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SERGIO ALVAREZ ARAUJO CORREIA

SEMIALGEBRAIC LIPSCHITZ EQUIVALENCE OF POLYNOMIAL FUNCTIONS

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SERGIO ALVAREZ ARAUJO CORREIA

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Thesis submitted to the Graduate Program of the Department of Mathematics at Universidade Federal do Ceará in partial fulfillment of the requirements for the Ph.D. degree in Mathematics. Area of expertise: Singularities

Advisor: Prof. Dr. Alexandre César Gurgel Fernandes.

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To my wife and my daughter.

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"If you can't solve a problem, then there is an easier problem you can solve: find it."

George Pólya, How to Solve It: A New Aspect of Mathematical Method

RESUMO

Mostramos como determinar, sob condições bastante gerais, se dois polinômios β-quasi-homogêneos em duas variáveis, com coeficientes reais, dados são $\mathcal{R}$-semialgebricamente Lipschitz equivalentes. Seguindo a estratégia usada em BIRBRAIR, FERNANDES, and PANAZZOLO (2009), mostramos primeiro como determinar se duas funções polinomiais reais de uma variável dadas são Lipschitz equivalentes comparando os valores e também as multiplicidades das funções polinomiais dadas nos seus pontos críticos, e então mostramos como reduzir, sob condições bastante gerais, o problema da $\mathcal{R}$-equivalência Lipschitz semi-algébrica de polinômios β-quasihomogêneos em duas variáveis, com coeficientes reais, ao problema da equivalência Lipschitz de funções polinomiais reais de uma variável. Como aplicação dos nossos resultados principais sobre $\mathcal{R}$-equivalência Lipschitz semialgébrica de polinômios β-quasihomogêneos em duas variáveis, investigamos as propriedades, no contexto da $\mathcal{R}$-equivalência Lipschitz semialgébrica, de uma família específica de polinômios quasihomogêneos, que foi usada antes em HENRY and PARUSIŃSKI (2004), para mostrar que a equivalência bi-Lipschitz de germes de funções analíticas $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ admite moduli contínuo. Das nossas conclusões decorre que a $\mathcal{R}$-equivalência Lipschitz semi-algébrica de polinômios β-quasihomogêneos em duas variáveis também admite moduli contínuo.

Palavras-chave: $\mathcal{R}$-equivalência Lipschitz semialgébrica; polinômios quasihomogêneos; moduli contínuo.

ABSTRACT

We show how to determine, under fairly general conditions, whether two given β-quasihomogeneous polynomials in two variables, with real coefficients, are $\mathcal{R}$-semialgebraically Lipschitz equivalent. Following the strategy used in BIRBRAIR, FERNANDES, and PANAZZOLO (2009), we first show how to determine whether two given real polynomial functions of a single variable are Lipschitz equivalent by comparing the values and also the multiplicities of the given polynomial functions at their critical points, and then we show how to reduce, under fairly general conditions, the problem of $\mathcal{R}$-semialgebraic Lipschitz equivalence of β-quasihomogeneous polynomials in two variables, with real coefficients, to the problem of Lipschitz equivalence of real polynomial functions of a single variable. As an application of our main results on $\mathcal{R}$-semialgebraic Lipschitz equivalence of β-quasihomogeneous polynomials in two variables, we investigate the properties, in the context of $\mathcal{R}$-semialgebraic Lipschitz equivalence, of a specific family of quasihomogeneous polynomials, which has been used before in HENRY and PARUSIŃSKI (2004), to show that the bi-Lipschitz equivalence of analytic function germs $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ admits continuous moduli. As a byproduct, our conclusions show that the $\mathcal{R}$-semialgebraic Lipschitz equivalence of real β-quasihomogeneous polynomials in two variables also admits continuous moduli.

Keywords: $\mathcal{R}$-semialgebraic Lipschitz equivalence; quasihomogeneous polynomials; continuous moduli.

LIST OF FIGURES

Figure 1 – Lipschitz equivalence classes of $t^3 + pt + q$, for $p < 0$ 36
Figure 2 – Graph of the function ϕ such that $g \circ \phi = f$, obtained by plotting the zeros of the equation $g(y) - f(t) = 0$, for $f(t) = t^3 + t^2 + t$ and $g(t) = t^3 - t^2 + t$ 41
Figure 3 – Graph of the function ϕ such that $g \circ \phi = f$, for $f(t) = t^3 + t^2 + t$ and $g(t) = t^3 - t^2 + t$, along with the asymptotic line $y = t + 2/3$ 46
Figure 4 – Graph of the function α, obtained by plotting the zeros of the equation $g((t + 2/3) + y) - f(t) = 0$, for $f(t) = t^3 + t^2 + t$ and $g(t) = t^3 - t^2 + t$. 47

LIST OF TABLES

Table 1 – Normal forms for the Lipschitz equivalence of quadratic functions 33
Table 2 – Normal forms for the Lipschitz equivalence of cubic functions 37

CONTENTS

1	**INTRODUCTION**	12
2	**PRELIMINARIES**	15
2.1	**Multiplicity of an analytic function at a point of its domain**	15
2.2	**Basic notions of semialgebraic sets and mappings**	17
2.3	**Transformation of paths by Lipschitz maps**	24
3	**LIPSCHITZ EQUIVALENCE OF POLYNOMIAL FUNCTIONS OF A SINGLE VARIABLE**	26
3.1	**Lipschitz classification of polynomial functions of a single variable**	26
3.2	**Normal forms for the Lipschitz equivalence of nonconstant polynomials of degree ≤ 3**	32
3.3	**On the bi-Lipschitz transformation ϕ**	37
4	**$\mathcal{R}$-SEMIALGEBRAIC LIPSCHITZ EQUIVALENCE OF β-QUASIHOMOGENEOUS POLYNOMIALS**	48
4.1	**β-isomorphisms and the β-transform**	51
4.2	**The group of proto-transitions**	65
4.3	**β-transitions and the inverse β-transform**	70
4.4	**Shifting from proto-transitions to β-transitions**	79
4.5	**Henry and Parusiński's example revisited**	92
5	**CONCLUSION**	95
	REFERENCES	98

1 INTRODUCTION

In HENRY and PARUSIŃSKI (2003), Henry and Parusiński showed that the bi-Lipschitz classification of complex analytic function germs admits continuous moduli. This fact had not been observed before and interestingly it contrasts with the fact that the bi-Lipschitz equivalence of complex analytic set germs does not admit moduli, see MOSTOWSKI (1985). The moduli space of bi-Lipschitz equivalence of function germs is not yet completely understood but it is worth noting that recently Câmara and Ruas have made progress in the study of the moduli space of bi-Lipschitz equivalence of quasi-homogeneous function germs, in the complex case, see CÂMARA and RUAS (2020).

In HENRY and PARUSIŃSKI (2004), Henry and Parusiński showed that the bi-Lipschitz classification of real analytic function germs admits continuous moduli. Then, in BIRBRAIR, FERNANDES, and PANAZZOLO (2009), Birbrair, Fernandes and Panazzolo described the semialgebraic bi-Lipschitz moduli in the "simplest possible case" (as they have called it): quasihomogeneous polynomial functions defined on the Hölder triangle $T_\beta := \{(x, y) \in \mathbb{R}^2 : 0 \leq x, 0 \leq y \leq x^\beta\}$. Independently, the particular case of weighted homogeneous polynomial functions of two real variables has been considered by Koike and Parusiński in KOIKE and PARUSIŃSKI (2013).

In this thesis, we consider the problem of classifying β-quasihomogeneous polynomials in two variables with real coefficients modulo $\mathcal{R}$-semialgebraic Lipschitz equivalence. Here and throughout the text, β always denotes a rational number > 1. (We define β-quasihomogeneous polynomials and $\mathcal{R}$-semialgebraic Lipschitz equivalence in the introduction to Chapter 4.) Our main goal is to extend the results obtained in BIRBRAIR, FERNANDES, and PANAZZOLO (2009) for the classification of germs of functions defined on the Hölder triangle to germs of functions defined on the whole plane.

Following the strategy used in BIRBRAIR, FERNANDES, and PANAZZOLO (2009), we solve the problem of determining whether two given real polynomial functions of a single variable are Lipschitz equivalent (Lipschitz equivalence of real polynomial functions of a single variable is defined in the introduction to Chapter 3), and then we try to reduce the problem of determining whether two given β-quasihomogeneous polynomials, with real coefficients, in two variables are $\mathcal{R}$-semialgebraically Lipschitz equivalent to the problem of determining whether two given polynomial functions of a single variable are Lipschitz equivalent.

The Lipschitz equivalence problem for polynomial functions of a single variable is solved in Chapter 3 (see Theorem 3.1a, Theorem 3.1b, and Theorem 3.1c). Again, we follow the approach taken in BIRBRAIR, FERNANDES, and PANAZZOLO (2009), which consists in comparing the values and also the multiplicities of the given polynomial functions at their critical points. The reduction to the single variable case is carried out in Chapter 4, under fairly general conditions. Still following the approach taken in BIRBRAIR, FERNANDES, and PANAZZOLO (2009), we associate with each β-quasihomogeneous polynomial $F(X, Y) \in \mathbb{R}[X, Y]$ a pair of polynomial functions $f_+, f_- : \mathbb{R} \to \mathbb{R}$, called the *height functions* of F, which encode a great deal of information about the original polynomial. Then, we consider the following questions:

1. Suppose that two given β-quasihomogeneous polynomials $F, G \in \mathbb{R}[X, Y]$ of degree $d \geq 1$ are $\mathcal{R}$-semialgebraically Lipschitz equivalent. Is it possible to arrange their height functions in pairs of Lipschitz equivalent functions (i.e. either $f_+ \cong g_+$ and $f_- \cong g_-$, or $f_+ \cong g_-$ and $f_- \cong g_+$)?
2. Suppose that the height functions of two given β-quasihomogeneous polynomials $F, G \in \mathbb{R}[X, Y]$ of degree $d \geq 1$ can be arranged in pairs of Lipschitz equivalent functions. Are F and G $\mathcal{R}$-semialgebraically Lipschitz equivalent?

We show that if the zero sets of the polynomials F and G have points both on the right half-plane and on the left half-plane then the answer to the first question is yes (see Corollary 4.3 and Remark 4.7). Also, we obtain some fairly general conditions under which the answer to the second question is affirmative (see Theorem 4.2, Corollary 4.11, and Corollary 4.12). These are our main results on $\mathcal{R}$-semialgebraic Lipschitz equivalence of β-quasihomogeneous polynomials. These results, along with those on Lipschitz equivalence of polynomial functions of a single variable (namely, Theorem 3.1a, Theorem 3.1b, and Theorem 3.1c) enable us to determine, under fairly general conditions, whether two given β-quasihomogeneous polynomials are $\mathcal{R}$-semialgebraically Lipschitz equivalent.

In BIRBRAIR, FERNANDES, and PANAZZOLO (2009), the questions stated above were both answered affirmatively in the case where the equivalence is restricted to the Hölder triangle T_β, assuming that the given β-quasihomogeneous polynomials vanish identically on ∂T_β and do not vanish at the interior points of T_β. Here, we generalize the methods devised by Lev Birbrair, Alexandre Fernandes, and Daniel Panazzolo. This generalization leads to the theory of β-transforms and inverse β-transforms developed in Sections 4.1, 4.2, 4.3, and 4.4. Our main results on $\mathcal{R}$-semialgebraic Lipschitz equivalence of β-quasihomogeneous polynomials in two variables are proved using this theory.

We provide some interesting applications of the main results obtained in this thesis. In Section 3.2, we apply our main results on Lipschitz equivalence of real polynomial functions of a single variable to construct a set of normal forms for the Lipschitz equivalence of polynomial functions of degree d, for $d = 1, 2, 3$. In Section 4.5, we apply our main results on $\mathcal{R}$-semialgebraic Lipschitz equivalence of β-quasihomogeneous polynomials in two variables to investigate the properties, in the context of $\mathcal{R}$-semialgebraic Lipschitz equivalence, of a specific family of quasihomogeneous polynomials, which has been used before in HENRY and PARUSIŃSKI (2004) to show that the bi-Lipschitz equivalence of analytic function germs $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ admits continuous moduli. As a byproduct, our conclusions show that the $\mathcal{R}$-semialgebraic Lipschitz equivalence of real β-quasihomogeneous polynomials in two variables also admits continuous moduli.

2 PRELIMINARIES

In this chapter, we discuss some basic concepts and results that are used in the subsequent chapters, where the research problem of this thesis is addressed.

Overview of this chapter. In Section 2.1, we introduce the concept of multiplicity of a nonconstant analytic function $f\colon I \to \mathbb{R}$ at a point $t_0 \in I$ that is used in this thesis. The main result of this section is Proposition 2.2, which provides a local form for nonconstant analytic functions $f\colon I \to \mathbb{R}$ around a point $t_0 \in I$, in terms of the multiplicity of f at t_0. This result is used in Chapter 3 (more precisely, in the proof of Lemma 3.3).

In Section 2.2, we review the fundamentals of semialgebraic sets and mappings, and prove some basic results that are used throughout this thesis. The main goals of this section are Proposition 2.4, which says that if a real semialgebraic function of a single real variable is differentiable then its derivative is also semialgebraic, and also Corollary 2.4 and Corollary 2.5, according to which one-sided limits and one-sided derivatives of real semialgebraic functions of a single variable are well-defined in the extended real line $\overline{\mathbb{R}} = [-\infty, +\infty]$, at the points where it makes sense to define them. These results are essential in providing a solid foundation for the next chapters.

In Section 2.3, we discuss the effect of a semialgebraic (bi-)Lipschitz transformation on the initial velocity of a continuous semialgebraic path in $\mathbb{R}^2$. Virtually all the results of this section are used in the subsequent chapters (more precisely, in Chapter 4).

2.1 Multiplicity of an analytic function at a point of its domain

A function $f\colon I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an open interval, is *analytic* if for each $t_0 \in I$, there exist $\delta > 0$ such that $(t_0 - \delta, t_0 + \delta) \subseteq I$ and a convergent power series[1] $\sum_{k=0}^{\infty} c_k(t - t_0)^k$ such that $f(t) = \sum_{k=0}^{\infty} c_k(t - t_0)^k$ for all $t \in (t_0 - \delta, t_0 + \delta)$. Let us recall the most basic facts about analytic functions:

(a) If $f, g\colon I \to \mathbb{R}$ are analytic functions, and $c \in \mathbb{R}$, then $f \pm g$, cf, and $f \cdot g$ are analytic functions. Also, if g does not vanish on I, then f/g is analytic.

(b) If $f\colon I \to \mathbb{R}$ and $g\colon J \to \mathbb{R}$ are analytic functions such that $f(I) \subseteq J$ then the composition $g \circ f\colon I \to \mathbb{R}$ is analytic.

(c) Every analytic function $f\colon I \to \mathbb{R}$ has derivatives of all orders, and for each $t_0 \in I$, the power series representation of f around t_0 is precisely its Taylor series around t_0:
$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(t_0)}{k!} \cdot (t - t_0)^k, \quad \text{for all } t \in (t_0 - \delta, t_0 + \delta).$$

[1] A power series $\sum_{k=0}^{\infty} c_k(t - t_0)^k$ is *convergent* if there exists $\eta > 0$ such that, for each $\tau \in (t_0 - \eta, t_0 + \eta)$, the infinite series $\sum_{k=0}^{\infty} c_k(\tau - t_0)^k$ is a convergent series of real numbers.

(d) If $f\colon I \to \mathbb{R}$ is an analytic function then, for each $k \geq 1$, the k-th derivative $f^{(k)}\colon I \to \mathbb{R}$ is analytic.

(e) If $\sum_{k=0}^{\infty} c_k(t-t_0)^k$ is a convergent power series, then $g\colon (t_0-\delta, t_0+\delta) \to \mathbb{R}$ defined by $g(t) := \sum_{k=0}^{\infty} c_k(t-t_0)^k$ is analytic. Moreover, the derivative g' can be found by differentiating the series term-by-term:

$$g'(t) = \sum_{k=0}^{\infty}(k+1)c_{k+1}(t-t_0)^k, \quad \text{for all } t \in (t_0-\delta, t_0+\delta).$$

(f) Let $f, g\colon I \to \mathbb{R}$ be analytic functions. If the set $\{t \in I : f(t) = g(t)\}$ has a limit point in I, then $f(t) = g(t)$ for all $t \in I$.

We refer to (LIMA, 1999, Chapter X) and (THOMSON, BRUCKNER, and BRUCKNER, 2008, Chapter 10) for a thorough introductory treatment of the fundamentals of the theory of real analytic functions, with proofs of the facts listed above. Now, we turn to the main topic of this section: the concept of multiplicity of an analytic function at a point of its domain.

Proposition 2.1. *If $f\colon I \to \mathbb{R}$ is a nonconstant analytic function defined on an open interval $I \subseteq \mathbb{R}$ then, for each $t_0 \in I$, there exist an integer $r \geq 1$ and an analytic function $g\colon (t_0-\delta, t_0+\delta) \to \mathbb{R}$ such that*

$$f(t) - f(t_0) = (t-t_0)^r \cdot g(t), \quad \textit{for all } t \in (t_0-\delta, t_0+\delta),$$

and $g(t_0) \neq 0$.

Proof. Let $f\colon I \to \mathbb{R}$ be a nonconstant analytic function and let t_0 be an arbitrary point of I. Since f is analytic, there exists a convergent power series $\sum_{k=0}^{\infty} c_k(t-t_0)^k$ such that $f(t) = \sum_{k=0}^{\infty} c_k(t-t_0)^k$ for all $t \in (t_0-\delta, t_0+\delta)$. Let $r := \min\{k \geq 1 : c_k \neq 0\}$, which is well-defined because $f(t) - f(t_0)$ is not identically zero. Then $f(t) - f(t_0) = (t-t_0)^r \cdot \sum_{k=0}^{\infty} c_{r+k}(t-t_0)^k$, for all $t \in (t_0-\delta, t_0+\delta)$. Now, we only need to note that the function $g\colon (t_0-\delta, t_0+\delta) \to \mathbb{R}$ given by $g(t) := \sum_{k=0}^{\infty} c_{r+k}(t-t_0)^k$ is analytic and that $g(t_0) = c_r$, so $g(t_0) \neq 0$. ■

Clearly, the integer $r \geq 1$ in the statement of Proposition 2.1 is uniquely determined by f and t_0. We call it the *multiplicity of f at t_0*.

Remark 2.1. *A straightforward computation shows that t_0 is a critical point of f if and only if the multiplicity of f at t_0 is ≥ 2.*

Proposition 2.2. *Let $f\colon I \to \mathbb{R}$ be a nonconstant analytic function defined on an open interval $I \subseteq \mathbb{R}$ and let $t_0 \in I$. If f has multiplicity k at t_0 then there exist an increasing* *analytic diffeomorphism* $u\colon I_0 \to (-\epsilon, \epsilon)$*, defined on an open subinterval* $I_0 \subseteq I$ *containing* t_0*, with* $u(t_0) = 0$*, and a constant* $\rho \in \mathbb{R} \setminus \{0\}$ *such that*

$$f \circ u^{-1}(t) = f(t_0) + \rho t^k, \quad \textit{for } |t| < \epsilon.$$

Proof. Suppose that f has multiplicity k at t_0. Then, by Proposition 2.1, there exists $\delta > 0$ such that $(t_0 - \delta, t_0 + \delta) \subseteq I$ and

$$f(t) - f(t_0) = (t - t_0)^k \cdot g(t), \quad \text{for all } t \in (t_0 - \delta, t_0 + \delta),$$

where $g\colon (t_0 - \delta, t_0 + \delta) \to \mathbb{R}$ is an analytic function such that $g(t_0) \neq 0$. Let us rewrite the above equation as

$$f(t) - f(t_0) = \rho \cdot (t - t_0)^k \cdot \hat{g}(t), \quad \text{for all } t \in (t_0 - \delta, t_0 + \delta),$$

where $\rho := g(t_0)$ and $\hat{g}(t) := g(t)/g(t_0)$. Since $\hat{g}(t_0) = 1$, we can assume (shrinking $\delta > 0$ if necessary) that $\hat{g}(t) > 0$ for all $t \in (t_0 - \delta, t_0 + \delta)$. Then, we have

$$f(t) - f(t_0) = \rho \cdot u^k(t),$$

where $u\colon (t_0 - \delta, t_0 + \delta) \to \mathbb{R}$, given by $u(t) = (t - t_0) \cdot \hat{g}^{\frac{1}{k}}(t)$, is a well-defined analytic function. Since $u'(t_0) = \hat{g}^{\frac{1}{k}}(t_0) = 1 > 0$, we can assume (again, shrinking $\delta > 0$ if necessary) that $u'(t) > 0$ for all $t \in (t_0 - \delta, t_0 + \delta)$. Thus, u is an increasing analytic diffeomorphism from $(t_0 - \delta, t_0 + \delta)$ to an open interval J containing $u(t_0) = 0$. Take an arbitrary $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subseteq J$. Then $I_0 := u^{-1}((-\epsilon, \epsilon))$ is an open subinterval of I containing t_0 and $u\colon I_0 \to (-\epsilon, \epsilon)$ is an increasing analytic diffeomorphism such that $u(t_0) = 0$ and

$$f \circ u^{-1}(t) = f(t_0) + \rho t^k, \quad \text{for } |t| < \epsilon.$$

■

2.2 Basic notions of semialgebraic sets and mappings

In this section, we review the fundamentals of semialgebraic sets and mappings, and prove some basic results that are used throughout this thesis. For the most part, the material presented in this section is drawn from COSTE (2002). In preparing this section, we have also found useful the article NEYMAN (2003). For an extensive treatment of semialgebraic sets and mappings, see COSTE (2002), BENEDETTI and RISLER (1990), or BOCHNAK, COSTE, and ROY (1998).

Let $\mathcal{SA}_n$ be the smallest collection of subsets of $\mathbb{R}^n$ satisfying the following conditions:

i. For each $P \in \mathbb{R}[X_1, \dots, X_n]$, the sets $\{x \in \mathbb{R}^n : P(x) = 0\}$ and $\{x \in \mathbb{R}^n : P(x) > 0\}$ belong to $\mathcal{SA}_n$.
ii. For any $A, B \in \mathcal{SA}_n$, each of the sets $A \cup B$, $A \cap B$, and $\mathbb{R}^n \setminus A$ belong to $\mathcal{SA}_n$.

The members of $\mathcal{SA}_n$ are called *semialgebraic subsets of* $\mathbb{R}^n$.

Proposition 2.3. *Every semialgebraic subset of* $\mathbb{R}^n$ *is the union of finitely many semialgebraic subsets of the form*

$$\{x \in \mathbb{R}^n : P(x) = 0 \text{ and } Q_1(x) > 0 \text{ and } \cdots \text{ and } Q_k(x) > 0\},$$

where $k \in \mathbb{N}_0$ *and* $P, Q_1, \dots, Q_k \in \mathbb{R}[X_1, \dots, X_n]$.

Proof. Clearly, the class $\mathcal{C}$ of all finite unions of such subsets satisfies properties (i) and (ii) above, so $\mathcal{SA}_n \subseteq \mathcal{C}$. It is also clear that every member of $\mathcal{C}$ is a semialgebraic set, so we also have $\mathcal{C} \subseteq \mathcal{SA}_n$. ■

Examples.

(a) The semialgebraic subsets of $\mathbb{R}$ are the finite unions of intervals (the empty set and the singletons are regarded as degenerate intervals).

(b) Every algebraic subset of $\mathbb{R}^n$ (i.e. a set defined as the zero set of a system of polynomial equations) is a semialgebraic set.

(c) Let $F \colon \mathbb{R}^m \to \mathbb{R}^n$ be a polynomial mapping: $F = (F_1, \dots, F_n)$, where $F_i \in \mathbb{R}[X_1, \dots, X_m]$, for $i = 1, \dots, n$. If A is a semialgebraic subset of $\mathbb{R}^n$, then $F^{-1}(A)$ is a semialgebraic subset of $\mathbb{R}^m$.

(d) If $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ are semialgebraic then $A \times B$ is a semialgebraic subset of $\mathbb{R}^{m+n}$.

Theorem 2.1 (Tarski-Seidenberg)**.** *Let* A *be a semialgebraic subset of* $\mathbb{R}^{n+1}$ *and let* $\pi \colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ *be the projection on the first* n *coordinates. Then* $\pi(A)$ *is a semialgebraic subset of* $\mathbb{R}^n$.

Proof. See (COSTE, 2002, Theorem 2.3), (BOCHNAK, COSTE, and ROY, 1998, Theorem 2.2.1) or (BENEDETTI and RISLER, 1990, Theorem 2.3.4). ■

Corollary 2.1. *If* A *is a semialgebraic subset of* $\mathbb{R}^{n+k}$*, then its image by the projection on the space of the first* n *coordinates is a semialgebraic subset of* $\mathbb{R}^n$.

Corollary 2.2. *If* A *is a semialgebraic subset of* $\mathbb{R}^m$ *and* $F \colon \mathbb{R}^m \to \mathbb{R}^n$ *is a polynomial mapping, then the direct image* $F(A)$ *is a semialgebraic subset of* $\mathbb{R}^n$.

Often, the most convenient way of showing that a given subset $A \subseteq \mathbb{R}^n$ is semialgebraic is to verify that A can be defined by a formula of the *language of ordered fields with parameters in* $\mathbb{R}$. Let us denote this language by $\mathcal{L}$. A formula of $\mathcal{L}$ is obtained by the following rules:

1. For each $P \in \mathbb{R}[X_1, \ldots, X_n]$, the formulas $P = 0$ and $P > 0$ belong to $\mathcal{L}$.
2. If Φ and Ψ are formulas of $\mathcal{L}$, then so are the formulas Φ and Ψ, Φ or Ψ, and not Φ (formally denoted by $\Phi \wedge \Psi$, $\Phi \vee \Psi$, and $\neg\Phi$, respectively).
3. If Φ is a formula of $\mathcal{L}$ and X is a variable ranging over $\mathbb{R}$, then so are the formulas $\exists X\, \Phi$ and $\forall X\, \Phi$.

Clearly, for each semialgebraic subset $A \subseteq \mathbb{R}^n$, there exists a formula $\Phi(X_1, \ldots, X_n) \in \mathcal{L}$ such that

$$A = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \Phi(x_1, \ldots, x_n)\}.$$

The next result shows that, conversely, every set of this form is semialgebraic.

Theorem 2.2 (Tarski-Seidenberg — logical formulation)**.** *If* $\Phi(X_1, \ldots, X_n)$ *is a formula of* $\mathcal{L}$*, then the set* $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \Phi(x_1, \ldots, x_n)\}$ *is semialgebraic.*

Proof. See (COSTE, 2002, Theorem 2.6), (BOCHNAK, COSTE, and ROY, 1998, Proposition 2.2.4) or (BENEDETTI and RISLER, 1990, Theorem 2.6). ∎

Example. If A is a semialgebraic subset of $\mathbb{R}^n$, then $\overline{A}$ (topological closure of A) is semialgebraic. By the logical formulation of Tarski-Seidenberg's Theorem, it suffices to note that

$$\overline{A} = \{x \in \mathbb{R}^n : \forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \Rightarrow \exists y \in \mathbb{R}^n, y \in A \text{ and } \|x - y\|^2 < \varepsilon^2\}.$$

Let $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ be semialgebraic sets. A mapping $f\colon A \to B$ is said to be *semialgebraic* if its graph

$$\Gamma_f := \{(x, y) \in A \times B : y = f(x)\}$$

is a semialgebraic subset of $\mathbb{R}^m \times \mathbb{R}^n$.

Examples.

(a) If $f\colon A \to B$ is a polynomial mapping (all its coordinates are polynomial), then it is semialgebraic.

(b) If $f\colon A \to B$ is a regular rational mapping (all its coordinates are rational fractions whose denominators do not vanish on A), then it is semialgebraic.

(c) If $f\colon A \to \mathbb{R}$ is a semialgebraic function, then $|f|$ is semialgebraic.

(d) If $f\colon A \to \mathbb{R}$ is semialgebraic and $f \geq 0$, then $\sqrt{f}$ is a semialgebraic function.

(e) If $A \subseteq \mathbb{R}^n$, is a nonempty semialgebraic subset, then the function

$$\begin{aligned} \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto d(x, A) = \inf\{\|x - y\| : y \in A\} \end{aligned}$$

is continuous semialgebraic.

From Tarski-Seidenberg Theorem, we can obtain some important properties of semialgebraic mappings.

Corollary 2.3.

i. The direct image and the inverse image of a semialgebraic set by a semialgebraic mapping are semialgebraic.

ii. The composition of two semialgebraic mappings is semialgebraic.

iii. The semialgebraic functions from A to $\mathbb{R}$ form a ring.

Proposition 2.4. *Let $A \subseteq \mathbb{R}$ be semialgebraic open set and let $f\colon A \to \mathbb{R}$ be a semialgebraic function. If f is differentiable, then the derivative $f'\colon A \to \mathbb{R}$ is semialgebraic.*

Proof. By definition,

$$f'(x) := \lim_{w \to x} \frac{f(w) - f(x)}{w - x},$$

so $f'(x)$ is the only real number y satisfying the following condition:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall w \quad 0 < |w - x| < \delta \Rightarrow \left| \frac{f(w) - f(x)}{w - x} - y \right| < \epsilon$$

Thus, the graph of f' is the set

$$\Gamma_{f'} := \Big\{ (x, y) \in \mathbb{R}^2 : \\ \forall \epsilon \left(\epsilon > 0 \Rightarrow \exists \delta \left(\delta > 0 \wedge \forall w \left(0 < |w - x| < \delta \Rightarrow \left| \frac{f(w) - f(x)}{w - x} - y \right| < \epsilon \right) \right) \right) \Big\}$$

The inequality $0 < |w - x| < \delta$ can be rewritten as a formula $\Phi(w, x) \in \mathcal{L}$:

$$\Phi(w, x) : (\delta - w + x > 0) \wedge (\delta + w - x > 0) \wedge \neg(w - x = 0)$$

We show that the inequality $\left|\frac{f(w)-f(x)}{w-x} - y\right| < \epsilon$ can also be rewritten as a formula of $\mathcal{L}$. Let $C := \{(w,x) \in A \times A : w \neq x\}$, which is a semialgebraic subset of $\mathbb{R}^2$, and let $F\colon C \times \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$F(w,x,y,\epsilon) := \left|\frac{f(w)-f(x)}{w-x} - y\right| - \epsilon.$$

Since f is semialgebraic, so is F. Then the set

$$F^{-1}((-\infty,0)) = \left\{(w,x,y,\epsilon) \in C \times \mathbb{R}^2 : \left|\frac{f(w)-f(x)}{w-x} - y\right| < \epsilon\right\}$$

is semialgebraic, and hence there exists a formula $\Psi(w,x,y,\epsilon) \in \mathcal{L}$ such that

$$\forall (w,x,y,\epsilon) \in C \times \mathbb{R}^2, \left|\frac{f(w)-f(x)}{w-x} - y\right| < \epsilon \Leftrightarrow \Psi(w,x,y,\epsilon).$$

Thus, $\Gamma_{f'}$ is defined by the following formula:

$$\forall \epsilon \, (\epsilon > 0 \Rightarrow \exists \delta \, (\delta > 0 \wedge \forall w \, (\Phi(w,x) < \delta \Rightarrow \Psi(w,x,y,\epsilon))))$$

Clearly, this formula can be rewritten as a formula of $\mathcal{L}$, so by Theorem 2.2, it follows that $\Gamma_{f'}$ is a semialgebraic set, and therefore f' is a semialgebraic function. ■

Proposition 2.5. *Let U be a semialgebraic open subset of $\mathbb{R}^n$, and let $f\colon U \to \mathbb{R}$ be a semialgebraic function. If f admits a partial derivative $\partial f/\partial x_i$ on U, then this derivative is semialgebraic.*

Proof. This result can be proved by an argument similar to the one used in Proposition 2.4: from the definition of the partial derivative $\partial f/\partial x_i$, we obtain a formula of $\mathcal{L}$ that describes its graph; then, by Theorem 2.2, it follows that the graph $\Gamma_{\partial f/\partial x_i}$ is a semialgebraic set, and therefore $\partial f/\partial x_i$ is a semialgebraic function. ■

Now, we set out to establish an important fact about semialgebraic functions that is tacitly used several times in this thesis, namely, that "semialgebraic functions of a single variable always have (possibly infinite) one-sided limits" (Corollary 2.4). As a consequence, we obtain another important fact that is also tacitly used several times in this thesis, namely, that "semialgebraic functions of a single variable always have (possibly infinite) one-sided derivatives" (Corollary 2.5). In order to obtain these results, we prove that "semialgebraic functions of a single variable are monotone on a small interval ending at any left limit point of its domain and also on a small interval starting at any right limit point of its domain " (Proposition 2.7). Before we can prove these facts, we need to show that the graph of any semialgebraic function is contained in the zero set of a nonzero polynomial (Proposition 2.6) — this is our starting point. The proofs of Proposition 2.6

and of Proposition 2.7 presented here, are based on the proofs given for these results in NEYMAN (2003).

Proposition 2.6. *For every semialgebraic map* $f\colon A \to \mathbb{R}^n$*,* $A \subseteq \mathbb{R}^m$ *semialgebraic, there exists a nonzero polynomial* $P \in \mathbb{R}[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$ *such that* $P(x, f(x)) = 0$ *for all* $x \in A$*.*

Proof. Let $f\colon A \to \mathbb{R}^n$ be a semialgebraic map. By Proposition 2.3, the graph of f is the union of finitely many nonempty sets G_i, of the form

$$G_i = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n : f_i(x, y) = 0 \text{ and } g_{ij}(x, y) > 0, \, j = 1, \ldots, k_i\},$$

with $f_i, g_{ij} \in \mathbb{R}[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$. Since the graph of a function cannot contain an open set, each one of the polynomials f_i is not identically zero, so $P = \prod_i f_i \in \mathbb{R}[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$ is a nonzero polynomial. Let us denote by Γ_f the graph of f. Since $\Gamma_f = \cup_i G_i \subseteq \cup_i f_i^{-1}(0) = P^{-1}(0)$, we have $P(x, f(x)) = 0$ for all $x \in A$. ■

Lemma 2.1. *Let* $A \subseteq \mathbb{R}$ *be a nonempty semialgebraic subset.*

i. If a *is a limit point of* $A \cap (a, +\infty)$*, then there exists* $\delta > 0$ *such that* $(a, a+\delta) \subseteq A$*.*

ii. If a *is a limit point of* $A \cap (-\infty, a)$*, then there exists* $\delta > 0$ *such that* $(a-\delta, a) \subseteq A$*.*

Proof. We prove only the first part of the lemma, the second part being analogous. Since A is a nonempty semialgebraic subset of $\mathbb{R}$, there exists a finite number $n \geq 1$ of disjoint intervals $I_1, \ldots, I_n$ such that $A = I_1 \cup \cdots \cup I_n$. (Some of these intervals may be actually singletons.) Suppose that a is a limit point of $A \cap (a, +\infty)$. Then, it is a limit point of $I_k \cap (a, +\infty)$, for a certain $k \in \{1, \ldots, n\}$. From this we can deduce that I_k is a nondegenerate interval, that a is less than the supremum of I_k and that a is greater than or equal to the infimum of I_k. Hence, there exists $\delta > 0$ such that $(a, a+\delta) \subseteq I_k \subseteq A$. ■

Proposition 2.7. *Let* $A \subseteq \mathbb{R}$ *be a nonempty semialgebraic subset, and let* $f\colon A \to \mathbb{R}$ *be a semialgebraic function.*

i. If a *is a limit point of* $A \cap (a, +\infty)$*, then there exists* $\delta > 0$ *such that* $(a, a+\delta) \subseteq A$ *and* f *is monotone on* $(a, a+\delta)$*.*

ii. If a *is a limit point of* $A \cap (-\infty, a)$*, then there exists* $\delta > 0$ *such that* $(a-\delta, a) \subseteq A$ *and* f *is monotone on* $(a-\delta, a)$*.*

Proof. As in the previous lemma, we only prove the first part, the second part being analogous. By Proposition 2.6, there exists a nonzero polynomial $P \in \mathbb{R}[X, Y]$ such that $P(x, f(x)) = 0$ for all $x \in A$. We prove the proposition by induction on $\deg_X P + \deg_Y P$, where $\deg_X P$ and $\deg_Y P$ stand for the degree of P with respect to the variables X and Y, respectively.

