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**LIPSCHITZ GEOMETRY AND COMBINATORICS OF ABNORMAL SURFACE  
GERMS**

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EMANOEL FERREIRA DE SOUZA

LIPSCHITZ GEOMETRY AND COMBINATORICS OF ABNORMAL SURFACE GERMS

Thesis submitted to the Doctoral Program of the Department of Mathematics of Universidade Federal do Ceará in partial fulfillment of the necessary requirements for the degree of Ph.D in Mathematics. Area of expertise: Singularities.

Advisor: Prof. Dr. Lev Birbrair.

Co-advisor: Prof. Dr. Andrei Gabriellov.

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I dedicate this work to my family and friends who gave me the emotional support to finish it. Specially my beloved wife, Jemima, for her love and patience, and my dear mother, Francisca Moreira de Souza, who sadly have not lived long enough to see me becoming a Doctor.

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“As a man sow, shall he reap. And I know that talk is cheap. But the heat of the battle is as sweet as the victory.”

(MARLEY, 1977)

## RESUMO

Estudamos a geometria Lipschitz exterior de superfícies semialgébricas ou, mais geralmente, superfícies definíveis em estruturas o-minimais sobre os reais polinomialmente limitadas. Em particular, qualquer triângulo de Hölder definível é ou Lipschitz normalmente mergulhado ou contém arcos anormais. Mostramos que arcos anormais constituem finitas zonas anormais no espaço de todos os arcos e investigamos propriedades geométricas e combinatórias de germes de superfícies anormais. Estabelecemos uma forte relação entre a geometria e a combinatória dos triângulos de Hölder anormais.

**Palavras-chave:** geometria Lipschitz; singularidades de superfícies; números de Catalão.



## ABSTRACT

We study outer Lipschitz geometry of real semialgebraic or, more general, definable in a polynomially bounded o-minimal structure over the reals, surface germs. In particular, any definable Hölder triangle is either Lipschitz normally embedded or contains some abnormal arcs. We show that abnormal arcs constitute finitely many abnormal zones in the space of all arcs, and investigate geometric and combinatorial properties of abnormal surface germs. We establish a strong relation between geometry and combinatorics of abnormal Hölder triangles.

**Keywords:** Lipschitz geometry; surface singularities; Catalan numbers.

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## 1 INTRODUCTION

This thesis explores Lipschitz geometry of germs of semialgebraic (or, more general, definable in a polynomially bounded o-minimal structure) real surfaces, with the goal towards effective bi-Lipschitz classification of definable surface singularities.

Lipschitz geometry of singularities attracted considerable attention for the last 50 years, as a natural approach to classification of singularities which is intermediate between their bi-regular (too fine) and topological (too coarse) equivalence. In particular, the finiteness theorems in (MOSTOWSKI, 1985) and (PARUSINSKI, 1994) suggest the possibility of effective bi-Lipschitz classification of definable real surface germs.

In the seminal paper of Pham and Teissier (PHAM; TEISSIER, 1969) on Lipschitz geometry of germs of complex algebraic plane curves, it was shown that two such germs are meromorphically (outer metric) bi-Lipschitz equivalent exactly when they are ambient topologically equivalent. This result was later improved by Fernandes in (FERNANDES, 2003), where the bi-Lipschitz homeomorphism between the curves was allowed to be subanalytic instead of meromorphic. In 2014, Neumann and Pichon (NEUMANN; PICHON, 2014) proved that the result still holds when the subanalyticity condition on the homeomorphism is removed. In 2018, Fernandes, Sampaio e Silva (FERNANDES *et al.*, 2018) generalized the result of Neumann and Pichon proving that one can only ask the homeomorphism between the curves to be bi- $\alpha$ -Hölder bi-Lipschitz, for  $\alpha$  sufficiently close to 1, instead of outer bi-Lipschitz. Then, bi-Lipschitz equivalence class of germs of complex plane curves is completely determined by essential Puiseux pairs of their irreducible branches, and by the orders of contact between the branches.

Later it became clear (BIRBRAIR; MOSTOWSKY, 2000) that any singular germ  $X$  inherits two metrics from the ambient space: the inner metric where the distance between two points of  $X$  is the length of the shortest path connecting them inside  $X$ , and the outer metric with the distance between two points of  $X$  being just their distance in the ambient space. This defines two classification problems, equivalence up to bi-Lipschitz homeomorphisms with respect to the inner and outer metrics, the inner metric classification being more coarse than the outer metric one.

Any semialgebraic surface germ with a link homeomorphic to a line segment is bi-Lipschitz equivalent with respect to the inner metric to the standard  $\beta$ -Hölder triangle  $\{0 \leq x \leq 1, 0 \leq y \leq x^\beta\}$ . Any semialgebraic surface with an isolated singularity and connected link is

bi-Lipschitz equivalent to a  $\beta$ -horn - surface of revolution of a  $\beta$ -cusp  $\{0 \leq x \leq 1, y = x^\beta\}$  (see (BIRBRAIR, 1999) and (BIRBRAIR, 2008)). For the Lipschitz normally embedded singularities the inner and outer metrics are equivalent, thus the two classifications are the same. It was proved in (KURDYKA; ORRO, 1997) that any semialgebraic set can be decomposed into finitely many normally embedded semialgebraic sets. Birbrair and Mostowski in (BIRBRAIR; MOSTOWSKY, 2000) used Kurdyka's construction to prove that any semialgebraic set is inner Lipschitz equivalent to a normally embedded semialgebraic set.

Classification of surface germs with respect to the outer metric is much more complicated. A singular germ  $X$  can be considered as the family  $X_t$  of its links (intersections with the spheres of a small radius  $t > 0$ ). Thus Lipschitz geometry of  $X$  can be understood as the dynamics of  $X_t$  as  $t \rightarrow 0$ . For this purpose we investigate the "Valette link" of  $X$ , the family of arcs in  $X$  parameterized by the distance to the origin. The outer Lipschitz invariants of  $X$  are described in terms of the tangency orders between those arcs.

The first step towards the outer metric classification of surface germs was made in (BIRBRAIR *et al.*, 2017) for the surfaces of a very special kind, each of them being the union of the real plane and a graph of a function defined on that plane. In the classical singularity theory, this corresponds to classification of functions with respect to bi-Lipschitz K-equivalence.

The present PhD thesis is the next step towards outer metric classification of surface germs. Using Kurdyka's normally embedded decomposition and the "pizza decomposition" from (BIRBRAIR *et al.*, 2017) for the distance functions, we identify basic "abnormal" parts of a surface germ, called snakes, and investigate their geometric and combinatorial properties.

In Chapter 2 we review some standard (and some less standard) definitions and technical tools of Lipschitz geometry of surface germs. The standard metric of  $\mathbb{R}^n$  induces two metrics on  $X$ : the outer and inner metrics. The distance between two points  $x$  and  $y$  of  $X$  in the outer metric is just the distance  $|x - y|$  between them in  $\mathbb{R}^n$ , while the distance in the inner metric is the infimum of the lengths of definable paths connecting  $x$  and  $y$  inside  $X$ . A surface  $X$  is normally embedded (or simply NE) if these two metrics are equivalent. An arc  $\gamma \subset X$  is the germ of a definable mapping  $[0, \varepsilon) \rightarrow X$  such that  $|\gamma(t)| = t$ . The outer (resp., inner) tangency order of two arcs  $\gamma$  and  $\gamma'$  is the exponent of the distance between  $\gamma(t)$  and  $\gamma'(t)$  in the outer (resp., inner) metric. This equips the set of all arcs in  $X$ , known as the Valette link  $V(X)$  of  $X$ , (see (VALETTE, 2007)) with a non-archimedean metric. The simplest surface germ is a  $\beta$ -Hölder triangle, which is bi-Lipschitz equivalent with respect to the inner metric to the germ of the set

$\{0 \leq x \leq 1, 0 \leq y \leq x^\beta\} \subset \mathbb{R}^2$ . A  $\beta$ -Hölder triangle  $T$  has two boundary arcs, corresponding to  $y = 0$  and  $y = x^\beta$ . All other arcs in  $T$  are interior arcs. An arc  $\gamma \subset X$  is Lipschitz non-singular if it is topologically non-singular and there is a normally embedded Hölder triangle  $T \subset X$  such that  $\gamma$  is an interior arc of  $T$ . There are finitely many Lipschitz singular arcs in any surface  $X$ . A Hölder triangle is non-singular if all its interior arcs are Lipschitz non-singular. An arc  $\gamma \subset X$  is generic if its inner tangency order with any Lipschitz singular arc of  $X$  is equal to the minimal tangency order of any two arcs in  $X$ .

In Section 2.4, we describe Kurdyka’s “pancake decomposition” of a surface germ (see Definition 2.4.1 and Remark 2.4.1) into NE Hölder triangles (“pancakes”).

In Section 2.5 we present the “pizza decomposition” from (BIRBRAIR *et al.*, 2017) in a suitable form. Together with pancake decomposition, it is our main technical tool for the study of Lipschitz geometry of surface germs. Further important developments of pizza decomposition for Lipschitz functions defined on a normally embedded Hölder triangle are established in Chapter 3.

Abnormal surfaces, the main object of this thesis, are introduced in Section 2.6 while fundamental examples of them are presented in Chapter 4. An arc  $\gamma \subset X$  is abnormal if there are two normally embedded non-singular Hölder triangles  $T$  and  $T'$  such that  $\gamma$  is their common boundary arc and  $T \cup T'$  is not normally embedded. Otherwise  $\gamma$  is a normal arc. A surface  $X$  is abnormal if all its generic arcs are abnormal. Note that the set of abnormal arcs in  $X$  is outer Lipschitz invariant. Abnormal surfaces are important building blocks of general surface germs. In particular, we study abnormal non-singular  $\beta$ -Hölder triangles, which we call  $\beta$ -snakes (see Fig. 3). Snakes are “weakly normally embedded” (see Definition 4.1.2): if  $X$  is a  $\beta$ -snake then any non-NE Hölder triangle  $T \subset X$  has the same exponent  $\beta$  (see Proposition 4.1.1). This fundamental property of snakes allows one to clarify the outer Lipschitz geometry of a surface germ by separating exponents associated with its different snakes. Another peculiar property of snakes is non-uniqueness of their reduced pancake decompositions (see Remark 4.1.2 and Fig. 4). Furthermore, there is a canonical (outer Lipschitz invariant) decomposition of the Valette link of a  $\beta$ -snake into finitely many normally embedded  $\beta$ -zones, segments and nodal zones (see Corollary 4.2.3 and Proposition 4.6.1).

In Chapter 5 we explain the role played by snakes in Lipschitz geometry of general surface germs. Theorem 5.0.1 states that each abnormal arc of a surface germ  $X$  belongs to one of the finitely many snakes and “non-snake bubbles” (see Fig. 10) contained in  $X$ .

In Chapter 6 we introduce snake names, combinatorial invariants associated with snakes, and investigate their non-trivial combinatorics. In particular, we show that any snake name can be reduced to a binary one, and derive recurrence relations for the numbers of distinct binary snake names of different lengths.

In Section 6.3 we present a strong relationship between geometry and combinatorics of snakes. We define “weakly outer bi-Lipschitz maps” (see Definition 6.3.1) between surface germs, and give combinatorial description of weak outer Lipschitz equivalence of snakes in terms of their snake names and some extra combinatorial data.

## 2 PRELIMINARIES

All sets, functions and maps in this text are assumed to be definable in a polynomially bounded o-minimal structure over  $\mathbb{R}$  with the field of exponents  $\mathbb{F}$  (see (DRIES; SPEISSEGER, 2002)), for example, real semialgebraic or subanalytic. Unless the contrary is explicitly stated, we consider germs at the origin of all sets and maps. Nevertheless, we will be mostly considering surface (two-dimensional) germs.

### 2.1 O-minimal structure

The definitions and results in this section are from (DRIES, 1998). Another interesting reading on this subject is (COSTE, 2000).

**Definition 2.1.1.** An *o-minimal structure over the reals* (or on  $\mathbb{R}$ ) is a sequence  $\mathcal{S} = (\mathcal{S}_n)_{n \in \mathcal{N}}$  such that for each  $n$ :

1.  $\mathcal{S}_n$  is a boolean algebra of subsets of  $\mathbb{R}^n$ , i.e.,  $\mathcal{S}_n$  is a collection of subsets of  $\mathbb{R}^n$  such that  $\emptyset \in \mathcal{S}_n$  and if  $A, B \in \mathcal{S}_n$  then  $A \cup B \in \mathcal{S}_n$  and  $\mathbb{R}^n \setminus A \in \mathcal{S}_n$ ;
2. if  $A \in \mathcal{S}_n$  then  $A \times \mathbb{R} \in \mathcal{S}_{n+1}$  and  $\mathbb{R} \times A \in \mathcal{S}_{n+1}$ ;
3.  $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i = x_j\} \in \mathcal{S}_n$  for  $1 \leq i < j \leq n$ ;
4. if  $A \in \mathcal{S}_{n+1}$  then  $\pi(A) \in \mathcal{S}_n$ , where  $\pi: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$  is the usual projection map;
5.  $r \in \mathcal{S}_1$  for each  $r \in \mathbb{R}$ , and  $\{(x, y) \in \mathbb{R}^2 \mid x < y\} \in \mathcal{S}_2$ ;
6. the only sets in  $\mathcal{S}_1$  are the finite unions of intervals (including the ones with infinite as endpoints) and points.

**Remark 2.1.1.** Item (or axiom) 5 of Definition 2.1.1 establish that finite unions of intervals and points are in  $\mathcal{S}_1$ . Item 6 says no further subsets of  $\mathbb{R}$  belong to  $\mathcal{S}_1$ . It is the minimality axiom that explains the term “o-minimal”.

**Example 1.** Examples of o-minimal structures over the reals are:

1. the semilinear sets;
2. the semialgebraic sets;
3. the subsets of the affine space  $\mathbb{R}^n$  for  $n = 0, 1, 2, \dots$  that are subanalytic in the larger projective space  $\mathbb{P}^n(\mathbb{R})$ .

**Definition 2.1.2.** Let  $A \subset \mathbb{R}^m$  and  $f: A \rightarrow \mathbb{R}^n$ . We say that  $A$  is a *definable set* if  $A \in \mathcal{S}_m$ . We say that  $f$  is a *definable map* if its graph,  $\text{graph}(f) \subset \mathbb{R}^{m+n}$  is a definable set. When a map  $f$  is

definable so are its domain and its image. The closure and the interior of a definable set are also definable (see (DRIES; MILLER, 1996)).

**Definition 2.1.3.** An o-minimal structure is said to be *polynomially bounded* at the origin if for any function  $f: \mathbb{R} \rightarrow \mathbb{R}$  belonging to the structure, there exist some positive rational number  $N$ , depending on  $f$ , such that  $f(t) = o(t^N)$  as  $t \rightarrow 0$ . In particular, when  $f \neq 0$  there are  $c \neq 0$  and  $N \in \mathbb{Q}_+$  such that  $f(t) = ct^N + o(t^N)$ .

**Remark 2.1.2.** The real semialgebraic sets in  $\mathbb{R}^n$  is an example of a polynomially bounded o-minimal structure over the reals. In this case, we have  $\mathbb{F} = \mathbb{Q}$ .

## 2.2 Normally embedded sets

**Definition 2.2.1.** Given a germ at the origin of a set  $X \subset \mathbb{R}^n$  we can define two metrics on  $X$ , the *outer metric*  $d(x, y) = |x - y|$  and the *inner metric*  $d_i(x, y) = \inf\{l(\alpha)\}$ , where  $l(\alpha)$  is the length of a rectifiable path  $\alpha$  from  $x$  to  $y$  in  $X$ . Note that such a path  $\alpha$  always exist, since  $X$  is definable. A set  $X \subset \mathbb{R}^n$  is *normally embedded* if the outer and inner metrics are equivalent (i.e., there is a positive constant  $C$  such that  $d_i(x, y) \leq Cd(x, y)$  for all  $x, y \in X$ ).

**Remark 2.2.1.** Note that the germ at the origin of a (closed) definable set  $X \subset \mathbb{R}^n$  is always connected. In particular, it is path-connected.

**Remark 2.2.2.** We do not know if the inner metric is definable (indeed, this is an open question!), but one can consider an equivalent definable metric as in (KURDYKA; ORRO, 1997), for example, the *pancake metric* defined in (BIRBRAIR; MOSTOWSKY, 2000).

**Example 2.** The set  $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 = y^3\}$  has two branches  $X_\pm = \{y \geq 0, x = \pm y^{3/2}\}$ . For two points  $p_+ = (y^{3/2}, y) \in X_+$  and  $p_- = (-y^{3/2}, y) \in X_-$ , we have  $d(p_+, p_-) = 2y^{3/2}$  and  $d_i(p_+, p_-) \geq y$ , thus  $d_i(p_+, p_-)/d(p_+, p_-) \rightarrow \infty$  as  $y \searrow 0$ . In particular,  $X$  is not normally embedded.

## 2.3 Hölder triangles

**Definition 2.3.1.** An *arc* in  $\mathbb{R}^n$  is a germ at the origin of a mapping  $\gamma: [0, \varepsilon) \rightarrow \mathbb{R}^n$  such that  $\gamma(0) = 0$ . Unless otherwise specified, we suppose that arcs are parameterized by the distance to the origin, i.e.,  $\|\gamma(t)\| = t$ . We usually identify an arc  $\gamma$  with its image in  $\mathbb{R}^n$ . For a germ at the



origin of a set  $X$ , the set of all arcs  $\gamma \subset X$  is denoted by  $V(X)$  (known as the Valette link of  $X$ , (VALETTE, 2007)).

**Definition 2.3.2.** The *tangency order* of two arcs  $\gamma_1$  and  $\gamma_2$  in  $V(X)$  (notation  $tord(\gamma_1, \gamma_2)$ ) is the exponent  $q$  where  $\|\gamma_1(t) - \gamma_2(t)\| = ct^q + o(t^q)$  with  $c \neq 0$ . By definition,  $tord(\gamma, \gamma) = \infty$ . For an arc  $\gamma$  and a set of arcs  $Z \subset V(X)$ , the tangency order of  $\gamma$  and  $Z$  (notation  $tord(\gamma, Z)$ ), is the supremum of  $tord(\gamma, \lambda)$  over all arcs  $\lambda \in Z$ . The tangency order of two sets of arcs  $Z$  and  $Z'$  (notation  $tord(Z, Z')$ ) is the supremum of  $tord(\gamma, Z')$  over all arcs  $\gamma \in Z$ . Similarly, we define the tangency orders in the inner metric, denoted by  $itord(\gamma_1, \gamma_2)$ ,  $itord(\gamma, Z)$  and  $itord(Z, Z')$ .

**Remark 2.3.1.** An interesting fact about the tangency order of arcs in  $\mathbb{R}^n$  is the so called “non-archimedean property” (it first appeared in (BIRBRAIR; FERNANDES, 2000) as “Isosceles property”): given arcs  $\gamma_1, \gamma_2, \gamma_3$  in  $\mathbb{R}^n$ , we have

$$tord(\gamma_2, \gamma_3) \geq \min(tord(\gamma_1, \gamma_2), tord(\gamma_1, \gamma_3)).$$

If  $tord(\gamma_1, \gamma_2) \neq tord(\gamma_1, \gamma_3)$  then  $tord(\gamma_2, \gamma_3) = \min(tord(\gamma_1, \gamma_2), tord(\gamma_1, \gamma_3))$ .

**Definition 2.3.3.** For  $\beta \in \mathbb{F}$ ,  $\beta \geq 1$ , the *standard  $\beta$ -Hölder triangle*  $T_\beta \subset \mathbb{R}^2$  is the germ at the origin of the set

$$T_\beta = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq x^\beta\}. \quad (2.1)$$

The curves  $\{x \geq 0, y = 0\}$  and  $\{x \geq 0, y = x^\beta\}$  are the *boundary arcs* of  $T_\beta$ .

**Definition 2.3.4.** A germ at the origin of a set  $T \subset \mathbb{R}^n$  that is bi-Lipschitz equivalent with respect to the inner metric to the standard  $\beta$ -Hölder triangle  $T_\beta$  is called a  *$\beta$ -Hölder triangle* (see (BIRBRAIR, 1999)). The number  $\beta \in \mathbb{F}$  is called the *exponent* of  $T$  (notation  $\beta = \mu(T)$ ). The arcs  $\gamma_1$  and  $\gamma_2$  of  $T$  mapped to the boundary arcs of  $T_\beta$  by the homeomorphism are the *boundary arcs* of  $T$  (notation  $T = T(\gamma_1, \gamma_2)$ ). All other arcs of  $T$  are *interior arcs*. The set of interior arcs of  $T$  is denoted by  $I(T)$ .

**Remark 2.3.2.** It follows from the Arc Selection Lemma that a Hölder triangle  $T$  is normally embedded if, and only if,  $tord(\gamma, \gamma') = itord(\gamma, \gamma')$  for any two arcs  $\gamma$  and  $\gamma'$  of  $T$  (see (BIRBRAIR; MENDES, 2018, Theorem 2.2)).

**Definition 2.3.5.** Let  $X$  be a surface (a two-dimensional set). An arc  $\gamma \subset X$  is *Lipschitz non-singular* if there exists a normally embedded Hölder triangle  $T \subset X$  such that  $\gamma$  is an interior arc

of  $T$  and  $\gamma \not\subset \overline{X \setminus T}$ . Otherwise,  $\gamma$  is *Lipschitz singular*. It follows from pancake decomposition (see Definition 2.4.1 and Remark 2.4.1) that a surface  $X$  contains finitely many Lipschitz singular arcs. The union of all Lipschitz singular arcs in  $X$  is denoted by  $Lsing(X)$ .

**Example 3.** If  $X$  is a surface such that its link is a circle with a radial segment intersecting this circle then  $X$  contains normally embedded Hölder triangles with an interior Lipschitz singular arc. Indeed, any Hölder triangle in  $X$  with link contained in the circle and containing the intersecting point (of the circle and radial segment) in its interior contains an arc  $\gamma$  not satisfying the condition  $\gamma \not\subset \overline{X \setminus T}$ .

**Definition 2.3.6.** A Hölder triangle  $T$  is *non-singular* if all interior arcs of  $T$  are Lipschitz non-singular.

**Example 4.** Let  $\alpha, \beta \in \mathbb{F}$  with  $1 \leq \beta < \alpha$ . Let  $\gamma_1, \gamma_2, \lambda \subset \mathbb{R}^3$  be arcs (not parameterized by the distance to the origin) such that  $\gamma_1(t) = (t, t^\beta, 0)$ ,  $\gamma_2(t) = (t, t^\beta, t^\alpha)$  and  $\lambda(t) = (t, 0, 0)$ . Consider the Hölder triangles  $T_1 = T(\gamma_1, \lambda) = \{(x, y, z) \mid x \geq 0, 0 \leq y \leq x^\beta, z = 0\}$  and  $T_2 = T(\lambda, \gamma_2) = \{(x, y, z) \mid x \geq 0, 0 \leq y \leq x^\beta, z = x^{\alpha-\beta}y\}$ . Let  $T = T_1 \cup T_2$ . Note that  $T_1$  and  $T_2$  are normally embedded  $\beta$ -Hölder triangles but  $T$  is not normally embedded, since  $tord(\gamma_1, \gamma_2) = \alpha > \beta = itord(\gamma_1, \gamma_2)$ . Thus every interior arc  $\gamma \neq \lambda$  of  $T$  is Lipschitz non-singular. Let us show that  $\lambda$  is a Lipschitz singular arc.

Consider the arcs  $\gamma'_1(t) = (t, t^p, 0) \subset T_1$  and  $\gamma'_2(t) = (t, t^p, t^{\alpha-\beta+p}) \subset T_2$ , where  $p > \beta$ ,  $p \in \mathbb{F}$ . We have  $tord(\gamma'_1, \lambda) = tord(\lambda, \gamma'_2) = p$  and  $tord(\gamma'_1, \gamma'_2) = \alpha - \beta + p > p = itord(\gamma'_1, \gamma'_2)$ . Thus Hölder triangles  $T'_1 = T(\gamma'_1, \lambda)$  and  $T'_2 = T(\lambda, \gamma'_2)$  are normally embedded but the Hölder triangle  $T_p = T'_1 \cup T'_2$  is not. If  $T' \subset T$  is any Hölder triangle such that  $\lambda \in I(T')$  then, for large enough  $p$ , the Hölder triangle  $T_p$  is contained in  $T'$ . Therefore,  $T'$  is not normally embedded, thus  $\lambda$  is a Lipschitz singular arc of  $T$ . Note also that any point of  $\lambda$  other than the origin has a normally embedded neighborhood in  $T$ .

**Definition 2.3.7.** Let  $X$  be a surface germ with connected link. The *exponent*  $\mu(X)$  of  $X$  is defined as  $\mu(X) = \min itord(\gamma, \gamma')$ , where the minimum is taken over all arcs  $\gamma, \gamma'$  of  $X$ . A surface  $X$  with exponent  $\beta$  is called a  $\beta$ -surface. An arc  $\gamma \subset X \setminus Lsing(X)$  is *generic* if  $itord(\gamma, \gamma') = \mu(X)$  for all arcs  $\gamma' \subset Lsing(X)$ . The set of generic arcs of  $X$  is denoted by  $G(X)$ .

**Remark 2.3.3.** Probably, for many readers, the natural way of defining  $\mu(X)$  would be  $\mu(X) = \inf_{\gamma, \gamma'} itord(\gamma, \gamma')$ . However, it follows from the Arc Selection Lemma that for a fixed  $\gamma \in V(X)$ ,

we have  $\inf_{\gamma'} itord(\gamma, \gamma') = itord(\gamma, \gamma_0)$  for some  $\gamma_0 \in V(X)$ . In particular,  $\inf_{\gamma'} itord(\gamma, \gamma') = \min_{\gamma'} itord(\gamma, \gamma')$ .

**Remark 2.3.4.** If  $X = T(\gamma_1, \gamma_2)$  is a non-singular  $\beta$ -Hölder triangle then an arc  $\gamma \subset X$  is *generic* if, and only if,  $itord(\gamma_1, \gamma) = itord(\gamma, \gamma_2) = \beta$ .

**Lemma 2.3.1.** *Let  $\gamma$  be an arc of a  $\beta$ -Hölder triangle  $T = T(\gamma_1, \gamma_2)$  such that  $tord(\gamma_1, \gamma) = itord(\gamma_1, \gamma)$  and  $tord(\gamma, \gamma_2) = itord(\gamma, \gamma_2)$ . If  $tord(\gamma_1, \gamma_2) > \beta$  then*

$$itord(\gamma_1, \gamma) = itord(\gamma, \gamma_2) = \beta.$$

*Proof.* Let  $\beta_1 = tord(\gamma_1, \gamma) = itord(\gamma_1, \gamma)$  and  $\beta_2 = tord(\gamma, \gamma_2) = itord(\gamma, \gamma_2)$ . Then

$$\beta = \min(itord(\gamma_1, \gamma), itord(\gamma, \gamma_2)) = \min(\beta_1, \beta_2).$$

If  $\beta_1 \neq \beta_2$  then, by the non-archimedean property, we have

$$tord(\gamma_1, \gamma_2) = \min(tord(\gamma_1, \gamma), tord(\gamma, \gamma_2)) = \min(\beta_1, \beta_2) = \beta,$$

a contradiction. □

## 2.4 Pancake decomposition

**Definition 2.4.1.** Let  $X \subset \mathbb{R}^n$  be the germ at the origin of a closed set. A *pancake decomposition* of  $X$  is a finite collection of closed normally embedded subsets  $X_k$  of  $X$  with connected links, called *pancakes*, such that  $X = \bigcup X_k$  and

$$\dim(X_j \cap X_k) < \min(\dim(X_j), \dim(X_k)) \quad \text{for all } j, k.$$

**Remark 2.4.1.** The term ‘‘pancake’’ was introduced in (BIRBRAIR; MOSTOWSKY, 2000), but this notion first appeared (with a different name) in (KURDYKA, 1992) and (KURDYKA; ORRO, 1997), where the existence of such decomposition was established.

**Remark 2.4.2.** If  $X$  is a Hölder triangle then each pancake  $X_k$  is also a Hölder triangle.

**Definition 2.4.2.** A pancake decomposition  $\{X_k\}$  of a set  $X$  is *reduced* if the union of any two adjacent pancakes  $X_j$  and  $X_k$  (such that  $X_j \cap X_k \neq \{0\}$ ) is not normally embedded.

**Remark 2.4.3.** When the union of two adjacent pancakes is normally embedded, they can be replaced by their union, reducing the number of pancakes. Thus, a reduced pancake decomposition always exists.

### 2.4.1 Bi-Lipschitz homeomorphisms between pancakes

**Proposition 2.4.1.** *Let  $T = T(\gamma_1, \gamma_2)$  and  $T' = T(\gamma'_1, \gamma'_2)$  be normally embedded  $\beta$ -Hölder triangles such that  $\text{tord}(\gamma_1, \gamma'_1) \geq \alpha$ ,  $\text{tord}(\gamma_2, \gamma'_2) \geq \alpha$ , and  $\text{tord}(\gamma, T') \geq \alpha$  for all arcs  $\gamma \subset T$ , for some  $\alpha > \beta$ . Then there is a bi-Lipschitz homeomorphism  $h : T \rightarrow T'$  such that  $h(\gamma_1) = \gamma'_1$ ,  $h(\gamma_2) = \gamma'_2$ , and  $\text{tord}(h(\gamma), \gamma) \geq \alpha$  for any arc  $\gamma \subset T$ .*

*Proof.* According to Theorem 4.5 from (BIRBRAIR *et al.*, 2021), we may assume, embedding  $T \cup T'$  into  $\mathbb{R}^n$  for some  $n \geq 5$ , that  $T' = T_\beta$  is a standard  $\beta$ -Hölder triangle (2.1) in the  $xy$ -plane  $\mathbb{R}^2 \subset \mathbb{R}^n$ ,  $\gamma'_1$  belongs to the positive  $x$ -axis and  $\gamma'_2$  to the graph  $y = x^\beta$ . Let  $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^2$  be orthogonal projection, and let  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}^{n-2}$  be orthogonal projection to the orthogonal complement of  $\mathbb{R}^2$  in  $\mathbb{R}^n$ . Orientation of  $\mathbb{R}^2$  defines orientation of  $T'$  such that a segment of the positive  $x$ -axis in its boundary is oriented in the positive direction. We are going to prove the following statement:

(\*) *There is a natural orientation of  $T$  such that, if  $S$  is the set of those points of  $T$  where  $\pi|_T$  is not a smooth, one-to-one, orientation-preserving map, then  $S$  is a union of finitely many  $\beta_j$ -Hölder triangles  $T_j$ , where  $\beta_j \geq \alpha$  for all  $j$ .*

Let  $V \subset \mathbb{R}^2$  be the union of the set of critical values of  $\pi|_T$  and the arcs  $\pi(\gamma_1)$ ,  $\pi(\gamma_2)$ ,  $\gamma'_1$  and  $\gamma'_2$ . The set  $W = \pi^{-1}(V) \cap T$  consists of finitely many isolated arcs and, possibly, some “vertical” Hölder triangles mapped by  $\pi$  to arcs in  $\mathbb{R}^2$ . Removing from  $W$  interiors of the vertical triangles, we obtain the set  $U \subset T$  consisting of finitely many arcs, all of them having tangency order at least  $\alpha$  with  $\mathbb{R}^2$ , since they have tangency order at least  $\alpha$  with  $T'$ . Let  $T_j \subset T$  be  $\beta_j$ -Hölder triangles bounded by arcs from  $U$  and containing no interior arcs from  $U$ . If  $T_j$  is a vertical triangle then  $\beta_j \geq \alpha$ . We may assume that  $\beta_1 = \beta$ , so triangle  $T_1$  is not vertical and  $\pi|_{T_1}$  defines orientation of  $T_1$ . We define orientation of  $T$  compatible with this orientation of  $T_1$ .

For any non-vertical triangle  $T_j$ , if  $\pi^{-1}(\pi(T_j)) \cap T$  contains more than one non-vertical triangle then, since  $T$  is normally embedded and all arcs of  $T$  have tangency order at least  $\alpha$  with  $\mathbb{R}^2$ , we have  $\beta_j \geq \alpha$ . If  $\pi|_{T_j}$  is orientation reversing then there is a  $\beta_j$ -Hölder triangle  $T_k \subset \pi^{-1}(\pi(T_j)) \cap T$  such that  $\pi|_{T_k}$  is orientation preserving, thus  $\beta_j \geq \alpha$  in that case, too. This completes the proof of (\*).

Note also that each of the sets  $\overline{T' \setminus \pi(T)}$  and  $\overline{\pi(T) \setminus T'}$  is either empty or consists of at most two Hölder triangles with exponents at least  $\alpha$ , since  $\text{tord}(\gamma_1, \gamma'_1) \geq \alpha$  and  $\text{tord}(\gamma_2, \gamma'_2) \geq \alpha$ .

Let now  $T_j \subset T$  be a  $\beta$ -Hölder triangle bounded by two arcs from  $U$  and containing

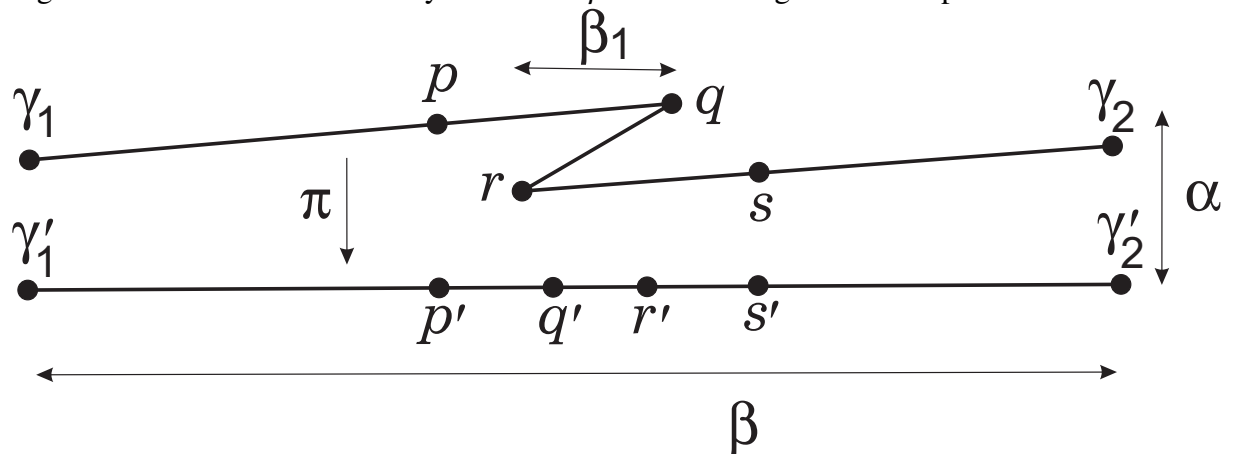
no interior arcs from  $U$ . Then  $T'_j = \pi(T_j) \subset T'$ ,  $\pi|_{T_j}$  is orientation preserving, and for each interior point  $P \in T_j$  we have  $\pi^{-1}(\pi(P)) = \{P\}$ . For  $(x, y) = \pi(P) \in T'_j$ , let  $f(x, y) = \rho(P)$  be a function  $f = (f_1, \dots, f_{n-2}) : T'_j \rightarrow \mathbb{R}^{n-2}$ . For  $c > 0$ , let  $T'_{j,c}$  be the set of points in  $T'_j$  where either  $f$  is not differentiable or  $|\partial f_k / \partial y| \geq c$  for some  $k$ . Since  $tord(\gamma, \mathbb{R}^2) \geq \alpha$  for each arc  $\gamma \subset T$ , each set  $T'_c$  is contained in the union of finitely many  $\alpha$ -Hölder triangles. Note that the mapping  $\pi : T_j \rightarrow T'_j$  is bi-Lipschitz outside these triangles.

