in \mathbb{R}^{n\times m}, C(\rho) \in \mathbb{R}^{p\times n}$ are polynomial or rational on $\rho(t)$ and affine with respect to $\delta(t)$.

Assumption 4.1 *The vector $\rho(t)$ lies inside a polytope Ω of $N = 2^r$ vertices, where r is the number of elements of ρ. For all $\rho(t) \in \Omega$ there exists $\alpha \in \Lambda_N$ (BRIAT, 2014). The polytope Λ_N is given by*

$$\Lambda_N = \{\alpha(\rho(t)) \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_i = 1; \alpha_i \geq 0; i = 1,\ldots,N\}, \tag{4.2}$$

where any point inside Λ_N can be represented by the convex combination of its vertices (BRIAT, 2014).

Assumption 4.2 *The vector $\delta(t)$ lies inside a polytope Π of $M = 2^{\nu}$ vertices. For all $\delta(t) \in \Pi$ there exists $\sigma \in \Lambda_M$ defined in* (4.2)*.*

4.1.1.1 DAR for Rational LPV systems

The LPV system (4.1) can be described by a Differential Algebraic Representation,

$$\begin{cases} \dot{x} = A_1(\delta)x + A_2(\delta)\pi + A_3(\delta)u, \\ y = C_1 x + C_2 \pi, \\ 0 = \Upsilon_1(\rho)x + \Upsilon_2(\rho)\pi + \Upsilon_3(\rho)u, \end{cases} \tag{4.3}$$

where $\pi(x,u,\rho,\delta) \in \mathbb{R}^{n_\pi}$ is an auxiliary vector that contains all nonlinear terms of (4.1) depending on ρ. $A_1(\delta) \in \mathbb{R}^{n\times n}, A_2(\delta) \in \mathbb{R}^{n\times n_\pi}, A_3(\delta) \in \mathbb{R}^{n\times m}$ are affine matrices with respect to δ, $C_1 \in \mathbb{R}^{p\times n}, C_2 \in \mathbb{R}^{p\times n_\pi}$ are constant matrices, and $\Upsilon_1(\rho) \in \mathbb{R}^{n_\pi \times n}$, $\Upsilon_2(\rho) \in \mathbb{R}^{n_\sigma \times n_\pi}$, $\Upsilon_3(\rho) \in \mathbb{R}^{n_\pi \times m}$ are affine matrices with respect to ρ, while $\Upsilon_2(\rho)$ is supposed to be a square full-rank matrix for all $(\rho) \in \Omega$. A general procedure to obtain the DAR of the LPV system can be found in Coutinho *et al.* (2009). Moreover, matrices $A_1(\delta)$,$A_2(\delta)$ and $A_3(\delta)$ can be represented in a polytopic domain,

$$\Theta = \{(A_1,A_2,A_3)(\delta) = \sum_{j=1}^{M} \sigma_j (A_1,A_2,A_3)_j,\ \delta(t) \in \Pi\}. \tag{4.4}$$

Remark 4.1 *The motivation to represent the LPV system in a DAR form is that, in* (4.3) *the dependency on ρ is transferred to the auxiliary matrices $\Upsilon_i(\rho)$, which depend only affinely on ρ. Moreover, matrices A_i's depend also only affinely on δ, which allows using techniques leading to convex design conditions expressed in the form of LMIs.*

4.1.1.2 Problem statement

Problem 4.1 *Find a gain scheduling static output feedback $K(\rho)$, i.e., a control law $u(t) = K(\rho)y(t)$, such that the closed-loop system given by*

$$\dot{x}(t) = (A(\rho,\delta) + B(\rho,\delta)K(\rho)C(\rho))x(t) \tag{4.5}$$

is asymptotically stable for all $\rho(t) \in \Omega$ and $\delta(t) \in \Pi$.

4.1.2 Controller design

The strategy proposed here consists in connecting Lemma 2.4 and strict QSR-dissipativity condition. Consider a quadratic Lyapunov function, a ρ-parameter dependent function $T(x,\rho)$ that can be defined in a polytopic domain

$$T(x,\rho) = x^\top H(\rho)x,\ \ H(\rho) = \sum_{i=1}^{N} \alpha_i H_i,\ H_i \succ 0, \tag{4.6}$$

where $H_i \in \mathbb{R}^{n\times n}$, and ρ-parameter dependent matrices $Q(\rho)$ and $S(\rho)$, that can also be defined in a polytopic domain,

$$Q(\rho) = \sum_{i=1}^{N} \alpha_i Q_i,\ \ S(\rho) = \sum_{i=1}^{N} \alpha_i S_i. \tag{4.7}$$

where $Q_i \in \mathbb{S}_p$ and $S_i \in \mathbb{R}^{p\times m}$. A version of the strict QSR-dissipativity condition (2.5), for the LPV system in a DAR form (4.3), can be rewritten as

$$t_d(x,u,\rho,\delta) = \nabla V^\top [A_1(\delta)x + A_2(\delta)\pi + A_3(\delta)u] + H(\rho) - y^\top Q(\rho)y - 2y^\top S(\rho)u - u^\top Ru \leq 0. \tag{4.8}$$

Observe that $t_d(x,u,\rho,\delta(t)) = \pi_d^\top Y(\rho,\delta)\pi_d$, with

$$\pi_d = \begin{bmatrix} x^\top & \pi^\top & u^\top \end{bmatrix}^\top,\ \ Y(\rho) = \sum_{i=1}^{N} \alpha_i \sum_{j=1}^{M} \sigma_j Y_{ij},$$

where Y_{ij} is a symmetric and linear matrix on all the unknown coefficients of (H_i, Q_i, S_i, R, P). In addition, consider

$$C_d(\rho) = \begin{bmatrix} \Upsilon_1(\rho) & \Upsilon_2(\rho) & \Upsilon_3(\rho) \end{bmatrix} \tag{4.9}$$

as a linear annihilator of π_d. Since matrices $\Upsilon_i(\rho)$ are affine on the parameter, it is convenient to represented them in a polytopic domain, as follows

$$C_d(\rho) = \sum_{i=1}^{N} \alpha_i C_{d_i} = \sum_{i=1}^{N} \alpha_i \begin{bmatrix} \Upsilon_{1_i} & \Upsilon_{2_i} & \Upsilon_{3_i} \end{bmatrix}. \tag{4.10}$$

Theorem 4.2 *Let Ω be a polytope of $\rho(t)$, Π a polytope of $\delta(t)$, and $C_d(\rho)$ a linear annihilator of π_d described by* (4.10)*. Assume there exist matrices $P \in \mathbb{S}_n^+$, $H_i \in \mathbb{S}_n^+$, $R \in \mathbb{S}_m^+$, Q_i, S_i, and $L \in \mathbb{R}^{n_q \times n_\pi}$, such that*

$$Y_{ij} + LC_{d_i} + C_{d_i}^\top L^\top \prec 0, \tag{4.11}$$

$$\Delta_{ii} \succ 0, \tag{4.12}$$

for $i = 1,\ldots,N$, $j = 1,\ldots,M$, *and*

$$\Delta_{ij} + \Delta_{ji} \succ 0, \tag{4.13}$$

for $1 \le i < j \le N$, *where* $\Delta_{ij} = S_i R^{-1} S_j^\top - Q_i$ *and*

$$Y_{ij} = \begin{bmatrix} PA_{1j} + A_{1j}^\top P - C_1^\top Q_i C_1 + H_i & \star & \star \\ \left(PA_{2j} - C_1^\top Q_i C_2\right)^\top & -C_2^\top Q_i C_2 & \star \\ \left(PA_{3j} - C_1^\top S_i\right)^\top & -S_i^\top C_2 & -R \end{bmatrix},$$

Then system (4.1) *is robust strictly QSR-dissipative and the gain scheduling SOF*

$$u = K(\rho)y,\ K(\rho) = \sum_{i=1}^{N} \alpha_i K_i,\ K_i = -R^{-1} S_i^\top, \tag{4.14}$$

asymptotically stabilizes (4.1)*, for all* $\rho(t) \in \Omega$ *and all* $\delta(t) \in \Pi$*, around the origin.*

Proof. First, consider the satisfaction of condition (4.11). Since $\sum_{i=1}^{N} \alpha_i = \sum_{j=1}^{M} \sigma_j = 1$ and $\alpha_i, \sigma_j \ge 0$ for $i = 1,\ldots,N$, $j = 1,\ldots,M$, note that, by multiplying all the terms of (4.11) by α_i and σ_j, and summing them up from $i = 1$ to $i = N$, and from $j = 1$ to $j = M$ we obtain

$$\sum_{i=1}^{N} \alpha_i \sum_{j=1}^{M} \sigma_j (Y_{ij} + \mathrm{He}\{LC_{d_i}\}) = Y(\rho) + \mathrm{He}\{LC_d(\rho)\} \prec 0. \tag{4.15}$$

Since $C_d(\rho)$ is an annihilator of π_d, from Lemma 2.4, satisfaction of (4.15) implies that $\pi_d^\top Y(\rho,\delta)\pi_d = t_d(x,u,\rho,\delta) < 0$ is also satisfied for all $\rho(t) \in \Omega$ and all $\delta(t) \in \Pi$, where $t_d(x,u,\rho,\delta)$ was first defined in (4.8). Thus, system (4.1) is *robust strictly QSR-dissipative* for all $\rho(t) \in \Omega$ and $\delta(t) \in \Pi$. In addition, note that as $H(\rho) \succ 0$, fulfilling

$$y^\top Q(\rho) y + 2y^\top S(\rho) u + u^\top R u \le 0 \tag{4.16}$$

is sufficient to guarantee $\nabla V^\top [A_1 x + A_2 \pi + A_3 u] < 0$, which ensures the asymptotic stability of system (4.1) about the origin. Let us recall that, we consider the gain scheduling static output feedback given by

$$u = K(\rho)y = -R^{-1} \sum_{i=1}^{N} \alpha_i S_i^\top y = -R^{-1} S^\top(\rho) y. \tag{4.17}$$

By substitution of (4.17) into (4.16), we obtain $-y^\top \Delta(\rho) y \ge 0$, where $\Delta(\rho) = S(\rho) R^{-1} S^\top(\rho) - Q(\rho)$. Since matrices $Q(\rho)$ and $S(\rho)$ can be represented as in (4.7), we have that

$$\Delta(\rho) = \sum_{i=1}^{N} \alpha_i \sum_{i=1}^{N} \alpha_i \Delta_{ii} \succeq 0 \tag{4.18}$$

is a sufficient condition for asymptotic stability, where $\Delta_{ii} = S_i R^{-1} S_i^\top - Q_i$. As $\sum_{i=1}^N \alpha_i = \sum_{i=j}^N \alpha_j = 1$, (4.18) can be rewritten as

$$\Delta(\rho) = \sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N} \alpha_j \Delta_{ij} \succeq 0 \tag{4.19}$$

where Δ_{ij} is given in Theorem 4.2. Therefore, by property 2.1, satisfaction of (4.12) and (4.13) implies fulfillment of (4.19) and system (4.1) is stabilized by a SOF gain scheduling given by (3.14), which completes the proof of Theorem 4.2. □

4.2 $\mathcal{L}_2$-gain Scheduling SOF Design

In this section, we consider that the system is affected by external disturbances. Then, the objective is to design the gain scheduling SOF at the same time guaranteeing an upper bound on the l_2 norm between a controlled output $z(t)$ and an external input $w(t)$.

