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IVALDO TRIBUTINO DE SOUSA

THEORETICAL CONTROLLABILITY RESULTS FOR SOME NONLINEAR PDEs FROM PHYSICS

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Tese apresentada ao Programa de Pós-graduação em Matemática do Departamento de Matemática da Universidade Federal do Ceará, como parte dos requisitos necessários para a obtenção do título de Doutor em Matemática. Área de concentração: Análise.

Orientador: Prof. Dr. José Fábio Bezerra Montenegro
Coorientador: Prof. Dr. Enrique Fernádez Cara.

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“Believe and act as if it were impossible to fail” - Charles Kettering.

ABSTRACT

This Thesis deals with the local null control of a free-boundary problem for the 1D semilinear heat equation with distributed controls (locally supported in space) or boundary controls (acting at $x = 0$). we prove that, if the final time T is fixed and the initial state is sufficiently small, there exists controls that drive the state exactly to rest at time $t = T$. Furthermore, we analyze the null controllability of a 1D nonlinear system which models the interaction of a fluid and its boundary. The fluid is governed by the viscous Burgers equation and the distributed controls. Lastly, we deal with the 3D Navier-Stokes and Boussinesq system, posed in a cube. In this context, we prove a result concerning its global approximate controllability by means of boundary controls which act in some part of cube faces.

Keywords: controllability, free-boundary, parabolic systems of PDEs.

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SUMÁRIO

1 INTRODUCTION

This Thesis deals with the controllability of several systems governed by nonlinear PDEs. First, the local null controllability of a free-boundary problem for the 1D semilinear heat equation. It is presented in Section 2, where the information are from the recent work, Fernández-Cara and De Sousa (in preparationa).

We also are interested local null controllability of a free-boundary problem for the viscous Burgers equation. This is explored in Section 3, where the material comes from Fernández-Cara and De Sousa (in preparationb).

Finally, some remarks concerning the global approximate controllability of the Boussinesq and Navier-Stokes systems are given in Section 4. This stems from Fernández-Cara, De Sousa and De Brito Viera (in preparation).

This section is devoted to the controllability of systems governed by linear heat equation. We will try to explain which is the meaning of controllability and which kind of controllability properties can be expected. The main related results, together with the main ideas in their proofs, will be recalled. Also, some controllability results for several nonlinear systems from fluid mechanics, like the Burgers, Navier-Stokes and Boussinesq systems, will be included. Finally, in Subsection 1.5, we review the main contributions in Sections 2, 3 and 4.

1.1 Basic results for the linear heat equation

This subsection is dedicated to the controllability of the linear heat equations. I will try to explain which is the meaning of controllability and which kind of controllability properties can be expected.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, with boundary Γ of class C^2. Let ω be an open and non-empty subset of Ω and $T > 0$. Let us consider the linear controlled heat equation in the cylinder $Q = \Omega \times (0,T)$

$$\begin{cases} y_t - \Delta y = v1_\omega & \text{in} \quad Q, \\ y = 0 & \text{in} \quad \Sigma, \\ y(x,0) = y^0(x) & \text{on} \quad \Omega. \end{cases} \tag{1}$$

In (1), $\Sigma = \Gamma \times (0,T)$ is the lateral boundary of Q, 1_ω is the characteristic function of the set ω, $y = y(x,t)$ is the state and $v = v(x,t)$ is the control. As v is multiplied by 1_ω, the action of the control is limited to $\omega \times (0,T)$. Assume That $y^0 \in L^2(\Omega)$ and $v \in L^2(\omega \times (0,T))$, so that (1) admits a unique solution

$$y \in C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)).$$

We will define $R(T;y^0) := \{y(\cdot,T) : v \in L^2(\omega \times (0,T))\}$. Then,

(a) It is said that system (1) is *approximately controllable* (at time T) if $R(T; y^0)$ is dense in $L^2(\Omega)$ for all $y^0 \in L^2(\Omega)$.

(b) It is said that system (1) is *exactly controllable* if $R(T; y^0) = L^2(\Omega)$ for all $y^0 \in L^2(\Omega)$.

(c) It is said that system (1) is *null controllable* if $0 \in R(T; y^0)$ for all $y^0 \in L^2(\Omega)$.

We will show below that approximate and null controllability hold for every non-empty open set $\omega \subset \Omega$ and every $T > 0$. On the other hand, it is clear that exact controllability cannot hold, except possible in the case in which $\omega = \Omega$. Indeed, due to the regularizing effect of the heat equation, the solution of (1) at time T are smooth in $\Omega \setminus \overline{\omega}$.

Our first main result is the following:

Theorem 1.1. *System* (1) *is approximately controllable for non-empty open set* $\omega \subset \Omega$ *and any* $T > 0$.

Proof. This is an easy consequence of Hahn-Banach theorem. For completeness, we will reproduce the argument here.

Let us fix ω and $T > 0$. Then, it is clear that 1 is approximately controllable if and only if $R(T; 0)$ is dense in $L^2(\Omega)$. But this is true if and only if φ^T in the orthogonal complement $R(T; 0)^\perp$ is necessarily zero.

Let $\varphi^T \in L^2(\Omega)$ be given and assume that belongs to $R(T; 0)^\perp$. Let us introduce the following backwards in time system:

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in} \quad Q, \\ \varphi = 0 & \text{in} \quad \Sigma, \\ \varphi(x, T) = \varphi^T(x) & \text{on} \quad \Omega. \end{cases} \tag{2}$$

Then, if $v \in L^2(\omega \times (0, T))$ is given and y is solution to (1) with $y^0 = 0$, we have

$$\iint_{\omega \times (0,T)} \varphi v \, dx \, dt = \int_\Omega \varphi^T(x) y(x, T) \, dx = 0.$$

Consequently, approximate controllability holds if and only if the following uniqueness property is true: If φ solve (2) and $\varphi = 0$ in $\omega \times (0, T)$, then necessarily $\varphi \equiv 0$, i.e. $\varphi^T = 0$.

But this is a well known uniqueness property for the heat equation, a consequence of the fact that the solution to (2) are analytic in space. This proves that approximate controllability holds for (1). $\square$

Let us now analyze the null controllability of (1). The other important result is the following:

Theorem 1.2. *The following observability inequality for the adjoint system* (2)

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \leq C \iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt \quad \forall \varphi^0 \in L^2(\Omega). \tag{3}$$

implies the null controllability of (1)

Proof. We divide the proof into two steps. First, we build a sequence of controls $v_\varepsilon \in L^2(\omega \times (0,T))$ with $\varepsilon > 0$ which provide the approximate controllability of (1). Second, we pass to the limit when ε tends to zero and we conclude.