Adding the sets  $T_{j,c} = \pi^{-1}(T'_{j,c}) \cap T_j$ , for some  $c > 0$  and each  $\beta$ -Hölder triangle  $T_j$ , to the set  $S$ , we can find a finite set of disjoint  $\alpha$ -Hölder triangles in  $T$  such that projection of each of them to  $\mathbb{R}^2$  is an  $\alpha$ -Hölder triangle either contained in  $T'$  or intersecting  $T'$  over an  $\alpha$ -Hölder triangle, and  $\pi|_T$  is a bi-Lipschitz mapping from  $T$  to  $T'$  outside these triangles.

We can now define  $h : T \rightarrow T'$  as any orientation preserving bi-Lipschitz homeomorphism from each of these  $\alpha$ -Hölder triangles to intersection of its projection with  $T'$ , and as  $\pi$  in the complement to all these triangles.  $\square$

**Example 5.** The links of two normally embedded  $\beta$ -Hölder triangles  $T = T(\gamma_1, \gamma_2)$  and  $T' = T(\gamma'_1, \gamma'_2)$  are shown in Fig. 1. A Hölder triangle  $T(q, r) \subset T$  is projected to  $T'$  with orientation reversed. Since its exponent is  $\beta_1 \geq \alpha > \beta$ , one can choose the points  $p, q, r, s$  so that  $p' = \pi(p), s' = \pi(s)$  and a mapping  $h : T' \rightarrow T$  such that  $h(q') = q$  and  $h(r') = r$  is an outer bi-Lipschitz homeomorphism.

Figure 1 – Links of two normally embedded  $\beta$ -Hölder triangles in Example 5



Source: elaborated by the author.

## 2.5 Pizza decomposition

In this subsection we use the definitions and results of (BIRBRAIR *et al.*, 2017).

**Definition 2.5.1.** Let  $f \not\equiv 0$  be a germ at the origin of a Lipschitz function defined on an arc  $\gamma$ . The *order* of  $f$  on  $\gamma$ , denoted by  $ord_\gamma f$ , is the value  $q \in \mathbb{F}$  such that  $f(\gamma(t)) = ct^q + o(t^q)$  as  $t \rightarrow 0$ , where  $c \neq 0$ . If  $f \equiv 0$  on  $\gamma$ , we set  $ord_\gamma f = \infty$ .

**Definition 2.5.2.** Let  $T \subset \mathbb{R}^n$  be a Hölder triangle, and let  $f: (T, 0) \rightarrow (\mathbb{R}, 0)$  be a Lipschitz function. We define

$$Q_f(T) = \bigcup_{\gamma \in V(T)} ord_\gamma f.$$

**Remark 2.5.1.** It was shown in (BIRBRAIR *et al.*, 2017) that  $Q_f(T)$  is a closed segment in  $\mathbb{F} \cup \{\infty\}$ .

**Definition 2.5.3.** A Hölder triangle  $T$  is *elementary* with respect to a Lipschitz function  $f$  if, for any two distinct arcs  $\gamma$  and  $\gamma'$  in  $T$  such that  $ord_\gamma f = ord_{\gamma'} f = q$ , the order of  $f$  is  $q$  on any arc in the Hölder triangle  $T(\gamma, \gamma') \subset T$ .

**Definition 2.5.4.** Let  $T \subset \mathbb{R}^n$  be a Hölder triangle and  $f: (T, 0) \rightarrow (\mathbb{R}, 0)$  a Lipschitz function. For each arc  $\gamma \subset T$ , the *width*  $\mu_T(\gamma, f)$  of  $\gamma$  with respect to  $f$  is the infimum of the exponents of Hölder triangles  $T' \subset T$  containing  $\gamma$  such that  $Q_f(T')$  is a point. For  $q \in Q_f(T)$  let  $\mu_{T,f}(q)$  be the set of exponents  $\mu_T(\gamma, f)$ , where  $\gamma$  is any arc in  $T$  such that  $ord_\gamma f = q$ . It was shown in (BIRBRAIR *et al.*, 2017) that the set  $\mu_{T,f}(q)$  is finite. This defines a multivalued *width function*  $\mu_{T,f}: Q_f(T) \rightarrow \mathbb{F} \cup \{\infty\}$ . When  $f$  is fixed, we write  $\mu_T(\gamma)$  instead of  $\mu_T(\gamma, f)$  and  $\mu_T$  instead of  $\mu_{T,f}$ . If  $T$  is an elementary Hölder triangle with respect to  $f$  then the function  $\mu_{T,f}$  is single valued.

**Definition 2.5.5.** Let  $T$  be a Hölder triangle and  $f: (T, 0) \rightarrow (\mathbb{R}, 0)$  a Lipschitz function. We say that  $T$  is a *pizza slice* associated with  $f$  if it is elementary with respect to  $f$  and  $\mu_{T,f}(q) = aq + b$  is an affine function.

**Lemma 2.5.1.** Let  $X = T(\gamma_1, \gamma_2)$  be a normally embedded Hölder triangle partitioned by an interior arc  $\gamma$  into two Hölder triangles  $X_1 = T(\gamma_1, \gamma)$  and  $X_2 = T(\gamma, \gamma_2)$ . Let  $f: (X_1, 0) \rightarrow (\mathbb{R}, 0)$  be the function given by  $f(x) = d(x, X_2)$ . Then, for every arc  $\theta \subset X_1$ , we have

$$ord_\theta f = \mu_{X_1}(\theta, f) = tord(\theta, \gamma).$$

*Proof.* Since  $X$  is normally embedded, we can assume that  $X$  is a standard Hölder triangle in  $\mathbb{R}^2$ . Then, for every arc  $\gamma' \subset X_1$ , since  $\gamma$  is the closest arc in  $X_2$  to  $\gamma'$ , we have  $\text{ord}_{\gamma'} f = \text{tord}(\gamma', \gamma)$ . Moreover, given an arc  $\theta \subset X_1$ , we write  $q_\theta = \text{ord}_\theta f$ , and if  $\theta' \in G(T(\theta, \gamma))$  then  $\text{ord}_{\gamma'} f = \text{tord}(\gamma', \gamma) = \text{tord}(\theta, \gamma) = q_\theta$  for every arc  $\gamma' \subset T(\theta, \theta')$ . Thus,  $\mu_{X_1}(q_\theta) \leq \mu(T(\theta, \theta')) = \text{tord}(\theta, \gamma) = q_\theta$ . However, if  $\mu_{X_1}(q_\theta) < q_\theta$  then there is an arc  $\gamma' \subset X_1$  such that  $\text{tord}(\theta, \gamma') < q_\theta$  and consequently,  $\text{tord}(\theta, \gamma) \neq q_\theta$ .  $\square$

**Proposition 2.5.1.** *Let  $T$  be a  $\beta$ -Hölder triangle,  $f$  a Lipschitz function on  $T$  and  $Q = Q_f(T)$ .*

*If  $T$  is a pizza slice associated with  $f$  then*

1.  $\mu_T$  is constant only when  $Q$  is a point;
2.  $\beta \leq \mu_T(q) \leq q$  for all  $q \in Q$ ;
3.  $\mu(\text{ord}_\gamma f) = \beta$  for all  $\gamma \in G(T)$ ;
4. *If  $Q$  is not a point, let  $\mu_0 = \max_{q \in Q} \mu_T(q)$ , and let  $\gamma_0$  be the boundary arc of  $T$  such that  $\mu_T(\gamma_0) = \mu_0$ . Then  $\mu_T(\gamma) = \text{itord}(\gamma_0, \gamma)$  for all arcs  $\gamma \subset T$  such that  $\text{itord}(\gamma_0, \gamma) \leq \mu_0$ .*

**Definition 2.5.6.** A decomposition  $\{T_i\}$  of a Hölder triangle  $X$  into  $\beta_i$ -Hölder triangles  $T_i = T(\lambda_{i-1}, \lambda_i)$  such that  $T_{i-1} \cap T_i = \lambda_i$  is a *pizza decomposition* of  $X$  (or just a *pizza* on  $X$ ) associated with  $f$  if each  $T_i$  is a pizza slice associated with  $f$ . We write  $Q_i = Q_f(T_i)$ ,  $\mu_i = \mu_{T_i, f}$  and  $q_i = \text{ord}_{\lambda_i} f$ .

**Remark 2.5.2.** The existence of a pizza associated with a function  $f$  was proved in (BIRBRAIR *et al.*, 2017) for a function defined in  $(\mathbb{R}^2, 0)$ . The same arguments prove the existence of a pizza associated with a function defined on a Hölder triangle as in Definition 2.5.6. The results mentioned in this subsection remain true when  $f$  is a Lipschitz function on a Hölder triangle  $T$  with respect to the inner metric, although in this paper we need them only for Lipschitz functions with respect to the outer metric.

**Definition 2.5.7.** A pizza  $\{T_i\}_{i=1}^p$  associated with a function  $f$  is *minimal* if, for any  $i \in \{2, \dots, p\}$ ,  $T_{i-1} \cup T_i$  is not a pizza slice associated with  $f$ .

**Remark 2.5.3.** From a given a pizza decomposition  $\{T_i\}_{i=1}^p$  associated with a function  $f$  one can obtain a minimal pizza decomposition  $\{\tilde{T}_i\}_{i=1}^{\tilde{p}}$  associated with  $f$  by replacing  $T_i, \dots, T_{i+k}$  by  $\tilde{T}_i = \bigcup_{i \leq j \leq i+k} T_j$  when this union is a pizza slice associated with  $f$ .

**Example 6.** Consider  $T_1$  and  $T_2$  as in Example 4. Let  $f: (T_1, 0) \rightarrow (\mathbb{R}, 0)$  be the function given by  $f(x, y, z) = x^{\alpha-\beta}y$ . Note that  $T_2$  is the graph of  $f$ . For each arc  $\gamma \subset T_1$ , we have

$\gamma(t) = (t, ct^p + o(t^p), 0)$ , where  $c > 0$  and  $p \geq \beta$ . Hence,  $f(\gamma(t)) = ct^{\alpha-\beta+p} + o(t^{\alpha-\beta+p})$  and, consequently,  $ord_\gamma f = \alpha - \beta + p$ . Moreover, if  $\gamma'(t) = c't^{p'} + o(t^{p'})$  is another arc in  $T_1$  then  $ord_\gamma f = ord_{\gamma'} f$  if and only if  $p = p'$ . Thus,  $T_1$  is elementary with respect to  $f$ , and a minimal pizza decomposition of  $T_1$  associated with  $f$  consists of the single pizza slice  $T_1$  with  $Q_1 = [\alpha, \infty)$ . Since  $ord_\lambda f = \infty$ , we have  $\mu(ord_\lambda f) = \infty = \max_{q \in Q_1} \mu(q)$ . Proposition 2.5.1 implies that  $\mu(ord_\gamma f) = itord(\gamma, \lambda)$  for every arc  $\gamma \subset T_1$ . For  $\gamma(t) = (t, ct^p + o(t^p), 0)$  we obtain  $itord(\gamma, \lambda) = tord(\gamma, \lambda) = p$ . Since  $q = ord_\gamma f = \alpha - \beta + p$ , we have  $p = q + \beta - \alpha$ , thus  $\mu(q) = q + \beta - \alpha$  for every  $q \in Q_1$ .

**Definition 2.5.8.** Consider the set of germs of Lipschitz functions  $f_l: (X, 0) \rightarrow (\mathbb{R}, 0)$ ,  $l = 1, \dots, m$ , defined on a Hölder triangle  $X$ . A *multipizza* on  $X$  associated with  $\{f_1, \dots, f_m\}$  is a decomposition  $\{T_i\}$  of  $X$  into  $\beta_i$ -Hölder triangles which is a pizza on  $X$  associated with  $f_l$  for each  $l$ .

**Remark 2.5.4.** The existence of a multipizza follows from the existence of a pizza associated with a single Lipschitz function  $f$ , since a refinement of a pizza associated with any function  $f$  is also a pizza associated with  $f$ .

## 2.6 Zones

In this subsection,  $(X, 0) \subset (\mathbb{R}^n, 0)$  is a surface germ. Some of the definitions below were first introduced in (BIRBRAIR; MENDES, 2018).

**Definition 2.6.1.** A nonempty set of arcs  $Z \subset V(X)$  is a *zone* if, for any two distinct arcs  $\gamma_1$  and  $\gamma_2$  in  $Z$ , there exists a non-singular Hölder triangle  $T = T(\gamma_1, \gamma_2) \subset X$  such that  $V(T) \subset Z$ . If  $Z = \{\gamma\}$  then  $Z$  is a *singular zone*.

**Definition 2.6.2.** Let  $B \subset V(X)$  be a nonempty set. A zone  $Z \subset B$  is *maximal in  $B$*  if, for any Hölder triangle  $T$  such that  $V(T) \subset B$ , one has either  $Z \cap V(T) = \emptyset$  or  $V(T) \subset Z$ .

**Remark 2.6.1.** A zone could be understood as an analog of a connected subset of  $V(X)$ , and a maximal zone in a set  $B$  is an analog of a connected component of  $B$ .

**Definition 2.6.3.** The *order*  $\mu(Z)$  of a zone  $Z$  is the infimum of  $tord(\gamma, \gamma')$  over all arcs  $\gamma$  and  $\gamma'$  in  $Z$ . If  $Z$  is a singular zone then  $\mu(Z) = \infty$ . A zone  $Z$  of order  $\beta$  is called a  $\beta$ -zone.

**Remark 2.6.2.** The tangency order can be replaced by the inner tangency order in Definition 2.6.3. Note that, for any arc  $\gamma \in Z$ ,  $\inf_{\gamma' \in Z} tord(\gamma, \gamma') = \inf_{\gamma' \in Z} itord(\gamma, \gamma') = \mu(Z)$ .



**Definition 2.6.4.** A zone  $Z$  is normally embedded if, for any two arcs  $\gamma$  and  $\gamma'$  in  $Z$ , there exists a normally embedded Hölder triangle  $T = T(\gamma, \gamma')$  such that  $V(T) \subset Z$ .

**Definition 2.6.5.** A  $\beta$ -zone  $Z$  is *closed* if there is a  $\beta$ -Hölder triangle  $T$  such that  $V(T) \subset Z$ . Otherwise,  $Z$  is *open*. A zone  $Z$  is *perfect* if, for any two arcs  $\gamma \neq \gamma'$  in  $Z$ , there is a Hölder triangle  $T$  such that  $V(T) \subset Z$  and both  $\gamma$  and  $\gamma'$  are generic arcs of  $T$ . By definition, any singular zone is perfect.

**Definition 2.6.6.** A closed  $\beta$ -zone  $Z \subset V(X)$  is  $\beta$ -*complete* if, for any  $\gamma \in Z$ ,

$$Z = \{\gamma' \in V(X) \mid itord(\gamma, \gamma') \geq \beta\}.$$

An open  $\beta$ -zone  $Z \subset V(X)$  is  $\beta$ -*complete* if, for any  $\gamma \in Z$ ,

$$Z = \{\gamma' \in V(X) \mid itord(\gamma, \gamma') > \beta\}.$$

**Remark 2.6.3.** Let  $Z$  and  $Z'$  be open  $\beta$ -complete zones. Then, either  $Z \cap Z' = \emptyset$  or  $Z = Z'$ . Moreover,  $Z \cap Z' = \emptyset$  implies  $itord(Z, Z') \leq \beta$ . The same holds when  $Z$  and  $Z'$  are closed  $\beta$ -complete zones, except  $Z \cap Z' = \emptyset$  implies  $itord(Z, Z') < \beta$ .

**Example 7.** If  $T$  is a non-singular  $\beta$ -Hölder triangle then the set  $V(T)$  of all arcs in  $T$ , the set  $I(T)$  of interior arcs of  $T$ , and the set  $G(T)$  of generic arcs of  $T$  are closed  $\beta$ -zones, but only  $V(T)$  is  $\beta$ -complete, and only  $G(T)$  is a perfect zone. The set  $V(T) \setminus G(T)$  consists of two open non-perfect  $\beta$ -complete zones. For any arc  $\gamma \in G(T)$ , the set of arcs  $\gamma' \in V(T)$  such that  $itord(\gamma, \gamma') > \beta$  is a perfect open zone.

**Definition 2.6.7.** Two zones  $Z$  and  $Z'$  in  $V(X)$  are *adjacent* if  $Z \cap Z' = \emptyset$  and there exist arcs  $\gamma \subset Z$  and  $\gamma' \subset Z'$  such that  $V(T(\gamma, \gamma')) \subset Z \cup Z'$ .

**Example 8.** Let  $T = T(\gamma_1, \gamma_2)$  be a non-singular  $\beta$ -Hölder triangle and consider  $H_i = \{\gamma \in V(T) \mid itord(\gamma, \gamma_i) > \beta\}$ , for  $i = 1, 2$ . The open non-perfect  $\beta$ -zones  $H_1$  and  $H_2$  are adjacent to the perfect  $\beta$ -zone  $G(T)$ .

**Lemma 2.6.1.** Let  $X$  be a Hölder triangle, and let  $Z$  and  $Z'$  be two zones in  $V(X)$  of orders  $\beta$  and  $\beta'$ , respectively. If either  $Z \cap Z' \neq \emptyset$  or  $Z$  and  $Z'$  are adjacent, then  $Z \cup Z'$  is a zone of order  $\min(\beta, \beta')$ .

*Proof.* One can easily check that in both cases  $Z \cup Z'$  is a zone.

If there is an arc  $\lambda \in Z \cap Z'$  then, for any arcs  $\gamma \in Z$  and  $\gamma' \in Z'$ , we have  $itord(\gamma, \gamma') \geq \min(itord(\gamma, \lambda), itord(\lambda, \gamma')) \geq \min(\beta, \beta')$ .

If  $Z$  and  $Z'$  are adjacent, let  $T = T(\lambda, \lambda')$  be a Hölder triangle such that  $\lambda \in Z$ ,  $\lambda' \in Z'$  and  $V(T) \subset Z \cup Z'$ . If  $\mu(T) < \min(\beta, \beta')$ , let us choose an arc  $\lambda'' \in G(T)$ . If  $\lambda'' \in Z$  (resp.,  $\lambda'' \in Z'$ ) then  $itord(\lambda, \lambda'') < \beta$  (resp.,  $itord(\lambda'', \lambda') < \beta'$ ), a contradiction. Thus  $\mu(T) \geq \min(\beta, \beta')$  and for any arcs  $\gamma \in Z$  and  $\gamma' \in Z'$  we have  $itord(\gamma, \gamma') \geq \min(itord(\gamma, \lambda), itord(\lambda', \gamma'), \mu(T)) \geq \min(\beta, \beta')$ , so  $\mu(Z \cup Z') = \min(\beta, \beta')$  in both cases.  $\square$

**Lemma 2.6.2.** *Let  $\{X_i\}$  be a finite decomposition of a Hölder triangle  $X$  into  $\beta_i$ -Hölder triangles. If  $Z \subset V(X)$  is a  $\beta$ -zone then  $Z_i = Z \cap V(X_i)$  is a  $\beta$ -zone for some  $i$ .*

*Proof.* Since  $Z = \bigcup_i Z_i$ , it follows from Lemma 2.6.1 that  $\mu(Z) = \min_i \mu(Z_i)$ . If  $\mu(Z_i) > \beta$  for all  $i$  then, by the non-archimedean property,  $\mu(Z) > \beta$ , a contradiction.  $\square$

**Lemma 2.6.3.** *Let  $X$  be a Hölder triangle. If  $Z$  and  $Z'$  are perfect  $\beta$ -zones in  $V(X)$ , then they are not adjacent.*

*Proof.* Suppose, by contradiction, that  $Z$  and  $Z'$  are adjacent. Definition 2.6.7 implies that there is a Hölder triangle  $T = T(\gamma, \gamma')$  such that  $\gamma \in Z$ ,  $\gamma' \in Z'$  and  $V(T) \subset Z \cup Z'$ . Since  $Z$  and  $Z'$  are adjacent  $\beta$ -zones,  $\mu(T) \geq \mu(Z \cup Z') = \beta$  by Lemma 2.6.1. If  $\mu(T) > \beta$ , since  $Z$  is perfect  $\beta$ -zone, we would have  $V(T) \subset Z$ , a contradiction with  $\gamma' \in Z'$ . Thus, we may assume that  $\mu(T) = \beta$ . Let  $h: T_\beta \rightarrow T$  be an inner bi-Lipschitz homeomorphism, where  $T_\beta = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq x^\beta\}$  is  $\beta$ -standard Hölder triangle, such that  $h(\{0 \leq x \leq 1, y = 0\}) = \gamma$ . Let  $c_0 = \sup\{c \in [0, 1] \mid h(\{0 \leq x \leq 1, y = cx^\beta\}) \in Z\}$  and let  $\gamma_0 = h(\{0 \leq x \leq 1, y = c_0x^\beta\})$ . If  $\gamma_0 \in Z$  then any arc  $\theta \subset (T \setminus T(\gamma, \gamma_0)) \cup \{0\}$  with  $itord(\theta, \gamma_0)$  belong to  $Z$ , a contradiction with  $Z$  being a perfect  $\beta$ -zone. Similarly, if  $\gamma_0 \in Z'$  then any arc  $\theta \subset (T \setminus T(\gamma', \gamma_0)) \cup \{0\}$  with  $itord(\theta, \gamma_0)$  belong to  $Z'$ , a contradiction.  $\square$

**Definition 2.6.8.** A Lipschitz non-singular arc  $\gamma$  of a surface germ  $X$  is *abnormal* if there are two normally embedded non-singular Hölder triangles  $T \subset X$  and  $T' \subset X$  such that  $T \cap T' = \gamma$  and  $T \cup T'$  is not normally embedded. Otherwise  $\gamma$  is *normal*. A zone is *abnormal* (resp., *normal*) if all of its arcs are abnormal (resp., normal). The sets of abnormal and normal arcs of  $X$  are denoted  $Abn(X)$  and  $Nor(X)$ , respectively.

**Remark 2.6.4.** It follows from Definition 2.6.8 that the property of an arc to be abnormal (resp., normal) is outer Lipschitz invariant: if  $h : X \rightarrow X'$  is an outer bi-Lipschitz homeomorphism then  $h(\gamma) \subset X'$  is an abnormal (resp., normal) arc for any abnormal (resp., normal) arc  $\gamma \subset X$ .

**Example 9.** Given two arcs  $\theta$  and  $\tilde{\theta}$  in  $\mathbb{R}^n$  let  $T(\theta, \tilde{\theta})$  be the Hölder triangle defined by (the germ of) the union of the straight line segments,  $[\theta(t), \tilde{\theta}(t)]$ , connecting  $\theta(t)$  and  $\tilde{\theta}(t)$  for all  $t \geq 0$ . Consider the set  $T = T_1 \cup T_2$  with  $T_1 = T(\gamma_1, \lambda_1)$  and  $T_2 = T(\lambda_1, \lambda_2) \cup T(\lambda_2, \gamma_2)$  where  $\gamma_i(t) = (t, (-1)^i t^{\frac{3}{2}}, 0)$  and  $\lambda_i(t) = (t, (-1)^i t, t)$ , for  $i = 1, 2$ , are arcs parameterized by the first coordinate, which is equivalent to the distance to the origin. The Hölder triangles  $T_1, T_2, T(\lambda_1, \lambda_2)$  and  $T(\lambda_2, \gamma_2)$  are normally embedded. In particular,  $T$  is non-singular.

Notice that  $tord(\gamma_1, \gamma_2) = \frac{3}{2} > 1$  and  $itord(\gamma_1, \gamma_2) = 1$ , since  $d_i(\gamma_1(t), \gamma_2(t)) \geq 2t$ . Therefore,  $T$  is a not normally embedded 1-Hölder triangle with  $Abn(T) = G(T)$  and  $Nor(T) = H_1 \cup H_2$ , where  $H_i$  is defined as in Example ?? for  $i = 1, 2$ . Thus,  $T$  is a not normally embedded abnormal surface (indeed,  $T$  is a bubble snake, see Definition 4.45 of (??)). Another example of not normally embedded abnormal surface, in this case with circular link, is the complex cusp  $\{(z, w) \in \mathbb{C}^2 \mid z^3 = w^2\}$  seen as a real surface in  $\mathbb{R}^4$ .

Examples of normally embedded abnormal surfaces with circular link are the cone  $\{(r \cos \phi, r \sin \phi, r) \mid r \geq 0, 0 \leq \phi \leq 2\pi\}$  and the  $\beta$ -horn of revolution.

**Definition 2.6.9.** A surface germ  $X$  is called *abnormal* if  $Abn(X) = G(X)$ , the set of generic arcs of  $X$ .

**Remark 2.6.5.** Given an abnormal arc  $\gamma \subset X$ , we can choose normally embedded non-singular Hölder triangles  $T = T(\lambda, \gamma) \subset X$  and  $T' = T(\gamma, \lambda') \subset X$  so that  $T \cap T' = \gamma$  and  $tord(\lambda, \lambda') > itord(\lambda, \lambda')$ . It follows from Lemma 2.3.1 that  $tord(\lambda, \gamma) = tord(\gamma, \lambda') = itord(\lambda, \lambda')$ .

**Definition 2.6.10.** Given an abnormal (resp., normal) arc  $\gamma \subset X$  the maximal abnormal zone (resp., maximal normal zone) in  $V(X)$  containing  $\gamma$  is the union of all abnormal (resp., normal) zones in  $V(X)$  containing  $\gamma$ . Alternatively, the maximal abnormal (resp., normal) zone containing  $\gamma$  is a maximal zone in  $Abn(X)$  (resp.,  $Nor(X)$ ) containing  $\gamma$ .

**Remark 2.6.6.** Since the property of an arc to be abnormal (resp., normal) is outer Lipschitz invariant (see Remark 2.6.4), maximal abnormal (resp., normal) zones in  $V(X)$  are also outer Lipschitz invariant: if  $h : X \rightarrow X'$  is an outer bi-Lipschitz homeomorphism then  $h(Z) \subset V(X')$  is a maximal abnormal (resp., normal) zone for any maximal abnormal (resp., normal) zone

$Z \subset V(X)$ . Here  $h : V(X) \rightarrow V(X')$  is the natural action of  $h$  on arcs in  $X$ . Classification of maximal abnormal zones in  $V(X)$  will be given in Chapter 5 below.

### 3 LIPSCHITZ FUNCTIONS ON A NORMALLY EMBEDDED $\beta$ -HÖLDER TRIANGLE

**Definition 3.0.1.** Let  $(T, 0) \subset (\mathbb{R}^n, 0)$  be a normally embedded  $\beta$ -Hölder triangle, and  $f: (T, 0) \rightarrow (\mathbb{R}, 0)$  a Lipschitz function such that  $\text{ord}_\gamma f \geq \beta$  for all  $\gamma \in V(T)$ . We define the following sets of arcs:

$$B_\beta = B_\beta(f) = \{\gamma \in G(T) \mid \text{ord}_\gamma f = \beta\}$$

and

$$H_\beta = H_\beta(f) = \{\gamma \in G(T) \mid \text{ord}_\gamma f > \beta\}.$$

In this section we study properties of these two sets. In particular, we are going to prove that each of them is a finite union of  $\beta$ -zones.

**Lemma 3.0.1.** *Let  $T$  and  $f$  be as in Definition 3.0.1, and let  $\gamma \in B_\beta$  and  $\gamma' \in H_\beta$ . Then  $\text{tord}(\gamma, \gamma') = \beta$ .*

*Proof.* Let  $\gamma \in B_\beta$  and  $\gamma' \in H_\beta$ . Since  $\text{ord}_{\gamma'} f > \text{ord}_\gamma f = \beta$ , we have  $|f(\gamma(t)) - f(\gamma'(t))| = ct^\beta + o(t^\beta)$ , where  $c \neq 0$ . Let  $K > 0$  be the Lipschitz constant of  $f$ . Thus  $|f(\gamma(t)) - f(\gamma'(t))| \leq K|\gamma(t) - \gamma'(t)|$ . Consequently, if  $|\gamma(t) - \gamma'(t)| = dt^{\text{tord}(\gamma, \gamma')} + o(t^{\text{tord}(\gamma, \gamma')})$ , where  $d \neq 0$ , then  $ct^\beta + o(t^\beta) \leq K(dt^{\text{tord}(\gamma, \gamma')} + o(t^{\text{tord}(\gamma, \gamma')}))$ . Hence,  $\text{tord}(\gamma, \gamma') \leq \beta$ . As  $T$  is a  $\beta$ -Hölder triangle,  $\text{tord}(\gamma, \gamma') = \beta$ .  $\square$

**Lemma 3.0.2.** *Let  $T$  and  $f$  be as in Definition 3.0.1, and let  $T' = T(\gamma_1, \gamma_2) \subset T$ . If  $\text{ord}_{\gamma_1} f = \beta$  and  $\text{ord}_{\gamma_2} f > \beta$  then  $\mu(T') = \beta$  and  $\mu_{T'}(\gamma_1, f) = \beta$ .*

*Proof.* If  $\text{ord}_{\gamma_1} f = \beta$  and  $\text{ord}_{\gamma_2} f > \beta$ , by Lemma 3.0.1, we have  $\text{tord}(\gamma_1, \gamma_2) = \beta$ . Since  $T$  is normally embedded,  $\mu(T') = \text{itord}(\gamma_1, \gamma_2) = \text{tord}(\gamma_1, \gamma_2) = \beta$ .

Then, if  $\text{tord}(\gamma_1, \gamma) > \beta$ , for any arc  $\gamma \subset T$ , then  $\text{ord}_\gamma f = \beta$ . Consequently,  $\mu_{T'}(\gamma_1, f) \leq \beta$ . As  $T$  is a  $\beta$ -Hölder triangle, we have  $\mu_{T'}(\gamma_1, f) \geq \beta$ . Hence,  $\mu_{T'}(\gamma_1, f) = \beta$   $\square$

**Lemma 3.0.3.** *Let  $T$  and  $f$  be as in Definition 3.0.1, and let  $\{T_i = T(\lambda_{i-1}, \lambda_i)\}_{i=1}^p$  be a minimal pizza on  $T$  associated with  $f$ . If  $p > 1$  then each  $T_i$  has at least one boundary arc  $\lambda$  such that  $\text{ord}_\lambda f > \beta$ .*

*Proof.* Note that, for each  $i < p$ , if  $\text{ord}_{\lambda_{i-1}} f = \text{ord}_{\lambda_i} f = \beta$  then  $\text{ord}_{\lambda_{i+1}} f > \beta$ . Indeed, if  $\text{ord}_{\lambda_{i-1}} f = \text{ord}_{\lambda_i} f = \text{ord}_{\lambda_{i+1}} f = \beta$  then  $Q_i = Q_{i+1} = \{\beta\}$  and  $T_i \cup T_{i+1}$  is a pizza slice, a con-

tradition with  $\{T_i\}$  being minimal. Similarly, for each  $i > 1$ , if  $\text{ord}_{\lambda_{i-1}}f = \text{ord}_{\lambda_i}f = \beta$  then  $\text{ord}_{\lambda_{i-2}}f > \beta$ .

Suppose, by contradiction, that there exists  $T_i$  such that  $\text{ord}_{\lambda_{i-1}}f = \text{ord}_{\lambda_i}f = \beta$ . Since  $p > 1$ , either  $i < p$  or  $i > 1$ . If  $i < p$  then  $\text{ord}_{\lambda_{i+1}}f > \beta$  and, by Lemma 3.2,  $T_i \cup T_{i+1}$  is a pizza slice, in contradiction with  $\{T_i\}$  being minimal. Similarly, if  $i > 1$  then  $T_{i-1} \cup T_i$  is a pizza slice, again a contradiction.  $\square$

**Lemma 3.0.4.** *Let  $T$  and  $f$  be as in Definition 3.0.1, and let  $\{T_i = T(\lambda_{i-1}, \lambda_i)\}_{i=1}^p$  be a minimal pizza associated with  $f$ . Then:*

1. *If  $B_\beta \cap V(T_i) \neq \emptyset$  then  $\beta_i = \beta$ .*
2. *If  $\lambda_i \in B_\beta$  then there exists a  $\beta$ -Hölder triangle  $T' \subset T_i \cup T_{i+1}$ , with  $V(T') \subset B_\beta$ , such that  $\lambda_i$  is a generic arc of  $T'$ .*

*Proof.* (1) Consider  $\gamma \in B_\beta \cap V(T_i)$ . If  $T_i = T$  the statement is obvious, since  $T$  has exponent  $\beta$ . Suppose that  $T_i \neq T$ . Since  $T_i$  is pizza slice (in particular,  $T_i$  is elementary with respect to  $f$ ) and  $\text{ord}_\gamma f = \beta$ , either  $\text{ord}_{\lambda_{i-1}}f = \beta$  or  $\text{ord}_{\lambda_i}f = \beta$ . Then, by Lemmas 3.0.2 and 3.0.3,  $\beta_i = \beta$ .