4.2.1 System description and problem formulation

First, consider an LPV system of the form

$$\begin{cases} \dot{x}(t) = A(\rho,\delta)x(t) + B(\rho,\delta)u(t) + B_w(\rho,\delta)w(t), \\ z(t) = A_z(\rho,\delta)x(t) + B_z(\rho,\delta)u(t) + D_z(\rho,\delta)w(t), \\ y(t) = C(\rho)x(t) + D(\rho)w(t), \end{cases} \tag{4.20}$$

that is the same system (4.1) with an additional external input $w(t) \in \mathbb{R}^q$ and a controlled output $z(t) \in \mathbb{R}^l$. As in (4.1), the matrices of the system can present rational or polynomial dependence on $\rho(t)$ and affine dependence on $\delta(t)$. This system in its DAR form is presented below

$$\begin{cases} \dot{x} = A_1(\delta)x + A_2(\delta)\pi + A_3(\delta)u + A_4(\delta)w, \\ z = B_1(\delta)x + B_2(\delta)\pi + B_3(\delta)u + B_4(\delta)w, \\ y = C_1 x + C_2 \pi + C_3 w, \\ 0 = \Upsilon_1(\rho)x + \Upsilon_2(\rho)\pi + \Upsilon_3(\rho)u + \Upsilon_4(\rho)w, \end{cases} \tag{4.21}$$

where matrices B_i's are affine on $\delta(t)$. Considering $u = K(\rho)y$, the closed loop form of this system is given by

$$\begin{cases} \dot{x} = \mathscr{A}_1 x + \mathscr{A}_2 \pi + \mathscr{A}_3 w, \\ z = \mathscr{B}_1 x + \mathscr{B}_2 \pi + \mathscr{B}_3 w, \\ 0 = \hat{\Upsilon}_1(\rho) x + \hat{\Upsilon}_2(\rho) \pi + \hat{\Upsilon}_3(\rho) w, \end{cases} \tag{4.22}$$

where

$$\begin{aligned} &\mathscr{A}_1 = (A_1 + A_3 K(\rho) C_1),\ \mathscr{A}_2 = (A_2 + A_3 K(\rho) C_2), \\ &\mathscr{A}_3 = (A_4 + A_3 K(\rho) C_3),\ \mathscr{B}_1 = (B_1 + B_3 K(\rho) C_1), \\ &\mathscr{B}_2 = (B_2 + B_3 K(\rho) C_2),\ \mathscr{B}_3 = (B_4 + B_3 K(\rho) C_3), \\ &\hat{\Upsilon}_1 = (\Upsilon_1 + \Upsilon_3 K(\rho) C_1),\ \hat{\Upsilon}_2 = (\Upsilon_2 + \Upsilon_3 K(\rho) C_2), \\ &\hat{\Upsilon}_3 = (\Upsilon_4 + \Upsilon_3 K(\rho) C_3). \end{aligned} \tag{4.23}$$

4.2.1.1 Problem statement

Problem 4.2 *Find a gain scheduling static output feedback $K(\rho)$, i.e., a control law $u(t) = K(\rho)y(t)$, such that the closed-loop system given by*

$$\dot{x}(t) = (A(\rho,\delta) + B(\rho,\delta)K(\rho)C(\rho))x(t) + B_w(\rho,\delta)w(t) \tag{4.24}$$

is stable for all $\rho(t) \in \Omega$ and all $\delta(t) \in \Pi$ with $\mathcal{L}_2$-gain bounded by γ.

4.2.2 $\mathcal{L}_2$-gain

The gain scheduling static output feedback control problem with $\mathcal{L}_2$-gain performance is equivalent to finding a control law $u(t) = K(\rho(t))y(t)$ such that the closed loop (4.22) is asymptotically stable in the absence of disturbance w and the l_2 norm of z is bounded such that

$$||z||_2 \leq \gamma ||w||_2 + \theta, \tag{4.25}$$

with positive scalars γ and θ, where θ is a bias term. When (4.25) is ensured, one can say that system (4.22) is input to output stable with $\mathcal{L}_2$-gain bounded by γ. In order to guarantee asymptotic stability at the same time satisfying relation (4.25), we have the following sufficient condition (COUTINHO *et al.*, 2008)

$$\dot{V} + \gamma^{-1} z^\top z - \gamma w^\top w < 0. \tag{4.26}$$

Note that by integrating both sides of (4.26), taking squares roots, and using the fact that $\sqrt{a+b} \leq a+b$, for $a,b \in \mathbb{R}^{+}$, one arrives at $\|z\|_2 \leq \gamma\|w\|_2 + \sqrt{\gamma V(x(0))}$, i.e., (4.25) with bias term $\theta = \sqrt{\gamma V(x(0))}$.

4.2.3 Controller design

Theorem 4.3 *If there exists a scalar $\gamma > 0$, such that conditions* (4.12)*, and* (4.13) *of Theorem 4.2 hold, and condition* (4.11) *holds replacing matrices* $(P,A_1,A_2,A_3,C_1,C_2,\Upsilon_1,\Upsilon_2,\Upsilon_3,L,H_i)$ *by* $(\bar{P},\bar{A}_1,\bar{A}_2,\bar{A}_3,\bar{C}_1,\bar{C}_2,\bar{\Upsilon}_1,\bar{\Upsilon}_2,\bar{\Upsilon}_3,\bar{L},\bar{H}_i)$*, respectively, where*

$$
\begin{aligned}
&\bar{P} = \begin{bmatrix} P & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & I_l \end{bmatrix}, \bar{A}_1 = \begin{bmatrix} A_1 & A_4 & 0_{n\times l} \\ 0_{q\times n} & -\frac{\gamma}{2}I_q & 0_{q\times l} \\ B_1 & B_4 & -\frac{\gamma}{2}I_l \end{bmatrix}, \\
&\bar{A}_2 = \begin{bmatrix} A_2 \\ 0_{q\times n_\pi} \\ B_2 \end{bmatrix}, \bar{A}_3 = \begin{bmatrix} A_3 \\ 0_{q\times m} \\ B_3 \end{bmatrix}, \bar{\Upsilon}_1^\top = \begin{bmatrix} \Upsilon_1 \\ \Upsilon_4 \\ 0_{n_\pi\times l} \end{bmatrix} \\
&\bar{C}_2 = C_2, \bar{C}_1 = \begin{bmatrix} C_1 & C_3 & 0_{r\times l} \end{bmatrix},, \bar{\Upsilon}_2 = \Upsilon_2, \\
&\bar{\Upsilon}_3 = \Upsilon_3, \bar{H}_i \in \mathbb{R}^{n_l\times n_l}, \bar{L} \in \mathbb{R}^{(n_l+n_\pi+m)\times n_\pi},
\end{aligned}
\tag{4.27}
$$

with $n_l = n+q+l$, then system (4.20) *is robust strictly QSR-dissipative for all $\rho(t) \in \Omega$, and the gain scheduling SOF*

$$
u = K(\rho)y,\ K(\rho) = \sum_{i=1}^{N} \alpha_i K_i,\ K_i = R^{-1}S_i^\top, \tag{4.28}
$$

asymptotically stabilizes system (4.20)*for all $\rho(t) \in \Omega$ and all $\delta(t) \in \Pi$ with $\mathcal{L}_2$-gain bounded by γ.*

Proof. First, consider system (4.20) in its DAR form (4.21). Considering $u = K(\rho)y$, we have that $\dot{V}(x) < 0$ that guarantees asymptotic stability for the closed-loop system (4.3), which is equivalently expressed as

$$
\begin{bmatrix} x \\ \pi \end{bmatrix}^\top \begin{bmatrix} He\{PA_1 + PA_3K(\rho)C_1\} & \star \\ A_2^\top P + C_2^\top K^\top(\rho)A_3^\top P & 0 \end{bmatrix} \begin{bmatrix} x \\ \pi \end{bmatrix} < 0. \tag{4.29}
$$

Since $u = K(\rho)y$ and also because of (4.3), matrix $[\Upsilon_1 + \Upsilon_3 K(\rho)C_1 \quad \Upsilon_2 + \Upsilon_3 K(\rho)C_2]$ is an annihilator of $[x^\top \quad \pi^\top]^\top$. Thus Lemma 2.4 can be applied. If there exists matrix L_a such that

$$\begin{aligned} &He\left\{\begin{bmatrix} PA_1 & PA_2 \\ 0 & 0 \end{bmatrix}\right\} + He\left\{\begin{bmatrix} PA_3KC_1 & PA_3KC_2 \\ 0 & 0 \end{bmatrix}\right\} \\ &+ He\left\{L_a \begin{bmatrix} \Upsilon_1 & \Upsilon_2 \end{bmatrix}\right\} + He\left\{L_a \Upsilon_3 K \begin{bmatrix} C_1 & C_2 \end{bmatrix}\right\} \prec 0, \end{aligned} \tag{4.30}$$

then (4.29) is satisfied for all $\rho(t) \in \Omega$ and all $\delta(t) \in \Pi$.

On the other hand, asymptotic stability of system (4.22) with $\mathcal{L}_2$-gain performance is guaranteed fulfilling (4.26), which is equivalent to $\pi_w^\top Y_w \pi_w < 0$, where $\pi_w = [x^\top \; w^\top \; \pi^\top]^\top$ and Y_w is given by

$$\begin{bmatrix} He\{P\mathscr{A}_1\} + \gamma^{-1}\mathscr{B}_1^\top \mathscr{B}_1 & \star & \star \\ \mathscr{A}_3^\top P + \gamma^{-1}\mathscr{B}_3^\top \mathscr{B}_3 & \gamma^{-1}\mathscr{B}_3^\top \mathscr{B}_3 - \gamma \mathrm{I} & \star \\ \mathscr{A}_2^\top P + \gamma^{-1}\mathscr{B}_2^\top \mathscr{B}_2 & \gamma^{-1}\mathscr{B}_2^\top \mathscr{B}_3 & \gamma^{-1}\mathscr{B}_2^\top \mathscr{B}_2 \end{bmatrix}.$$

By noting from (4.22) that $\hat{\Upsilon}_w = [\hat{\Upsilon}_1 \; \hat{\Upsilon}_3 \; \hat{\Upsilon}_2]$ is an annihilator of π_w, Lemma 2.4 can also be applied. If there exists a matrix $L_w = [L_1^\top \; L_3^\top \; L_2^\top]^\top$ such that

$$Y_w + L_w \hat{\Upsilon}_w + \hat{\Upsilon}_w^\top L_w^\top \prec 0, \tag{4.31}$$

then $\pi_w^\top Y_w \pi_w < 0$ is satisfied for all $\rho(t) \in \Omega$ and all $\delta(t) \in \Pi$. Next, applying Schur complement in (4.31) followed by a congruence transformation with $diag(I_2, J_2)$, we obtain

$$\begin{bmatrix} He\{P\mathscr{A}_1\} & * & * & * \\ \mathscr{A}_3^\top P & -\gamma \mathrm{I} & * & * \\ \mathscr{B}_1 & \mathscr{B}_3 & -\gamma \mathrm{I} & * \\ \mathscr{A}_2^\top P & 0 & \mathscr{B}_2^\top & 0 \end{bmatrix} + He\left\{L_b \bar{\Upsilon}_w\right\} \prec 0, \tag{4.32}$$

where $L_b = [L_1^\top \; L_3^\top \; 0 \; L_2^\top]^\top$ and $\bar{\Upsilon}_w = \begin{bmatrix} \hat{\Upsilon}_1 & \hat{\Upsilon}_3 & 0 & \hat{\Upsilon}_2 \end{bmatrix}$. By taking into account the definitions of matrices $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3, \mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3, \hat{\Upsilon}_1, \hat{\Upsilon}_2, \hat{\Upsilon}_3$ in (4.23), the following equivalent expression for (4.32) is obtained in terms of the matrices $\bar{P}, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{C}_1, \bar{C}_2, \bar{\Upsilon}_1, \bar{\Upsilon}_2, \bar{\Upsilon}_3$ given in (4.27)[1]

$$\begin{aligned} &He\left\{\begin{bmatrix} \bar{P}\bar{A}_1 & \bar{P}\bar{A}_2 \\ 0 & 0 \end{bmatrix}\right\} + He\left\{\begin{bmatrix} \bar{P}\bar{A}_3K\bar{C}_1 & \bar{P}\bar{A}_3K\bar{C}_2 \\ 0 & 0 \end{bmatrix}\right\} \\ &+ He\left\{L_b \begin{bmatrix} \bar{\Upsilon}_1 & \bar{\Upsilon}_2 \end{bmatrix}\right\} + He\left\{L_b \bar{\Upsilon}_3 K \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix}\right\} \prec 0. \end{aligned} \tag{4.33}$$

[1] Dependence on ρ and δ was omitted for simplicity of notation.