Step 1. Let $y^0 \in L^2(\Omega)$ and $\varepsilon > 0$ be given. Let us introduce the function J_ε, with

$$J_\varepsilon(\varphi^0) = \frac{1}{2}\iint_{\omega\times 0,T} |\varphi|^2\,dx\,dt + \varepsilon\|\varphi^T\|_{L^2(\Omega)} + (\varphi(0), y^0)_{L^2(\Omega)} \tag{4}$$

for every $\varphi^T \in L^2(\Omega)$. Here, φ is solution of (2) associated to the initial condition φ^T. Using (3), it is not difficult to check that J_ε is strictly convex, continuous, and coercive in $L^2(\Omega)$, so it possesses a unique minimum $\varphi_\varepsilon^T \in L^2(\Omega)$, whose associated solution is denoted by φ_ε. Let us now introduce the control $v_\varepsilon = \varphi_\varepsilon 1_\omega$, and let us denote by y_ε the solution (1) associated to v_ε.

Let y_1 be the final state of the solution to (1) with vanishing control. Let us remark that the unique interesting case to be studied turns out to be when $\|y_1\|_{L^2(\Omega)} > \varepsilon$ since this is equivalent to $\varphi_\varepsilon^T \neq 0$. See (FABRE, PUEL, and ZUAZUA, 1995) for more details. Under this assumption, we can differentiate the functional J_ε at φ_ε^T and obtain a necessary condition for J_ε to reach a minimum at φ_ε^T. Consequently,

$$\iint_{\omega\times(0,T)} \varphi_\varepsilon\varphi\,dx\,dt + \varepsilon\left(\frac{\varphi_\varepsilon^T}{\|\varphi_\varepsilon^T\|_{L^2(\Omega)}}, \varphi^T\right) + (\varphi(0), y^0)_{L^2(\Omega)} = 0 \tag{5}$$

for every $\varphi^T \in L^2(\Omega)$.

Using the above equation and (3) for $\varphi^T = \varphi_\varepsilon^T$, we obtain $\|v_\varepsilon\|_{\omega\times(0,T)} \leq \sqrt{C}\|y^0\|_{L^2(\Omega)}$, where C is the observability constant of (3).

Since system (1) and (2) are in duality, we have

$$\iint_{\omega\times(0,T)} \varphi_\varepsilon\varphi\,dx\,dt = (y_\varepsilon(T), \varphi^T)_{L^2(\Omega)} - (y^0, \varphi(0))_{L^2(\Omega)}, \tag{6}$$

which, combined with (5), yields

$$\|y_\varepsilon(T)\|_{L^2(\Omega)} \leq \varepsilon. \tag{7}$$

Step 2. Since the sequence $\{v_\varepsilon\}$ is bounded in $L^2(\omega \times (0,T))$, it possesses a weakly convergent subsequence to certain $v \in L^2(\omega \times (0,T))$. Using classical parabolic estimates we deduce that, at least for a subsequence,

$$y_\varepsilon \to y \text{ weakly in } L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \tag{8}$$

where y is the solution of (1) with control v. In particular, this gives weak convergence for $\{y_\varepsilon(t)\}(t \in [0,T])$ in $L^2(\Omega)$ so we have $y(T) = 0$. □

Three important comments,

1. We have proved that (3) implies null controllability with a control that satisfies

$$\|v\|_{\omega\times(0,T)} \leq \sqrt{C}\|y^0\|_{L^2(\Omega)},$$

   where C is the observability constant.
   Conversely, if we have null controllability with controls $v \in L^2(\omega \times (0,T))$ that satisfy

$$\|v\|_{\omega\times(0,T)} \leq \sqrt{C}\|y^0\|_{L^2(\Omega)}$$

   for some constant $C > 0$, then it can be checked that we have (3) with the same constant C.
2. It is possible to present a similar argument in a general frame. Let us consider three Hilbert spaces U, H, E and two linear continuous operators $L \in \mathcal{L}(U;E)$ and $M \in \mathcal{L}(H;E)$. Then we have

$$\|m^*\varphi^T\|_H \leq C\|L^*\varphi^T\|_{U'}, \ \ \forall\varphi^T \in E'$$

   for some positive constant C if and only if $R(M) \subset R(L)$ and, moreover,

$$\forall y^0 \in H, \exists v \in U; Lv = My^0, \|v\|_U \leq C\|y^0\|_H.$$

   Properties of this kind have been established and analyzed for the first time in the framework of control theory in (RUSSELL, 1973). They have been successfully used in many different contexts in recent years.
3. We have to know that the estimates (3) are implied by the so called global Carleman inequalities. These have been introduced in the context of the controllability of PDEs by Fursikov and Imanuvilov; see IMANUVILOV (1995); FURSIKOV and IMANUVILOV (1996a). When they are applied to the solutions to the adjoint system (2), they take the form

$$\iint_{\Omega\times(0,T)} \rho^{-2}|\varphi|^2\,dx\,dt \leq K \iint_{\omega\times(0,T)} \rho^{-2}|\varphi|^2\,dx\,dt, \quad \forall\varphi^T \in L^2(\Omega), \tag{9}$$

   where $\rho = \rho(x,t)$ is an appropriate weight depending on Ω, ω and T and the constant K only depends on Ω and ω. Combining (9) and the and the dissipative backwards heat equation (2), it is not difficult to deduce (3) for some C only depending on Ω, ω and T.

1.2 Positive and negative controllability results for the Burgers equation

In this subsection, we will be concerned with the null controllability of the following system for the viscous Burgers equation:

$$\begin{cases} y_t + yy_x - y_{xx} = v1_\omega, & (x,t) \in (0,1) \times (0,T) \\ y(0,t) = y(1,t) = 0, & t \in (0,T) \\ y(x,0) = y^0(x), & x \in (0,1) \end{cases} \tag{10}$$

Some controllability properties of (10) have been studied in (FURSIKOV and IMANUVILOV, 1996a) (see Chapter 1, theorems 6.3 and 6.4). There, it is shown that, in general, a stationary solution of (10) with large L^2-norm cannot be reached (not even approximately) at any time T. In other words, with the help of one control, the Burgers solutions cannot go anywhere at any time.