(2) As  $\lambda_i \in B_\beta \subset G(T)$ , it is not one of the boundary arcs of  $T$ . In particular,  $0 < i < p$ . Item (1) of this Lemma implies that  $\beta_i = \beta_{i+1} = \beta$ . Thus,  $G(T_i \cup T_{i+1}) \subset B_\beta$  and one can define  $T' = T(\gamma', \gamma'')$  where  $\gamma' \in G(T_i)$  and  $\gamma'' \in G(T_{i+1})$ .  $\square$

**Proposition 3.0.1.** *Let  $T$  and  $f$  be as in Definition 3.0.1, and let  $\{T_i\}_{i=1}^p$  be a minimal pizza on  $T$  associated with  $f$ . Let  $B_0 = G(T_1)$ ,  $B_p = G(T_p)$  and, for  $0 < i < p$ ,  $B_i = G(T_i \cup T_{i+1})$ . Then*

1. *If  $\text{ord}_{\lambda_i}f = \beta$  then  $B_i$  is a perfect  $\beta$ -zone maximal in  $B_\beta$ .*
2. *If  $p > 1$  then the set  $B_\beta$  is the disjoint union of all perfect  $\beta$ -zones  $B_i$  such that  $\text{ord}_{\lambda_i}f = \beta$ .*

*Proof.* (1) When  $p = 1$  and  $B_\beta \neq \emptyset$  then  $B_0 = B_p = B_\beta = G(T)$  and the result is trivially true. Thus, assume that  $p > 1$ . Consider  $0 \leq i < p$  such that  $\text{ord}_{\lambda_i}f = \beta$ . Lemma 3.0.4 implies that  $\beta_{i+1} = \beta$ . If  $i = 0$  then  $\text{ord}_{\lambda_1}f > \beta$ , by Lemma 3.0.3. Proposition 2.5.1 implies that  $B_0 = G(T_1)$  is a perfect  $\beta$ -zone in  $B_\beta$ . Furthermore, also by Proposition 2.5.1,  $B_0$  is maximal in  $B_\beta$ , since for every arc  $\gamma \in V(T_1) \cap G(T)$ ,  $\text{ord}_\gamma f = \beta$  if and only if  $\text{tord}(\gamma, \lambda_1) = \beta$ . Thus, when  $i = 0$ ,  $B_0$  is a perfect  $\beta$ -zone maximal in  $B_\beta$ . Similarly, if  $\text{ord}_{\lambda_p}f = \beta$  then  $\text{ord}_{\lambda_{p-1}}f > \beta$ ,  $\beta_p = \beta$  and  $B_p = G(T_p)$  is a maximal perfect  $\beta$ -zone in  $B_\beta$ . Finally, suppose that  $0 < i < p$ . Then, Lemma 3.0.4 implies that  $\beta_i = \beta_{i+1} = \beta$  and Lemma 3.0.3 implies that  $\text{ord}_{\lambda_{i-1}}f > \beta$  and  $\text{ord}_{\lambda_{i+1}}f > \beta$ . Therefore, by Proposition 2.5.1,  $B_i = G(T_i \cup T_{i+1})$  is a perfect  $\beta$ -zone maximal in  $B_\beta$ .

(2) Consider  $\mathcal{S} = \{i_0 < i_1 < \dots < i_m\} = \{l \in \mathbb{Z} \mid \text{ord}_{\lambda_l} f = \beta\}$ . Then, by item (1) of this Proposition, each  $B_{i_j}$  is a perfect  $\beta$ -zone maximal in  $B_\beta$ . Moreover, by Lemma 3.0.3, unless  $p = 1$ , the set  $\mathcal{S}$  does not contain consecutive integers and consequently,  $B_{i_0}, \dots, B_{i_m}$  are disjoint, since there are arcs in  $H_\beta$  in between each two such zones. Hence,  $B_{i_0}, \dots, B_{i_m}$  are perfect  $\beta$ -zones maximal in  $B_\beta$  such that

$$\bigcup_{l=0}^m B_{i_l} \subset B_\beta.$$

Finally, given an arc  $\gamma \in B_\beta$ , there exists  $1 \leq i \leq p$  such that  $\gamma \in T_i$ . Thus, by Lemma 3.0.4,  $\beta_i = \beta_{i+1} = \beta$  and either  $B_{i-1}$  or  $B_i$  is a perfect  $\beta$ -zone maximal in  $B_\beta$  containing  $\gamma$ , since we have either  $\text{ord}_{\lambda_{i-1}} f = \beta$  or  $\text{ord}_{\lambda_i} f = \beta$ . So,

$$B_\beta = \bigcup_{l=0}^m B_{i_l}.$$

□

**Proposition 3.0.2.** *Let  $T$  and  $f$  be as in Definition 3.0.1, and let  $\{T_i = T(\lambda_{i-1}, \lambda_i)\}_{i=1}^p$  be a minimal pizza associated with  $f$ . Then*

1. *For each  $i \in \{1, \dots, p-1\}$  such that  $\lambda_i \in G(T)$  and  $\text{ord}_{\lambda_i} f > \beta$ ,  
 $H_i = \{\gamma \in G(T) \mid \text{tord}(\gamma, \lambda_i) > \beta\}$  is an open  $\beta$ -complete zone in  $H_\beta$ .*
2. *For each  $i \in \{1, \dots, p\}$ , if  $\beta_i = \beta$  and  $\text{ord}_{\lambda_l} f > \beta$  for  $l = i-1, i$  then  $H'_i = G(T_i)$  is a perfect  $\beta$ -zone in  $H_\beta$ .*
3. *Each maximal zone  $Z \subset H_\beta$  is the union of some zones as in items (1) and (2).*
4. *The set  $H_\beta$  is a finite union of maximal  $\beta$ -zones.*

*Proof.* (1) This is an immediate consequence of Lemma 3.0.1.

(2) This follows from Proposition 2.5.1.

(3) We will explicitly define all the perfect maximal  $\beta$ -zones  $Z_1, \dots, Z_m$  in  $H_\beta$ . To explicitly define such zones consider the sequence  $\{0 = i_0 < \dots < i_m = p\}$  such that

$$\{i_0, \dots, i_m\} = \{l \in \mathbb{Z} \mid \text{ord}_{\lambda_l} f = \beta\} \cup \{0, p\}.$$

Note that we do not necessarily have  $\text{ord}_{\lambda_j} f = \beta$  for  $j = 0, m$ .

For each  $j \in \{1, \dots, m\}$  we define  $T'_j = T(\lambda_{i_{j-1}}, \lambda_{i_j})$ . Note that each  $T'_j$  is a  $\beta$ -Hölder triangle. We further define the set of indices  $I_j = \{l \in \mathbb{Z} \mid \lambda_l \in G(T'_j), \text{ord}_{\lambda_l} f > \beta\}$  and the integer numbers  $a_j = \min I_j$  and  $b_j = \max I_j$ . Note that if  $\text{ord}_{\lambda_{i_{j-1}}} f = \text{ord}_{\lambda_{i_j}} f = \beta$  then, by Lemma 3.0.3,  $i_{j-1}$  and  $i_j$  are not consecutive integers and  $I_j$  is nonempty.

First, assume that  $m > 1$  and define the sets of arcs  $Z_j \subset V(T'_j)$  as follows (see Fig. 2).

$$Z_j = H_{a_j} \cup V(T(\lambda_{a_j}, \lambda_{b_j})) \cup H_{b_j}, \text{ for each } 1 < j < m,$$

$$Z_1 = \begin{cases} H_{a_1} \cup V(T(\lambda_{a_1}, \lambda_{b_1})) \cup H_{b_1}, & \text{if } \text{ord}_{\lambda_0} f = \beta \\ H'_{a_1} \cup H_{a_1} \cup V(T(\lambda_{a_1}, \lambda_{b_1})) \cup H_{b_1}, & \text{if } \text{ord}_{\lambda_0} f > \beta \text{ and } I_1 \neq \emptyset \\ \emptyset, & \text{if } I_1 = \emptyset \end{cases}$$

and

$$Z_m = \begin{cases} H_{a_m} \cup V(T(\lambda_{a_m}, \lambda_{b_m})) \cup H_{b_m}, & \text{if } \text{ord}_{\lambda_p} f = \beta \\ H_{a_m} \cup V(T(\lambda_{a_m}, \lambda_{b_m})) \cup H_{b_m} \cup H'_{b_{m+1}}, & \text{if } \text{ord}_{\lambda_p} f > \beta \text{ and } I_m \neq \emptyset \\ \emptyset, & \text{if } I_m = \emptyset \end{cases} .$$

In any of the cases above, if  $a_j = b_j$  we set  $T(\lambda_{a_j}, \lambda_{b_j}) = \lambda_{a_j}$ .

Now we are going to prove that, for each  $1 \leq j \leq m$ , if  $Z_j \neq \emptyset$  then it is a  $\beta$ -zone maximal in  $H_\beta$ . We consider three cases:  $1 < j < m$ ,  $j = 1$  and  $j = m$ .

*Case*  $1 < j < m$ . In this case we have  $\text{ord}_{\lambda_{i_{j-1}}} f = \text{ord}_{\lambda_{i_j}} f = \beta$ . Thus,  $I_j$  is nonempty. So, the numbers  $a_j$  and  $b_j$  exist and  $Z_j$  is also nonempty. Finally, note that if  $a_j \neq b_j$  then  $H_{a_j} \cap V(T(\lambda_{a_j}, \lambda_{b_j})) \neq \emptyset$  and  $V(T(\lambda_{a_j}, \lambda_{b_j})) \cap H_{b_j} \neq \emptyset$  (see Fig. 2a), and if  $a_j = b_j$  then  $Z_j = H_{a_j} = H_{b_j}$ . In any case  $Z_j$  is a zone, since the union of a sequence of finitely many zones, such that the intersection of any two consecutive such zones is nonempty, is a zone. Moreover, Proposition 2.5.1 and Lemma 3.0.1 imply that  $Z_j$  is maximal in  $H_\beta$  since from the definition of  $a_j$  and  $b_j$ , if  $V(T'') \cap Z_j \neq \emptyset$  for a Hölder triangle  $T''$  with  $V(T'') \subset H_\beta$ , the boundary arcs of  $T''$  must both belong to  $Z_j$ .

*Case*  $j = 1$ . We have three options:  $\text{ord}_{\lambda_0} f = \beta$ ,  $\text{ord}_{\lambda_0} f > \beta$  and  $I_1 \neq \emptyset$ , and  $I_1 = \emptyset$ .

If  $\text{ord}_{\lambda_0} f = \beta$  then, using the same arguments as in case 1, we obtain that  $Z_1$  is a maximal  $\beta$ -zone in  $H_\beta$ .

Suppose that  $\text{ord}_{\lambda_0} f > \beta$  and  $I_1 \neq \emptyset$ . Since  $a_1$  and  $b_1$  exist, note that,  $H'_{a_1}$  and  $H_{a_1} \cup V(T(\lambda_{a_1}, \lambda_{b_1})) \cup H_{b_1}$  are adjacent zones (see Fig. 2b and Fig. 2c). Then,  $Z_1$  is a zone. Moreover, by the definitions of  $a_1$  and  $b_1$ , Proposition 2.5.1 and Lemma 3.0.1 imply that every arc in  $G(T) \cap H_\beta$  must belong to  $Z_1$ . So, again  $Z_1$  is a maximal  $\beta$ -zone in  $H_\beta$ .

If  $I_1 = \emptyset$  then, by Proposition 2.5.1,  $H_\beta \cap G(T'_1) = \emptyset$ .

*Case*  $j = m$ . This case is very similar to the case  $j = 1$  and its proof is omitted.



Second, if  $m = 1$  we have four options:  $ord_{\lambda_0}f > \beta$  and  $ord_{\lambda_p}f > \beta$ ,  $ord_{\lambda_0}f = ord_{\lambda_p}f = \beta$ ,  $ord_{\lambda_0}f > \beta$  and  $ord_{\lambda_p}f = \beta$ , and  $ord_{\lambda_0}f = \beta$  and  $ord_{\lambda_p}f > \beta$ .

If  $ord_{\lambda_0}f > \beta$  and  $ord_{\lambda_p}f > \beta$  then  $H_\beta = G(T)$ .

If  $ord_{\lambda_0}f = ord_{\lambda_p}f = \beta$  then, similarly as shown above in case  $1 < j < m$ ,  $Z = H_{a_1} \cup V(T(\lambda_{a_1}, \lambda_{b_1})) \cup H_{b_1}$  is a perfect  $\beta$ -zone maximal in  $H_\beta$ .

If  $ord_{\lambda_0}f > \beta$  and  $ord_{\lambda_p}f = \beta$  then either  $H_\beta = \emptyset$  if  $I_1 = \emptyset$  or  $H'_{a_1} \cup H_{a_1} \cup V(T(\lambda_{a_1}, \lambda_{b_1})) \cup H_{b_1}$  is the perfect  $\beta$ -zone maximal in  $H_\beta$  otherwise.

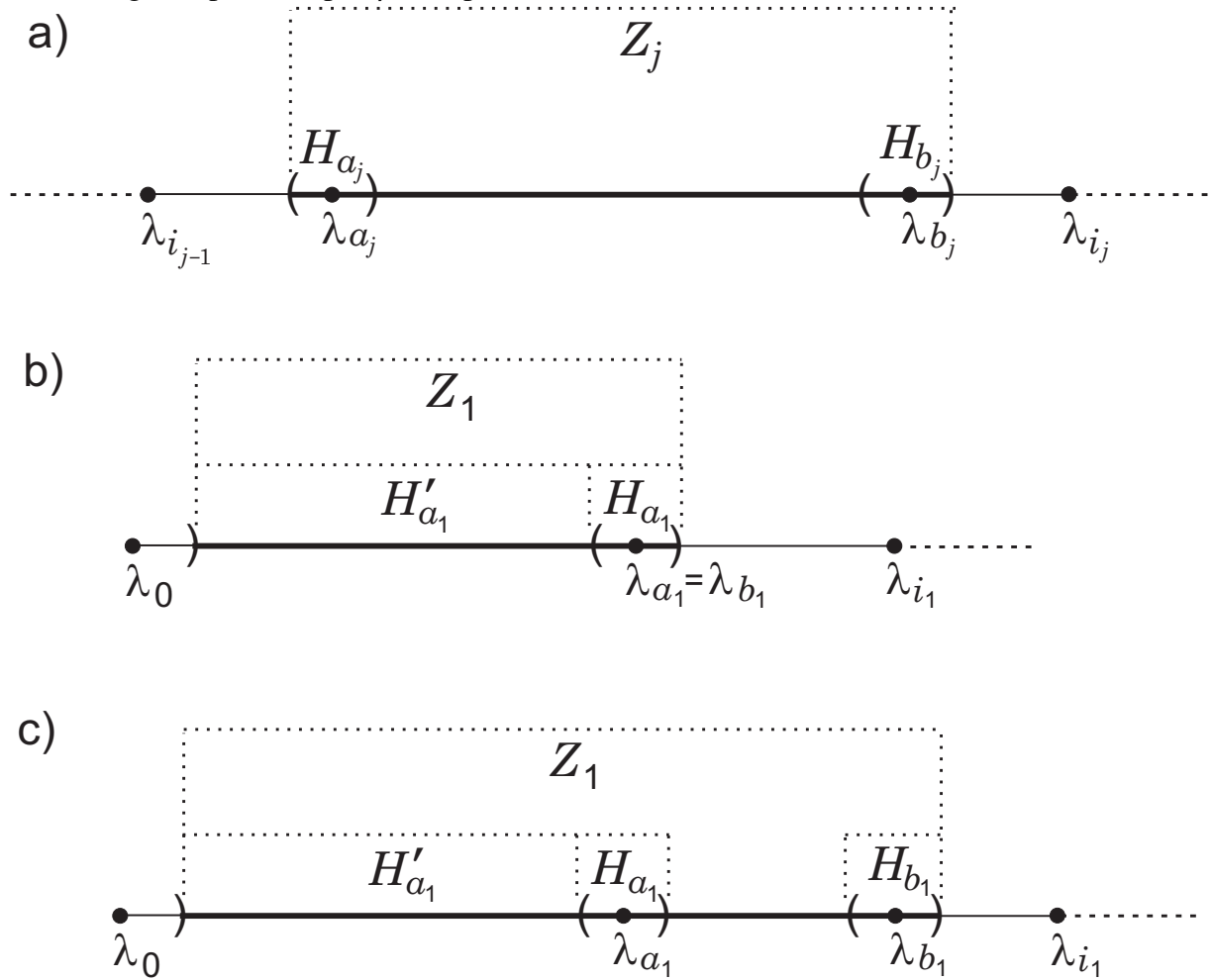
If  $ord_{\lambda_0}f = \beta$  and  $ord_{\lambda_p}f > \beta$  then either  $H_\beta = \emptyset$  if  $I_1 = \emptyset$  or  $H_{a_1} \cup V(T(\lambda_{a_1}, \lambda_{b_1})) \cup H_{b_1} \cup H'_{b_1+1}$  is the perfect  $\beta$ -zone maximal in  $H_\beta$  otherwise.

Finally, since  $Z_1, \dots, Z_m$  are disjoint zones maximal in  $H_\beta$ , any maximal zone in  $H_\beta$  coincide with one of those.

(4) By item (3) of this Proposition,  $H_\beta = \bigcup_{j=1}^m Z_j$ .

□

Figure 2 – Several cases in the proof of Proposition 3.7: a)  $1 < j < m$ ; b)  $j = 1, a_1 = b_1$ ; c)  $j = 1, a_1 < b_1$ . Zones  $Z_j, H_i$  and  $H'_i$  are indicated by dotted lines. The “open intervals” containing  $\lambda_i$  represent open  $\beta$ -complete zones.



Source: elaborated by the author.

## 4 SNAKES

In this chapter we define snakes, one of the main objects of this paper. A  $\beta$ -snake is an abnormal surface germ which is a  $\beta$ -Hölder triangle. We define a canonical partition of the Valette link of a  $\beta$ -snake into segments and nodal zones. All segments of a  $\beta$ -snake are closed perfect  $\beta$ -zones, and all its nodal zones are open  $\beta$ -complete. A node of a  $\beta$ -snake is defined as the union of its nodal zones having tangency order higher than  $\beta$ . We consider relations between pancake decompositions of a snake and its segments and nodes.

### 4.1 Snakes and their pancake decomposition

**Definition 4.1.1.** A non-singular  $\beta$ -Hölder triangle  $T$  is called a  $\beta$ -snake if  $T$  is an abnormal surface (see Definition 2.6.9).

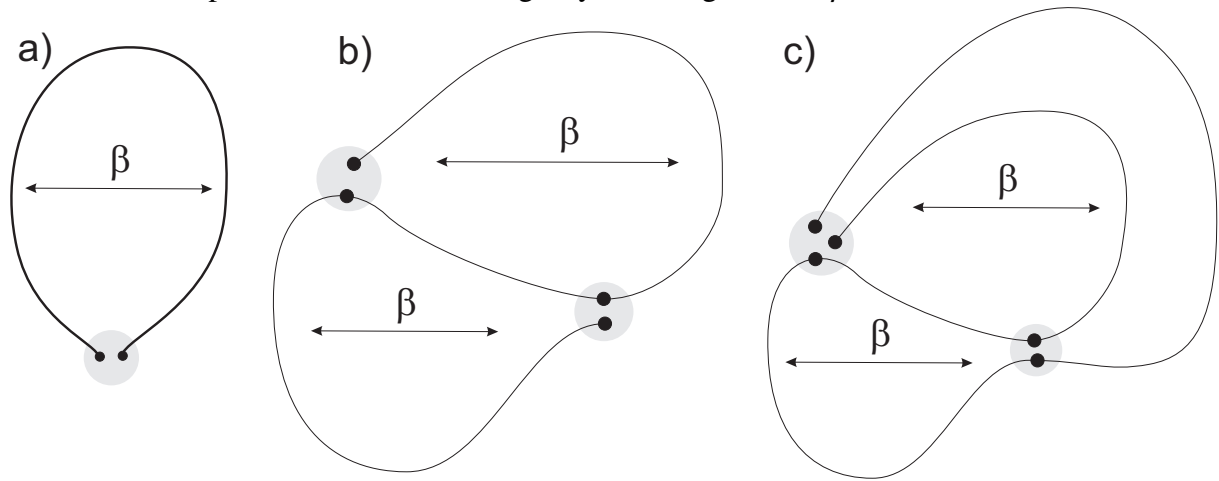
**Remark 4.1.1.** It follows from Definition 4.1.1 and Remark 2.3.4 that each normal arc in  $T$  has inner tangency order higher than  $\beta$  with one of its boundary arcs, and each abnormal arc in  $T$  has inner tangency order  $\beta$  with both boundary arcs.

**Example 10.** A snake with the link as in Fig. 3a is a bubble snake (see Definition 4.5.1 below). A snake with the link as in Fig. 3b is a binary snake, while a snake with the link as in Fig. 3c is not (see Definition 6.4.1 below). We use planar pictures to represent the links of snakes. Points in the picture correspond to arcs in a snake with the given link. Although the Euclidean distance in the link's picture does not accurately translate the tangency order of arcs in the snake with the given link, we will often use it so that points with smaller Euclidean distance in the picture correspond to arcs in the snake with higher tangency order. For example, points inside the shaded disks correspond to arcs with the tangency order higher than  $\beta$ .

**Remark 4.1.2.** Note that minimal generic pancake decompositions of a snake may have different number of pancakes. For example, one of the two reduced pancake decompositions of the snake Fig. 3b shown in Fig. 4 has two pancakes while the other one has three.

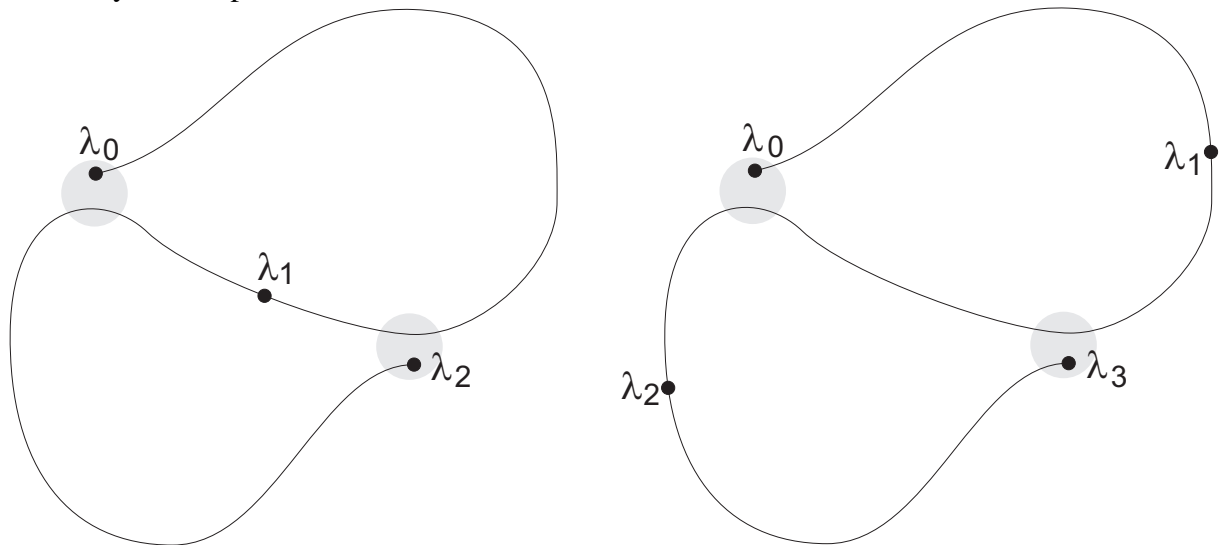
**Remark 4.1.3.** One can also define a *circular snake* as a surface with connected link such that any arc in it is abnormal (in particular, each of its arcs is Lipschitz non-singular). The simplest circular snake is a normally embedded surface germ with the link homeomorphic to a circle. A link of a circle snake that is not normally embedded is shown in Fig. 5. Circular snakes are not discussed in this chapter, although they appear in Example 12, and in Theorem 5.0.1.

Figure 3 – Three links of  $\beta$ -snakes: a) a bubble snake; b) a binary snake; c) a non-binary snake. Shaded disks represent arcs with the tangency order higher than  $\beta$ .



Source: elaborated by the author.

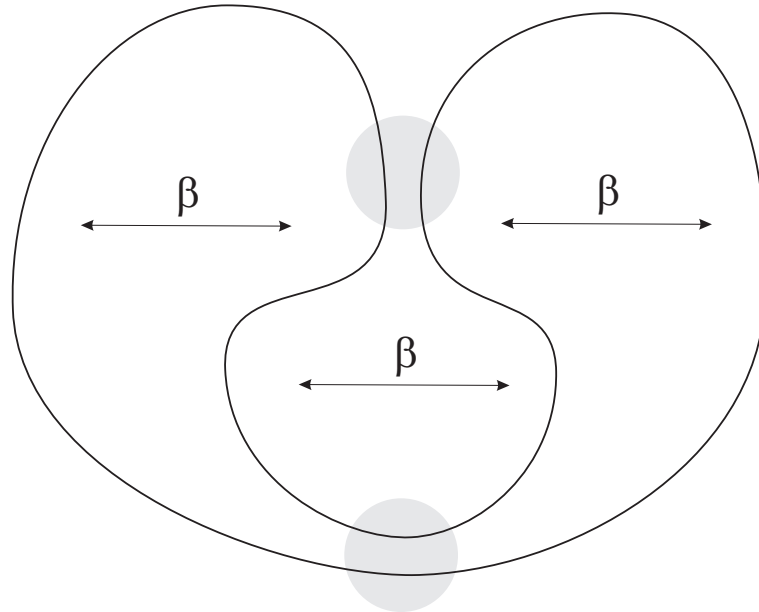
Figure 4 – Two reduced pancake decompositions of the snake in Fig. 3b. Black dots indicate the boundary arcs of pancakes.



Source: elaborated by the author.

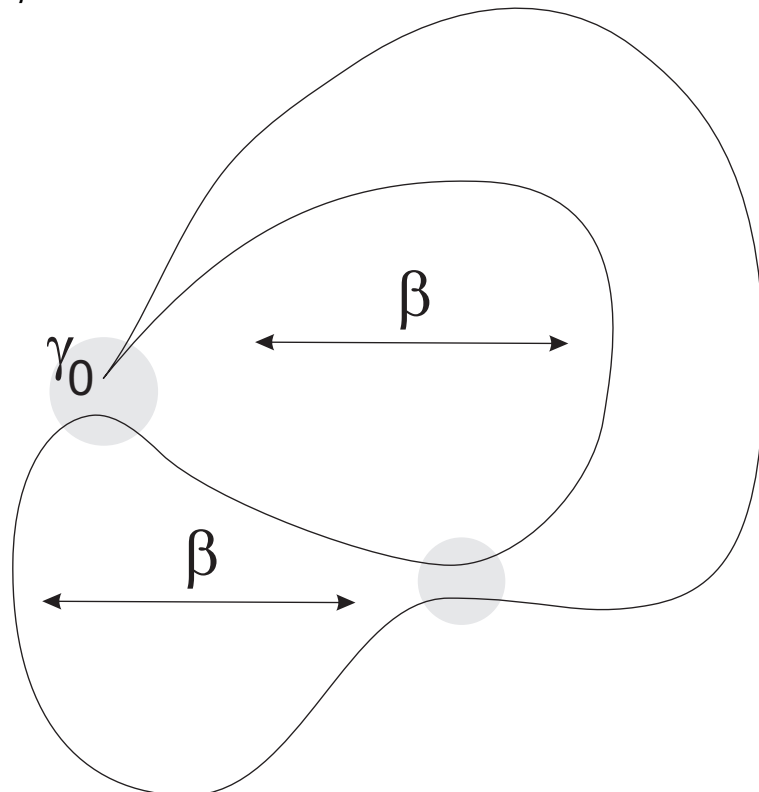
**Example 11.** A circular link (bi-Lipschitz homeomorphic to a circle with respect to the inner metric) of an abnormal  $\beta$ -surface  $X$  with  $Lsing(X) = \gamma_0$  is shown in Fig. 6. Note that  $X$  is not a snake, since it is not even a Hölder triangle. Despite its circular link,  $X$  is not a circular snake as well, since it contains the singular arc  $\gamma_0$ . One can obtain a snake  $T \subset X$  as follows. Consider arcs  $\gamma_1 \neq \gamma_0$  and  $\gamma_2 \neq \gamma_0$  in  $X$  such that  $itord(\gamma_0, \gamma_1) = itord(\gamma_0, \gamma_2) = \alpha > \beta$  and  $T = T(\gamma_1, \gamma_2) \subset (X \setminus \gamma_0) \cup \{0\}$  is a  $\beta$ -Hölder triangle. Then  $T$  is a  $\beta$ -snake with link as shown in

Figure 5 – An example of a circular  $\beta$ -snake. Shaded disks represent arcs with the tangency order higher than  $\beta$ .



Source: elaborated by the author.

Figure 6 – The link of a cusp snake with a singular arc  $\gamma_0$ . Shaded disks represent arcs with tangency order higher than  $\beta$ .



Source: elaborated by the author.

Fig. 3c. This surface  $X$  may be considered as a snake with both boundary arcs equal the singular arc  $\gamma_0$ . We call such a surface a cusp snake.

The next Lemma is fundamental to the understanding of the geometric nature of abnormal surfaces.

**Lemma 4.1.1.** *Let  $X$  be a surface germ and  $A \subset V(X)$  a maximal abnormal  $\beta$ -zone. Let  $\gamma \in A$ , and let  $T = T(\lambda, \gamma) \subset X$  and  $T' = T(\gamma, \lambda') \subset X$  be normally embedded non-singular  $\alpha$ -Hölder triangles such that  $T \cap T' = \gamma$  and  $tord(\lambda, \lambda') > itord(\lambda, \lambda')$ . Then  $\alpha \leq \beta$ .*

*Proof.* Suppose, by contradiction, that  $\alpha > \beta$ . Since  $A$  has order  $\beta$ , we can assume, without loss of generality, that there is a non-singular  $\beta$ -Hölder triangle  $T_0 = T(\gamma_0, \gamma) \subset X$  such that  $T_0 \cap T' = \gamma$  and  $V(T_0) \cap A$  is a  $\beta$ -zone. In particular, if  $\theta \subset T_0$  is any arc such that  $itord(\theta, \gamma) > \beta$ , then  $\theta \in A$ . Since  $itord(\lambda, \gamma) = \alpha > \beta$ , we have  $\lambda \in A$ . Let  $\theta_0 = \lambda$  and  $\theta'_0 = \lambda'$ . Then, there are normally embedded  $\alpha_1$ -Hölder triangles  $T_1 = T(\theta_1, \theta_0) \subset X$  and  $T'_1 = T(\theta_0, \theta'_1) \subset X$  such that  $T(\gamma_0, \theta_0) \cap T'_1 = T_1 \cap T'_1 = \theta_0$  and  $tord(\theta_1, \theta'_1) > itord(\theta_1, \theta'_1)$ . Since  $T \cup T'$  is not normally embedded, we have  $\theta'_1 \subset T \cup T'$ .

Note that  $\alpha_1 > \beta$ . Indeed, if  $\theta'_1 \subset T$  then, by non-archimedean property,  $\alpha_1 = itord(\theta_0, \theta'_1) \geq itord(\theta_0, \gamma) = \alpha > \beta$ . If  $\theta'_1 \subset T'$  we use that  $\alpha = itord(\gamma, \theta'_0)$  and apply non-archimedean property again. Thus,  $\theta_1 \in A$  and there are normally embedded  $\alpha_2$ -Hölder triangles  $T_2 = T(\theta_2, \theta_1) \subset X$  and  $T'_2 = T(\theta_1, \theta'_2) \subset X$  such that  $T(\gamma_0, \theta_1) \cap T'_2 = T_2 \cap T'_2 = \theta_1$  and  $tord(\theta_2, \theta'_2) > itord(\theta_2, \theta'_2)$ . Since  $T_1 \cup T'_1$  is not normally embedded, we have  $\theta'_2 \subset T_1 \cup T'_1$ . Similarly, we prove that  $\alpha_2 \geq \alpha_1 > \beta$  and obtain that  $\theta_2 \in A$ .

Continuing this procedure for  $i > 2$ , we see that  $\alpha_i \geq \dots \geq \alpha_1 > \beta$ , thus  $\theta_i \in A$  and there are normally embedded  $\alpha_{i+1}$ -Hölder triangles  $T_{i+1} = T(\theta_{i+1}, \theta_i)$  and  $T'_{i+1} = T(\theta_i, \theta'_{i+1})$  such that  $T_{i+1} \cap T'_{i+1} = \theta_i$  and  $tord(\theta_{i+1}, \theta'_{i+1}) > itord(\theta_{i+1}, \theta'_{i+1})$  (see Fig. 7).

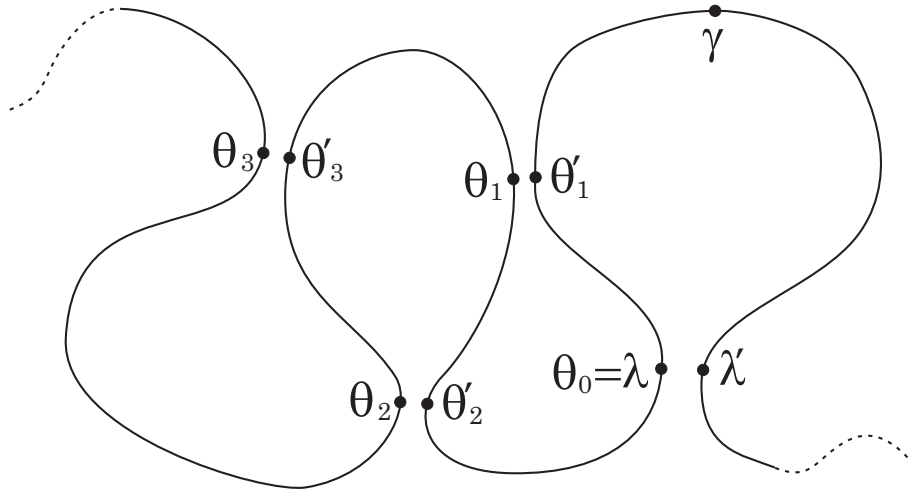
Given a reduced pancake decomposition of  $X$ , for each  $i \geq 0$ ,  $\theta'_i$  and  $\theta_{i+2}$  belong to different pancakes, since  $T(\theta_i, \theta'_i) \subset T(\theta_{i+2}, \theta'_i)$  is not normally embedded. Since there are only finitely many pancakes in a reduced pancake decomposition, our procedure must stop after finitely many steps, in contradiction with  $\alpha > \beta$ .  $\square$

**Lemma 4.1.2.** *Let  $X$  be a  $\beta$ -snake, and let  $\{X_k\}_{k=1}^p$  be a reduced pancake decomposition of  $X$ . Then each  $X_k$  is a  $\beta$ -Hölder triangle.*

*Proof.* We may assume, by Remark 2.4.2, that  $X_k = T(\lambda_{k-1}, \lambda_k)$ , thus  $X_k \cap X_{k+1} = \lambda_k$  and  $X = T(\lambda_0, \lambda_p)$ . Let  $\mu(X_k) = tord(\lambda_{k-1}, \lambda_k) = \beta_k$ .

We prove first that  $\beta_1 = \beta$ . Suppose  $\beta_1 > \beta$ . Then, by the definition of a  $\beta$ -snake,  $\lambda_1$  must be normal. However,  $\lambda_1$  is also abnormal, since  $X_1$  and  $X_2$  are two normally embedded

Figure 7 – Construction in the proof of Lemma 4.1.1. Each pair  $\theta_i, \theta'_i$  has tangency order higher than its inner tangency order.