Note that condition (4.33) has the same form of condition (4.30). Thus, by applying Theorem 4.2 with the bar matrices, one ensures satisfaction of $\pi_w^\top Y_w \pi_w < 0$, $\forall \pi_w \in \mathbb{R}^{n+q+n_\pi} : \hat{\Upsilon}_w \pi_w = 0$, for all $\rho(t) \in \Omega$ and all $\delta(t) \in \Pi$ with the designed SOF gain scheduling control (4.28), which in turn guarantees (4.26) along the trajectories of the closed-loop perturbed system (4.22) with $\mathcal{L}_2$-gain bounded by γ. □

4.3 Iterative Algorithm

Note that condition (4.11) is an LMI condition and can be efficiently solved as a linear problem. However, conditions (4.12) and (4.13) present a nonlinearity on the term $S_x R^{-1} S_x^\top$ and cannot be easily solved as a linear problem. In this case, we can also apply the iterative algorithm proposed by Alves Lima *et al.* (2022) and presented in section 3.3. The complete iterative algorithm adapted for the LPV case is summarized in Algorithm 2.

Algoritmo 2: Control design algorithm.

```
input  : k_max
output : K_i and γ for i = 1,...,N
k ← 1, S_0i ← 0, and R_0 ← I;
while k ≤ k_max do
    L_si ← [−R_0^{-1} S_0i^⊤   −I_m]^⊤;
    minimize λ s.t. (4.11), (3.25);
    if λ_i ≤ 0 or (Δ_ii ⪰ 0 and Δ_ij + Δ_ji ⪰ 0) then
        return K_i = −R^{-1} S_i^⊤ and γ;
    end
    k ← k + 1, S_0i ← S_i, and R_0 ← R;
end
```

An optimization problem is also formulated by Alves Lima *et al.* (2022), and the purpose is to find the control gain while minimizing an objective function. We utilize this optimization problem to find the gain scheduling SOF gain while minimizing the $\mathcal{L}_2$-gain γ. The Algorithm for the optimization problem is summarized in Algorithm 3. From Alves Lima *et al.* (2022), for a given $\varepsilon > 0$, Algorithm 3 always returns an output for a sufficient amount of iterations k_{max}. The proof of this claim is shown in Corollary 4.1 from Alves Lima *et al.* (2022). Moreover, smaller values of ε can lead to lower values of γ at the cost of more iterations.

Corollary 4.1 *For a given $\varepsilon > 0$, Algorithm 2 always returns an output for a sufficient amount of iterations k_{max}.*

Proof. Given any solution set of matrices $\mathcal{Y}_s = \{P_s(\rho), H_s(\rho), R_s, Q_s(\rho), S_s(\rho)\}$ obtained with Algorithm 2, there exists at least one feasible solution to Algorithm 3 at iteration $k = 0$ given by the same set of matrices $\mathcal{Y}_0 = \mathcal{Y}_s$. To see this, note that satisfaction of (4.11) with this trivial solution is guaranteed. Then, by Schur complement, one obtains that (3.25) is equivalent to $R_s^{-1} > 0$ and $Q_s(\rho) - S_s(\rho)R_s^{-1}S_s(\rho)^\top \preceq 0$, which is also certified since $\mathcal{Y}_s$ is a solution to Algorithm 2. The same reasoning can be applied to the subsequent steps, where there always exists at least the solution $\mathcal{Y}_{k+1} = \mathcal{Y}_k$, which leads to $|\gamma_{k+1} - \gamma_k| = 0 < \varepsilon$, $\forall \varepsilon > 0$. $\square$

Remark 4.4 *In case that the $\mathcal{L}_2$-gain is considered, condition* (4.11) *is applied with the bar matrices* (4.27) *as in Theorem 4.3. However, if the $\mathcal{L}_2$-gain is not considered, as in Theorem 4.2, Algorithm 2 can be applied with the original condition* (4.11) *and without γ as an output of the Algorithm, and Algorithm 3 does not apply.*

Algoritmo 3: Minimization of γ.

---

input : k_{max}, ε and $\{R, S_i, \gamma\}$ solution to Algorithm 2.
output : K_i and γ
$k \leftarrow 1$, $S_{0i} \leftarrow S_i$, $R_0 \leftarrow R$, and $\gamma_0 \leftarrow \gamma$;
while $k \leq k_{max}$ **do**
  $L_{si} \leftarrow \begin{bmatrix} -R_0^{-1}S_{0i}^\top & -I_m \end{bmatrix}^\top$;
  minimize γ s.t. (4.11), (3.24);
  if $|\gamma - \gamma_0| \leq \varepsilon$ **then**
    return $K_i = -R^{-1}S_i^\top$ and γ;
  end
  $k \leftarrow k+1$, $S_{0i} \leftarrow S_i$, $R_0 \leftarrow R$, and $\gamma_0 \leftarrow \gamma$;
end

4.4 Numerical Examples

In this section, we present two numerical examples to illustrate the effectiveness of our strategy. One is an affine LPV system with uncertainties that are unavailable for measurement, and the other is a rational LPV system. Here, the comparisons are in terms of a minimum $\mathcal{L}_2$-gain obtained.

4.4.1 *Example 1- Uncertain LPV system*

Consider the following uncertain LPV system from Rotondo *et al.* (2014)

$$\dot{x}(t) = \begin{bmatrix} \delta_1 & \delta_2\rho(t) \\ -2 & -4\delta_2 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ \delta_1 \end{bmatrix} w(t),$$
$$z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t), \tag{4.34}$$

where $\rho \in [2\ \ 4]$ is the time-varying parameter available for measurement and $\delta_1,\ \delta_2 \in [0.9\ \ 1.1]$ are the uncertainties unavailable for measurement. In view of comparisons with the results from Rotondo *et al.* (2014), the purpose is to find a gain scheduling static state feedback gain such that the closed-loop is stable and the $\mathcal{L}_2$-gain from $z(t)$ to $w(t)$ is bounded by γ, then we have to consider $y(t) = Cx(t)$, where $C = I_2$. The DAR form of this system is given by

$$\pi = \rho(t)x_2,\ A_1 = \begin{bmatrix} \delta_1 & 0 \\ -2 & -4\delta_2 \end{bmatrix},\ A_2 = \begin{bmatrix} \delta_2 \\ 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A_4 = \begin{bmatrix} 0 \\ \delta_1 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 & 0 \end{bmatrix},\ B_2 = 0,\ B_3 = 0_{1\times 2},\ B_4 = 0,\ C_1 = I_2,\ C_2 = 0_{2\times 1},$$
$$C_3 = 0_{2\times 1},\ \Upsilon_1 = \begin{bmatrix} 0 & \rho(t) \end{bmatrix},\ \Upsilon_2 = -1,\ \Upsilon_3 = 0,\ \Upsilon_4 = 0.$$

Applying Algorithms 2-3, after 2 iterations in 2.9601s in Algorithm 2 and 4 iterations in 2.3380s in Algorithm 3, we find gains

$$K_1 = \begin{bmatrix} -88.8399 & -2.6823 \\ -65.6109 & -3.0164 \end{bmatrix},\ K_2 = \begin{bmatrix} -92.8209 & -5.1411 \\ -68.3447 & -5.1964 \end{bmatrix}$$

which characterize the following control law

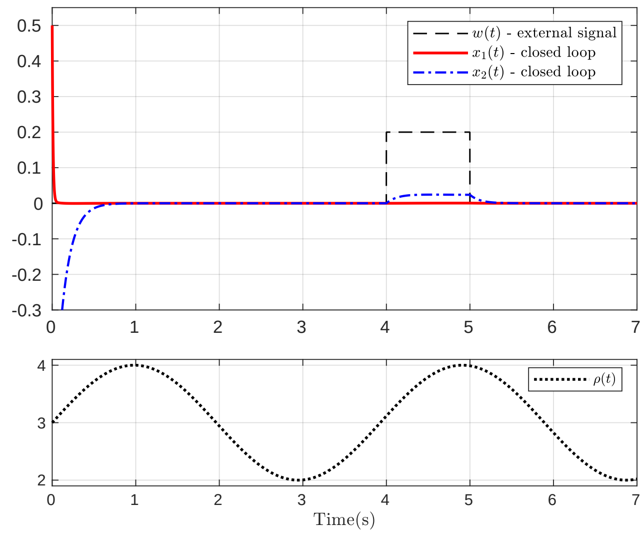
$$u(t) = (\alpha_1 K_1 + \alpha_2 K_2)x(t),$$

that guarantees stability for the closed-loop system with $\gamma = 0.012$, while Rotondo *et al.* (2014) guarantees stability with a $\gamma = 10$. As presented in Assumption 4.1, for any $\rho \in \Omega$ there exist $\alpha \in \Lambda_N$, then we can find scalars α_i's from,

$$\rho(t) = \alpha_1 v_1 + \alpha_2 v_2,$$

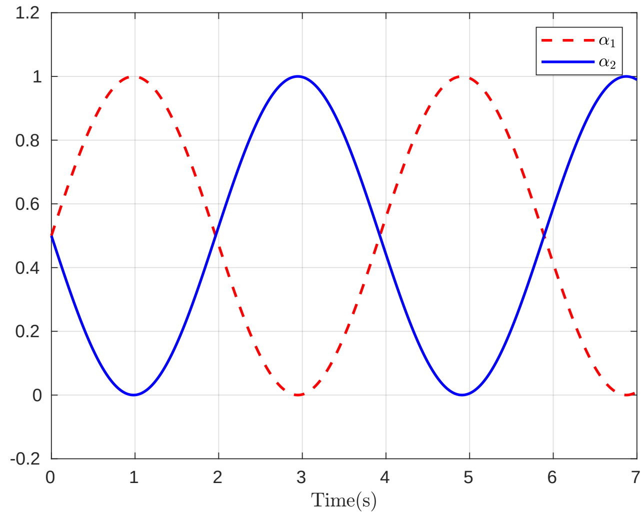
where $v_1, v_2 \in \mathcal{V}(\Omega)$ and $\alpha_1 + \alpha_2 = 1$. Simulations of the closed-loop system were performed considering $\rho(t) = 3 + sin(1.6t)$, $\delta_1 = \delta_2 = 1$, $w(t) = 0.2$ for $4 \leq t \leq 5$, and initial conditions $x(0) = [0.5\ \ -0.3]^\top$. Figure 6 presents the state trajectories of the system, which shows that the closed-loop system is stable even in the presence of an external disturbance. Figure 7 shows the values of $\alpha_1(t)$ and $\alpha_2(t)$ that were computed from $\rho(t)$. As expected, $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$, $\forall t \geq 0$.

Figure 6 – States of the closed-loop system and time-varying parameter $\rho(t)$.



Fonte: The author.

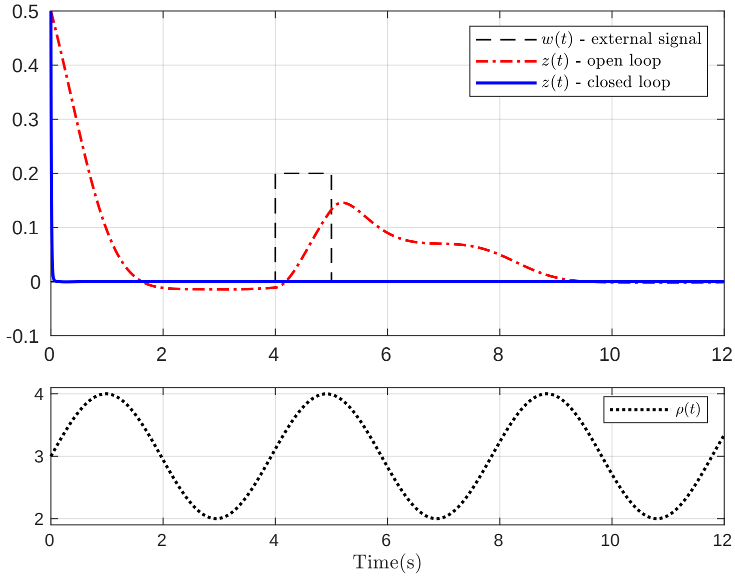
Figure 7 – Values of the time-varying parameter α.