In the base case, $\deg_X P + \deg_Y P = 1$, we have $\deg_X P = 0$ or $\deg_Y P = 0$. Let us show that $\deg_Y P = 0$ is not possible. Suppose that $\deg_Y P = 0$. Then, we have

$P(x, y) = mx + q$, where $m, q \in \mathbb{R}$, $m \neq 0$. Since $P(x, f(x)) = 0$ for all $x \in A$, it follows that $A = \{-q/m\}$. But this contradicts our assumption that A has a limit point. Hence, we have $\deg_X P = 0$. Then $P(x, y) = my + q$, where $m, q \in \mathbb{R}$, $m \neq 0$. Since $P(x, f(x)) = 0$ for all $x \in A$, it follows that $f(x) = -q/m$ for all $x \in A$. Therefore, f is constant. By hypothesis, a is a limit point of $A \cap (a, +\infty)$, so by Lemma 2.1 there exists $\delta > 0$ such that $(a, a + \delta) \subseteq A$. Then it follows that $f|_{(a,a+\delta)}$ is constant, and therefore it is monotone.

Now we prove the induction step. For every choice of signs $\epsilon = (\epsilon_1, \epsilon_2) \in \{-1, 0, 1\}^2$, the set

$$A_\epsilon = \left\{ x \in A \cap (a, +\infty) : \text{sign}\, \frac{\partial P}{\partial X}(x, f(x)) = \epsilon_1 \;\text{ and }\; \text{sign}\, \frac{\partial P}{\partial Y}(x, f(x)) = \epsilon_2 \right\}$$

is semialgebraic. Also, note that $\cup_\epsilon A_\epsilon = A \cap (a, +\infty)$. Then, since a is a limit point of $A \cap (a, +\infty)$, it follows that a is a limit point of $A_{\overline{\epsilon}}$ for a certain $\overline{\epsilon} = (\overline{\epsilon}_1, \overline{\epsilon}_2) \in \{-1, 0, 1\}^2$. If $\overline{\epsilon}_1 = 0$ or $\overline{\epsilon}_2 = 0$, the existence of $\delta > 0$ such that $(a, a+\delta) \subseteq A$ and $f|_{(a,a+\delta)}$ is monotone follows from the induction hypothesis. (For example, if $\overline{\epsilon}_1 = 0$ then $\partial P/\partial X(x, f(x)) = 0$ for all $x \in A_{\overline{\epsilon}}$, and $\deg_X \partial P/\partial X + \deg_Y \partial P/\partial X < \deg_X P + \deg_Y P$.)

Now, suppose that $\overline{\epsilon}_1 \neq 0$ and $\overline{\epsilon}_2 \neq 0$. By Lemma 2.1, there exists $\delta > 0$ such that $(a, a + \delta) \subseteq A_{\overline{\epsilon}} \subseteq A$. We claim that the restriction $f|_{(a,a+\delta)}$ is decreasing if $\overline{\epsilon}_1 \cdot \overline{\epsilon}_2 > 0$ and it is increasing if $\overline{\epsilon}_1 \cdot \overline{\epsilon}_2 < 0$. We prove only that if $\overline{\epsilon}_1 > 0$ and $\overline{\epsilon}_2 > 0$ then $f|_{(a,a+\delta)}$ is decreasing, the other cases being analogous. So assume that $\overline{\epsilon}_1 > 0$ and $\overline{\epsilon}_2 > 0$. We begin by proving that for each point $x_0 \in (a, a + \delta)$ there exists $\theta > 0$ such that $(x_0 - \theta, x_0 + \theta) \subseteq (a, a+\delta)$ and $f|_{(x_0-\theta,x_0+\theta)}$ is decreasing. Fix an arbitrary point $x_0 \in (a, a+\delta)$. By hypothesis, $\partial P/\partial X(x_0, f(x_0)) > 0$ and $\partial P/\partial Y(x_0, f(x_0)) > 0$. Since the partial derivatives $\partial P/\partial X(x, y)$ and $\partial P/\partial Y(x, y)$ are continuous, there exist $\theta > 0$ and $\eta > 0$ such that $\partial P/\partial X(x, y) > 0$ and $\partial P/\partial Y(x, y) > 0$ for all $(x, y) \in [x_0 - \theta, x_0 + \theta] \times [f(x_0) - \eta, f(x_0) + \eta]$. Shrinking θ, if necessary, we can assume that $(x_0 - \theta, x_0 + \theta) \subseteq (a, a+\delta)$ and $f((x_0 - \theta, x_0 + \theta)) \subseteq (f(x_0) - \eta, f(x_0) + \eta)$. Now, given two points $x_1, x_2 \in (x_0 - \theta, x_0 + \theta)$, if $x_1 < x_2$ then $P(x_2, f(x_2)) = 0 = P(x_1, f(x_1)) < P(x_2, f(x_1))$, and hence $f(x_2) < f(x_1)$. This shows that $f|_{(x_0-\theta,x_0+\theta)}$ is decreasing. Now, we prove that $f|(a, a + \delta)$ is decreasing. Take any two points $x_1, x_2 \in (a, a + \delta)$, with $x_1 < x_2$. For each point $x \in [x_1, x_2]$, take $\theta_x > 0$ such that $(x - \theta_x, x + \theta_x) \subseteq (a, a + \delta)$ and $f|_{(x-\theta_x,x+\theta_x)}$ is decreasing. Then, take a partition $x_1 = t_0 < t_1 < \cdots < t_{n-1} < t_n = x_2$, whose norm is smaller than some Lebesgue number of the open cover $[x_1, x_2] \subseteq \cup_{x\in[x_1,x_2]}(x-\theta_x, x+\theta_x)$, so that for each $i \in \{1, \ldots, n\}$, there exists $x \in [x_1, x_2]$ for which $[t_{i-1}, t_i] \subseteq (x-\theta_x, x+\theta_x)$. Since $f|_{(x-\theta_x,x+\theta_x)}$ is decreasing for each $x \in [x_1, x_2]$, we have: $f(x_1) = f(t_0) > f(t_1) > \cdots > f(t_{n-1}) > f(t_n) = f(x_2)$. Therefore, $f|_{(a,a+\delta)}$ is decreasing. ∎

Corollary 2.4. *Let $A \subseteq \mathbb{R}$ be a nonempty semialgebraic subset, and let $f\colon A \to \mathbb{R}$ be a semialgebraic function.*

i. If a is a limit point of $A \cap (a, +\infty)$, then the right-hand limit $\lim_{x\to a^+} f(x)$ is well-defined in the extended real line $\overline{\mathbb{R}} = [-\infty, +\infty]$.

ii. If a is a limit point of $A \cap (-\infty, a)$, then the left-hand limit $\lim_{x\to a^-} f(x)$ is well-defined in the extended real line $\overline{\mathbb{R}} = [-\infty, +\infty]$.

Corollary 2.5. *Let $A \subseteq \mathbb{R}$ be a nonempty semialgebraic subset, and let $f\colon A \to \mathbb{R}$ be a semialgebraic function. For any point $a \in A$, we have:*

i. If a is a limit point of $A\cap(a, +\infty)$, then the right derivative $f'_+(a) := \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}$ is well-defined in the extended real line $\overline{\mathbb{R}} = [-\infty, +\infty]$.

ii. If a is a limit point of $A\cap(-\infty, a)$, then the left derivative $f'_-(a) := \lim_{x\to a^-} \frac{f(x)-f(a)}{x-a}$ is well-defined in the extended real line $\overline{\mathbb{R}} = [-\infty, +\infty]$.

2.3 Transformation of paths by Lipschitz maps

Let $\gamma\colon [0, \epsilon) \to \mathbb{R}^2$ be a continuous semialgebraic path such that $\gamma(0) = 0$. By Corollary 2.5, the right derivative $\gamma'_+(0) := \lim_{t\to 0^+} \frac{\gamma(t)}{t}$ is a well-defined element of $\overline{\mathbb{R}}^2$ (even without the assumption that γ is continuous). We call $\gamma'_+(0)$ the *initial velocity* of γ. Given a germ of semialgebraic Lipschitz map $\Phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, we can transform γ into another continuous semialgebraic path $\tilde{\gamma} := \Phi \circ \gamma$. In this section, we investigate the effect of such a transformation on the initial velocity $\gamma'_+(0)$.

Lemma 2.2. *Let $\Phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a germ of semialgebraic Lipschitz map, let $\gamma\colon [0, \epsilon) \to \mathbb{R}^2$ be a continuous semialgebraic path such that $\gamma(0) = 0$, and let $\tilde{\gamma}(t) := \Phi(\gamma(t))$. If γ has finite initial velocity then $\tilde{\gamma}$ also has finite initial velocity, that is, if $\lim_{t\to 0^+} |\gamma(t)/t| < \infty$ then $\lim_{t\to 0^+} |\tilde{\gamma}(t)/t| < \infty$.*

Proof. First, note that the initial velocity $\tilde{\gamma}'_+(0)$ is well-defined, because $\tilde{\gamma} = \Phi \circ \gamma$ is a semialgebraic path. Since Φ is Lipschitz and $\Phi(0) = 0$, there exists $K > 0$ such that $|\Phi(x, y)| \leq K\,|(x, y)|$. Then,

$$\left|\frac{\tilde{\gamma}(t)}{t}\right| = \left|\frac{\Phi(\gamma(t))}{t}\right| \leq K \left|\frac{\gamma(t)}{t}\right|,$$

which implies that $\tilde{\gamma}'_+(0)$ is finite, given that $\gamma'_+(0)$ is finite, by hypothesis. ∎

Lemma 2.3. *Let $\Phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a germ of semialgebraic Lipschitz map. Let $\gamma_1, \gamma_2\colon [0, \epsilon) \to \mathbb{R}^2$ be two continuous semialgebraic paths, with finite initial velocity, such that $\gamma_1(0) = \gamma_2(0) = 0$, and let $\tilde{\gamma}_i(t) := \Phi(\gamma_i(t))$, $i = 1, 2$. If $\gamma'_{1+}(0) = \gamma'_{2+}(0)$, then $\tilde{\gamma}'_{1+}(0) = \tilde{\gamma}'_{2+}(0)$.*

Proof. Let $K > 0$ be a Lipschitz constant for Φ. Then $|\tilde{\gamma}_1(t) - \tilde{\gamma}_2(t)| \leq K\,|\gamma_1(t) - \gamma_2(t)|$, and we have:

$$\left|\frac{\tilde{\gamma}_1(t)}{t} - \frac{\tilde{\gamma}_2(t)}{t}\right| \leq K\left|\frac{\gamma_1(t)}{t} - \frac{\gamma_2(t)}{t}\right|, \qquad \text{for } t > 0.$$

Also, from $\lim_{t\to 0^+} \gamma_i(t)/t = \gamma'_{i+}(0)$ and $\gamma'_{1+}(0) = \gamma'_{2+}(0)$, we see that

$$\lim_{t\to 0^+}\left|\frac{\gamma_1(t)}{t} - \frac{\gamma_2(t)}{t}\right| = 0,$$

since both γ_1 and γ_2 have finite initial velocity.

Then, by the Squeeze Theorem,

$$\lim_{t\to 0^+}\left|\frac{\tilde{\gamma}_1(t)}{t} - \frac{\tilde{\gamma}_2(t)}{t}\right| = 0.$$

Hence, $\tilde{\gamma}'_{1+}(0) = \tilde{\gamma}'_{2+}(0)$. ■

Corollary 2.6. *Let $\Phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a germ of semialgebraic Lipschitz map, let $\gamma_1, \gamma_2\colon [0, \epsilon) \to \mathbb{R}^2$ be two continuous semialgebraic paths, with finite initial velocity, such that $\gamma_1(0) = \gamma_2(0) = 0$, and let $\tilde{\gamma}_i(t) := \Phi(\gamma_i(t))$, $i = 1, 2$. If $\gamma'_{2+}(0) = c \cdot \gamma'_{1+}(0)$, with $c > 0$, then $\tilde{\gamma}'_{2+}(0) = c \cdot \tilde{\gamma}'_{1+}(0)$.*

Proof. Consider the path $\gamma_0(t) := \gamma_1(ct)$, $0 \leq t < \epsilon/c$, and let $\tilde{\gamma}_0(t) := \Phi(\gamma_0(t))$. By Lemma 2.3, since $\gamma'_{0+}(0) = c \cdot \gamma'_{1+}(0) = \gamma'_{2+}(0)$, we have $\tilde{\gamma}'_{0+}(0) = \tilde{\gamma}'_{2+}(0)$. On the other hand, $\tilde{\gamma}_0(t) = \Phi(\gamma_0(t)) = \Phi(\gamma_1(ct)) = \tilde{\gamma}_1(ct)$, so $\tilde{\gamma}'_{0+}(0) = c \cdot \tilde{\gamma}'_{1+}(0)$. Hence, $\tilde{\gamma}'_{2+}(0) = c \cdot \tilde{\gamma}'_{1+}(0)$. ■

Corollary 2.7. *Let $\Phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a germ of semialgebraic bi-Lipschitz map. Let $\gamma_1, \gamma_2\colon [0, \epsilon) \to \mathbb{R}^2$ be two continuous semialgebraic paths, with finite initial velocity, such that $\gamma_1(0) = \gamma_2(0) = 0$, and let $\tilde{\gamma}_i(t) := \Phi(\gamma_i(t))$, $i = 1, 2$. The initial velocities of the paths γ_1 and γ_2 have the same direction*[2] *if and only if the initial velocities of the paths $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ have the same direction.*

[2] We say that two vectors u and v *have the same direction* if $u = c \cdot v$, for some $c > 0$.

3 LIPSCHITZ EQUIVALENCE OF POLYNOMIAL FUNCTIONS OF A SINGLE VARIABLE

Two polynomial functions $f, g\colon \mathbb{R} \to \mathbb{R}$ are *Lipschitz equivalent*, written $f \cong g$, if there exist a bi-Lipschitz homeomorphism $\phi\colon \mathbb{R} \to \mathbb{R}$ and a constant $c > 0$ such that $g \circ \phi = cf$. In this chapter, we address some questions regarding this equivalence relation that are relevant for the solution of the problem considered in the next chapter.

Overview of this chapter. In Section 3.1, we show how to determine whether any two given polynomial functions $f, g\colon \mathbb{R} \to \mathbb{R}$ are Lipschitz equivalent. Following BIRBRAIR, FERNANDES, and PANAZZOLO (2009), we show that if two polynomial functions $f, g\colon \mathbb{R} \to \mathbb{R}$ are Lipschitz equivalent then they have the same degree and there is a 1-1 correspondence between the critical points of f and the critical points of g which preserves multiplicity (see Lemma 3.2 and Lemma 3.3). Then, we provide effective criteria to determine whether any two polynomial functions $f, g\colon \mathbb{R} \to \mathbb{R}$ of the same degree $d \geq 1$ are Lipschitz equivalent, considering the following cases separately: (a) f and g have no critical points (Theorem 3.1a) (b) f and g have only one critical point (Theorem 3.1b) (c) f and g have the same number $p \geq 2$ of critical points (Theorem 3.1c).

In Section 3.2, we apply these results to find a complete set of normal forms for polynomial functions of degree d, along with criteria to determine the corresponding canonical form for each polynomial function of degree d, for $d = 1, 2, 3$ (see Proposition 3.1, Proposition 3.2, and Proposition 3.3).

In Section 3.3, we discuss some qualitative properties of bi-Lipschitz homeomorphisms $\phi\colon \mathbb{R} \to \mathbb{R}$ that satisfy an equation of the form $g \circ \phi = cf$, where $f, g\colon \mathbb{R} \to \mathbb{R}$ are nonconstant polynomial functions of the same degree, and $c > 0$ is a constant. We begin by noting that such bi-Lipschitz homeomorphisms are actually Nash diffeomorphisms and that they are not necessarily affine functions. Then, as we investigate the asymptotic behavior of such bi-Lipschitz homeomorphisms $\phi\colon \mathbb{R} \to \mathbb{R}$, we find that $\phi(t) = (\lambda t + k) + \alpha(t)$, where $\lambda, k \in \mathbb{R}$, $\lambda \neq 0$, and $\alpha\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz Nash function that can be analytically extended to $\mathbb{R} \cup \{\infty\} \cong \mathbb{P}^1(\mathbb{R})$ and satisfies $\lim_{|t| \to +\infty} \alpha(t) = 0$ (see Proposition 3.5). We show how to compute the constants λ, k, in general (see Proposition 3.6 and Remark 3.2), and then we work out an example to illustrate our formulas.

3.1 Lipschitz classification of polynomial functions of a single variable

In this section, we provide effective criteria to determine whether any two given polynomial functions $f, g\colon \mathbb{R} \to \mathbb{R}$ are Lipschitz equivalent. The case of constant functions is trivial: if two polynomial functions are Lipschitz equivalent, then one of them is constant if and only if the other one is constant; furthermore, two constant functions are Lipschitz equivalent if and only if they have the same sign (positive, negative or zero). So we focus on nonconstant polynomial functions. In this case, ϕ is necessarily semialgebraic. This follows from the next lemma, which is an adapted version of (BIRBRAIR, FERNANDES, and PANAZZOLO, 2009, Lemma 3.1).

Lemma 3.1. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be nonconstant polynomial functions. If $\phi\colon \mathbb{R} \to \mathbb{R}$ is a homeomorphism such that $g \circ \phi = f$, then ϕ is semialgebraic.*

Proof. Let $s_1 < \ldots < s_p$ be all the critical points of g, so we have

$$\mathbb{R} = (-\infty, s_1] \cup [s_1, s_2] \cup \cdots \cup [s_{p-1}, s_p] \cup [s_p, +\infty),$$

and g is monotone and injective on each of the intervals $(-\infty, s_1], [s_1, s_2], \ldots, [s_{p-1}, s_p]$, and $[s_p, +\infty)$. Let $t_i \coloneqq \phi^{-1}(s_i)$, for $i = 1, \ldots, p$.

Suppose that ϕ is an increasing homeomorphism (the case where ϕ is a decreasing homeomorphism can be treated similarly). Then, we have $t_1 < \cdots < t_p$. On each of the intervals $(-\infty, t_1], [t_1, t_2], \ldots, [t_{p-1}, t_p]$, and $[t_p, +\infty)$, we have $\phi = g^{-1} \circ f$. More precisely:

$$\begin{aligned}
\phi|_{(-\infty,t_1]} &= \left(g|_{(-\infty,s_1]}\right)^{-1} \circ f|_{(-\infty,t_1]},\\
\phi|_{[t_i,t_{i+1}]} &= \left(g|_{[s_i,s_{i+1}]}\right)^{-1} \circ f|_{[t_i,t_{i+1}]}, \text{ for } 1 \le i < p,\\
\phi|_{[t_p,+\infty)} &= \left(g|_{[s_p,+\infty)}\right)^{-1} \circ f|_{[t_p,+\infty)}.
\end{aligned}$$

Since f and g are polynomial functions, we see that each of the restrictions $\phi|_{(-\infty,t_1]}$, $\phi|_{[t_1,t_2]}, \ldots, \phi|_{[t_{p-1},t_p]}$, and $\phi|_{[t_p,+\infty)}$ is a semialgebraic function (being the composition of two semialgebraic functions). Thus, ϕ is a semialgebraic function. ∎

Lemma 3.2. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be nonconstant polynomial functions. If f and g are Lipschitz equivalent, then* $\deg f = \deg g$.

Proof. Let $f(t) = \sum_{i=0}^{d} a_i t^i$ and $g(t) = \sum_{i=0}^{e} b_i t^i$, where $a_d, b_e \neq 0$. Suppose that f and g are Lipschitz equivalent, so that $g \circ \phi = cf$, for some bi-Lipschitz function $\phi\colon \mathbb{R} \to \mathbb{R}$ and some constant $c > 0$. We must show that $d = e$.

Let $\lambda \coloneqq \lim_{t\to+\infty} \phi(t)/t$. This limit is well-defined in the extended real line, because ϕ is semialgebraic (see Lemma 3.1). Also, since ϕ is bi-Lipschitz, λ is a nonzero real number.

Since $\lim_{|t|\to+\infty} |\phi(t)| = +\infty$, we have:

$$\lim_{|t|\to+\infty} \frac{g(\phi(t))}{\phi(t)^e} = \lim_{|t|\to+\infty} \frac{g(t)}{t^e} = b_e \tag{1}$$

On the other hand, since $cf = g \circ \phi$, we have:

$$c \cdot \lim_{t \to +\infty} \frac{f(t)}{t^e} = \lim_{t \to +\infty} \frac{g(\phi(t))}{t^e} = \lim_{t \to +\infty} \frac{g(\phi(t))}{\phi(t)^e} \cdot \lim_{t \to +\infty} \left(\frac{\phi(t)}{t}\right)^e \tag{2}$$

From (1) and (2), we obtain:

$$\lim_{t \to +\infty} \frac{f(t)}{t^e} = \frac{b_e \cdot \lambda^e}{c}$$

Since $b_e \cdot \lambda^e / c$ is a nonzero real number, it follows that $d = e$. ∎

Lemma 3.3. *Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be two polynomial functions of the same degree $d \geq 1$, and suppose that $\phi \colon \mathbb{R} \to \mathbb{R}$ is a bijective function such that $g \circ \phi = cf$, for some constant $c > 0$. The following conditions are equivalent:*

i. ϕ is bi-Lipschitz;

ii. The multiplicity of f at t is equal to the multiplicity of g at $\phi(t)$, for all $t \in \mathbb{R}$;

iii. ϕ is bi-analytic.

Proof. $(i) \Rightarrow (ii)$: Pick any point $t_0 \in \mathbb{R}$. Let k be the multiplicity of f at t_0, and let l be the multiplicity of g at $\phi(t_0)$. For any pair of functions $u, v \colon \mathbb{R} \to \mathbb{R}$, we write $u \sim v$ near t_0 to indicate that there exist constants $A, B > 0$ such that $A\,|v(t)| \leq |u(t)| \leq B\,|v(t)|$, for t sufficiently close to t_0. Then, $f(t) - f(t_0) \sim (t - t_0)^k$ near t_0 and $g(s) - g(\phi(t_0)) \sim (s - \phi(t_0))^l$ near $\phi(t_0)$. Since $g \circ \phi = cf$, this implies that $(t - t_0)^k \sim (\phi(t) - \phi(t_0))^l$ near t_0. And since we are assumig that ϕ is bi-Lipschitz, it follows that $(t - t_0)^k \sim (t - t_0)^l$ near t_0. Therefore, $k = l$.

$(ii) \Rightarrow (iii)$: Pick any point $t_0 \in \mathbb{R}$. Suppose that $\hat{f} := cf$ has multiplicity k at t_0. By Proposition 2.2, there exist an increasing analytic diffeomorphism $u \colon I \to (-\epsilon, \epsilon)$, with $t_0 \in I$, and a constant $\rho \in \mathbb{R} \setminus \{0\}$, such that $u(t_0) = 0$ and $\hat{f} \circ u^{-1}(t) = a + \rho t^k$, for $|t| < \epsilon$; where $a := \hat{f}(t_0) = g \circ \phi(t_0)$. Since we are assuming that condition (ii) holds, the multiplicity of g at the point $\phi(t_0)$ is also k. Then, as before, there exist an increasing analytic diffeomorphism $v \colon J \to (-\epsilon', \epsilon')$, with $\phi(t_0) \in J$, and a constant $\sigma \in \mathbb{R} \setminus \{0\}$, such that $v(\phi(t_0)) = 0$ and $g \circ v^{-1}(t) = a + \sigma t^k$, for $|t| < \epsilon'$. Shrinking the interval I, if necessary, we can assume that $\phi(I) \subseteq J$. Hence, we can write $\hat{f} \circ u^{-1}(t) = g \circ v^{-1}(\bar{\phi}(t))$, where $\bar{\phi} := v \circ \phi \circ u^{-1} \colon (-\epsilon, \epsilon) \to (-\epsilon', \epsilon')$; and then it follows that $\bar{\phi}(t) = \nu t$, where $\nu = \pm \left|\frac{\rho}{\sigma}\right|^{\frac{1}{k}}$, depending on whether ϕ is increasing (positive sign) or decreasing (negative sign). In particular, this shows that $\bar{\phi}$ is analytic. Therefore, $\phi|_I = v^{-1} \circ \bar{\phi} \circ u$ is analytic; so ϕ is analytic at t_0. Since the point $t_0 \in \mathbb{R}$ is arbitrary, it follows that ϕ is an analytic function. In order to show that ϕ^{-1} is also analytic, note that $\phi^{-1} \colon \mathbb{R} \to \mathbb{R}$ is a bijective function such that $f \circ \phi^{-1} = c^{-1} g$, satisfying condition (ii) with f and g interchanged: the multiplicity of g at t is equal to the multiplicity of f at $\phi^{-1}(t)$, for all $t \in \mathbb{R}$. Then, by what we have already proved, it follows that ϕ^{-1} is analytic.

(iii) ⇒ *(i)*: Suppose that ϕ is bi-analytic. Then, in particular, ϕ is a homeomorphism, so ϕ is monotone and $\lim_{|t|\to+\infty} |\phi(t)| = +\infty$. Also, by Lemma 3.1, ϕ is a semialgebraic function. Let $f(t) = \sum_{i=0}^{d} a_i t^i$ and $g(t) = \sum_{i=0}^{d} b_i t^i$, with $a_d, b_d \neq 0$. Since $g \circ \phi = cf$ and $\lim_{|t|\to+\infty} |\phi(t)| = +\infty$, we have

$$\lim_{t\to-\infty} \left(\frac{\phi(t)}{t}\right)^d = \lim_{t\to+\infty} \left(\frac{\phi(t)}{t}\right)^d = c \cdot \frac{a_d}{b_d} \tag{3}$$

Let $l_+ := \lim_{t\to+\infty} \phi(t)/t$ and $l_- := \lim_{t\to-\infty} \phi(t)/t$. Both these limits are well-defined in the extended real line because ϕ is semialgebraic. It follows from (3) that we actually have $l_+, l_- \in \mathbb{R} \setminus \{0\}$ and $|l_+| = |l_-|$. (Notice that to obtain this last equality from (3), we use the fact that $d > 0$.) By L'Hôpital's rule, $\lim_{t\to+\infty} \phi'(t) = l_+$ and $\lim_{t\to-\infty} \phi'(t) = l_-$. (The existence of these limits in the extended real line is guaranteed by the fact that ϕ' is semialgebraic — see Proposition 2.4, so L'Hôpital's rule can be applied.) Thus, $\lim_{t\to+\infty} |\phi'(t)| = \lim_{t\to-\infty} |\phi'(t)| > 0$; so that $|\phi'|$ can be continuously extended to a positive function defined on the compact space $\mathbb{R} \cup \{\infty\} \cong \mathbb{P}^1(\mathbb{R})$. Hence, there exist constants $A, B > 0$ such that $A \leq |\phi'(t)| \leq B$, for all $t \in \mathbb{R}$. Therefore, ϕ is bi-Lipschitz.

■

From the last two lemmas, it follows that if two polynomial functions $f, g\colon \mathbb{R} \to \mathbb{R}$ are Lipschitz equivalent, then they have the same degree and the same number of critical points. The first assertion is precisely the content of Lemma 3.2. The second assertion is an immediate consequence of Lemma 3.3: if $\phi\colon \mathbb{R} \to \mathbb{R}$ is a bi-Lipschitz homemorphism such that $g \circ \phi = cf$, for some constant $c > 0$, then ϕ induces a 1-1 correpondence between the critical points of f and the critical points of g, because it preserves multiplicity (see Remark 2.1).

The next results provide effective criteria to determine whether any two given nonconstant polynomial functions $f, g\colon \mathbb{R} \to \mathbb{R}$, of the same degree, are Lipschitz equivalent. In Theorem 3.1a, we address the case where f and g have no critical points; in Theorem 3.1b, the case where both f and g have only one critical point; and in Theorem 3.1c, the case where f and g have the same number $p \geq 2$ of critical points.

Theorem 3.1a. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be polynomial functions of the same degree $d \geq 1$. If f and g have no critical points, then f and g are Lipschitz equivalent.*

Proof. If f and g have no critical points, then they are both bi-analytic diffeomorphisms. Hence, $f = g \circ \phi$, where $\phi := g^{-1} \circ f$ is a bi-analytic diffeomorphism. By Lemma 3.3, ϕ is bi-Lipschitz. ■

Theorem 3.1b. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be polynomial functions of the same degree $d \geq 1$. Suppose that f has only one critical point t_0, with multiplicity k, and that g has only one* *critical point s_0, with the same multiplicity k. Also, suppose that $f(t_0)$ and $g(s_0)$ have the same sign*[3]*(positive, negative or zero). We have:*

i. If d is odd, then f and g are Lipschitz equivalent.

ii. If d is even, then f and g are Lipschitz equivalent if and only if t_0 and s_0 are either both minimum points, or both maximum points of f and g, respectively.

Proof. First, consider the case where d is odd. If a real polynomial function of a single variable, of odd degree, has only one critical point, then it is a homeomorphism. Thus, under the assumption that d is odd, f and g are homeomorphisms. Choose a constant $c > 0$ such that $g(s_0) = cf(t_0)$, and define $\phi := g^{-1} \circ \hat{f} \colon \mathbb{R} \to \mathbb{R}$, where $\hat{f} := cf$. The function ϕ is a bijection such that $g \circ \phi = cf$, and the multiplicity of f at t is equal to the multiplicity of g at $\phi(t)$ for all $t \in \mathbb{R}$. By Lemma 3.3, ϕ is bi-Lipschitz.

Now, suppose that d is even. If a real polynomial function of a single variable, of even degree, has only one critical point, then this critical point is a point of global extremum. Applying this to f and g, we see that t_0 is a point of global extremum of f and s_0 is a point of global extremum of g. If f and g are Lipschitz equivalent, then t_0 and s_0 are either both minimum points or both maximum points of f and g, respectively. Otherwise, we would have $\hat{f}(\mathbb{R}) \cap g(\mathbb{R}) = \{\hat{f}(t_0)\} = \{g(s_0)\}$, which is absurd, since the equation $g \circ \phi = cf$ implies that $\hat{f}(\mathbb{R}) = g(\mathbb{R})$. Conversely, suppose that t_0 and s_0 are either both minimum points or both maximum points of f and g, respectively. Pick any constant $c > 0$ for which $g(s_0) = cf(t_0)$, and define $\phi \colon \mathbb{R} \to \mathbb{R}$ by

$$\phi|_{(-\infty,t_0]} := \left(g|_{(-\infty,s_0]}\right)^{-1} \circ \hat{f}|_{(-\infty,t_0]}, \quad \phi|_{[t_0,+\infty)} := \left(g|_{[s_0,+\infty)}\right)^{-1} \circ \hat{f}|_{[t_0,+\infty)}.$$

Clearly, ϕ is a bijection such that $g \circ \phi = cf$, and the multiplicity of f at t is equal to the multiplicity of g at $\phi(t)$ for all $t \in \mathbb{R}$. Hence, by Lemma 3.3, ϕ is bi-Lipschitz. ■

For the case in which f and g have the same number $p \geq 2$ of critical points, we introduce an adapted version of the notion of *multiplicity symbol* defined in BIRBRAIR, FERNANDES, and PANAZZOLO (2009).

Let $f \colon \mathbb{R} \to \mathbb{R}$ be a polynomial function of degree $d \geq 1$, having exactly p critical points, with $p \geq 2$. Let $t_1 < \ldots < t_p$ be the critical points of f, with multiplicities $\mu_1, \ldots, \mu_p$, respectively. The *multiplicity symbol* of f is the ordered pair (a, μ) whose first entry is the p-tuple $a = (f(t_1), \ldots, f(t_p))$, and second entry is the p-tuple $\mu = (\mu_1, \ldots, \mu_p)$.

Let $g \colon \mathbb{R} \to \mathbb{R}$ be another polynomial function of degree $d \geq 1$, having exactly the same number $p \geq 2$ of critical points. Let $s_1 < \ldots < s_p$ be the critical points of g, with multiplicities $\nu_1, \ldots, \nu_p$, respectively. The multiplicity symbol of g is the ordered pair (b, ν), where $b = (g(s_1), \ldots, g(s_p))$ and $\nu = (\nu_1, \ldots, \nu_p)$.

The multiplicity symbols (a, μ) and (b, ν) are said to be:

[3]Clearly, this is a necessary condition for f and g to be Lipschitz equivalent.

i. *directly similar*, if there exists a constant $c > 0$ such that $b = c \cdot a$, and $\nu = \mu$;

ii. *reversely similar*, if there exists a constant $c > 0$ such that $b = c \cdot \overline{a}$, and $\nu = \overline{\mu}$.

For any p-tuple $x = (x_1, \ldots, x_p)$, $\overline{x} := (x_p, \ldots, x_1)$ is the p-tuple x written in reverse order.

The multiplicity symbols (a, μ) and (b, ν) are said to be *similar* if they are either directly similar or reversely similar.

Theorem 3.1c. *Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be polynomial functions of the same degree $d \geq 1$, having the same number $p \geq 2$ of critical points. Then, f and g are Lipschitz equivalent if and only if their multiplicity symbols are similar.*

Proof. First, suppose that f and g are Lipschitz equivalent. Then there exist a bi-Lipschitz function $\phi \colon \mathbb{R} \to \mathbb{R}$ and a constant $c > 0$ such that $g \circ \phi = cf$. Let $t_1, \ldots, t_p$ be the critical points of f, with multiplicities $\mu_1, \ldots, \mu_p$, and let $s_1, \ldots, s_p$ be the critical points of g, with multiplicities $\nu_1, \ldots, \nu_p$. (As we noted just after the proof of Lemma 3.3, if two real polynomial functions of a single variable are Lipschitz equivalent, then they have the same number of critical points.) Let (a, μ) be the multiplicity symbol of f and (b, ν) the multiplicity symbol of g. By Lemma 3.3, ϕ preserves multiplicities. Thus, if ϕ is increasing, then we have $\phi(t_i) = s_i$ and $\mu_i = \nu_i$, for $i = 1, \ldots, p$. Also, since $g \circ \phi = cf$, it follows that $b_i = g(s_i) = c \cdot f(t_i) = c \cdot a_i$, for $i = 1, \ldots, p$. Hence, if ϕ is increasing then the multiplicity symbols of f and g are directly similar. On the other hand, if ϕ is decreasing, then we have $\phi(t_{p+1-i}) = s_i$ and $\mu_{p+1-i} = \nu_i$, for $i = 1, \ldots, p$. Also, since $g \circ \phi = cf$, it follows that $b_i = g(s_i) = c \cdot f(t_{p+1-i}) = c \cdot a_{p+1-i}$, for $i = 1, \ldots, p$. Hence, if ϕ is decreasing then the multiplicity symbols of f and g are reversely similar. In either case, we conclude that the multiplicity symbols of f and g are similar.