Source: elaborated by the author.

non-singular Hölder triangles such that  $X_1 \cap X_2 = \lambda_1$  and  $X_1 \cup X_2$  is not normally embedded, which is a contradiction. By a similar argument we can prove that  $\beta_p = \beta$ .

Finally, as  $\beta_1 = \beta_p = \beta$ , if  $1 < j < p$  then  $\lambda_j \in G(X)$ . As  $X$  is  $\beta$ -snake,  $G(X)$  is an abnormal  $\beta$ -zone. Thus, since  $\{X_k\}_{k=1}^p$  is a reduced pancake decomposition,  $X_j \cup X_{j+1}$  is not normally embedded and there are (non-singular) normally embedded  $\alpha$ -Hölder triangles  $T = T(\lambda, \lambda_j) \subset X_j$  and  $T' = T(\lambda_j, \lambda') \subset X_{j+1}$  such that  $tord(\lambda, \lambda') > itord(\lambda, \lambda')$ . Then, by Lemma 4.1.1, we have  $\alpha \leq \beta$ . In particular, by the non-archimedean property,  $\beta \leq \beta_i \leq \alpha \leq \beta$  for  $i = j, j+1$ .  $\square$

**Definition 4.1.2.** A  $\beta$ -Hölder triangle  $X$  is *weakly normally embedded* if, for any two arcs  $\gamma$  and  $\gamma'$  in  $V(X)$  such that  $tord(\gamma, \gamma') > itord(\gamma, \gamma')$ , we have  $itord(\gamma, \gamma') = \beta$ .

**Proposition 4.1.1.** *Let  $X$  be a  $\beta$ -snake. Then  $X$  is weakly normally embedded.*

*Proof.* Let  $\gamma$  and  $\gamma'$  be two arcs in  $V(X)$ . Consider a reduced pancake decomposition  $\{X_k\}_{k=1}^p$  of  $X$ . Since each pancake is normally embedded,  $\gamma$  and  $\gamma'$  do not belong to the same pancake. If  $\gamma$  and  $\gamma'$  are not in adjacent pancakes, then Lemma 4.1.2 implies that  $itord(\gamma, \gamma') = \beta$ . Let us assume that  $\gamma \subset X_{j-1}$  and  $\gamma' \subset X_j$  for some  $j \in \{2, \dots, p\}$ . Consider  $T = T(\gamma, \lambda_{j-1})$  and  $T' = T(\lambda_{j-1}, \gamma')$ . Note that both  $T$  and  $T'$  are normally embedded and non-singular, since each of them is contained in a pancake. Let  $\alpha = itord(\gamma, \gamma')$ . As we have  $tord(\gamma, \gamma') > itord(\gamma, \gamma')$ , Lemma 2.3.1 implies that  $\alpha = tord(\gamma, \lambda_{j-1}) = tord(\lambda_{j-1}, \gamma')$ . Finally, Lemma 4.1.1 that  $\alpha \leq \beta$ . Hence, by the non-archimedean property,  $\beta \leq \alpha \leq \beta$ .  $\square$

## 4.2 Segments and nodes

**Definition 4.2.1.** Let  $X$  be a surface and  $\gamma \subset X$  an arc. For  $a > 0$  and  $1 \leq \alpha \in \mathbb{F}$ , the  $(a, \alpha)$ -horn neighborhood of  $\gamma$  in  $X$  is defined as follows:

$$HX_{a,\alpha}(\gamma) = \bigcup_{0 \leq t \ll 1} X \cap S(0,t) \cap \bar{B}(\gamma(t), at^\alpha),$$

where  $S(0,t) = \{x \in \mathbb{R}^n \mid \|x\| = t\}$  and  $\bar{B}(y,R) = \{x \in \mathbb{R}^n \mid \|x - y\| \leq R\}$ .

**Remark 4.2.1.** When there is no confusion about the surface  $X$  being considered, one writes  $H_{a,\alpha}(\gamma)$  instead of  $HX_{a,\alpha}(\gamma)$ .

**Definition 4.2.2.** If  $X$  is a  $\beta$ -snake and  $\gamma$  an arc in  $X$ , the *multiplicity* of  $\gamma$ , denoted by  $m_X(\gamma)$  (or just  $m(\gamma)$ ), is defined as the number of connected components of  $HX_{a,\beta}(\gamma) \setminus \{0\}$  for  $a > 0$  small enough.

**Remark 4.2.2.** Since  $X$  is definable in an o-minimal structure, the family of sets  $\{HX_{a,\beta}(\gamma)\}_{a>0}$  in Definition 4.2.2 is also definable. In particular, the number of connected component of this set is constant for small  $a > 0$ .

**Lemma 4.2.1.** *Let  $X$  be a surface,  $\gamma \subset X$  an arc and  $Y \subset X$  a closed set. If, for  $a > 0$  sufficiently small,  $Y \cap HX_{a,\alpha}(\gamma) \neq \{0\}$ , then there is an arc  $\gamma' \subset Y$  such that  $\text{tord}(\gamma, \gamma') > \alpha$ .*

*Proof.* Let  $Y_t = S(0,t) \cap Y$  and  $M_t = \{x \in Y_t \mid d(\gamma(t), Y_t) = d(\gamma(t), x)\}$ . Each set  $M_t$  is definable, and so is  $M = \bigcup_{0 \leq t} M_t$ . By the Arc Selection Lemma there exists an arc  $\gamma' \subset M \subset Y$ .

If for each arc  $\gamma' \subset M$  we have  $\text{tord}(\gamma, \gamma') = \alpha$  then, for  $a > 0$  sufficiently small,  $\gamma' \not\subset Y \cap H_{a,\alpha}(\gamma)$ , a contradiction with  $Y \cap H_{a,\alpha}(\gamma) \neq \{0\}$ .  $\square$

**Proposition 4.2.1.** *Let  $X$  be a surface,  $T \subset X$  a normally embedded  $\beta$ -Hölder triangle and  $\gamma \subset X$  an arc. Then, for  $1 \leq \alpha \in \mathbb{F}$  and  $a > 0$  sufficiently small, the link of  $T \cap HX_{a,\alpha}(\gamma)$  is connected.*

*Proof.* Let  $H = HX_{a,\alpha}(\gamma)$ . If  $\alpha < \beta$  and  $T \cap H \neq \{0\}$  for  $a > 0$  sufficiently small, then there is an arc  $\gamma' \subset T$  such that  $\text{tord}(\gamma', \gamma'') > \alpha$ , by Lemma 4.2.1. This implies that  $T \subset H$ , thus the link of  $T \cap H = T$  is connected.

Suppose that  $\alpha \geq \beta$  and, for  $a > 0$  sufficiently small, the link of  $T \cap H$  is not connected. Let  $C$  and  $C'$  be two distinct connected components of  $(T \cap H) \setminus \{0\}$ . By Lemma



4.2.1, for small enough  $a$ , there exist arcs  $\gamma' \subset C$  and  $\gamma'' \subset C'$  such that

$$\text{tord}(\gamma', \gamma'') \geq \min(\text{tord}(\gamma, \gamma'), \text{tord}(\gamma, \gamma'')) > \alpha.$$

Consider  $T' = T(\gamma', \gamma'') \subset T$ . As  $\gamma'$  and  $\gamma''$  are in different connected components, there exists an arc  $\lambda \subset (T' \setminus H) \cup \{0\}$ . Thus,  $\text{itord}(\gamma', \gamma'') \leq \alpha$ , a contradiction with  $T$  being normally embedded.  $\square$

**Corollary 4.2.1.** *Let  $X$  be a surface,  $T \subset X$  a normally embedded Hölder triangle and  $\gamma \subset X$  an arc. Then, for  $1 \leq \alpha \in \mathbb{F}$  and  $a > 0$  sufficiently small, either  $T \cap HX_{a,\alpha}(\gamma) = \{0\}$  or it is a Hölder triangle.*

**Definition 4.2.3.** Let  $X$  be a  $\beta$ -snake and  $Z \subset V(X)$  a zone. We say that  $Z$  is a *constant zone* of multiplicity  $q$  (notation  $m(Z) = q$ ) if all arcs in  $Z$  have the same multiplicity  $q$ .

**Definition 4.2.4.** Let  $X$  be a  $\beta$ -snake and  $\gamma \subset X$  an arc. We say that  $\gamma$  is a *segment arc* if there exists a  $\beta$ -Hölder triangle  $T \subset X$  such that  $\gamma$  is a generic arc of  $T$  and  $V(T)$  is a constant zone. Otherwise  $\gamma$  is a *nodal arc*. We denote the set of segment arcs and the set of nodal arcs in  $X$  by  $\mathbf{S}(X)$  and  $\mathbf{N}(X)$ , respectively. A *segment* of  $X$  is a maximal zone in  $\mathbf{S}(X)$ . A *nodal zone* of  $X$  is a maximal zone in  $\mathbf{N}(X)$ . We write  $\text{Seg}_\gamma$  or  $\text{Nod}_\gamma$  for a segment or a nodal zone containing a given arc  $\gamma$ .

**Proposition 4.2.2.** *If  $X$  is a  $\beta$ -snake then each segment of  $X$  is a closed perfect  $\beta$ -zone (see Definition 2.6.5).*

*Proof.* Given arcs  $\gamma$  and  $\gamma'$  in a segment  $S$  of  $X$ , by Definition 4.2.4, there exist  $\beta$ -Hölder triangles  $T = T(\gamma_1, \gamma_2)$  and  $T' = T(\gamma'_1, \gamma'_2)$  such that  $\gamma$  and  $\gamma'$  are generic arcs of  $T$  and  $T'$ , respectively, and  $V(T) \subset S$  and  $V(T') \subset S$  are constant zones.

This immediately implies that  $S$  is a closed  $\beta$ -zone. In order to prove that  $S$  is a perfect zone, assume that  $\gamma_2$  and  $\gamma'_1$  are in  $T(\gamma, \gamma')$ . Thus,  $T(\gamma_1, \gamma'_2)$  is a  $\beta$ -Hölder triangle containing both  $\gamma$  and  $\gamma'$  as generic arcs such that  $V(T(\gamma_1, \gamma'_2)) \subset S$ , since  $S$  is a zone.  $\square$

**Lemma 4.2.2.** *Let  $X$  be a  $\beta$ -snake and  $\{X_k\}_{k=1}^p$  a pancake decomposition of  $X$ . Let  $T = X_j$  be one of the pancakes and consider the set of germs of Lipschitz functions  $f_l: (T, 0) \rightarrow (\mathbb{R}, 0)$  given by  $f_l(x) = d(x, X_l)$ . If  $\{T_i\}$  is a multipizza on  $T$  associated with  $\{f_1, \dots, f_p\}$  then, for each  $i$ , the following holds:*

1.  $\mu_{il}(\text{ord}_\gamma f_l) = \beta_i$  for all  $l$  and all  $\gamma \in G(T_i)$ , thus  $G(T_i)$  is a constant zone.

2.  $V(T_i)$  intersects at most one segment of  $X$ .
3. If  $V(T_i)$  is contained in a segment then it is a constant zone.

*Proof.* (1). This is an immediate consequence of Definition 2.5.6 and Proposition 2.5.1.

(2). If  $\beta_i > \beta$  and  $V(T_i)$  intersects a segment  $S$ , then  $V(T_i) \subset S$ , since  $S$  is a perfect  $\beta$ -zone, by Proposition 4.2.2.

Let  $\beta_i = \beta$ . Suppose that  $V(T_i)$  intersects distinct segments  $S$  and  $S'$ . As each segment is a perfect  $\beta$ -zone, we can choose arcs  $\lambda \in S$  and  $\lambda' \in S'$  so that  $\lambda, \lambda' \in G(T_i)$ . Let  $T' = T(\lambda, \lambda')$ . By item (1) of this Lemma, all arcs in  $G(T')$  have the same multiplicity. It follows from Definition 4.2.4 that each arc in  $T'$  is a segment arc. Thus,  $S$  and  $S'$  belong to the same segment, a contradiction.

(3) This is a consequence of Definition 4.2.4 and item (1) of this Lemma.  $\square$

**Proposition 4.2.3.** *Let  $X$  be a  $\beta$ -snake. Then*

1. *There are no adjacent segments in  $X$ .*
2.  *$X$  has finitely many segments.*

*Proof.* (1) This is an immediate consequence of Proposition 4.2.2 and Lemma 2.6.3.

(2) Let  $\{X_k\}_{k=1}^p$  be a pancake decomposition of  $X$ . It is enough to show that, for each pancake  $X_j$ ,  $V(X_j)$  intersects with finitely many segments. But this follows from Lemma 4.2.2, since there are finitely many Hölder triangles in a multipizza.  $\square$

**Lemma 4.2.3.** *Let  $X$  be a  $\beta$ -snake. Then, any two arcs in  $V(X)$  with inner tangency order higher than  $\beta$  have the same multiplicity.*

*Proof.* Let  $\{X_k\}_{k=1}^p$  be a pancake decomposition of  $X$ ,  $T = X_j$  one of the pancakes and  $\{T_i\}$  a multipizza associated with  $\{f_1, \dots, f_p\}$  as in Lemma 4.2.2. Consider arcs  $\gamma$  and  $\gamma'$  in  $V(X)$  such that  $\text{itord}(\gamma, \gamma') > \beta$  and  $\gamma \in V(T)$ . We can suppose that  $\gamma, \gamma' \in V(T)$ , otherwise we can just replace  $\gamma'$  by the boundary arc of  $T$  in  $T(\gamma, \gamma')$ .

It is enough to show that for each  $l$  we have  $\text{ord}_\gamma f_l > \beta$  if and only if  $\text{ord}_{\gamma'} f_l > \beta$ . This follows from Lemma 3.0.1.  $\square$

**Corollary 4.2.2.** *Let  $X$  be a  $\beta$ -snake. Then, all segments and all nodal zones of  $X$  are constant zones.*

*Proof.* Let  $\{X_k\}$  be a reduced pancake decomposition of  $X$ , and  $\{T_i\}$  a multipizza on  $T = T_j$  associated with  $\{f_1, \dots, f_p\}$  as in Lemma 4.2.2.

Let  $S$  be a segment of  $X$ . Consider two arcs  $\gamma, \gamma' \in S$ . Replacing, if necessary, one of the arcs  $\gamma, \gamma'$  by one of the boundary arcs of  $T$ , we can assume that  $\gamma, \gamma' \in V(T)$ . Thus, if  $\gamma \in T_i$  and  $\gamma' \in T_{i+l}$ , for some  $l \geq 0$ , it follows from Lemma ?? that  $m(V(T_{i+1})) = \dots = m(V(T_{i+l-1}))$ , since  $V(T_{i+1}), \dots, V(T_{i+l-1})$  are subsets of  $S$ . Finally, as  $m(\lambda_i) = m(V(T_{i+1})) = m(\lambda_{i+l-2})$  and  $\lambda_i, \lambda_{i+l-2} \in S$ , it follows that  $m(G(T_i)) = m(G(T_{i+l}))$ . Consequently, since  $\gamma$  and  $\gamma'$  are segment arcs,  $m(\gamma) = m(G(T_i)) = m(V(T_{i+1})) = m(V(T_{i+l-1})) = m(G(T_{i+l})) = m(\gamma')$ .

Let now  $N$  be a nodal zone of  $X$ . Consider two arcs  $\gamma, \gamma' \in N$  and assume, without loss of generality, that  $\gamma, \gamma' \in V(T)$ . If  $itord(\gamma, \gamma') = \beta$  then  $G(T) \cap G(T_i) \neq \emptyset$  for some  $i$  such that  $\beta_i = \beta$ , where  $T = T(\gamma, \gamma')$ . As  $G(T)$  and  $G(T_i)$  are perfect  $\beta$ -zones,  $G(T) \cap G(T_i)$  is also a perfect  $\beta$ -zone. Lemma 4.2.2 implies that  $G(T) \cap G(T_i)$  contains a segment arc, since  $G(T_i)$  is a constant zone, a contradiction with  $V(T) \subset N$ . Thus,  $itord(\gamma, \gamma') > \beta$  and  $m(\gamma) = m(\gamma')$  by Lemma 4.2.3.  $\square$

**Remark 4.2.3.** If  $X$  is a  $\beta$ -snake then any open zone  $Z$  in  $V(X)$ , and any zone  $Z'$  of order  $\beta' > \beta$ , is a constant zone.

**Proposition 4.2.4.** *Let  $X$  be a  $\beta$ -snake. Then*

1. *For any nodal arc  $\gamma$  we have  $Nod_\gamma = \{\gamma' \in V(X) \mid itord(\gamma, \gamma') > \beta\}$ . In particular, a nodal zone is an open  $\beta$ -complete zone.*
2. *There are no adjacent nodal zones.*
3. *There are finitely many nodal zones in  $V(X)$ .*

*Proof.* (1) Let  $\gamma \in V(X)$  be a nodal arc. Given  $\gamma' \in V(X)$ , if  $itord(\gamma, \gamma') = \beta$  then  $\gamma' \notin Nod_\gamma$ . Indeed, if  $\gamma' \in Nod_\gamma$  and  $itord(\gamma, \gamma') = \beta$  then, since  $V(T(\gamma, \gamma')) \subset Nod_\gamma$  and  $Nod_\gamma$  is a constant zone, by Corollary 4.2.2, every arc in  $G(T(\gamma, \gamma'))$  is a segment arc, a contradiction with  $Nod_\gamma$  being a zone. Thus, a nodal zone is completely determined by any one of its arcs, i.e.,  $Nod_\gamma = \{\gamma' \in V(X) \mid itord(\gamma, \gamma') > \beta\}$ . Therefore, any nodal zone is an open  $\beta$ -complete zone.

(2) This is an immediate consequence of (1) and Remark 2.6.3.

(3) It is a consequence of Proposition 4.2.3 and item (2) of this Proposition.  $\square$

**Corollary 4.2.3.** *If  $X$  is a snake then  $V(X)$  is a disjoint union of finitely many segments and nodal zones.*

**Definition 4.2.5.** Let  $X = T(\gamma_1, \gamma_2)$  be a  $\beta$ -snake. By Definition 4.2.4, the boundary arcs  $\gamma_1$  and  $\gamma_2$  of  $X$  are nodal arcs. The nodal zones  $Nod_{\gamma_1}$  and  $Nod_{\gamma_2}$  are called the *boundary nodal zones*. All other nodal zones are called *interior nodal zones*.

**Proposition 4.2.5.** *Let  $X$  be a  $\beta$ -snake. Then, each interior nodal zone in  $X$  has exactly two adjacent segments, and each segment in  $X$  is adjacent to exactly two nodal zones. Moreover, if  $N$  and  $N'$  are the nodal zones adjacent to a segment  $S$ , then for any arcs  $\gamma \subset N$  and  $\gamma' \subset N'$ , we have  $S = G(T(\gamma, \gamma'))$ .*

*Proof.* Propositions 4.2.4 and 4.2.3 imply that each nodal zone in  $V(X)$  could only be adjacent to a segment  $S$ , and vice versa.

Finally, let  $N$  and  $N'$  be the two nodal zones adjacent to  $S$  and let  $\gamma \in N$  and  $\gamma' \in N'$ . Since each arc in  $T(\gamma, \gamma')$  which has tangency order higher than  $\beta$  with one of the boundary arcs is a nodal arc, by Proposition 4.2.4, each segment arc in  $T(\gamma, \gamma')$  must be in  $G(T(\gamma, \gamma'))$ , and vice versa.  $\square$

**Definition 4.2.6.** Let  $X$  be a  $\beta$ -snake. A *node*  $\mathcal{N}$  in  $X$  is a union of nodal zones in  $X$  such that for any nodal zones  $N, N'$  with  $N \subset \mathcal{N}$  then  $N' \subset \mathcal{N}$  if and only if  $tord(N, N') > \beta$ . Given a node  $\mathcal{N} = \bigcup_{i=1}^m N_i$ , where  $N_i$  are the nodal zones in  $\mathcal{N}$ , the set  $Spec(\mathcal{N}) = \{q_{ij} = tord(N_i, N_j) \mid i \neq j\}$  is called the *spectrum* of  $\mathcal{N}$ .

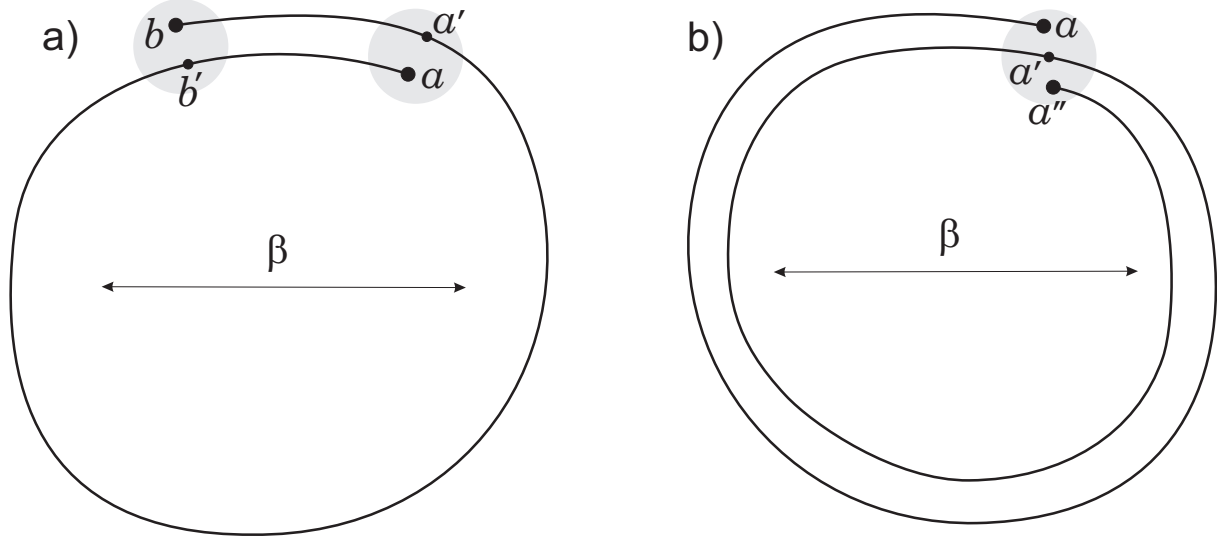
### 4.3 Clusters and cluster partitions

**Definition 4.3.1.** Let  $\mathcal{N}$  and  $\mathcal{N}'$  be nodes of a  $\beta$ -snake  $X$ , and let  $\mathcal{S}(\mathcal{N}, \mathcal{N}')$  be the (possibly empty) set of all segments of  $X$  having adjacent nodal zones in the nodes  $\mathcal{N}$  and  $\mathcal{N}'$  (see Proposition 4.2.5). Two segments  $S$  and  $S'$  in  $\mathcal{S}(\mathcal{N}, \mathcal{N}')$  belong to the same *cluster* if  $tord(S, S') > \beta$ . This defines a *cluster partition* of  $\mathcal{S}(\mathcal{N}, \mathcal{N}')$ . The size of each cluster  $C$  of this partition is equal to the multiplicity of each segment  $S \in C$  (see Definition 4.2.3).

**Remark 4.3.1.** Proposition 4.5.1 below implies that all segments of a spiral snake  $X$  belong to the same cluster. If  $X$  is not a spiral snake, then Proposition 4.6.3 below implies that any two segments of  $X$  adjacent to the same nodal zone do not belong to the same cluster.

**Example 12.** Given relatively prime natural numbers  $p$  and  $q$ , where  $1 < p < q$ , the germ at the origin of the complex curve  $X = \{y^p = x^q\} \subset \mathbb{C}^2$ , considered as a real surface in  $\mathbb{R}^4$ , is an example of a circular 1-snake with a single segment and no nodes. Removing from  $X$  the Hölder triangle  $T = \{(x, y) \in \mathbb{C}^2 : 0 \leq \arg(x) \leq \pi/q, 0 \leq \arg(y) \leq \pi/p\}$ , and taking the closure, one obtains a 1-snake  $X'$  with  $p$  segments of multiplicity  $p$  and  $p - 1$  segments of multiplicity  $p - 1$ . Each of the two nodes  $\mathcal{N}$  and  $\mathcal{N}'$  of  $X'$  has multiplicity  $p$ , and its spectrum consists of a single exponent  $q/p$ . The set  $\mathcal{S}(\mathcal{N}, \mathcal{N}')$  is partitioned into two clusters of sizes  $p$  and  $p - 1$ .

Figure 8 – Links of snakes with segments of multiplicity two. a) has two nodes; b) a spiral snake and its single node. Shaded disks represent arcs with tangency order higher than  $\beta$ .



Source: elaborated by the author.

**Example 13.** Fig. 8a represents the link of a  $\beta$ -snake  $X$  with three segments,  $S = G(T(a', b'))$ ,  $S' = G(T(b', a))$  and  $S'' = G(T(b, a'))$ , such that  $m(S) = 1$  and  $m(S') = m(S'') = 2$ , and two nodes  $\mathcal{N} = \text{Nod}_a \cup \text{Nod}_{a'}$  and  $\mathcal{N}' = \text{Nod}_b \cup \text{Nod}_{b'}$ . If  $\beta = 1$ ,  $\text{tord}(\gamma, T(b, a')) = 3/2$  for all arcs  $\gamma \subset T(b, a')$  and  $\text{tord}(\gamma', T(b', a)) = 3/2$  for all arcs  $\gamma' \subset T(b', a)$  then the link of  $X$  is outer metric equivalent to the link of the snake  $X'$  in Example 12 with  $p = 2$  and  $q = 3$ .

**Example 14.** Fig. 8b represents the link of a  $\beta$ -snake  $X'$  containing two segments,  $S = G(T(a, a'))$  and  $S' = G(T(a', a''))$  such that  $m(S) = m(S') = 2$ , and a single node  $\mathcal{N} = \text{Nod}_a \cup \text{Nod}_{a'} \cup \text{Nod}_{a''}$ . All three segments of  $X'$  belong to a single cluster in  $\mathcal{S}(\mathcal{N}, \mathcal{N})$ .

#### 4.4 Segments and nodal zones with respect to a pancake

**Definition 4.4.1.** Let  $X$  be a  $\beta$ -snake, and  $\{X_k\}_{k=1}^p$  a pancake decomposition of  $X$ . If  $\mu(X_j) = \beta$  we define the functions  $f_1, \dots, f_p$ , where  $f_l: (X_j, 0) \rightarrow (\mathbb{R}, 0)$  is given by  $f_l(x) = d(x, X_l)$ . For each  $l$  we define  $m_l: V(X_j) \rightarrow \{0, 1\}$  as follows:  $m_l(\gamma) = 1$  if and only if  $\text{ord}_\gamma f_l > \beta$  and  $m_l(\gamma) = 0$  otherwise. In particular,  $m_j \equiv 1$ .

**Remark 4.4.1.** Consider  $m_1, \dots, m_p$  as in Definition 4.4.1. For each  $\gamma \in G(X_j)$  we have  $m(\gamma) = \sum_{l=1}^p m_l(\gamma)$ .

**Definition 4.4.2.** Consider  $m_1, \dots, m_p$  as in Definition 4.4.1. A zone  $Z \subset V(X_j)$  is *constant with respect to  $X_l$*  if  $m_l|_Z$  is constant.

**Definition 4.4.3.** Let  $m_1, \dots, m_p$  be as in Definition 4.4.1. Consider an arc  $\gamma \in G(X_j)$ . For each  $l$  we say that  $\gamma$  is a *segment arc with respect to  $X_l$*  if there exists a  $\beta$ -Hölder triangle  $T$  such that  $\gamma$  is a generic arc of  $T$  and  $V(T)$  is constant with respect to  $X_l$ . Otherwise  $\gamma$  is a *nodal arc with respect to  $X_l$* . The set of segment arcs in  $G(X_j)$  with respect to  $X_l$  and the set of nodal arcs in  $G(X_j)$  with respect to  $X_l$  are denoted by  $\mathbf{S}_l(X_j)$  and  $\mathbf{N}_l(X_j)$ , respectively. Furthermore, a *segment with respect to  $X_l$*  is a zone  $S_{l,j}$  maximal in  $\mathbf{S}_l(X_j)$ , and a *nodal zone with respect to  $X_l$*  is a zone  $N_{l,j}$  maximal in  $\mathbf{N}_l(X_j)$ . We write  $Seg_\gamma^{l,j}$  or  $Nod_\gamma^{l,j}$  for a segment or a nodal zone with respect to  $X_l$  in  $G(X_j)$  containing an arc  $\gamma$ .

**Remark 4.4.2.** Let  $f_1, \dots, f_p$  be as in Definition 4.4.1. Propositions 4.2.2, 4.2.3 and 4.2.4 remain valid for segments and nodal zones in  $G(X_j)$  with respect to  $X_l$ .

In particular, taking  $f = f_l$  and  $T = X_j$ , segments in  $G(X_j)$  with respect to  $X_l$  are in a one-to-one correspondence with the maximal perfect zones in  $B_\beta(f_l)$  and  $H_\beta(f_l)$ . Similarly, the nodal zones with respect to  $X_l$  are in a one-to-one correspondence with the open  $\beta$ -complete zones in  $H_\beta(f_l)$  (see Propositions 3.0.1 and 3.0.2).

**Lemma 4.4.1.** Let  $X$  be a  $\beta$ -snake, and let  $f_1, \dots, f_p$  be as in Definition 4.4.1.

(1) Let  $l, l' \in \{1, \dots, p\} \setminus \{j\}$  with  $l \neq l'$ . Then, any two nodal zones  $N_{l,j}$  and  $N_{l',j}$  either coincide or are disjoint. If  $N_{l,j} \cap N_{l',j} = \emptyset$  then  $\text{tord}(N_{l,j}, N_{l',j}) = \beta$ .

(2) If  $\gamma \in G(X_j)$  is a segment arc of  $X$  then  $\gamma$  is a segment arc with respect to  $X_l$ .

*Proof.* (1) This is an immediate consequence of Remark 4.4.2 and Remark 2.6.3.

(2) Suppose that  $\gamma \in G(X_j)$  belong to a segment  $S$  of  $X$ , and there exists  $l \neq j$  such that  $\gamma$  is a nodal arc with respect to  $X_l$ . Remark 4.4.2 implies that

$$Nod_\gamma^{l,j} = \{\gamma' \in V(X_j) : \text{tord}(\gamma, \gamma') > \beta\} \subset H_\beta(f_l).$$

There is a perfect  $\beta$ -zone  $B_l \subset B_\beta(f_l)$  adjacent to  $Nod_\gamma^{l,j}$ . Let  $\lambda_1 \in S \cap B_l$ . Then,  $m_l(\gamma) = 1$  and  $m_l(\lambda_1) = 0$ . As  $\gamma, \lambda_1 \in S$ , it follows that  $m(\gamma) = m(\lambda_1)$ . Thus, Remark 4.4.1 implies that there is  $l_1 \in \{1, \dots, p\} \setminus \{l, j\}$  such that  $m_{l_1}(\lambda_1) = 1$  and  $m_{l_1}(\gamma) = 0$ .

As  $\gamma \in B_\beta(f_{l_1})$ , by Proposition 3.0.1, there is a perfect  $\beta$ -zone  $B_{l_1}$  maximal in  $B_\beta(f_{l_1})$  containing  $\gamma$ . Thus, there is  $\lambda_2 \in (B_l \cap B_{l_1}) \cap V(T(\lambda_1, \gamma))$ . In particular  $\lambda_2 \in S$ .

As  $\gamma, \lambda_2 \in S$  it follows that  $m(\gamma) = m(\lambda_2)$ . Then, as  $m_l(\lambda_2) = m_{l_1}(\lambda_2) = 0$ , by Remark 4.4.1, there is  $l_2 \in \{1, \dots, p\} \setminus \{l, l_1, j\}$  such that  $m_{l_2}(\lambda_2) = 1$  and  $m_{l_2}(\gamma) = m_{l_2}(\lambda_1) = 0$ .

Similarly, as  $\gamma \in B_\beta(f_{l_2})$ , there are a perfect  $\beta$ -zone  $B_{l_2}$  maximal in  $B_\beta(f_{l_2})$  containing  $\gamma$  and an arc  $\lambda_3 \in (B_l \cap B_{l_1} \cap B_{l_2}) \cap V(T(\lambda_1, \gamma))$ . In particular  $\lambda_3 \in S$ .

Continuing with this process, after at most  $p - 1$  steps, we get a contradiction.  $\square$

**Corollary 4.4.1.** *Let  $X$  be a  $\beta$ -snake,  $\{X_k\}_{k=1}^p$  a pancake decomposition of  $X$ , and  $S \subset V(X)$  a segment. If  $\gamma, \lambda \in S \cap G(X_j)$  then  $m_l(\gamma) = m_l(\lambda)$  for all  $l$ .*

*Proof.* Given arcs  $\gamma, \lambda \in S \cap G(X_j)$ , by Lemma 4.4.1,  $\gamma \in \text{Seg}_\lambda^{l,j}$  for all  $l$ . As a segment in  $G(X_j)$  with respect to  $X_j$  is a constant zone, it follows that  $m_l(\gamma) = m_l(\lambda)$ .  $\square$

**Proposition 4.4.1.** *Let  $X$  be a  $\beta$ -snake. Then*

1. *If  $\text{tord}(\gamma, \gamma') > \text{itord}(\gamma, \gamma')$  for some  $\gamma, \gamma' \in V(X)$ , and  $\gamma$  is a nodal arc, then  $\gamma'$  is also a nodal arc.*
2. *Each node of  $X$  has at least two nodal zones.*

*Proof.* (1) Consider  $\gamma, \gamma' \in V(X)$  such that  $\text{tord}(\gamma, \gamma') > \text{itord}(\gamma, \gamma')$  and  $\gamma$  is a nodal arc. Clearly,  $\gamma' \notin \text{Nod}_\gamma$ , since  $\text{itord}(\gamma, \gamma') = \beta$ . Suppose, by contradiction, that  $\gamma'$  is a segment arc, say  $\gamma' \in S$  where  $S = \text{Seg}_{\gamma'}$ . Let  $\{X_k\}$  be a reduced pancake decomposition of  $X$ . Assume that  $\gamma \in X_j$  and  $\gamma' \in X_{j'}$ . Since each pancake is normally embedded,  $j \neq j'$ . As  $\gamma'$  is a segment arc we can assume that  $\gamma' \in G(X_{j'})$ . Consider arcs  $\theta'_1 \in S \cap G(T(\lambda_{j'-1}, \gamma'))$  and  $\theta'_2 \in S \cap G(T(\gamma', \lambda_{j'}))$ . Since  $\theta'_1, \theta'_2, \gamma' \in S$  and  $m_j(\gamma') = 1$ , by Corollary 4.4.1,  $m_j(\theta'_1) = m_j(\theta'_2) = 1$ . Thus, there exist arcs  $\theta_1, \theta_2 \in V(X_j)$  such that  $\text{tord}(\theta_i, \theta'_i) > \beta$ , for  $i = 1, 2$ ,  $T = T(\theta_1, \theta_2)$  is a  $\beta$ -Hölder triangle and  $\gamma$  is a generic arc of  $T$ . Then,  $V(T) \subset H_\beta(f_{j'})$ , what implies that  $\gamma$  is segment arc with respect to  $X_{j'}$ , a contradiction with Remark 4.4.2, since if a nodal arc belongs to a zone contained in  $H_\beta(f_{j'})$ , this zone should be an open  $\beta$ -complete zone.