Fonte: The author.

To finish, Figure 8 shows the trajectories of the controlled output for the open- and closed-loop system. The $\mathcal{L}_2$-gain performance intends to minimizing the effect of the external signal on the controlled output. In Figure 8, the controlled output of the open-loop system is very affected by the external signal, while the controlled output of the closed-loop system is almost unaffected, showing the advantage of the design considering the $\mathcal{L}_2$-gain performance.

Figure 8 – Controlled output.



Fonte: The author.

4.4.2 *Example 2 - Rational LPV system*

Consider the rational LPV system from Bouali *et al.* (2006)

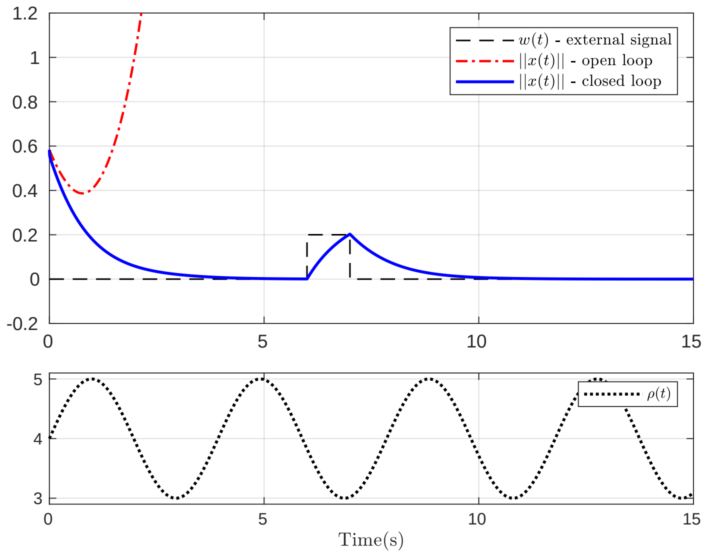
$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & \frac{1}{\rho(t)} \end{bmatrix} x(t) + \begin{bmatrix} \rho \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} w(t),$$
$$z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) - w(t), \tag{4.35}$$

where $\rho \in [3 \quad 5]$ is the time-varying parameter available for measurement. In this case, in order to illustrate the effectiveness of dealing with gain scheduling SOF design, we consider $y(t) = \begin{bmatrix} 2 & 3 \end{bmatrix} x(t)$. The DAR form of this system is given by

$$\pi = \begin{bmatrix} \frac{x_2}{\rho(t)} & \rho(t)u(t) \end{bmatrix}^\top, A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, A_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, B_2 = [0 \ \ 0], B_3 = 0, B_4 = -1, C_1 = [2 \ \ 3], C_2 = [0 \ \ 0],$$
$$C_3 = 0, \Upsilon_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \Upsilon_2 = \begin{bmatrix} -\rho(t) & 0 \\ 0 & -1 \end{bmatrix}, \Upsilon_3 = \begin{bmatrix} 0 \\ \rho(t) \end{bmatrix}, \Upsilon_4 = 0_{2\times 1}.$$

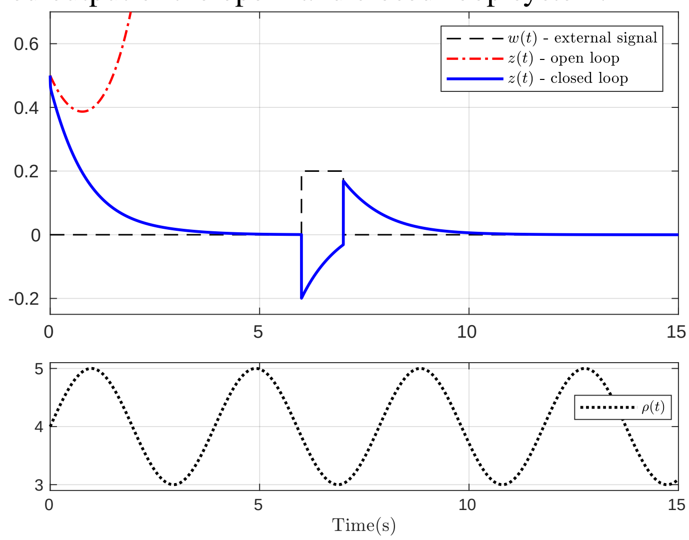
Applying Algorithms 2-3, after 12 iterations in 5.4196s in Algorithm 2 and 120 iterations in 22.3914s in Algorithm 3, we find gains $K_1 = -23.3227,\ K_2 = -23.3227$ which guarantees stability of the closed-loop system with $\mathcal{L}_2$-gain bounded by $\gamma = 1$. Bouali *et al.* (2006) achieves stability with a gain scheduling static state feedback controller and $\mathcal{L}_2$-gain bounded by $\gamma = 1$. In our approach, we achieve stability with the same bound for the $\mathcal{L}_2$-gain ($\gamma = 1$) designing only a gain scheduling static output feedback controller.

Figure 9 – 2-Norm of the states of the open- and closed-loop system.



Fonte: The author.

Figure 10 – Controlled output of the open- and closed-loop system.



Fonte: The author.

Simulations of the closed-loop system were performed considering $\rho(t) = 4 + sin(1.6t)$, $w(t) = 0.2$ for $6 \leq t \leq 7$, and initial conditions $x(0) = [0.5 \quad -0.3]^\top$. Figure 9 presents the Euclidean norm of the states of the open- and closed-loop system, confirming that the LPV system was stabilized by the SOF gain. Figure 10 shows the controlled output of the open- and closed-loop system. The SOF gain designed have stabilized the system at the same time minimizing the effect of the external signal on the controlled output.

5 STATIC OUTPUT FEEDBACK CONTROL OF INPUT SATURATED NONLINEAR SYSTEMS

In the context of nonlinear systems, designing a control law such that the closed-loop system is stable for a set of initial conditions is a challenging problem. Some important works dealt only with stability analysis of nonlinear systems (COUTINHO *et al.*, 2008; COUTINHO; GOMES DA SILVA JR., 2010; CHESI, 2004; CHESI, 2010; TROFINO; DEZUO, 2014). Coutinho and Gomes da Silva Jr. (2010) and Trofino and Dezuo (2014) dealt with rational nonlinear systems using techniques based on Differential algebraic representation (DAR). Both works developed analysis conditions based on LMIs for computing estimates of the Region of attraction (RA) of the closed-loop system. One of the advantages of DARs is the possibility of using well-known methods for the manipulation of the conditions. Aiming at less conservative results, rational Lyapunov functions have been used in Coutinho and Gomes da Silva Jr. (2010), Trofino and Dezuo (2014). Moreover, Coutinho and Gomes da Silva Jr. (2010) considers the nonlinear system as input saturated.

Concerning the control synthesis for nonlinear systems, some strategies have been proposed using Sum of Squares (SOS) techniques (ICHIHARA, 2009; VALMORBIDA *et al.*, 2013; JENNAWASIN; BANJERDPONGCHAI, 2018; JENNAWASIN; BANJERDPONGCHAI, 2021; VALMORBIDA; PAPACHRISTODOULOU, 2021). In the last decade, some works have proposed LMI-based strategies for computing estimates of the region of attraction of nonlinear systems. Oliveira *et al.* (2012) proposed a Static state feedback (SSF) design for single-input rational nonlinear systems with saturation on the input. Azizi *et al.* (2018) have also proposed the design of a state feedback control law for input saturated nonlinear systems, however with the consideration of Multiple-input multiple-output (MIMO) systems with parametric uncertainties.

Regarding output feedback controllers, Madeira and Adamy (2016) applied a notion of passivity to stabilize nonlinear systems via dynamic output feedback. Gomes da Silva Jr. *et al.* (2014) proposed the design of a static anti-windup gain that ensures stability for the input saturated system, assuming that a dynamic output feedback controller is previously designed to stabilize the nonlinear system. Castro *et al.* (2021) also presented a new strategy based on anti-windup design, now for output regulation of rational systems subject to control input saturation. Nevertheless, none of these works have provided tools for the SOF stabilization of rational nonlinear systems. More recently, Madeira (2018), Madeira and Viana (2020), and Alves Lima *et al.* (2022) have proposed the stabilization of rational nonlinear systems via static output feedback using dissipativity. However, these works consider only quadratic Lyapunov functions, leading to more conservative results then those using more generic Lyapunov functions, as polynomial and rational ones. Moreover, Madeira (2018) and Madeira and Viana (2020) do not consider that the system is input saturated.

The motivation for considering the system to be saturated at the input arises from the fact that saturation is a constraint presented in most controlled physical systems. Due to physical or safety restrictions, the magnitude of the signal delivered by the actuator is limited. The nonlinear behavior of the saturation affects the closed-loop system. Even if the open-loop system is considered linear, the presence of the saturation makes the closed-loop system nonlinear (TARBOURIECH *et al.*, 2011). Not taking into account the presence of saturation in the system model in the control design can be the reason for closed-loop instability in practical situations. Then, many techniques have been proposed to deal with the stabilization problem for systems with the saturation constraint, as shown in the last paragraphs.

Therefore, in this chapter, the static output feedback stabilization of input saturated uncertain nonlinear systems is proposed by employing an LMI-based iterative algorithm. Unlike the previous SOF design chapter for LPV systems, the uncertainty vector of the parameters δ is constant in this case. This problem was recently trackled in Alves Lima *et al.* (2022). However, Alves Lima *et al.* (2022) employed a quadratic Lyapunov function which can lead to conservative conditions. Then, aiming at less conservative results, the proposed approach considers more generic Lyapunov functions such as polynomial or rational ones. In addition, this strategy deals with the design of linear and nonlinear control laws. As in the last chapters, we also use the definition of strict QSR-dissipativity and Finsler's Lemma together with a generalized sector condition that allows dealing with dead-zone nonlinearities. Furthermore, we apply the iterative algorithm from Alves Lima *et al.* (2022), allowing solving the developed conditions with SDP tools. And also an optimization algorithm to maximize the estimates of the region of attraction of the closed-loop system.

5.1 System Description and Problem Formulation

Consider an uncertain input saturated nonlinear system

$$\begin{cases} \dot{x}(t) = f(x(t),\delta) + g(x(t),\delta)sat(v(t)), \\ y(t) = h(x(t),\delta), \end{cases} \tag{5.1}$$

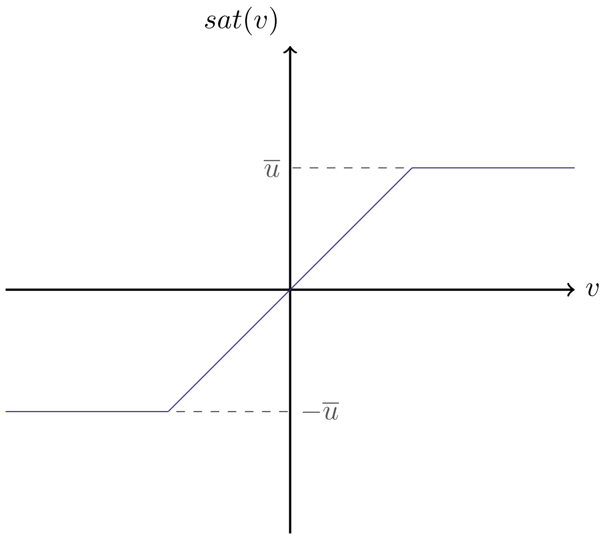
where $x(t) \in \mathbb{R}^n$ is the state vector with initial conditions $x(0) \in \mathcal{X} \subseteq \mathbb{R}^n$, $\delta \in \mathcal{D} \subset \mathbb{R}^l$ is an uncertain bounded vector of constant parameters which accounts for deviations of the model description around its nominal part, $y(t) \in \mathbb{R}^p$ is the system output. $f(x,\delta),\ g(x,\delta),\ h(x,\delta)$ are polynomial or rational functions on $x(t)$ and δ such that $(f,g,h) \in \mathcal{C}^1$, $(f(0,\delta),h(0,\delta)) = (0,0)$ for all $\delta \in \mathcal{D}$, and the origin $(x(t),u(t)) = (0,0)$ is an equilibrium point of (5.1). Moreover, $v(t)$ is the input of the system, $sat(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is the classical unit saturation function given by

$$sat(v_i(t)) = \begin{cases} \bar{u}_i & \text{if } v_i > \bar{u}_i \\ v_i & \text{if } -\bar{u}_i \leq v_i \leq \bar{u}_i \\ -\bar{u}_i & \text{if } v_i < -\bar{u}_i \end{cases} \tag{5.2}$$

where $i = \{1,\dots,m\}$ and $\bar{u}_i$ is the saturation bound of each actuator. Figure 11 presents the illustration of the saturation function. Furthermore, for systems subject to input saturation, it is essential to consider a model for the saturation $sat(v(t))$ to facilitate the design of a stability procedure to avoid its undesirable effects in the closed-loop system. Then, we consider the so-called dead zone nonlinearity φ, defined in 5.3, which will be helpful later in this work.

$$\varphi(v(t)) = sat(v(t)) - v(t). \tag{5.3}$$

Figure 11 – Saturation function.