For each $y^0 \in L^2(0,1)$, let us define

$$T(y^0) = \inf\{T > 0 : (10) \text{ is null controllable at time } T\}.$$

Then, for each $r > 0$, let us define the quantity

$$T^*(r) = \sup\{T(y^0) : \|y^0\|_{L^2(0,1)} \leq r\}.$$

Our main intention is to show that $T^*(r) > 0$, with explicit sharp estimates from above and from below. In particular, this will imply that (global) null controllability at any positive time does not hold for (10). Indeed, let us set $\phi(r) = (\log \frac{1}{r})^{-1}$. We have the following result from (FERNÁNDEZ-CARA and GUERRERO, 2007):

Theorem 1.3. *One has*

$$C_0\phi(r) \leq T^*(r) \leq C_1\phi(r) \text{ as } r \to 0, \tag{11}$$

for some positive constants C_0 and C_1 not depending of r.

Remark 1.1. The same estimates hold when the control v acts on system (10) through the boundary only at $x = 1$ (or $x = 0$). Indeed, it is easy to transform the boundary controlled system

$$\begin{cases} y_t + yy_x - y_{xx} = 0, & (x,t) \in (0,1) \times (0,T) \\ y(0,t) = 0, \;\; y(1,t) = w(t), & t \in (0,T) \\ y(x,0) = y^0(x), & x \in (0,1) \end{cases} \tag{12}$$

into a system of the kind (10) The boundary controllability of the Burgers equation with two controls ($x = 0$ and $x = 1$) was analyzed in (GUERRERO and IMANUVILOV, 2007). This paper show that even in this more favorable situation null controllability does not hold for small time. Moreover, it is proved in that work that exact controllability does not hold for large time. □

Remark 1.2. It is proved in (CHAPOULY, 2009) that the Burgers equation is *globally* null controllable when we act on the system through two boundary controls and an additional right hand side only depending on t. In other words, for any $y^0 \in L^2(0,1)$, there exist w_1, w_2 and h in $L^2(0,T)$ such that the solution to

$$\begin{cases} y_t + yy_x - y_{xx} = h(t), & (x,t) \in (0,1) \times (0,T) \\ y(0,t) = w_1(t), \ \ y(1,t) = w_2(t), & t \in (0,T) \\ y(x,0) = y^0(x), & x \in (0,1) \end{cases} \tag{13}$$

satisfies

$$y(x,T) = 0, \ \ x \in (0,1).$$

However, it is unknown whether this global property is conserved when one of the boundary controls w_1 or w_2 is eliminated. □

The proof of the estimate (13) can be obtained by solving the null controllability problem for (10) via a standard fixed point argument, global Carleman inequalities to estimate the control and energy inequalities to estimate the state and being very careful with the role of T in these inequalities.

The proof of the below estimate was inspired by the arguments from (ANITA and TATARU, 2002) and implied by the following property: there exist positive constants C_0 and C_0' such that, for any sufficiently small $r > 0$, we can find initial data y^0 and associated states y satisfying $\|y^0\| \leq r$ and

$$|y(x,t)| \geq C_0' r \ \text{ for some } \ x \in (0,1) \ \text{ and } \ 0 < t < C_0 \phi(r).$$

For more details, see (FERNÁNDEZ-CARA and GUERRERO, 2007).

1.3 Controllability results for Navier-Stokes and Boussinesq systems

In this subsection, $N = 2$ or $N = 3$. The controllability properties of the Navier-Stokes system have been the subject of intensive research these last years. The question was first treated by Lions in (1990), where *approximate controllability* was conjectured. This was followed by several papers, where many partial answers were furnished such as (CORON, 1996), (CORON and FURSIKOV, 1996) and (LIONS and ZUAZUA, 1998). Concerning null controllability and exact controllability to the trajectories, the first local results were given in (FURSIKOV and IMANUVILOV, 1996b). In this part we deal with the local exact controllability of the Navier-Stokes system with distributed controls.

Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set. Let $\omega \subset \Omega$ be non-empty open subset and let $T > 0$. We will use the notation $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$ and we will denote by $n(x)$ the outward unit normal to Ω at the point $x \in \partial\Omega$. We have the well known Navier-Stokes equations:

$$\begin{cases} u_t - \Delta u + (u,\nabla)u + \nabla p = v1_\omega, & \text{in} \quad Q, \\ \nabla \cdot u = 0, & \text{in} \quad Q, \\ u = 0, & \text{on} \quad \Sigma, \\ u(0,x) = u_0(x), & \text{in} \quad \Omega. \end{cases} \tag{14}$$

The pair (u,p) is the state (the velocity field and the pressure distribution) and v is the control (a field forces applied to the fluid particles located at ω).

Let us recall the definition of some usual spaces in the context of Navier-Stokes system.

$$V = \{u \in H_0^1(\Omega)^N : \nabla \cdot u = 0 \text{ in } \Omega\}$$

and

$$H = \{u \in L^2(\Omega)^N : \nabla \cdot u = 0 \text{ in } \Omega, \;\; y \cdot n = 0 \text{ on } \partial\Omega\}.$$

Let us now introduce the concept of *exact controllability to the trajectories* for the Navier-Stokes system. For any trajectory $(\overline{u},\overline{p})$, i.e. any solution of the uncontrolled Navier-Stokes system

$$\begin{cases} \overline{u}_t - \Delta \overline{u} + (\overline{u},\nabla)\overline{u} + \nabla \overline{p} = 0, & \text{in} \quad Q \\ \nabla \cdot \overline{u} = 0, & \text{in} \quad Q \\ \overline{u} = 0, & \text{on} \quad \Sigma \\ \overline{u}(0,x) = \overline{u}_0(x), & \text{in} \quad \Omega. \end{cases} \tag{15}$$

Here, $u_0 \in H$, there exist controls $v \in L^2(\omega \times (0,T))^N$ and associated solution (u,p) such that

$$u(x,T) = \overline{u}(x,T), \;\; x \in \Omega. \tag{16}$$

At the moment, we do not know any global result concerning exact controllability to the trajectories for (14). However, the following local result holds:

Theorem 1.4. *Let* $(\overline{u},\overline{p})$ *be a strong solution of* (15), *with*

$$\overline{u}\in L^{\infty}(Q)^N,\ \ \overline{u}(\cdot,0)\in V.$$

Then, there exists $\delta>0$ *such that, for any* $u^0\in H\cap L^{2N-2}(\Omega)^N$ *satisfying*

$$\|u_0-\overline{u}_0\|_{L^{2N-2}(\Omega)^N}\le\delta,$$

we can find a control $v\in L^2(\omega\times(0,T))^N$ *and an associated solution* (u,p) *to* (14) *such that* (16) *holds.*

The proof of theorem (1.4) can be obtained as an application of *Liusternik's inverse mapping theorem* in an appropriate framework.