Now we prove the converse. Suppose that the multiplicity symbols of f and g are similar. Replacing (if necessary) g with $g \circ \alpha$, where $\alpha := -\mathrm{id} \colon \mathbb{R} \to \mathbb{R}$ (which is bi-Lipschitz), we may assume that the multiplicity symbols of f and g are directly similar. Let $t_1 < \ldots < t_p$ be the critical points of f, with multiplicities $\mu_1, \ldots, \mu_p$, respectively; and let $s_1 < \ldots < s_p$ be the critical points of g, with multiplicities $\nu_1, \ldots, \nu_p$, respectively. Since we are assuming that the multiplicity symbols of f and g are directly similar, there exists a constant $c > 0$ such that $g(s_i) = cf(t_i)$, for $i = 1, \ldots, p$; and $\mu_i = \nu_i$, for $i = 1, \ldots, p$. Let $\hat{f} := cf$ and $c_i := g(s_i) = \hat{f}(t_i)$. (Note that $c_i \neq c_{i+1}$, for $1 \leq i < p$.) The functions $\hat{f}|_{[t_i, t_{i+1}]} \colon [t_i, t_{i+1}] \to [c_i, c_{i+1}]$ and $g|_{[s_i, s_{i+1}]} \colon [s_i, s_{i+1}] \to [c_i, c_{i+1}]$ are both monotone and injective, for $1 \leq i < p$. The same is true for the functions $\hat{f}|_{(-\infty, t_1]}$ and $g|_{(-\infty, s_1]}$, and also for the functions $\hat{f}|_{[t_p, +\infty)}$ and $g|_{[s_p, +\infty)}$. Moreover, the functions $\hat{f}|_{(-\infty, t_1]}$ and $g|_{(-\infty, s_1]}$ are either both increasing or both decreasing because $\hat{f}(t_1) = g(s_1)$, $\hat{f}(t_2) = g(s_2)$, and $\mu_1 = \nu_1$. Since $\hat{f}(t_1) = g(s_1)$ and $|\hat{f}(t)|, |g(t)| \to +\infty$, as $|t| \to +\infty$, this implies that $\hat{f}((-\infty, t_1]) = g((-\infty, s_1])$. Similarly, the functions $\hat{f}|_{[t_p, +\infty)}$ and $g|_{[s_p, +\infty)}$

are either both increasing or both decreasing because $\hat{f}(t_{p-1}) = g(s_{p-1})$, $\hat{f}(t_p) = g(s_p)$, and $\mu_p = \nu_p$. Since $\hat{f}(t_p) = g(s_p)$, and $|\hat{f}(t)|, |g(t)| \to +\infty$, as $|t| \to +\infty$, this implies that $\hat{f}([t_p, +\infty)) = g([s_p, +\infty))$. Define $\phi\colon \mathbb{R} \to \mathbb{R}$ by

$$\phi|_{(-\infty,t_1]} := \left(g|_{(-\infty,s_1]}\right)^{-1} \circ \hat{f}|_{(-\infty,t_1]}$$

$$\phi|_{[t_i,t_{i+1}]} := \left(g|_{[s_i,s_{i+1}]}\right)^{-1} \circ \hat{f}|_{[t_i,t_{i+1}]}, \text{ for } 1 \leq i < p$$

$$\phi|_{[t_p,+\infty)} := \left(g|_{[s_p,+\infty)}\right)^{-1} \circ \hat{f}|_{[t_p,+\infty)}$$

Clearly, ϕ is a bijection such that $g \circ \phi = cf$, and it takes t_i to s_i, for $i = 1, \ldots, p$. Since the multiplicity symbols of f and g are directly similar, it follows that the multiplicity of f at t is equal to the multiplicity of g at $\phi(t)$ for all $t \in \mathbb{R}$. Hence, by Lemma 3.3, ϕ is bi-Lipschitz. ■

3.2 Normal forms for the Lipschitz equivalence of nonconstant polynomials of degree ≤ 3

A set of polynomial functions of degree d is called a *complete set of normal forms for the Lipschitz equivalence of polynomial functions of degree d* if it contains exactly one element of each Lipschitz equivalence class of polynomial functions of degree d. In this section, we find a complete set of normal forms for the Lipschitz equivalence of polynomial functions of degree d, for $d = 1, 2, 3$, and we provide criteria to determine the corresponding normal form of each polynomial function of degree d, for $d = 1, 2, 3$. We begin with the trivial case of the polynomial functions of degree 1.

Proposition 3.1. *Every polynomial function $f\colon \mathbb{R} \to \mathbb{R}$ of degree 1 is Lipschitz equivalent to the identity function $\mathrm{id}_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}$.*

Proof. Given any polynomial function $f\colon \mathbb{R} \to \mathbb{R}$ of degree 1, we have $f = \mathrm{id}_{\mathbb{R}} \circ \phi$, where $\phi = f$ is a bi-Lipschitz function (because every affine function is bi-Lipschitz). Hence the result. ■

The next result provides a complete set of canonical forms for polynomial functions of degree 2, along with criteria for determining the corresponding canonical form of each quadratic function.

Proposition 3.2. *Every polynomial function $f\colon \mathbb{R} \to \mathbb{R}$ of degree 2 is Lipschitz equivalent to exactly one of the polynomial functions listed on the left column of Table 1 (the one on the same row as the only distinctive feature that it possesses among those listed on the right column of the table).*

Proof. Every polynomial function from $\mathbb{R}$ to $\mathbb{R}$, of degree 2, has only one critical point, which has multiplicity 2. Given two polynomial functions $f, g\colon \mathbb{R} \to \mathbb{R}$, of degree 2, denote by t_0 the critical point of f, and by s_0 the critical point of g. By Theorem 3.1b, f and g are Lipschitz equivalent if and only if the following conditions hold:

i. $f(t_0)$ and $g(s_0)$ have the same sign.

ii. t_0 and s_0 are either both minimum points, or both maximum points of f and g, respectively.

Since the quadratic functions listed on the left column of Table 1 cover all possibilities ($t^2 + 1$ has a positive minimum, t^2 has a minimum equal to zero, $t^2 - 1$ has a negative minimum, $-t^2 + 1$ has a positive maximum, $-t^2$ has a maximum equal to zero, and $-t^2 - 1$ has a negative maximum), it follows that each polynomial function $f : \mathbb{R} \to \mathbb{R}$ of degree 2 is Lipschitz equivalent to exactly one of these functions (the one whose distinctive feature coincides with the one possessed by f). ■

Table 1 – Normal forms for the Lipschitz equivalence of quadratic functions

Normal form	Distinctive feature
$t^2 + 1$	positive minimum
t^2	minimum equal to zero
$t^2 - 1$	negative minimum
$-t^2 + 1$	positive maximum
$-t^2$	maximum equal to zero
$-t^2 - 1$	negative maximum

Source: Elaborated by the author.

Now we set out to find a complete set of normal forms for the Lipschitz equivalence of cubic functions. The next lemma facilitates this task by ensuring that such a complete set of normal forms can be found whose members are all cubic functions of a specific simple form.

Lemma 3.4. *Every polynomial function $f : \mathbb{R} \to \mathbb{R}$ of degree 3 is Lipschitz equivalent to a cubic function of the form $t^3 + pt + q$.*

Proof. If $f(t) = a_3t^3 + a_2t^2 + a_1t + a_0$, with $a_3 \neq 0$, then

$$\frac{1}{|a_3|} f\left(t - \frac{a_2}{3a_3}\right)$$

has the form $\pm t^3 + pt + q$, where the coefficient of t^3 is equal to $+1$ if $a_3 > 0$ and it is equal to -1 if $a_3 < 0$. So if $a_3 > 0$ then the transformation above turns $f(t)$ into a function of the form $t^3 + pt + q$, as we wanted. On the other hand, if $a_3 < 0$ then this transformation turns $f(t)$ into a function of the form $-t^3 + pt + q$, but this can be turned into a function of the desired form by substituting $-t$ for t. In any case, the resulting function is Lipschitz equivalent to $f(t)$. ■

The next result provides a complete set of canonical forms for polynomial functions of degree 3, along with criteria for determining the corresponding canonical form of each cubic function.

Proposition 3.3. *Every polynomial function $f\colon \mathbb{R} \to \mathbb{R}$ of degree 3 is Lipschitz equivalent to exactly one of the cubic functions listed on the left column of Table 2 (the one on the same row as the only set of distinctive features that it possesses among those listed on the right column of the table).*[4,5]

Proof. By Lemma 3.4, there exists a complete set of normal forms for the Lipschitz equivalence of cubic functions among the polynomial functions of the form $t^3 + pt + q$. So we begin by classifying the cubic functions of this reduced form. First, we note that a cubic function of the form $t^3 + pt + q$ has no critical points if $p > 0$, it has only one critical point (of multiplicity 3) if $p = 0$, and it has exactly two critical points if $p < 0$.

By Theorem 3.1a, all polynomial functions of the form $t^3 + pt + q$, with $p > 0$, are Lipschitz equivalent. Thus, all such functions are Lipschitz equivalent to $t^3 + t$. By Theorem 3.1b, $t^3 + q$ and $t^3 + \tilde{q}$ are Lipschitz equivalent if and only if q and $\tilde{q}$ have the same sign. Thus, each polynomial function of the form $t^3 + q$ is Lipschitz equivalent to exactly one of the functions: $t^3 + 1$, t^3, or $t^3 - 1$, according as q is positive, zero, or negative, respectively.

Now, let $f(t) = t^3 + pt + q$ and $g(t) = t^3 + \tilde{p}t + \tilde{q}$, with $p < 0$ and $\tilde{p} < 0$. In order to determine whether these cubic functions are Lipschitz equivalent, we need to compute their multiplicity symbols. The critical points of $f(t)$ are

$$t_1 = -\left(\frac{|p|}{3}\right)^{\frac{1}{2}} \quad \text{and} \quad t_2 = \left(\frac{|p|}{3}\right)^{\frac{1}{2}},$$

each of multiplicity 2. Hence, the multiplicity symbol of $f(t)$ is

$$((k + q, -k + q), (2, 2)),$$

where $k = 2 \cdot \left(\frac{|p|}{3}\right)^{\frac{3}{2}}$. Similarly, we find that the multiplicity symbol of $g(t)$ is

$$((\tilde{k} + \tilde{q}, -\tilde{k} + \tilde{q}), (2, 2)),$$

where $\tilde{k} = 2 \cdot \left(\frac{|\tilde{p}|}{3}\right)^{\frac{3}{2}}$.

[4] The last row of the table should be regarded as an infinite family of rows indexed by the parameter θ.

[5] In the last row of the table, gcv denotes the greatest critical value of the function, lcv denotes the least critical value of the function, and the equality (gcv : lcv) $= (\theta + 2 : \theta - 2)$ is meant to be understood as an equality of ratios.

Since $k+q > -k+q$ and $\tilde{k}+\tilde{q} > -\tilde{k}+\tilde{q}$, the multiplicity symbols of f and g are similar if and only if they are directly similar. Hence, by Theorem 3.1c, f and g are Lipschitz equivalent if and only if

$$\tilde{k}+\tilde{q} = \lambda \cdot (k+q) \quad \text{and} \quad -\tilde{k}+\tilde{q} = \lambda \cdot (-k+q),$$

for some $\lambda > 0$. Equivalently, f and g are Lipschitz equivalent if and only if

$$\tilde{k} = \lambda k \quad \text{and} \quad \tilde{q} = \lambda q,$$

for some $\lambda > 0$.

For $\lambda > 0$, we have:

$$\begin{aligned} & \tilde{k} = \lambda k \quad \text{and} \quad \tilde{q} = \lambda q \\ \Leftrightarrow\ & |\tilde{p}|^{3/2} = \lambda |p|^{3/2} \quad \text{and} \quad \tilde{q} = \lambda q \\ \Leftrightarrow\ & \tilde{p} = \lambda^{2/3} p \quad \text{and} \quad \tilde{q} = \lambda q \\ \Leftrightarrow\ & \begin{cases} q^2\tilde{p}^3 - p^3\tilde{q}^2 = 0 \\ q \text{ and } \tilde{q} \text{ have the same sign.} \end{cases} \end{aligned}$$

Hence, if f and g are Lipschitz equivalent, then q and $\tilde{q}$ are either both equal to zero or both nonzero. Moreover, if q and $\tilde{q}$ are both equal to zero, then f and g are Lipschitz equivalent; and if q and $\tilde{q}$ are both nonzero, then f and g are Lipschitz equivalent if and only if the following conditions are satisfied:

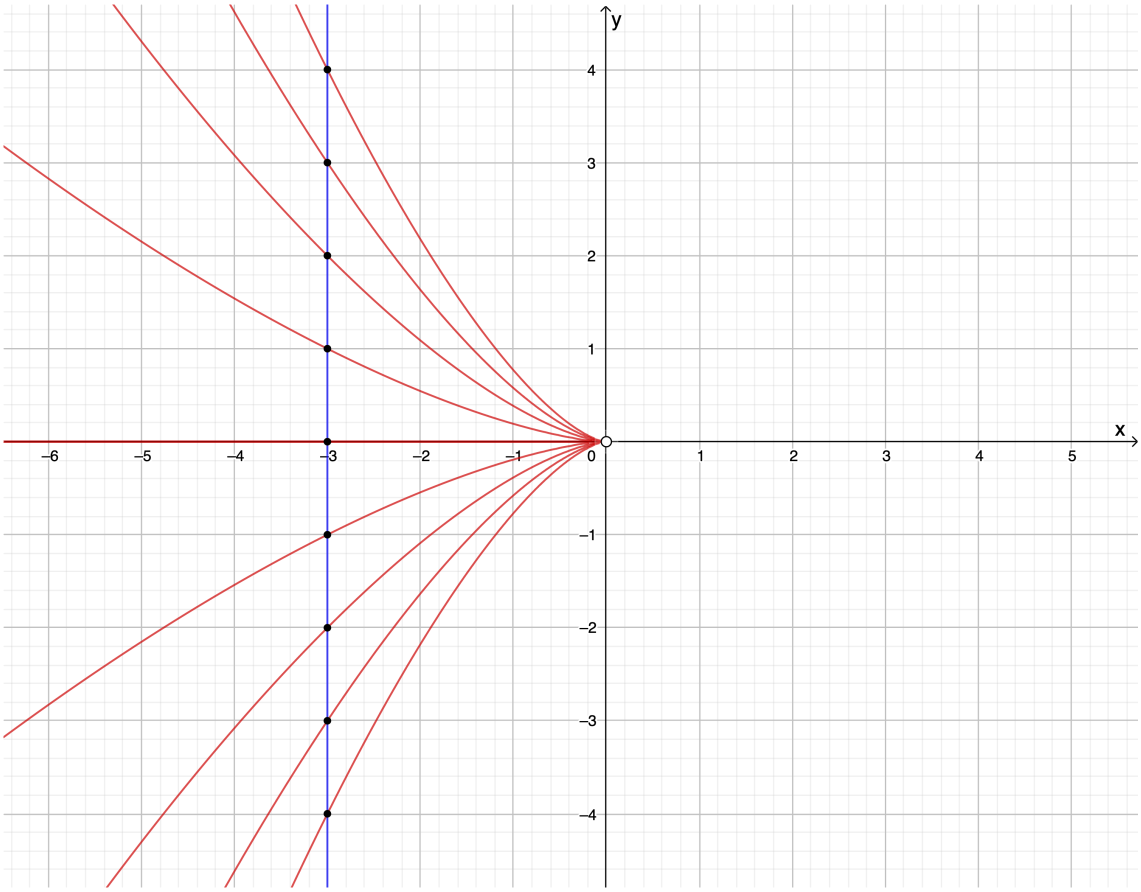
i. $(\tilde{p}, \tilde{q})$ belongs to the cusp $q^2X^3 - p^3Y^2 = 0$.

ii. q and $\tilde{q}$ have the same sign.

If we identify the cubic function t^3+pt+q with the point $(p,q) \in \mathbb{R}^2$, then the Lipschitz equivalence class of t^3+pt+q is represented by: the cusp branch $\{(x,y) \in \mathbb{R}^2 : q^2x^3 - p^3y^2 = 0, y > 0\}$ if $q > 0$, the cusp branch $\{(x,y) \in \mathbb{R}^2 : q^2x^3 - p^3y^2 = 0, y < 0\}$ if $q < 0$, the negative x-axis $\{(x,0) \in \mathbb{R}^2 : x < 0\}$ if $q = 0$. Since any fixed vertical line $x = x_0$, with $x_0 < 0$, intersects each of these sets on a single point (see Figure 1), we can obtain a single representative for the Lipschitz equivalence class of each cubic function of the form t^3+pt+q, with $p < 0$, by fixing an arbitrary negative value for p and letting q run through all real values. Choosing $p = -3$, we see that the family of cubic functions $\{t^3 - 3t + \theta : \theta \in \mathbb{R}\}$ contains a single representative for the Lipschitz equivalence class of each cubic function of the form t^3+pt+q, with $p < 0$.

By Theorem 3.1c, t^3+pt+q, with $p < 0$, is Lipschitz equivalent to $t^3 - 3t + \theta$ if and only if the multiplicity symbols of these functions are similar. Note that both these functions have two critical points, each of multiplicity 2, so we only need to compare their

Figure 1 – Lipschitz equivalence classes of $t^3 + pt + q$, for $p < 0$



Source: Elaborated by the author.

critical values if we want to decide whether their multiplicity symbols are similar. The critical values of $t^3 - 3t + \theta$ are $\theta + 2$ and $\theta - 2$. Thus, $t^3 + pt + q$, with $p < 0$, is Lipschitz equivalent to $t^3 - 3t + \theta$ if and only if $(\mathrm{gcv} : \mathrm{lcv}) = (\theta + 2 : \theta - 2)$, where gcv and lcv denote, respectively, the greatest and the least critical value of $t^3 + pt + q$, and the equality is meant to be understood as an equality of ratios.

By the argument above, the cubic functions listed on the left column of Table 2, form a complete set of normal forms for the Lipschitz equivalence of cubic functions. Now, we show that the distinctive features listed on the right column of the table allow us to determine the corresponding normal form of any given cubic function. Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary cubic function. By Lemma 3.4, f is Lipschitz equivalent to a cubic function $g : \mathbb{R} \to \mathbb{R}$ of the form $g(t) = t^3 + pt + q$. Note that f and g have the same number of critical points (see the remark just after the proof of Lemma 3.3). First, suppose that f has no critical points. Then, g has no critical points either, so $p > 0$ and hence g is Lipschitz equivalent to $t^3 + t$. Therefore, f is Lipschitz equivalent to $t^3 + t$. Now, suppose that f has only one critical point. Then g has only one critical point too, so $p = 0$ and hence $g(t) = t^3 + q$ is Lipschitz equivalent to exactly one of the functions: $t^3 + 1, t^3, t^3 - 1$, according as q is positive, zero, or negative, respectively. Since f and g are Lipschitz equivalent, the critical value of f and the critical value of g, which is q, have the same sign. Therefore, f is Lipschitz equivalent to one of the functions: t^3+1, t^3, t^3-1, according as the critical value of f is positive, zero, or negative, respectively. Finally, suppose that f has two critical points. Then g has two critical points too, so $p < 0$ and hence g is Lipschitz equivalent to the cubic function $t^3-3t+\theta$, for which $(\mathrm{gcv}(g) : \mathrm{lcv}(g)) = (\theta+2 : \theta-2)$, where $\mathrm{gcv}(g)$ and $\mathrm{lcv}(g)$ denote the greatest and the least of the critical values of g, respectively. Since f and g are Lipschitz equivalent, we have $(\mathrm{gcv}(f) : \mathrm{lcv}(f)) = (\mathrm{gcv}(g) : \mathrm{lcv}(g))$, where $\mathrm{gcv}(f)$ and $\mathrm{lcv}(f)$ denote the greatest and the least of the critical values of f, respectively. Therefore, f is Lipschitz equivalent to the cubic function $t^3 - 3t + \theta$ for which $(\mathrm{gcv}(f) : \mathrm{lcv}(f)) = (\theta + 2 : \theta - 2)$. ■

Table 2 – Normal forms for the Lipschitz equivalence of cubic functions

Normal form	Distinctive features
$t^3 + t$	no critical points
$t^3 + 1$	only one critical point, critical value > 0
t^3	only one critical point, critical value $= 0$
$t^3 - 1$	only one critical point, critical value < 0
$t^3 - 3t + \theta \quad (\theta \in \mathbb{R})$	two critical points, $(\mathrm{gcv} : \mathrm{lcv}) = (\theta + 2 : \theta - 2)$

Source: Elaborated by the author.

3.3 On the bi-Lipschitz transformation ϕ

Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be any two nonconstant polynomial functions that are Lipschitz equivalent. By definition, there exist a bi-Lipschitz homeomorphism $\phi\colon \mathbb{R} \to \mathbb{R}$ and a constant $c > 0$ such that $g \circ \phi = cf$. What can we say about the function ϕ? Clearly, it is an algebraic function and, by Lemma 3.3, it is also bi-analytic; so ϕ is a bi-Lipschitz Nash diffeomorphism. It can be an affine function, for example. Indeed, as the next proposition shows, this is exactly what happens whenever the polynomial functions f and g have degree 1 or 2.

Remark 3.1. *A Nash function on $\mathbb{R}$ is an analytic algebraic function on $\mathbb{R}$, that is, an analytic function $f\colon \mathbb{R} \to \mathbb{R}$ that satisfies an equation $a_d(t)(f(t))^d + \cdots + a_1(t)f_1(t) + a_0(t) = 0$, where $a_d, \ldots, a_0$ are real polynomial functions of a single variable, and $a_d \neq 0$. A Nash diffeomorphism on $\mathbb{R}$ is a bijective Nash function on $\mathbb{R}$ whose inverse is also a Nash function.*

Proposition 3.4. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be Lipschitz equivalent polynomial functions, with $g \circ \phi = cf$, where $\phi\colon \mathbb{R} \to \mathbb{R}$ is a bi-Lipschitz homeomorphism and $c > 0$ is constant. If the degree of the polynomial functions f and g is equal to 1 or 2, then ϕ is an affine function.*

Proof. Clearly, if f and g are polynomial functions of degree 1 then $\phi = g^{-1} \circ (cf)$ is an affine function. Now, suppose that f and g are polynomial functions of degree 2, say $f(t) = a_2t^2+a_1t+a_0$ and $g(t) = b_2t^2+b_1t+b_0$, with $a_2, a_1, a_0, b_2, b_1, b_0 \in \mathbb{R}$, $a_2, b_2 \neq 0$. Since f and g are quadratic functions, each of them has only one critical point (of multiplicity 2). Thus, f has only one critical value, which is $-\Delta_f/4a_2$, and g has only one critical value, which is $-\Delta_g/4b_2$, where $\Delta_f = a_1^2 - 4a_2a_0$ and $\Delta_g = b_1^2 - 4b_2b_0$. By Theorem 3.1b, the critical points of f and g are either both minimum values or both maximum values of f and g. This means that a_2 and b_2 have the same sign. Moreover, since ϕ sends the critical point of f to the critical point of g, the equality $g \circ \phi = cf$ implies that

$$-\frac{\Delta_g}{4b_2} = c \cdot \left(-\frac{\Delta_f}{4a_2}\right). \tag{4}$$

Having gathered all this information, we can now prove that ϕ is indeed an affine function. By completing the square, the equation $g \circ \phi = cf$ can be written in the form:

$$b_2 \cdot \left(\phi(t) + \frac{b_1}{2b_2}\right)^2 - \frac{\Delta_g}{4b_2} = c \cdot \left(a_2 \cdot \left(t + \frac{a_1}{2a_2}\right)^2 - \frac{\Delta_f}{4a_2}\right)$$

By (4), it follows that

$$b_2 \cdot \left(\phi(t) + \frac{b_1}{2b_2}\right)^2 = c \cdot a_2 \cdot \left(t + \frac{a_1}{2a_2}\right)^2.$$

Since a_2 and b_2 have the same sign, we can write $c \cdot a_2/b_2 = k^2$, with $k \neq 0$, and then we can rewrite the above equation as

$$(\phi(t) - s_0)^2 = k^2 \cdot (t - t_0)^2,$$

where $s_0 = -b_1/2b_2$ and $t_0 = -a_1/2a_2$. Moving all terms to the left side and factoring the difference of squares, we obtain:

$$((\phi(t) - s_0) + k \cdot (t - t_0)) \cdot ((\phi(t) - s_0) - k \cdot (t - t_0)) = 0.$$

Since ϕ is analytic, it follows that

$$\phi(t) = -kt + (kt_0 + s_0) \quad \text{or} \quad \phi(t) = kt + (-kt_0 + s_0).$$

In either case, ϕ is an affine function. ■

Of course, ϕ is not always an affine function. For example, suppose that f and g have $p \geq 3$ critical points. Denote by $t_1, \ldots, t_p$ the critical points of f, and by $s_1, \ldots, s_p$ the critical points of g. By the remark just after the proof of Lemma 3.3, $\phi(t_1), \ldots, \phi(t_p)$

are exactly the critical points of g, so reordering the s_i if necessary we have $\phi(t_i) = s_i$, for $i = 1, \dots, p$. But there is no reason why the points $(t_1, s_1), \dots, (t_p, s_p)$ should all lie on a straight line; so ϕ is not necessarily an affine function.

Note, however, that ϕ might not be an affine function even if f and g had less than 3 critical points. We illustrate this point by giving an example of two Lipschitz equivalent polynomial functions f and g having no critical points for which ϕ is not an affine function. By Proposition 3.4, any two such polynomial functions f and g must have degree ≥ 3. Consider, for example[6], the polynomial functions $f, g \colon \mathbb{R} \to \mathbb{R}$ given by

$$f(t) = t^3 + t^2 + t \quad \text{and} \quad g(t) = t^3 - t^2 + t.$$

Both these functions have no critical points, so by Theorem 3.1a, they are Lipschitz equivalent.

Actually, there exist infinitely many bi-Lipschitz functions $\phi \colon \mathbb{R} \to \mathbb{R}$ such that $g \circ \phi = cf$, for some constant $c > 0$. More precisely, for each constant $c > 0$, there exists a unique bi-Lipschitz function $\phi \colon \mathbb{R} \to \mathbb{R}$ such that $g \circ \phi = cf$ (such function ϕ is given by $\phi := g^{-1} \circ \hat{f}$, where $\hat{f} := cf$). None of these functions is affine. In order to see this, we show that there is no affine function $\phi \colon \mathbb{R} \to \mathbb{R}$ such that $g \circ \phi = cf$, for some constant $c > 0$.

Suppose that there exist $a, b, c \in \mathbb{R}$ satisfying the following conditions:

$$\begin{cases} g(at + b) = cf(t), \text{ for all } t \in \mathbb{R} \\ a \neq 0, \, c > 0 \end{cases}$$

Since f and g are polynomial functions of degree 3, the condition

$$g(at + b) = cf(t), \text{ for all } t \in \mathbb{R}$$

is equivalent to the following system of equations:

$$\begin{cases} g(b) = cf(0) \\ a \cdot g'(b) = cf'(0) \\ a^2 \cdot g''(b) = cf''(0) \\ a^3 \cdot g'''(b) = cf'''(0) \end{cases}$$

[6]This example has been suggested to me by Prof. Vincent Grandjean.

Thus, we can restate our hypothesis by saying that there exist $a, b, c \in \mathbb{R}$ such that

$$\begin{cases} b^3 - b^2 + b = 0 \\ a \cdot (3b^2 - 2b + 1) = c \\ a^2 \cdot (3b - 1) = c \\ a^3 = c \\ a \neq 0,\ c > 0 \end{cases} \tag{5}$$

Since $c = a^3$, the second and third equations of the system above can be rewritten as:

$$a \cdot (3b^2 - 2b + 1) = a^3 \quad \text{and} \quad a^2 \cdot (3b - 1) = a^3.$$

And since $a \neq 0$, it follows that

$$3b^2 - 2b + 1 = a^2 \quad \text{and} \quad 3b - 1 = a.$$

Hence,

$$3b^2 - 2b + 1 = (3b - 1)^2.$$

Equivalently,

$$b \cdot (3b - 2) = 0.$$

Therefore,

$$b = 0 \ \text{ or } \ 3b - 2 = 0. \tag{6}$$

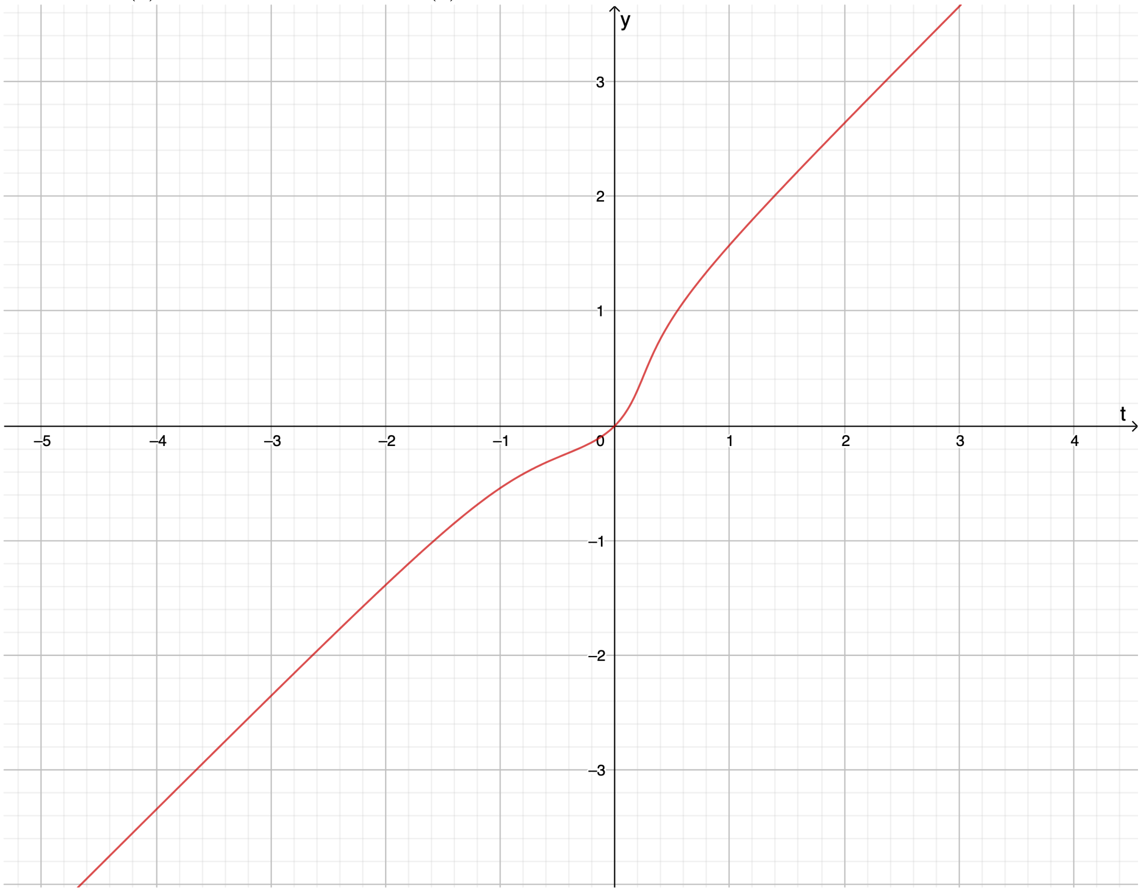
On the other hand, the first equation of the system, tells us that

$$b = 0 \ \text{ or } \ b^2 - b + 1 = 0. \tag{7}$$

From (6) and (7), it follows that $b = 0$. Since $a = 3b - 1$ and $c = a^3$, it follows that $a = -1$ and $c = -1$. This shows that the only triple of real numbers that could possibly satisfy (5) is $(a, b, c) = (-1, 0, -1)$. However, $(a, b, c) = (-1, 0, -1)$ is not a solution of (5), since it does not satisfy the condition $c > 0$. So the system does not admit any solutions, and hence there is no affine function $\phi\colon \mathbb{R} \to \mathbb{R}$ such that $g \circ \phi = cf$, for some $c > 0$.

Now we investigate the asymptotic behavior of the bi-Lipschitz transformation ϕ relating any two given Lipschitz equivalent polynomial functions $f, g\colon \mathbb{R} \to \mathbb{R}$. We begin with an auxiliary result, which shows that the function $\phi(t)/t$, defined on $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$, can be extended to an analytic function defined on $\mathbb{R}^* \cup \{\infty\} \subseteq \mathbb{P}^1(\mathbb{R})$, whose value at ∞ is a nonzero real number.

Figure 2 – Graph of the function ϕ such that $g \circ \phi = f$, obtained by plotting the zeros of the equation $g(y) - f(t) = 0$, for $f(t) = t^3 + t^2 + t$ and $g(t) = t^3 - t^2 + t$



Source: Elaborated by the author.

Lemma 3.5. *Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a bi-Lipschitz homeomorphism that satisfies an equation of the form $g \circ \phi = cf$, where $f, g\colon \mathbb{R} \to \mathbb{R}$ are nonconstant polynomial functions of the same degree, and $c > 0$ is a constant. We have:*

i. $\lim_{t\to+\infty} \phi(t)/t = \lim_{t\to-\infty} \phi(t)/t = \lambda$, *where* $\lambda \in \mathbb{R} \setminus \{0\}$

ii. The function $\psi\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\psi(t) := \begin{cases} t\phi(t^{-1}), & \text{if } \ t \in \mathbb{R} \setminus \{0\} \\ \lambda, & \text{if } \ t = 0 \end{cases} \tag{8}$$

is analytic.[7]

Proof.

[7]This fact and the approach taken here to prove it have been suggested to me by Prof. Maria Michalska.

i. Let $f(t) = \sum_{i=0}^{d} a_i t^i$ and $g(t) = \sum_{i=0}^{d} b_i t^i$, where $a_d, b_d \neq 0$, and let

$$l_+ := \lim_{t\to+\infty} \frac{\phi(t)}{t} \quad \text{and} \quad l_- := \lim_{t\to-\infty} \frac{\phi(t)}{t}.$$

Both of these limits are well-defined in the extended real line because ϕ is semialgebraic (see Lemma 3.1). Also, since ϕ is bi-Lipschitz, l_+ and l_- are nonzero real numbers.

Claim. $l_+^d = l_-^d = c \cdot \dfrac{a_d}{b_d}$

Proof of the Claim. Since $cf = g \circ \phi$, we have

$$c \cdot \lim_{t\to+\infty} \frac{f(t)}{t^d} = \lim_{t\to+\infty} \frac{g(\phi(t))}{t^d} = \lim_{t\to+\infty} \frac{g(\phi(t))}{\phi(t)^d} \cdot \lim_{t\to+\infty} \left(\frac{\phi(t)}{t}\right)^d. \tag{9}$$

On the other hand, since $\lim_{|t|\to+\infty} |\phi(t)| = +\infty$, we have

$$\lim_{|t|\to+\infty} \frac{g(\phi(t))}{\phi(t)^d} = \lim_{|t|\to+\infty} \frac{g(t)}{t^d} = b_d. \tag{10}$$

From (9) and (10), we obtain $c \cdot a_d = b_d \cdot l_+^d$. Hence, $l_+^d = c \cdot a_d/b_d$. The proof of the equality $l_-^d = c \cdot a_d/b_d$ is completely analogous. ■

By the claim above, we have $l_+^d = l_-^d$, which implies that $|l_+| = |l_-|$ (note that $d \geq 1$). On the other hand, since ϕ is monotone and $\lim_{|t|\to+\infty} |\phi(t)| = +\infty$, either $\phi(t)$ and t have the same sign for large values of $|t|$, or $\phi(t)$ and t have opposite signs for large values of $|t|$. In any case, l_+ and l_- have the same sign, and therefore $l_+ = l_-$.

ii. By Lemma 3.3, ψ is analytic at every point $t \in \mathbb{R} \setminus \{0\}$, so we only need to prove that ψ is analytic at $t = 0$. Let $P(X,Y) := g(Y) - cf(X)$ and let $P^*(X,Y,Z)$ be the homogeneization of P. Also, let $f(t) = \sum_{i=0}^{d} a_i t^i$ and $g(t) = \sum_{i=0}^{d} b_i t^i$, where $a_d, b_d \neq 0$ and $d \geq 1$, so

$$P(X,Y) = \sum_{i=0}^{d} b_i Y^i - c \cdot \sum_{i=0}^{d} a_i X^i$$

and

$$P^*(X,Y,Z) = \sum_{i=0}^{d} b_i Y^i Z^{d-i} - c \cdot \sum_{i=0}^{d} a_i X^i Z^{d-i}.$$

Since $g(\phi(t)) = cf(t)$, we have $P(t, \phi(t)) = 0$ for all $t \in \mathbb{R}$. Equivalently, $P^*(t, \phi(t), 1) = 0$ for all $t \in \mathbb{R}$. Since P^* is a homogeneous polynomial, it follows that $P^*(1, \phi(t)/t, 1/t) = 0$ for all $t \in \mathbb{R} \setminus \{0\}$. Equivalently,

$$P^*(1, t\phi(t^{-1}), t) = 0 \quad \text{for all } t \in \mathbb{R} \setminus \{0\}.$$

Let $\widetilde{P}(Y, Z) := P^*(1, Y, Z)$. From the computations above, it follows that

$$\widetilde{P}(\psi(t), t) = 0 \quad \text{for all } t \in \mathbb{R}.$$

Now, we show that the equation $\widetilde{P}(y, z) = 0$ determines y as an analytic function of z in a neighborhood of $(\lambda, 0)$, and from this we deduce that ψ is analytic at 0. The partial derivatives of $\widetilde{P}$ at $(y, 0)$ are given by

$$\frac{\partial \widetilde{P}}{\partial y}(y, 0) = d \cdot b_d \cdot y^{d-1} \quad \text{and} \quad \frac{\partial \widetilde{P}}{\partial z}(y, 0) = b_{d-1} \cdot y^{d-1} - c \cdot a_{d-1}.$$

Since $\widetilde{P}(\lambda, 0) = 0$ and $\frac{\partial \widetilde{P}}{\partial y}(\lambda, 0) = d \cdot b_d \cdot \lambda^{d-1} \neq 0$, the Implicit Function Theorem guarantees that there exists an analytic function $\widetilde{\psi} \colon I \to J$ from an open interval I containing 0 to an open interval J containing λ such that

$$\widetilde{P}(y, z) = 0 \Leftrightarrow y = \widetilde{\psi}(z), \quad \text{for all } y \in J, z \in I.$$

Since ψ is continuous and $\psi(0) = \lambda$, there exists an open interval $I_0 \subseteq I$ containing 0 such that $\psi(t) \in J$ for all $t \in I_0$. And since $\widetilde{P}(\psi(t), t) = 0$ for all $t \in I_0$, it follows that $\psi(t) = \widetilde{\psi}(t)$ for all $t \in I_0$. Hence, ψ is analytic at $t = 0$.