Fonte: The author.

5.1.1 *DAR for the uncertain input saturated nonlinear system*

By taking into account the DAR (2.33), with $u(t) = sat(v(t))$, and the identity (5.3), the following equivalent representation for an uncertain nonlinear system with input saturation is obtained

$$\begin{cases} \dot{x} = A_1x(t) + A_2\pi(t) + A_3v(t) + A_3\varphi(v(t)) \\ 0 = \Upsilon_1x(t) + \Upsilon_2\pi(t) + \Upsilon_3v(t) + \Upsilon_3\varphi(v(t)) \\ y = C_1x(t) + C_2\pi(t), \end{cases} \tag{5.4}$$

where $\pi(x, sat(v), \delta) \in \mathbb{R}^{n_\pi}$ is a suitably chosen vector of nonlinear functions. Matrices $A_1(x,\delta) \in \mathbb{R}^{n\times n}$, $A_2(x,\delta) \in \mathbb{R}^{n\times n_\pi}$, $A_3(x,\delta) \in \mathbb{R}^{n\times m}$, $\Upsilon_1(x,\delta) \in \mathbb{R}^{n_\pi\times n}$, $\Upsilon_2(x,\delta) \in \mathbb{R}^{n_\pi\times n_\pi}$, $\Upsilon_3(x,\delta) \in \mathbb{R}^{n_\pi\times m}$ are affine functions with respect to (x,δ), while $\Upsilon_2(x,\delta) \in \mathbb{R}^{n_\pi\times n_\pi}$ is supposed to be a square full-rank matrix for all vectors $(x,\delta) \in \mathcal{X} \times \mathcal{D}$, and C_1, C_2 are constant matrices of appropriate dimensions.

5.1.2 Problem statement

The problem to which we propose a solution to in this chapter can be summarized as follows.

Problem 5.1 *Given the DAR matrices $A_1(x,\delta)$, $A_2(x,\delta)$, $A_3(x,\delta)$, $\Upsilon_1(x,\delta)$, $\Upsilon_2(x,\delta)$, $\Upsilon_3(x,\delta)$, C_1, and C_2 of the input saturated nonlinear system* (5.1)*, we intend to design a static gain K such that the control law given by*

$$v(t) = Ky(t) \tag{5.5}$$

asymptotically stabilizes the closed-loop system yielded from the connection (5.1)-(5.5) *for some set of initial conditions $\mathcal{H} \subseteq \mathcal{X} \subset \mathbb{R}^n$, $\forall \delta \in \mathcal{D}$. Furthermore, we want to maximize estimates of the region of attraction of the closed-loop system.*

5.2 Generalized Sector Condition

Concerning input saturated systems, a sector condition can be employed together with Lyapunov theory to develop sufficient conditions to achieve stability. Here, we present the sector condition that is used in this work. First, consider the following set

$$\mathcal{L}(\bar{u}) = \{v \in \mathbb{R}^m; \theta \in \mathbb{R}^m; -\bar{u} \leq v - \theta \leq \bar{u}\}, \tag{5.6}$$

where $\theta = f_\theta(x(t))$ is an auxiliary vector to be defined, and the deadzone nonlinearity φ defined in (5.3). We then recall the following Lemma from Tarbouriech *et al.* (2011, p. 43).

Lemma 5.1 *If v and θ belong to the set $\mathcal{L}(\bar{u})$, then the deadzone nonlinearity $\varphi(v)$ satisfies the following inequality, which is true for any diagonal positive definite matrix $W \in \mathbb{R}^{m\times m}$*

$$\varphi^\top(v)W(\varphi(v)+\theta) \leq 0. \tag{5.7}$$

Here, we consider the application of Lemma 5.1 with

$$\theta = f_\theta(x(t)) = v - Gx - G_\zeta \zeta, \tag{5.8}$$

where the auxiliary matrices $G(x,\delta) \in \mathbb{R}^{m\times n}$ and $G_\zeta(x,\delta) \in \mathbb{R}^{m\times n_\zeta}$ are affine in their arguments. Then, for the resulting set

$$\mathcal{L}(\bar{u}) = \{x \in \mathbb{R}^n; |G_{(i)}x + G_{\zeta(i)}\zeta| \leq \bar{u}_{(i)}, i = 1,\ldots,m, \forall\delta \in \mathcal{D}\} \tag{5.9}$$

the inequality

$$B(\varphi,v,x,\zeta) = -2\varphi^\top(v)W(\varphi^\top(v)+v-Gx-G_\zeta\zeta) \geq 0, \tag{5.10}$$

holds if $(x,\delta) \in \mathcal{L}(\bar{u}) \times \mathcal{D}$.

5.3 Lyapunov Function Candidate

In this case, we consider the following class of Lyapunov functions

$$V(x) = x^\top \mathbf{P}(x,\delta)x = \Theta^\top(x,\delta)P(x,\delta)\Theta(x,\delta), \tag{5.11}$$

where $P(x,\delta)$ is affine with respect to (x,δ) and $\Theta(x,\delta) = [x^\top \ \ \zeta^\top]^T$, with ζ being a chosen vector of nonlinear functions, such that ζ depends only on x and δ (TROFINO; DEZUO, 2014). Moreover, the following relation applies

$$0 = \Omega_1(x,\delta)x + \Omega_2(x,\delta)\zeta, \tag{5.12}$$

where Ω_2 is invertible, such that $\zeta = -\Omega_2^{-1}\Omega_1 x$, and $\Omega_1(x,\delta)$, $\Omega_2(x,\delta)$ are affine functions of x and δ, that can be decomposed in the following manner

$$\begin{aligned}
&\Omega_1(x,\delta) = \Omega_1^0 + \Omega_1^x(x) + \Omega_1^\delta(\delta),\\
&\Omega_1^x(x) = \sum_{i=1}^n \Omega_{1_i}^x x_i,\ \Omega_1^\delta(\delta) = \sum_{i=1}^n \Omega_{1_i}^\delta \delta_i,\\
&\Omega_2(x,\delta) = \Omega_2^0 + \Omega_2^x(x) + \Omega_2^\delta(\delta),\\
&\Omega_{x_2}(x) = \sum_{i=1}^n \Omega_{2_i}^x x_i,\ \Omega_2^\delta(\delta) = \sum_{i=1}^n \Omega_{2_i}^\delta \delta_i,
\end{aligned} \tag{5.13}$$

where $\Omega_1^x(x)$ and $\Omega_2^x(x)$ are the terms of $\Omega_1(x,\delta)$ and $\Omega_2(x,\delta)$, respectively, depending only on x, $\Omega_1^\delta(\delta)$ and $\Omega_2^\delta(\delta)$ the terms of $\Omega_1(x,\delta)$ and $\Omega_2(x,\delta)$, respectively, depending only on δ, and Ω_1^0, Ω_2^0 the constant terms of $\Omega_1(x,\delta)$ and $\Omega_2(x,\delta)$, respectively. Moreover, $\Omega_{1_i}^x$, $\Omega_{1_i}^\delta$, $\Omega_{2_i}^x$, $\Omega_{2_i}^\delta$ are constant matrices coming from the decomposition of $\Omega_1^x(x)$, $\Omega_1^\delta(\delta)$, $\Omega_2^x(x)$, $\Omega_2^\delta(\delta)$, respectively.

Therefore, $\mathbf{P}(x,\delta)$ is a polynomial or rational function that can be defined as

$$\mathbf{P}(x,\delta) = \begin{bmatrix} I_n \\ -\Omega_2^{-1}\Omega_1 \end{bmatrix}^T P(x,\delta) \begin{bmatrix} I_n \\ -\Omega_2^{-1}\Omega_1 \end{bmatrix}. \tag{5.14}$$

For this Lyapunov function candidate, an estimate of the region of attraction of the system is given by

$$\mathcal{E} = \{x \in \mathbb{R}^n : x^\top \mathbf{P}(x,\delta)x \leq 1,\ \forall \delta \in \mathcal{D}\}. \tag{5.15}$$

Remark 5.1 *The vector ζ of the Lyapunov function can be chosen based on the dynamics of the system. Normally, it can be considered equal to the nonlinear vector of the DAR π. However, π depends on x, δ, and u, while ζ depends only on x and δ. Then, a partition on vector π can be applied such $\pi = [\pi_{x,\delta}^\top \ \ \pi_u^\top]^\top$ and, in this case, $\zeta = \pi_{x,\delta}$. In a more generic case, when vector π is not as complex as desired, ζ can be arbitrarily chosen given considering more complex Lyapunov functions.*

5.4 Static Output Feedback Design

The proposed strategy here consists in connecting strict QSR-dissipativity condition, Lemma 2.4, and the generalized sector condition provided in Lemma 5.1 for regional stabilization and estimation of the region of attraction of system (5.1). The following Theorem summarizes the strategy that provides a solution to Problem 5.1.2.