A key in the proof is a related null controllability result for the linearized Navier-Stokes system at $(\overline{u},\overline{p})$, this is to say:

$$\begin{cases} u_t-\Delta u+(\overline{u},\nabla)u+(u,\nabla)\overline{u}+\nabla p=v1_{\omega}, & \text{in} \quad Q,\\ \nabla\cdot u=0, & \text{in} \quad Q,\\ u=0, & \text{on} \quad \Sigma,\\ u(0,x)=u_0(x), & \text{in} \quad \Omega. \end{cases} \tag{17}$$

This is implied by a global Carleman inequality of the kind (9) that can be established for the solution to the adjoint of (17), which is the following

$$\begin{cases} -\varphi_t-\Delta\varphi+(\nabla\varphi+\nabla^t\varphi)\overline{u}+\nabla\pi=g, & \text{in} \quad Q,\\ \nabla\cdot\varphi=0, & \text{in} \quad Q,\\ \varphi=0, & \text{on} \quad \Sigma,\\ \varphi(0,T)=\varphi^T(x), & \text{in} \quad \Omega. \end{cases} \tag{18}$$

The details can be found in (FERNÁNDEZ-CARA *et al.*, 2004).

Similar result can found in (GUERRERO, 2006) for The Boussinesq equations

$$\begin{cases} u_t-\Delta u+(u,\nabla)u+\nabla p=v1_{\omega}+\theta e_N,\ \ \nabla\cdot u=0 & \text{in} \quad Q,\\ \theta_t-\Delta\theta+u\cdot\nabla\theta=h1_{\omega}, & \text{in} \quad Q,\\ u=0,\ \ \theta=0, & \text{on} \quad \Sigma,\\ u(0,x)=u_0(x),\ \ \theta(0,x)=\theta_0(x) & \text{in} \quad \Omega. \end{cases} \tag{19}$$

Here, the state is the triplet (u,θ,p) (θ is interpreted as a temperature distribution) and the control is (v,h) (as before, v is a field of external forces; h is an external heat source).

Again, a crucial point to prove the null controllability of certain linearized systems, this time *modified* controls. For instance, when dealing with (14) the task is reduced to prove that, for some appropriate weights ρ, ρ_0 and some $K > 0$, the solution (18) satisfy the following Carleman estimate

$$\iint_{\Omega\times(0,T)} \rho^{-2}|\varphi|^2\,dx\,dt \le K \iint_{\omega\times(0,T)} \rho_0^{-2}(|\varphi_1|^2 + |\varphi_2|^2)\,dx\,dt,\ \ \forall\varphi^T \in H.$$

1.4 Free-boundary system and controllability of fluid-rigid body system

Free boundary problem is a partial differential equation to be solved for both an unknown function y and an unknown domain Ω. The segment Γ of the boundary of Ω which is not known at the outset of the problem is the free boundary.

The classic example is the melting of ice (two-phase Stefan problem). Given a ice block, one can solve the heat equation given initial and boundary conditions to determine its temperature. However, if in any region the temperature is higher than the melting point of ice, this domain will be occupied by liquid water instead. The boundary formed from the ice-liquid interface is controlled dynamically by the solution of the PDE. We assume the melting point of ice to be a constant equal 0.

The melting of ice is a problem formulated as follows. Consider a medium occupying a region Ω consisting of two phases, the liquid phase and the solid phase. Let the two phases have diffusivity coefficient α_1 and α_2, where $\alpha_1 > 0$ is the diffusivity coefficient in water, and $\alpha_2 > 0$ is the diffusivity coefficient in ice (in principle $\alpha_1 \neq \alpha_2$).

In the regions consisting solely of one phase, the temperature is determined by the heat equation: in the liquid phase region (temperature> 0),

$$y_t - \alpha_1 \Delta y = f,$$

while in the solid phase region (temperature< 0),

$$y_t - \alpha_2 \Delta y = f.$$

Here, f represents sources or sinks of heat. Let $s(t)$ be the position of the interface at time t (temperature $= 0$). Let n denote the unit outward normal vector to the solid phase. The *Stefan condition* determines the evolution of the surface s by giving an equation governing the velocity s' of the free surface in the direction n, specifically

$$Ls'(t) = \alpha_2 \partial_n y(s(t)+, t) - \alpha_1 \partial_n y(s(t)-, t), \tag{20}$$

where L is the latent heat of melting.

In this problem, we know beforehand the whole region Ω but we only know the ice-liquid interface s at time $t = 0$. To solve the Stefan problem we not only have to solve the heat equation in each region, but we must also track the free boundary

$s(t),\ t \in (0, T)$.

Note that, we can have the case where one of the phases has at constant temperature. Let assume the solid phase in at constant temperature equal 0. Thus (20) reduces to

$$Ls'(t) = -\alpha_1 \partial_n y(s(t)-, t). \tag{21}$$

It is therefore evident that on the free boundary both conditions $y(s(t)) = 0$ and (21) should be prescribed in order to have a well posed problem. The basic idea to prove the existence of solutions is we assign arbitrarily a free boundary S and consider the solution y to the problem where $y(s(t)) = 0,\ t \in (0, T)$. Then we define a *transformed* boundary $\overline{s}$ such that

$$L\overline{s}'(t) = -\alpha_1 \partial_n y(s(t)-, t).$$

We may assume the problem is one dimensional. Assuming that $0 < s(t) < B,\ t \in (0, T)$ and denoting by x the space variable, the complete two-phases problem can be written as:

$$\begin{cases} y_t - \alpha_1 y_{xx} = 0, & (x,t) \in Q_1 \\ y_t - \alpha_2 y_{xx} = 0, & (x,t) \in Q_2 \\ y(s(t)+, t) = y(s(t)-, t) = k, & t \in (0, T) \\ -\alpha_1 y_x(0, t) = h_1(t), & t \in (0, T) \\ -\alpha_2 y_x(0, t) = h_2(t), & t \in (0, T) \\ -\alpha_1 y_x(s(t)-, t) + \alpha_2 y_x(s(t)+, t) = Ls'(t), & t \in (0, T) \\ y(x, 0) = y^0(x), & x \in (0, B) \\ s(0) = s_0 \end{cases} \tag{22}$$

Here, $0 < s_0 < B$, $T > 0$ and α_1, α_2, k are given positive number. The liquid phase occupies at the initial time $t = 0$ the interval $(0, s_0)$, while the solid phase occupies (s_0, B). The problem is posed in the interval $(0, T)$. Moreover we have set

$$Q_1 = \{(x,t) : 0 < x < s(t),\ t \in (0, T)\},$$

$$Q_2 = \{(x,t) : s(t) < x < B,\ t \in (0, T)\}.$$

Actually, we will deal mainly with the one phase version of (22) where the solid phase is at constant temperature. In other words, we will deal with the following problem:

$$\begin{cases} y_t - y_{xx} = 0, & (x,t) \in Q_s, \\ y(s(t),t) = 0, & t \in (0,T), \\ -y_x(0,t) = h(t), & t \in (0,T), \\ -y_x(s(t),t) = s'(t), & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,B), \\ s(0) = s_0 \end{cases} \tag{23}$$

Here, we have used the notation

$$Q_s = \{(x,t) : x \in (0,s(t)),\ \ t \in (0,T)\},$$

where $s \in C^0([0,T])$.