(2) Let  $\mathcal{N}$  be a node of  $X$ , and  $N$  a nodal zone of  $\mathcal{N}$ . By Remark 4.4.2, given  $\gamma \in N$  there exists  $\gamma' \in V(X) \setminus N$  such that  $\text{tord}(\gamma, \gamma') > \text{itord}(\gamma, \gamma')$ , since  $N \subset H_\beta(f_l)$  for some  $l$ . By item (1) of this Proposition,  $\gamma'$  is a nodal arc and  $\text{Nod}_{\gamma'} \neq N$  is a nodal zone of  $\mathcal{N}$ .  $\square$

## 4.5 Bubbles, bubble snakes and spiral snakes

**Definition 4.5.1.** A  $\beta$ -bubble is a non-singular  $\beta$ -Hölder triangle  $X = T(\gamma_1, \gamma_2)$  such that there exists an interior arc  $\theta$  of  $X$  with both  $X_1 = T(\gamma_1, \theta)$  and  $X_2 = T(\theta, \gamma_2)$  normally embedded and  $\text{tord}(\gamma_1, \gamma_2) > \text{itord}(\gamma_1, \gamma_2)$ . If  $X$  is a snake then it is called a  $\beta$ -bubble snake.

**Remark 4.5.1.** It follows from Lemma 2.3.1 that if  $X$  is a  $\beta$ -bubble then  $X_1$  and  $X_2$  are  $\beta$ -Hölder triangles.

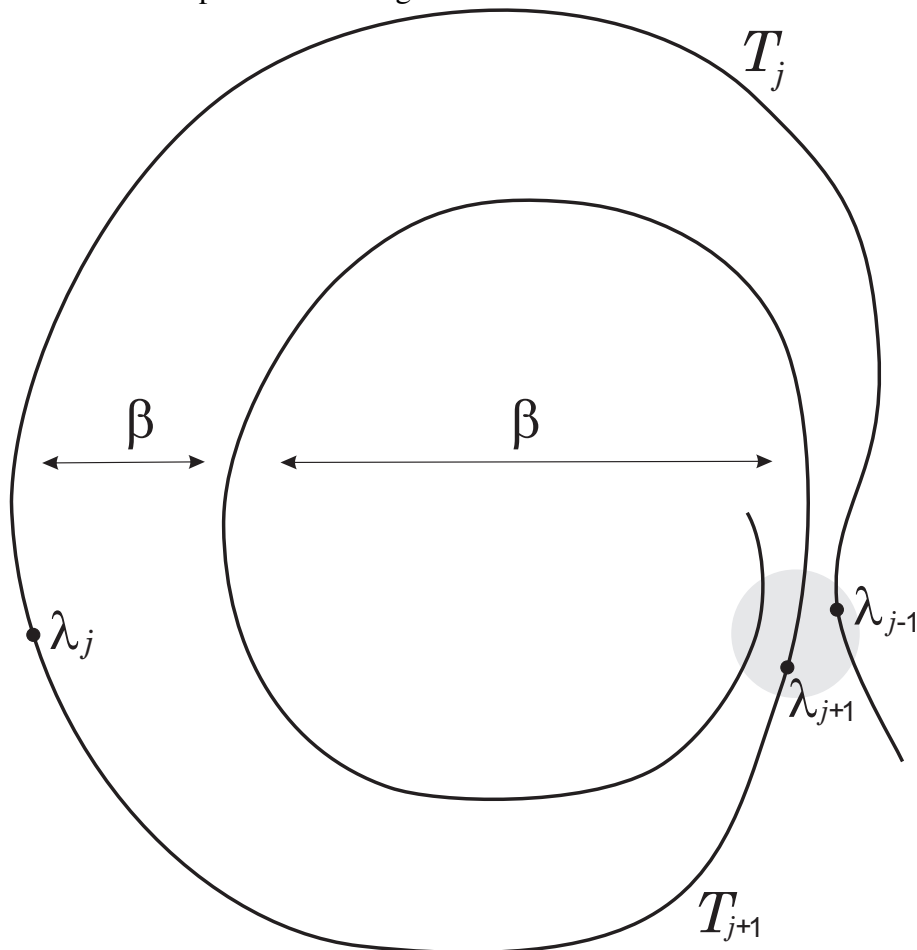
**Definition 4.5.2.** A *spiral  $\beta$ -snake*  $X$  is a  $\beta$ -snake with a single node and two or more segments (see Fig. 8b).

**Example 15.** Instead of removing the Hölder triangle  $T$  from a complex curve as in Example 12, remove an  $\alpha$ -Hölder triangle  $T'$  with  $\alpha > 1$  contained in  $X$ . Then  $X'' = \overline{X \setminus T'}$  is a spiral snake with  $p$  segments.

**Remark 4.5.2.** Any snake with a single node and  $p$  segments is either a bubble snake if  $p = 1$  or a spiral snake if  $p > 1$ .

**Proposition 4.5.1.** Let  $X$  be a spiral  $\beta$ -snake. Then, for each segment arc  $\gamma$  in  $X$  and for each segment  $S \neq \text{Seg}_\gamma$  of  $X$ ,  $\text{tord}(\gamma, S) > \beta$ .

Figure 9 – Contradictory case in the proof of Proposition 4.5.1. The shaded disk represents the single node of  $X$ .



Source: elaborated by the author.

*Proof.* First, we are going to prove that if  $X$  is a spiral  $\beta$ -snake and  $S, S'$  are consecutive segments of  $X$ , then  $\text{tord}(\gamma, S') > \beta$  for each  $\gamma \in S$ . Let  $N$  be the nodal zone adjacent to both  $S$  and  $S'$ ,



and  $\tilde{N}, \tilde{N}'$  the other nodal zones adjacent to  $S$  and  $S'$ , respectively. Consider arcs  $\lambda \in N, \tilde{\lambda} \in \tilde{N}, \tilde{\lambda}' \in \tilde{N}'$ , and the  $\beta$ -Hölder triangles  $T = T(\tilde{\lambda}, \lambda)$  and  $T' = T(\lambda, \tilde{\lambda}')$ . Proposition 4.2.5 implies that  $S = G(T)$  and  $S' = G(T')$ . Consider the germ of the function  $f: (T, 0) \rightarrow (\mathbb{R}, 0)$  given by  $f(x) = d(x, T')$ . Let  $\{T_i = T(\lambda_{i-1}, \lambda_i)\}_{i=1}^p$  be a minimal pizza on  $T$  associated with  $f$ . It is enough to show that  $\text{ord}_{\lambda_i} f > \beta$  for each  $i = 0, \dots, p$ .

Suppose, by contradiction, that there is  $j \in \{0, \dots, p\}$  such that  $\text{ord}_{\lambda_j} f = \beta$ . Since a spiral snake has a single node  $\mathcal{N}$ , both  $\lambda_0$  and  $\lambda_p$  belong to  $\mathcal{N}$ . Thus,  $\text{ord}_{\lambda_0} f > \beta, \text{ord}_{\lambda_p} f > \beta$  and  $0 < j < p$ . Then  $p > 1$  and Lemma 3.0.4 implies that  $\text{ord}_{\lambda_{j-1}} f > \beta$  and  $\text{ord}_{\lambda_{j+1}} f > \beta$ . We claim that both  $\lambda_{j-1}$  and  $\lambda_{j+1}$  do not belong to  $S$  and consequently, since  $X$  is a spiral snake, are nodal arcs (see Fig. 9). Assume that  $\lambda_{j+1} \in S$  (if  $\lambda_{j-1} \in S$  we obtain a similar contradiction). Let  $\{X_k\}$  be a pancake decomposition of  $X$  such that  $\lambda_{j+1} \in G(X_k), X_k \subset T$  and  $\mu(X_k) = \beta$ . As  $\text{ord}_{\lambda_{j+1}} f > \beta$ ,  $T$  is not normally embedded and  $\{X_k\}$  is a pancake decomposition, there exists a pancake  $X_l, X_l \neq X_k$ , such that  $X_l \cap T' \neq \emptyset$  and  $\text{tord}(\lambda_{j+1}, X_l \cap T') > \beta$ . Since  $\lambda_{j+1} \in S$ , Lemma 4.4.1 implies that  $\lambda_{j+1}$  is a segment arc in  $G(X_k)$  with respect to  $X_l$ . Thus, since  $\text{ord}_{\lambda_{j+1}} f_l = \text{ord}_{\lambda_{j+1}} f > \beta$ , Remark 4.4.2 implies that  $\lambda_{j+1}$  is contained in a perfect  $\beta$ -zone  $H$  maximal in  $H_\beta(f_l)$ . Hence,  $H \cap G(T_{j+1}) \neq \emptyset$ , a contradiction with  $G(T_{j+1}) \subset B_\beta(f_l) \subset B_\beta(f)$ , by Proposition 2.5.1.

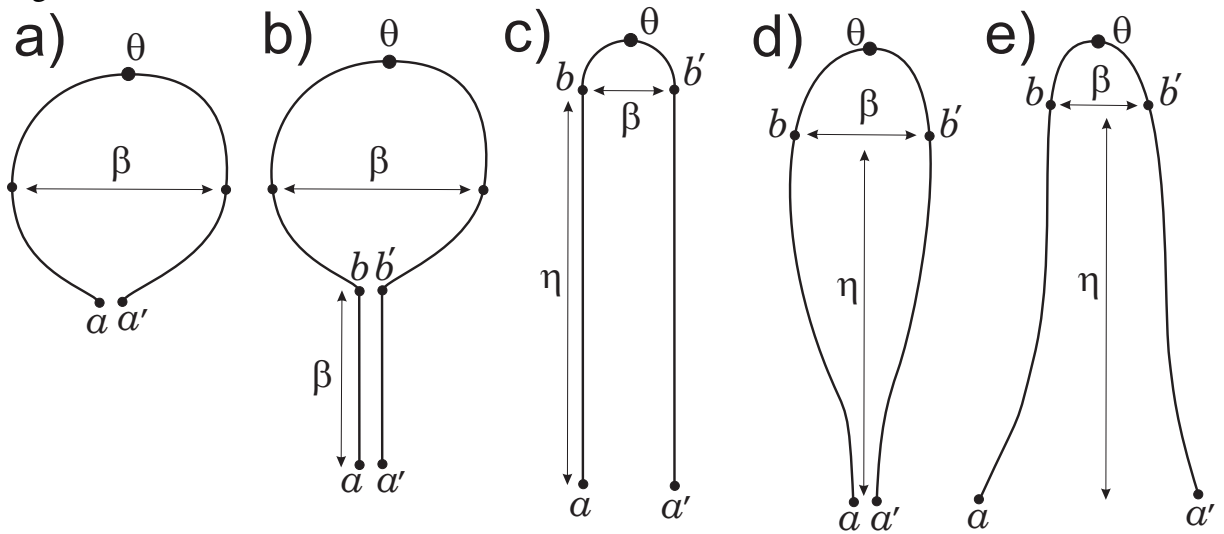
Then, for every  $\lambda' \in S, \text{ord}_{\lambda'} f = \beta$  and consequently,  $\text{tord}(S, S') = \beta$ , a contradiction with the arc  $\theta = \lambda_{j+1}$ , in an interior nodal zone, being abnormal. To show this, suppose that  $\theta$  is normal and consider arcs  $\theta_1 \in V(T(\gamma_1, \theta))$  and  $\theta_2 \in V(T(\theta, \gamma_2))$ , where  $X = T(\gamma_1, \gamma_2)$ , such that  $T(\theta_1, \theta)$  and  $T(\theta, \theta_2)$  are normally embedded  $\beta$ -Hölder triangles which intersection is  $\theta$  and  $\text{tord}(\theta_1, \theta_2) > \text{itord}(\theta_1, \theta_2)$ . Since  $T(\theta_1, \theta)$  and  $T(\theta, \theta_2)$  are normally embedded and  $X$  has a single node, since it is a spiral snake, both  $\theta_1$  and  $\theta_2$  are in  $S \cup S'$ , say  $\theta_1 \in S$  and  $\theta_2 \in S'$ . However,  $\text{tord}(S, S') = \beta$  thus  $\text{tord}(\theta_1, \theta_2) = \beta$ , a contradiction with  $\text{tord}(\theta_1, \theta_2) > \beta = \text{itord}(\theta_1, \theta_2)$ .

Finally, given two non-necessarily consecutive segments  $S$  and  $S' = \text{Seg}_\gamma$  as in the Proposition 4.5.1, the result follows from the non-archimedean property.  $\square$

**Example 16.** Fig. 10a shows the link of a  $\beta$ -bubble snake  $X_a$ , with  $\text{tord}(a, a') > \beta$ .

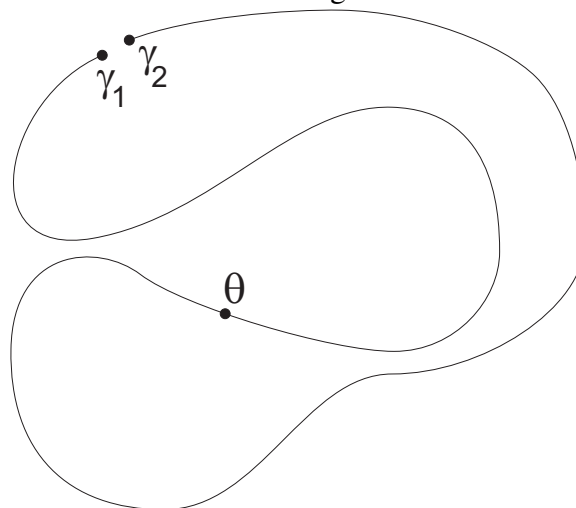
Fig. 10b shows the link of a  $\beta$ -bubble  $X_b$  with a “neck” consisting of two normally embedded  $\beta$ -Hölder triangles  $T = T(a, b)$  and  $T' = T(a', b')$  such that  $\text{tord}(\gamma, T') > \beta$  for all arcs  $\gamma \in V(T)$  and  $\text{tord}(\gamma, T) > \beta$  for all arcs  $\gamma \in V(T')$ . Since all arcs in  $T$  and  $T'$  are normal,  $X_b$  is not a snake, although it does contain a  $\beta$ -bubble snake  $X_a$ .

Figure 10 – Links of a bubble snake and non-snake bubbles.



Source: elaborated by the author.

Figure 11 – Link of a non-snake bubble containing a non-bubble snake.



Source: elaborated by the author.

Figs. 10c, 10d and 10e show the links of non-snake  $\eta$ -bubbles  $X_c$ ,  $X_d$  and  $X_e$ , respectively, with  $tord(a, a') > \eta$ . The set of abnormal arcs in each of them is a perfect  $\beta$ -complete abnormal zone  $Z$ . In each of these three figures  $T(b, b')$  is a normally embedded  $\beta$ -Hölder triangle, while  $T = T(a, b)$  and  $T' = T(a', b')$  are normally embedded  $\eta$ -Hölder triangles where  $\eta < \beta$ . For each arc  $\gamma \in V(T) \setminus Z$  we have  $tord(\gamma, T') = \beta$  in  $X_c$ ,  $tord(\gamma, T') > \beta$  in  $X_d$ , and  $tord(\gamma, T') < \beta$  in  $X_e$ .

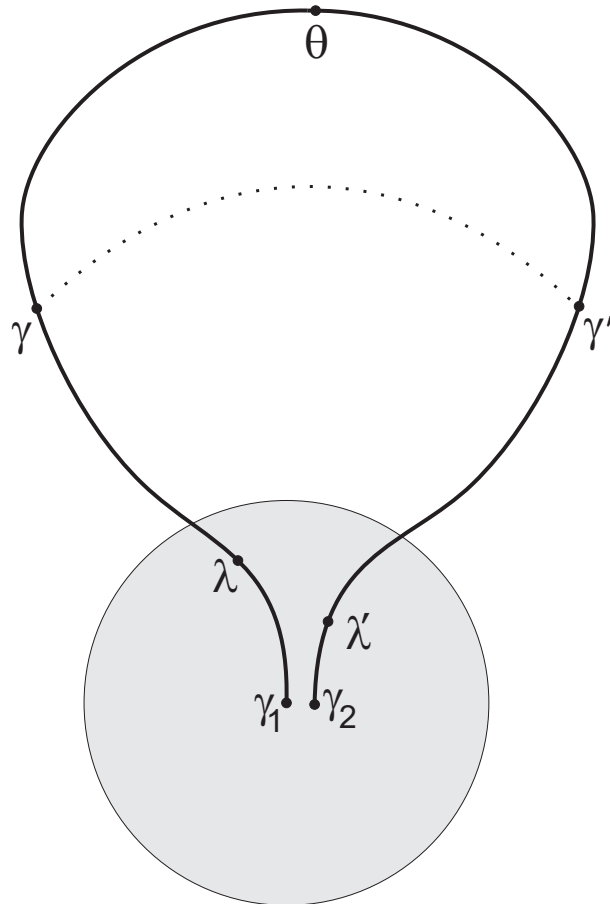
**Example 17.** Fig. 11 shows the link of a non-snake  $\beta$ -bubble containing a non-bubble  $\beta$ -snake with the same link as in Fig. 3b.

**Proposition 4.5.2.** *Let  $X$  be a  $\beta$ -bubble snake as in Definition 4.5.1. If  $\gamma \in G(X_1)$  then*

$$\text{tord}(\gamma, X_2) = \beta.$$

*Proof.* Suppose, by contradiction, that  $\text{tord}(\gamma, X_2) > \beta$ . Then there is an arc  $\gamma' \in G(X_2)$  such that  $\text{tord}(\gamma, \gamma') > \beta$ . Choose  $b > 0$  (by Corollary 4.2.1 such a real number exists) so that  $T = X_1 \cap H_{b, \beta}(\gamma_1)$  and  $T' = X_2 \cap H_{b, \beta}(\gamma_1)$  are Hölder triangles (in particular,  $T$  and  $T'$  are connected),  $\gamma \notin T$ ,  $\gamma' \notin T'$ . Next, choose  $\lambda \in G(T)$  so that  $T(\gamma_1, \lambda) \subset H_{b/2, \beta}(\gamma_1)$  (see Fig. 12). Then any arc  $\lambda' \subset X_2$  such that  $\text{tord}(\lambda', T(\gamma_1, \lambda)) > \beta$  must belong to  $T'$ . Note that  $T$  has exponent  $\beta$ . Thus, since  $X$  is a snake and  $\lambda \in G(T) \subset G(X)$ , the arc  $\lambda$  is abnormal: there exist normally embedded triangles  $\tilde{T} \subset T(\gamma_1, \lambda)$  and  $\tilde{T}' \subset T(\lambda, \gamma_2)$  such that  $\tilde{T} \cup \tilde{T}'$  is not normally embedded. Since  $X_1$  is normally embedded,  $\theta \in \tilde{T}'$ , thus  $\gamma \in \tilde{T}'$  and both  $\tilde{T}$  and  $\tilde{T}'$  are  $\beta$ -Hölder triangles. Since  $\tilde{T} \cup \tilde{T}'$  is not normally embedded, there exists an arc  $\lambda' \subset \tilde{T}' \cap X_2$  such that  $\text{tord}(\lambda', T(\gamma_1, \lambda)) > \beta$ . Then  $\lambda' \subset T'$ , which implies that  $\gamma' \in \tilde{T}'$ , in contradiction to  $\tilde{T}'$  being normally embedded, as  $\text{itord}(\gamma, \gamma') = \beta$  and  $\text{tord}(\gamma, \gamma') > \beta$ .

Figure 12 – The shaded disk represents a  $\beta$ -horn neighborhood of  $\gamma_1$  in the proof of Proposition 4.5.2.



Source: elaborated by the author.

□

**Proposition 4.5.3.** *If  $X$  is a  $\beta$ -bubble snake as in Definition 4.5.1 then*

1.  $V(X)$  consists of a single segment  $S$  of multiplicity 1 and a single node  $\mathcal{N}$  with two boundary nodal zones.
2. For any generic arc  $\gamma$  of  $X$ , both  $T(\gamma_1, \gamma)$  and  $T(\gamma, \gamma_2)$  are normally embedded.
3. Any reduced pancake decomposition of  $X$  has exactly two pancakes.

*Proof.* (1) Let  $f : (X_1, 0) \rightarrow (\mathbb{R}, 0)$  be the germ of the Lipschitz function given by  $f(x) = d(x, X_2)$ . Note that if  $\gamma \in G(X_1)$  then  $\text{ord}_\gamma f > \beta$  if and only if  $\text{tord}(\gamma, X_2) > \beta$ . Thus, by Proposition 4.5.2, the result follows.

(2) Let  $\gamma$  be a generic arc of  $X$ . Let  $\tilde{T}_1 = T(\gamma_1, \gamma)$  and  $\tilde{T}_2 = T(\gamma, \gamma_2)$ . From the definition of a bubble, there exists a generic arc  $\theta$  of  $X$  such that  $X_1 = T(\gamma_1, \theta)$  and  $X_2 = T(\theta, \gamma_2)$  are normally embedded.

Suppose that  $\tilde{T}_2 \subset X_2$ . We are going to prove that  $\tilde{T}_1$  is normally embedded. The case when  $\tilde{T}_1 \subset X_1$  and  $\tilde{T}_2$  is not normally embedded is similar.

If  $\tilde{T}_1$  is not normally embedded then there exist arcs  $\lambda \in V(X_1)$  and  $\lambda' \in V(\tilde{T}_1) \setminus V(X_1)$  such that  $\text{tord}(\lambda, \lambda') > \text{itord}(\lambda, \lambda')$ . Note that  $\lambda'$  is generic and consequently abnormal. This implies that  $\lambda$  is also abnormal.

Thus,  $\lambda$  and  $\lambda'$  must be generic arcs of  $X$ . but this implies, by (1), that  $m(\lambda) = m(\lambda') = 1$ , a contradiction with  $\text{tord}(\lambda, \lambda') > \text{itord}(\lambda, \lambda')$ .

(3) This is an immediate consequence of item (2) of this Proposition. □

#### 4.6 Pancake decomposition defined by segments and nodal zones

**Proposition 4.6.1.** *Let  $X = T(\gamma_1, \gamma_2)$  be a snake,  $S$  a segment of  $X$  and  $N \neq N'$  two nodal zones of  $X$  adjacent to  $S$ . Then*

1. If  $\gamma$  and  $\gamma'$  are two arcs in  $N$  then  $T(\gamma, \gamma')$  is normally embedded.
2. If  $\gamma$  and  $\gamma'$  are two arcs in  $S$  then  $T(\gamma, \gamma')$  is normally embedded.
3. If  $\gamma \in S$  and  $\gamma' \in N$  then  $T(\gamma, \gamma')$  is normally embedded.
4. If  $\gamma \in N$  and  $\gamma' \in N'$  then  $T(\gamma, \gamma')$  is normally embedded, unless  $X$  is either a bubble snake or a spiral snake.

*In particular, nodal zones and segments are normally embedded zones (see Definition 2.6.4).*

*Proof.* (1) Let  $\gamma$  and  $\gamma'$  be two arcs in  $N$ . Note that, by Proposition 4.2.4,  $T(\gamma, \gamma')$  has exponent greater than  $\beta$ . Then, there are no arcs  $\lambda$  and  $\lambda'$  in  $N$  such that  $tord(\lambda, \lambda') > itord(\lambda, \lambda')$ , otherwise, by Proposition 4.1.1,  $itord(\lambda, \lambda') = \beta$ , a contradiction with exponent of  $T(\gamma, \gamma')$  greater than  $\beta$ .

To prove the next two items it is enough to show that there are no arcs  $\gamma \in S$  and  $\gamma' \in S \cup N$  such that  $tord(\gamma, \gamma') > itord(\gamma, \gamma')$ . Let us assume that  $\gamma' \subset T(\gamma_1, \gamma)$ .

(2) and (3) Suppose, by contradiction, that there exist such arcs  $\gamma$  and  $\gamma'$ . As  $\gamma \in S$  it is abnormal and then there are arcs  $\lambda \subset T(\gamma, \gamma')$  and  $\lambda' \subset T(\gamma, \gamma_2)$  such that  $T = T(\lambda, \gamma)$  and  $T' = T(\gamma, \lambda')$  are normally embedded (non-singular)  $\beta$ -Hölder triangles such that  $T \cap T' = \gamma$  and  $tord(\lambda, \lambda') > itord(\lambda, \lambda')$  (see Fig. 13). Let  $\{X_k\}$  be a reduced pancake decomposition of  $X$ . We can assume that  $\lambda \in V(X_j)$  and  $\lambda' \in V(X_{j+1})$ , since none of these arcs is in a nodal boundary zone and consequently, if necessary, we could enlarge the pancake attaching a  $\beta$ -Hölder triangle to one of its boundaries.

Since  $T$  is normally embedded,  $\lambda \in S$ . So, we can assume that  $\lambda \in G(X_j)$ . We can further assume that  $\gamma \in G(X_j)$ , since  $\gamma \in S$ . Thus, since, by Corollary 4.4.1,  $m_{j+1}(\gamma) = m_{j+1}(\lambda) = 1$ , there exists  $\gamma'' \in V(\lambda_j, \lambda')$  such that  $tord(\gamma, \gamma'') > \beta = itord(\gamma, \gamma'')$ , a contradiction with  $T'$  being normally embedded.

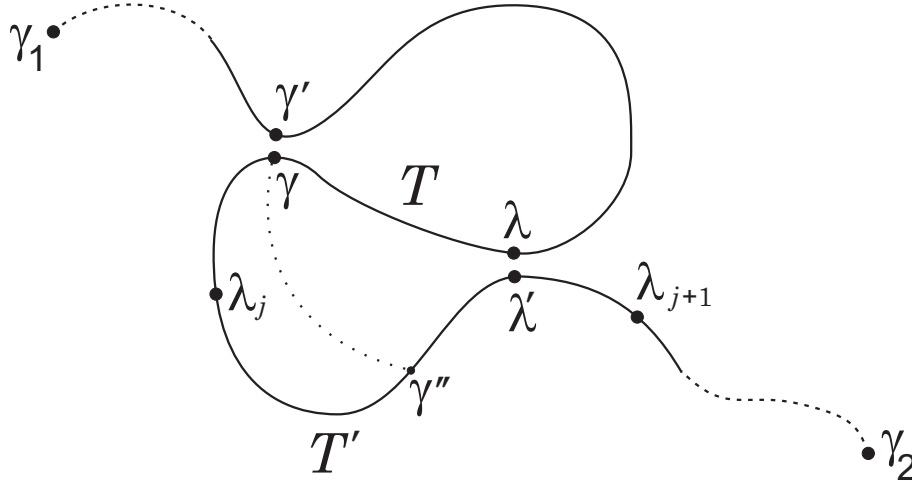
(4) Note that as a spiral snake has a single node, the result is trivially false in this case. Thus, assume that  $X$  is not a  $\beta$ -spiral snake. Suppose, by contradiction, that there exist arcs  $\gamma \in N$  and  $\gamma' \in N'$  such that  $tord(\gamma, \gamma') > itord(\gamma, \gamma')$ . If  $X$  is not a bubble snake then we can assume that one of the arcs  $\gamma, \gamma'$ , say  $\gamma$ , is abnormal. As  $\gamma$  is abnormal there are arcs  $\lambda \subset T(\gamma, \gamma')$  and  $\lambda' \subset T(\gamma, \gamma_2)$  such that  $T = T(\lambda, \gamma)$  and  $T' = T(\gamma, \lambda')$  are normally embedded  $\beta$ -Hölder triangles such that  $T \cap T' = \gamma$  and  $tord(\lambda, \lambda') > itord(\lambda, \lambda')$ . Let  $\{X_k\}$  be a reduced pancake decomposition of  $X$ . We can assume that  $\lambda \in V(X_j)$  and  $\lambda' \in V(X_{j+1})$ .

Since  $T$  is normally embedded, we have  $\lambda \in S$ . Thus, we can assume that  $\lambda \in G(X_j)$ . As  $m_{j+1}(\lambda) = 1$ , Lemma 4.4.1 and Proposition 3.0.2 imply that  $N \subset H_\beta(f_{j+1})$  and  $\lambda$  belong to a perfect  $\beta$ -zone maximal in  $H_\beta(f_{j+1})$  (the segment with respect to  $X_{j+1}, S_\lambda^{j+1}$ ) adjacent to  $N$ . Then, there exists  $\gamma'' \in V(\lambda_j, \lambda')$  such that  $tord(\gamma, \gamma'') > \beta = itord(\gamma, \gamma'')$ , a contradiction with  $T'$  normally embedded.

□

**Proposition 4.6.2.** *The following decomposition of a snake  $X$  other than the bubble and the spiral into Hölder triangles determines a pancake decomposition of  $X$ : the boundary arcs of the*

Figure 13 – Position of arcs in the proof of Proposition 4.6.1



Source: elaborated by the author.

*Hölder triangles in the decomposition are the two boundary arcs of  $X$  together with one arc in each nodal zone. The segments of  $X$  are in one-to-one correspondence with the sets of generic arcs of its pancakes.*

*Proof.* This is an immediate consequence of Proposition 4.6.1. □

**Remark 4.6.1.** In general, the pancake decomposition described in Proposition 4.6.2 is not minimal. Moreover, if  $\{X_k = T(\gamma_{k-1}, \gamma_k)\}_{k=1}^p$  is the pancake decomposition defined in Proposition 4.6.2 then  $X$  has exactly  $p$  segments  $S_i = G(X_i)$ ,  $i = 1, \dots, p$  and  $p + 1$  nodal zones  $Nod_{\gamma_j}$ ,  $j = 0, \dots, p$ .

**Proposition 4.6.3.** *Let  $X$  be a snake other than the spiral,  $N$  a nodal zone of  $X$  and  $S \neq S'$  two segments of  $X$  adjacent to  $N$ . If  $\gamma \in S$  and  $\gamma' \in S'$  then  $T(\gamma, \gamma')$  is normally embedded.*

*Proof.* Let  $\tilde{N}$  and  $\tilde{N}'$  be the nodal zones, distinct from  $N$ , adjacent to  $S$  and  $S'$ , respectively. Consider arcs  $\tilde{\gamma} \in \tilde{N}$ ,  $\tilde{\gamma}' \in \tilde{N}'$  and  $\theta \in N$ . As  $X$  is not a spiral snake, by item (4) of Proposition 4.6.2, we can assume that  $T = T(\tilde{\gamma}, \theta)$  and  $T' = T(\theta, \tilde{\gamma}')$  are pancakes of a reduced pancake decomposition. Proposition 4.2.5 implies that  $S = G(T)$  and  $S' = G(T')$ .

Then, if there were arcs  $\gamma \in S$  and  $\gamma' \in S'$  such that  $tord(\gamma, \gamma') > itord(\gamma, \gamma')$ , by Corollary 4.4.1, we would have that for each arc  $\lambda \in S$  there should exist  $\lambda' \in S'$  such that  $tord(\lambda, \lambda') > itord(\lambda, \lambda')$ . This implies that  $tord(\tilde{N}, \tilde{N}') > \beta$ , but the arcs in  $S$  should be abnormal, a contradiction. □

**Proposition 4.6.4.** *Let  $S$  be a segment of a  $\beta$ -snake  $X$  such that the nodal zones adjacent to  $S$  belong to distinct nodes  $\mathcal{N}$  and  $\widetilde{\mathcal{N}}$ . If  $S'$  is another segment of  $X$  such that  $\text{tord}(S, S') > \beta$  then the nodal zones adjacent to  $S'$  belong to the same nodes  $\mathcal{N}$  and  $\widetilde{\mathcal{N}}$ .*

*Proof.* Let  $N, \widetilde{N}$  and  $N', \widetilde{N}'$  be the nodal zones adjacent to  $S$  and  $S'$ , respectively. Assume that  $N \subset \mathcal{N}$  and  $\widetilde{N} \subset \widetilde{\mathcal{N}}$ . Consider the arcs  $\lambda \in N$ ,  $\widetilde{\lambda} \in \widetilde{N}$ ,  $\lambda' \in N'$ ,  $\widetilde{\lambda}' \in \widetilde{N}'$  and the  $\beta$ -Hölder triangles  $T = T(\lambda, \widetilde{\lambda})$  and  $T' = T(\lambda', \widetilde{\lambda}')$ . Proposition 4.2.5 implies that  $S = G(T)$  and  $S' = G(T')$ . Moreover, Proposition 4.6.2 implies that  $T$  and  $T'$  are pancakes from a pancake decomposition of  $X$ . Then, as  $\text{tord}(S, S') > \beta$ , Corollary 4.4.1 implies that  $\text{tord}(\gamma, S') > \beta$  for all arcs  $\gamma \in S$ .

We now prove that either  $N' \subset \mathcal{N}$  or  $\widetilde{N}' \subset \widetilde{\mathcal{N}}$ . Suppose, by contradiction, that  $\text{tord}(N, N') = \text{tord}(N, \widetilde{N}') = \beta$ . Let  $f: (T, 0) \rightarrow (\mathbb{R}, 0)$  be the function given by  $f(x) = d(x, T')$  and let  $\{T_i\}$  be a pizza on  $T$  associated with  $f$ . As  $\text{tord}(N, N') = \text{tord}(N, \widetilde{N}') = \beta$ , Proposition 4.4.1 implies that  $\text{ord}_\lambda f = \beta$ . Then, Proposition 2.5.1 implies that there is an arc  $\theta \in G(T)$  such that  $\text{ord}_\theta f = \beta$ , a contradiction with  $\text{tord}(\theta, S') > \beta$ , since  $G(T) = S$ .

Finally, if, for example,  $N' \subset \mathcal{N}$  then  $\widetilde{N}' \subset \widetilde{\mathcal{N}}$ . Indeed,  $N' \subset \mathcal{N}$  implies that  $\text{tord}(\widetilde{N}, N') = \beta$ . If  $\text{tord}(\widetilde{N}, \widetilde{N}') = \beta$  then, similarly,  $\text{ord}_{\widetilde{\lambda}} f = \beta$  and we obtain an arc  $\theta \in G(T)$  such that  $\text{ord}_\theta f = \beta$ , a contradiction. Hence,  $\widetilde{N}' \subset \widetilde{\mathcal{N}}$ .  $\square$

## 5 MAIN THEOREM

In this section we investigate the role played by abnormal zones and snakes in the Lipschitz Geometry of surface germs. The main result of this section, Theorem 5.0.1, was the original motivation for this paper. We use definitions and notations of the pizza decomposition from subsection 2.5. In particular,  $\beta_i$ ,  $Q_i$ ,  $\mu_i$  and  $q_i$  are as in Definition 2.5.6.