Theorem 5.2 *Assume that there exist matrices $N \in \mathbb{S}_n^+$, $R \in \mathbb{S}_m^+$, $Q \in \mathbb{S}_p$, diagonal matrix $W \in \mathbb{S}_m^+$, matrices $S \in \mathbb{R}^{p\times m}$, $\mathfrak{I} \in \mathbb{R}^{(4n_\zeta+4n+2m+n_\pi)\times(3n+4n_\zeta+n_\pi)}$, $L_s \in \mathbb{R}^{(p+m)\times m}$, $L_d \in \mathbb{R}^{(n+n_\zeta)\times n_\zeta}$, $Z \in \mathbb{R}^{n\times n_\zeta}$, $Z_s \in \mathbb{R}^{n_\zeta\times n_\zeta}$, and matrices $\overline{G}(x,\delta) \in \mathbb{R}^{m\times n}$, $\overline{G}_\zeta(x,\delta) \in \mathbb{R}^{m\times n_\zeta}$, $P(x,\delta) \in \mathbb{S}_{(n+n_\zeta)}$ with affine dependence in (x,δ) such that for all (x,δ) at the vertices of $\mathcal{X}\times\mathcal{D}$*

$$P + L_d C_d + C_d^\top L_d \succ 0, \tag{5.16}$$

$$\Phi + \mathfrak{I}\Gamma + \Gamma^\top \mathfrak{I}^\top \prec 0, \tag{5.17}$$

$$\begin{bmatrix} P_1 + He\{Z\Omega_1\} & P_2 + Z\Omega_2 & a_k \\ \star & P_3 & 0 \\ \star & \star & 1 \end{bmatrix} \succeq 0, \quad k = 1, \ldots, n_{xe}, \tag{5.18}$$

$$\begin{bmatrix} P_1 & P_2 + \Omega_1^\top Z_s^\top & \bar{G}_{(i)}^\top \\ \star & P_3 + He\{Z_s\Omega_2\} & \bar{G}_{\zeta(i)}^\top \\ \star & \star & 2W_{(i,i)} - \bar{u}_i^{-2} \end{bmatrix} \succeq 0, \quad i = 1, \ldots, m, \tag{5.19}$$

and

$$\Delta = SR^{-1}S^\top - Q \succeq 0 \tag{5.20}$$

hold with $C_d(x,\delta) = [\Omega_1(x,\delta) \;\; \Omega_2(x,\delta)]$ *and*

$$\Phi = \begin{bmatrix} N - C_1^\top Q_1 C_1 & \star & \star & \star & \star & \star & \star & \star & \star \\ -C_2^\top Q_1 C_1 & -C_2^\top Q_1 C_2 & \star & \star & \star & \star & \star & \star & \star \\ -S_1^\top C_1 & -S_1^\top C_2 & -R & \star & \star & \star & \star & \star & \star \\ \bar{G} & 0 & -W & -2W & \star & \star & \star & \star & \star \\ P_1 & 0 & 0 & 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \bar{G}_\zeta^\top & P_2^\top & 0 & \star & \star & \star \\ P_2^\top & 0 & 0 & 0 & 0 & P_3 & 0 & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} \Upsilon_1 & \Upsilon_2 & \Upsilon_3 & \Upsilon_3 & 0 & 0 & 0 & 0 & \\ A_1 & A_2 & A_3 & A_3 & -I_n & 0 & 0 & 0 & 0 \\ \Omega_1 & 0 & 0 & 0 & 0 & \Omega_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (\Omega_1 + \bar{\Omega}_{1a}) & 0 & \Omega_2 & 0 & \bar{\Omega}_{2a} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\Omega_{1b} & \Omega_{2b} \\ 0 & 0 & 0 & 0 & E_b & 0 & 0 & I_{n^2} & 0 \end{bmatrix}$$

where

$$P(x,\delta) = \begin{bmatrix} P_1(x,\delta) & P_2(x,\delta) \\ P_2^\top(x,\delta) & P_3(x,\delta) \end{bmatrix}, \; Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^\top & Q_3 \end{bmatrix}, \; S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix},$$

$\Omega_{1b}(x,\delta) = [\Omega_1(x,\delta)]_{diag}^{\mathbb{I}_n}$, $\Omega_{2b}(x,\delta) = [\Omega_2(x,\delta)]_{diag}^{\mathbb{I}_n}$,

$E_b(x) = [xE_i]_{col}^{i \in \mathbb{I}_n}, E_i = row_i(I_n)$, *and, from decomposition* (5.13)*,*

$$\overline{\Omega}_{1a}(x) = \sum_{i=1}^{n} \Omega_{1_i}^{x} x E_i,\ \overline{\Omega}_{2a} = [\Omega_{2_i}^{x}]_{row}^{i\in\mathbb{I}_n}.$$

Then, the SOF gain $K = -R^{-1}S^{\top}$ asymptotically stabilizes the closed-loop system (5.1)-(5.5) *around the origin, and $\mathcal{E} \subset \mathcal{H} \subset \mathbb{R}^n$, where $\mathcal{H} = \mathcal{X} \cap \mathcal{L}(\bar{u})$ and $\mathcal{E} = \{x \in \mathbb{R}^n; x^{\top}\boldsymbol{P}(x,\delta)x \leq 1,\ \forall \delta \in \mathcal{D}\}$ is an estimate of the closed-loop domain of attraction.*

Proof. In this proof, the objective is to show that the estimated domain of attraction $\mathcal{E}$ is inside both the polytope of the states $\mathcal{X}$ and the polyhedral set $\mathcal{L}(\bar{u})$ for all $\delta \in \mathcal{D}$, i.e., $\mathcal{E} \subset \mathcal{X} \cap \mathcal{L}(\bar{u})$. Moreover, we want to guarantee that for a positive definite function $V(x) > 0$, $\dot{V}(x) < 0$ along the trajectories of the closed-loop system (5.1)-(5.5) for $x \in \mathcal{X} \cap \mathcal{L}(\bar{u})$, and all $\delta \in \mathcal{D}$, which characterizes $\mathcal{X} \cap \mathcal{L}(\bar{u})$ as a region of asymptotic stability. The following steps detail the complete proof.

First, consider (5.11), note that $C_d(x,\delta)\Theta(x,\delta) = 0$, then Lemma 2.4 can be applied, which leads to (5.16). If there exists a matrix $L_d \in \mathbb{R}^{(n+n_\zeta)\times n_\zeta}$ such that $P + \mathrm{He}\{L_d C_d\} \succ 0$, then $V(x) = x^{\top}\mathbf{P}(x,\delta)x > 0$ is guaranteed for all $(x,\delta) \in \mathcal{X} \times \mathcal{D}$. Then, consider definitions (5.15) and (2.4), the inclusion $\mathcal{E} \subset \mathcal{X}$ can be cast by the standard condition $x^{\top}\mathbf{P}(x,\delta)x - (x^{\top}a_k)(a_k^{\top}x) \geq 0$. However, in our case, we use the fact that $x^{\top}Z(\Omega_1 x + \Omega_2\zeta) = 0$ for the terms of the anti-diagonal not to be zeros, then $\mathcal{E} \subset \mathcal{X}$ can be cast as $x^{\top}\mathbf{P}(x,\delta)x - (x^{\top}a_k)(a_k^{\top}x) + 2x^{\top}Z(\Omega_1 x + \Omega_2\zeta) \geq 0$ which is equivalent to

$$\begin{bmatrix} x \\ \zeta \end{bmatrix}^{\top} \begin{bmatrix} P_1 - a_k a_k^{\top} + \mathrm{He}\{Z\Omega_1\} & P_2 + Z\Omega_2 \\ \star & P_3 \end{bmatrix} \begin{bmatrix} x \\ \zeta \end{bmatrix} \geq 0 \tag{5.21}$$

for $x \in \mathcal{V}(\mathcal{X})$, $\delta \in \mathcal{V}(\mathcal{D})$, and $k = 1,\ldots,n_{xe}$. Applying Schur complement in (5.21) we obtain condition (5.18) of the Theorem which, then, ensures $\mathcal{E} \subset \mathcal{X}$. Now, consider relation (5.19), using the fact that

$$\left(\bar{u}_{(i)}^{-2} - W_{(i.i)}\right)^{\top} \bar{u}_{(i)}^{2} \left(\bar{u}_{(i)}^{-2} - W_{(i,i)}\right) \geq 0, \tag{5.22}$$

we obtain the following inequality $2W_{(i,i)} - \bar{u}_{(i)}^{-2} \leq W_{(i,i)}^{\top}\bar{u}_{(i)}^{2}W_{(i,i)}$. Thus, satisfaction of (5.19) implies the fulfillment of

$$\begin{bmatrix} P_1 & P_2 + \Omega_1^{\top}Z_s^{\top} & \bar{G}_{(i)}^{\top} \\ \star & P_3 + He\{Z_s\Omega_2\} & \bar{G}_{\zeta(i)}^{\top} \\ \star & \star & W_{(i,i)}^{\top}\bar{u}_{(i)}^{2}W_{(i,i)} \end{bmatrix} \succeq 0, \quad i = 1,\ldots,m$$

Then, pre- and post-multiply the last inequality by $diag\left(I, I, W_{(i,i)}^{-1}\right)$, apply a Schur complement, and use pre- and post-multiplication by $\begin{bmatrix} x^\top & \zeta^\top \end{bmatrix}^\top$ to obtain

$$(G_{(i)}x + G_{\zeta(i)}\zeta)^2\,\bar{u}^{-2} \leq x^\top \mathbf{P}(x,\delta)x + 2\zeta^\top Z_s(\Omega_1 x + \Omega_2\zeta), \quad i = 1,\ldots,m$$

which (by taking into account relation $\Omega_1 x + \Omega_2\zeta = 0$ and $x^\top \mathbf{P}(x,\delta)x \leq 1$) ensures the inclusion of the ellipsoid $\mathcal{E}$ in the polyhedral set $\mathcal{L}(\bar{u})$. Therefore, satisfying both (5.18) and (5.19), one ensures $\mathcal{E} \subset \mathcal{H}$, where $\mathcal{H} = \mathcal{X} \cap \mathcal{L}(\bar{u})$. Then, if we can ensure that $\dot{V}(x) < 0$ along the trajectories of the closed-loop system (5.1)-(5.5) for all initial conditions at the vertices of $\mathcal{X} \times \mathcal{D}$, then it follows that $\mathcal{E} \subset \mathcal{H}$ is an estimation on the domain of attraction, while $\mathcal{H}$ is a region of guaranteed asymptotic stability of the system. Consider condition (5.10) and the following relation

$$\dot{V}(x) + T(x) + B(\varphi, v, x, \zeta) \leq r(u(t), y(t)), \tag{5.23}$$

since $T(x) = x^\top N x$, $y = C_1 x(t) + C_2\pi(t)$, (5.23) can be equivalently rewritten as $\pi_d^\top \Phi \pi_d \leq 0$, where Φ has been defined in Theorem 5.2 and $\pi_d = [x^\top\ \pi^\top\ v^\top\ \varphi^\top\ \dot{x}^\top\ \zeta^\top\ \dot{\zeta}^\top\ \mu^\top\ \eta^\top]^\top$, μ and η are auxiliary vectors that are going to be helpful in the construction of the annihilator of vector π_d. Now, consider relation $0 = \Omega_1(x,\delta)x + \Omega_2(x,\delta)\zeta$, deriving we have

$$0 = \dot{\Omega}_1(x,\delta)x + \Omega_1(x,\delta)\dot{x} + \dot{\Omega}_2(x,\delta)\zeta + \Omega_2(x,\delta)\dot{\zeta}. \tag{5.24}$$

As the uncertainty δ is time invariant, $\dot{\Omega}_1(x,\delta)$ and $\dot{\Omega}_2(x,\delta)$ are linear functions of $\dot{x}$. From decomposition (5.13), $\dot{\Omega}_1(x,\delta) = \Omega_1^x(\dot{x})$, $\dot{\Omega}_2(x,\delta) = \Omega_2^x(\dot{x})$, which leads to

$$0 = \Omega_1^x(\dot{x})x + \Omega_1(x,\delta)\dot{x} + \Omega_2^x(\dot{x})\zeta + \Omega_2(x,\delta)\dot{\zeta}. \tag{5.25}$$