Theorem 1.5. *Let us assume $(y_0, h) \in C^0([0,T])^2$ and $h > 0$ and $0 < y_0 < s_0$. Then, there exists a unique solution (y, s) to problem* (23) *such that*

$$s \in C^1((0,T]) \cap C^0([0,T]),\ \ s(0) = s_0,\ \ s(t) > 0,\ \ t \in (0,T);$$

$$y \in C^0(\overline{Q_s}) \cap C^{2,1}(Q_s),\ \ y_x \in C^0(\overline{Q_s} - \{t = 0\}),$$

For further reading on the one-phase Stefan problem, see (ANDREUCCI, 2002; CANNON, 1984).

Now, we will present some control results for models of fluid-solid interaction, Those results can be found in (IMANUVILOV and TAKAHASKI, 2007), (DOUBOVA and FERNÁNDEZ-CARA, 2005), and (LIU, 2011). This part is devoted to the controllability of fluid-structure (or fluid-solid) system. Our goals is obtain knowledge to resolve our controllability problem of a free-boundary Burgers system. The fluid is viscous and incompressible and its motion is modeled by the Navier-Stokes equation where the structure is a rigid ball, which satisfies Newton's laws. We show results the local null controllability for the velocities of the fluid and of the solid mass and the exact controllability for the solid mass position.

Let $\mathcal{C}$ be an open boundary set of $\mathbb{R}^2$, containing the open ball $S(t)$ of radius 1 moving into a viscous fluid which is occupying the domain $\Omega(t) = \mathcal{C} \setminus S(t)$. Let ω be an open subset with $\overline{\omega} \subset \Omega(t)$. The fluid-rigid body (fluid-solid mass) system is controlled by a force field supported in ω. Then, the equation of motion of the fluid-structure system are:

$$u_t - \nu\Delta u + (u \cdot \nabla)u + \nabla p = v1_\omega,\ \ x \in \Omega(t),\ \ t \in [0,T], \tag{24}$$

$$\nabla \cdot u = 0, \;\; x \in \Omega(t), \;\; t \in [0,T], \tag{25}$$

$$u = 0, \;\; x \in \partial\mathcal{C}, \;\; t \in [0,T], \tag{26}$$

$$u(t,x) = h'(t) + \theta'(t)(x - h(t))^{\perp}, \;\; x \in \partial S(t) \;\; t \in [0,T], \tag{27}$$

$$Mh''(t) = -\int_{\partial S(t)} \sigma(u,p) n \, d\Gamma, \;\; t \in (0,T), \tag{28}$$

$$J\theta''(t) = -\int_{\partial S(t)} (x - h(t))^{\perp} \cdot \sigma(u,p) n \, d\Gamma, \;\; t \in (0,T), \tag{29}$$

$$u(x,0) = u^0(x), \;\; x \in \Omega(0), \tag{30}$$

$$h(0) = h^0, \;\; h'(0) = h^1, \;\; \theta(0) = \theta^0, \;\; \theta'(0) = \theta^1, \tag{31}$$

where

$$\sigma(v,p) = -p\mathrm{Id} + 2\nu D(u) \;\text{ and }\; D(u)_{i,j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}, \frac{\partial u_i}{\partial x_j}\right).$$

In the above system the unknowns are $u(x,t)$ (the velocity field of the fluid), $p(x,t)$ (the pressure of the fluid), $h(t)$ (the position of the center of the rigid ball) and $\theta(t)$ (the angular of the rigid body). The function $v(x,t)$ is the control of the system. The domain $S(t)$ is defined by:

$$S(t) = B(h(t)),$$

where $B = \{x \in \mathbb{R}^2 : |x - c| < 1\}$ denotes the open ball of $\mathbb{R}^2$. The constants M and J are the mass and the moment of inertia of the rigid body. For sake of simplicity, assume that the rigid body is homogeneous and thus we have that

$$M = 2\pi\delta, \quad J = \delta \int_S |y|^2 dy,$$

where $\delta > 0$ is the rigid body density. The positive constant ν is the viscosity of the fluid.

For all $x = \binom{x_1}{x_2}$, we denote by $x^{\perp}$ the vector $x^{\perp} = \binom{x_2}{-x_1}$. Moreover we denote by $\partial S(t)$ the boundary of rigid body and by $n(x,t)$ the unit normal to $\partial S(t)$ at the point x directed to the interior of the rigid body.

Assume that

$$\overline{S(0)} \subset \mathcal{C} \setminus \overline{\omega}, \tag{32}$$

then, for $|h_T - h^0|$ small enough, we have that $B(h_T) \subset \mathcal{C} \setminus \overline{\omega}$. Therefore, it is natural to wonder if with some control v we can have $S(T) = B(h_T)$. Indeed, we have a control such that the velocities of the fluid and of the rigid body are equal to at time T. This is the main result of (IMANUVILOV and TAKAHASKI, 2007):

Theorem 1.6. *Let* $T > 0$ *and assume that* (32) *holds true. Suppose also that* $u^0 \in H^1(\Omega(0))$ *and that*

$$\begin{cases} \nabla \cdot u = 0, \;\; x \in \Omega(0), \\ u^0(x) = h^1 + \theta^1(x - h^0)^{\perp}, \;\; x \in \partial S(0), \\ u^0(x) = 0, \;\; x \in \partial \mathcal{C}. \end{cases} \tag{33}$$