■

Proposition 3.5. *Let $\phi \colon \mathbb{R} \to \mathbb{R}$ be a bi-Lipschitz function that satisfies an equation of the form $g \circ \phi = cf$, where $f, g \colon \mathbb{R} \to \mathbb{R}$ are nonconstant polynomial functions of the same degree, and $c > 0$ is a constant. Then there exist $\lambda, k \in \mathbb{R}$, with $\lambda \neq 0$, such that*

$$\phi(t) = (\lambda t + k) + \alpha(t), \tag{11}$$

where $\alpha \colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz Nash function such that:

i. $\lim_{|t| \to +\infty} \alpha(t) = 0$

ii. *α can be analytically extended to $\mathbb{R} \cup \{\infty\} \cong \mathbb{P}^1(\mathbb{R})$.*

Proof. Let $\lambda \in \mathbb{R} \setminus \{0\}$ and $\psi \colon \mathbb{R} \to \mathbb{R}$ be as in Lemma 3.5. We prove that the limit $\lim_{|t| \to +\infty} \phi(t) - \lambda t$ is a well-defined real number. For $t \neq 0$, we have

$$\phi(t) - \lambda t = t\psi(t^{-1}) - \lambda t = \frac{\psi(t^{-1}) - \lambda}{t^{-1}}.$$

Hence,

$$\lim_{|t|\to+\infty} \phi(t) - \lambda t = \lim_{t\to 0} \phi(t^{-1}) - \lambda t^{-1} = \lim_{t\to 0} \frac{\psi(t) - \lambda}{t} = \psi'(0) \in \mathbb{R}.$$

Now, let $k \coloneqq \lim_{|t|\to+\infty} \phi(t) - \lambda t$. Obviously, the only function $\alpha\colon \mathbb{R} \to \mathbb{R}$ satisfying (11) is the one given by $\alpha(t) \coloneqq \phi(t) - (\lambda t + k)$, which is a Lipschitz Nash function (because it is the difference of two Lipschitz Nash functions). From the definition of k, it follows that $\lim_{|t|\to+\infty} \alpha(t) = 0$. It remains to show that α can be analytically extended to $\mathbb{R} \cup \{\infty\} \cong \mathbb{P}^1(\mathbb{R})$.

For $t \in \mathbb{R} \setminus \{0\}$, we have

$$\begin{aligned} \alpha(t^{-1}) &= \phi(t^{-1}) - (\lambda t^{-1} + k) \\ &= (\psi(t) - \lambda) \cdot t^{-1} - k \end{aligned}$$

On the other hand, since $\psi(0) = \lambda$ and $\psi'(0) = k$, we have

$$\psi(t) = \lambda + kt + \sum_{k=2}^{\infty} c_k t^k,$$

for $|t|$ sufficiently small. Hence,

$$\alpha(t^{-1}) = \sum_{k=1}^{\infty} c_{k+1} t^k,$$

for $|t|$ sufficiently small.

Therefore, the function $\widehat{\alpha}\colon \mathbb{R} \cup \{\infty\} \to \mathbb{R}$ defined by

$$\widehat{\alpha}(t) \coloneqq \begin{cases} \alpha(t), & \text{if } t \in \mathbb{R} \\ 0, & \text{if } t = \infty \end{cases}$$

is analytic. ■

The next proposition shows how to compute the constants λ and k appearing in equation (11).

Proposition 3.6. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be nonconstant polynomial functions that are Lipschitz equivalent,*[8] *so that $g \circ \phi = cf$, for some bi-Lipschitz function $\phi\colon \mathbb{R} \to \mathbb{R}$ and some constant $c > 0$. Let $\lambda \coloneqq \lim_{|t|\to+\infty} \phi(t)/t$ and $k \coloneqq \lim_{|t|\to+\infty} \phi(t) - \lambda t$ (λ is well-defined, by Lemma 3.5; and k is well-defined, by Proposition 3.5). If $f(t) = \sum_{i=0}^{d} a_i t^i$ and $g(t) = \sum_{i=0}^{d} b_i t^i$, with $a_d, b_d \neq 0$, then we have:*

i. $b_d \cdot \lambda^d - c \cdot a_d = 0$

[8] By Lemma 3.2, f and g have the same degree.

ii. $k = \dfrac{c \cdot a_{d-1} - b_{d-1} \cdot \lambda^{d-1}}{d \cdot b_d \cdot \lambda^{d-1}}$

Remark 3.2. *It follows from* (i) *that*

$$\lambda = \left(\frac{c \cdot a_d}{b_d}\right)^{\frac{1}{d}}, \quad \text{if } d \text{ is odd.}$$

But if d *is even, we have only the equality of absolute values*

$$|\lambda| = \left|\frac{c \cdot a_d}{b_d}\right|^{\frac{1}{d}},$$

so we still need to determine the sign of λ *if we want to determine its value. This can be done by analyzing the behavior of* ϕ*: if* ϕ *is increasing then* $\lambda > 0$*, if* ϕ *is decreasing then* $\lambda < 0$*.*

Proof. Retaining the notation used in the proof of Lemma 3.5, we have

$$\tilde{P}(\psi(t), t) = 0, \quad \text{for all } t \in \mathbb{R}, \tag{12}$$

where $\tilde{P}(y, z) = \sum_{i=0}^{d} b_i y^i z^{d-i} - c \cdot \sum_{i=0}^{d} a_i z^{d-i}$.

Setting $t = 0$ in (12), we get $\tilde{P}(\lambda, 0) = 0$. This proves (i).

Now, differentiating both sides of (12), we get

$$\frac{\partial \tilde{P}}{\partial y}(\psi(t), t) \cdot \psi'(t) + \frac{\partial \tilde{P}}{\partial z}(\psi(t), t) = 0, \quad \text{for all } t \in \mathbb{R}.$$

Setting $t = 0$ in this equation, we obtain[9]

$$\frac{\partial \tilde{P}}{\partial y}(\lambda, 0) \cdot k + \frac{\partial \tilde{P}}{\partial z}(\lambda, 0) = 0$$

Performing explicit computations, we get

$$d \cdot b_d \cdot \lambda^{d-1} \cdot k + (b_{d-1}\lambda^{d-1} - c \cdot a_{d-1}) = 0.$$

From this, (ii) follows. ■

Let us revisit the example of the polynomial functions $f(t) = t^3 + t^2 + t$ and $g(t) = t^3 - t^2 + t$. Figure 2 shows the graph of the bi-Lipschitz function $\phi\colon \mathbb{R} \to \mathbb{R}$ such that $g \circ \phi = f$. Now, we obtain the graph of the function $\alpha\colon \mathbb{R} \to \mathbb{R}$ given by Proposition

[9] In the proof of Proposition 3.5, we showed that $k = \psi'(0)$.

3.5. We begin by computing the coefficients λ and k that appear in equation (11). By Proposition 3.6, we have:

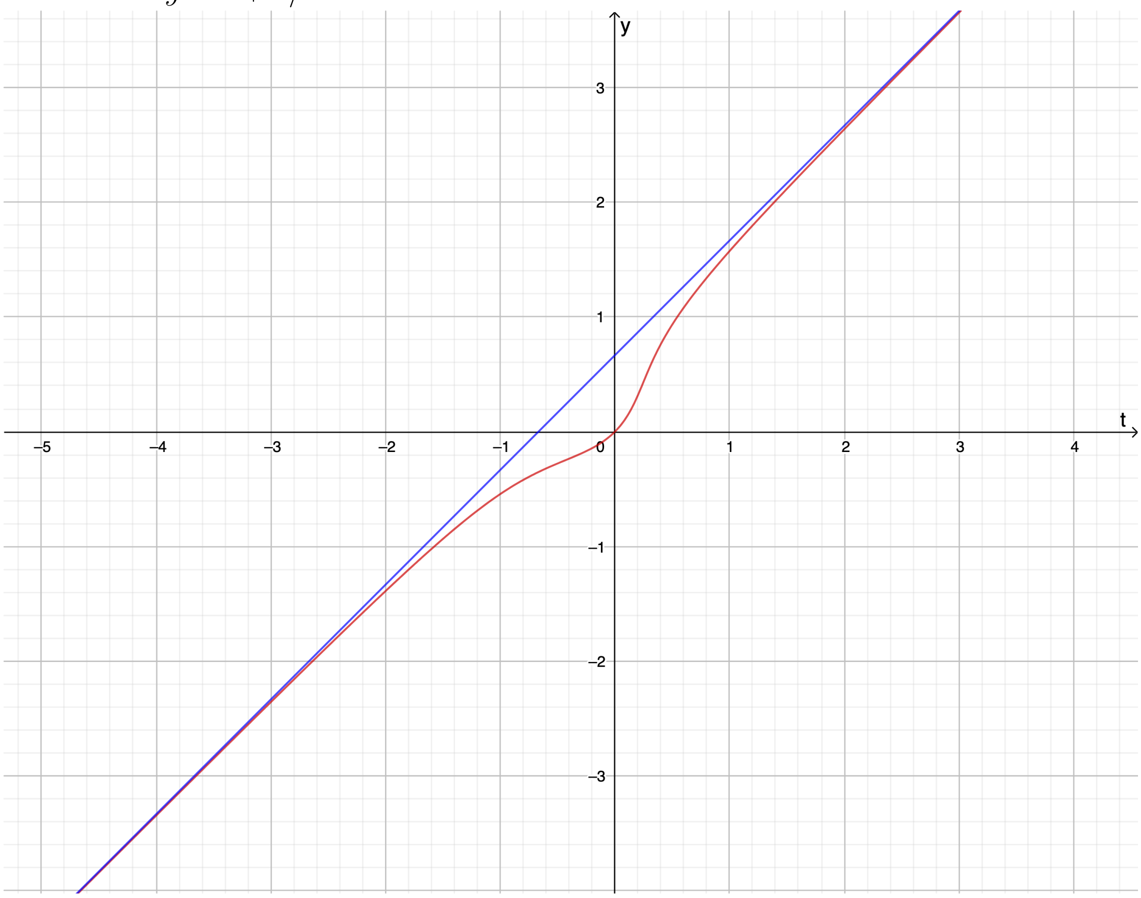
$$\lambda = \left(\frac{c \cdot a_3}{b_3}\right)^{\frac{1}{3}} \quad \text{and} \quad k = \frac{c \cdot a_2 - b_2 \cdot \lambda^2}{3 \cdot b_3 \cdot \lambda^2} \tag{13}$$

In the example under consideration, we have: $c = a_3 = b_3 = a_2 = 1$ and $b_2 = -1$. Substituting these values into (13), we obtain: $\lambda = 1$ and $k = 2/3$. Thus,

$$\phi(t) = \left(t + \frac{2}{3}\right) + \alpha(t), \quad \text{for all } t \in \mathbb{R}. \tag{14}$$

Since $\lim_{|t|\to+\infty} \alpha(t) = 0$, the above equation shows that the line $y = t + 2/3$ is asymptotic to the graph of ϕ.

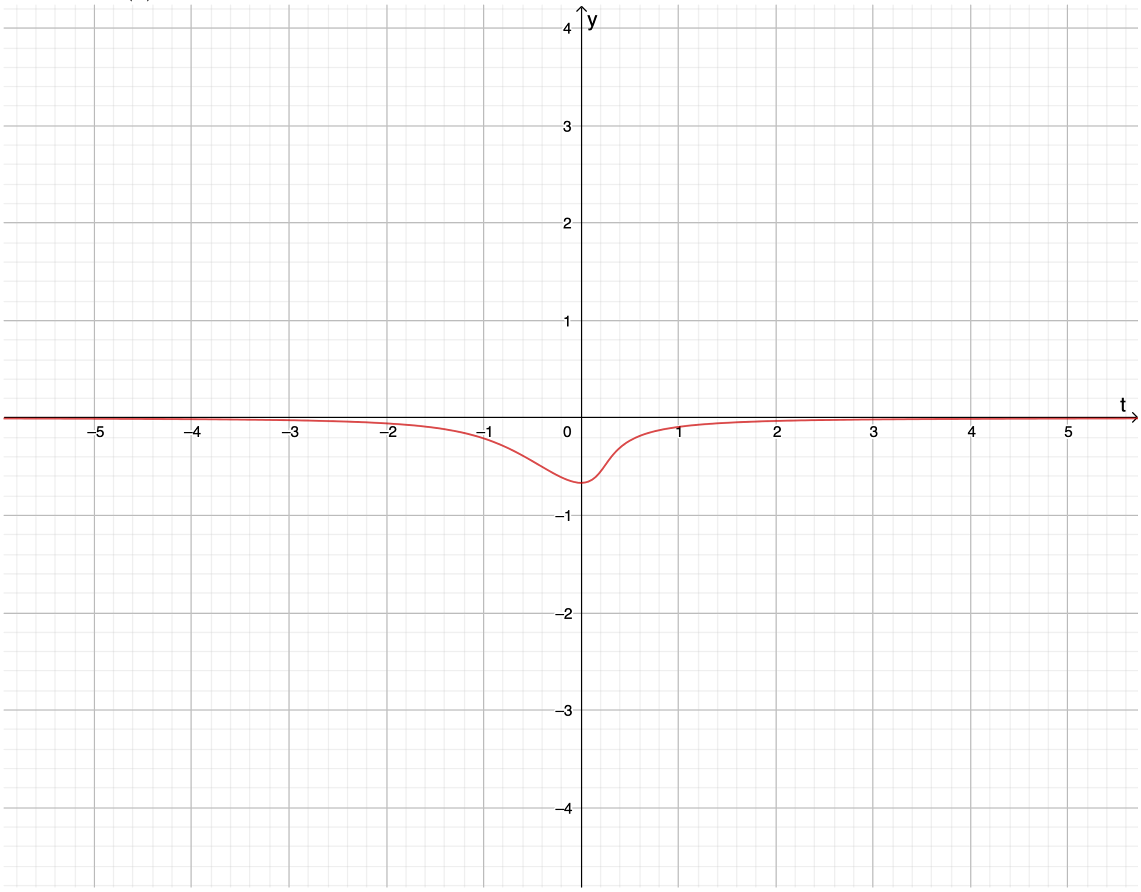
Figure 3 – Graph of the function ϕ such that $g \circ \phi = f$, for $f(t) = t^3 + t^2 + t$ and $g(t) = t^3 - t^2 + t$, along with the asymptotic line $y = t + 2/3$



Source: Elaborated by the author.

Also by (14), the graph of α is the zero set of the polynomial $g((t + 2/3) + y) - f(t)$. Since we do not have a simple explicit expression for α, this provides a convenient means for graphing this function.

Figure 4 – Graph of the function α, obtained by plotting the zeros of the equation $g((t+2/3)+y)-f(t)=0$, for $f(t)=t^3+t^2+t$ and $g(t)=t^3-t^2+t$



Source: Elaborated by the author.

4 $\mathcal{R}$-SEMIALGEBRAIC LIPSCHITZ EQUIVALENCE OF β-QUASIHOMOGENEOUS POLYNOMIALS

Let β be a rational number > 1 and let d be a positive integer. A polynomial $F(X, Y) \in \mathbb{R}[X, Y]$ is said to be *β-quasihomogeneous of degree d* if

$$F(tX, t^\beta Y) = t^d F(X, Y) \text{ for all } t > 0.$$

The positive integer d is called *β-quasihomogeneous degree* of $F(X, Y)$.

Remark 4.1. *According to this definition, the zero polynomial is β-quasihomogeneous of degree d, for all rational numbers $\beta > 1$ and all positive integers d. But we restrict our attention to nonzero β-quasihomogeneous polynomials, so whenever we say that $F(X, Y)$ is a β-quasihomogeneous polynomial of degree d, it is implied that F is a nonzero polynomial.*

Remark 4.2. *In this thesis, we consider only polynomials with real coefficients. So, throughout the text, the word "polynomial" is meant to be understood as "real polynomial".*

If $\beta = r/s$, where $r > s > 0$ and $\gcd(r, s) = 1$, then the β-quasihomogeneous polynomials of degree d are those of the form

$$F(X, Y) = \sum_{k=0}^{m} c_k X^{d-rk} Y^{sk},$$

where the coefficients c_k are real numbers, $c_m \neq 0$, and $m \leq \lfloor d/r \rfloor$.

In this chapter, we address the problem of determining whether any two given β-quasihomogeneous polynomials are $\mathcal{R}$-semialgebraically Lipschitz equivalent. Two β-quasihomogeneous polynomials $F(X, Y)$ and $G(X, Y)$ are said to be *$\mathcal{R}$-semialgebraically Lipschitz equivalent* if there exists a germ of semialgebraic bi-Lipschitz homeomorphism $\Phi \colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $G \circ \Phi = F$.

Following the approach taken in BIRBRAIR, FERNANDES, and PANAZZOLO (2009), we reduce this problem to the simpler one of Lipschitz classification of polynomial functions of a single variable (which has been solved in Chapter 3). In order to do this, we associate two polynomial functions of a single variable with each β-quasihomogeneous polynomial $F(X, Y)$: the *right height function* $f_+ \colon \mathbb{R} \to \mathbb{R}$, given by $f_+(t) := F(1, t)$, and the *left height function* $f_- \colon \mathbb{R} \to \mathbb{R}$, given by $f_-(t) := F(-1, t)$. Since

$$F(x, t\,|x|^\beta) = \begin{cases} |x|^d f_+(t), & \text{if } x > 0 \\ |x|^d f_-(t), & \text{if } x < 0 \end{cases},$$

it is clear that the height functions f_+ and f_- encode a great deal of information about the behavior of the function given by $(x, y) \mapsto F(x, y)$, for all $(x, y) \in \mathbb{R}^2$. So, we consider the following questions:

1. Can we say that if two given β-quasihomogeneous polynomials $F(X, Y)$ and $G(X, Y)$ are $\mathcal{R}$-semialgebraically Lipschitz equivalent, then their height functions can be arranged in pairs of Lipschitz equivalent functions (i.e. either $f_+ \cong g_+$ and $f_- \cong g_-$, or $f_+ \cong g_-$ and $f_- \cong g_+$)?
2. Can we say that if the height functions of two given β-quasihomogeneous polynomials can be arranged in pairs of Lipschitz equivalent functions then these polynomials are $\mathcal{R}$-semialgebraically Lipschitz equivalent?

In this chapter, we obtain fairly general conditions under which each of these questions is answered affirmatively. So, under appropriate conditions, the problem of determining whether two given β-quasihomogeneous polynomials are $\mathcal{R}$-semialgebraically Lipschitz equivalent will be reduced to the problem of determining whether their height functions (which are polynomial functions of a single variable) can be arranged in pairs of Lipschitz equivalent functions. Since the problem of determining whether two given polynomial functions of a single variable are Lipschitz equivalent has already been solved in Chapter 3, this will enable us to determine, under fairly general conditions, whether two given β-quasihomogeneous polynomials are $\mathcal{R}$-semialgebraically Lipschitz equivalent.

Overview of this chapter. In Section 4.1, we show that if two β-quasihomogeneous polynomials $F(X, Y)$ and $G(X, Y)$ of the same β-quasihomogeneous degree d, whose zero sets contain points both on the left half-plane and on the right half-plane, are $\mathcal{R}$-semialgebraically Lipschitz equivalent, then their height functions can be arranged in pairs of Lipschitz equivalent functions. (See Corollary 4.3, Remark 4.7.) We begin by introducing a special type of germ of semialgebraic map $\Phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ called *β-isomorphism*, and the so-called *β-transform*, which turns each β-isomorphism Φ into a pair (λ, ϕ), where $\lambda = (\lambda_+, \lambda_-)$ is an ordered pair of nonzero real numbers having the same sign and $\phi = (\phi_+, \phi_-)$ is an ordered pair of bi-Lipschitz functions from $\mathbb{R}$ to $\mathbb{R}$ (see Proposition 4.8). We distinguish two kinds of β-isomorphisms: *direct* and *reverse* (see Proposition 4.2 and the paragraph following it). We show that if $F(X, Y)$ and $G(X, Y)$ are two β-quasihomogeneous polynomials such that $G \circ \Phi = F$, where Φ is a β-isomorphism, then

$$\begin{cases} g_+ \circ \phi_+ = |\lambda_+|^{-d} f_+ \quad \text{and} \quad g_- \circ \phi_- = |\lambda_-|^{-d} f_- \quad \text{if } \Phi \text{ is direct} \\ g_- \circ \phi_+ = |\lambda_+|^{-d} f_+ \quad \text{and} \quad g_+ \circ \phi_- = |\lambda_-|^{-d} f_- \quad \text{if } \Phi \text{ is reverse} \end{cases},$$

where f_+, f_- are the height functions of F, g_+, g_- are the height functions of G, and $(\lambda, \phi) = ((\lambda_+, \lambda_-), (\phi_+, \phi_-))$ is the β-transform of Φ (see Proposition 4.9). Then, we show that if two β-quasihomogeneous polynomials $F(X, Y)$ and $G(X, Y)$ of the same β-quasihomogeneous degree d whose zero sets contain points both on the left half-plane and on the right half-plane, are $\mathcal{R}$-semialgebraically Lipschitz equivalent, then there exists a β-isomorphism Φ such that $G \circ \Phi = F$ (see Theorem 4.1); hence, by applying Proposition 4.9, we conclude that the height functions of F and G can be arranged in pairs of Lipschitz equivalent functions (see Theorem 4.1 and Corollary 4.3).

In the next three sections, we address the problem of determining appropriate conditions to ensure that if the height functions of two given β-quasihomogeneous polynomials of the same β-quasihomogeneous degree d can be arranged in pairs of Lipschitz equivalent functions, then these polynomials are $\mathcal{R}$-semialgebraically Lipschitz equivalent.

We begin by expressing the above hypothesis on the height functions in terms of a certain group action. In Section 4.2, we introduce a group, called the *group of proto-transitions*, whose elements are ordered pairs (λ, ϕ), where $\lambda = (\lambda_1, \lambda_2)$ is an ordered pair of nonzero real numbers having the same sign and $\phi = (\phi_1, \phi_2)$ is an ordered pair of bi-Lipschitz Nash diffeomorphisms on $\mathbb{R}$; then we define a family of group actions of the group of proto-transitions on $C^\omega \times C^\omega$, depending on an integer parameter $d \geq 1$. For each $d \geq 1$, we denote by $(g_1, g_2) \circ (\lambda, \phi)$ the action of the proto-transition (λ, ϕ) on the element $(g_1, g_2) \in C^\omega \times C^\omega$. By the end of Section 4.2, we show that the height functions f_+, f_- and g_+, g_- of two given β-quasihomogeneous polynomials $F(X, Y)$ and $G(X, Y)$ can be arranged in pairs of Lipschitz equivalent functions if and only if $(g_+, g_-) \circ (\lambda, \phi) = (f_+, f_-)$, for some proto-transition (λ, ϕ) (see Corollary 4.4).

In Section 4.3, we introduce a special type of proto-transition, called *β-transition*, and we show how to construct from any given β-transition (λ, ϕ) a map germ $\Phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ called the *inverse β-transform* of (λ, ϕ). We show that Φ is actually a germ of semialgebraic bi-Lipschitz map (see Corollary 4.7); also, we show that if $F(X, Y)$ and $G(X, Y)$ are β-quasihomogeneous polynomials of the same β-quasihomogeneous degree d such that $(g_+, g_-) \circ (\lambda, \phi) = (f_+, f_-)$, where (λ, ϕ) is a β-transition, then $G \circ \Phi = F$, where Φ is the inverse β-transform of (λ, ϕ) (see Proposition 4.17), and then it follows that F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent (see Corollary 4.8).

All this brings us to the following problem: suppose that the height functions f_+, f_- and g_+, g_- of two given β-quasihomogeneous polynomials $F(X, Y)$ and $G(X, Y)$ of the same β-quasihomogeneous degree d can be arranged in pairs of Lipschitz equivalent functions, so that $(g_+, g_-) \circ (\lambda, \phi) = (f_+, f_-)$ for some proto-transition (λ, ϕ). Under what conditions, can we choose this proto-transition among the β-transitions? In Section 4.4, we obtain fairly general conditions under which such a choice is possible (see Proposition 4.21 and Corollary 4.10), so that we can apply Corollary 4.8 to conclude that F and G are

$\mathcal{R}$-semialgebraically Lipschitz equivalent (see Theorem 4.2, Corollary 4.11, and Corollary 4.12).

Finally, in Section 4.5, we consider a specific family of quasihomogeneous polynomials, which has been used before in HENRY and PARUSIŃSKI (2004) to show that the bi-Lipschitz equivalence of analytic function germs $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ admits continuous moduli. Using results obtained in this thesis, we investigate the properties of this family in the context of $\mathcal{R}$-semialgebraic Lipschitz equivalence. As a byproduct, our conclusions show that the $\mathcal{R}$-semialgebraic Lipschitz equivalence of real β-quasihomogeneous polynomials in two variables admits continuous moduli.

4.1 β-isomorphisms and the β-transform

In this section, we show that if two β-quasihomogeneous polynomials of the same β-quasihomogeneous degree d whose zero sets contain points both on the left half-plane and on the right half-plane are $\mathcal{R}$-semialgebraically Lipschitz equivalent, then their height functions can be arranged in pairs of Lipschitz equivalent functions. (See Theorem 4.1 and Corollary 4.3 at the end of this section.)

A germ of semialgebraic bi-Lipschitz map $\Phi = (\Phi_1, \Phi_2)\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is said to be a *β-isomorphism* if the following conditions are satisfied:

i. There exist β-quasihomogeneous polynomials $F(X, Y)$ and $G(X, Y)$ of the same β-quasihomogeneous degree d such that $G \circ \Phi = F$.

ii. $\lim_{x\to 0^+} \Phi_1(x, 0)/x \neq 0$ and $\lim_{x\to 0^-} \Phi_1(x, 0)/x \neq 0$

Remark 4.3. *For any germ of semialgebraic bi-Lipschitz homeomorphism $\Phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, the path $\Phi(x, 0)$, $0 \leq x < \epsilon$, has finite nonzero initial velocity.*

In fact, it is immediate from Lemma 2.2 that the path $\Phi(x, 0)$, $0 \leq x < \epsilon$, has finite initial velocity. On the other hand, there exists $A > 0$ such that $\Phi(x, 0) \geq A\,|x|$, because Φ is bi-Lipschitz and $\Phi(0, 0) = 0$; so we have $|\Phi(x, 0)/x| \geq A$, for $x \neq 0$. Hence, $\lim_{x\to 0^+} \Phi(x, 0)/x \neq 0$. Similarly, we can prove that the path $\Phi(-x, 0)$, $0 \leq x < \epsilon$, also has finite nonzero initial velocity.

Remark 4.4. *Let $\Phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be any germ of semialgebraic bi-Lipschitz map. It is immediate from Lemma 2.3 that for all $t \in \mathbb{R}$, we have:*

i. The initial velocity of the path $\Phi(x, tx^\beta)$, $0 \leq x < \epsilon$, is equal to the initial velocity of the path $\Phi(x, 0)$, $0 \leq x < \epsilon$.

ii. The initial velocity of the path $\Phi(-x, tx^\beta)$, $0 \leq x < \epsilon$, is equal to the initial velocity of the path $\Phi(-x, 0)$, $0 \leq x < \epsilon$.

Hence, for all $t \in \mathbb{R}$,

$$\lim_{x\to 0^+} \frac{\Phi_1(x,tx^\beta)}{x} = \lim_{x\to 0^+} \frac{\Phi_1(x,0)}{x} \quad \text{and} \quad \lim_{x\to 0^+} \frac{\Phi_1(-x,tx^\beta)}{x} = \lim_{x\to 0^+} \frac{\Phi_1(-x,0)}{x}. \tag{15}$$

Also, for all $t \in \mathbb{R}$,

$$\lim_{x\to 0^+} \frac{\Phi_2(x,tx^\beta)}{x} = \lim_{x\to 0^+} \frac{\Phi_2(x,0)}{x} \quad \text{and} \quad \lim_{x\to 0^+} \frac{\Phi_2(-x,tx^\beta)}{x} = \lim_{x\to 0^+} \frac{\Phi_2(-x,0)}{x}. \tag{16}$$

Proposition 4.1. *Let $\Phi\colon (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ be a germ of semialgebraic bi-Lipschitz homeomorphism. Suppose that there exist β-quasihomogeneous polynomials $F(X,Y)$ and $G(X,Y)$ of degree d such that $G \circ \Phi = F$. We have:*

i. If Φ is a β-isomorphism then, for each $t \in \mathbb{R}$, $\Phi_2(x,t\,|x|^\beta) = O(|x|^\beta)$ as $x \to 0$.

ii. If there exist $t_0, t_1 \in \mathbb{R}$ such that

$$\Phi_2(x,t_0\,|x|^\beta) = O(|x|^\beta) \text{ as } x \to 0^+ \quad \text{and} \quad \Phi_2(x,t_1\,|x|^\beta) = O(|x|^\beta) \text{ as } x \to 0^-,$$

then Φ is a β-isomorphism.

Proof. Suppose that Φ is a β-isomorphism. Let $\beta = r/s$, where $r > s > 0$ and $\gcd(r,s) = 1$. Then we have

$$G(X,Y) = \sum_{k=0}^{m} c_k X^{d-rk} Y^{sk},$$

where the coefficients c_k are real numbers, $c_m \neq 0$, and $m \leq \lfloor d/r \rfloor$.

By hypothesis, $G(\Phi(x,y)) = F(x,y)$. Thus, for any $t \in \mathbb{R}$ and $x \neq 0$ sufficiently small,

$$G(\Phi(x,t\,|x|^\beta)) = F(x,t\,|x|^\beta).$$

Since the polynomials F and G are β-quasihomogeneous of the same β-quasihomogeneous degree d, this implies that

$$G\left(\frac{\Phi_1(x,t\,|x|^\beta)}{|x|}, \frac{\Phi_2(x,t\,|x|^\beta)}{|x|^\beta}\right) = f(t),$$

where

$$f(t) = \begin{cases} F(1,t), & \text{if } x > 0 \\ F(-1,t), & \text{if } x < 0 \end{cases}.$$

Hence, for each $t \in \mathbb{R}$ and $x \neq 0$ sufficiently small, $y = \Phi_2(x,t\,|x|^\beta)/\,|x|^\beta$ is a zero of the nonconstant polynomial

$$H_{t,x}(y) \coloneqq G(\tilde{x},y) - f(t) \in \mathbb{R}[y],$$

where $\tilde{x} := \Phi_1(x, t\,|x|^\beta)/\,|x|$. Applying Cauchy's bound on the roots of a polynomial, we obtain

$$\left|\frac{\Phi_2(x,t\,|x|^\beta)}{|x|^\beta}\right| \leq 1 + \max\left\{\left|\frac{c_{m-1}}{c_m}\right| |\tilde{x}|^r, \ldots, \left|\frac{c_1}{c_m}\right| |\tilde{x}|^{r(m-1)}, \left|\frac{c_0\tilde{x}^d - f(t)}{c_m\tilde{x}^{d-rm}}\right|\right\}. \tag{17}$$

Since Φ is a β-isomorphism, we have

$$\lim_{x\to 0^+} \frac{\Phi_1(x,0)}{x} \neq 0 \quad \text{and} \quad \lim_{x\to 0^+} \frac{\Phi_1(-x,0)}{x} \neq 0$$

Then, by (15), we obtain

$$\lim_{x\to 0^+} \frac{\Phi_1(x,t\,|x|^\beta)}{|x|} \neq 0 \quad \text{and} \quad \lim_{x\to 0^-} \frac{\Phi_1(x,t\,|x|^\beta)}{|x|} \neq 0, \quad \text{for all } t \in \mathbb{R}. \tag{18}$$

From (17) and (18), it follows that, for each $t \in \mathbb{R}$, $\Phi_2(x, t\,|x|^\beta)/\,|x|^\beta$ is bounded for $x \neq 0$ sufficiently small. This proves (i).

Now, suppose that for certain $t_0, t_1 \in \mathbb{R}$, $\Phi_2(x, t_0\,|x|^\beta)/\,|x|^\beta$ is bounded for $x > 0$ sufficiently small, and $\Phi_2(x, t_1\,|x|^\beta)/\,|x|^\beta$ is bounded for $x < 0$ sufficiently small. Then

$$\lim_{x\to 0^+} \frac{\Phi_2(x,t_0x^\beta)}{x} = \lim_{x\to 0^+} \frac{\Phi_2(x,t_0x^\beta)}{x^\beta}\cdot x^{\beta-1} = 0$$

and

$$\lim_{x\to 0^+} \frac{\Phi_2(-x,t_1x^\beta)}{x} = \lim_{x\to 0^+} \frac{\Phi_2(-x,t_1x^\beta)}{x^\beta}\cdot x^{\beta-1} = 0$$

Thus, by (16),

$$\lim_{x\to 0^+} \frac{\Phi_2(x,0)}{x} = 0 \quad \text{and} \quad \lim_{x\to 0^+} \frac{\Phi_2(-x,0)}{x} = 0\,.$$

On the other hand, by Remark 4.3, the paths $\Phi(x,0)$, $0 \leq x < \epsilon$, and $\Phi(-x,0)$, $0 \leq x < \epsilon$, both have (finite) nonzero initial velocity. Hence,

$$\lim_{x\to 0^+} \frac{\Phi_1(x,0)}{x} \neq 0 \quad \text{and} \quad \lim_{x\to 0^+} \frac{\Phi_1(-x,0)}{x} \neq 0\,.$$

Therefore, Φ is a β-isomorphism. ■

Proposition 4.2. *If $\Phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is a β-isomorphism then the initial velocities of the paths $\Phi(x,0)$, $0 \leq x < \epsilon$, and $\Phi(-x,0)$, $0 \leq x < \epsilon$, are horizontal*[10] *and have opposite directions.*

Proof. Let Φ be a β-isomorphism. By Proposition 4.1, we have $\Phi_2(x,0) = O(|x|^\beta)$ as $x \to 0$. Then, $\lim_{x\to 0^+} \Phi_2(x,0)/x = \lim_{x\to 0^+} \Phi_2(-x,0)/x = 0$. Since the paths $\Phi(x,0)$, $0 \leq x < \epsilon$, and $\Phi(-x,0)$, $0 \leq x < \epsilon$, both have nonzero finite initial velocity (Remark

[10] We say that a vector $(v_1, v_2) \in \mathbb{R}^2$ is *horizontal* if $v_1 \neq 0$ and $v_2 = 0$.

4.3), it follows that both of them have horizontal initial velocity. By Corollary 2.7, the initial velocities of the paths $\Phi(x,0)$, $0 \le x < \epsilon$, and $\Phi(-x,0)$, $0 \le x < \epsilon$, do not have the same direction. Since they are both horizontal, they have opposite directions. ■

It follows from Proposition 4.2 that each β-isomorphism $\Phi\colon (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ satisfies one of the following conditions:

(A) $\lim_{x\to 0^+} \Phi_1(x,0)/x > 0$ and $\lim_{x\to 0^-} \Phi_1(x,0)/x > 0$

(B) $\lim_{x\to 0^+} \Phi_1(x,0)/x < 0$ and $\lim_{x\to 0^-} \Phi_1(x,0)/x < 0$

A β-isomorphism Φ is said to be *direct* if it satisfies (A), and it is said to be *reverse* if it satisfies (B).