**Lemma 5.0.1.** *Let  $X = T(\gamma_1, \gamma_2)$  be a non-singular Hölder triangle partitioned by an interior arc  $\gamma$  into two normally embedded Hölder triangles  $X_1 = T(\gamma_1, \gamma)$  and  $X_2 = T(\gamma, \gamma_2)$ . Let  $f: (X_1, 0) \rightarrow (\mathbb{R}, 0)$  be the function given by  $f(x) = d(x, X_2)$ , and let  $\{T_i = T(\lambda_{i-1}, \lambda_i)\}_{i=1}^p$  be a pizza on  $X_1$  associated with  $f$  such that  $\lambda_0 = \gamma$ . Then,  $\mu_1(q_\theta) = \text{itord}(\theta, \gamma)$  for every arc  $\theta \subset T_1$ . Moreover,  $\mu_1(q) = q$  for all  $q \in Q_1$ .*

*Proof.* Since the maximum of  $\mu_1$  is  $\mu_1(q_\gamma) = \mu_1(\infty) = \infty$ , by Proposition 2.5.1, we have  $\mu_1(q_\theta) = \text{itord}(\theta, \gamma)$  for every arc  $\theta \subset T_1$ .

As  $\gamma$  is Lipschitz non-singular, there is a normally embedded  $\alpha$ -Hölder triangle  $X' = T(\tilde{\gamma}_1, \tilde{\gamma}_2) \subset X$ , with  $\tilde{\gamma}_1 \subset X_1$ , such that  $\gamma \in G(X')$ . We are going to prove that, for each arc  $\theta \subset T_1$  such that  $\text{itord}(\theta, \gamma) > \alpha$ , we have  $\mu_1(q_\theta) = q_\theta$ . Indeed, given such an arc  $\theta \subset T_1$ , by the Arc Selection Lemma, there is an arc  $\theta' \subset X_2$  such that  $q_\theta = \text{tord}(\theta, \theta')$ . We claim that  $\text{tord}(\theta, \theta') = \text{itord}(\theta, \theta')$ . Suppose, by contradiction, that  $\text{tord}(\theta, \theta') > \text{itord}(\theta, \theta')$ . As  $X_1$  and  $X_2$  are normally embedded,  $\text{tord}(\theta, \gamma) = \text{itord}(\theta, \gamma)$  and  $\text{tord}(\gamma, \theta') = \text{itord}(\gamma, \theta')$ . Thus, Lemma 2.3.1 implies that  $\text{itord}(\theta, \theta') = \text{tord}(\gamma, \theta') = \text{tord}(\theta, \gamma) > \alpha$  and consequently, since  $\gamma \in G(X')$ ,  $\theta' \subset X'$ , a contradiction with  $X'$  being normally embedded. Then, since  $T(\theta, \gamma) \cup X_2$  is normally embedded, Lemma 2.5.1 implies that  $q_\theta = \text{itord}(\theta, \gamma)$ . Finally,

$$q_\theta = \text{tord}(\theta, \theta') = \text{itord}(\theta, \theta') = \text{itord}(\theta, \gamma) = \mu_1(q_\theta).$$

Hence, since  $\mu_1$  is linear, we have  $\mu_1(q) = q$  for all  $q \in Q_1$ . □

**Lemma 5.0.2.** *Let  $X$ ,  $X_1$ ,  $X_2$ ,  $f$  and  $\{T_i\}$  be as in Lemma 5.0.1. Then,*

1.  $T_1 \cup X_2$  is normally embedded.
2. If  $p > 1$  and  $\{T_i\}$  is a minimal pizza then  $(T_1 \cup T_2) \cup X_2$  is not normally embedded.

*Proof.* (1) If  $T_1 \cup X_2$  is not normally embedded then there are arcs  $\theta \subset T_1$  and  $\theta' \subset X_2$  such that  $\text{tord}(\theta, \theta') > \text{itord}(\theta, \theta')$ . Thus,  $q_\theta \geq \text{tord}(\theta, \theta') > \text{itord}(\theta, \theta')$ . However,  $\mu_1(q_\theta) = \text{itord}(\theta, \gamma)$  and, since  $X_1$  and  $X_2$  are normally embedded, by Lemma 2.3.1,  $\text{itord}(\theta, \gamma) = \text{itord}(\theta, \theta')$ . Then,  $\mu_1(q_\theta) = \text{itord}(\theta, \theta') < q_\theta$ , a contradiction with Lemma 5.0.1.



(2) If  $(T_1 \cup T_2) \cup X_2$  is normally embedded, Lemmas 2.5.1 and 5.0.1 imply that  $T_1 \cup T_2$  is a pizza slice, a contradiction with  $\{T_i\}$  being a minimal pizza.  $\square$

**Lemma 5.0.3.** *Let  $X = T(\gamma_1, \gamma_2)$  be a non-singular Hölder triangle and  $\gamma$  an interior arc of  $X$ . Let  $T = T(\lambda, \gamma)$  and  $T' = T(\gamma, \lambda')$  be normally embedded Hölder triangles in  $X$  such that  $T \cap T' = \gamma$  and  $tord(\lambda, \lambda') > itord(\lambda, \lambda')$ . Let  $f: (T, 0) \rightarrow (\mathbb{R}, 0)$  be the function given by  $f(x) = d(x, T')$ , and let  $\{T_i = T(\lambda_{i-1}, \lambda_i)\}_{i=1}^p$  be a minimal pizza on  $T$  associated with  $f$  such that  $\lambda_0 = \gamma$ . Then*

1. *If  $\beta_2 < \beta_1$  then, for every  $\sigma \in \mathbb{F}$  such that  $\beta_2 < \sigma < \beta_1$ , there are arcs  $\theta \subset T_2$  and  $\theta' \subset T'$  such that  $itord(\theta, \gamma) = \sigma$  and  $tord(\theta, \theta') > itord(\theta, \theta')$ .*
2. *If  $\beta_2 = \beta_1$  then, for every arc  $\theta \subset T_2$  such that  $tord(\theta, \lambda_2) > \beta_2$ , there is an arc  $\theta' \subset T'$  such that  $tord(\theta, \theta') > itord(\theta, \theta')$ .*

*Proof.* For both items (1) and (2) we shall consider the following three cases:

*Case 1 -  $Q_2$  is not a point and  $\mu_2(q_2) = M$  is the maximum of  $\mu_2$ :* Proposition 2.5.1 and Lemma 3.0.1 imply that the minimum of  $\mu_2$  is  $\mu_2(q_1) = \beta_2 = \mu_2(q_{\gamma'})$  for every arc  $\gamma' \subset T_2$  such that  $itord(\gamma', \lambda_2) = \beta_2$ . Then, since  $q_1 = \beta_1$ , by Lemma 5.0.1, we have  $\beta_1 = q_{\gamma'}$ , for every  $\gamma' \subset T_2$  such that  $itord(\gamma', \lambda_2) = \beta_2$ .

*Case 2 -  $Q_2$  is a point:* Since by Lemma 5.0.1,  $q_1 = \beta_1$ , for every  $\gamma' \subset T_2$  we have  $\beta_1 = q_{\gamma'}$ .

*Case 3 -  $Q_2$  is not a point and  $\mu_2(q_1) = M$  is the maximum of  $\mu_2$ :* Proposition 2.5.1 implies that  $itord(\gamma', \lambda_1) = \mu_2(q_{\gamma'})$  for every arc  $\gamma' \subset T_2$  such that  $itord(\gamma', \lambda_1) \leq M \leq \beta_1$ . Moreover, if  $\gamma' \subset T_2$  and  $itord(\gamma', \gamma) < \beta_1$  then  $itord(\gamma', \lambda_1) = itord(\gamma', \gamma) = \mu_2(q_{\gamma'})$ . Since  $itord(\gamma', \gamma) = \mu_1(q_{\gamma'}) = q_{\gamma'}$  for each arc  $\gamma' \subset T_1$  and  $\mu_2(q) \leq q$  for each  $q \in Q_2$ , we have  $itord(\gamma', \lambda_1) = \mu_2(q_{\gamma'}) < q_{\gamma'}$  for every  $\gamma' \subset T_2$  such that  $itord(\gamma', \gamma) < \beta_1$ . Otherwise  $T_1 \cup T_2$  would be a pizza slice, a contradiction with  $\{T_i\}$  being a minimal pizza.

(1) Consider  $\sigma \in \mathbb{F}$  such that  $\beta_2 < \sigma < \beta_1$  and an arc  $\theta \subset T_2$  such that  $\sigma = itord(\theta, \gamma)$ . Let  $\theta' \subset T'$  be an arc such that  $q_{\theta} = tord(\theta, \theta')$ .

Suppose that  $Q_2$  is as in case 1 or 2. Note that  $\sigma > \beta_2$  implies that  $itord(\theta, \lambda_2) = \beta_2$ . Thus, by cases 1 and 2 considered above, we have  $q_{\theta} = \beta_1$ . As  $\sigma < \beta_1$ , we have

$$tord(\theta, \theta') = q_{\theta} = \beta_1 > \sigma = itord(\theta, \gamma) \geq itord(\theta, \theta').$$

If  $Q_2$  is as in case 3 then

$$tord(\theta, \theta') = q_{\theta} > itord(\theta, \lambda_1) = itord(\theta, \gamma) \geq itord(\theta, \theta').$$

(2) Suppose that  $\beta_2 = \beta_1$  and consider an arc  $\theta \subset T_2$  such that  $tord(\theta, \lambda_2) > \beta_2$ . Let  $\theta' \subset T'$  be an arc such that  $q_\theta = tord(\theta, \theta')$ . Lemma 5.0.2 implies that  $tord(T_2, T') > \beta_2$ . Let  $\tilde{\gamma} \subset T_2$  be an arc such that  $tord(\tilde{\gamma}, T') > \beta_2$ . Note that  $itord(\tilde{\gamma}, \lambda_1) = \beta_2$ , otherwise, by the non-archimedean property, we would have  $T_1 \cup T'$  not normally embedded, a contradiction with Lemma 5.0.2.

If  $Q_2$  is as in case 1 or 2 then  $\beta_1 = q_1 = ord_{\tilde{\gamma}} f > \beta_2$ , a contradiction. Then, it is enough to consider  $Q_2$  as in case 3. Thus, we have  $ord_{\tilde{\gamma}} f = q_\theta$  and consequently,

$$tord(\theta, \theta') = q_\theta = q_{\tilde{\gamma}} > \beta_2 = itord(\theta, \gamma) \geq itord(\theta, \theta').$$

□

**Lemma 5.0.4.** *Let  $A, T, T'$  be as in Lemma 4.1.1 and let  $\{T_i = T(\lambda_{i-1}, \lambda_i)\}_{i=1}^p$  be a minimal pizza on  $T$  associated with the function  $f: (T, 0) \rightarrow (\mathbb{R}, 0)$  given by  $f(x) = d(x, T')$ , such that  $\lambda_0 = \gamma$ . Then  $p > 1$ , and one can choose the arcs  $\lambda$  and  $\lambda'$  in Lemma 4.1.1 so that  $p = 2$  and  $\lambda = \lambda_2$ . Moreover,  $\mu(T_2) \leq \mu(T_1) = \beta$ .*

*Proof.* Lemma 5.0.2 implies that  $p > 1$ , since otherwise we would have  $T \cup T'$  normally embedded. Since  $(T_1 \cup T_2) \cup T'$  is not normally embedded, we can choose  $\lambda = \lambda_2$  and have  $p = 2$ . Moreover, as  $T_1 \cup T'$  is normally embedded,  $V(T_1) \subset A$  thus  $\beta \leq \beta_1 = \mu(T_1)$ .

Since  $\{T_i\}$  is a minimal pizza,  $\mu(T_2) = \beta_2 \leq \beta_1$ . From now on we assume that  $\lambda_2 = \lambda$ . We can further assume that any arc  $\gamma' \subset T$  such that  $tord(\gamma', \gamma) > \beta$  belongs to  $A$ . If  $\beta_2 > \beta$  then  $\lambda_2 = \lambda \in A$  and we obtain the same contradiction as in the proof of Lemma 4.1.1. Thus we have  $\beta_2 \leq \beta \leq \beta_1$ . It remains to prove that  $\beta = \beta_1$ .

If  $\beta_1 > \beta$  then, in particular,  $\beta_1 > \beta_2$ , since  $\beta_2 \leq \beta$ . Lemma 5.0.3 implies that we can find arcs  $\theta \subset T_2$  and  $\theta' \subset T'$  such that  $\beta < itord(\theta, \gamma) < \beta_1$  and  $tord(\theta, \theta') > itord(\theta, \theta')$ . Replacing  $\lambda = \lambda_2$  with  $\theta$  and  $\lambda'$  with  $\theta'$ , we obtain a minimal pizza  $\{T_1, T_2\}$  such that  $\beta_2 > \beta$ , a contradiction with  $\beta_2 \leq \beta$  for any minimal pizza  $\{T_1, T_2\}$  of  $T$ . □

**Corollary 5.0.1.** *Let  $X$  be a non-singular Hölder triangle, and  $A \subset V(X)$  a maximal abnormal  $\beta$ -zone. Then  $A$  is a perfect zone.*

*Proof.* This is an immediate consequence of Lemmas 5.0.2 and 5.0.4. □

**Remark 5.0.1.** The links of bubbles shown in Figs. 10 and 11 are examples of the possibilities for the minimal pizza decomposition in Lemmas 5.0.3 and 5.0.4:

1. In Fig. 10a, the triangle  $T = T(a, \theta)$  has exactly two pizza slices with  $\lambda_0 = \theta$ ,  $\lambda_1$  being any generic arc of  $T$ , and  $\lambda_2 = a$ . Moreover,  $\beta_1 = \beta_2 = \beta$  and  $Q_2$  is not a point. Also, the maximum of  $\mu_2$  is  $\mu_2(q_2)$ .

2. In Fig. 10b, the triangle  $T = T(a, \theta)$  has exactly three pizza slices with the same exponent  $\beta$ , where  $\lambda_0 = \theta$ ,  $\lambda_1$  is any generic arc of  $T(b, \theta)$ ,  $\lambda_2 = b$  and  $\lambda_3 = a$ . Moreover,  $Q_2$  is not a point, maximum of  $\mu_2$  is  $\mu_2(q_2)$ , and  $Q_3$  is a point with  $q_3 > \beta$ .

3. In Fig. 10c, the triangle  $T = T(a, \theta)$  has exactly two pizza slices with  $\lambda_0 = \theta$ ,  $\lambda_1$  being any arc in  $T$  having exponent  $\beta$  with  $\theta$ , and  $\lambda_2 = a$ . Moreover,  $\beta_2 = \eta < \beta = \beta_1$  and  $Q_2$  is a point.

4. In both Fig. 10d and Fig. 10e, the triangle  $T = T(a, \theta)$  has exactly two pizza slices with  $\lambda_0 = \theta$ ,  $\lambda_1$  being any arc in  $T$  having exponent  $\beta$  with  $\theta$ , and  $\lambda_2 = a$ . Moreover,  $\beta_2 = \eta < \beta = \beta_1$ ,  $q_2 > \beta$  and  $\mu_2(q_1)$  may be either the maximum or the minimum of  $\mu_2$ . If  $\max \mu_2 = \mu_2(q_1)$  then  $\max \mu_2 \leq \beta$  and the slope of  $\mu_2$  is negative in the case of Fig. 10d and positive in the case of Fig. 10e. Otherwise,  $\max \mu_2 < \beta$ . In both cases, if  $\mu_2(\lambda_1) < \beta$  then the bubble contains the bubble in Fig. 10d with  $\eta = \mu(\lambda_1)$ . If  $\max \mu_2 = \mu_2(q_2)$  then the slope of  $\mu_2$  is positive in Fig. 10d and negative in Fig. 10e.

5. In Fig. 11, the minimal pizza on  $T = T(\gamma_1, \theta)$  such that  $\lambda_0 = \theta$ , has exactly four pizza slices, each of them with exponent  $\beta$ .

**Lemma 5.0.5.** *Let  $X$  be a non-singular Hölder triangle and  $\{X_k = T(\theta_{k-1}, \theta_k)\}_{k=1}^p$  a reduced pancake decomposition of  $X$  with  $\beta_k = \mu(X_k)$ . If  $A \subset V(X)$  is a maximal abnormal  $\beta$ -zone then:*

1. *the zone  $A$  has non-empty intersection with at least two of the zones  $V(X_k)$ ;*
2. *if  $V(X_k) \cap A \neq \emptyset$  then  $\beta_k \leq \beta$ ;*

*Proof.* (1) Suppose, by contradiction, that  $A$  intersects only a single zone  $V(X_k)$ . Then  $A \subset V(X_k)$  and  $\mu(X_k) \leq \beta$ . Given an arc  $\gamma \in A$  there exist arcs  $\lambda, \lambda' \in V(X)$  such that  $T = T(\lambda, \gamma)$  and  $T' = T(\gamma, \lambda')$  are normally embedded Hölder triangles with a common boundary arc  $\gamma$  and  $tord(\lambda, \lambda') > itord(\lambda, \lambda')$ . Let  $\alpha = itord(\lambda, \lambda')$ . Lemma 2.3.1 implies that  $\mu(T) = \mu(T') = \alpha$ .

Since  $X_k$  is normally embedded, one of the arcs  $\lambda$  and  $\lambda'$ , say  $\lambda$ , is not contained in  $X_k$ . Assume that  $\theta_k \subset T(\lambda, \gamma)$ . As  $\{X_k\}$  is a reduced pancake decomposition, we can assume that  $\lambda$  and  $\lambda'$  are in adjacent pancakes,  $\lambda' \in V(X_k)$  and  $\lambda \in V(X_{k+1})$ . Then,  $\theta_k$  is abnormal, since  $T(\lambda', \theta_k)$  and  $T(\theta_k, \lambda)$  are normally embedded. However, by Lemma 5.0.2, there exist arcs of  $A$  in  $V(X_{k+1})$ , a contradiction with  $A \subset V(X_k)$ .

(2) Suppose, by contradiction, that  $V(X_k) \cap A \neq \emptyset$  and  $\mu(X_k) > \beta$ . Corollary 5.0.1 implies that  $V(X_k) \subset A$ . In particular,  $\theta_k$  is abnormal. Thus, there are arcs  $\lambda, \lambda' \in V(X)$  such that  $T = T(\lambda, \theta_k)$  and  $T' = T(\theta_k, \lambda')$  are normally embedded  $\alpha$ -Hölder triangles with a common boundary arc  $\theta_k$  and  $tord(\lambda, \lambda') > itord(\lambda, \lambda')$ . As  $\{X_k\}$  is a reduced pancake decomposition, we may assume that  $\lambda$  and  $\lambda'$  are in adjacent pancakes. Hence,  $\alpha \geq \mu(X_k) > \beta$  and consequently,  $\lambda, \lambda' \in A$ , a contradiction with Lemma 5.0.4.  $\square$

**Lemma 5.0.6.** *Let  $A, T$  and  $T'$  be as in Lemma 4.1.1, and let  $\{T_1, T_2\}$  be a minimal pizza on  $T$  associated with  $f: (T, 0) \rightarrow (\mathbb{R}, 0)$ , given by  $f(x) = d(x, T')$ , such that  $\lambda_0 = \gamma$  and  $\lambda_2 = \lambda$  (see Lemma 5.0.4). Then*

1. *If  $\mu(T_2) = \beta$  in Lemma 5.0.4 then  $\gamma$  is contained in a  $\beta$ -bubble snake and  $A \subset V(Y)$  where  $Y$  is a  $\beta$ -snake.*
2. *If  $\mu(T_2) < \beta$  then  $\gamma$  is not contained in any snake.*

*Proof.* (1) If  $\beta_2 = \mu(T_2) = \beta$  then, by Lemmas 5.0.3 and 5.0.4, we have  $\beta_1 = \beta_2 = \beta$ . We claim that  $T \cup T'$  is a  $\beta$ -bubble snake. Indeed, since  $tord(\lambda_1, T') = \beta$  and  $tord(\lambda_2, T') > \beta$ , we have  $\min \mu_2 = \mu_2(q_1) = \beta$  and  $\max \mu_2 = \mu_2(q_2)$ . Proposition 2.5.1 implies that  $q_{\gamma'} = tord(\gamma', T') = \beta$  for every  $\gamma' \in G(T)$ . Then, every arc in  $G(T)$  is abnormal and similarly we can prove that every arc in  $G(T')$  is also abnormal. Finally, since  $\gamma$  is in a perfect abnormal  $\beta$ -zone by Corollary 5.0.1, it follows that  $G(T \cup T') = Abn(T \cup T')$ .

Now we are going to prove that, when  $\beta_1 = \beta_2 = \beta$ , there is a  $\beta$ -snake  $Y$  such that  $A \subset V(Y)$ . We already proved that  $T \cup T'$  is a  $\beta$ -bubble snake. If  $\lambda \notin A$  and  $\lambda' \notin A$  then we can take  $Y = T \cup T'$ . Suppose that  $\lambda \in A$  and  $\lambda' \notin A$ . Since  $\lambda \in A$ , there are normally embedded  $\alpha_1$ -Hölder triangles  $T'_1 = T(\theta_1, \lambda)$  and  $T''_1 = T(\lambda, \theta'_1)$  such that  $T'_1 \cap T = T'_1 \cap T''_1 = \lambda$  and  $tord(\theta_1, \theta'_1) > itord(\theta_1, \theta'_1)$ . Then  $\alpha_1 \leq \beta$  according to Lemma 4.1.1. As  $T \cup T'$  is not normally embedded, we have  $\theta'_1 \subset T \cup T'$ . Thus,  $\alpha_1 = tord(\lambda, \theta'_1) \geq itord(\lambda, \lambda') = \beta$ . Hence  $\alpha_1 = \beta$  and  $T'_1 \cup T''_1$  is a  $\beta$ -bubble snake. If  $\theta_1 \notin A$ , then  $Y = (T'_1 \cup T''_1) \cup (T \cup T')$  is a  $\beta$ -snake and  $A \subset V(Y)$ . If  $\theta_1 \in A$  we apply the same argument to find normally embedded  $\beta$ -Hölder triangles  $T'_2 = T(\theta_2, \theta_1)$  and  $T''_2 = T(\theta_1, \theta'_2)$  such that  $T'_2 \cap T''_2 = \theta_1$  and  $tord(\theta_2, \theta'_2) > itord(\theta_2, \theta'_2)$ . If  $\theta_2 \notin A$  then  $Y = (T'_2 \cup T''_2) \cup (T'_1 \cup T''_1) \cup (T \cup T')$  is a  $\beta$ -snake and  $A \subset V(Y)$ . If  $\theta_2 \in A$  we continue applying the same argument. This procedure must stop after finitely many steps, otherwise there would be infinitely many pancakes in a reduced pancake decomposition of  $X$ , a contradiction. Thus, after finitely many steps, we find an integer  $p$  such that  $\theta_p \notin A$ , thus  $Y = (\bigcup_{i=1}^p Y_i) \cup (T \cup T')$  is a  $\beta$ -snake, where  $Y_i = T'_i \cup T''_i$ , and  $A \subset Y$ .

(2) It is enough to prove that if  $\beta_2 = \mu(T_2) < \beta = \beta_1$  then  $T \cup T'$  is a non-snake  $\beta$ -bubble. Let  $Y = T \cup T'$ . Suppose, by contradiction, that  $Y$  is a  $\beta_2$ -bubble snake. Consider  $\alpha' \in \mathbb{F}$  such that  $\beta_2 < \alpha' < \beta_1$ . By Lemma 5.0.3, there is an arc  $\theta \subset T_2$  such that  $tord(\theta, T') > \beta_2$ , a contradiction with Proposition 4.5.2.  $\square$

**Theorem 5.0.1.** *Let  $X$  be a surface germ. Then  $V(X)$  is the union of finitely many maximal normal (possibly singular) zones and finitely many maximal abnormal zones. Moreover, each maximal abnormal zone  $A$  in  $V(X)$  is perfect, and if its order is  $\beta$  then it satisfies one of the three following conditions:*

1.  $A = V(Y)$ , where  $Y$  is a circular  $\beta$ -snake;
2.  $A = G(Y)$ , where  $Y$  is a  $\beta$ -snake;
3.  $A \subset V(T_\eta)$ , where  $T_\eta$  is a non-snake  $\eta$ -bubble with  $\eta = \beta - \varepsilon$  for any small  $\varepsilon > 0$ .

(see Figures 10c, 10d, 10e).

*Proof.* It is enough to prove the statement for a surface germ  $X$  with connected link. If the link of  $X$  is a circle and all arcs in  $X$  are abnormal then  $X$  is a circular snake. Otherwise,  $X$  must contain some normal (possibly, Lipschitz singular) arcs.

Note that  $tord(\gamma, Abn(X)) < \infty$  for each normal arc  $\gamma$  of  $X$ . It is enough to prove that  $itord(\gamma, Abn(X)) < \infty$  for each normal arc  $\gamma$  of  $X$ . If  $itord(\gamma, Abn(X)) = \infty$  for some normal arc  $\gamma$  of  $X$ , by the existence of pancake decomposition of  $X$ , there should exist a maximal abnormal zone  $A$  of  $X$  such that for each sufficiently large  $\eta \in \mathbb{F}$ , there will exist  $\lambda \in A$  such that  $itord(\gamma, \lambda) > \eta$ . However, if  $\mu(A) = \beta$ , for  $\eta > \beta$ , item 1 of Lemma 5.0.2 and Lemma 5.0.4 imply that  $\gamma$  is abnormal, a contradiction.

If  $\alpha > tord(\gamma, Abn(X))$  then an  $\alpha$ -horn neighborhood  $H_{a,\alpha}(\gamma)$  of  $\gamma$  in  $X$  (see Definition 4.2.1 and Remark 4.2.1) consists of normal arcs. Removing  $H_{a,\alpha}(\gamma)$  from  $X$  does not change  $Abn(X)$ . Indeed, let  $\lambda \in Abn(X)$  be an arc such that  $\lambda = T_1 \cap T_2$  where  $T_1 = T(\lambda_1, \lambda)$  and  $T_2 = T(\lambda, \lambda_2)$  are normally embedded non-singular  $\beta$ -Hölder triangles such that  $tord(\lambda_1, \lambda_2) > \beta$  and  $\gamma \subset T_1$ . In particular,  $\beta < \alpha$ . If  $T(\gamma, \lambda) \cup T_2$  is not normally embedded then  $\lambda \in Abn(X \setminus H_{a,\alpha}(\gamma))$ . Otherwise, we would have  $tord(\gamma, \lambda_1) = \beta$ , thus  $\gamma$  would be an abnormal arc, a contradiction.

Removing from  $X$ , if necessary, open horn neighborhoods of Lipschitz singular arcs and finitely many normal arcs, we get finitely many disjoint non-singular Hölder triangles  $X_i \subset X$  such that  $Abn(X) \subset \cup Abn(X_i)$ . Thus it is enough to prove the statement for a non-singular Hölder triangle  $X$ .

By definition, abnormal zones do not intersect with normal zones. Moreover (see Definition 2.6.10) there are no adjacent maximal abnormal zones and adjacent maximal normal zones.

Let  $\{X_k = T(\theta_{k-1}, \theta_k)\}$  be a reduced pancake decomposition of  $X$ . To prove that there are finitely many abnormal zones in  $V(X)$  it is enough to prove that each zone  $V(X_k)$  intersects finitely many maximal abnormal zones in  $V(X)$ . Suppose, by contradiction, that there are infinitely many maximal abnormal zones  $A_1, A_2, \dots$  in  $V(X)$  such that  $V(X_k) \cap A_i \neq \emptyset$  for all  $i = 1, 2, \dots$ . Lemma 5.0.5 implies that  $\mu(X_k) \leq \mu(A_i)$  for all  $i = 1, 2, \dots$ . One of the boundary arcs of  $X_k$  must belong to  $A_i$  for some  $i$ , otherwise, since  $\mu(X_k) \leq \mu(A_i)$  for all  $i = 1, 2, \dots$ , we would have  $A_i \subset V(X_k)$  for all  $i$ , a contradiction with Lemma 5.0.5. Assume that  $\theta_k \in A_i$  and consider  $A_j$  for  $j \neq i$ . Thus,  $\theta_{k-1} \in A_j$ , otherwise we would have  $A_j \subset V(X_k)$ , since  $A_i \cap A_j = \emptyset$ , a contradiction with Lemma 5.0.5. Then,  $\theta_{k-1} \in A_j$ ,  $\theta_k \in A_i$  and  $A_l \subset V(X_k)$  for all  $l = 1, 2, \dots$  with  $l \neq i$  and  $l \neq j$ , a contradiction with Lemma 5.0.5. Since there exist finitely many abnormal zones it follows that there are finitely many maximal normal zones in  $V(X)$ .

Finally, let  $A$  be a maximal abnormal  $\beta$ -zone in  $V(X)$ . Corollary 5.0.1 implies that  $X$  is a perfect zone. Consider an arc  $\gamma \in A$  and arcs  $\lambda, \lambda' \in V(X)$  such that  $T = T(\lambda, \gamma)$  and  $T' = T(\gamma, \lambda')$  are normally embedded non-singular Hölder triangles with  $tord(\lambda, \lambda') > itord(\lambda, \lambda')$ . Let  $f: (T, 0) \rightarrow (\mathbb{R}, 0)$  be the function given by  $f(x) = d(x, T')$ , and  $\{T_i\}$  a minimal pizza on  $T$  associated with  $f$ . By Lemma 5.0.4, we can assume that  $p = 2$  and  $\beta_2 \leq \beta_1 = \beta$ . If  $\beta_2 = \beta_1 = \beta$  then, by Lemma 5.0.6,  $A$  is contained in a  $\beta$ -snake. If  $\beta_2 < \beta_1$  then, also by Lemma 5.0.6,  $A$  is contained in the non-snake bubble  $Y = T \cup T'$  and, by Lemma 5.0.3, for any  $\varepsilon > 0$  such that  $\beta_2 < \eta = \beta - \varepsilon < \beta_1 = \beta$ ,  $A \subset V(T_\eta)$  where  $T_\eta \subset Y$  is a non-snake  $\eta$ -bubble.  $\square$

## 6 COMBINATORICS OF SNAKES

In this section we assign a word to a snake. It is a combinatorial invariant of the snake reflecting the order, with respect to a fixed orientation, in which nodal zones belonging to each of its nodes appear.

### 6.1 Words and partitions

**Definition 6.1.1.** Consider an alphabet  $A = \{x_1, \dots, x_n\}$ . A *word*  $W$  of length  $m = |W|$  over  $A$  is a finite sequence of  $m$  letters in  $A$ , i.e.,  $W = [w_1 \cdots w_m]$  with  $w_i \in A$  for  $1 \leq i \leq m$ . One also considers the *empty word*  $\varepsilon = []$  of length 0. Given a word  $W = [w_1 \cdots w_m]$ , the letter  $w_i$  is called the  *$i$ -th entry* of  $W$ . If  $w_i = x$  for some  $x \in A$ , it is called a *node entry* of  $W$  if it is the first occurrence of  $x$  in  $W$ . Alternatively,  $w_i$  is a node entry of  $W$  if  $w_j \neq w_i$  for all  $j < i$ .

**Definition 6.1.2.** Given a word  $W = [w_1 \cdots w_m]$ , a *subword* of  $W$  is either an empty word or a word  $[w_j \cdots w_k]$  formed by consecutive entries of  $W$  in positions  $j, \dots, k$ , for some  $j \leq k$ . We also consider *open* subwords  $(w_j \cdots w_k)$  formed by the entries of  $W$  in positions  $j+1, \dots, k-1$ , for some  $j < k$ , and *semi-open* subwords  $(w_j \cdots w_k]$  and  $[w_j \cdots w_k)$  formed by the entries of  $W$  in positions  $j+1, \dots, k$  and  $j, \dots, k-1$ , respectively.

**Definition 6.1.3.** Let  $W = [w_1 \cdots w_m]$  be a word of length  $m$  containing  $n$  distinct letters  $x_1, \dots, x_n$ . We associate with  $W$  a partition  $P(W) = \{I_1, \dots, I_n\}$  of the set  $\{1, \dots, m\}$  where  $i \in I_j$  if  $w_i = x_j$ .

**Remark 6.1.1.** Note that  $P(W)$  does not depend on the alphabet, only on positions where the same letters appear. For convenience we often assign  $a$  (or  $x_1$ ) to the first letter of the word  $W$ ,  $b$  (or  $x_2$ ) to the first letter of  $W$  other than  $a$ , and so on. Two words  $W$  and  $W'$  are equivalent if  $P(W) = P(W')$ . In particular, equivalent words have the same length and the same number of distinct letters. For example, the words  $X = abcdacbd$ ,  $Y = bcdabdca$  and  $Z = xyzwxzyw$  are equivalent, since  $P(X) = P(Y) = P(Z) = \{\{1, 5\}, \{2, 7\}, \{3, 6\}, \{4, 8\}\}$ .

**Definition 6.1.4.** A word  $W$  is *primitive* if it contains no repeated letters, i.e., if each part of  $P(W)$  contains a single entry. We say that  $W = [w_1 \cdots w_m]$  is *semi-primitive* if  $w_1 = w_m$  and the subword  $[w_1 \cdots w_m)$  of  $W$  is primitive, i.e., if each part of  $P(W)$  except  $\{w_1, w_m\}$  contains a single entry. A word  $W$  is *binary* if each of its letters appears in  $W$  exactly twice, i.e., if each part of  $P(W)$  contains exactly two entries.

## 6.2 Snake names

**Definition 6.2.1.** Given a non-empty word  $W = [w_1 \cdots w_m]$ , we say that  $W$  is a *snake name* if the following conditions hold:

- (i) Each of the letters of  $W$  appears in  $W$  at least twice;
- (ii) For any  $k \in \{2, \dots, m-1\}$ , there is a semi-primitive subword  $[w_j \cdots w_l]$  of  $W$  such that  $j < k < l$ .

**Remark 6.2.1.** Note that every word equivalent to a snake name  $W$  is also a snake name.

**Remark 6.2.2.** The word  $[aa]$  (or any equivalent word) is the only snake name of length two. No snake name of length greater than two contains the same letter in consecutive positions. There are no snake names of length three, and the words  $[abab]$  and  $[ababa]$  are the only snake names, up to equivalence, of length four and five, respectively.

**Example 18.** The word  $W = [abcdacbd]$  is a snake name, while the word  $W' = [abacdcbd]$  is not, since the entry  $w_3 = a$  of  $W'$  does not satisfy condition (ii) of Definition 6.2.1. There may be more than one subword in a snake name satisfying condition (ii) of Definition 6.2.1 for a fixed position  $k$ . For example both subwords  $[abcda]$  and  $[cdac]$  of  $W$  satisfy condition (ii) for its entry  $w_4 = d$ .