Then there exists $\varepsilon > 0$ *such that if*

$$\|u^0\|_{H^1(\Omega(0))} + |h^0 - h_T| + |h^1| + |\theta^0 - \theta_T| + |\theta^1| < \varepsilon,$$

then the system (24)-(31) *is null controllable at time* T *in velocity and exactly controllable at time* T *for the position of the rigid body. More precisely, there exists* $v \in L^2(0,T; L^2(\omega))$ *such that*

$$u(T) = 0, \quad h'(T) = 0, \quad \theta(T) = 0,$$

and

$$h(T) = h_T, \quad \theta(T) = \theta_T.$$

Another result is the null controllability of a one-dimensional nonlinear system which models the interaction of a fluid and a particle. The velocity of the fluid is governed by the Burqers equation at both sides of the point mass location $y = h(t)$ and the control is exerted at the boundary points. For simplicity, the fluid density was defined constant and the solid particle has unit mass. The system is thus the following:

$$\begin{cases} y_t + yy_x - y_{xx} = 0, & (x,t) \in Q \setminus h(t), \\ y(-1,t) = \alpha(t), \;\; y(1,t) = \beta(t), & t \in (0,T), \\ y(h(t),t) = h'(t), \;\; [y_x](h(t),t) = h''(t), & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (-1,1), \\ h(0) = h^0, \;\; h'(0) = h_1. \end{cases} \tag{34}$$

Here, $Q = (-1,1) \times (0,T)$, $T > 0$, $y(x,t)$ is the velocity of the fluid particle located at x at time t, $h(t)$ is the position occupied by the particle at time t, α and β are the controls (two functions at least in $L^\infty(0,T)$) and the initial data satisfy

$$y^0 \in H^1(-1,1), \;\; h^0 \in (-1,1) \;\text{ and }\; h^1 = y^0(h^0), \tag{35}$$

and $[f(x)]$ denotes the *jump* of the function f at point x.

In (34), the spatial domain depends on t. $|h(t)| \leq 1 - b$ where b is a positive small constant, we can introduce the following change of variable: For any $x \in (-1, h(t)) \cup (h(t), 1)$, we put $\xi = (x-h)/(1-\kappa h)$, where ξ is the sign of ξ. The change of variables allows us to rewrite the problem as

$$\begin{cases} (1-\kappa h)y_t - (1-\kappa\xi)h'y_x + yy_x - \dfrac{1}{1-\kappa h}y_{xx} = 0, & (\xi,t) \in Q, \ \ \xi \neq 0, \\ y(-1,t) = \alpha(t), \ \ y(1,t) = \beta(t), & t \in (0,T), \\ y(0,t) = h'(t), \ \ \left[\dfrac{1}{1-\kappa h}y_x\right](h(t),t) = h''(t), & t \in (0,T), \\ y(\xi,0) = y^0(\xi), & \xi \in (-1,1), \\ h(0) = h^0, \ \ h'(0) = h_1. \end{cases} \tag{36}$$

In (DOUBOVA and FERNÁNDEZ-CARA, 2005), one of the main results is the following:

Theorem 1.7. *The nonlinear system* (36) *is locally null controllable. More precisely, there exists* $\varepsilon > 0$ *depending on* $T > 0$ *such that, wherever the initial data satisfy* (35) *and*

$$\|y^0\|_{H^1(-1,1)} + |h^0| + |h^1| \leq \varepsilon,$$

we can find controls $\alpha, \beta \in L^\infty(0,T)$ *and associated states*

$$(y,h) \in C^0([0,T]; L^2(-1,1)) \times C^1([0,T]) \ \ \textit{satisfying}$$

$$y(\xi,t) = 0 \ \ \textit{in} \ \ (-1,1), \ \ h(T) = 0 \ \ h'(T) = 0.$$

Finally, in (LIU, 2011) one of the main results is a local null controllability of system (34) with only one boundary control. In order to clarify the situation, let us present the result:

Theorem 1.8. *Let* $T > 0$*. There exists* $\varepsilon > 0$ *such that, wherever the initial data satisfy* (35) *and*

$$\|y^0\|_{H^1(-1,1)} + |h^0| + |h^1| \leq \varepsilon,$$

we can find a control $\alpha \in C^0[0,T]$ *and associated states*

$$y \in L^2([0,T]; H^2((-1,1) \setminus \{h(t)\})) \cap C^0([0,T]; H^1(-1,1)), \ \ h \in C^1([0,T]) \ \ \textit{satisfying}$$

$$y(x,t) = 0 \ \ \textit{in} \ \ (-1,1), \ \ h(T) = 0 \ \ h'(T) = 0.$$

It has considered a one-dimensional model for a compressible fluid containing a solid represented as a point particle of mass $m > 0$ which floats with fluid. The fluid density was assumed to be constant and the fluid velocity governed by the viscous Burgers equation on both sides of the point mass location $x = h(t)$ and the boundary control $\alpha \in C^0([0,T])$. The complete system of equation and data is

$$\begin{cases} y_t + yy_x - y_{xx} = 0, & (x,t) \in Q \setminus h(t), \\ y(-1,t) = \alpha(t), \;\; y(1,t) = 0, & t \in (0,T), \\ y(h(t),t) = h'(t), \;\; [y_x](h(t),t) = mh''(t), & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (-1,1), \\ h(0) = h^0, \;\; h'(0) = h_1. \end{cases} \tag{37}$$

1.5 Main results of the Thesis

1.5.1 *Main result of section 2*

Let $T > 0$ be given and $L \in C^1([0,T])$ be a function with

$$0 < L_* \leq L(t) \leq B, \;\; t \in (0,T).$$

Define $Q_L = \{(x,t) : x \in (0, L(t)), \;\; t \in (0,T)\}$ and assume that $f : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz continuous function. We will consider free-boundary problems for semilinear parabolic systems of the form

$$\begin{cases} y_t - y_{xx} + f(y) = v1_\omega, & (x,t) \in Q_L, \\ y(0,t) = 0, \;\; y(L(t),t) = 0, & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0, L_0), \\ L(0) = L_0, \end{cases} \tag{38}$$

with the additional boundary condition

$$L'(t) = -y_x(L(t),t), \;\; t \in (0,T). \tag{39}$$

Here, $y = y(x,t)$ is the state and $v = v(x,t)$ is a control; it acts on the system at any time through the nonempty open set $\omega = (a,b)$ with $0 < a < b < L_*$; 1_ω denotes the characteristic function of the set ω; we will assume that $y^0 \in H_0^1(0, L_0)$ and $L(0) = L_0$.