Proposition 4.3. *Let $\Phi\colon (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ be a germ of semialgebraic bi-Lipschitz homeomorphism. Suppose that there exist β-quasihomogeneous polynomials $F(X,Y)$ and $G(X,Y)$ of the same β-quasihomogeneous degree d, such that $G \circ \Phi = F$. The following assertions are equivalent:*

i. Φ is a β-isomorphism

ii. For any continuous semialgebraic path $\gamma\colon [0,\epsilon) \to \mathbb{R}^2$ such that $\gamma(0) = 0$, if the initial velocity of γ is horizontal, then the initial velocity of $\tilde{\gamma} := \Phi \circ \gamma$ is horizontal.

Proof. First, we prove that $(i) \Rightarrow (ii)$. Suppose that Φ is a β-isomorphism. Take any continuous semialgebraic path $\gamma\colon [0,\epsilon) \to \mathbb{R}^2$ such that $\gamma(0) = 0$, whose initial velocity is horizontal. Since the initial velocities of the paths $\alpha_+, \alpha_-\colon [0,\epsilon) \to \mathbb{R}^2$, given by $\alpha_+(t) = (t,0)$ and $\alpha_-(t) = (-t,0)$, are both horizontal and have opposite directions, the initial velocity of γ has either the same direction as the initial velocity of α_+, or the same direction as the initial velocity of α_-. By Corollary 2.7, this implies that the initial velocity of $\tilde{\gamma} := \Phi \circ \gamma$ has either the same direction as the initial velocity of $\tilde{\alpha}_+ := \Phi \circ \alpha_+$, or the same direction as the initial velocity of $\tilde{\alpha}_- := \Phi \circ \alpha_-$. In any case, the initial velocity of $\tilde{\gamma}$ is horizontal: by Proposition 4.2, the initial velocities of $\tilde{\alpha}_+$ and $\tilde{\alpha}_-$ are both horizontal (and have opposite directions) because Φ is a β-isomorphism.

Now, we prove that $(ii) \Rightarrow (i)$. Assume that condition (ii) is satisfied. We show that $\lim_{x\to 0^+} \Phi_1(x,0)/x \neq 0$ and $\lim_{x\to 0^-} \Phi_1(x,0)/x \neq 0$. Since the initial velocity of the path α_+ is horizontal, it follows from our assumption that the initial velocity of the path $\tilde{\alpha}_+ = \Phi \circ \alpha_+$ is horizontal. Hence, $\lim_{x\to 0^+} \Phi_1(x,0)/x \neq 0$. Similarly, since the initial velocity of the path α_- is horizontal, it follows from our assumption that the initial velocity of the path $\tilde{\alpha}_- = \Phi \circ \alpha_-$ is horizontal. Hence, $\lim_{x\to 0^+} \Phi_1(-x,0)/x \neq 0$. And since $\lim_{x\to 0^-} \Phi_1(x,0)/x = -\lim_{x\to 0^+} \Phi_1(-x,0)/x$, we have $\lim_{x\to 0^-} \Phi_1(x,0)/x \neq 0$. ■

Let Φ and Ψ be β-isomorphisms. We say that Ψ is *composable* with Φ if there exist β-quasihomogeneous polynomials $F(X,Y), G(X,Y), H(X,Y)$, all of the same β-quasihomogeneous degree d, such that $H \circ \Psi = G$ and $G \circ \Phi = F$.

Proposition 4.4. *Let $\Phi, \Psi \colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be β-isomorphisms. If Ψ is composable with Φ, then $\Psi \circ \Phi$ is a β-isomorphism.*

Proof. Suppose that Ψ is composable with Φ. Then, there exist β-quasihomogeneous polynomials $F(X,Y), G(X,Y), H(X,Y)$ of the same β-quasihomogeneous degree d such that $H \circ \Psi = G$ and $G \circ \Phi = F$, so $H \circ (\Psi \circ \Phi) = F$. Now, take any continuous semialgebraic path $\gamma \colon [0, \epsilon) \to \mathbb{R}^2$ such that $\gamma(0) = 0$, with horizontal initial velocity. Since Φ is a β-isomorphism, the initial velocity of the continuous semialgebraic path $\tilde{\gamma} := \Phi \circ \gamma$ is horizontal; and since Ψ is a β-isomorphism, the initial velocity of the continuous semialgebraic path $(\Psi \circ \Phi) \circ \gamma = \Psi \circ \tilde{\gamma}$ is horizontal. By Proposition 4.3, $\Psi \circ \Phi$ is a β-isomorphism. ■

Proposition 4.5. *The germ of the identity map $I \colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is a β-isomorphism. Moreover, for every β-isomorphism $\Phi \colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, I is composable with Φ, Φ is composable with I, and $I \circ \Phi = \Phi \circ I = \Phi$.*

Proof. Clearly, for any β-quasihomogeneous polynomial $F(X,Y)$, we have $F \circ I = F$, and $\lim_{x \to 0} I(x,0)/x = 1$, so I is a β-isomorphism. Now, if $\Phi \colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is a β-isomorphism, then there exist β-quasihomogeneous polynomials of the same β-quasihomogeneous degree d such that $G \circ \Phi = F$. Since $G \circ I = G$ and $G \circ \Phi = F$, I is composable with Φ; and since $G \circ \Phi = F$ and $F \circ I = F$, Φ is composable with I. Obviously, $I \circ \Phi = \Phi \circ I = \Phi$. ■

Proposition 4.6. *Let $\Phi \colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a germ of semialgebraic bi-Lipschitz map. If Φ is a β-isomorphism then Φ^{-1} is a β-isomorphism. Moreover, Φ is composable with Φ^{-1}, Φ^{-1} is composable with Φ, and $\Phi \circ \Phi^{-1} = \Phi^{-1} \circ \Phi = I$.*

Proof. Since Φ is a β-isomorphism, there exist β-quasihomogeneous polynomials $F(X,Y)$, $G(X,Y)$ of the same β-quasihomogeneous degree d, such that $G \circ \Phi = F$. Hence, there exist β-quasihomogeneous polynomials $\tilde{F}(X,Y), \tilde{G}(X,Y)$ of degree d such that $\tilde{G} \circ \Phi^{-1} = \tilde{F}$: for example, take $\tilde{G} = F$ and $\tilde{F} = G$. Now, we show that for any continuous semialgebraic path $\tilde{\gamma} \colon [0, \epsilon) \to \mathbb{R}^2$, with $\tilde{\gamma}(0) = 0$, whose initial velocity is horizontal, the initial velocity of $\gamma := \Phi^{-1} \circ \tilde{\gamma}$ is horizontal. Fix one such path $\tilde{\gamma} \colon [0, \epsilon) \to \mathbb{R}^2$. Let $\alpha_+, \alpha_- \colon [0, \epsilon) \to \mathbb{R}^2$ be the continuous semialgebraic paths defined by $\alpha_+(t) = (t, 0)$ and $\alpha_-(t) = (-t, 0)$. Since Φ is a β-isomorphism, the initial velocities of the paths $\tilde{\alpha}_+ := \Phi \circ \alpha_+$ and $\tilde{\alpha}_- := \Phi \circ \alpha_-$ are both horizontal and have opposite directions. So, the initial velocity of $\tilde{\gamma}$ (which is horizontal) has either the same direction as the initial velocity of $\tilde{\alpha}_+$ or the same direction as the initial velocity of $\tilde{\alpha}_-$. Then, by Corollary 2.7, the initial velocity of $\gamma = \Phi^{-1} \circ \tilde{\gamma}$ has either the same direction as the initial velocity of $\alpha_+ = \Phi^{-1} \circ \tilde{\alpha}_+$ or the same direction as the initial velocity of $\alpha_- = \Phi^{-1} \circ \tilde{\alpha}_-$. In any case, the initial velocity of γ is horizontal. By Proposition 4.3, Φ^{-1} is a β-isomorphism. Since $G \circ \Phi = F$ and

$F \circ \Phi^{-1} = G$, it is clear that both Φ is composable with Φ^{-1} and Φ^{-1} is composable with Φ. Also, it is obvious that $\Phi \circ \Phi^{-1} = \Phi^{-1} \circ \Phi = I$. ■

Now, we introduce the β-transform. Given a β-isomorphism $\Phi = (\Phi_1, \Phi_2)\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, we define[11] $\lambda := (\lambda_+, \lambda_-)$ and $\phi := (\phi_+, \phi_-)$, where:

$$\lambda_+ := \lim_{x \to 0^+} \frac{\Phi_1(x,0)}{x}, \quad \lambda_- := \lim_{x \to 0^-} \frac{\Phi_1(x,0)}{x}$$

$$\phi_+(t) := |\lambda_+|^{-\beta} \cdot \lim_{x \to 0^+} \frac{\Phi_2(x, t\,|x|^\beta)}{|x|^\beta}, \quad \phi_-(t) := |\lambda_-|^{-\beta} \cdot \lim_{x \to 0^-} \frac{\Phi_2(x, t\,|x|^\beta)}{|x|^\beta}$$

We define the *β-transform* of Φ to be the ordered pair (λ, ϕ).

From the definitions above, it follows that, for each $t \in \mathbb{R}$:

$$\Phi_1(x, t\,|x|^\beta) = \lambda x + o(x) \quad \text{as } x \to 0 \tag{19}$$

$$\Phi_2(x, t\,|x|^\beta) = |\lambda|^\beta\, \phi(t)\, |x|^\beta + o(|x|^\beta) \quad \text{as } x \to 0 \tag{20}$$

where

$$\begin{cases} \lambda = \lambda_+ \text{ and } \phi = \phi_+, & \text{if } x > 0 \\ \lambda = \lambda_- \text{ and } \phi = \phi_-, & \text{if } x < 0 \end{cases}.$$

Remark 4.5. *To obtain (19), we used (15).*

Proposition 4.7. *Let $\Phi = (\Phi_1, \Phi_2)$ and $\Psi = (\Psi_1, \Psi_2)$ be β-isomorphisms such that Ψ is composable with Φ, and let $Z := \Psi \circ \Phi$. Let (λ, ϕ), (μ, ψ) be the β-transforms of Φ, Ψ, respectively. We have:*

i. Asympotic formula for $Z_1(x, 0)$.

$$Z_1(x,0) = \lambda\mu x + o(x) \ \text{ as } x \to 0, \tag{21}$$

where

$$\begin{cases} \lambda = \lambda_+,\ \mu = \mu_+, & \text{if } x > 0 \\ \lambda = \lambda_-,\ \mu = \mu_-, & \text{if } x < 0 \end{cases} \quad \text{or} \quad \begin{cases} \lambda = \lambda_+,\ \mu = \mu_-, & \text{if } x > 0 \\ \lambda = \lambda_-,\ \mu = \mu_+, & \text{if } x < 0 \end{cases},$$

according as Φ is direct or reverse, respectively.

[11] By Proposition 4.2, λ_+ and λ_- are well-defined nonzero real numbers. By Proposition 4.1 and Corollary 2.4, $\phi_+(t)$ and $\phi_-(t)$ are well-defined real numbers, for each $t \in \mathbb{R}$.

ii. Asympotic formula for $Z_2(x, t\,|x|^\beta)$*, with t fixed.*

$$Z_2(x, t\,|x|^\beta) = |\lambda\mu|^\beta\, \psi(\phi(t))\,|x|^\beta + o(|x|^\beta) \;\; \textit{as } x \to 0, \tag{22}$$

where

$$\begin{cases} \lambda = \lambda_+,\ \mu = \mu_+,\ \phi = \phi_+,\ \psi = \psi_+, & \textit{if } x > 0 \\ \lambda = \lambda_-,\ \mu = \mu_-,\ \phi = \phi_-,\ \psi = \psi_-, & \textit{if } x < 0 \end{cases}$$

or

$$\begin{cases} \lambda = \lambda_+,\ \mu = \mu_-,\ \phi = \phi_+,\ \psi = \psi_-, & \textit{if } x > 0 \\ \lambda = \lambda_-,\ \mu = \mu_+,\ \phi = \phi_-,\ \psi = \psi_+, & \textit{if } x < 0 \end{cases},$$

according as Φ *is direct or reverse, respectively.*

Proof. By definition,

$$Z_1(x, 0) = \Psi_1(\Phi(x, 0)).$$

Then, by (19) and (20),

$$Z_1(x, 0) = \Psi_1(\lambda x, |\lambda|^\beta\, \phi(0)\,|x|^\beta) + o(x),$$

where

$$\begin{cases} \lambda = \lambda_+ \text{ and } \phi = \phi_+, & \text{if } x > 0 \\ \lambda = \lambda_- \text{ and } \phi = \phi_-, & \text{if } x < 0 \end{cases}. \tag{23}$$

And then, by applying (19) to Ψ_1 in the equation above, we obtain:

$$Z_1(x, 0) = \lambda\mu x + o(x),$$

where

$$\begin{cases} \mu = \mu_+, & \text{if } \lambda x > 0 \\ \mu = \mu_-, & \text{if } \lambda x < 0 \end{cases}. \tag{24}$$

Since

$$\begin{cases} \lambda_+ > 0 \text{ and } \lambda_- > 0, & \text{if } \Phi \text{ is direct} \\ \lambda_+ < 0 \text{ and } \lambda_- < 0, & \text{if } \Phi \text{ is reverse} \end{cases},$$

it follows from (23) and (24) that

$$\begin{cases} \lambda = \lambda_+,\ \mu = \mu_+, & \text{if } x > 0 \\ \lambda = \lambda_-,\ \mu = \mu_-, & \text{if } x < 0 \end{cases} \quad \text{or} \quad \begin{cases} \lambda = \lambda_+,\ \mu = \mu_-, & \text{if } x > 0 \\ \lambda = \lambda_-,\ \mu = \mu_+, & \text{if } x < 0 \end{cases},$$

according as Φ is direct or reverse, respectively.

Now, let $t \in \mathbb{R}$ be fixed. By definition,

$$Z_2(x, t\,|x|^\beta) = \Psi_2(\tilde{x}, \Phi_2(x, t\,|x|^\beta)), \tag{25}$$

where $\tilde{x} \coloneqq \Phi_1(x, t\,|x|^\beta)$. By (20), we have

$$\Phi_2(x, t\,|x|^\beta) = \phi(t)\,|\lambda x|^\beta + o(|x|^\beta), \tag{26}$$

where

$$\begin{cases} \lambda = \lambda_+ \ \text{ and } \ \phi = \phi_+, & \text{if } \ x > 0 \\ \lambda = \lambda_- \ \text{ and } \ \phi = \phi_-, & \text{if } \ x < 0 \end{cases}. \tag{27}$$

By (19), we have

$$\lim_{x \to 0} \frac{\tilde{x}}{\lambda x} = 1,$$

so that

$$\lim_{x \to 0} \frac{|\tilde{x}|^\beta}{|\lambda x|^\beta} = 1 \tag{28}$$

and hence

$$|\lambda x|^\beta = |\tilde{x}|^\beta + o(|x|^\beta). \tag{29}$$

From (26) and (29), we get

$$\Phi_2(x, t\,|x|^\beta) = \phi(t)\,|\tilde{x}|^\beta + o(|x|^\beta).$$

Thus,

$$\Psi_2(\tilde{x}, \Phi_2(x, t\,|x|^\beta)) = \Psi_2(\tilde{x}, \phi(t)\,|\tilde{x}|^\beta) + o(|x|^\beta). \tag{30}$$

By applying (20) to Ψ_2 and using (28), we obtain

$$\Psi_2(\tilde{x}, \phi(t)\,|\tilde{x}|^\beta) = |\mu|^\beta\,\psi(\phi(t))\,|\tilde{x}|^\beta + o(|x|^\beta),$$

where

$$\begin{cases} \mu = \mu_+ \ \text{ and } \ \psi = \psi_+, & \text{if } \ \tilde{x} > 0 \\ \mu = \mu_- \ \text{ and } \ \psi = \psi_-, & \text{if } \ \tilde{x} < 0 \end{cases}. \tag{31}$$

By (29), it follows that

$$\Psi_2(\tilde{x}, \phi(t)\,|\tilde{x}|^\beta) = |\lambda\mu|^\beta\,\psi(\phi(t))\,|x|^\beta + o(|x|^\beta). \tag{32}$$

Then, by (25), (30), and (32), we have

$$Z_2(x, t\,|x|^\beta) = |\lambda\mu|^\beta\,\psi(\phi(t))\,|x|^\beta + o(|x|^\beta).$$

Since $\tilde{x}$ and λx have the same sign for small values of x, it follows from (31) that

$$\begin{cases} \mu = \mu_+ \ \text{ and } \ \psi = \psi_+, & \text{if } \lambda x > 0 \\ \mu = \mu_- \ \text{ and } \ \psi = \psi_-, & \text{if } \lambda x < 0 \end{cases}. \tag{33}$$

And since

$$\begin{cases} \lambda_+ > 0 \text{ and } \lambda_- > 0, & \text{if } \Phi \text{ is direct} \\ \lambda_+ < 0 \text{ and } \lambda_- < 0, & \text{if } \Phi \text{ is reverse} \end{cases},$$

it follows from (27) and (33) that

$$\begin{cases} \lambda = \lambda_+,\ \mu = \mu_+,\ \phi = \phi_+,\ \psi = \psi_+, & \text{if } x > 0 \\ \lambda = \lambda_-,\ \mu = \mu_-,\ \phi = \phi_-,\ \psi = \psi_-, & \text{if } x < 0 \end{cases}$$

or

$$\begin{cases} \lambda = \lambda_+,\ \mu = \mu_-,\ \phi = \phi_+,\ \psi = \psi_-, & \text{if } x > 0 \\ \lambda = \lambda_-,\ \mu = \mu_+,\ \phi = \phi_-,\ \psi = \psi_+, & \text{if } x < 0 \end{cases},$$

according as Φ is direct or reverse, respectively. ■

Corollary 4.1. *Let $\Phi = (\Phi_1, \Phi_2)$ and $\Psi = (\Psi_1, \Psi_2)$ be β-isomorphisms such that Ψ is composable with Φ, and let $Z := \Psi \circ \Phi$. Let (λ, ϕ), (μ, ψ), and (ν, ζ) be the β-transforms of Φ, Ψ, and Z, respectively. We have:*

$$(\nu, \zeta) = \begin{cases} ((\lambda_+\mu_+, \lambda_-\mu_-), (\psi_+ \circ \phi_+, \psi_- \circ \phi_-)), & \textit{if } \Phi \textit{ is direct} \\ ((\lambda_+\mu_-, \lambda_-\mu_+), (\psi_- \circ \phi_+, \psi_+ \circ \phi_-)), & \textit{if } \Phi \textit{ is reverse} \end{cases}$$

Proof. Dividing both sides of (21) by x and then successively letting $x \to 0^+$ and $x \to 0^-$, we obtain

$$\begin{cases} \nu_+ = \lambda_+\mu_+ \ \text{ and } \ \nu_- = \lambda_-\mu_-, & \text{if } \Phi \text{ is direct} \\ \nu_+ = \lambda_+\mu_- \ \text{ and } \ \nu_- = \lambda_-\mu_+, & \text{if } \Phi \text{ is reverse} \end{cases}$$

So, equation (22) can be rewritten as

$$Z_2(x, t\,|x|^\beta) = |\nu|^\beta\, \psi(\phi(t))\, |x|^\beta + o(|x|^\beta) \ \text{ as } x \to 0, \tag{34}$$

where

$$\begin{cases} \nu = \nu_+,\ \phi = \phi_+,\ \psi = \psi_+, & \text{if } x > 0 \\ \nu = \nu_-,\ \phi = \phi_-,\ \psi = \psi_-, & \text{if } x < 0 \end{cases} \quad \text{or} \quad \begin{cases} \nu = \nu_+,\ \phi = \phi_+,\ \psi = \psi_-, & \text{if } x > 0 \\ \nu = \nu_-,\ \phi = \phi_-,\ \psi = \psi_+, & \text{if } x < 0 \end{cases},$$

according as Φ is direct or reverse, respectively.

Dividing both sides of (34) by $|x|^\beta$ and then successively letting $x \to 0^+$ and $x \to 0^-$, we obtain:

$$\begin{cases} \zeta_+ = \psi_+ \circ \phi_+ \text{ and } \zeta_- = \psi_- \circ \phi_-, & \text{if } \Phi \text{ is direct} \\ \zeta_+ = \psi_- \circ \phi_+ \text{ and } \zeta_- = \psi_+ \circ \phi_-, & \text{if } \Phi \text{ is reverse} \end{cases}$$

■

Corollary 4.2. *Let* $\Phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *be a* β*-isomorphism, and let* (λ, ϕ) *be the* β*-transform of* Φ*. The* β*-transform of* Φ^{-1} *is given by:*

$$\begin{cases} \left((\lambda_+^{-1}, \lambda_-^{-1}), (\phi_+^{-1}, \phi_-^{-1})\right), & \textit{if } \Phi \textit{ is direct} \\ \left((\lambda_-^{-1}, \lambda_+^{-1}), (\phi_-^{-1}, \phi_+^{-1})\right), & \textit{if } \Phi \textit{ is reverse} \end{cases}$$

Proof. By Proposition 4.6, Φ^{-1} is a β-isomorphism, Φ is composable with Φ^{-1}, Φ^{-1} is composable with Φ, and $\Phi \circ \Phi^{-1} = \Phi^{-1} \circ \Phi = I$, where $I\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is the germ of the identity map on $\mathbb{R}^2$. Suppose that Φ is a direct β-isomorphism. Let $(\mu, \psi) = ((\mu_+, \mu_-), (\psi_+, \psi_-))$ be the β-transform of Φ^{-1}. We must show that $\mu_+ = \lambda_+^{-1}$, $\mu_- = \lambda_-^{-1}$, $\psi_+ = \phi_+^{-1}$, and $\psi_- = \phi_-^{-1}$.

By Corollary 4.1, the β-transform of $\Phi^{-1} \circ \Phi$ is given by

$$((\lambda_+\mu_+, \lambda_-\mu_-), (\psi_+ \circ \phi_+, \psi_- \circ \phi_-)).$$

On the other hand, since $\Phi^{-1} \circ \Phi = I$, the β-transform of $\Phi^{-1} \circ \Phi$ is equal to the β-transform of I, which is

$$((1, 1), (\mathrm{id}_\mathbb{R}, \mathrm{id}_\mathbb{R})).$$

Thus,

$$\mu_+ = \lambda_+^{-1}, \quad \mu_- = \lambda_-^{-1} \tag{35}$$

and

$$\psi_+ \circ \phi_+ = \psi_- \circ \phi_- = \mathrm{id}_\mathbb{R}. \tag{36}$$

Since Φ is a direct β-isomorphism, we have $\lambda_+ > 0$ and $\lambda_- > 0$, so by (35), $\mu_+ > 0$ and $\mu_- > 0$ and hence Φ^{-1} is a direct β-isomorphism too. Then, by Corollary 4.1, the β-transform of $\Phi \circ \Phi^{-1}$ is given by

$$((\lambda_+\mu_+, \lambda_-\mu_-), (\phi_+ \circ \psi_+, \phi_- \circ \psi_-)),$$

and since $\Phi \circ \Phi^{-1} = I$, it follows that

$$\phi_+ \circ \psi_+ = \phi_- \circ \psi_- = \mathrm{id}_\mathbb{R}. \tag{37}$$

From (36) and (37), it follows that

$$\psi_+ = \phi_+^{-1} \text{ and } \psi_- = \phi_-^{-1}.$$

The case where Φ is a reverse β-isomorphism is analogous. ■

Proposition 4.8. *Let $\Phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a β-isomorphism, and let (λ, ϕ) be the β-transform of Φ. We have:*

(a) *λ_+ and λ_- are nonzero real numbers and they have the same sign.*

(b) *ϕ_+ and ϕ_- are both bi-Lipschitz functions.*

Proof. As pointed out just after the proof of Proposition 4.2, $\lambda_+ = \lim_{x\to 0^+} \Phi_1(x,0)/x$ and $\lambda_- = \lim_{x\to 0^-} \Phi_1(x,0)/x$ are nonzero real numbers which have the same sign. This establishes the first part of the proposition. Now, let us prove the second part.

By (20), for any fixed t and t', we have:

$$\phi_+(t) - \phi_+(t') = |\lambda_+|^{-\beta} \cdot \frac{\Phi_2(x, t\,|x|^\beta) - \Phi_2(x, t'\,|x|^\beta)}{|x|^\beta} + \frac{o(|x|^\beta)}{|x|^\beta}, \quad \text{as } x \to 0^+$$

and

$$\phi_-(t) - \phi_-(t') = |\lambda_-|^{-\beta} \cdot \frac{\Phi_2(x, t\,|x|^\beta) - \Phi_2(x, t'\,|x|^\beta)}{|x|^\beta} + \frac{o(|x|^\beta)}{|x|^\beta}, \quad \text{as } x \to 0^-.$$

On the other hand, since Φ_2 is Lipschitz, there exists $K > 0$ (independent of x, t, t') such that

$$\left|\Phi_2(x, t\,|x|^\beta) - \Phi_2(x, t'\,|x|^\beta)\right| \le K\,|t - t'|\,|x|^\beta.$$

Thus,

$$|\phi_+(t) - \phi_+(t')| \le |\lambda_+|^{-\beta}\,K\,|t - t'| + \frac{o(|x|^\beta)}{|x|^\beta} \quad \text{as } x \to 0^+ \tag{38}$$

and

$$|\phi_-(t) - \phi_-(t')| \le |\lambda_-|^{-\beta}\,K\,|t - t'| + \frac{o(|x|^\beta)}{|x|^\beta} \quad \text{as } x \to 0^-. \tag{39}$$

By letting $x \to 0^+$ in (38), and $x \to 0^-$ in (39), we obtain:

$$|\phi_+(t) - \phi_+(t')| \le |\lambda_+|^{-\beta}\,K\,|t - t'| \quad \text{and} \quad |\phi_-(t) - \phi_-(t')| \le |\lambda_-|^{-\beta}\,K\,|t - t'|\,.$$

Therefore, both ϕ_+ and ϕ_- are Lipschitz functions.

Up to this point, our argument shows that, if Φ is a β-isomorphism, then the functions ϕ_+ and ϕ_- are both Lipschitz. Since Φ^{-1} is a β-isomorphism (see Proposition 4.6), this conclusion can also be applied to Φ^{-1}. Then, by Corollary 4.2, it follows that the functions ϕ_+^{-1} and ϕ_-^{-1} are both Lipschitz too. ■

Proposition 4.9. *Let $F(X,Y)$ and $G(X,Y)$ be β-quasihomogeneous polynomials of degree d. Suppose that $G \circ \Phi = F$, for a certain β-isomorphism $\Phi\colon (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$. Let f_+, f_- be the height functions of F, g_+, g_- the height functions of G, and let (λ,ϕ) be the β-transform of Φ. We have:*

$$\begin{cases} g_+ \circ \phi_+ = |\lambda_+|^{-d} f_+ \quad \textit{and} \quad g_- \circ \phi_- = |\lambda_-|^{-d} f_-, & \textit{if } \Phi \textit{ is direct} \\ g_- \circ \phi_+ = |\lambda_+|^{-d} f_+ \quad \textit{and} \quad g_+ \circ \phi_- = |\lambda_-|^{-d} f_-, & \textit{if } \Phi \textit{ is reverse} \end{cases}$$

Proof. By hypothesis, $G(\Phi(x,y)) = F(x,y)$. Thus, for all $t \in \mathbb{R}$ and $x \neq 0$ sufficiently small,

$$G(\Phi(x, t\,|x|^\beta)) = F(x, t\,|x|^\beta)\,.$$

Since F is β-quasihomogeneous, this implies that

$$G(\Phi(x, t\,|x|^\beta)) = |x|^d f(t),$$

where

$$f = \begin{cases} f_+, & \text{if } x > 0 \\ f_-, & \text{if } x < 0 \end{cases}.$$

Multiplying both sides of this equation by $|x|^{-d}$, we obtain:

$$G\left(\frac{\Phi_1(x, t\,|x|^\beta)}{|x|}, \frac{\Phi_2(x, t\,|x|^\beta)}{|x|^\beta}\right) = f(t)$$

Letting $x \to 0^+$, it follows that

$$G\left(\lambda_+, \lim_{x\to 0^+} \frac{\Phi_2(x, t\,|x|^\beta)}{|x|^\beta}\right) = f_+(t)\,.$$

Hence,

$$g(\phi_+(t)) = |\lambda_+|^{-d} f_+(t), \tag{40}$$

where

$$g = \begin{cases} g_+, & \text{if } \lambda_+ > 0 \\ g_-, & \text{if } \lambda_+ < 0 \end{cases}.$$

Similarly, letting $x \to 0^-$, it follows that

$$G\left(-\lambda_-, \lim_{x\to 0^-} \frac{\Phi_2(x, t\,|x|^\beta)}{|x|^\beta}\right) = f_-(t)\,.$$

Hence,

$$g(\phi_-(t)) = |\lambda_-|^{-d} f_-(t), \tag{41}$$

where

$$g = \begin{cases} g_-, & \text{if } \lambda_- > 0 \\ g_+, & \text{if } \lambda_- < 0 \end{cases}.$$

Clearly, equations (40) and (41) yield the result. ■

Remark 4.6. *We can remove the assumption that the β-quasihomogeneous polynomials F and G have the same β-quasihomogeneous degree d from the hypotheses of Proposition 4.9, and then obtain it as a consequence of the fact that $G \circ \Phi = F$, where Φ is a β-isomorphism:*

Let $F(X,Y)$ be a β-quasihomogeneous polynomial of degree d_F, and let $G(X,Y)$ be a β-quasihomogeneous polynomial of degree d_G. If $G \circ \Phi = F$, where Φ is a β-isomorphism, then $d_F = d_G$.

In order to prove this, we suppose, for the sake of contradiction, that $d_F \neq d_G$. Without loss of generality, assume that $d_F > d_G$. As usual, we assume that F and G are not identically zero. By applying the argument used in the proof of Proposition 4.9, we conclude that

$$\begin{cases} g_+ \circ \phi_+ \equiv 0 \ \textit{ and } \ g_- \circ \phi_- \equiv 0, & \textit{if } \Phi \textit{ is direct} \\ g_- \circ \phi_+ \equiv 0 \ \textit{ and } \ g_+ \circ \phi_- \equiv 0, & \textit{if } \Phi \textit{ is reverse} \end{cases}.$$

Since ϕ_+ and ϕ_- are bijective, it follows that $g_+ \equiv 0$ and $g_- \equiv 0$. And since

$$G(x, t\,|x|^\beta) = \begin{cases} |x|^{d_G}\, g_+(t), & \textit{if } x > 0 \\ |x|^{d_G}\, g_-(t), & \textit{if } x < 0 \end{cases},$$

this implies that $G \equiv 0$, a contradiction.

Theorem 4.1. *Let $F(X,Y)$ and $G(X,Y)$ be β-quasihomogeneous polynomials of degree d. Suppose that F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent, and let $\Phi\colon (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ be a germ of semialgebraic bi-Lipschitz homeomorphism such that $G \circ \Phi = F$. If $F^{-1}(0) \cap \{x > 0\} \neq \varnothing$ and $F^{-1}(0) \cap \{x < 0\} \neq \varnothing$, then Φ is a β-isomorphism. Consequently, we have:*

$$\begin{cases} g_+ \circ \phi_+ = |\lambda_+|^{-d} f_+ \quad \textit{and} \quad g_- \circ \phi_- = |\lambda_-|^{-d} f_-, & \textit{if } \Phi \textit{ is direct} \\ g_+ \circ \phi_- = |\lambda_-|^{-d} f_- \quad \textit{and} \quad g_- \circ \phi_+ = |\lambda_+|^{-d} f_+, & \textit{if } \Phi \textit{ is reverse} \end{cases},$$

where f_+, f_- are the height functions of F, g_+, g_- are the height functions of G, and (λ, ϕ) is the β-transform of Φ.

Proof. Suppose that $F^{-1}(0) \cap \{x > 0\} \neq \varnothing$ and $F^{-1}(0) \cap \{x < 0\} \neq \varnothing$. Since

$$F^{-1}(0) \cap \{x > 0\} = \bigcup_{t \in f_+^{-1}(0)} \left\{(x, tx^\beta) : x > 0\right\},$$

the condition $F^{-1}(0) \cap \{x > 0\} \neq \varnothing$ implies that $f_+^{-1}(0) \neq \varnothing$. Take $t_0 \in \mathbb{R}$ such that $f_+(t_0) = 0$. In the notation used in the proof of Proposition 4.1, we have:

$$\left|\frac{\Phi_2(x, t_0 |x|^\beta)}{|x|^\beta}\right| \leq 1 + \max\left\{\left|\frac{c_{m-1}}{c_m}\right| |\tilde{x}|^r, \ldots, \left|\frac{c_1}{c_m}\right| |\tilde{x}|^{r(m-1)}, \left|\frac{c_0}{c_m}\right| |\tilde{x}|^{rm}\right\},$$

for $x > 0$ sufficiently small. Since $\lim_{x \to 0^+} \left|\Phi_1(x, t_0 |x|^\beta)/x\right| < \infty$ (this is guaranteed by Remark 4.3 along with (15)), it follows that $\Phi_2(x, t_0 |x|^\beta)/|x|^\beta$ is bounded, for $x > 0$ sufficiently small.

Similarly, the assumption that $F^{-1}(0) \cap \{x < 0\} \neq \varnothing$ ensures the existence of $t_1 \in \mathbb{R}$ such that $f_-(t_1) = 0$ and then, by adapting the argument above, we can prove that $\Phi_2(x, t_1 |x|^\beta)/|x|^\beta$ is bounded, for $x < 0$ sufficiently small. By Proposition 4.1, it follows that Φ is a β-isomorphism. Now, the final statement is an immediate consequence of Proposition 4.9. ■

Corollary 4.3. *Let $F(X, Y)$ and $G(X, Y)$ be β-quasihomogeneous polynomials of degree d. Suppose that $F^{-1}(0) \cap \{x > 0\} \neq \varnothing$ and $F^{-1}(0) \cap \{x < 0\} \neq \varnothing$. If F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent, then*

$$\begin{cases} f_+ \cong g_+ \text{ and } f_- \cong g_-, & \text{if } \Phi \text{ is direct} \\ f_+ \cong g_- \text{ and } f_- \cong g_+, & \text{if } \Phi \text{ is reverse} \end{cases}.$$

Remark 4.7. *Let $F(X, Y)$ and $G(X, Y)$ be β-quasihomogeneous polynomials of degree d. Suppose that $G \circ \Phi = F$, for a certain β-isomorphism $\Phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. Then, either both of the following conditions hold or none of them hold:*

i. $F^{-1}(0) \cap \{x > 0\} \neq \varnothing$ *and* $F^{-1}(0) \cap \{x < 0\} \neq \varnothing$

ii. $G^{-1}(0) \cap \{x > 0\} \neq \varnothing$ *and* $G^{-1}(0) \cap \{x < 0\} \neq \varnothing$

We prove that conditions (i) and (ii) are equivalent in the case where Φ is a direct β-isomorphism; the other case (where Φ is a reverse β-isomorphism) being analogous. Thus, suppose that Φ is a direct β-isomorphism. By Proposition 4.9, we have

$$g_+ \circ \phi_+ = |\lambda_+|^{-d} f_+ \text{ and } g_- \circ \phi_- = |\lambda_-|^{-d} f_-, \tag{42}$$

where f_+, f_- are the height functions of F, g_+, g_- are the height functions of G, and $(\lambda, \phi) = ((\lambda_+, \lambda_-), (\phi_+, \phi_-))$ is the β-transform of Φ. Suppose that $F^{-1}(0) \cap \{x > 0\} \neq \varnothing$ and $F^{-1}(0) \cap \{x < 0\} \neq \varnothing$. Then it follows that $f_+(t_0) = 0$ for some $t_0 \in \mathbb{R}$, and $f_-(t_1) = 0$ for some $t_1 \in \mathbb{R}$. Hence, by (42), $g_+(s_0) = g_-(s_1) = 0$, where $s_0 = \phi_+(t_0)$ and $s_1 = \phi_-(t_1)$. Thus, $G(1, s_0) = g_+(s_0) = 0$ and $G(-1, s_1) = g_-(s_1) = 0$, from which we see that $G^{-1}(0) \cap \{x > 0\} \neq \varnothing$ and $G^{-1}(0) \cap \{x < 0\} \neq \varnothing$. Conversely, suppose that $G^{-1}(0) \cap \{x > 0\} \neq \varnothing$ and $G^{-1}(0) \cap \{x < 0\} \neq \varnothing$. Then it follows that $g_+(s_0) = 0$ for some $s_0 \in \mathbb{R}$, and $g_-(s_1) = 0$ for some $s_1 \in \mathbb{R}$. Hence, by (42), $f_+(t_0) = f_-(t_1) = 0$, where $t_0 = \phi_+^{-1}(s_0)$ and $t_1 = \phi_-^{-1}(s_1)$. Thus, $F(1, t_0) = f_+(t_0) = 0$ and $F(-1, t_1) = f_-(t_1) = 0$, from which we see that $F^{-1}(0) \cap \{x > 0\} \neq \varnothing$ and $F^{-1}(0) \cap \{x < 0\} \neq \varnothing$.