By a similar procedure done in Trofino and Dezuo (2014), consider decomposition (5.13), we have then

$$\Omega_2^x(\dot{x})\zeta = \sum_{i=1}^{n} \Omega_{2_i}^x \dot{x}_i \zeta, \tag{5.26}$$

since $\zeta = -\Omega_2^{-1}\Omega_1 x$,

$$\Omega_2^x(\dot{x})\zeta = -\sum_{i=1}^{n} \Omega_{2_i}^x \dot{x}_i \Omega_2^{-1}\Omega_1 x = \sum_{i=1}^{n} \Omega_{2_i}^x \Omega_2^{-1}\Omega_1 \mu_i \tag{5.27}$$

where $\mu_i = -x\dot{x}_i = -xE_i\dot{x}$, $E_i = row_i(I_n)$. Introducing the following change of variables

$$\eta_i = \zeta\dot{x}_i = \Omega_2^{-1}\Omega_1\mu_i, \tag{5.28}$$

we obtain

$$\Omega_2^x(\dot{x})\zeta = \sum_{i=1}^{n} \Omega_{2_i}^x \eta_i = \overline{\Omega}_{2a}\eta, \tag{5.29}$$

where $\eta = [\eta_i]_{col}^{i\in\mathbb{I}_n}$. Moreover, the term $\Omega_1^x(\dot{x})x$ can be represented as follows

$$\Omega_1^x(\dot{x})x = \sum_{i=1}^{n} \Omega_{1_i}^x \dot{x}_i x = \sum_{i=1}^{n} \Omega_{1_i}^x x E_i \dot{x} = \overline{\Omega}_{1a}\dot{x}, \tag{5.30}$$

according to the last steps, (5.24) is equivalent to

$$0 = (\Omega_1 + \overline{\Omega}_{1a})\dot{x} + \Omega_2\dot{\zeta} + \overline{\Omega}_{2a}\eta, \tag{5.31}$$

where $\overline{\Omega}_{1a}$ and $\overline{\Omega}_{2a}$ are given in Theorem 5.2. Furthermore, since $\mu_i = -xE_i\dot{x}$ and $\eta_i = \Omega_2^{-1}\Omega_1\mu_i$, we have that

$$\mu = -E_b(x)\dot{x}, \tag{5.32}$$

$$\Omega_{1b}(x,\delta)\mu = \Omega_{2b}(x,\delta)\eta, \tag{5.33}$$

where $\mu = [\mu_i]_{col}^{i\in\mathbb{I}_n}$, and $E_b(x), \Omega_{1b}$ and Ω_{2b} are given in Theorem 5.2. From (5.4), (5.12), (5.26), (5.32), (5.31), we can conclude that the matrix function $\Gamma(x,\delta)$ defined in Theorem 5.2 is an annihilator of π_d, i.e., $\Gamma\pi_d = 0$. Then, from Lemma 2.4, $\pi_d^\top \Phi \pi_d \leq 0$ is satisfied for all $(x,\delta) \in \mathcal{H} \times \mathcal{D}$ if (5.17) is fulfilled at all vertices of the polytope $\mathcal{X} \times \mathcal{D}$ for some matrix $\mathfrak{I}$. Furthermore, if at the same time we guarantee that

$$r(v(t),y(t)) = y^\top Qy + 2y^\top Sv + v^\top Rv \leq 0, \tag{5.34}$$

then $\dot{V} < 0$ is also ensured for all $(x,\delta) \in \mathcal{H} \times \mathcal{D}$. By considering the control law

$$v(t) = -R^{-1}S^\top y(t), \tag{5.35}$$

substitution of (5.35) into (5.34), we obtain

$$r(v(t),y(t)) = -y^\top \Delta y, \tag{5.36}$$

where $\Delta = SR^{-1}S^\top - Q$. Then, $\Delta \succeq 0$ is a sufficient condition for the asymptotic stability for all $(x,\delta) \in \mathcal{H} \times \mathcal{D}$, completing the proof. □

5.4.1 Iterative algorithm

Note that conditions (5.16), (5.17), (5.18), and (5.19) are LMI conditions and can be efficiently solved as a linear problem. However, condition (5.20) presents a nonlinearity on the term $SR^{-1}S^{\top}$ and cannot be easily solved as a linear problem. In this case, we can also apply the iterative algorithm proposed by Alves Lima *et al.* (2022) and presented in section 3.3. The complete iterative algorithm adapted for the SOF design for nonlinear systems is summarized in Algorithm 4.

Here, we also apply the optimization problem formulated in Alves Lima *et al.* (2022) that was presented in section 4.3. In this case, the purpose is to find the SOF gain while maximizing the estimates of the domain of attraction $\mathcal{E}$. In Alves Lima *et al.* (2022), it is considered a quadratic lyapunov function, which characterizes $\mathcal{E}$ as an ellipsoidal set. Then, they use the size criterion of minimizing the trace of P for maximizing $\mathcal{E}$. However, we consider here a more generic rational Lyapunov function, where the level set is not necessarily an ellipsoid. Then, a solution to the problem of maximizing $\mathcal{E}$ can be indirectly addressed by the following convex optimization problem from Coutinho *et al.* (2008),

$$\begin{aligned} &minimize \;\; \gamma, \\ &\text{subject to } \gamma - tr(P + L_d C_d + C_d^{\top} L_d^{\top}) > 0, \end{aligned} \tag{5.37}$$

at all vertices of the polytope $\mathcal{X} \times \mathcal{D}$. The complete Algorithm for the optimization problem is summarized in Algorithm 5.

Algoritmo 4: Control design algorithm.

---

input : k_{max}
output : K and P
$k \leftarrow 0$, $S_0 \leftarrow 0$, and $R_0 \leftarrow I$;
while $k < k_{max}$ **do**
    $L_s \leftarrow \begin{bmatrix} -R_0^{-1}S_0^{\top} & -I_m \end{bmatrix}^{\top}$;
    minimize λ s.t. (5.16), (5.17), (5.18), (5.19), (3.25);
    if $\lambda \leq 0$ *or* $\Delta \succeq 0$ **then**
        return $K = -R^{-1}S^{\top}$ and P;
    end
    $k \leftarrow k+1$, $S_0 \leftarrow S$, and $R_0 \leftarrow R$;
end

Algoritmo 5: Maximization of $\mathcal{E}$.

input : k_{max}, ε, and $\{R,S,P\}$ solution to Algorithm 4.
output : K and P
$k \leftarrow 0$, $S_0 \leftarrow S$, $R_0 \leftarrow R$, and $P_0 \leftarrow P$;
while $k < k_{max}$ **do**
  $L_s \leftarrow \begin{bmatrix} -R_0^{-1}S_0^\top & -I_m \end{bmatrix}^\top$;
  minimize γ s.t. $\gamma > 0$, (5.16), (5.17), (5.18), (5.19), (5.37), (3.24);
  if $k \geq 1$ **then**
    if $|\gamma - \gamma_0| \leq \varepsilon$ **then**
      return $K = -R^{-1}S^\top$ and P;
    end
    $\gamma_0 \leftarrow \gamma$;
  end
  $k \leftarrow k+1$, $S_0 \leftarrow S$, and $R_0 \leftarrow R$;
end

5.5 Numerical Examples

In this section, we present two numerical examples to illustrate the effectiveness of the proposed method. First, a comparison with an iterative method from the literature to design scheduled static output feedback controllers for polynomial systems is presented. Then, a second example dealing with an uncertain rational nonlinear system from the literature is detailed. The comparisons are in terms of estimated region of attraction obtained.

5.5.1 *Example 1 - Polynomial nonlinear system*

Consider the input saturated polynomial system analysed in Example 1 of Jennawasin and Banjerdpongchai (2021),

$$\begin{cases} \dot{x}_1 = -x_1 + x_1^2 - \frac{3}{2}x_1^3 - \frac{3}{4}x_2^2 x_1 + \frac{1}{4}x_2 - x_1^2 x_2 - \frac{1}{2}x_2^3, \\ \dot{x}_2 = sat(v(t)), \\ y = x_1 - x_2. \end{cases} \tag{5.38}$$

A DAR of this system is given by

$$\pi = \begin{bmatrix} x_1^2 & x_2^2 \end{bmatrix}^\top, \ \Upsilon_2 = -I_2, \ A_3 = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top, \ A_1 = \begin{bmatrix} -1 & \frac{1}{4} \\ 0 & 0 \end{bmatrix}, \ \Upsilon_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^\top, \ C_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}^\top, \Upsilon_1 = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 - \frac{3}{2}x_1 - x_2 & -\frac{3}{4}x_1 - \frac{1}{2}x_2 \\ 0 & 0 \end{bmatrix}.$$

In this case, we consider a polynomial Lyapunov function with $\zeta = \pi$, leading to $\Omega_1 = \Upsilon_1$, $\Omega_2 = \Upsilon_2$, and

$$\bar{\Omega}_{2a} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \bar{\Omega}_{1a} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} E_1 + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} E_2,$$

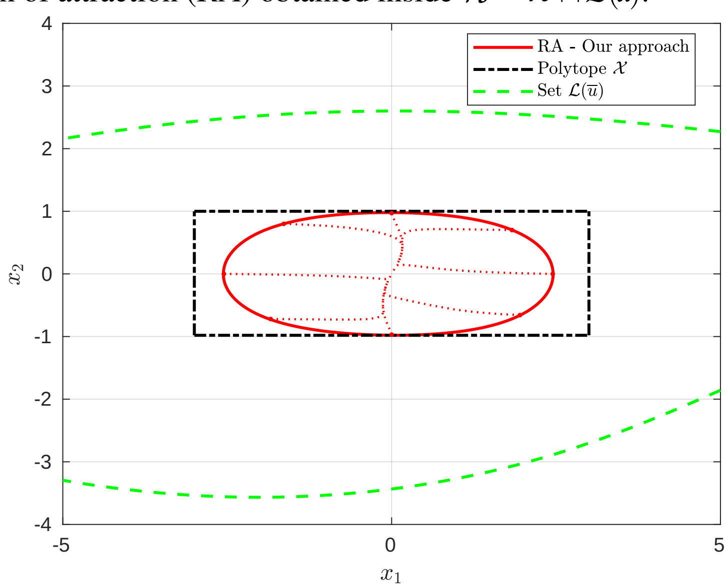
where $E_1 = [1 \;\; 0]$, $E_2 = [0 \;\; 1]$. Moreover, we consider affine matrices $P_i(x,\delta) = P_{0i} + P_{1i}x_1 + P_{2i}x_2$, $i = \{1,2,3\}$. We consider $\bar{u} = 1.5$, which is the input saturation value used in Jennawasin and Banjerdpongchai (2021). By running Algorithms 4-5 with polytope $\mathcal{X}$ defined with the limits $|x_1| \leq 3, -0.98 \leq x_2 \leq 1$ we obtain the stabilizing gain $K = 0.1312$, after 8 iterations in 12.7059s in Algorithm 4 and 2 iterations in 2.6406s in Algorithm 5. Table 2 presents the maximum radius of the estimated regions of attraction obtained with some strategies from the literature. The result of Jennawasin and Banjerdpongchai (2021) presented here considers a polynomial Lyapunov function of degree 3, while Alves Lima *et al.* (2022) considers a quadratic Lyapunov function. As we can see, using a polynomial Lyapunov function of degree 5, our approach provided the best result.

Table 2 – Maximum radius of the region of attraction.

Our approach	Alves Lima *et al.* (2022)	Jennawasin and Banjerdpongchai (2021)
2.45	0.9001	0.5816

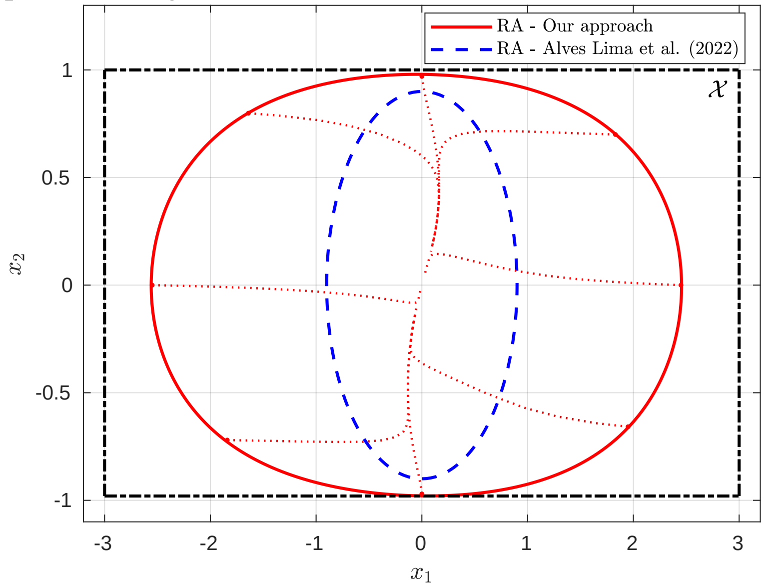
Fonte: The author.

Figure 12 – Region of attraction (RA) obtained inside $\mathcal{H} = \mathcal{X} \cap \mathcal{L}(\bar{u})$.



Fonte: The author.

Figure 13 – Comparison of regions of attraction (RA).



Fonte: The author.