The main goal of Section 2 is to analyze the null controllability of (38). It will be said that (38) is null-controllable at time T if, for each $y^0 \in H_0^1(0,T)$, there exists $v \in L^2(\omega \times (0,T)$, a function $L \in C^1([0,T])$ and an associated solution $y = y(x,t)$ satisfying (38), (39) and

$$y(x,T) = 0, \quad x \in (0, L(T)). \tag{40}$$

Denote by y^* the extension of y by 0. The main result in the Section 3 is the following:

Theorem 1.9. *Assume that f is globally Lipschitz continuous, $f(0) = 0$, $T > 0$ and $B > 0$. Also, assume that $0 < a < b < L_* < L_0 < B$. Then* (38) *is locally null-controllable. More precisely, there exists $\varepsilon > 0$ such that, if $\|y^0\|_{H_0^1(0,L_0)} \le \varepsilon$ there exists triplets (L, v, y) with*

$$\begin{cases} L \in C^1([0,T]), \quad L_* \le L(t) \le B, \\ v \in L^2(\omega \times (0,T)), \quad y^* \in C^0([0,T]; H_0^1(0,B)), \end{cases} \tag{41}$$

satisfying (38), (39) *and* (40).

1.5.2 *Main result of section 3*

In Section 3, we will consider a 1D nonlinear system which models the interaction between a fluid and its boundary. We will assume that the velocity of the fluid is governed by the viscous Burgers equation and, for simplicity, that the fluid density is constant. Thus, the proposed system is the following:

$$\begin{cases} y_t + y y_x - y_{xx} = v 1_\omega, & (x,t) \in Q_\ell, \\ y(0,t) = 0, \quad y(\ell(t),t) = \ell'(t), & t \in (0,T), \\ y_x(\ell(t),t) = -\ell''(t) & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,\ell_0), \\ \ell(0) = l_0, \quad \ell'(0) = l_1, \end{cases} \tag{42}$$

Here, $T > 0$, $0 < a < b < l_* < l_0 < B$ and $Q_\ell = \{(x,t) : x \in (0,\ell(t)),\ t \in (0,T)\}$. Also, 1_ω is the characteristic of $\omega = (a,b)$, $y(x,t)$ is the velocity of fluid particle located at x at time t, v is a distributed control with support in the cylinder $\omega \times (0,T)$ which can be interpreted as an external force field acting on the fluid, ℓ is a function in the set

$$X := \{\ell \in C^2([0,T]) : 0 < l_* < \ell(t) < B, \forall t \in (0,T)\},$$

and the initial data satisfy

$$y^0 \in H^1(0,l_0), \quad 0 < l_* < \ell_0 < B \text{ and } y^0(l_0) = l_1.$$

We define $Q = (0,B) \times (0,T)$. Let $\widehat{y}$ be the extension of y defined below:

$$\widehat{y}(x,t) = \begin{cases} y(x,t), & \text{in } Q_\ell \\ \\ \ell'(t), & \text{in } Q \setminus Q_\ell \end{cases}$$

It will be said that (42) is null controllable at time T if, for every $(y^0, l_0, l_1) \in H^1(0, l_0) \times (l_*, B) \times \mathbb{R}$, there exist $v \in L^2(\omega \times (0,T))$ and an associated solution $(\widehat{y}, l) \in C^0([0,T]; L^2(0,B)) \times C^2([0,T])$ satisfying

$$\widehat{y}(x,T) = 0, \quad x \in (0,B).$$

The first main result of the section 3 is the following:

Theorem 1.10. *Let us assume $0 < a < b < \ell_* < \ell_0 < B$. Then (46) is locally null-controllable. More precisely, there exists $\varepsilon > 0$ such that, if $(y^0, \ell_1) \in H^1(0,\ell_0) \times \mathbb{R}$ and $\|(y^0,\ell_1)\|_{H^1(0,\ell_0)\times\mathbb{R}} \le \varepsilon$, we can find controls v and associated solutions (y,ℓ) satisfying*

$$v \in L^2(\omega \times (0,T)), \;\; \widehat{y} \in C^0([0,T]; H^1(0,B)). \;\; \ell \in X$$

and

$$\widehat{y}(x,T) = 0, \quad x \in (0,B). \tag{43}$$

1.5.3 *Main result of section 4*

Let $T > 0$ and let Ω be the open set

$$\Omega = \{x \in \mathbb{R}^3 : x_1, x_2, x_3 \in (0,1)\},$$

whose boundary is denoted by $\partial\Omega$. We will use the notation $Q := \Omega \times (0,T)$ and $\Sigma := \partial\Omega \times (0,T)$.

Let us introduce the Hilbert spaces

$$H(\Omega) = \{w \in L^2(\Omega)^3 : \nabla \cdot w = 0 \text{ in } \Omega, \;\; w \cdot n = 0 \text{ on } \partial\Omega\}$$

(where $n = n(x)$ is the outward unit normal vector at $x \in \partial\Omega$) and

$$V_0(\Omega) = \{w \in H_0^1(\Omega)^3 : \nabla \cdot w = 0 \text{ in } \Omega\}.$$

We consider the three-dimensional Navier-Stokes and Boussinesq systems

$$\begin{cases} u_t - \Delta u + (u,\nabla)u + \nabla p = f, \;\; \nabla \cdot u = 0 & \text{in } \; Q, \\ u(0,x_2,x_3,t) = 0, & \text{in } \; (0,1)^2 \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \; \Omega \end{cases} \tag{44}$$

and

$$\begin{cases} u_t - \Delta u + (u,\nabla)u + \nabla p = \theta e_N + f, \;\; \nabla\cdot u = 0 & \text{in} \;\; Q \\ \theta_t - \Delta\theta + u\cdot\nabla\theta = g & \text{in} \;\; Q \\ u(0,x_2,x_3,t) = 0, \;\; \theta(0,x_2,x_3,t) = 0 & \text{in} \;\; (0,1)^2\times(0,T) \\ (u(x,0),\theta(x,0)) = (u_0(x),\theta_0(x)) & \text{in} \;\; \Omega. \end{cases} \tag{45}$$

Here, $f \in L^2(0,T;L^2(\Omega)^3)$, $g \in L^2(0,T;L^2(\Omega))$ are given source terms, $u_0 \in H(\Omega)$ and $\theta_0 \in L^2(\Omega)$.