4.2 The group of proto-transitions

Let $\mathbb{R}^*$ be the multiplicative group of all nonzero real numbers, and let $\mathcal{L}$ be the group of all bi-Lipschitz Nash diffeomorphisms on $\mathbb{R}$. Let $H \coloneqq \{(\lambda_1, \lambda_2) \in \mathbb{R}^* \times \mathbb{R}^* : \lambda_1 \lambda_2 > 0\}$ (considered as a subgroup of the direct product $\mathbb{R}^* \times \mathbb{R}^*$), and let $K \coloneqq \mathcal{L} \times \mathcal{L}$ (direct product). Define a binary operation on $H \times K$ by setting:

$$(\mu, \psi) \circ (\lambda, \phi) \coloneqq \begin{cases} ((\lambda_1 \mu_1, \lambda_2 \mu_2), (\psi_1 \circ \phi_1, \psi_2 \circ \phi_2)), & \text{if } \lambda > 0 \\ ((\lambda_1 \mu_2, \lambda_2 \mu_1), (\psi_2 \circ \phi_1, \psi_1 \circ \phi_2)), & \text{if } \lambda < 0 \end{cases}$$

for all $(\lambda, \phi) = ((\lambda_1, \lambda_2), (\phi_1, \phi_2))$ and $(\mu, \psi) = ((\mu_1, \mu_2), (\psi_1, \psi_2))$, where $\lambda > 0$ means that $\lambda_1 > 0$ and $\lambda_2 > 0$, and $\lambda < 0$ means that $\lambda_1 < 0$ and $\lambda_2 < 0$.

Proposition 4.10. *$(H \times K, \circ)$ is a group. We call it the group of proto-transitions.*

Proof. Let $(H \times K, \cdot)$ be the direct product of H and K, so that

$$(\mu, \psi) \cdot (\lambda, \phi) = ((\lambda_1 \mu_1, \lambda_2 \mu_2), (\psi_1 \circ \phi_1, \psi_2 \circ \phi_2)).$$

We express the operation $\circ$ in terms of the operation $\cdot$, and then we use this expression to show that $(H \times K, \circ)$ is a group. Let $\iota\colon H \times K \to H \times K$ be the identity map and $\tau\colon H \times K \to H \times K$ be given by $\tau((\lambda_1, \lambda_2), (\phi_1, \phi_2)) = ((\lambda_2, \lambda_1), (\phi_2, \phi_1))$.

Define $\theta\colon H \to \mathrm{Aut}(H \times K)$ by

$$\theta(\lambda) \coloneqq \begin{cases} \iota, & \text{if } \lambda > 0 \\ \tau, & \text{if } \lambda < 0 \end{cases}$$

Clearly, θ is a group homomorphism.

Let $\pi\colon H \times K \to H$ be the projection homomorphism. Then, $\alpha \coloneqq \theta \circ \pi\colon H \times K \to \mathrm{Aut}(H \times K)$ is a group homomorphism such that:

i. $\alpha \circ \varphi = \alpha$ for all $\varphi \in \operatorname{Im}\alpha$;

ii. $\operatorname{Im}\alpha$ is an abelian subgroup of $\operatorname{Aut}(H \times K)$.

Also, we have:

$$(\mu, \psi) \circ (\lambda, \phi) = (\alpha(\lambda, \phi)(\mu, \psi)) \cdot (\lambda, \phi).$$

Hence, the result follows from the following lemma.

Lemma 4.1. *Let $(G, \cdot)$ be a group and let $\alpha \colon G \to Aut(G)$ be a group homomorphism satisfying the following conditions:*

i. $\alpha \circ \varphi = \alpha$ for all $\varphi \in \operatorname{Im}\alpha$;

ii. $Im\,\alpha$ is an abelian subgroup of $Aut(G)$.

Define a new operation $\circ$ on G by setting $g \circ h := (\alpha(h)(g)) \cdot h$. Then, $(G, \circ)$ is a group.

Proof of Lemma 4.1. First, we prove that the new operation is associative. For all $g_1, g_2, g_3 \in G$, we have:

$$\begin{aligned}(g_1 \circ g_2) \circ g_3 &= (\alpha(g_2)(g_1) \cdot g_2) \circ g_3 \\ &= \alpha(g_3)\,(\alpha(g_2)(g_1) \cdot g_2) \cdot g_3 \\ &= \alpha(g_3)\,(\alpha(g_2)(g_1)) \cdot \alpha(g_3)(g_2) \cdot g_3 \end{aligned} \tag{43}$$

On the other hand,

$$\begin{aligned}g_1 \circ (g_2 \circ g_3) &= g_1 \circ (\alpha(g_3)(g_2) \cdot g_3) \\ &= \alpha(\alpha(g_3)(g_2) \cdot g_3)(g_1) \cdot \alpha(g_3)(g_2) \cdot g_3 \\ &= \alpha(\alpha(g_3)(g_2))(\alpha(g_3)(g_1)) \cdot \alpha(g_3)(g_2) \cdot g_3 \\ &\overset{(i)}{=} \alpha(g_2)(\alpha(g_3)(g_1)) \cdot \alpha(g_3)(g_2) \cdot g_3 \\ &\overset{(ii)}{=} \alpha(g_3)(\alpha(g_2)(g_1)) \cdot \alpha(g_3)(g_2) \cdot g_3 \end{aligned} \tag{44}$$

From (43) and (44), it follows that $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$.

Now we prove that the identity element 1 of the group $(G, \cdot)$ is also an identity element of G with respect to the operation $\circ$. In fact, for all $g \in G$, we have:

$$\begin{aligned}1 \circ g &= \alpha(g)(1) \cdot g = 1 \cdot g = g \\ g \circ 1 &= \alpha(1)(g) \cdot 1 = \operatorname{id}(g) = g\end{aligned}$$

Finally, we prove that each element $g \in G$ has an inverse with respect to the operation $\circ$. First, notice that for all $g, h \in G$,

$$h \circ g = 1 \Leftrightarrow \alpha(g)(h) \cdot g = 1 \Leftrightarrow \alpha(g)(h) = g^{-1} \Leftrightarrow h = \alpha(g)^{-1}(g^{-1}) \Leftrightarrow h = \alpha(g^{-1})(g^{-1})\,.$$

Thus, $h = \alpha(g^{-1})(g^{-1})$ is a left inverse of g with respect to the operation $\circ$. Let us show that h is also a right inverse of g. In fact,

$$\begin{aligned} g \circ \alpha(g^{-1})(g^{-1}) &= \alpha(\alpha(g^{-1})(g^{-1}))(g) \cdot \alpha(g^{-1})(g^{-1}) \overset{(i)}{=} \alpha(g^{-1})(g) \cdot \alpha(g^{-1})(g^{-1}) \\ &= \alpha(g^{-1})(g \cdot g^{-1}) = \alpha(g^{-1})(1) = 1\,. \end{aligned}$$

Hence, $h = \alpha(g^{-1})(g^{-1})$ is the inverse of g with respect to the operation $\circ$. ■

Now, we define a family of actions of the group of proto-transitions on the set $C^\omega \times C^\omega$, where C^ω is the set of all real analytic functions on $\mathbb{R}$.

Proposition 4.11. *For each integer $d \geq 1$, the map $\circ\colon (C^\omega \times C^\omega) \times (H \times K) \to C^\omega \times C^\omega$ defined by*

$$(g_1, g_2) \circ (\lambda, \phi) := \begin{cases} \left(|\lambda_1|^d\, g_1 \circ \phi_1, |\lambda_2|^d\, g_2 \circ \phi_2\right), & \textit{if} \quad \lambda > 0 \\ \left(|\lambda_1|^d\, g_2 \circ \phi_1, |\lambda_2|^d\, g_1 \circ \phi_2\right), & \textit{if} \quad \lambda < 0 \end{cases}$$

is an action of the group of proto-transitions on $C^\omega \times C^\omega$.

Proof. First, we notice that the map $\bullet : (C^\omega \times C^\omega) \times (H \times K) \to C^\omega \times C^\omega$ given by

$$(g_1, g_2) \bullet (\lambda, \phi) := (|\lambda_1|^d\, g_1 \circ \phi_1, |\lambda_2|^d\, g_2 \circ \phi_2)$$

is an action of the direct product $(H \times K, \cdot)$ on $C^\omega \times C^\omega$. In fact, we have:

$$\begin{aligned} ((g_1, g_2) \bullet (\mu, \psi)) \bullet (\lambda, \phi) &= (|\mu_1|^d\, g_1 \circ \psi_1, |\mu_2|^d\, g_2 \circ \psi_2) \bullet (\lambda, \phi) \\ &= (|\lambda_1|^d\, |\mu_1|^d\, (g_1 \circ \psi_1) \circ \phi_1, |\lambda_2|^d\, |\mu_2|^d\, (g_2 \circ \psi_2) \circ \phi_2) \\ &= (|\lambda_1 \mu_1|^d\, g_1 \circ (\psi_1 \circ \phi_1), |\lambda_2 \mu_2|^d\, g_2 \circ (\psi_2 \circ \phi_2)) \\ &= (g_1, g_2) \bullet ((\mu, \psi) \cdot (\lambda, \phi)) \end{aligned}$$

and

$$(g_1, g_2) \bullet ((\mathrm{id}_\mathbb{R}, \mathrm{id}_\mathbb{R}), (1, 1)) = (|1|^d\, g_1 \circ \mathrm{id}_\mathbb{R}, |1|^d\, g_2 \circ \mathrm{id}_\mathbb{R}) = (g_1, g_2).$$

Now, we express the map $\circ$ in terms of the action $\bullet$, and then we use this expression to show that the map $\circ$ is an action of the group of proto-transitions on $C^\omega \times C^\omega$.

Denote by $\mathrm{Bij}(C^\omega \times C^\omega)$ the group of all bijections on $C^\omega \times C^\omega$. Let $I\colon C^\omega \times C^\omega \to C^\omega \times C^\omega$ be the identity map and $T\colon C^\omega \times C^\omega \to C^\omega \times C^\omega$ be given by $T(g_1, g_2) = (g_2, g_1)$. Define $\Theta\colon H \to \mathrm{Bij}(C^\omega \times C^\omega)$ by

$$\Theta(\lambda) := \begin{cases} I, & \text{if} \quad \lambda > 0 \\ T, & \text{if} \quad \lambda < 0 \end{cases}.$$

Clearly, Θ is a group homomorphism. Let $\pi\colon H \times K \to H$ be the projection homomorphism. Then $A := \Theta \circ \pi\colon (H \times K, \circ) \to \mathrm{Bij}(C^\omega \times C^\omega)$ is a group homomorphism such that:

I. $A(\mu, \psi)((g_1, g_2) \bullet (\lambda, \phi)) = (A(\mu, \psi)(g_1, g_2)) \bullet \alpha(\mu, \psi)(\lambda, \phi)$, where $\alpha = \theta \circ \pi\colon H \times K \to \mathrm{Aut}(H \times K, \cdot)$ is the group homomorphism defined in the proof of Proposition 4.10;

II. $\mathrm{Im}\, A$ is an abelian subgroup of $\mathrm{Bij}(C^\omega \times C^\omega)$.

Also, we have:

$$(g_1, g_2) \circ (\lambda, \phi) = (A(\lambda, \phi)(g_1, g_2)) \bullet (\lambda, \phi).$$

Hence, the result follows from the following lemma.

Lemma 4.2. *Let us use the notation of Lemma 4.1. Also, let X be a set, $\bullet\colon X \times (G, \cdot) \to X$ a group action, and $A\colon (G, \circ) \to Bij(X)$ a group homomorphism satisfying the following conditions:*

I. $A(h)(x \bullet g) = (A(h)(x)) \bullet \alpha(h)(g)$ for all $x \in X$ and $g, h \in G$;

II. $Im\, A$ is an abelian subgroup of $Bij(X)$.

Then, the map $\circ\colon X \times (G, \circ) \to X$ defined by

$$x \circ g := (A(g)(x)) \bullet g$$

is a group action.

Proof of Lemma 4.2. For all $x \in X$ and $g, h \in G$, we have:

$$\begin{aligned} (x \circ g) \circ h &= (A(h)(x \circ g)) \bullet h = (A(h)((A(g)(x)) \bullet g)) \bullet h \\ &\overset{(I)}{=} ((A(h)(A(g)(x))) \bullet \alpha(h)(g)) \bullet h = ((A(h) \circ A(g))(x)) \bullet (\alpha(h)(g) \cdot h) \\ &\overset{(II)}{=} ((A(g) \circ A(h))(x)) \bullet (\alpha(h)(g) \cdot h) = (A(g \circ h)(x)) \bullet (g \circ h) \\ &= x \circ (g \circ h) \end{aligned}$$

Also, for all $x \in X$,

$$x \circ 1 = (A(1)(x)) \bullet 1 = A(1)(x) = \mathrm{id}_X(x) = x.$$

■

Proposition 4.12. *Let $f_1, f_2, g_1, g_2 \colon \mathbb{R} \to \mathbb{R}$ be polynomial functions, and let $d \geq 1$ be a fixed integer. The following conditions are equivalent:*

i. There exist constants $c_1, c_2 > 0$ and bi-Lipschitz functions $\phi_1, \phi_2 \colon \mathbb{R} \to \mathbb{R}$ such that

$$g_1 \circ \phi_1 = c_1 f_1 \quad \textit{and} \quad g_2 \circ \phi_2 = c_2 f_2$$

or

$$g_2 \circ \phi_1 = c_1 f_1 \quad \textit{and} \quad g_1 \circ \phi_2 = c_2 f_2.$$

ii. There exists a proto-transition (λ, ϕ) such that

$$(g_1, g_2) \circ (\lambda, \phi) = (f_1, f_2).$$

Proof. We prove only that (i) implies (ii), the other implication being immediate. Suppose that there exist constants $c_1, c_2 > 0$ and bi-Lipschitz functions $\phi_1, \phi_2 \colon \mathbb{R} \to \mathbb{R}$ such that

$$g_1 \circ \phi_1 = c_1 f_1 \quad \text{and} \quad g_2 \circ \phi_2 = c_2 f_2$$

or

$$g_2 \circ \phi_1 = c_1 f_1 \quad \text{and} \quad g_1 \circ \phi_2 = c_2 f_2.$$

Note that since f_1, f_2, g_1, g_2 are polynomial functions, we can assume that ϕ_1 and ϕ_2 are bi-Lipschitz Nash diffeomorphisms. Indeed, as pointed out in the beginning of Section 3.3, if $f, g \colon \mathbb{R} \to \mathbb{R}$ are nonconstant polynomial functions, and $\phi \colon \mathbb{R} \to \mathbb{R}$ is a bi-Lipschitz function such that $g \circ \phi = cf$, where c is a positive constant, then ϕ is a bi-Lipschitz Nash diffeomorphism. If instead, f and g are constant functions, then this is not necessarily true but, in this case, ϕ can be replaced with any function — for example, we could take $\phi = \mathrm{id}_{\mathbb{R}}$. So, even if f and g are constant functions, we can assume that ϕ is a bi-Lipschitz Nash diffeomorphism.

If $g_1 \circ \phi_1 = c_1 f_1$ and $g_2 \circ \phi_2 = c_2 f_2$, then $(g_1, g_2) \circ (\lambda, \phi) = (f_1, f_2)$, where $\lambda = (c_1^{-1/d}, c_2^{-1/d})$ and $\phi = (\phi_1, \phi_2)$. Otherwise, if $g_2 \circ \phi_1 = c_1 f_1$ and $g_1 \circ \phi_2 = c_2 f_2$, then $(g_1, g_2) \circ (\lambda, \phi) = (f_1, f_2)$, where $\lambda = (-c_1^{-1/d}, -c_2^{-1/d})$ and $\phi = (\phi_1, \phi_2)$. ■

Corollary 4.4. *Let $F(X, Y)$ and $G(X, Y)$ be β-quasihomogeneous polynomials of degree d. Let f_+, f_- be the height functions of F, and g_+, g_- the height functions of G. The* *following conditions are equivalent:*

i. $\begin{cases} f_+ \cong g_+ \\ f_- \cong g_- \end{cases}$ *or* $\begin{cases} f_+ \cong g_- \\ f_- \cong g_+ \end{cases}$.

ii. $(g_+, g_-) \circ (\lambda, \phi) = (f_+, f_-)$, *for some proto-transition* (λ, ϕ).

4.3 β-transitions and the inverse β-transform

Denote by $\mathcal{P}$ the set of all real polynomial functions on $\mathbb{R}$. A proto-transition (λ, ϕ) is said to be a *β-transition* if the following conditions are satisfied:

i. There exist pairs of nonconstant polynomial functions $(f_1, f_2), (g_1, g_2) \in \mathcal{P} \times \mathcal{P}$ such that

$$(g_1, g_2) \circ (\lambda, \phi) = (f_1, f_2);$$

ii. $|\lambda_1|^\beta \cdot \lim_{|t| \to +\infty} \dfrac{\phi_1(t)}{t} = |\lambda_2|^\beta \cdot \lim_{|t| \to +\infty} \dfrac{\phi_2(t)}{t}$

Remark 4.8. *By Lemma 3.5, the limit* $\lim_{|t| \to +\infty} \phi_i(t)/t$ *is a well-defined nonzero real number, for* $i = 1, 2$.

Let (λ, ϕ) and (μ, ψ) be β-transitions. We say that (μ, ψ) is *composable* with (λ, ϕ) if there exist pairs of nonconstant polynomial functions $(f_1, f_2), (g_1, g_2), (h_1, h_2) \in \mathcal{P} \times \mathcal{P}$ such that $(h_1, h_2) \circ (\mu, \psi) = (g_1, g_2)$ and $(g_1, g_2) \circ (\lambda, \phi) = (f_1, f_2)$.

Lemma 4.3. *Let* $\phi, \psi \colon \mathbb{R} \to \mathbb{R}$ *be functions for which the limits* $\lim_{|t| \to +\infty} \phi(t)/t$ *and* $\lim_{|t| \to +\infty} \psi(t)/t$ *are nonzero real numbers. Then,*

$$\lim_{|t| \to +\infty} \frac{\psi(\phi(t))}{t} = \lim_{|t| \to +\infty} \frac{\psi(t)}{t} \cdot \lim_{|t| \to +\infty} \frac{\phi(t)}{t}.$$

Proof. Since $\lim_{|t| \to +\infty} |\phi(t)/t| > 0$, we have $\lim_{|t| \to +\infty} |\phi(t)| = +\infty$. Then,

$$\lim_{|t| \to +\infty} \frac{\psi(t)}{t} = \lim_{|t| \to +\infty} \frac{\psi(\phi(t))}{\phi(t)}.$$

Hence,

$$\lim_{|t| \to +\infty} \frac{\psi(\phi(t))}{t} = \lim_{|t| \to +\infty} \frac{\psi(t)}{t} \cdot \lim_{|t| \to +\infty} \frac{\phi(t)}{t}.$$

■

Proposition 4.13. *Let* (λ, ϕ) *and* (μ, ψ) *be β-transitions. If* (μ, ψ) *is composable with* (λ, ϕ) *then* $(\mu, \psi) \circ (\lambda, \phi)$ *is a β-transition.*

Proof. Since (μ,ψ) is composable with (λ,ϕ), there exist pairs of nonconstant polynomial functions $(f_1,f_2),(g_1,g_2),(h_1,h_2)\in\mathcal{P}\times\mathcal{P}$ such that $(h_1,h_2)\circ(\mu,\psi)=(g_1,g_2)$ and $(g_1,g_2)\circ(\lambda,\phi)=(f_1,f_2)$. Hence, $(h_1,h_2)\circ((\mu,\psi)\circ(\lambda,\phi))=(f_1,f_2)$.

Now, let $(\lambda,\phi)=((\lambda_1,\lambda_2),(\phi_1,\phi_2))$ and $(\mu,\psi)=((\mu_1,\mu_2),(\psi_1,\psi_2))$. By definition,

$$(\mu,\psi)\circ(\lambda,\phi)=\begin{cases}((\lambda_1\mu_1,\lambda_2\mu_2),(\psi_1\circ\phi_1,\psi_2\circ\phi_2)), & \text{if } \lambda>0\\ ((\lambda_1\mu_2,\lambda_2\mu_1),(\psi_2\circ\phi_1,\psi_1\circ\phi_2)), & \text{if } \lambda<0\end{cases}.$$

Suppose that $\lambda>0$. By Lemma 4.3,

$$\lim_{|t|\to+\infty}\frac{\psi_i\circ\phi_i(t)}{t}=\lim_{|t|\to+\infty}\frac{\psi_i(t)}{t}\cdot\lim_{|t|\to+\infty}\frac{\phi_i(t)}{t}\qquad\text{for } i=1,2.$$

Thus,

$$\begin{aligned}|\lambda_1\mu_1|^\beta\lim_{|t|\to+\infty}\frac{\psi_1(\phi_1(t))}{t}&=|\lambda_1\mu_1|^\beta\lim_{|t|\to+\infty}\frac{\psi_1(t)}{t}\cdot\lim_{|t|\to+\infty}\frac{\phi_1(t)}{t}\\&=\left(|\mu_1|^\beta\frac{\psi_1(t)}{t}\right)\cdot\left(|\lambda_1|^\beta\frac{\phi_1(t)}{t}\right)\\&=\left(|\mu_2|^\beta\frac{\psi_2(t)}{t}\right)\cdot\left(|\lambda_2|^\beta\frac{\phi_2(t)}{t}\right)\\&=|\lambda_2\mu_2|^\beta\lim_{|t|\to+\infty}\frac{\psi_2(t)}{t}\cdot\lim_{|t|\to+\infty}\frac{\phi_2(t)}{t}\\&=|\lambda_2\mu_2|^\beta\lim_{|t|\to+\infty}\frac{\psi_2(\phi_2(t))}{t},\end{aligned}$$

so that $(\mu,\psi)\circ(\lambda,\phi)$ is indeed a β-transition. The proof for $\lambda<0$ is analogous. ■

Proposition 4.14. *The identity element $\hat{\imath}:=((1,1),(\mathrm{id}_\mathbb{R},\mathrm{id}_\mathbb{R}))$ of the group of the proto-transitions is a β-transition, which we call the identity β-transition. For every β-transition (λ,ϕ), $\hat{\imath}$ is composable with (λ,ϕ), (λ,ϕ) is composable with $\hat{\imath}$, and $\hat{\imath}\circ(\lambda,\phi)=(\lambda,\phi)\circ\hat{\imath}=(\lambda,\phi)$.*

Proof. Clearly, for any pair of nonconstant polynomial functions $(f_1,f_2)\in\mathcal{P}\times\mathcal{P}$, we have $(f_1,f_2)\circ\hat{\imath}=(f_1,f_2)$, and

$$|\lambda_1|^\beta\cdot\lim_{|t|\to+\infty}\frac{\phi_1(t)}{t}=|\lambda_2|^\beta\cdot\lim_{|t|\to+\infty}\frac{\phi_2(t)}{t},$$

for $\lambda_1=\lambda_2=1$ and $\phi_1=\phi_2=\mathrm{id}_\mathbb{R}$; so $\hat{\imath}$ is a β-transition.

Now, if (λ,ϕ) is a β-transition, then there exist pairs of nonconstant polynomial functions $(f_1,f_2),(g_1,g_2)\in\mathcal{P}\times\mathcal{P}$ such that $(g_1,g_2)\circ(\lambda,\phi)=(f_1,f_2)$. Since $(g_1,g_2)\circ\hat{\imath}=(g_1,g_2)$ and $(g_1,g_2)\circ(\lambda,\phi)=(f_1,f_2)$, $\hat{\imath}$ is composable with (λ,ϕ); and since $(g_1,g_2)\circ$

$(\lambda,\phi)=(f_1,f_2)$ and $(f_1,f_2)\circ\hat{\imath}=(f_1,f_2)$, (λ,ϕ) is composable with $\hat{\imath}$. Obviously, $\hat{\imath}\circ(\lambda,\phi)=(\lambda,\phi)\circ\hat{\imath}=(\lambda,\phi)$. ■

Lemma 4.4. *Let $\phi\colon\mathbb{R}\to\mathbb{R}$ be a bijective continuous function for which the limit $\lim_{|t|\to+\infty}\phi(t)/t$ is a nonzero real number. Then,*

$$\lim_{|t|\to+\infty}\frac{\phi^{-1}(t)}{t}=\left(\lim_{|t|\to+\infty}\frac{\phi(t)}{t}\right)^{-1}.$$

Proof. Note that ϕ is monotone (because it is an injective continuous function whose domain is an interval), so ϕ^{-1} is also monotone. Since $\phi^{-1}(\mathbb{R})=\mathbb{R}$, it follows that $\lim_{|t|\to+\infty}|\phi^{-1}(t)|=+\infty$. Then,

$$\lim_{|t|\to+\infty}\frac{\phi(t)}{t}=\lim_{|t|\to+\infty}\frac{\phi(\phi^{-1}(t))}{\phi^{-1}(t)}=\lim_{|t|\to+\infty}\frac{t}{\phi^{-1}(t)}.$$

Hence the result. ■

Proposition 4.15. *If (λ,ϕ) is a β-transition then $(\lambda,\phi)^{-1}$ is a β-transition. Moreover, (λ,ϕ) is composable with $(\lambda,\phi)^{-1}$, $(\lambda,\phi)^{-1}$ is composable with (λ,ϕ), and $(\lambda,\phi)\circ(\lambda,\phi)^{-1}=(\lambda,\phi)^{-1}\circ(\lambda,\phi)=\hat{\imath}$.*

Proof. Let (λ,ϕ) be a β-transition. Then, there exist pairs of nonconstant polynomial functions $(f_1,f_2),(g_1,g_2)\in\mathcal{P}\times\mathcal{P}$ such that $(g_1,g_2)\circ(\lambda,\phi)=(f_1,f_2)$, and hence $(f_1,f_2)\circ(\lambda,\phi)^{-1}=(g_1,g_2)$.

Now, let $(\lambda,\phi)=((\lambda_1,\lambda_2),(\phi_1,\phi_2))$. We have:

$$(\lambda,\phi)^{-1}=\begin{cases}((\lambda_1^{-1},\lambda_2^{-1}),(\phi_1^{-1},\phi_2^{-1})), & \text{if } \lambda>0\\ ((\lambda_2^{-1},\lambda_1^{-1}),(\phi_2^{-1},\phi_1^{-1})), & \text{if } \lambda<0\end{cases}.$$

Suppose that $\lambda>0$. By Lemma 4.4,

$$\lim_{|t|\to+\infty}\frac{\phi_i^{-1}(t)}{t}=\left(\lim_{|t|\to+\infty}\frac{\phi_i(t)}{t}\right)^{-1},\quad \text{for } i=1,2.$$

Thus,

$$\begin{aligned}\left|\lambda_1^{-1}\right|^{\beta}\lim_{|t|\to+\infty}\frac{\phi_1^{-1}(t)}{t}&=|\lambda_1|^{-\beta}\left(\lim_{|t|\to+\infty}\frac{\phi_1(t)}{t}\right)^{-1}\\&=\left(|\lambda_1|^{\beta}\lim_{|t|\to+\infty}\frac{\phi_1(t)}{t}\right)^{-1}\\&=\left(|\lambda_2|^{\beta}\lim_{|t|\to+\infty}\frac{\phi_2(t)}{t}\right)^{-1}\end{aligned}$$

$$= |\lambda_2|^{-\beta} \left(\lim_{|t| \to +\infty} \frac{\phi_2(t)}{t} \right)^{-1}$$
$$= \left|\lambda_2^{-1}\right|^{\beta} \lim_{|t| \to +\infty} \frac{\phi_2^{-1}(t)}{t},$$

so that $(\lambda, \phi)^{-1}$ is indeed a β-transition. The proof for $\lambda < 0$ is analogous.

Since $(g_1, g_2) \circ (\lambda, \phi) = (f_1, f_2)$ and $(f_1, f_2) \circ (\lambda, \phi)^{-1} = (g_1, g_2)$, it is clear that both (λ, ϕ) is composable with $(\lambda, \phi)^{-1}$ and $(\lambda, \phi)^{-1}$ is composable with (λ, ϕ). Also, it is immediate that $(\lambda, \phi) \circ (\lambda, \phi)^{-1} = (\lambda, \phi)^{-1} \circ (\lambda, \phi) = \hat{\imath}$. ■

Given a β-transition (λ, ϕ), we define a map $\Phi \colon \mathbb{R}^2 \to \mathbb{R}^2$ by setting:

- $\Phi(x, t\,|x|^{\beta}) := \left(\lambda_1 x, |\lambda_1|^{\beta}\, \phi_1(t)\,|x|^{\beta}\right), \quad$ for $x > 0$, $t \in \mathbb{R}$
- $\Phi(x, t\,|x|^{\beta}) := \left(\lambda_2 x, |\lambda_2|^{\beta}\, \phi_2(t)\,|x|^{\beta}\right), \quad$ for $x < 0$, $t \in \mathbb{R}$
- $\Phi(0, y) := \left(0, |\lambda_1|^{\beta} \lim_{|t| \to +\infty} \frac{\phi_1(t)}{t}\, y\right) = \left(0, |\lambda_2|^{\beta} \lim_{|t| \to +\infty} \frac{\phi_2(t)}{t}\, y\right), \quad$ for all $y \in \mathbb{R}$

The *inverse β-transform* of (λ, ϕ) is the germ $\Phi \colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ determined by the map Φ.

Proposition 4.16. *Let (λ, ϕ) and (μ, ψ) be β-transitions such that (μ, ψ) is composable with (λ, ϕ), let Φ and Ψ be their respective inverse β-transforms, and let Z be the inverse β-transform of $(\mu, \psi) \circ (\lambda, \phi)$. Then, $Z = \Psi \circ \Phi$.*

Proof. For all $x > 0, t \in \mathbb{R}$,

$$\begin{aligned}
\Psi(\Phi(x, t\,|x|^{\beta})) &= \Psi\left(\lambda_1 x, |\lambda_1|^{\beta}\, \phi_1(t)\,|x|^{\beta}\right) \\
&= \begin{cases} \left(\lambda_1\mu_1 x, |\lambda_1\mu_1|^{\beta}\, \psi_1(\phi_1(t))\,|x|^{\beta}\right), & \text{if } \lambda > 0 \\ \left(\lambda_1\mu_2 x, |\lambda_1\mu_2|^{\beta}\, \psi_2(\phi_1(t))\,|x|^{\beta}\right), & \text{if } \lambda < 0 \end{cases} \\
&= Z(x, t\,|x|^{\beta})\,.
\end{aligned}$$

For all $x < 0, t \in \mathbb{R}$,

$$\begin{aligned}
\Psi(\Phi(x, t\,|x|^{\beta})) &= \Psi\left(\lambda_2 x, |\lambda_2|^{\beta}\, \phi_2(t)\,|x|^{\beta}\right) \\
&= \begin{cases} \left(\lambda_2\mu_2 x, |\lambda_2\mu_2|^{\beta}\, \psi_2(\phi_2(t))\,|x|^{\beta}\right), & \text{if } \lambda > 0 \\ \left(\lambda_2\mu_1 x, |\lambda_2\mu_1|^{\beta}\, \psi_1(\phi_2(t))\,|x|^{\beta}\right), & \text{if } \lambda < 0 \end{cases} \\
&= Z(x, t\,|x|^{\beta})\,.
\end{aligned}$$

Now, we prove that $Z(0,y) = \Psi(\Phi(0,y))$, for all $y \in \mathbb{R}$. For $\lambda > 0$, we have:

$$\begin{aligned}
Z(0,y) &= \left(0, |\lambda_1\mu_1|^\beta \cdot \lim_{|t|\to+\infty} \frac{\psi_1 \circ \phi_1(t)}{t} \cdot y\right) \\
&= \left(0, |\lambda_1\mu_1|^\beta \cdot \lim_{|t|\to+\infty} \frac{\psi_1(t)}{t} \cdot \lim_{|t|\to+\infty} \frac{\phi_1(t)}{t} \cdot y\right) \\
&= \left(0, \left(|\mu_1|^\beta \lim_{|t|\to+\infty} \frac{\psi_1(t)}{t}\right) \cdot \left(|\lambda_1|^\beta \lim_{|t|\to+\infty} \frac{\phi_1(t)}{t}\right) \cdot y\right) \\
&= \Psi(\Phi(0,y)).
\end{aligned}$$

And for $\lambda < 0$, we have:

$$\begin{aligned}
Z(0,y) &= \left(0, |\lambda_1\mu_2|^\beta \cdot \lim_{|t|\to+\infty} \frac{\psi_2 \circ \phi_1(t)}{t} \cdot y\right) \\
&= \left(0, |\lambda_1\mu_2|^\beta \cdot \lim_{|t|\to+\infty} \frac{\psi_2(t)}{t} \cdot \lim_{|t|\to+\infty} \frac{\phi_1(t)}{t} \cdot y\right) \\
&= \left(0, \left(|\mu_2|^\beta \lim_{|t|\to+\infty} \frac{\psi_2(t)}{t}\right) \cdot \left(|\lambda_1|^\beta \lim_{|t|\to+\infty} \frac{\phi_1(t)}{t}\right) \cdot y\right) \\
&= \Psi(\Phi(0,y)).
\end{aligned}$$

■

Remark 4.9. *The inverse β-transform of the identity β-transition $\hat{\imath}$ is the germ of the identity map $I\colon (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$.*

Corollary 4.5. *Let (λ,ϕ) be a β-transition and Φ its inverse β-transform. Then, the inverse β-transform of $(\lambda,\phi)^{-1}$ is Φ^{-1}.*

Now, we prove that the inverse β-transform of a β-transition is a germ of bi-Lipschitz map.

Lemma 4.5. *Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a bi-Lipschitz function such that $g \circ \phi = f$ for some nonconstant polynomial functions $f, g\colon \mathbb{R} \to \mathbb{R}$. Then ϕ is bi-analytic and $\phi(t) - t\phi'(t)$ is bounded.*

Proof. First, note that by Lemma 3.2, $\deg f = \deg g$, so we can apply Lemma 3.3 to conclude that ϕ is bi-analytic. Now, by Lemma 3.5, the function $\psi\colon \mathbb{R} \to \mathbb{R}$ given by

$$\psi(t) := \begin{cases} t \cdot \phi(t^{-1}), & \text{if } t \in \mathbb{R} \setminus \{0\} \\ \lim_{|t|\to+\infty} \phi(t)/t, & \text{if } t = 0 \end{cases}$$

is analytic.