**Definition 6.2.2.** Let  $T = T(\gamma_1, \gamma_2)$  be a  $\beta$ -snake with  $n$  nodes  $\mathcal{N}_1, \dots, \mathcal{N}_n$  and  $m$  nodal zones  $N_1, \dots, N_m$ . From now on we assume that the link of  $T$  is oriented from  $\gamma_1$  to  $\gamma_2$ , and the nodal zones of  $T$  are enumerated in the order in which they appear when we move along the link of  $T$  from  $\gamma_1$  to  $\gamma_2$ . We enumerate the nodes of  $T$  similarly, starting with the node  $\mathcal{N}_1$  containing  $\gamma_1$ , skipping the nodes for which the numbers were already assigned. In particular,  $\gamma_1 \in N_1 \subset \mathcal{N}_1$  and  $\gamma_2 \in N_m$ , but  $N_m$  does not necessarily belong to  $\mathcal{N}_n$ .

Consider an alphabet  $A = \{x_1, \dots, x_n\}$  where each letter  $x_j$  is assigned to the node  $\mathcal{N}_j$  of  $T$ . A word  $W = [w_1 \cdots w_m]$  over  $A$  is associated with the snake  $T = T(\gamma_1, \gamma_2)$  (notation  $W = W(T)$ ) if, while moving along the link of  $T$  from  $\gamma_1$  to  $\gamma_2$ , the  $i$ -th entry  $w_i$  of  $W$  is the letter  $x_j$  assigned to the node  $\mathcal{N}_j$  to which the nodal zone  $N_i$  belongs.

**Proposition 6.2.1.** Let  $T = T(\gamma_1, \gamma_2)$  be a snake, other than a spiral snake, and let  $W = W(T)$  be the word associated with  $T$ . Then  $W$  is a snake name satisfying conditions (i) and (ii) of Definition 6.2.1.



*Proof.* Condition (i) of Definition 6.2.1 holds since each node of  $T$  contains at least two nodal zones (see Proposition 4.4.1).

For a bubble snake condition (ii) of Definition 6.2.1 is empty, thus we may assume that  $T$  is not a bubble snake. Consider  $w_k$ , with  $1 < k < m$ , and the nodal zone  $N_k$  associated with  $w_k$ . Since  $N_k$  is an interior nodal zone, every arc in  $N_k$  is abnormal. Let  $\gamma \in N_k$ , and let  $\lambda_1 \subset T(\gamma_1, \gamma)$  and  $\lambda_2 \subset T(\gamma, \gamma_2)$  be two arcs such that  $T(\lambda_1, \gamma)$  and  $T(\gamma, \lambda_2)$  are normally embedded non-singular Hölder triangles with  $tord(\lambda_1, \lambda_2) > itord(\lambda_1, \lambda_2)$  (see Remark 2.6.5). Propositions 4.4.1 and 4.6.1 imply that  $\lambda_1$  and  $\lambda_2$  belong either to distinct nodal zones in the same node or to distinct segments.

We can assume, replacing the arcs  $\lambda_1$  and  $\lambda_2$  if necessary, that  $\lambda_1$  and  $\lambda_2$  belong to different nodal zones in the same node. Indeed, suppose that  $\lambda_1$  and  $\lambda_2$  belong to segments  $S_1$  and  $S_2$ , respectively. Let  $\mathcal{N}$  and  $\mathcal{N}'$  be the nodes containing the nodal zones adjacent to  $S_1$  and  $S_2$  (see Proposition 4.6.4). We can assume that  $T(\lambda_1, \gamma)$  and  $T(\gamma, \lambda_2)$  do not contain arcs in nodal zones of the same node. Otherwise,  $\lambda_1$  and  $\lambda_2$  can be replaced by those arcs. Then,  $T(\lambda_1, \gamma)$  contains arcs in a nodal zone in one of these nodes, say  $N \subset \mathcal{N}$ , and  $T(\gamma, \lambda_2)$  contains arcs in a nodal zone  $N' \subset \mathcal{N}'$ . This implies that  $T(\gamma, \lambda_2)$  does not contain arcs in  $N$ . If  $T(\gamma, \lambda_2)$  contains arcs of some other nodal zone  $N''$  in  $\mathcal{N}$  other than  $N$  then  $\lambda_1$  and  $\lambda_2$  can be replaced by arcs in  $N$  and  $N''$ , respectively. Thus, assume that  $T(\gamma, \lambda_2)$  does not contain arcs in  $\mathcal{N}$  and consider arcs  $\lambda'_1 \in N$  and  $\lambda'_2 \in N'$ . Since Hölder triangles  $T(\lambda'_1, \gamma)$  and  $T(\gamma, \lambda'_2)$  are normally embedded, we can replace  $\lambda_i$  by  $\lambda'_i$  for  $i = 1, 2$ . In fact,  $T(\lambda'_1, \gamma)$  is normally embedded because  $T(\lambda'_1, \gamma) \subset T(\lambda_1, \gamma)$  where  $T(\lambda_1, \gamma)$  is normally embedded. If  $T(\gamma, \lambda'_2)$  is not normally embedded, we get a contradiction with  $T(\gamma, \lambda_2)$  containing no arcs in  $\mathcal{N}$  (see Proposition 4.4.1 item (1)).

Assume now that  $\lambda_1$  and  $\lambda_2$  belong to nodal zones  $N_j$  and  $N_l$ , respectively, in the same node  $\mathcal{N}$ , where  $j < l$ . Since  $T(\lambda_1, \gamma)$  and  $T(\gamma, \lambda_2)$  are normally embedded, the nodal zone  $N_k$  does not belong to  $\mathcal{N}$ , and consequently,  $j < k < l$ . Furthermore, we may assume that  $V(T(\lambda_1, \lambda_2))$  does not contain distinct nodal zones in the same node other than  $\mathcal{N}$ .

Let  $w_j$  and  $w_l$  be the entries associated with  $N_j$  and  $N_l$ , respectively. By our assumption for  $V(T(\lambda_1, \lambda_2))$ , the only letters common to the primitive subwords  $[w_j \cdots w_k]$  and  $[w_k \cdots w_l]$  are  $w_k$  and  $w_j = w_l$ . Hence, the subword  $[w_j \cdots w_l]$  is semi-primitive, and condition (ii) of Definition 6.2.1 is satisfied.  $\square$

**Proposition 6.2.2.** *Let  $T$  be a snake, and  $W = [w_1 \cdots w_m]$  the word associated with  $T$ . Let  $T' \subset T$  be a Hölder triangle with the boundary arcs in the nodal zones  $N_j$  and  $N_k$  of  $T$ , where*

$j < k$ . Then  $T'$  is normally embedded if, and only if, the subword  $[w_j \cdots w_k]$  of  $W$  is primitive.

*Proof.* If two of the nodal zones  $N_j, \dots, N_k$  belong to the same node then, by Proposition 4.1.1,  $T'$  is not normally embedded.

Conversely, if  $T'$  is not normally embedded then there are arcs  $\lambda, \lambda' \subset T'$  such that  $tord(\lambda, \lambda') > itord(\lambda, \lambda') = \beta$ . By the same argument as in the proof of Proposition 6.2.1, we can assume that  $\lambda \in N_{j'}$  and  $\lambda' \in N_{k'}$  where  $j \leq j' < k' \leq k$ . Hence  $N_{j'}$  and  $N_{k'}$  belong to the same node,  $w_{j'} = w_{k'}$  and the subword  $[w_j \cdots w_k]$  is not primitive.  $\square$

**Corollary 6.2.1.** *Let  $T, W$  and  $T'$  be as in Proposition 6.2.2. Then  $T'$  is a bubble snake if, and only if, the subword  $[w_j \cdots w_k]$  of  $W$  is semi-primitive.*

*Proof.* If  $T'$  is a bubble snake then  $N_j$  and  $N_k$  belong to the same node. If  $[w_j \cdots w_k]$  is not semi-primitive then at least one of the words  $[w_j \cdots w_k]$  and  $(w_j \cdots w_k]$  is not primitive. If  $[w_j \cdots w_k]$  is not primitive (the case when  $(w_j \cdots w_k]$  is not primitive is similar) then there are entries  $w_{j'}$  and  $w_{k'}$  with  $j \leq j' < k' < k$  such that  $w_{j'} = w_{k'}$ . Consequently, there are nodal zones  $N_{j'}$  and  $N_{k'}$  of  $T$  such that  $tord(N_{j'}, N_{k'}) > \beta$ . As  $j \leq j' < k' < k$ , we have  $N_{j'} \cap V(T') \neq \emptyset$  and  $N_{k'} \subset G(T')$ , a contradiction with Proposition 4.5.2.

Conversely, if  $[w_j \cdots w_k]$  is semi-primitive then  $N_j$  and  $N_k$  are the only nodal zones of  $T$  having nonempty intersection with  $V(T')$  which belong to the same node. Proposition 4.6.4 implies that  $T'$  is a bubble snake.  $\square$

**Definition 6.2.3.** Let  $W = [w_1 \cdots w_m]$  be a snake name. If  $w_j$  is not a node entry, for some  $j = 2, \dots, m$ , we define  $r(j)$  so that  $w_{r(j)}$  is a node entry and  $w_{r(j)} = w_j$ . If  $w_j$  is a node entry then  $r(j) = j$ .

**Definition 6.2.4.** Given arcs  $\gamma, \gamma' \subset \mathbb{R}^p$  we define the set  $\Delta(\gamma, \gamma')$  as the union of straight line segments,  $[\gamma(t), \gamma'(t)]$ , connecting  $\gamma(t)$  and  $\gamma'(t)$  for any  $t \geq 0$ .

**Definition 6.2.5.** Let  $W = [w_1, \dots, w_m]$  be a snake name of length  $m > 2$ . Consider the space  $\mathbb{R}^{2m-1}$  with the standard basis  $\mathbf{e}_1, \dots, \mathbf{e}_{2m-1}$ . Let  $\alpha, \beta \in \mathbb{F}$ , with  $1 \leq \beta < \alpha$ , and let  $\delta_1, \dots, \delta_m$  and  $\sigma_1, \dots, \sigma_{m-1}$  be arcs in  $\mathbb{R}^{2m-1}$  (parameterized by the first coordinate, which is equivalent to the distance to the origin) such that:

1.  $\delta_1(t) = t\mathbf{e}_1$ ;
2. for  $1 < j \leq m$ , if  $w_j$  is a node entry then  $\delta_j(t) = t\mathbf{e}_1 + t^\beta \mathbf{e}_j$ . Otherwise,  $\delta_j(t) = \delta_{r(j)}(t) + t^\alpha \mathbf{e}_j$ ;

3. for any  $j = 1, \dots, m-1$ , we define  $\sigma_j(t) = t\mathbf{e}_1 + t^\beta \mathbf{e}_{m+j}$ .

Consider the  $\beta$ -Hölder triangles  $T_j = \Delta(\delta_j, \sigma_j) \cup \Delta(\sigma_j, \delta_{j+1})$  for  $j = 1, \dots, m-1$ , and  $T'_j = \Delta(\sigma_j, \delta_{j+1}) \cup \Delta(\delta_{j+1}, \sigma_{j+1})$  for  $j = 1, \dots, m-2$  (see Definition 6.2.4). Let  $T_W = \bigcup_{j=1}^{m-1} T_j$ . The Hölder triangle  $T_W = T(\delta_1, \delta_m)$  is the  $\beta$ -Hölder triangle associated with the snake name  $W$ . Assuming that the link of  $T_W$  is oriented from  $\delta_1$  to  $\delta_m$ , the arcs  $\delta_j$  and  $\sigma_k$  appear in  $T_W$  in the following order  $\delta_1, \sigma_1, \delta_2, \dots, \delta_{m-1}, \sigma_{m-1}, \delta_m$ .

The following Lemma is a consequence of Definition 6.2.5.

**Lemma 6.2.1.** *The arcs  $\delta_1, \dots, \delta_m, \sigma_1, \dots, \sigma_{m-1}$  of Definition 6.2.5 satisfy the following:*

- (i)  $tord(\delta_i, \delta_j) = \begin{cases} \alpha & \text{if } w_i = w_j \\ \beta & \text{otherwise} \end{cases}$  for all  $i \neq j$ ,
- (ii)  $tord(\sigma_i, \delta_j) = \beta$  for all  $i$  and  $j$ ,
- (iii)  $tord(\sigma_i, \sigma_j) = \beta$  for all  $i$  and  $j$  with  $i \neq j$ .

**Lemma 6.2.2.** *Each  $T_j$  in Definition 6.2.5 is a normally embedded  $\beta$ -Hölder triangle.*

*Proof.* Note that, for each  $j = 1, \dots, m-1$ ,

$$T_j = \bigcup_{t \geq 0} ([\delta_j(t), \sigma_j(t)] \cup [\sigma_j(t), \delta_{j+1}(t)])$$

where  $[\delta_j(t), \sigma_j(t)]$  and  $[\sigma_j(t), \delta_{j+1}(t)]$  are straight line segments with a common endpoint. As  $n > 1$ , any consecutive letters of  $W$  are distinct, thus item (i) of Lemma 6.2.1 implies that  $tord(\delta_j, \delta_{j+1}) = \beta$ . Also, item (ii) of Lemma 6.2.1 implies that  $tord(\delta_j, \sigma_j) = \beta$  and  $tord(\sigma_j, \delta_{j+1}) = \beta$ . Then, the family of angles  $\phi(t)$  formed by the straight line segments  $[\delta_j(t), \sigma_j(t)]$  and  $[\sigma_j(t), \delta_{j+1}(t)]$  is bounded from below by a positive constant.

This implies that  $T_j$  is normally embedded. Indeed, given two arcs  $\gamma \subset \Delta(\delta_j, \sigma_j)$  and  $\gamma' \subset \Delta(\sigma_j, \delta_{j+1})$ , such that  $\gamma(t) \in [\delta_j(t), \sigma_j(t)]$  and  $\gamma'(t) \in [\sigma_j(t), \delta_{j+1}(t)]$ , we have  $|\gamma'(t) - \gamma(t)| > C \max(|\sigma_j(t) - \gamma(t)|, |\gamma'(t) - \sigma_j(t)|)$  for some constant  $C > 0$ , thus

$$itord(\gamma, \gamma') = \min(tord(\gamma, \sigma_j), tord(\sigma_j, \gamma')) = tord(\gamma, \gamma').$$

□

**Lemma 6.2.3.** *Each  $T'_j$  in Definition 6.2.5 is a normally embedded  $\beta$ -Hölder triangle.*

*Proof.* Note that  $T'_j = \bigcup_{0 \leq t} ([\sigma_j(t), \delta_{j+1}(t)] \cup [\delta_{j+1}(t), \sigma_{j+1}(t)])$ , with  $[\sigma_j(t), \delta_{j+1}(t)]$  and  $[\delta_{j+1}(t), \sigma_{j+1}(t)]$  being straight line segments with a common endpoint. The same argument as in the proof of Lemma 6.2.2 implies that  $T'_j$  is a normally embedded  $\beta$ -Hölder triangle. □

**Corollary 6.2.2.** *Let  $W$  be a snake name of length  $m > 2$ , and let  $T_W$  be the Hölder triangle associated with  $W$  in Definition 6.2.5. Then  $T_W$  is a non-singular Hölder triangle.*

*Proof.* Since, by Lemma 6.2.2, each  $T_j$  is non-singular, it is enough to prove that  $\delta_j$  is Lipschitz non-singular for each  $j = 2, \dots, m-1$ . But  $\delta_j \in I(T'_{j-1})$ , where  $T'_{j-1}$  is normally embedded by Lemma 6.2.3.  $\square$

**Lemma 6.2.4.** *Let  $W = [w_1 \cdots w_m]$  and  $T_W$  be as in Corollary 6.2.2. If  $W' = [w_j \cdots w_l]$  is a primitive subword of  $W$  then  $T(\delta_j, \delta_l) \subset T_W$  is a normally embedded  $\beta$ -Hölder triangle.*

*Proof.* Consider constants  $c_j, c_{j+1}, \dots, c_l$  and  $s_j, s_{j+1}, \dots, s_{l-1}$  such that

$$c_j < s_j < c_{j+1} < \dots < c_{l-1} < s_{l-1} < c_l, \quad (6.1)$$

and  $c_i = 0$  if  $r(i) = 1$  for some  $i$  (see Definition 6.2.3). Consider the ordered sequence of basis vectors

$$\mathcal{E} = \{\mathbf{e}_{r(j)}, \mathbf{e}_{m+j}, \mathbf{e}_{r(j+1)}, \mathbf{e}_{m+j+1}, \dots, \mathbf{e}_{m+l-1}, \mathbf{e}_{r(l)}\},$$

where each vector  $\mathbf{e}_{r(i)}$  is associated with the arc  $\delta_i$  and each vector  $\mathbf{e}_{m+i}$  is associated with the arc  $\sigma_i$  (see Definition 6.2.5).

We define the linear mapping  $\pi: \mathbb{R}^{2m-1} \rightarrow \mathbb{R}^2$  given by  $\pi(\mathbf{e}_1) = (1, 0)$ ,  $\pi(\mathbf{e}_{r(i)}) = (0, c_i)$ ,  $\pi(\mathbf{e}_{m+i}) = (0, s_i)$  and  $\pi(\mathbf{e}_i) = (0, 0)$  if  $\mathbf{e}_i \notin \mathcal{E} \cup \{\mathbf{e}_1\}$ . We claim that  $\pi$  maps  $T(\delta_j, \delta_l)$  one-to-one to the  $\beta$ -Hölder triangle  $T(\pi(\delta_j), \pi(\delta_l))$ . Indeed, for each  $i = j, \dots, l$ , we have

$$\pi(\delta_i(t)) = \pi(\delta_{r(i)}(t) + t^\alpha \mathbf{e}_i) = t \pi(\mathbf{e}_1) + t^\beta \pi(\mathbf{e}_{r(i)}) = (t, c_i t^\beta).$$

Similarly,  $\pi(\sigma_i) = \pi(t \mathbf{e}_1 + t^\beta \mathbf{e}_{m+i}) = (t, s_i t^\beta)$  for each  $i = j, \dots, l-1$ . Inequality (6.1) implies that the arcs  $\pi(\delta_j), \pi(\sigma_j), \pi(\delta_{j+1}), \dots, \pi(\sigma_{l-1}), \pi(\delta_l)$  are ordered in  $\mathbb{R}^2$  in the same way as  $\delta_j, \sigma_j, \delta_{j+1}, \dots, \sigma_{l-1}, \delta_l$  are ordered in  $T(\delta_j, \delta_l)$  (see Definition 6.2.5). Then, as each Hölder triangle  $\Delta(\delta_i, \sigma_i)$  and  $\Delta(\sigma_i, \delta_{i+1})$  is a union of straight line segments and  $\pi$  is a linear mapping, it follows that  $\pi: T(\delta_j, \delta_l) \rightarrow T(\pi(\delta_j), \pi(\delta_l))$  is one-to-one.

One can easily check that

$$\text{tord}(\pi(\delta_i), \pi(\sigma_k)) = \text{tord}(\pi(\delta_i), \pi(\delta_p)) = \text{tord}(\pi(\sigma_i), \pi(\sigma_p)) = \beta$$

for all  $i, k, p$  with  $i \neq p$ . We want to prove that given two arcs  $\gamma, \gamma' \subset T(\delta_j, \delta_l)$  we have  $\text{tord}(\pi(\gamma), \pi(\gamma')) \geq \text{tord}(\gamma, \gamma')$ . First, note that  $\pi$  is Lipschitz, since it is linear. Thus, there is

$K > 0$  such that  $|\pi(x) - \pi(y)| \leq K|x - y|$  for every  $x, y \in \mathbb{R}^{2m-1}$ . Given an arc  $\gamma \subset T(\delta_j, \delta_l)$ , we may assume that  $\gamma \subset T(\delta_i, \sigma_i)$  (if  $\gamma \subset T(\sigma_i, \delta_{i+1})$  the argument is the same). Reparameterizing  $\gamma$ , if necessary, we can assume that  $\gamma(t) \in [\delta_i(t), \sigma_i(t)]$  for any  $t \geq 0$ . Then, as  $\delta_i$  and  $\sigma_i$  are both parameterized by the first coordinate,  $\gamma$  is also parameterized by the first coordinate  $t$ . So, since  $\pi$  maps the first coordinate  $t$  of  $\delta_i$  and  $\sigma_i$  to the first coordinate  $t$  of  $\pi(\delta_i)$  and  $\pi(\sigma_i)$ , it follows that  $\pi(\gamma)$  is also parameterized by the first coordinate  $t$ . Hence, given two arcs  $\gamma, \gamma' \subset T(\delta_j, \delta_l)$  we have  $tord(\pi(\gamma), \pi(\gamma')) \geq tord(\gamma, \gamma')$ , since  $|\pi(\gamma(t)) - \pi(\gamma'(t))| \leq K|\gamma(t) - \gamma'(t)|$ .

Now we can finally prove that  $T(\delta_j, \delta_l)$  is normally embedded. Suppose, by contradiction, that there are arcs  $\gamma, \gamma' \subset T(\delta_j, \delta_l)$  such that  $tord(\gamma, \gamma') > itord(\gamma, \gamma')$ . Lemmas 6.2.2 and 6.2.3 imply that  $\gamma$  and  $\gamma'$  cannot be both contained in  $T_i$  or  $T'_k$  for every  $i = 1, \dots, m-1$  and  $k = 1, \dots, m-2$  (in particular,  $itord(\gamma, \gamma') = \beta$ ). Then, as the arcs  $\pi(\delta_j), \pi(\sigma_j), \pi(\delta_{j+1}), \dots, \pi(\sigma_{l-1}), \pi(\delta_l)$  are ordered as described above and  $tord(\pi(\delta_i), \pi(\sigma_k)) = tord(\pi(\delta_i), \pi(\delta_p)) = tord(\pi(\sigma_i), \pi(\sigma_p)) = \beta$  for all  $i, k, p$  with  $i \neq p$ , we have  $tord(\pi(\gamma), \pi(\gamma')) = \beta$ . However, we should have  $\beta = itord(\gamma, \gamma') < tord(\gamma, \gamma') \leq tord(\pi(\gamma), \pi(\gamma')) = \beta$ , a contradiction.  $\square$

**Corollary 6.2.3.** *Let  $W$  and  $T_W$  be as in Lemma 6.2.4. Then  $G(T_W) \subset Abn(T_W)$ .*

*Proof.* Note that each arc  $\delta_k$  of  $T_W$  is abnormal, for  $k = 2, \dots, m-1$ . Indeed, since  $W$  is a snake name and  $1 < k < m$ , there is a semi-primitive subword  $[w_j \cdots w_l]$  of  $W$  with  $j < k < l$ . In particular,  $[w_j \cdots w_k]$  and  $[w_k \cdots w_l]$  are also primitive. Thus, Lemma 6.2.4 and Corollary 6.2.2 imply that the Hölder triangles  $T(\delta_j, \delta_k)$  and  $T(\delta_k, \delta_l)$  are normally embedded and non-singular. As  $w_j = w_l$  we have  $tord(\delta_j, \delta_l) = \alpha > \beta = itord(\delta_j, \delta_l)$ . Hence,  $\delta_k$  is abnormal.

Now, consider an arc  $\gamma \in G(T_W)$ . Let  $\gamma \subset T_{k-1}$  and assume that  $k < m$ . As  $1 < k < m$ , we have  $\delta_k$  abnormal. Let  $\delta_j$  and  $\delta_l$  be arcs such that the Hölder triangles  $T(\delta_j, \delta_k)$  and  $T(\delta_k, \delta_l)$  are normally embedded and  $tord(\delta_j, \delta_l) = \alpha > \beta = itord(\delta_j, \delta_l)$ . If  $k-1 > 1$  then, as  $[w_j \cdots w_k]$  and  $[w_{k-1} \cdots w_l]$  are also primitive words, Lemma 6.2.4 and Corollary 6.2.2 imply that  $T(\delta_j, \gamma)$  and  $T(\gamma, \delta_l)$  are normally embedded and non-singular, since  $T(\delta_j, \gamma) \subset T(\delta_j, \delta_k)$  and  $T(\gamma, \delta_l) \subset T(\delta_{k-1}, \delta_l)$ . Thus,  $\gamma$  is abnormal. If  $k-1 = 1$  then  $j = 1$ . Hence, as  $\mu(T(\delta_1, \delta_k)) = \beta$ , Lemma 5.0.6 implies that  $\delta_k$  is contained in  $\beta$ -snake where  $\delta_1$  is a boundary arc. So, as  $itord(\gamma, \delta_l) = \beta$ , by Remark 4.1.1,  $\gamma$  is abnormal.

If  $k = m$  the argument to prove that  $\gamma$  is abnormal is similar (regarding  $\delta_{k-1}$  instead of  $\delta_k$ ) and will be omitted.  $\square$

**Theorem 6.2.3.** *Given a snake name  $W$ , there exists a snake  $T$  such that  $W = W(T)$  (see Definition 6.2.2).*

*Proof.* Let  $W = [w_1 \dots w_m]$  be a snake name with  $n$  distinct letters. If  $m = 2$  then  $W$  is the word associated with a bubble snake. Thus, assume that  $m > 2$ . Let  $T = T_W$  be the  $\beta$ -Hölder triangle associated with  $W$  (see Definition 6.2.5). We claim that  $T$  is a  $\beta$ -snake such that  $W = W(T)$ .

Corollary 6.2.2 implies that  $T$  is a non-singular  $\beta$ -Hölder triangle. So, to show that  $T$  is a  $\beta$ -snake it remains to prove that  $G(T) = Abn(T)$ . The inclusion  $Abn(T) \subset G(T)$  is obvious, and the inverse inclusion is given by Corollary 6.2.3.

Finally, as the link of  $T$  is oriented from  $\delta_1$  to  $\delta_m$  (see Definition 6.2.5), the  $i$ -th nodal zone of  $T$  is  $N_i = \{\gamma \in V(T) : itord(\gamma, \delta_i) > \beta\}$ , for  $i = 1, \dots, m$ , and the  $k$ -th node of  $T$  is  $\mathcal{N}_k = \bigcup_{i \in I_k} N_i$  for each  $k = 1, \dots, n$  (here  $I_k$  is as in Definition 6.1.3). In particular,  $T$  is a  $\beta$ -snake with  $m$  nodal zones and  $n$  nodes, such that  $W = W(T)$ .  $\square$

**Remark 6.2.3.** If we would consider  $T_W$  as in Definition 6.2.5 for  $m = 2$ , so that  $n = 1$ , we would obtain a Hölder triangle outer bi-Lipschitz equivalent to the Hölder triangle  $T$  in Example 4. Then  $T_W$  would contain a Lipschitz singular arc and would not be a snake.

**Remark 6.2.4.** The triangle  $T_W$  in Definition 6.2.5 is the simplest kind of a  $\beta$ -snake associated with the snake name  $W$ . All segments of  $T_W$  have multiplicity one, and the spectrum of each of its nodes consists of a single exponent  $\alpha$ . Moreover, if we consider a pancake decomposition  $\{X_k\}$  of  $T$  defined in Proposition 4.6.2, then a minimal pizza on any pancake  $X_k$ , for the distance function from  $X_k$  to any other pancake, has at most two pizza slices  $T_i$ , such that either  $Q_i = \{\beta\}$  is a point and  $\mu_i = \beta$  or  $Q_i = [\beta, \alpha]$  and  $\mu_i(q) = q$  for all  $q \in Q_i$ . Note that construction in Definition 6.2.5 can be slightly modified to obtain a snake with the given snake name  $W$  and prescribed cluster partitions of the sets  $\mathcal{S}(\mathcal{N}, \mathcal{N}')$  of its segments (see Remark 4.3.1 for conditions satisfied by such partitions).

**Remark 6.2.5.** The snake name ignores many geometric properties of a snake, such as pizza decompositions for the distance functions on pancakes associated with its segments, and the spectra of its nodes.

### 6.3 Weakly bi-Lipschitz maps and weak Lipschitz equivalence

In this Subsection we consider combinatorial and geometric significance of the cluster partitions of the sets  $\mathcal{S}(\mathcal{N}, \mathcal{N}')$  in Definition 4.3.1.

**Definition 6.3.1.** Let  $h : X \rightarrow X'$  be a homeomorphism of two  $\beta$ -Hölder triangles  $X$  and  $X'$ , bi-Lipschitz with respect to the inner metrics of  $X$  and  $X'$ . We say that  $h$  is *weakly outer bi-Lipschitz* when  $\text{tord}(h(\gamma), h(\gamma')) > \beta$  for any two arcs  $\gamma$  and  $\gamma'$  of  $X$  if, and only if,  $\text{tord}(\gamma, \gamma') > \beta$ . If such a homeomorphism exists, we say that  $X$  and  $X'$  are *weakly outer Lipschitz equivalent*.

**Theorem 6.3.1.** *Two  $\beta$ -snakes  $X$  and  $X'$  are weakly outer Lipschitz equivalent if, and only if, they can be oriented so that*

- (i) *Their snake names are equivalent, the nodes  $\mathcal{N}_1, \dots, \mathcal{N}_n$  of  $X$  are in one-to-one correspondence with the nodes  $\mathcal{N}'_1, \dots, \mathcal{N}'_n$  of  $X'$ , and the nodal zones  $N_1, \dots, N_m$  of  $X$  are in one-to-one correspondence with the nodal zones  $N'_1, \dots, N'_m$  of  $X'$ ;*
- (ii) *For any two nodes  $\mathcal{N}_j$  and  $\mathcal{N}_k$  of  $X$ , and the corresponding nodes  $\mathcal{N}'_j$  and  $\mathcal{N}'_k$  of  $X'$ , each cluster of the cluster partition of the set  $\mathcal{S}(\mathcal{N}'_j, \mathcal{N}'_k)$  (see Definition 4.3.1) consists of the segments of  $X'$  corresponding to the segments of  $X$  contained in a cluster of the cluster partition of the set  $\mathcal{S}(\mathcal{N}_j, \mathcal{N}_k)$ .*

*Proof.* It follows from the definition 6.3.1 that a weakly outer bi-Lipschitz homeomorphism  $h : X \rightarrow X'$  defines equivalence of the snake names  $W = W(X)$  and  $W' = W(X')$ , and identifies cluster partitions of the sets  $\mathcal{S}(N_j, N_k)$  and  $\mathcal{S}(N'_j, N'_k)$  for any  $j$  and  $k$ . Thus we have to prove that conditions (i) and (ii) of Theorem 6.3.1 imply weak outer Lipschitz equivalence of the snakes  $X$  and  $X'$ .

Let us assume first that  $X$  and  $X'$  are not bubble or spiral snakes, so any segment of each of them has two adjacent nodal zones in two distinct nodes. Since the snake names  $W$  and  $W'$  are equivalent, each nodal zone  $N_j$  of  $X$  corresponds to the  $j$ -th entry  $w_j$  of  $W$  and each nodal zone  $N'_j$  of  $X'$  corresponds to the  $j$ -th entry  $w'_j$  of  $X'$ . Also, nodal zones  $N_j$  and  $N_k$  of  $X$  (resp.,  $N'_j$  and  $N'_k$  of  $X'$ ) belong to the same node if, and only if,  $w_j = w_k$  (resp.,  $w'_j = w'_k$ ). Selecting an arc  $\gamma_j$  in each nodal zone  $N_j$  of  $X$  (a boundary arc if  $N_j$  is a boundary nodal zone of  $X$ ) and an arc  $\gamma'_j$  in each nodal zone  $N'_j$  of  $X'$  (a boundary arc if  $N'_j$  is a boundary nodal zone of  $X'$ ) we obtain, according to Proposition 4.6.2, pancake decompositions of  $X$  and  $X'$ , such that each pancake  $X_j = T(\gamma_j, \gamma_{j+1})$  of  $X$  (resp., pancake  $X'_j = T(\gamma'_j, \gamma'_{j+1})$  of  $X'$ ) is a  $\beta$ -Hölder triangle corresponding to a segment of  $X$  with adjacent nodal zones  $N_j$  and  $N_{j+1}$  (resp., to a segment of  $X'$  with adjacent nodal zones  $N'_j$  and  $N'_{j+1}$ ).

We construct a weakly outer bi-Lipschitz homeomorphism  $h : X \rightarrow X'$  as follows.

First, we define  $h$  on each arc  $\gamma_j$  as the map  $\gamma_j \rightarrow \gamma'_j$  consistent with the parameterisations of both arcs by the distance to the origin. Next, for each nodes  $\mathcal{N}$  and  $\mathcal{N}'$  of  $X$ , if the set

$\mathcal{S} = \mathcal{S}(\mathcal{N}, \mathcal{N}')$  is not empty, we choose one pancake  $X_j = T(\gamma_j, \gamma_{j+1})$  corresponding to a segment from each cluster of the cluster partition of  $\mathcal{S}$ , and define a bi-Lipschitz homeomorphism  $h_j : X_j \rightarrow X'_j$  consistent with the previously defined mappings for the arcs  $\gamma_j$  and  $\gamma_{j+1}$ . Finally, for any cluster of  $\mathcal{S}$  containing a segment with the homeomorphism  $h = h_j$  defined on the corresponding pancake  $X_j$ , if there is another segment in that cluster, we define  $h$  on the pancake  $X_k$  corresponding to that segment as follows. Since pancakes  $X_j$  and  $X_k$  correspond to segments in the same cluster, pancakes  $X'_j$  and  $X'_k$  also correspond to segments in the same cluster. It follows from Proposition 2.4.1 that there is a bi-Lipschitz homeomorphism  $h_{kj} : X_k \rightarrow X_j$  such that  $tord(\gamma, h_{kj}(\gamma)) > \beta$  for each arc  $\gamma \subset X_k$ , and a bi-Lipschitz homeomorphism  $h'_{jk} : X'_j \rightarrow X'_k$  such that  $tord(\gamma', h'_{jk}(\gamma')) > \beta$  for each arc  $\gamma' \subset X'_j$ . Then  $h : X_k \rightarrow X'_k$  is defined as the composition of  $h_{kj}$ ,  $h_j$  and  $h'_{jk}$ . This defines an outer bi-Lipschitz homeomorphism  $h : X \rightarrow X'$ .