Figure 12 presents the set $\mathcal{L}(\bar{u})$, polytope $\mathcal{X}$, the estimated region of attraction $\mathcal{E}$ of the closed-loop system, and the state trajectories with initial conditions inside $\mathcal{E}$. As expected, the domain of attraction is inside of the set $\mathcal{H} = \mathcal{X} \cap \mathcal{L}(\bar{u})$ and all trajectories starting inside $\mathcal{E}$ converge asymptotically to the equilibrium point $x = 0$. Figure 13 shows two level curves which corresponds to the estimated region of attraction obtained here, with a polynomial Lyapunov function, and in Alves Lima *et al.* (2022) with a quadratic Lyapunov function. As we can see, we obtain a much larger region of attraction.

5.5.2 Example 2 - Rational nonlinear system

Consider the uncertain rational nonlinear input saturated plant analysed in Example 5.4 of Azizi *et al.* (2018),

$$\begin{cases} \dot{x_1} = x_2, \\ \dot{x_2} = -\frac{b_0\delta_2 x_2}{M_o(1+\delta_1)} + \frac{2\delta_1 x_1 x_2^2}{(1+\delta_1)(1+x_1^2)} + \frac{g\delta_1 x_1}{l(1+\delta_1)} + \frac{sat(v(t))}{(1+\delta_1)}, \end{cases} \tag{5.39}$$

which corresponds to an inverted pendulum system. The DAR for the system is the same from Azizi *et al.* (2018),

$$\pi = \begin{bmatrix} \frac{x_1}{1+\delta_1} & \frac{x_2}{1+\delta_1} & \frac{x_1x_2\delta_1}{(1+\delta_1)(1+x_1^2)} & \frac{x_1\delta_1}{1+x_1^2} & \frac{x_1^2\delta_1}{1+x_1^2} & \frac{sat(v(t))}{1+\delta_1} \end{bmatrix}^\top,$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{g}{l}\delta_1 & \frac{b_0}{M_0}\delta_2 & 2x_2 & 0 & 0 & -\delta_1 \end{bmatrix}, A_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Upsilon_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \delta_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \Upsilon_2 = \begin{bmatrix} -(1+\delta_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & -(1+\delta_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & -(1+\delta_1) & x_2 & 0 & 0 \\ 0 & 0 & 0 & -1 & -x_1 & 0 \\ 0 & 0 & 0 & x_1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(1+\delta_1) \end{bmatrix}, \Upsilon_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

In this example, we consider a rational Lyapunov function, with vector ζ and matrices Ω_1, Ω_2 given by

$$\zeta = \begin{bmatrix} \frac{x_1}{1+x_1} & \frac{x_2}{1+x_1} & \frac{x_1^2 x_2}{(1+x_1)(1+x_1^2)} & \frac{x_1^2}{1+x_1^2} & \frac{x_1^3}{1+x_1^2} \end{bmatrix}^\top,$$

$$\Omega_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ x_1 & 0 \\ 0 & 0 \end{bmatrix}, \ \Omega_2 = \begin{bmatrix} -(1+x_1) & 0 & 0 & 0 & 0 \\ 0 & -(1+x_1) & 0 & 0 & 0 \\ 0 & 0 & -(1+x_1) & x_2 & 0 \\ 0 & 0 & 0 & -1 & -x_1 \\ 0 & 0 & 0 & x_1 & -1 \end{bmatrix},$$

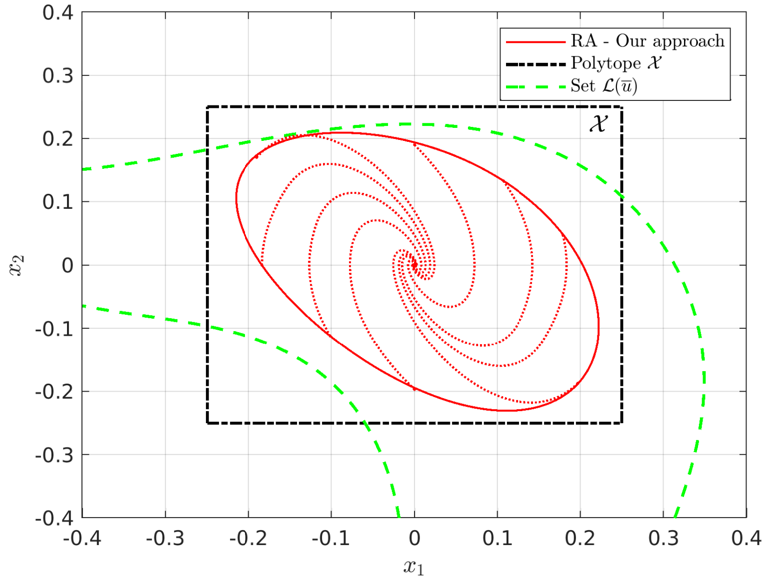
leading to

$$\bar{\Omega}_{2a} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \bar{\Omega}_{1a} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} E_1 + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} E_2,$$

where $E_1 = [1 \ \ 0]$, $E_2 = [0 \ \ 1]$. Moreover, we consider affine matrices $P_i(x,\delta) = P_{0i} + P_{1i}x_1 + P_{2i}x_2 + P_{3i}\delta_1 + P_{4i}\delta_2, \ i = \{1,2,3\}$. Then, in this case, we have a Lyapunov function such that the numerator is a polynomial of degree 5 in x and degree 3 in δ, and the denominator is a polynomial of degree 4 in x and degree 2 in δ. First, we intend to provide comparisons, in terms of size of the estimated domain of attraction, with some papers from the literature such as Azizi *et al.* (2018) and Alves Lima *et al.* (2022). Since these papers have designed static state feedback gains for stabilization of system (5.39), we have to consider $y(t) = x(t)$, leading to $C_1 = I_2, \ C_2 = 0_{2\times 5}$. In our approach, we consider the same saturation limit $\bar{u} = 0.25$ from Azizi *et al.* (2018), a bigger polytope given by $|x_1| \leq 0.25$, $|x_2| \leq 0.25$, and uncertainties $|\delta_1| \leq 0.1$, $|\delta_2| \leq 0.99$. By running Algorithms 4-5, we obtain the stabilizing gain $K = \begin{bmatrix} -2.0824 & -3.2234 \end{bmatrix}$, after 45 iterations in

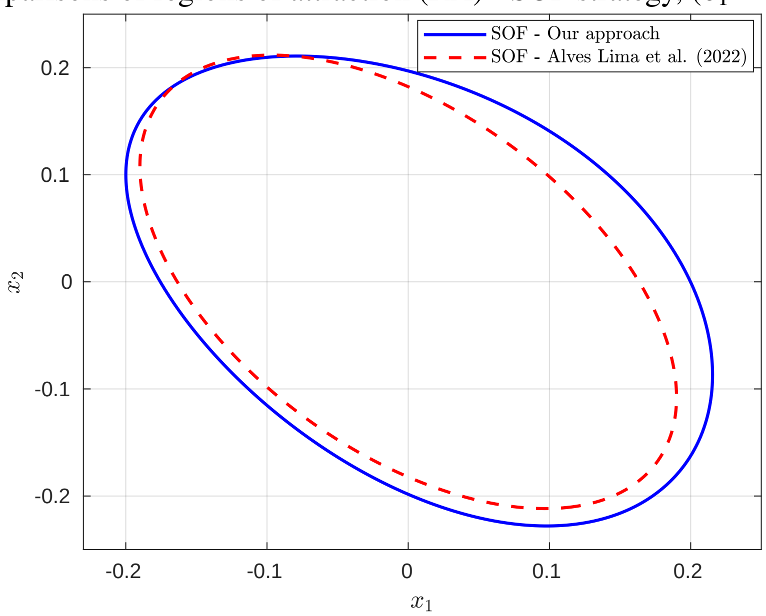
16.33min in Algorithm 4 and 10 iterations in 4.80min in Algorithm 5, which leads to results presented in Figure 14. The estimated region of attraction of the closed-loop system is inside the set $\mathcal{H} = \mathcal{X} \cap \mathcal{L}(\bar{u})$. Moreover, all state trajectories starting inside the region of attraction converges asymptotically to the equilibrium point.

Figure 14 – State trajectories and region of attraction (RA) inside $\mathcal{H} = \mathcal{X} \cap \mathcal{L}(\bar{u})$, ($\delta_1 = \delta_2 = 0$).



Fonte: The author.

Figure 15 – Comparisons of regions of attraction (RA) - SOF strategy, ($\delta_1 = \delta_2 = 0$).



Fonte: The author.

We have also employed a static output feedback design to compare with the SOF obtained by Alves Lima *et al.* (2022), which considers $y(t) = [1\ 1]x(t)$, leading to $C_1 = [1\ \ 1]$, $C_2 =$

$0_{1\times 5}$. Then, applying Algorithms 4-5 with the same parameters considered in the SSF case, we obtain a stabilizing gain $K = -2.0182$, after 133 iterations in 47.5617min in Algorithm 4 and 2 iterations in 53.3029s in Algorithm 5. Figure 15 presents the estimated regions of attraction, with a SOF control, from Alves Lima *et al.* (2022) and from our work. As we can see, our approach leads to a better estimated region of attraction.

Table 3 presents a comparative summary of the area of the regions of attraction $\mathcal{E}$ considering both the SSF and SOF design cases.

Table 3 – Comparisons in terms of area of $\mathcal{E}$.

Proposed - SSF & SOF	Alves Lima *et al.* (2022) - SSF & SOF	Azizi *et al.* (2018) - SSF
0.1338 & 0.1296	0.1054 & 0.1088	0.1046

Fonte: The author.

6 CONCLUSIONS

This work proposes new conditions for the design of stabilizing static output feedback gains for three classes of dynamical systems, such as uncertain LTI systems, uncertain LPV systems, and uncertain input saturated nonlinear systems. We use the definition of strict QSR-dissipativity, which is known as a necessary and sufficient condition for SOF stabilizability (MADEIRA, 2022), to formulate sufficient LMI conditions for the SOF design.

In chapter 3, a new strategy for static output feedback stabilization of uncertain linear systems with a lower bound on the decay rate has been proposed. An iterative algorithm recently developed is used to obtain LMI conditions for the SOF design. Some numerical examples from the literature have demonstrated the effectiveness of the proposed strategy that has led to less conservative results in comparison with other strategies from the literature as Felipe and Oliveira (2021), Agulhari *et al.* (2012).

In chapter 4, an approach for gain-scheduled SOF stabilization of LPV systems with $\mathcal{L}_2$-gain performance is proposed. A differential algebraic representation for rational LPV systems and Finsler's Lemma have been applied to formulate LMI conditions for dissipativity analysis and gain-scheduled SOF design. The main contribution is that the system matrices can present polynomial or rational dependence, not only affine as it is common in the literature. In addition, differently from some strategies in the field, the formulated solution does not need to solve a static state feedback problem as an initial stage to design the gain-scheduled static output feedback. We successfully applied the strategy in some numerical examples that present unmeasured uncertainties and rational dependence on the time-varying parameter, providing comparisons, in terms of a minimum bound for the $\mathcal{L}_2$-gain, with some works from the literature.

In chapter 5, new conditions for the design of stabilizing static output feedback gains for uncertain input saturated nonlinear systems are proposed. The nonlinear system can present polynomial and rational dependence on (x,δ). As in the LPV case, a differential algebraic representation for rational nonlinear systems, Finsler's Lemma, and also a generalized sector condition have been applied to formulate LMI conditions for SOF design. It is important to highlight that, in this approach, we consider rational Lyapunov functions which are known as more generic functions that can lead to less conservative conditions. A recent iterative algorithm, from the literature, is used to find the stabilizing feedback gains that maximize the closed-loop region of attraction. The effectiveness of the approach is demonstrated by comparisons, in terms of the obtained estimations on the closed-loop region of attraction, with some nonlinear systems from the literature.

Future research includes the development of a procedure to obtain the least conservative DAR. Moreover, in the LPV case, we intend to use parameter-dependent Lyapunov functions with the objective of decreasing the conservatism of conditions. Finally, the application of equilibrium independent dissipativity for feedback stabilization of nonzero equilibria of rational nonlinear systems using SDP strategies is scheduled.

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