From the definition of ψ, it is immediate that

$$\frac{\phi(t)}{t} = \psi(t^{-1}) \quad \text{for all } t \in \mathbb{R} \setminus \{0\}.$$

Differentiating both sides of this equation, we get

$$\frac{\phi'(t) \cdot t - \phi(t)}{t^2} = -\frac{\psi'(t^{-1})}{t^2} \quad \text{for all } t \in \mathbb{R} \setminus \{0\}.$$

Equivalently, we have

$$\phi(t) - t\phi'(t) = \psi'(t^{-1}) \quad \text{for all } t \in \mathbb{R} \setminus \{0\}.$$

Hence,

$$\lim_{|t| \to +\infty} \phi(t) - t\phi'(t) = \psi'(0).$$

Since the function $\phi(t) - t\phi'(t)$ is continuous on $\mathbb{R}$, the existence of this limit implies that this function is bounded. ■

Lemma 4.6. *Let $\phi \colon \mathbb{R} \to \mathbb{R}$ be a bi-Lipschitz function such that $g \circ \phi = f$ for some nonconstant polynomial functions $f, g \colon \mathbb{R} \to \mathbb{R}$, and let λ be a nonzero real number. Then, the map $\Phi \colon \mathcal{H} \to \mathbb{R}^2$, defined on the right half-plane $\mathcal{H} := \{(x, y) \in \mathbb{R}^2 : x > 0\}$ by*

$$\Phi(x, tx^\beta) := (\lambda x, |\lambda|^\beta \phi(t) x^\beta)$$

for all $x > 0$ and $t \in \mathbb{R}$, is Lipschitz on the strip $\mathcal{H}_\delta := \{(x, y) \in \mathbb{R}^2 : 0 < x < \delta\}$, for each $\delta > 0$.

Proof. Let $\delta > 0$ be fixed. We prove that Φ is Lipschitz on both the upper half-strip $\mathcal{H}_\delta \cap \{y > 0\}$ and the lower half-strip $\mathcal{H}_\delta \cap \{y < 0\}$. Let us see that this implies the result. Assuming this claim, and using the fact that Φ is continuous, we see that there exists a constant $C > 0$ such that

$$|\Phi(x_1, y_1) - \Phi(x_2, y_2)| \leq C \, |(x_1, y_1) - (x_2, y_2)| \,, \tag{45}$$

whenever (x_1, y_1) and (x_2, y_2) both belong to $\mathcal{H}_\delta \cap \{y \geq 0\}$ or to $\mathcal{H}_\delta \cap \{y \leq 0\}$. We show that (45) still holds for $(x_1, y_1) \in \mathcal{H}_\delta \cap \{y \geq 0\}$ and $(x_2, y_2) \in \mathcal{H}_\delta \cap \{y \leq 0\}$. Let $(\bar{x}, 0)$ be the point at which the line segment whose endpoints are (x_1, y_1) and (x_2, y_2) intersects the x-axis. By our assumptions, we have:

$$|\Phi(x_1, y_1) - \Phi(\bar{x}, 0)| \leq C \, |(x_1, y_1) - (\bar{x}, 0)|$$

and

$$|\Phi(\bar{x},0) - \Phi(x_2,y_2)| \leq C\,|(\bar{x},0) - (x_2,y_2)|\,.$$

Hence,

$$\begin{aligned} |\Phi(x_1,y_1) - \Phi(x_2,y_2)| &\leq |\Phi(x_1,y_1) - \Phi(\bar{x},0)| + |\Phi(\bar{x},0) - \Phi(x_2,y_2)| \\ &\leq C\,(|(x_1,y_1) - (\bar{x},0)| + |(\bar{x},0) - (x_2,y_2)|) \\ &= C\,|(x_1,y_1) - (x_2,y_2)|\,, \end{aligned}$$

where the last equality holds because the point $(\bar{x},0)$ lies in the segment whose endpoints are (x_1,y_1) and (x_2,y_2). Therefore, our initial claim implies that Φ is Lipschitz on the strip $\mathcal{H}_\delta$.

In order to establish our initial claim, we first show that for each fixed pair of points $(x_1, t_1 x_1^\beta)$ and $(x_2, t_2 x_2^\beta)$, either both on $\mathcal{H}_\delta \cap \{y > 0\}$ or both on $\mathcal{H}_\delta \cap \{y < 0\}$, with $x_1 \neq x_2$ and $t_1 \neq t_2$, there exist ω between x_1 and x_2, and τ between t_1 and t_2 such that[12]

$$\phi(t_2)x_2^\beta - \phi(t_1)x_1^\beta = (\phi(\tau) - \tau\phi'(\tau))\cdot\beta\omega^{\beta-1}\cdot(x_2 - x_1) + \frac{\phi(t_2)-\phi(t_1)}{t_2 - t_1}\cdot(t_2 x_2^\beta - t_1 x_1^\beta) \tag{46}$$

In fact,

$$\begin{aligned} \phi(t_2)x_2^\beta - \phi(t_1)x_1^\beta &= (\phi(t_1) + q\cdot(t_2 - t_1))x_2^\beta - \phi(t_1)x_1^\beta, \text{ where } q = \frac{\phi(t_2)-\phi(t_1)}{t_2 - t_1} \\ &= \phi(t_1)\cdot(x_2^\beta - x_1^\beta) + q\cdot(t_2 x_2^\beta - t_1 x_2^\beta) \\ &= \phi(t_1)\cdot(x_2^\beta - x_1^\beta) + q\cdot(t_2 x_2^\beta - t_1 x_1^\beta) + q\cdot t_1\cdot(x_1^\beta - x_2^\beta) \\ &= (\phi(t_1) - q\cdot t_1)\cdot(x_2^\beta - x_1^\beta) + q\cdot(t_2 x_2^\beta - t_1 x_1^\beta) \\ &= \frac{t_2\phi(t_1) - t_1\phi(t_2)}{t_2 - t_1}\cdot(x_2^\beta - x_1^\beta) + \frac{\phi(t_2)-\phi(t_1)}{t_2 - t_1}\cdot(t_2 x_2^\beta - t_1 x_1^\beta) \end{aligned} \tag{47}$$

Since the points $(x_1, t_1 x_1^\beta)$ and $(x_2, t_2 x_2^\beta)$ are either both on $\mathcal{H}_\delta \cap \{y > 0\}$ or both on $\mathcal{H}_\delta \cap \{y < 0\}$, the real numbers t_1, t_2 are either both positive or both negative. Thus, by Pompeiu's Mean Value Theorem (DRAGOMIR, 2015, p. 1 – 2), there exists a real number τ between t_1 and t_2 such that

$$\frac{t_2\phi(t_1) - t_1\phi(t_2)}{t_2 - t_1} = \phi(\tau) - \tau\phi'(\tau). \tag{48}$$

Also, by Lagrange's Mean Value Theorem, there exists a real number ω between x_1 and x_2 such that

$$x_2^\beta - x_1^\beta = \beta\omega^{\beta-1}\cdot(x_2 - x_1). \tag{49}$$

[12] By Lemma 4.5, ϕ is bi-analytic. So, in particular, $\phi' \colon \mathbb{R} \to \mathbb{R}$ is a well-defined continuous function.

Substituting (48) and (49) in (47), we obtain (46).

Now, since ϕ is Lipschitz, there exists a constant $C_1 > 0$ such that

$$\left|\frac{\phi(t_2) - \phi(t_1)}{t_2 - t_1}\right| \leq C_1,$$

for $t_1 \neq t_2$. On the other hand, by Lemma 4.5, $\phi(t) - t\phi'(t)$ is bounded, so there exists a constant $C_2 > 0$ such that

$$\left|(\tau\phi'(\tau) - \phi(\tau)) \cdot \beta\omega^{\beta-1}\right| \leq C_2,$$

provided that $0 < x_1, x_2 < \delta$.

Applying these bounds to (46), we obtain:

$$\left|\phi(t_2)x_2^\beta - \phi(t_1)x_1^\beta\right| \leq C \cdot \left(|x_2 - x_1| + \left|t_2x_2^\beta - t_1x_1^\beta\right|\right),$$

where $C = \max\{C_1, C_2\}$, for any pair of points $(x_1, t_1x_1^\beta)$ and $(x_2, t_2x_2^\beta)$, either both on $\mathcal{H}_\delta \cap \{y > 0\}$ or both on $\mathcal{H}_\delta \cap \{y < 0\}$, thereby proving our initial claim. ■

Corollary 4.6. *Let (λ, ϕ) be a β-transition, and let $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the map defined by:*

- $\Phi(x, t\,|x|^\beta) := \left(\lambda_1 x, |\lambda_1|^\beta\,\phi_1(t)\,|x|^\beta\right)$, *for $x > 0$, $t \in \mathbb{R}$*
- $\Phi(x, t\,|x|^\beta) := \left(\lambda_2 x, |\lambda_2|^\beta\,\phi_2(t)\,|x|^\beta\right)$, *for $x < 0$, $t \in \mathbb{R}$*
- $\Phi(0, y) := \left(0, |\lambda_1|^\beta \lim\limits_{|t|\to+\infty} \frac{\phi_1(t)}{t}\,y\right) = \left(0, |\lambda_2|^\beta \lim\limits_{|t|\to+\infty} \frac{\phi_2(t)}{t}\,y\right)$, *for all $y \in \mathbb{R}$*

Then, Φ is Lipschitz on the strip $\{(x, y) \in \mathbb{R}^2 : |x| < \delta\}$, for each $\delta > 0$.

Proof. Let $\delta > 0$ be fixed arbitrarily. By Lemma 4.6, there exists a constant $C_+ > 0$ such that $\Phi|_{\mathcal{H}_\delta}\colon \mathcal{H}_\delta \to \mathbb{R}^2$ is C_+-Lipschitz. Since $\Phi|_{\mathcal{H}_\delta}$ is uniformly continuous and takes values in $\mathbb{R}^2$, it has a unique continuous extension $\widetilde{\Phi}$ to $\widetilde{\mathcal{H}}_\delta := \{(x, y) \in \mathbb{R}^2 : 0 \leq x < \delta\}$. Let us show that $\widetilde{\Phi} = \Phi|_{\widetilde{\mathcal{H}}_\delta}$. Obviously, $\widetilde{\Phi}(x, y) = \Phi(x, y)$ for all $(x, y) \in \mathcal{H}_\delta$; and for all $y \in \mathbb{R}$, we have:

$$\begin{aligned}
\widetilde{\Phi}(0, y) &= \lim_{x\to0^+} \Phi(x, y) \\
&= \lim_{x\to0^+} \left(\lambda_1 x, |\lambda_1|^\beta \cdot \frac{\phi_1(t)}{t} \cdot y\right), \quad \text{where } t = \frac{y}{x^\beta} \\
&= \left(0, |\lambda_1|^\beta \cdot \lim_{|t|\to+\infty} \frac{\phi_1(t)}{t} \cdot y\right) \\
&= \Phi(0, y).
\end{aligned}$$

Hence, $\Phi|_{\widetilde{\mathcal{H}}_\delta}$ is the continuous extension of $\Phi|_{\mathcal{H}_\delta}$ to $\widetilde{\mathcal{H}}_\delta$. And since $\Phi|_{\mathcal{H}_\delta}$ is C_+-Lipschitz, it follows that $\Phi|_{\widetilde{\mathcal{H}}_\delta}$ is C_+-Lipschitz too.

Similarly, we can prove that there exists $C_- > 0$ such that $\Phi|_{-\widetilde{\mathcal{H}}_\delta} \colon -\widetilde{\mathcal{H}}_\delta \to \mathbb{R}^2$ is C_--Lipschitz. Therefore, Φ is C-Lipschitz on the strip $\{(x,y) \in \mathbb{R}^2 : |x| < \delta\}$, where $C = \max\{C_+, C_-\}$. ■

Corollary 4.7. *The inverse β-transform $\Phi\colon (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ of every β-transition (λ,ϕ) is a germ of semialgebraic bi-Lipschitz map.*

Proof. Since ϕ_1 and ϕ_2 are both semialgebraic functions, it is immediate from the definition of the inverse β-transform that Φ is a germ of semialgebraic map. Also, by Corollary 4.6, both Φ and Φ^{-1} are germs of Lipschitz maps. (Note that Φ^{-1} is the inverse β-transform of $(\lambda,\phi)^{-1}$, by Corollary 4.5.) Hence the result. ■

Proposition 4.17. *Let $F, G \in \mathbb{R}[X,Y]$ be β-quasihomogeneous polynomials of degree d, and let f_+, f_- be the height functions of F and g_+, g_- the height functions of G. Suppose that $(g_+, g_-) \circ (\lambda,\phi) = (f_+, f_-)$ for some β-transition (λ,ϕ). Then, $G \circ \Phi = F$, where Φ is the inverse β-transform of (λ,ϕ).*

Proof. First, we show that

$$G(\lambda_1, |\lambda_1|^\beta \, \phi_1(t)) = F(1,t) \quad \text{and} \quad G(-\lambda_2, |\lambda_2|^\beta \, \phi_2(t)) = F(-1,t). \tag{50}$$

We consider separately the cases $\lambda > 0$ and $\lambda < 0$.

If $\lambda > 0$, we have

$$|\lambda_1|^d \cdot g_+ \circ \phi_1 = f_+ \quad \text{and} \quad |\lambda_2|^d \cdot g_- \circ \phi_2 = f_-.$$

Equivalently,

$$|\lambda_1|^d \cdot G(1, \phi_1(t)) = F(1,t) \quad \text{and} \quad |\lambda_2|^d \cdot G(-1, \phi_2(t)) = F(-1,t).$$

Thus, using the fact that the polynomials F and G are β-quasihomogeneous of degree d, we obtain (50).

If $\lambda < 0$, we have

$$|\lambda_1|^d \cdot g_- \circ \phi_1 = f_+ \quad \text{and} \quad |\lambda_2|^d \cdot g_+ \circ \phi_2 = f_-.$$

Equivalently,

$$|\lambda_1|^d \cdot G(-1, \phi_1(t)) = F(1,t) \quad \text{and} \quad |\lambda_2|^d \cdot G(1, \phi_2(t)) = F(-1,t).$$

Again, using the fact that the polynomials F and G are β-quasihomogeneous of degree d, we obtain (50).

Now, by using once more the fact that the polynomials F and G are β-quasihomogeneous of degree d, we obtain from (50):

$$G(\lambda_1 x, |\lambda_1|^\beta \phi_1(t)\, |x|^\beta) = F(x, t\,|x|^\beta) \quad \text{for } x > 0, t \in \mathbb{R}$$

and

$$G(\lambda_2 x, |\lambda_2|^\beta \phi_2(t)\, |x|^\beta) = F(x, t\,|x|^\beta) \quad \text{for } x < 0, t \in \mathbb{R}.$$

In other words,

$$G(\Phi(x,y)) = F(x,y) \quad \text{for all } (x,y) \in \mathbb{R}^2, \text{ with } x \neq 0,$$

where Φ is the inverse β-transform of (λ, ϕ). Since Φ is continuous[13], we have $G(\Phi(x,y)) = F(x,y)$ for all $(x,y) \in \mathbb{R}^2$. ∎

Corollary 4.8. *Let $F, G \in \mathbb{R}[X,Y]$ be β-quasihomogeneous polynomials of degree d, and let f_+, f_- be the height functions of F and g_+, g_- the height functions of G. Suppose that $(g_+, g_-) \circ (\lambda, \phi) = (f_+, f_-)$ for some β-transition (λ, ϕ). Then, F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent.*

Proof. Let (λ, ϕ) be a β-transition such that $(g_+, g_-) \circ (\lambda, \phi) = (f_+, f_-)$. By Proposition 4.17, $G \circ \Phi = F$, where Φ is the inverse β-transform of (λ, ϕ). Since, by Corollary 4.7, $\Phi \colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is a germ of semialgebraic bi-Lipschtiz map, this shows that F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent. ∎

4.4 Shifting from proto-transitions to β-transitions

Let $F, G \in \mathbb{R}[X,Y]$ be β-quasihomogeneous polynomials of degree d. Suppose that the height functions of F and G can be arranged in pairs of Lipschitz equivalent functions. Our goal is to find conditions under which this assumption implies that F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent. By Corollary 4.4, the height functions f_+, f_- of F and the height functions g_+, g_- of G can be arranged in pairs of Lipschitz equivalent functions if and only if $(g_+, g_-) \circ (\lambda, \phi) = (f_+, f_-)$ for some proto-transition (λ, ϕ). In general, such a proto-transition (λ, ϕ) is not necessarily a β-transition, but since it is not uniquely determined by (g_+, g_-) and (f_+, f_-) we can still ask whether it may replaced with a β-transition $(\tilde{\lambda}, \tilde{\phi})$. In this section, we are interested in finding conditions under which the answer to this question is affirmative. Then, we can apply Corollary 4.8 to conclude that F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent.

[13] Clearly, Φ is continuous on the right half-plane $\{(x,y) \in \mathbb{R}^2 : x > 0\}$ and also on the left half-plane $\{(x,y) \in \mathbb{R}^2 : x < 0\}$. By Corollary 4.6, Φ is Lipschitz (and therefore continuous) on a strip $\{(x,y) \in \mathbb{R}^2 : |x| < \delta\}$. Since the right half-plane, the left half-plane, and the strip around the y-axis form an open cover of the plane, it follows that Φ is continuous.

We consider separately the case where F and G are both of the form cX^d and the case where neither F nor G is of this form.[14,15] In the first case, we can easily determine directly from first principles whether F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent. In the second case, we follow the strategy sketched above: assuming that the height functions of F and G can be arranged in pairs of Lipschitz equivalent functions, so that $(g_+, g_-) \circ (\lambda, \phi) = (f_+, f_-)$, where (λ, ϕ) is a proto-transition, we find conditions under which we can construct from the proto-transition (λ, ϕ) a β-transition $(\tilde{\lambda}, \tilde{\phi})$ such that $(g_+, g_-) \circ (\tilde{\lambda}, \tilde{\phi}) = (f_+, f_-)$. But first of all, we show that if the height functions of F and G can be arranged in pairs of Lipschitz equivalent functions, then either both F and G are of the form cX^d or none of them is of this form. This follows from the next two propositions (see Corollary 4.9).

Proposition 4.18. *Let $P \in \mathbb{R}[X,Y]$ be a β-quasihomogeneous polynomial of degree d and let e be the multiplicity of X as a factor of P in $\mathbb{R}[X,Y]$. Then $e \leq d$, with equality if and only if P is of the form cX^d.*

Proof. Since P is a β-quasihomogeneous polynomial of degree $d \geq 1$, we have $P(X,Y) = \sum_{k=0}^{n} c_k X^{d-rk} Y^{sk}$, where $c_n \neq 0$ and $0 \leq n \leq \lfloor d/r \rfloor$. Then, clearly, $e = d - rn$. So $e \leq d$, with equality if and only if $n = 0$. Since $n = 0$ if and only if P is of the form cX^d, the result follows. ∎

Proposition 4.19. *Let $F, G \in \mathbb{R}[X,Y]$ be β-quasihomogeneous polynomials of degree d. Denote by e_F the multiplicity of X as a factor of F, and by e_G the multiplicity of X as a factor of G, so that $F(X,Y) = X^{e_F} \cdot \widetilde{F}(X,Y)$ and $G(X,Y) = X^{e_G} \cdot \widetilde{G}(X,Y)$, where $X \nmid \widetilde{F}(X,Y)$ and $X \nmid \widetilde{G}(X,Y)$. If the height functions of F and G can be arranged in pairs of Lipschitz equivalent functions, then:*

i. $e_F = e_G$

ii. *For all $t > 0$, we have:*

$$\widetilde{F}(tX, t^\beta Y) = t^{d-e} \widetilde{F}(X,Y) \quad \text{and} \quad \widetilde{G}(tX, t^\beta Y) = t^{d-e} \widetilde{G}(X,Y),$$

where $e = e_F = e_G$.

[14] We show that if the height functions of F and G can be arranged in pairs of Lipschitz equivalent functions, then either both F and G are of the form cX^d or none of them is of this form (see Corollary 4.9).

[15] The main reason for us to consider these two cases separately is that only in the second case the height functions of F and G are nonconstant.

Proof. Let $\beta = r/s$, where $r > s > 0$ and $\gcd(r,s) = 1$. Since F and G are β-quasihomogeneous polynomials of degree d, we have:

$$F(X,Y) = \sum_{k=0}^{m} a_k X^{d-rk} Y^{sk} \quad \text{and} \quad G(X,Y) = \sum_{k=0}^{n} b_k X^{d-rk} Y^{sk},$$

where $a_m \neq 0$, $b_n \neq 0$, and $0 \leq m, n \leq \lfloor d/r \rfloor$. Then, $e_F = d - rm$ and $e_G = d - rn$.

We prove that $m = n$ — by the equations above, this implies that $e_F = e_G$. Since $f_+(t) = \sum_{k=0}^{m} a_k t^{sk}$ and $f_-(t) = \sum_{k=0}^{m} (-1)^{d-rk} a_k t^{sk}$, we have $\deg f_+ = \deg f_- = sm$; and since $g_+(t) = \sum_{k=0}^{n} b_k t^{sk}$ and $g_-(t) = \sum_{k=0}^{n} (-1)^{d-rk} b_k t^{sk}$, we have $\deg g_+ = \deg g_- = sn$. Thus, since f_+ and f_- are Lipschitz equivalent to g_+ and g_- in some order (by hypothesis), it follows that $sm = sn$. Therefore, $m = n$.

From now on, we drop the subscript and denote simply by e the multiplicity of X as a factor of either F or G. Let us prove the second part of the proposition. Since F is a β-quasihomogeneous polynomial of degree d,

$$\begin{aligned} F(tX, t^\beta Y) &= t^d F(X,Y) \\ &= t^d X^e \widetilde{F}(X,Y). \end{aligned}$$

On the other hand, since the multiplicity of X as a factor of F is equal to e,

$$\begin{aligned} F(tX, t^\beta Y) &= (tX)^e \widetilde{F}(tX, t^\beta Y) \\ &= t^e X^e \widetilde{F}(tX, t^\beta Y). \end{aligned}$$

Hence,

$$t^e X^e \widetilde{F}(tX, t^\beta Y) = t^d X^e \widetilde{F}(X,Y).$$

Therefore,

$$\widetilde{F}(tX, t^\beta Y) = t^{d-e} \widetilde{F}(X,Y).$$

Obviously, the deduction above with F replaced by G yields the other equation. ■

Corollary 4.9. *Let $F, G \in \mathbb{R}[X,Y]$ be β-quasihomogeneous polynomials of degree d. If the height functions of F and G can be arranged in pairs of Lipschitz equivalent functions, then either both F and G are of the form cX^d or none of them is of this form.*

Proof. Denote by e_F the multiplicity of X as a factor of F, and by e_G the multiplicity of X as a factor of G. By Proposition 4.18, F is of the form cX^d if and only if $e_F = d$, and G is of the form cX^d if and only if $e_G = d$. Suppose that the height functions of F and G can be arranged in pairs of Lipschitz equivalent functions. Then, by Proposition 4.19,

$e_F = e_G$, so that $e_F = d$ if and only if $e_G = d$, and therefore F is of the form cX^d if and only if G is of the form cX^d. ■

The next proposition shows how to determine whether any two polynomials of the form cX^d, with $c \neq 0$ and $d \geq 1$, are $\mathcal{R}$-semialgebraically Lipschitz equivalent.

Proposition 4.20. *Let $F(X,Y) = aX^d$ and $G(X,Y) = bX^d$, where $a, b \in \mathbb{R} \setminus \{0\}$ and $d \geq 1$.*

i. If d is even, then F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent if and only if a and b have the same sign.

ii. If d is odd, then F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent.

Proof. (i) Suppose that d is even. If there exists a germ of semialgebraic bi-Lipschitz homeomorphism $\Phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $G \circ \Phi = F$ then $b \cdot \Phi_1(x,y)^d = ax^d$ in a neighborhood of the origin, which implies that a and b have the same sign, since d is even. Now, assuming that a and b have the same sign, we have $G \circ \Phi = F$, where $\Phi(x,y) = \left(\left(\frac{a}{b}\right)^{\frac{1}{d}} \cdot x, y\right)$.

(ii) If d is odd, then $G \circ \Phi = F$, where $\Phi(x,y) = \left(\left(\frac{a}{b}\right)^{\frac{1}{d}} \cdot x, y\right)$. ■

Now, we turn our attention to the case where none of the polynomials F and G is of the form cX^d.

Lemma 4.7. *Let $F, G \in \mathbb{R}[X,Y]$ be β-quasihomogeneous polynomials of degree d, none of which being of the form cX^d. Suppose that the height functions f_+, f_- of F and the height functions g_+, g_- of G can be arranged in pairs of Lipschitz equivalent functions, so that $(g_+, g_-) \circ (\lambda, \phi) = (f_+, f_-)$, where $(\lambda, \phi) = ((\lambda_1, \lambda_2), (\phi_1, \phi_2))$ is a proto-transition. We have:*

$$|\lambda_1|^{\frac{d\beta}{d-e}} \cdot \lim_{|t| \to +\infty} \left|\frac{\phi_1(t)}{t}\right| = |\lambda_2|^{\frac{d\beta}{d-e}} \cdot \lim_{|t| \to +\infty} \left|\frac{\phi_2(t)}{t}\right|. \tag{51}$$

Proof. We begin with some preliminary remarks. By hypothesis, the height functions of F and G can be arranged in pairs of Lipschitz equivalent functions so, by Proposition 4.19, the multiplicity of X as a factor of F is equal to the multiplicity of X as a factor of G. Let e be the multiplicity of X as a factor of F, and also as a factor of G, so that $F(X,Y) = X^e \cdot \widetilde{F}(X,Y)$ and $G(X,Y) = X^e \cdot \widetilde{G}(X,Y)$, where $X \nmid \widetilde{F}(X,Y)$ and $X \nmid \widetilde{G}(X,Y)$. By Proposition 4.18, we have $e \leq d$, with equality if and only if F and G are of the form cX^d. Since, by hypothesis, F and G are not of this form, we have $e < d$.

Now, we proceed to the proof of (51). Throughout the rest of the proof, we assume that $t > 0$. We begin with the case where $\lambda > 0$. In this case, we have

$$|\lambda_1|^d \cdot g_+ \circ \phi_1 = f_+ \quad \text{and} \quad |\lambda_2|^d \cdot g_- \circ \phi_2 = f_-.$$

Note that

$$\begin{aligned}|\lambda_1|^d \cdot g_+(\phi_1(t)) = f_+(t) &\Rightarrow |\lambda_1|^d \cdot G(1, \phi_1(t)) = F(1,t)\\ &\Rightarrow |\lambda_1|^d \cdot \widetilde{G}(1, \phi_1(t)) = \widetilde{F}(1,t)\\ &\Rightarrow |\lambda_1|^d \cdot \widetilde{G}\left(t^{-\frac{1}{\beta}}, \frac{\phi_1(t)}{t}\right) = \widetilde{F}\left(t^{-\frac{1}{\beta}}, 1\right).\end{aligned}$$

In the last implication, we used the fact that $\widetilde{F}$ and $\widetilde{G}$ are β-quasihomogeneous of the same degree: by the second part of Proposition 4.19, both $\widetilde{F}$ and $\widetilde{G}$ are β-quasihomogeneous of degree $d-e$. Letting $t \to +\infty$, we obtain:

$$|\lambda_1|^d \cdot \widetilde{G}\left(0, \lim_{|t|\to+\infty} \frac{\phi_1(t)}{t}\right) = \widetilde{F}(0,1)$$

Since, $\widetilde{G}$ is β-quasihomogeneous of degree $d-e$, it follows that

$$\widetilde{G}\left(0, |\lambda_1|^{\frac{d\beta}{d-e}} \cdot \lim_{|t|\to+\infty} \frac{\phi_1(t)}{t}\right) = \widetilde{F}(0,1). \tag{52}$$

Similarly,

$$\begin{aligned}|\lambda_2|^d \cdot g_-(\phi_2(t)) = f_-(t) &\Rightarrow |\lambda_2|^d \cdot G(-1, \phi_2(t)) = F(-1,t)\\ &\Rightarrow |\lambda_2|^d \cdot \widetilde{G}(-1, \phi_2(t)) = \widetilde{F}(-1,t)\\ &\Rightarrow |\lambda_2|^d \cdot \widetilde{G}\left(-t^{-\frac{1}{\beta}}, \frac{\phi_2(t)}{t}\right) = \widetilde{F}\left(-t^{-\frac{1}{\beta}}, 1\right).\end{aligned}$$

Letting $t \to +\infty$, we obtain:

$$|\lambda_2|^d \cdot \widetilde{G}\left(0, \lim_{|t|\to+\infty} \frac{\phi_2(t)}{t}\right) = \widetilde{F}(0,1)$$

Since $\widetilde{G}$ is β-quasihomogeneous of degree $d-e$, it follows that

$$\widetilde{G}\left(0, |\lambda_2|^{\frac{d\beta}{d-e}} \cdot \lim_{|t|\to+\infty} \frac{\phi_2(t)}{t}\right) = \widetilde{F}(0,1). \tag{53}$$

From (52) and (53), we obtain:

$$\widetilde{G}\left(0, |\lambda_1|^{\frac{d\beta}{d-e}} \cdot \lim_{|t|\to+\infty} \frac{\phi_1(t)}{t}\right) = \widetilde{G}\left(0, |\lambda_2|^{\frac{d\beta}{d-e}} \cdot \lim_{|t|\to+\infty} \frac{\phi_2(t)}{t}\right).$$

Since[16] $\widetilde{G}(X,Y) = \sum_{k=0}^{n} b_k X^{r(n-k)} Y^{sk}$, with $b_n \neq 0$, it follows that

$$b_n \cdot \left(|\lambda_1|^{\frac{d\beta}{d-e}} \cdot \lim_{|t|\to+\infty} \frac{\phi_1(t)}{t} \right)^{sn} = b_n \cdot \left(|\lambda_2|^{\frac{d\beta}{d-e}} \cdot \lim_{|t|\to+\infty} \frac{\phi_2(t)}{t} \right)^{sn}.$$

Hence,

$$|\lambda_1|^{\frac{d\beta}{d-e}} \cdot \lim_{|t|\to+\infty} \left| \frac{\phi_1(t)}{t} \right| = |\lambda_2|^{\frac{d\beta}{d-e}} \cdot \lim_{|t|\to+\infty} \left| \frac{\phi_2(t)}{t} \right|.$$

Now, we consider the case where $\lambda < 0$. In this case, we have

$$|\lambda_1|^d \cdot g_- \circ \phi_1 = f_+ \quad \text{and} \quad |\lambda_2|^d \cdot g_+ \circ \phi_2 = f_-.$$

Note that

$$\begin{aligned} |\lambda_1|^d \cdot g_-(\phi_1(t)) = f_+(t) &\Rightarrow |\lambda_1|^d \cdot G(-1, \phi_1(t)) = F(1,t) \\ &\Rightarrow |\lambda_1|^d \cdot (-1)^e \cdot \widetilde{G}(-1, \phi_1(t)) = \widetilde{F}(1,t) \\ &\Rightarrow |\lambda_+|^d \cdot (-1)^e \cdot \widetilde{G}\left(-t^{-\frac{1}{\beta}}, \frac{\phi_1(t)}{t}\right) = \widetilde{F}\left(t^{-\frac{1}{\beta}}, 1\right). \end{aligned}$$

Letting $t \to +\infty$, we obtain:

$$|\lambda_1|^d \cdot (-1)^e \cdot \widetilde{G}\left(0, \lim_{|t|\to+\infty} \frac{\phi_1(t)}{t}\right) = \widetilde{F}(0,1)$$

Since $\widetilde{G}$ is β-quasihomogeneous of degree $d-e$, it follows that

$$\widetilde{G}\left(0, |\lambda_1|^{\frac{d\beta}{d-e}} \cdot \lim_{|t|\to+\infty} \frac{\phi_1(t)}{t}\right) = (-1)^e \cdot \widetilde{F}(0,1). \tag{54}$$

Similarly,

$$\begin{aligned} |\lambda_2|^d \cdot g_+(\phi_2(t)) = f_-(t) &\Rightarrow |\lambda_2|^d \cdot G(1, \phi_2(t)) = F(-1,t) \\ &\Rightarrow |\lambda_2|^d \cdot \widetilde{G}(1, \phi_2(t)) = (-1)^e \cdot \widetilde{F}(-1,t) \\ &\Rightarrow |\lambda_2|^d \cdot \widetilde{G}\left(t^{-\frac{1}{\beta}}, \frac{\phi_2(t)}{t}\right) = (-1)^e \cdot \widetilde{F}\left(-t^{-\frac{1}{\beta}}, 1\right). \end{aligned}$$

Letting $t \to +\infty$, we obtain:

$$|\lambda_2|^d \cdot \widetilde{G}\left(0, \lim_{|t|\to+\infty} \frac{\phi_2(t)}{t}\right) = (-1)^e \cdot \widetilde{F}(0,1)$$

[16] As we have seen in the proof of Proposition 4.19, $d - e = rn$.

Since $\widetilde{G}$ is β-quasihomogeneous of degree $d-e$, it follows that

$$\widetilde{G}\left(0, |\lambda_2|^{\frac{d\beta}{d-e}} \cdot \lim_{|t|\to+\infty} \frac{\phi_2(t)}{t}\right) = (-1)^e \cdot \widetilde{F}(0,1). \tag{55}$$

From (54) and (55), we obtain:

$$\widetilde{G}\left(0, |\lambda_1|^{\frac{d\beta}{d-e}} \cdot \lim_{|t|\to+\infty} \frac{\phi_1(t)}{t}\right) = \widetilde{G}\left(0, |\lambda_2|^{\frac{d\beta}{d-e}} \cdot \lim_{|t|\to+\infty} \frac{\phi_2(t)}{t}\right).$$

Then, it follows that

$$|\lambda_1|^{\frac{d\beta}{d-e}} \cdot \lim_{|t|\to+\infty} \left|\frac{\phi_1(t)}{t}\right| = |\lambda_2|^{\frac{d\beta}{d-e}} \cdot \lim_{|t|\to+\infty} \left|\frac{\phi_2(t)}{t}\right|.$$

■

Proposition 4.21. *Let $F, G \in \mathbb{R}[X,Y]$ be β-quasihomogeneous polynomials of degree d, none of which being of the form cX^d, and let f_+, f_- be the height functions of F and g_+, g_- the height functions of G. Suppose that $(g_+, g_-) \circ (\lambda, \phi) = (f_+, f_-)$ for some proto-transition (λ, ϕ). Then (λ, ϕ) is a β-transition if and only if the following conditions hold:*

i. ϕ_1 and ϕ_2 are coherent[17]

ii. None of the polynomials F, G has X as a factor, or $\lambda_1 = \lambda_2$.

Proof. First, suppose that (λ, ϕ) is a β-transition. Then, we have

$$|\lambda_1|^{\beta} \cdot \lim_{|t|\to+\infty} \frac{\phi_1(t)}{t} = |\lambda_2|^{\beta} \cdot \lim_{|t|\to+\infty} \frac{\phi_2(t)}{t}. \tag{56}$$

Since $|\lambda_1| > 0$ and $|\lambda_2| > 0$, it follows that $\lim_{|t|\to+\infty} \phi_1(t)/t$ and $\lim_{|t|\to+\infty} \phi_2(t)/t$ have the same sign. And since ϕ_1 and ϕ_2 are monotone, this implies that they are coherent. Hence, condition (i) is satisfied.

Now, since $(g_+, g_-) \circ (\lambda, \phi) = (f_+, f_-)$, where (λ, ϕ) is a proto-transition, the multiplicity of X as a factor of F is equal to the multiplicity of X as a factor of G (see the first part of Proposition 4.19). Let us denote by e the multiplicity of X both as a factor of F and as a factor of G. By Proposition 4.18, we have $e \leq d$, with equality if and only if F and G are of the form cX^d. Since, by hypothesis, F and G are not of this form, we have $e < d$. So, by Lemma 4.7,

$$|\lambda_1|^{\frac{d\beta}{d-e}} \lim_{|t|\to+\infty} \left|\frac{\phi_1(t)}{t}\right| = |\lambda_2|^{\frac{d\beta}{d-e}} \lim_{|t|\to+\infty} \left|\frac{\phi_2(t)}{t}\right|. \tag{57}$$

[17] For the sake of convenience, we say that two monotone functions $\phi_1, \phi_2 \colon \mathbb{R} \to \mathbb{R}$ are *coherent* if they are either both increasing or both decreasing.