If  $X$  and  $X'$  are either bubble snakes or spiral snakes then the Hölder triangles  $X_j$  and  $X'_j$  are not normally embedded. Thus, the above construction should be slightly modified by adding extra arcs  $\lambda_j$  in each segment of  $X$  and  $\lambda'_j$  in each segment of  $X'$  so that  $tord(\lambda_j, \lambda_k) > \beta$  and  $tord(\lambda'_j, \lambda'_k) > \beta$  for all  $j$  and  $k$ .  $\square$

**Remark 6.3.1.** The sets of segments  $\mathcal{S}(\mathcal{N}, \mathcal{N}')$  in Definition 4.3.1 can be recovered from the snake name  $W = W(X)$  of a snake  $X$  as follows. Let  $\mathcal{N}$  and  $\mathcal{N}'$  be two nodes of  $X$  associated with the letters  $x$  and  $x'$  of  $W$ . Then the set  $\mathcal{S}(\mathcal{N}, \mathcal{N}')$  can be identified with the set  $\mathcal{S}(x, x')$  of pairs of consecutive entries  $(w_j, w_{j+1})$  of  $W$  such that either  $(w_j, w_{j+1}) = (x, x')$  or  $(w_j, w_{j+1}) = (x', x)$ . Accordingly, a cluster partition of the set  $\mathcal{S}(\mathcal{N}, \mathcal{N}')$  in Definition 4.3.1 can be identified with a partition of  $\mathcal{S}(x, x')$ . Remark 4.3.1 implies that, if  $X$  is a spiral snake, then  $\mathcal{N} = \mathcal{N}'$  and the partition of  $\mathcal{S}(\mathcal{N}, \mathcal{N})$  consists of a single cluster. Also, if  $w_{j-1} = w_{j+1}$  in  $W(X)$  then the pairs  $(w_{j-1}, w_j)$  and  $(w_j, w_{j+1})$  cannot belong to the same cluster of partition.

## 6.4 Binary snakes and their names

In this subsection we consider binary snakes (see Definition 6.4.1). They play important role in the combinatorial classification of snakes since any snake name can be reduced to a binary one (see Definition 6.4.2).

**Definition 6.4.1.** A *binary snake name* is a snake name  $W$  which is also a binary word (see Definition 6.1.4). A snake  $T$  is *binary* if  $W(T)$  is a binary snake name. Alternatively, a snake  $T$  is binary if each of its nodes contains exactly two nodal zones.



**Definition 6.4.2.** Let  $W$  be a snake name and  $x$  a letter of  $W$ . If  $x$  appears  $p > 2$  times in  $W$  and  $W = X_0xX_1x \cdots xX_{p-1}xX_p$ , we replace  $x$  by  $p - 1$  distinct new letters  $x_1, \dots, x_{p-1}$ , and define the *binary reduction* of  $W$  with respect to  $x$  as the word

$$W_x = X_0x_1X_1x_1x_2X_2x_2x_3 \cdots x_{p-2}x_{p-1}X_{p-1}x_{p-1}X_p. \quad (6.2)$$

Note that the first and last entries of  $x$  are replaced by a single letter each, while every other entry of  $x$  is replaced by two letters.

**Proposition 6.4.1.** *The word  $W_x$  in Definition 6.4.2 is a snake name.*

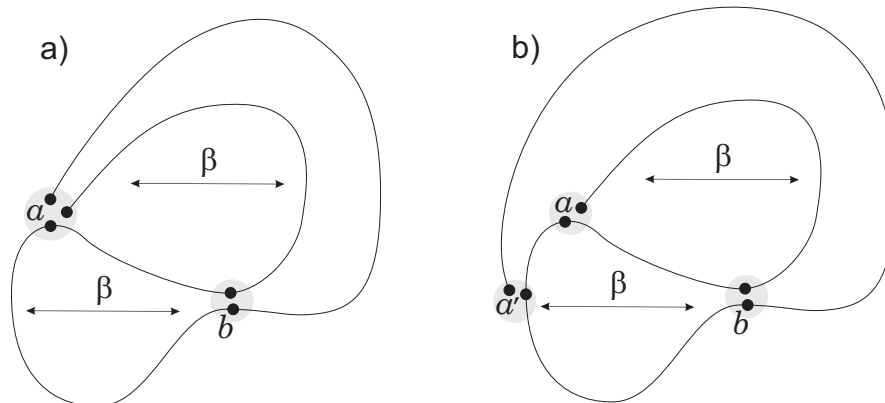
*Proof.* Note that  $W_x$  satisfies condition (i) of Definition 6.2.1 because each of the new letters  $x_i$  in  $W_x$  appears exactly twice, and each other letter appears at least twice, since  $W$  is a snake name. It remains to prove that  $W_x$  satisfies condition (ii) of Definition 6.2.1.

Let  $w$  be an entry of  $W$  other than  $x$  such that there is a semi-primitive subword  $[w_j \cdots w_l]$  of  $W$  containing  $w$ , where  $w_j = w_l \neq w$ . If  $w_j = w_l = x$  then  $w$  belongs to one of the subwords  $X_k$  of  $W$  and  $x_kX_kx_k$  is a semi-primitive subword of  $W_x$  containing  $w$ . Otherwise  $[w_j \cdots w_l]$  contains at most one entry of  $x$ , and replacing that entry with one or two new letters results in a semi-primitive subword of  $W_x$  containing  $w$ .

If  $w = x$  then  $[w_j \cdots w_l]$  does not contain other entries of  $x$ , and replacing  $x$  with one or two new letters results in a semi-primitive subword of  $W_x$  containing the new entries.  $\square$

**Remark 6.4.1.** The binary reduction could be geometrically interpreted as splitting a node with more than two nodal zones (see Fig. 14).

Figure 14 – Reducing a non-binary snake (a) to a binary snake (b)



Source: elaborated by the author.

**Remark 6.4.2.** Any non-binary snake name  $W$  could be reduced to a binary snake name by applying binary reduction to each letter that appears in  $W$  more than twice. If there are several such letters, the resulting binary snake name does not depend on the order of the letters to which binary reduction is applied.

## 6.5 Recursion for the number of binary snake names

**Proposition 6.5.1.** *If  $W = aXaZ$  is a binary snake name then  $aXa$  is semi-primitive.*

*Proof.* Since  $W$  is a snake name, there is a semi-primitive subword  $[w_j \cdots w_l]$  of  $W$  such that  $j < 2 < l$ . Thus  $j = 1$  and  $w_j = w_l = a$ . As  $W$  is binary,  $aXa$  is the only option for such a subword.  $\square$

**Definition 6.5.1.** Given a word  $W$  and a letter  $x$  of  $W$  that appears exactly twice, we write  $x^-$  and  $x^+$  to denote the first and second entries of  $x$  in  $W$ , respectively. If  $W = X_1x^-X_2x^+X_3$  we write  $W - \{x\}$  to denote the word  $X_1X_2X_3$  representing *deletion* of the letter  $x$  from  $W$ .

**Lemma 6.5.1.** *Let  $W = abZ$  be a binary snake name, and  $W' = W - \{a\}$ . Then,  $W'$  is a snake name if and only if  $[b^- \cdots b^+]$  is a semi-primitive subword of  $W$ .*

*Proof.* Given a letter  $x$  of  $W$ , let  $W'(x)$  be the subword of  $W'$  obtained by deleting the letter  $a$  from the subword  $[x^- \cdots x^+]$  of  $W$ .

If  $W'$  is a snake name then Proposition 6.5.1 applied to  $W'$  (note that  $b$  is the first letter of  $W'$ ) implies that  $W'(b)$  is a semi-primitive subword of  $W'$ . As  $W'(b)$  does not contain  $a$ , the subword  $[b^- \cdots b^+]$  of  $W$ , obtained by inserting the second entry of  $a$  into  $W'(b)$ , is also semi-primitive.

Conversely, suppose that  $[b^- \cdots b^+]$  is a semi-primitive subword of  $W$ . Since  $W'$  is a binary word, it satisfies condition (i) of Definition 6.2.1, and we have only to check that condition (ii) is satisfied. Since  $[b^- \cdots b^+]$  is a semi-primitive subword of  $W$ , the subword  $W'(b)$  of  $W'$  is also semi-primitive. Thus any entry  $w \neq b$  of  $W'$  contained in  $W'(b)$  satisfies condition (ii) of Definition 6.2.1. Let  $w$  be an entry of  $W'$ , other than the last one, not contained in the subword  $W'(a)$ . Since  $W$  is a snake name, there exists a semi-primitive subword  $[x^- \cdots x^+]$  of  $W$  containing the corresponding entry  $w$  of  $W$ . Then  $w \neq x \neq a$ , and  $W'(x)$  is a semi-primitive subword of  $W'$  containing  $w$ . Since any entry of  $W'$  either belongs to  $W'(b)$  or does not belong to  $W'(a)$ , this implies that all entries of  $W'$ , except the first and last ones, satisfy condition (ii) of Definition 6.2.1. Thus  $W'$  is a snake name.  $\square$

**Lemma 6.5.2.** *Let  $W$  be a snake name where a letter  $x$  appears exactly twice. If the subword  $[x^- \cdots x^+]$  of  $W$  is not semi-primitive then  $W - \{x\}$  is a snake name.*

*Proof.* Since  $W$  is a snake name,  $W - \{x\}$  satisfies condition (i) of Definition 6.2.1. Let  $w \neq x$  be an entry of  $W$  such that there is a semi-primitive subword  $[w_j \cdots w_l]$  of  $W$  containing  $w$ , with  $w_j = w_l \neq w$ . Since  $[x^- \cdots x^+]$  is not semi-primitive,  $w_j \neq x$ , and deleting  $x$  from  $W$  results in a semi-primitive subword  $[w_j \cdots w_l]$  of  $W - \{x\}$  containing  $w$ . This implies that all entries of  $W - \{x\}$ , except the first and last ones, satisfy condition (ii) of Definition 6.2.1. Then  $W - \{x\}$  is a snake name.  $\square$

**Proposition 6.5.2.** *Let  $W = abZ$  be a binary snake name. If  $W - \{a\}$  is not a snake name then  $W - \{b\}$  is a snake name.*

*Proof.* If  $W - \{a\}$  is not a snake name then, by Lemma 6.5.1,  $[b^- \cdots b^+]$  is not semi-primitive. Then, Lemma 6.5.2 implies that  $W - \{b\}$  is a snake name.  $\square$

**Remark 6.5.1.** We can (similarly) prove a symmetric version of Proposition 6.5.2, i.e., if  $W = XYZ$  is a binary snake name and  $W - \{z\}$  is not a snake name then  $W - \{y\}$  is a snake name.

**Definition 6.5.2.** Given a binary snake name  $W = aXaZ$  of length  $2m > 2$ , we define its *parameters* as the numbers  $j$  and  $k$  where  $j$  is the position of  $a^+$  and  $w_k$  is the first entry of  $W$  such that  $[w_2 \cdots w_k]$  is not a primitive subword. For  $m > 1$ , we define  $\mathscr{W}_m(j, k)$  as the set of all binary snake names of length  $2m$  with parameters  $j$  and  $k$ .

**Remark 6.5.2.** Note that parameter  $k$  is not defined for the bubble snake name  $[aa]$ . The word  $[abab] \in \mathscr{W}_2(3, 4)$  is the only binary snake name of length 4. For  $m \geq 3$ , the set  $\mathscr{W}_m(j, k)$  is nonempty only when  $3 \leq j < k$  and  $5 \leq k \leq m + 2$ . In particular,  $[abacbc] \in \mathscr{W}_3(3, 5)$  and  $[abcabc] \in \mathscr{W}_3(4, 5)$  are the only binary snake names of length 6.

**Definition 6.5.3.** Given a binary snake name  $W = [w_1 \cdots w_{2m}] \in \mathscr{W}_m(j, k)$ , we can obtain new binary words of length  $2m + 2$  inserting a new letter at two positions in  $W$  as follows:

- (A) For  $l = 2, \dots, j$ , insert the first copy of a new letter  $a$  to  $W$  in front of  $w_1$ , and a second copy between  $w_{l-1}$  and  $w_l$ .
- (B) For  $l = k + 1, \dots, 2m$ , insert the first copy of a new letter  $b$  to  $W$  between  $w_1$  and  $w_2$ , and a second copy between  $w_{l-1}$  and  $w_l$ .

**Example 19.** The binary snake names  $[abacbc] \in \mathcal{W}_3(3,5)$  and  $[abcabc] \in \mathcal{W}_3(4,5)$  can be obtained from the binary snake name  $[abab] \in \mathcal{W}_2(3,4)$  by applying operation (A) with  $l = 2$  and  $l = 3$ , respectively, and renaming the letters. Applying operations (A) with  $l = 2, 3$  and (B) with  $l = 6$  to  $W = [abacbc]$  we obtain, renaming the letters, the words  $[abacbdcd] \in \mathcal{W}_4(3,5)$ ,  $[abcabdcd] \in \mathcal{W}_4(4,5)$  and  $[abcadcbd] \in \mathcal{W}_4(4,6)$ . Applying operations (A) with  $l = 2, 3, 4$  and (B) with  $l = 6$  to  $W = [abcabc]$  we obtain, renaming the letters, the words  $[abacdbcd] \in \mathcal{W}_4(3,6)$ ,  $[abcadbcd] \in \mathcal{W}_4(4,6)$ ,  $[abcdabcd] \in \mathcal{W}_4(5,6)$  and  $[abcdacbd] \in \mathcal{W}_4(5,6)$ . Note that all these words are binary snake names, and that all 7 binary snake names of length 8 are thus obtained (see Propositions 6.5.3 and 6.5.4 and Theorem 6.5.5 below).

**Proposition 6.5.3.** *If  $W = [w_1 \cdots w_{2m}] \in \mathcal{W}_m(j, k)$  is a binary snake name then the words obtained from  $W$  by applying any operations (A) and (B) in Definition 6.5.3 are also binary snake names.*

*Proof.* Let  $W_A$  be the word obtained by applying operation (A) in Definition 6.5.3 to  $W$  for some  $l \in \{2, \dots, j\}$ . Since  $W_A$  is a binary word, condition (i) of Definition 6.2.1 is satisfied. As the first entry  $a^-$  of the letter  $a$  is the first letter of  $W_A$ , we have to check condition (ii) of Definition 6.2.1 for the second entry  $a^+$  of  $a$ , and for any entry  $w \neq a$  of  $W_A$  other than its last entry. Since  $W \in \mathcal{W}_m(j, k)$ , we have  $w_1 = w_j$  and  $[w_1 \cdots w_j]$  is a semi-primitive subword of  $W$ , by Proposition 6.5.1. Since  $l \leq j$ , the corresponding subword  $[w_1 \cdots a^+ \cdots w_j]$  of  $W_A$  is also semi-primitive. Since  $W$  is a snake name, any entry  $w \neq a$  of  $W_A$ , other than its last entry, corresponds to an entry of  $W$  contained in some semi-primitive subword  $[w_p \cdots w \cdots w_q]$  of  $W$ , where  $w_p = w_q \neq w$ . The corresponding subword  $[w_p \cdots w \cdots w_q]$  of  $W_A$  is also semi-primitive (it is either the same as in  $W$  or contains one extra entry  $a^+$ ). Thus condition (ii) of Definition 6.2.1 is satisfied for any entry  $w \neq a$  of  $W_A$ . Then  $W_A$  is a snake name.

Let now  $W_B$  be the word obtained by applying operation (B) in Definition 6.5.3 to  $W$  for some  $l \in \{k+1, \dots, 2m\}$ . Since  $W_B$  is a binary word, condition (i) of Definition 6.2.1 is satisfied. The first entry  $b^-$  of the letter  $b$  is contained in the semi-primitive subword  $[w_1 b^- \cdots w_j]$  of  $W_B$ , and its second entry  $b^+$ , inserted between the entries  $w_{l-1}$  and  $w_l$  of  $W$ , belongs to the semi-primitive subword of  $W_B$  corresponding to a semi-primitive subword  $[w_p \cdots w_{l-1} \cdots w_q]$  of  $W$  containing  $w_{l-1}$ . Note that, as  $l > k > j$ , we have  $w_p = w_q \neq w_1$ , thus the subword  $[w_p \cdots b^+ \cdots w_q]$  of  $W_B$  cannot contain  $b^-$  and remains semi-primitive. The same argument as for  $W_A$  shows that condition (ii) of Definition 6.2.1 is satisfied for any entry  $w \neq b$  of  $W_B$ . Then  $W_B$  is a snake name.  $\square$

**Remark 6.5.3.** Note that a word  $W_B$ , obtained by applying operation (B) in Definition 6.5.3 to a binary snake name  $W$ , would be a binary snake name even if  $l > j$  instead of  $l > k$  was allowed. However, condition  $l > k$  in Definition 6.5.3 implies that the subword  $[b^- \cdots b^+]$  of  $W_B$  is not semi-primitive, thus  $W_B$  cannot be obtained applying the operation (A) to any binary snake name. Similarly, the word  $W_A$  cannot be obtained applying the operation (B) to any binary snake name.

**Remark 6.5.4.** If  $W_A$  (resp.,  $W_B$ ) is obtained from a binary snake name  $W$  by applying operation (A) (resp., (B)) then the first (resp., second) letter of  $W_A$  (resp.,  $W_B$ ) can be deleted, resulting in the original word  $W$ . Note that “deletion” operations are unique, while “insertion” operations are not.

**Proposition 6.5.4.** Any binary snake name of length  $2m + 2$  could be obtained from a binary snake name of length  $2m$  by applying exactly one of the operations (A) and (B) as in Definition 6.5.3.

*Proof.* Let  $W = abZ$  be a binary snake name of length  $2m + 2$ . If  $W - \{a\}$  is a snake name then  $W$  can be obtained from  $W - \{a\}$  by applying operation (A) to add back the deleted letter  $a$ . If  $W - \{a\}$  is not a snake name then, by Proposition 6.5.2,  $W - \{b\}$  is a snake name and, similarly,  $W$  can be obtained from  $W - \{b\}$  by applying operation (B).

Finally, if  $W$  was obtained from a word of length  $2m$  by applying operation (A) (resp., (B)) then  $W$  cannot be obtained from any word of length  $2m$  by applying operation (B) (resp., (A)) (see Remark 6.5.3).  $\square$

**Theorem 6.5.5.** Let  $M_m$  be the number of all binary snake names of length  $2m$ , and let  $M_m(j, k) = |\mathcal{W}_m(j, k)|$  be the number of binary snake names of length  $2m > 2$  with parameters  $j$  and  $k$  (see Definition 6.5.2). Then  $M_1 = 1$ ,  $M_2 = M_2(3, 4) = 1$  and, for  $m \geq 2$ ,

$$M_{m+1}(j, k) = M_{m,A}(j, k) + M_{m,B}(j, k), \quad (6.3)$$

where

$$M_{m,A}(j, k) = \sum_{l=k-1}^{m+2} M_m(k-2, l) \quad (6.4)$$

and

$$M_{m,B}(j, k) = (2m - k + 1)M_m(j-1, k-1). \quad (6.5)$$

Consequently,

$$M_{m+1} = \sum_{3 \leq j < k, 5 \leq k \leq m+3} M_{m+1}(j, k). \quad (6.6)$$

*Proof.* Since the bubble snake name  $[aa]$  is the only binary snake name of length 2, and the word  $[abab]$  is the only binary snake name of length 4, we have  $M_1 = 1$ ,  $M_2 = M_2(3, 4) = 1$ . For  $m \geq 2$ , Proposition 6.5.4 implies that it is enough to count separately the binary snake names of length  $2m + 2$  obtained by applying operations (A) and (B) from the binary snake names of length  $2m$ .

Note that  $M_{m,A}(j, k)$  denotes the number of binary snake names of length  $2m + 2$  with parameters  $j$  and  $k$  obtained from binary snake names of length  $2m$  by applying operation (A). Each such binary snake name  $W'$  of length  $2m$  must have parameters  $j' = k - 2$  and  $k' \in \{k - 1, \dots, m + 2\}$ . This implies (6.4).

Similarly,  $M_{m,B}$  denotes the number of binary snake names of length  $2m + 2$  with parameters  $j$  and  $k$  obtained from binary snake names of length  $2m$  by applying operation (B). Each such snake name  $W'$  of length  $2m$  must have parameters  $j' = j - 1$  and  $k' = k - 1$ . For each of them we have  $2m - k' = 2m - (k - 1) = 2m - k + 1$  possibilities to place the second entry of the new letter. This implies (6.5).

Adding up these two numbers, we obtain the formula (6.3). Remark 6.5.2 implies (6.6). □

## 6.6 Binary snake names and standard Young tableaux

In this subsection we assign a standard Young tableau (SYT) of shape  $(m - 1, m - 1)$  to a binary snake name of length  $2m$ .

**Definition 6.6.1.** A *Young diagram*, or *shape*,  $\lambda = (\lambda_1, \lambda_2, \dots)$  of size  $n$ , where  $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$  and  $\lambda_1 + \lambda_2 + \dots = n$  (see, e.g., (FULTON, 1997) pp. 1-2) is a collection of cells arranged in left-justified rows of lengths  $\lambda_j$ . A *filling* of  $\lambda$  means placing positive integers in each of its cells. A *standard Young tableau* (SYT) of shape  $\lambda$  is a filling of  $\lambda$  with the numbers from 1 to  $n$ , each of them occurring exactly once, so that the numbers in each row and each column of  $\lambda$  are strictly increasing.

**Definition 6.6.2.** Let  $W = [w_1 \cdots w_{2m}]$  be a binary snake name. We assign to  $W$  the following filling  $\mathcal{T}(W)$  of shape  $\lambda = (m - 1, m - 1)$ : for  $i = 2, \dots, 2m - 1$ , we place the number  $i - 1$  into the first empty cell of the first row of  $\lambda$  if  $w_i \neq w_j$  for all  $j < i$ , and into the first empty cell of the second row of  $\lambda$  otherwise. Alternatively,  $i - 1$  is inserted into the first row of  $\lambda$  if  $w_i$  is a node entry of  $W$ , and into the second row otherwise.

**Proposition 6.6.1.** *The filling  $\mathcal{T}(W)$  assigned to a binary snake name  $W = [w_1 \cdots w_{2m}]$  in Definition 6.6.2 is a standard Young tableau.*

*Proof.* By Definition 6.6.2, each number  $1, \dots, 2m - 2$  appears in  $\mathcal{T}(W)$  exactly once, and the numbers in each row are strictly increasing. To check that the numbers are increasing in columns, suppose that  $W \in \mathcal{W}_{j,k}$  for some  $j$  and  $k$ , and that the  $\ell$ -th cell of the second row of  $\mathcal{T}(W)$  contains the number  $i - 1$ . This means that  $w_i$  is the second entry of some letter of  $W$ , and that exactly  $\ell$  distinct letters appear twice in the subword  $[w_1 \cdots w_i]$  of  $W$ . Note that at least one letter of  $W$  must appear only once in the subword  $[w_1 \cdots w_i]$  (Proposition 6.5.1 implies that  $j \leq i$ , thus the first letter of  $W$  appears twice in  $[w_1 \cdots w_i]$ ). Otherwise  $i = 2\ell$  would be even,  $i + 1 < 2m$ , and there will be no semi-primitive subword  $[x^- \cdots w_{i+1} \cdots x^+]$  of  $W$  containing  $w_{i+1}$ , in contradiction to  $W$  being a snake name. This implies that the subword  $[w_2 \cdots w_{i-1}]$  contains at least  $\ell$  node entries of  $W$ . Thus the number in the  $\ell$ -th cell of the first row of  $\mathcal{T}(W)$  is strictly less than  $i - 1$ . This completes the proof.  $\square$

**Remark 6.6.1.** Note that Proposition 6.6.1 does not necessarily hold for binary words which are not snake names. For example, it is not true for the binary words  $W = [aabb]$  and  $W = [ababcdcd]$ .

**Remark 6.6.2.** The empty SYT of shape  $(0,0)$  is assigned to the bubble snake

name  $[aa]$ , and the single SYT  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  of shape  $(1,1)$  is assigned to the binary snake

name  $[abab]$ . Two SYTs  $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  of shape  $(2,2)$  are assigned to the

binary snake names  $[abacbc]$  and  $[abcabc]$ , respectively. Consider next the SYT

$\lambda = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{bmatrix}$  of shape  $(3,3)$ . The words  $W = [abcadbcd]$  and  $W' = [abcadcbd]$

are distinct binary snake names such that  $\mathcal{T}(W) = \mathcal{T}(W') = \lambda$ . Thus the same SYT may be assigned to several binary snake names.

**Definition 6.6.3.** Let  $\mathcal{T}$  be a standard Young tableau of shape  $(m - 1, m - 1)$ . We define a binary word  $W = W(\mathcal{T}) = [w_1 \cdots w_{2m}]$  with  $m$  distinct letters  $x_1, \dots, x_m$  as follows. If  $m = 1$  and  $\mathcal{T}$  is empty then  $W(\mathcal{T}) = [x_1 x_1]$ . If  $m > 1$ , we set  $w_1 = x_1$ ,  $w_{2m} = x_m$  and, for  $1 < i < 2m$ ,  $w_i = x_{k+1}$  (resp.,  $w_i = x_k$ ) if the  $k$ -th cell of the first row (resp., second row) of  $\mathcal{T}$  contains the number  $i - 1$ .

**Remark 6.6.3.** If  $1 < i < 2m$  and the  $k$ -th cell of the first row of  $\mathcal{T}$  contains  $i - 1$ , it follows from Definition 6.6.3 that the subword  $[w_1 \cdots w_i]$  of  $W(\mathcal{T})$  contains exactly  $k + 1$  first entries of the letters  $x_1, \dots, x_{k+1}$ , with  $w_i = x_{k+1}$ , and at most  $k - 1$  second entries of letters  $x_j$  for some  $j < k$ . In particular, there are at least two more first entries than second entries of the letters in  $[w_1 \cdots w_i]$ .

If the  $k$ -th cell of the second row of  $\mathcal{T}$  contains  $i - 1$ , then the subword  $[w_1 \cdots w_i]$  of  $W(\mathcal{T})$  contains  $\ell \geq k + 1$  first entries of the letters  $x_1, \dots, x_\ell$  and exactly  $k$  second entries of the letters  $x_1, \dots, x_k$ , with  $w_i = x_k$ . In particular, there are more first entries than second entries of the letters in  $[w_1 \cdots w_i]$ .

This implies that the first entries of all letters  $x_j$  appear in  $W(\mathcal{T})$  in increasing order of their indices  $j$ . Similarly, the second entries of all letters  $x_j$  appear in  $W(\mathcal{T})$  in increasing order of their indices  $j$ .

**Definition 6.6.4.** An *inversion* in a binary word  $W$  is a pair of distinct letters  $x$  and  $y$  contained in  $W$  such that the subword  $[x^- \cdots x^+]$  of  $W$  contains both entries of  $y$ . We say that a binary word  $W$  is *inversion free* if it has no inversions.

**Lemma 6.6.1.** *If  $\mathcal{T}$  is a standard Young tableau of shape  $(m - 1, m - 1)$  then  $W(\mathcal{T})$  in Definition 6.6.3 is an inversion free binary word.*

*Proof.* For  $m = 1$  the statement is true since  $\mathcal{T}$  is empty and  $W(\mathcal{T}) = [x_1 x_1]$ , thus we may assume that  $m > 1$ .

Let us show first that  $W(\mathcal{T})$  is binary. The letter  $x_1$  is the first letter of  $W(\mathcal{T})$ , and  $w_i = x_1$  for  $i > 1$  only if the first cell of the second row of  $\mathcal{T}$  contains  $i - 1$ . Thus  $x_1$  appears in  $W(\mathcal{T})$  exactly twice. Similarly,  $x_m$  is the last letter of  $W(\mathcal{T})$ , and  $w_i = x_m$  for  $i < 2m$  only if the last cell of the first row of  $\mathcal{T}$  contains  $i - 1$ . Thus  $x_m$  appears in  $W(\mathcal{T})$  exactly twice. If  $1 < k < m$  then  $w_i = w_j = x_k$  for  $i < j$  only when the cell  $(k - 1)$  of the first row contains  $i - 1$  and the cell  $k$  of the second row contains  $j - 1$ . Thus  $x_k$  appears in  $W(\mathcal{T})$  exactly twice. This proves that  $W(\mathcal{T})$  is a binary word.

To prove that  $W(\mathcal{T})$  is inversion free, consider the entries in  $W(\mathcal{T})$  of two letters  $x_k$  and  $x_\ell$  for  $k < \ell$ . If  $1 < k < m$  then the two entries of  $x_k$  are  $w_i$  and  $w_j$  where  $i - 1$  is in the cell  $k - 1$  of the first row of  $\mathcal{T}$  and  $j - 1$  is in the cell  $k$  of its second row, while the two entries of  $x_\ell$  are  $w_{i'}$  and  $w_{j'}$  where  $i' - 1$  is in the cell  $\ell - 1$  of the first row of  $\mathcal{T}$  and  $j' - 1$  is in the cell  $\ell$  of



its second row. Since  $\mathcal{T}$  is a standard Young tableau, we have  $i < i'$  and  $j < j'$ , thus  $x_k$  and  $x_\ell$  is not an inversion.

The proofs for the cases  $k = 1$  and  $\ell = m$  are similar.  $\square$

**Proposition 6.6.2.** *The word  $W(\mathcal{T})$  in Definition 6.6.3 is an inversion free binary snake name.*

*Proof.* By Lemma 6.6.1,  $W(\mathcal{T})$  is an inversion free binary word. In particular, condition (i) of Definition 6.2.1 is satisfied. We are going to prove that condition (ii) of Definition 6.2.1 is also satisfied. For  $m = 1$  the statement is true since  $\mathcal{T}$  is empty and  $W(\mathcal{T}) = [x_1x_1]$ , thus we may assume that  $m > 1$ .

Note first that any subword  $[x^- \cdots x^+]$  of an inversion free binary word is semi-primitive. Let  $w_i$  be an entry of  $W(\mathcal{T})$  where  $1 < i < 2m$  which is the first entry of some of its letters. Remark 6.6.3 implies that the subword  $[w_1 \cdots w_{i-1}]$  of  $W(\mathcal{T})$  contains only one entry of some letter  $x$ . Since  $W(\mathcal{T})$  is inversion free,  $[x^- \cdots x^+]$  is its semi-primitive subword containing  $w_i$ . The proof for the case when  $w_i$  is the second entry of some letter is similar.  $\square$

**Lemma 6.6.2.** *Let  $\mathcal{T}$  be a standard Young tableau of shape  $(m-1, m-1)$ , and let  $W = W(\mathcal{T})$  be the word of length  $2m$  associated with  $\mathcal{T}$  in Definition 6.6.3, which is an inversion free binary snake name by Proposition 6.6.2. If  $\mathcal{T}(W)$  is the standard Young tableau associated with  $W$  in Definition 6.6.2 then  $\mathcal{T}(W) = \mathcal{T}$ .*

*Proof.* If  $w_i$  is an entry of  $W$  such that  $i-1$  is in the  $k$ -th cell of the first row of  $\mathcal{T}$ , then  $i > 1$  and, by Remark 6.6.3, the subword  $[w_1 \cdots w_i]$  of  $W$  contains exactly  $k+1$  first entries of the letters  $x_1, \dots, x_{k+1}$  of  $W$ . By Definition 6.6.2, the  $k$ -th cell of the first row of  $\mathcal{T}(W)$  contains the same number  $i-1$  as the  $k$ -th cell of the first row of  $\mathcal{T}$ .

If  $w_i$  is an entry of  $W$  such that  $i-1$  is in the  $k$ -th cell of the second row of  $\mathcal{T}$ , by Remark 6.6.3, the subword  $[w_1 \cdots w_i]$  of  $W$  contains exactly  $k$  second entries of the letters  $x_1, \dots, x_k$  of  $W$ . By Definition 6.6.2, the  $k$ -th cell of the second row of  $\mathcal{T}(W)$  contains the same number  $i-1$  as the  $k$ -th cell of the second row of  $\mathcal{T}$ .  $\square$

**Lemma 6.6.3.** *Let  $W$  be an inversion free binary snake name of length  $2m$  containing  $m$  letters  $x_1, \dots, x_m$ , so that their first entries in  $W$  appear in the same order as their indices. Let  $\mathcal{T} = \mathcal{T}(W)$  be the standard Young tableau of shape  $(m-1, m-1)$  associated with  $W$  in Definition 6.6.2. If  $W(\mathcal{T})$  is the word associated with  $\mathcal{T}$  in Definition 6.6.3 then  $W(\mathcal{T}) = W$ .*

*Proof.* Since  $W$  and  $W(\mathcal{T})$  are inversion free words, second entries of all letters  $x_j$  in each of them appear in the same order as their first entries, and in the same order as their indices. In particular, the first entry of  $W(\mathcal{T})$  is  $x_1$ , same as the first entry of  $W$ , and the last entry of  $W(\mathcal{T})$  is  $x_m$ , same as the last entry of  $W$ .

Let  $w_i = x_k^-$  and  $w_j = x_k^+$  be two entries of the letter  $x_k$  in  $W$ , where  $1 < k < m$ . Since  $w_i$  is the  $k$ -th first entry of a letter in  $W$ ,  $i - 1$  is in the cell  $k - 1$  of the first row of  $\mathcal{T}$ . Similarly, since  $w_j$  is the  $k$ -th second entry of a letter in  $W$ ,  $j - 1$  is in the cell  $k$  of the second row of  $\mathcal{T}$ . Definition 6.6.3 implies that  $x_k$  appears in  $W(\mathcal{T})$  also as its  $i$ -th and  $j$ -th entries. The proofs for the second entry of  $x_1$  and the first entry of  $x_m$  are similar. Thus all entries of these two words are the same.  $\square$

**Theorem 6.6.3.** *There is a bijection between the set of standard Young tableaux of shape  $(m - 1, m - 1)$  and the set of equivalence classes of inversion free binary snake names of length  $2m$ , for each  $m \geq 1$ .*

*Proof.* Definition 6.6.2 defines the map  $f$  from the set of equivalence classes of inversion free binary snake names of length  $2m$  to the set of standard Young tableaux of shape  $(m - 1, m - 1)$ , and Definition 6.6.3 defines a map in the opposite direction. It follows from Lemmas 6.6.2 and 6.6.3 that these two maps are inverses of each other, thus they are bijective.  $\square$

**Corollary 6.6.1.** (See (STANLEY, 1999) p. 226 Exercise 6.19 ww, p. 230 Exercise 6.20.) *The number of equivalence classes of inversion free binary snake names of length  $2m + 2$  is the  $m$ -th Catalan number*

$$C_m = \frac{1}{m+1} \binom{2m}{m}.$$

## 7 CONCLUSION

It was proved that if  $X$  is a  $\beta$ -surface germ then  $V(X)$  can be decomposed into finitely many normal zones and abnormal zones. Moreover, each abnormal arc in  $X$  is contained in one of the finitely many  $\beta'$ -snakes and  $\beta''$ -non-snake bubbles contained in  $X$ , in particular  $\beta', \beta'' \geq \beta$ . Nevertheless, the Valette link of a snake can be decomposed into finitely many normally embedded zones which are Lipschitz invariants, called segments and nodal zones.

In addition, it was proved that there are combinatorial invariants associated with snakes called snake names. Those combinatorial objects allowed a strong relationship between geometry and combinatorics of snakes. It was also proved that for each snake name  $W$  there is a snake  $T$  such that  $W$  is a word associated with  $T$ . Finally, we defined “weakly outer bi-Lipschitz maps” between surface germs and gave a combinatorial description of weak outer Lipschitz equivalence of snakes in terms of their snake names and cluster partitions.

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