Since the limits $\lim_{|t|\to+\infty}\phi_1(t)/t$ and $\lim_{|t|\to+\infty}\phi_2(t)/t$ have the same sign, it follows that

$$|\lambda_1|^{\frac{d\beta}{d-e}}\lim_{|t|\to+\infty}\frac{\phi_1(t)}{t}=|\lambda_2|^{\frac{d\beta}{d-e}}\lim_{|t|\to+\infty}\frac{\phi_2(t)}{t}. \tag{58}$$

Since the limits $\lim_{|t|\to+\infty}\phi_1(t)/t$ and $\lim_{|t|\to+\infty}\phi_2(t)/t$ are both nonzero (because ϕ_1 and ϕ_2 are bi-Lipschitz), it follows from equations (56) and (58) that

$$\frac{|\lambda_1|^{\frac{d\beta}{d-e}}}{|\lambda_1|^{\beta}}=\frac{|\lambda_2|^{\frac{d\beta}{d-e}}}{|\lambda_2|^{\beta}}. \tag{59}$$

Equivalently,

$$|\lambda_1|^{\frac{e\beta}{d-e}}=|\lambda_2|^{\frac{e\beta}{d-e}}. \tag{60}$$

Furthermore, this equality holds if and only if $e=0$ or $|\lambda_1|=|\lambda_2|$. And since λ_1 and λ_2 have the same sign, this is equivalent to condition (ii). Since (60) actually holds, condition (ii) is satisfied.

Now, in order to prove the converse, suppose that conditions (i) and (ii) hold. Since $(g_+,g_-)\circ(\lambda,\phi)=(f_+,f_-)$, where (λ,ϕ) is a proto-transition, (57) still holds for this part of the argument. Thus, condition (i) implies (58). On the other hand, as we have just proved, condition (ii) is equivalent to (59). Since we are assuming that condition (ii) is satisfied, (59) holds. From (58) and (59), we obtain (56). Therefore, (λ,ϕ) is a β-transition. ■

Corollary 4.10. *Let $F,G\in\mathbb{R}[X,Y]$ be β-quasihomogeneous polynomials of degree d, none of which being of the form cX^d, and let f_+,f_- be the height functions of F and g_+,g_- the height functions of G. Also, let $\beta=r/s$, where $r>s>0$ and $\gcd(r,s)=1$. Suppose that $(g_+,g_-)\circ(\lambda,\phi)=(f_+,f_-)$ for some proto-transition (λ,ϕ). Then, we have:*

(a) *If r is even or s is odd, then there exists a β-transition $(\tilde\lambda,\tilde\phi)$ such that $(g_+,g_-)\circ(\tilde\lambda,\tilde\phi)=(f_+,f_-)$.*

(b) *If s is even, then there exists $\tilde\phi=(\tilde\phi_1,\tilde\phi_2)$, with $\tilde\phi_1$ and $\tilde\phi_2$ coherent, such that $(g_+,g_-)\circ(\lambda,\tilde\phi)=(f_+,f_-)$.*

Proof. Let $F(X,Y)=\sum_{k=0}^{n}a_kX^{d-rk}Y^{sk}$ and $G(X,Y)=\sum_{k=0}^{n}b_kX^{d-rk}Y^{sk}$, with $a_n,b_n\neq 0$, $n\geq 1$. (In the proof of Proposition 4.19, we showed that the upper limit of summation n is the same for F and G, provided that the height functions of F and G can be arranged in pairs of Lipschitz equivalent functions — which is the case, since $(g_+,g_-)\circ(\lambda,\phi)=(f_+,f_-)$ for some proto-transition (λ,ϕ). Also, we have $n\geq 1$ because none of the polynomials F,G is of the form cX^d.) Let us proceed to the proof of parts (a) and (b).

(a) **Case 1.** r is even

In this case, we have $f_-(t) = (-1)^d \cdot f_+(t)$ and $g_-(t) = (-1)^d \cdot g_+(t)$. In fact,

$$f_-(t) = \sum_{k=0}^{n} a_k \cdot (-1)^{d-rk} \cdot t^{sk} = (-1)^d \cdot \sum_{k=0}^{n} a_k t^{sk} = (-1)^d \cdot f_+(t).$$

The same reasoning, with f replaced by g, gives the other equation.
By hypothesis, there exists a proto-transition (λ, ϕ) such that $(g_+, g_-) \circ (\lambda, \phi) = (f_+, f_-)$. We claim that $(g_+, g_-) \circ (\tilde{\lambda}, \tilde{\phi}) = (f_+, f_-)$, where $\tilde{\lambda} = (\lambda_1, \lambda_1)$ and $\tilde{\phi} = (\phi_1, \phi_1)$. In fact, if $\lambda > 0$ then $|\lambda_1|^d \cdot g_+ \circ \phi_1 = f_+$ and hence

$$|\lambda_1|^d \cdot g_-(\phi_1(t)) = (-1)^d \cdot |\lambda_1|^d \cdot g_+(\phi_1(t)) = (-1)^d \cdot f_+(t) = f_-(t),$$

so we also have $|\lambda_1|^d \cdot g_- \circ \phi_1 = f_-$. If $\lambda < 0$ then $|\lambda_1|^d \cdot g_- \circ \phi_1 = f_+$ and hence

$$|\lambda_1|^d \cdot g_+(\phi_1(t)) = (-1)^d \cdot |\lambda_1|^d \cdot g_-(\phi_1(t)) = (-1)^d \cdot f_+(t) = f_-(t),$$

so we also have $|\lambda_1|^d \cdot g_+ \circ \phi_1 = f_-$. By Proposition 4.21, $(\tilde{\lambda}, \tilde{\phi})$ is a β-transform.

Case 2. r and s are both odd
In this case, we have $f_-(t) = (-1)^d \cdot f_+(-t)$ and $g_-(t) = (-1)^d \cdot g_+(-t)$. In fact,

$$f_-(t) = \sum_{k=0}^{n} (-1)^{d-rk} \cdot a_k \cdot t^{sk} = (-1)^d \cdot \sum_{k=0}^{n} (-1)^k a_k t^{sk} = (-1)^d \cdot f_+(-t).$$

The same reasoning, with f replaced by g, gives the other equation.
By hypothesis, there exists a proto-transition (λ, ϕ) such that $(g_+, g_-) \circ (\lambda, \phi) = (f_+, f_-)$. We claim that $(g_+, g_-) \circ (\tilde{\lambda}, \tilde{\phi}) = (f_+, f_-)$, where $\tilde{\lambda} = (\lambda_1, \lambda_1)$ and $\tilde{\phi}(t) = (\phi_1(t), -\phi_1(-t))$. In fact, if $\lambda > 0$ then $|\lambda_1|^d \cdot g_+ \circ \phi_1 = f_+$ and hence

$$|\lambda_1|^d \cdot g_-(-\phi_1(-t)) = (-1)^d \cdot |\lambda_1|^d \cdot g_+(\phi_1(-t)) = (-1)^d \cdot f_+(-t) = f_-(t),$$

so we also have $|\lambda_1|^d \cdot g_-(-\phi_1(-t)) = f_-(t)$. If $\lambda < 0$ then $|\lambda_1|^d \cdot g_- \circ \phi_1 = f_+$ and hence

$$|\lambda_1|^d \cdot g_+(-\phi_1(-t)) = (-1)^d \cdot |\lambda_1|^d \cdot g_-(\phi_1(-t)) = (-1)^d \cdot f_+(-t) = f_-(t),$$

so we also have $|\lambda_1|^d \cdot g_+(-\phi_1(-t)) = f_-(t)$. By Proposition 4.21, $(\tilde{\lambda}, \tilde{\phi})$ is a β-transform.

(b) Suppose that s is even. Then, we have $g_+(-t) = g_+(t)$ and $g_-(-t) = g_-(t)$. In fact,

$$g_+(-t) = \sum_{k=0}^{n} b_k \cdot (-t)^{sk} = \sum_{k=0}^{n} b_k t^{sk} = g_+(t)$$

and

$$g_-(-t)=\sum_{k=0}^{n}(-1)^{d-rk}\cdot b_k\cdot(-t)^{sk}=\sum_{k=0}^{n}(-1)^{d-rk}\cdot b_k t^{sk}=g_-(t).$$

By hypothesis, there exists a proto-transition (λ,ϕ) such that $(g_+,g_-)\circ(\lambda,\phi)=(f_+,f_-)$. We claim that $(g_+,g_-)\circ(\lambda,\overline{\phi})=(f_+,f_-)$, where $\overline{\phi}=(\phi_1,-\phi_2)$. In fact, if $\lambda>0$ then $|\lambda_2|^d\cdot g_-\circ\phi_2=f_-$ and hence $|\lambda_2|^d\cdot g_-\circ(-\phi_2)=f_-$. If $\lambda<0$ then $|\lambda_2|^d\cdot g_+\circ\phi_2=f_-$ and hence $|\lambda_2|^d\cdot g_+\circ(-\phi_2)=f_-$. Finally, notice that ϕ_1 is coherent with either ϕ_2 or $-\phi_2$. If ϕ_1 and ϕ_2 are coherent, we take $\tilde{\phi}=\phi$. If ϕ_1 and $-\phi_2$ are coherent, we take $\tilde{\phi}=\overline{\phi}$.

■

Theorem 4.2. *Let $F,G\in\mathbb{R}[X,Y]$ be β-quasihomogeneous polynomials of degree d, none of which being of the form cX^d, and let f_+,f_- be the height functions of F and g_+,g_- the height functions of G. Also, let $\beta=r/s$, where $r>s>0$ and $\gcd(r,s)=1$. Suppose that $(g_+,g_-)\circ(\lambda,\phi)=(f_+,f_-)$ for some proto-transition $(\lambda,\phi)=((\lambda_1,\lambda_2),(\phi_1,\phi_2))$. If any of the following conditions is satisfied then F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent:*

(a) *r is even or s is odd.*

(b) $\lambda_1=\lambda_2$

(c) *None of the polynomials F,G has X as a factor.*

Proof. If r is even or s is odd then, by Corollary 4.10, there exists a β-transition $(\tilde{\lambda},\tilde{\phi})$ such that $(g_+,g_-)\circ(\tilde{\lambda},\tilde{\phi})=(f_+,f_-)$. Hence, by Corollary 4.8, F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent.

Now, assume that either (b) or (c) holds. If s is odd, then condition (a) is satisfied, and therefore F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent, as we have just proved. If s is even then, by Corollary 4.10, there exists $\tilde{\phi}=(\tilde{\phi}_1,\tilde{\phi}_2)$, with $\tilde{\phi}_1$ and $\tilde{\phi}_2$ coherent, such that $(g_+,g_-)\circ(\lambda,\tilde{\phi})=(f_+,f_-)$. Since we are assuming that either (b) or (c) holds, Proposition 4.21 guarantees that $(\lambda,\tilde{\phi})$ is a β-transition. Then, by Corollary 4.8, it follows that F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent. ■

Corollary 4.11. *Let $F,G\in\mathbb{R}[X,Y]$ be β-quasihomogeneous polynomials of degree d, none of which being of the form cX^d. Suppose that the height functions f_+,f_- of F and the height functions g_+,g_- of G can be arranged in pairs of Lipschitz equivalent functions. If one of the height functions f_+,f_-,g_+,g_- has no critical points, then F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent.*

Proof. By Corollary 4.4, since the height functions of F and G can be arranged in pairs of Lipschitz equivalent functions, there exists a proto-transition $(\lambda,\phi)=((\lambda_1,\lambda_2),(\phi_1,\phi_2))$

such that $(g_+, g_-) \circ (\lambda, \phi) = (f_+, f_-)$. We prove that there exists a proto-transition $(\tilde{\lambda}, \tilde{\phi}) = ((\tilde{\lambda}_1, \tilde{\lambda}_2), (\tilde{\phi}_1, \tilde{\phi}_2))$, with $\tilde{\lambda}_1 = \tilde{\lambda}_2$, such that $(g_+, g_-) \circ (\tilde{\lambda}, \tilde{\phi}) = (f_+, f_-)$. From this, by Theorem 4.2, it follows that F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent.

Let us consider the case where $\lambda > 0$. In this case, we have

$$|\lambda_1|^d \, g_+ \circ \phi_1 = f_+ \quad \text{and} \quad |\lambda_2|^d \, g_- \circ \phi_2 = f_- \, .$$

Since $f_+ \cong g_+$ and $f_- \cong g_-$, we see that f_+ and g_+ have the same number of critical points, and also that f_- and g_- have the same number of critical points. Thus, we only need to consider the following possibilities: (A) Both f_+ and g_+ have no critical points. (B) Both f_- and g_- have no critical points.

Suppose that both f_+ and g_+ have no critical points. The proof of Theorem 3.1a shows that if $f, g\colon \mathbb{R} \to \mathbb{R}$ are Lipschitz equivalent polynomial functions of degree $d \geq 1$ with no critical points, then there exists a bi-Lipschitz homeomorphism $\phi\colon \mathbb{R} \to \mathbb{R}$ such that $g \circ \phi = f$. Note that, by Lemma 3.3, ϕ is actually a bi-Lipschitz Nash diffeomorphism. Applying this to f_+ and $|\lambda_2|^d \, g_+$, we obtain a bi-Lipschitz Nash diffeomorphism $\tilde{\phi}_1\colon \mathbb{R} \to \mathbb{R}$ such that $|\lambda_2|^d \, g_+ \circ \tilde{\phi}_1 = f_+$. Thus, we have $(g_+, g_-) \circ (\tilde{\lambda}, \tilde{\phi}) = (f_+, f_-)$, where $\tilde{\lambda} = (\lambda_2, \lambda_2)$ and $\tilde{\phi} = (\tilde{\phi}_1, \phi_2)$.

Similarly, if both f_- and g_- have no critical points then we can obtain a bi-Lipschitz Nash diffeomorphism $\tilde{\phi}_2\colon \mathbb{R} \to \mathbb{R}$ such that $|\lambda_1|^d \, g_- \circ \tilde{\phi}_2 = f_-$, so that $(g_+, g_-) \circ (\tilde{\lambda}, \tilde{\phi}) = (f_+, f_-)$, where $\tilde{\lambda} = (\lambda_1, \lambda_1)$ and $\tilde{\phi} = (\phi_1, \tilde{\phi}_2)$. ■

Proposition 4.22. *Let $F, G \in \mathbb{R}[X, Y]$ be β-quasihomogeneous polynomials of degree d, none of which being of the form cX^d, and let f_+, f_- be the height functions of F and g_+, g_- the height functions of G. Also, let $\beta = r/s$, where $r > s > 0$ and $\gcd(r, s) = 1$. Suppose that $(g_+, g_-) \circ (\lambda, \phi) = (f_+, f_-)$ for some proto-transition $(\lambda, \phi) = ((\lambda_1, \lambda_2), (\phi_1, \phi_2))$. If r is odd and s is even, then:*

i. Either both of the polynomials F, G has Y as a factor, or none of them has Y as a factor.

ii. If none of the polynomials F, G has Y as a factor then $\lambda_1 = \lambda_2$.

iii. If Y is a factor of both F and G, and one of the height functions f_+, f_-, g_+, g_- has only one critical point, then there exists a proto-transition $(\tilde{\lambda}, \tilde{\phi}) = ((\tilde{\lambda}_1, \tilde{\lambda}_2), (\tilde{\phi}_1, \tilde{\phi}_2))$, with $\tilde{\lambda}_1 = \tilde{\lambda}_2$, such that $(g_+, g_-) \circ (\tilde{\lambda}, \tilde{\phi}) = (f_+, f_-)$.

Proof. Let $F(X, Y) = \sum_{k=0}^{n} a_k X^{d-rk} Y^{sk}$ and $G(X, Y) = \sum_{k=0}^{n} b_k X^{d-rk} Y^{sk}$, with $a_n, b_n \neq 0$, $n \geq 1$. (In the proof of Proposition 4.19, we showed that the upper limit of the summation n is the same for F and G, provided that the height functions of F and G can be arranged in pairs of Lipschitz equivalent functions. Also, we have $n \geq 1$ because none of the polynomials F, G is of the form cX^d.)

Suppose that r is odd and s is even. Then, we have:

$$f_+(t) = \sum_{k=0}^{n} a_k t^{sk}\,, \quad f_-(t) = (-1)^d \sum_{k=0}^{n} (-1)^k a_k t^{sk} \tag{61}$$

$$g_+(t) = \sum_{k=0}^{n} b_k t^{sk}\,, \quad g_-(t) = (-1)^d \sum_{k=0}^{n} (-1)^k b_k t^{sk} \tag{62}$$

We show that

$$|\lambda_1|^d\, b_0 = a_0 \quad \text{and} \quad |\lambda_2|^d\, b_0 = a_0. \tag{63}$$

First, note that since s is even, equations (61) and (62) show that f_+, f_-, g_+, g_- are all even functions. Thus, for each of these height functions, 0 is a critical point and the number of negative critical points is equal to the number of positive critical points (more precisely, the map $t \mapsto -t$ establishes a 1-1 correspondence between the positive critical points and the negative critical points). Also, note that each of the functions f_+, f_-, g_+, g_- has only a finite number of critical points, since they are all nonconstant polynomial functions (each of them has degree $2n$ because $a_n, b_n \neq 0$, and $n \geq 1$).

Now, suppose that $\lambda > 0$. Then,

$$|\lambda_1|^d\, g_+ \circ \phi_1 = f_+ \quad \text{and} \quad |\lambda_2|^d\, g_- \circ \phi_2 = f_-\,.$$

Let $-t_p < \cdots < -t_1 < 0 < t_1 < \cdots < t_p$ be the critical points of f_+, and let $-s_p < \cdots < -s_1 < 0 < s_1 < \cdots < s_p$ be the critical points of g_+ (f_+ and g_+ have the same number of critical points because they are Lipschitz equivalent). Since ϕ_1 is monotone, injective, and takes critical points of f_+ to critical points of g_+, it follows that $\phi_1(0) = 0$. Consequently, since $|\lambda_1|^d\, g_+ \circ \phi_1 = f_+$, we have $|\lambda_1|^d\, g_+(0) = f_+(0)$. Equivalently, since $f_+(0) = a_0$ and $g_+(0) = b_0$, we have $|\lambda_1|^d\, b_0 = a_0$. Similarly, we can show that $\phi_2(0) = 0$, and then we can use this along with the equation $|\lambda_2|^d\, g_- \circ \phi_2 = f_-$ to conclude that $|\lambda_2|^d\, b_0 = a_0$. This shows that (63) holds for $\lambda > 0$. The proof for $\lambda < 0$ is analogous.

Now, we proceed to the proof of the proposition itself. From (63), we see that either $a_0 = b_0 = 0$, or $a_0 \neq 0$ and $b_0 \neq 0$. Clearly, Y is a factor of F if and only if $a_0 = 0$, and Y is a factor of G if and only if $b_0 = 0$. Hence, either both F and G have Y as a factor or none of them has Y as a factor. This proves the first part of the proposition.

For the second part, suppose that none of the polynomials F, G has Y as a factor, so that $a_0 \neq 0$ and $b_0 \neq 0$. Then, from (63), it follows that $|\lambda_1| = |\lambda_2|$. Since λ_1 and λ_2 have the same sign, we actually have $\lambda_1 = \lambda_2$.

Now, we prove the third part. Suppose that Y is a factor of both F and G, and that one of the height functions f_+, f_-, g_+, g_- has only one critical point. Let us consider the case where $\lambda > 0$. In this case, we have:

$$|\lambda_1|^d g_+ \circ \phi_1 = f_+ \quad \text{and} \quad |\lambda_2|^d g_- \circ \phi_2 = f_- \,.$$

Since $f_+ \cong g_+$ and $f_- \cong g_-$, we see that f_+ and g_+ have the same number of critical points, and also that f_- and g_- have the same number of critical points. Thus, we only need to consider the following possibilities: (A) Both f_+ and g_+ have only one critical point. (B) Both f_- and g_- have only one critical point.

Suppose that both f_+ and g_+ have only one critical point. We have already seen that 0 is a critical point of both f_+ and g_+. Hence, 0 is the only critical point of f_+, and also the only critical point of g_+. Recall that $f_+(0) = a_0$ and $g_+(0) = b_0$. Since, by hypothesis, Y is a factor of both F and G, we have $a_0 = b_0 = 0$. Therefore, $f_+(0) = g_+(0) = 0$. From the proof of Theorem 3.1b, it is clear that if $f, g \colon \mathbb{R} \to \mathbb{R}$ are Lipschitz equivalent polynomial functions of degree ≥ 1 such that f has only one critical point t_0 and g has only one critical point s_0, then given a constant $c > 0$ such that $g_+(s_0) = cf_+(t_0)$, there exists a bi-Lipschitz homeomorphism $\phi \colon \mathbb{R} \to \mathbb{R}$ such that $g \circ \phi = cf$. In particular, if $g(s_0) = f(t_0) = 0$, then for any constant $c > 0$, there exists a bi-Lipschitz homeomorphism $\phi \colon \mathbb{R} \to \mathbb{R}$ such that $g \circ \phi = cf$. Note that, by Lemma 3.3, ϕ is actually a bi-Lipschitz Nash diffeomorphism. Applying this to f_+ and g_+, we obtain a bi-Lipschitz Nash diffeomorphism $\tilde{\phi}_1 \colon \mathbb{R} \to \mathbb{R}$ such that $|\lambda_2|^d g_+ \circ \tilde{\phi}_1 = f_+$. Thus, we have $(g_+, g_-) \circ (\tilde{\lambda}, \tilde{\phi}) = (f_+, f_-)$, where $\tilde{\lambda} = (\lambda_2, \lambda_2)$ and $\tilde{\phi} = (\tilde{\phi}_1, \phi_2)$.

Similarly, if both f_- and g_- have only one critical point then we can obtain a bi-Lipschitz Nash diffeomorphism $\tilde{\phi}_2 \colon \mathbb{R} \to \mathbb{R}$ such that $|\lambda_1|^d g_- \circ \tilde{\phi}_2 = f_-$, whence $(g_+, g_-) \circ (\tilde{\lambda}, \tilde{\phi}) = (f_+, f_-)$, where $\tilde{\lambda} = (\lambda_1, \lambda_1)$ and $\tilde{\phi} = (\phi_1, \tilde{\phi}_2)$. This proves the third part of the proposition for $\lambda > 0$. The proof for $\lambda < 0$ is analogous. ∎

Corollary 4.12. *Let $F, G \in \mathbb{R}[X, Y]$ be β-quasihomogeneous polynomials of degree d, none of which being of the form cX^d. Suppose that the height functions f_+, f_- of F and the height functions g_+, g_- of G can be arranged in pairs of Lipschitz equivalent functions. If any of the following conditions is satisfied then F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent:*

(a) *None of the polynomials F, G has Y as a factor.*

(b) *One of the height functions f_+, f_-, g_+, g_- has only one critical point.*

Proof. Let $\beta = r/s$, where $r > s > 0$ and $\gcd(r, s) = 1$. By hypothesis, the height functions of F and G can be arranged in pairs of Lipschitz equivalent functions. So, by Corollary 4.4, there exists a proto-transition $(\lambda, \phi) = ((\lambda_1, \lambda_2), (\phi_1, \phi_2))$ such that $(g_+, g_-) \circ (\lambda, \phi) = (f_+, f_-)$. Suppose that none of the polynomials F, G has Y as a factor.

If r is odd and s is even, then $\lambda_1 = \lambda_2$ (by Proposition 4.22), otherwise r is even or s is odd. In any case, by Theorem 4.2, F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent. This proves part (a) of the corollary. Let us prove part (b).

Suppose that one of the height functions f_+, f_-, g_+, g_- has only one critical point. By Theorem 4.2(a), if r is even or s is odd, then F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent. So, let us assume that r is odd and s is even. By the (already proved) part (a) of this corollary, if none of the polynomials F, G has Y as a factor, then F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent. So, let us add to our assumptions that Y is a factor of both F and G. (Since we are assuming that r is odd and s is even, Proposition 4.22(i) ensures that either both F and G have Y as a factor, or none of them has Y as a factor.) Then, by Proposition 4.22(iii), there exists a proto-transition $(\tilde{\lambda}, \tilde{\phi}) = ((\tilde{\lambda}_1, \tilde{\lambda}_2), (\tilde{\phi}_1, \tilde{\phi}_2))$, with $\tilde{\lambda}_1 = \tilde{\lambda}_2$, such that $(g_+, g_-) \circ (\tilde{\lambda}, \tilde{\phi}) = (f_+, f_-)$ and, by Theorem 4.2(b), F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent. ∎

4.5 Henry and Parusiński's example revisited

In HENRY and PARUSIŃSKI (2004), the authors constructed an invariant of the bi-Lipschitz equivalence of analytic function germs $(\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ that varies continuously in many analytic families, thereby showing that the bi-Lipschitz equivalence of analytic function germs admits continuous moduli. As an example, they show that the one-parameter family of germs $f_t(x, y)\colon (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, given by

$$f_t(x, y) = x^3 - 3txy^4 + y^6, \quad t \in \mathbb{R} \tag{64}$$

satisfies the following properties:

i. Given $t, t' > 0$, if $t \neq t'$ then there exists no germ of bi-Lipschitz homeomorphism $h\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $f_t \circ h = f_{t'}$.

ii. Given $t, t' < 0$, there exists a germ of bi-Lipschitz homeomorphism $h\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $f_t \circ h = f_{t'}$.

Note that property (i) shows, in particular, that the bi-Lipschitz classification of real analytic function germs admits continuous moduli.

Now, we analyze this example in the context of $\mathcal{R}$-semialgebraic Lipschitz equivalence, using only results obtained in this thesis. But before we do that, we need to make a small adjustment. For each $t \in \mathbb{R}$, we have

$$f_t(\lambda x, \lambda^{1/2} y) = \lambda^3 f_t(x, y), \text{ for all } \lambda > 0,$$

so that f_t is β-quasihomogeneous of degree 3, with $\beta = 1/2$. This is a problem, if we want to apply our results, because we have developed the whole theory of $\mathcal{R}$-semialgebraic

Lipschitz equivalence for β-quasihomogeneous polynomials in two variables under the assumption that $\beta > 1$. Fortunately, we can fix this simply by interchanging the variables x and y, so we consider the family

$$F_\lambda(X,Y) = X^6 - 3\lambda X^4 Y + Y^3, \quad \lambda \in \mathbb{R} \tag{65}$$

which is a family of β-quasihomogeneous polynomials of degree 6, with $\beta = 2$. We show that this family satisfies the following properties:[18]

i'. Given $\lambda, \mu > 0$, if $\lambda \neq \mu$ then there exists no germ of semialgebraic bi-Lipschitz homeomorphism $\Phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $F_\mu \circ \Phi = F_\lambda$.

ii'. Given $\lambda, \mu < 0$, there exists a germ of semialgebraic bi-Lipschitz homeomorphism $\Phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $F_\mu \circ \Phi = F_\lambda$.

First, note that the height functions of F_λ are given by

$$(f_\lambda)_+(t) = 1 - 3\lambda t + t^3 \quad \text{and} \quad (f_\lambda)_-(t) = 1 - 3\lambda t + t^3\,.$$

So, we drop the subscript sign and write simply

$$f_\lambda(t) = t^3 - 3\lambda t + 1\,.$$

Now, to prove (i'), fix any two real numbers $\lambda, \mu > 0$; we show that if F_λ and F_μ are $\mathcal{R}$-semialgebraically Lipschitz equivalent then $\lambda = \mu$. We proceed in two steps. First, we show that if F_λ and F_μ are $\mathcal{R}$-semialgebraically Lipschitz equivalent then $f_\lambda \cong f_\mu$. Second, we show that if $f_\lambda \cong f_\mu$ then $\lambda = \mu$.

Since f_λ has at least one real zero t_0, we have

$$F_\lambda(1, t_0) = f_\lambda(t_0) = 0 \quad \text{and} \quad F_\lambda(-1, t_0) = f_\lambda(t_0) = 0\,.$$

Thus,

$$F_\lambda^{-1}(0) \cap \{x > 0\} \neq \varnothing \quad \text{and} \quad F_\lambda^{-1}(0) \cap \{x < 0\} \neq \varnothing\,.$$

Hence, by Theorem 4.1, if F_λ and F_μ are $\mathcal{R}$-semialgebraically Lipschitz equivalent then $f_\lambda \cong f_\mu$.

For the second step, note that f_λ has exactly two distinct critical points $t_1 = -\lambda^{1/2}$ and $t_2 = \lambda^{1/2}$ and its multiplicity symbol is $((1 + 2\lambda^{3/2}, 1 - 2\lambda^{3/2}), (2,2))$. Also, f_μ has exactly two distinct critical points $s_1 = -\mu^{1/2}$ and $s_2 = \mu^{1/2}$ and its multiplicity symbol is $((1 + 2\mu^{3/2}, 1 - 2\mu^{3/2}), (2,2))$. Now, suppose that $f_\lambda \cong f_\mu$. By Theorem 3.1c, the multiplicity symbols of f_λ and f_μ are similar. Since $1 + 2\lambda^{3/2} > 1 - 2\lambda^{3/2}$ and

[18] Since the map $\mathbb{R}^2 \to \mathbb{R}^2$ given by $(X,Y) \mapsto (Y,X)$ is a semialgebraic bi-Lipschitz homeomorphism, these properties are also satisfied by the family (64).

$1 + 2\mu^{3/2} > 1 - 2\mu^{3/2}$, it follows that, actually, the multiplicity symbols of f_λ and f_μ are directly similar, and hence

$$\begin{vmatrix} 1 + 2\lambda^{3/2} & 1 - 2\lambda^{3/2} \\ 1 + 2\mu^{3/2} & 1 - 2\mu^{3/2} \end{vmatrix} = 0\,.$$

Since this equality holds if and only if $\lambda = \mu$, we conclude that if $f_\lambda \cong f_\mu$ then $\lambda = \mu$.

To prove (ii'), fix any two real numbers $\lambda, \mu < 0$. In this case, both f_λ and f_μ have no critical points. Hence, by Corollary 4.11, F_λ and F_μ are $\mathcal{R}$-semialgebraically Lipschitz equivalent.

Finally, we note that property (i') shows, in particular, that the $\mathcal{R}$-semialgebraic Lipschitz equivalence of real β-quasihomogeneous polynomials in two variables admits continuous moduli.

5 CONCLUSION

In this thesis, we generalize the methods from BIRBRAIR, FERNANDES, and PANAZZOLO (2009) to address the following two main problems:

1. **Lipschitz equivalence problem for nonconstant real polynomial functions of a single variable.** Given two nonconstant polynomial functions $f, g\colon \mathbb{R} \to \mathbb{R}$, determine whether f and g are Lipschitz equivalent.
2. **$\mathcal{R}$-semialgebraic Lipschitz equivalence problem for nonzero real β-quasihomogeneous polynomials in two variables.** Given two nonzero real β-quasihomogeneous polynomials $F(X,Y)$ and $G(X,Y)$, determine whether F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent.

The Lipschitz equivalence problem for nonconstant real polynomial functions of a single variable is completely solved by the main results of Chapter 3, namely, Theorem 3.1a, Theorem 3.1b, and Theorem 3.1c. These results allow us to determine whether any two given polynomial functions $f, g\colon \mathbb{R} \to \mathbb{R}$, of the same degree $d \geq 1$, having the same number of critical points, are Lipschitz equivalent.[19] Theorem 3.1a deals with the case where f and g have no critical points, Theorem 3.1b deals with the case where f and g have only one critical point, and Theorem 3.1c deals with the case where f and g have at least two critical points.

Then, in Chapter 4, we try to reduce the second problem to the first one by addressing the following questions:

2.1. Suppose that two given β-quasihomogeneous polynomials $F, G \in \mathbb{R}[X,Y]$ of degree $d \geq 1$ are $\mathcal{R}$-semialgebraically Lipschitz equivalent. Is it possible to arrange their height functions in pairs of Lipschitz equivalent functions (i.e. either $f_+ \cong g_+$ and $f_- \cong g_-$, or $f_+ \cong g_-$ and $f_- \cong g_+$)?

2.2. Suppose that the height functions of two given β-quasihomogeneous polynomials $F, G \in \mathbb{R}[X,Y]$ of degree $d \geq 1$ can be arranged in pairs of Lipschitz equivalent functions. Are F and G $\mathcal{R}$-semialgebraically Lipschitz equivalent?

This reduction is accomplished for wide classes of β-quasihomogeneous polynomials. We show that if the zero sets of the polynomials F and G have points both on the right half-plane and on the left half-plane then the answer to the first question is yes (see Corollary 4.3 and Remark 4.7). Also, we obtain some fairly general conditions under which the answer to the second question is affirmative (see Theorem 4.2, Corollary 4.11, and Corollary 4.12). These are our main results on $\mathcal{R}$-semialgebraic Lipschitz equivalence of β-quasihomogeneous polynomials. These results, along with those on Lipschitz equivalence of polynomial functions of a single variable enable us to determine, under fairly general

[19] This solves the problem because Lipschitz equivalent polynomial functions have the same degree and the same number of critical points.

conditions, whether two given β-quasihomogeneous polynomials are $\mathcal{R}$-semialgebraically Lipschitz equivalent.

Chapter 4 ends with an application of our results. In Section 4.5, we revisit an example given in HENRY and PARUSIŃSKI (2004) showing that the bi-Lipschitz classification of real analytic function germs admits continuous moduli. We analyze this example in the context of $\mathcal{R}$-semialgebraic Lipschitz equivalence, using only results obtained in this thesis. As a byproduct of this analysis, we find that the $\mathcal{R}$-semialgebraic Lipschitz equivalence of real β-quasihomogeneous polynomials in two variables admits continuous moduli.

Finally, as suggestions for future research, we list some interesting questions on the subject of this thesis that still remain unanswered.

1. Are there β-quasihomogeneous polynomials $F(X,Y)$ and $G(X,Y)$, of the same β-quasihomogeneous degree $d \geq 1$, which are $\mathcal{R}$-semialgebraically Lipschitz equivalent, whose height functions cannot be arranged in pairs of Lipschitz equivalent functions? If such an example of β-quasihomogeneous polynomials $F(X,Y)$ and $G(X,Y)$ does exist then, by Corollary 4.3, the set $F^{-1}(0)$ is contained in one of the closed half-planes $\{x \leq 0\}$ or $\{x \geq 0\}$ (and, by symmetry, the same holds for the set $G^{-1}(0)$).

2. Are there β-quasihomogeneous polynomials $F(X,Y)$ and $G(X,Y)$, of the same β-quasihomogeneous degree $d \geq 1$, which are not $\mathcal{R}$-semialgebraically Lipschitz equivalent, whose height functions can be arranged in pairs of Lipschitz equivalent functions? Let $\beta = r/s$, where $r > s > 0$ and $\gcd(r,s) = 1$. If such an example of β-quasihomogeneous polynomials $F(X,Y)$ and $G(X,Y)$ does exist then, by Theorem 4.2, Corollary 4.11, and Corollary 4.12, r is odd, s is even, both F and G are multiples of XY, and each of the height functions f_+, f_-, g_+, g_- has at least two critical points.

3. Let $F(X,Y)$ and $G(X,Y)$ be β-quasihomogeneous polynomials of degree $d \geq 1$. Suppose that F and G are $\mathcal{R}$-semialgebraically Lipschitz equivalent, so that there is a germ of semialgebraic bi-Lipschitz map $\Phi\colon (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ such that $G \circ \Phi = F$. What is the level of regularity of Φ? Under what conditions is it true that $\Phi \in C^1$?

4. Show how to find sets of normal forms for the Lipschitz equivalence of real polynomial functions of a single variable of fixed degree $d \geq 1$. In Section 3.2, we solved this problem for $d = 1, 2, 3$.

5. Show how to find sets of normal forms for the $\mathcal{R}$-semialgebraic Lipschitz equivalence of β-quasihomogeneous polynomials in two variables of fixed β-quasihomogeneous degree $d \geq 1$.

6. Show how to determine, for any two given any two polynomials $F(X,Y)$ and $G(X,Y)$ (not necessarily β-quasihomogeneous), with real coefficients, whether there

exists a germ of semialgebraic bi-Lipschitz map $\Phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $G \circ \Phi = F$.

7. Show how to determine, for any two given polynomial maps $P, Q\colon \mathbb{R}^2 \to \mathbb{R}^2$, whether there exists a germ of semialgebraic bi-Lipschitz map $\Phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $Q \circ \Phi = P$.

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