The first main result in the section 4 is the following:

Theorem 1.11. *Assume that* $(u_0,\theta_0) \in V_0(\Omega)\times H^1(\Omega)$ *and* $(f,g) \in L^2(Q)^3\times L^2(Q)$. *Then, there exists a sequence* $\{(f_\varepsilon,g_\varepsilon)\}_{\varepsilon>0}$ *in* $L^2(Q)^3\times L^2(Q)$ *such that*

$$(f_\varepsilon,g_\varepsilon) \to (f,g) \;\; in \;\; L^r(0,T;H^{-1}(\Omega)^3)\times L^r(0,T;H^{-1}(\Omega))$$

for all $r\in(1,4/3)$ *and there exist solutions* $(u_\varepsilon,p_\varepsilon,\theta_\varepsilon)$ *to the null controllability problems*

$$\begin{cases} u_{\varepsilon,t} - \Delta u_\varepsilon + (u_\varepsilon,\nabla)u_\varepsilon + \nabla p_\varepsilon = \theta_\varepsilon e_N + f_\varepsilon, \;\; \nabla\cdot u_\varepsilon = 0 & in\; Q, \\ \theta_{\varepsilon,t} - \Delta\theta_\varepsilon + u_\varepsilon\cdot\nabla\theta = g_\varepsilon & in\; Q \\ u_\varepsilon(0,x_2,x_3,t) = 0, \;\; \theta_\varepsilon(0,x_2,x_3,t) = 0 & in\; (0,1)^2\times(0,T), \\ (u_\varepsilon(x,0),\theta_\varepsilon(x,0)) = (u_0(x),\theta_0(x)) & in\; \Omega, \\ (u_\varepsilon(x,T),\theta_\varepsilon(x,T)) = (0,0) & in\; \Omega, \end{cases}$$

with

$$u_\varepsilon \in L^2(0,T;V(\Omega))\cap L^\infty([0,T];H(\Omega))$$

and

$$\theta_\varepsilon \in L^2(0,T;H_0^1(\Omega))\cap L^\infty([0,T];L^2(\Omega)).$$

As in (GUERRERO, IMANUVILOV, and PUEL, 2012), the proof of Theorem 4.1 will take four steps. Thus, we divide our time interval $(0,T)$ in four subintervals, where different strategies are used.

The following two results concern generalizations of Theorem 1 in (GUERRERO, IMANUVILOV, and PUEL, 2012). In the first one, we prove that the approximate boundary controllability can also be obtained with controls acting only on three faces of the unit cube. In the second one, we show that Ω can be a much more general set, namely a bounded domain of $\mathbb{R}^3$ whose boundary contains a piece of a plane entirely located inside one of the associated semispaces.

Theorem 1.12. *Assume that* $u_0 \in H(\Omega)$ *and* $f \in L^2(\Omega\times(0,T))$ *are given. Then, there exists* *a sequence* $\{(f_\varepsilon)\}_{\varepsilon>0}$ *in* $L^2(Q)^3$ *such that*

$$f_\varepsilon \to f \;\; in \;\; L^r(0,T;H^{-1}(\Omega)^3)$$

for all $r \in (1, 4/3)$ *and there exist solutions* $(u_\varepsilon, p_\varepsilon)$ *to the null controllability problems*

$$\begin{cases} u_{\varepsilon,t} - \Delta u_\varepsilon + (u_\varepsilon, \nabla)u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon & in \;\; \Omega \times (0,T), \\ \nabla \cdot u_\varepsilon = 0 & in \;\; \Omega \times (0,T), \\ u_\varepsilon(0,x_2,x_3,t) = u_\varepsilon(1,x_2,x_3,t) = u_\varepsilon(x_1,x_2,0,t) = 0 & in \;\; (0,1)^2 \times (0,T), \\ u_\varepsilon(x,0) = u_0(x), \;\; u_\varepsilon(x,T) = 0 & in \;\; \Omega. \end{cases}$$

Now, let Π be a plane in $\mathbb{R}^3$, let Π^+ be one of the semispaces determined by Π and let $\Omega_\Pi \subset \mathbb{R}^3$ be a bounded domain satisfying

$$\Omega_\Pi \subset \Pi^+, \;\; \Omega_\Pi \cap \Pi \;\text{ is a non-empty open set}$$

and let us consider the Navier-Stokes system

$$\begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = f & \text{in} \;\; \Omega_\Pi \times (0,T), \\ \nabla \cdot u = 0 & \text{in} \;\; \Omega_\Pi \times (0,T), \\ u(x,t) = 0 & \text{in} \;\; (\partial\Omega_\Pi \cap \Pi) \times (0,T), \\ u(x,0) = u_0(x) & \text{in} \;\; \Omega_\Pi. \end{cases}$$

Theorem 1.13. *Assume that* $u_0 \in H(\Omega_\Pi)$ *and* $f \in L^2(\Omega_\Pi \times (0,T))$*. Then, there exists a sequence* $\{(f_\varepsilon)\}_{\varepsilon>0}$ *in* $L^2(\Omega_\Pi \times (0,T))^3$ *such that*

$$f_\varepsilon \to f \;\; in \;\; L^r(0,T;H^{-1}(\Omega_\Pi)^3)$$

for all $r \in (1, 4/3)$ *and there exist solutions* $(u_\varepsilon, p_\varepsilon)$ *to the null controllability problems*

$$\begin{cases} u_{\varepsilon,t} - \Delta u_\varepsilon + (u_\varepsilon, \nabla)u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon & in \;\; \Omega_\Pi \times (0,T), \\ \nabla \cdot u_\varepsilon = 0 & in \;\; \Omega_\Pi \times (0,T), \\ u_\varepsilon(x,t) = 0 & on \;\; (\partial\Omega_\Pi \cap \Pi) \times (0,t), \\ u_\varepsilon(x,0) = u_0(x), \;\; u_\varepsilon(x,T) = 0 & in \;\; \Omega_\Pi. \end{cases}$$

A similar result can be deduced for the Boussinesq system in $\Omega_\Pi \times (0,T)$.

2 LOCAL NULL CONTROLLABILITY OF A FREE-BOUNDARY PROBLEM FOR THE SEMILINEAR 1D HEAT EQUATION

2.1 Introduction

Let $T > 0$ be given and $L \in C^1([0,T])$ be a function with

$$0 < L_* \leq L(t) \leq B, \quad t \in (0,T).$$

Define $Q_L = \{(x,t) : x \in (0, L(t)), \quad t \in (0,T)\}$ and assume that $f : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz continuous function. We will consider free-boundary problems for semilinear parabolic systems of the form

$$\begin{cases} y_t - y_{xx} + f(y) = v1_\omega, & (x,t) \in Q_L, \\ y(0,t) = 0, \quad y(L(t),t) = 0, & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0, L_0), \\ L(0) = L_0, & \end{cases} \tag{46}$$

with the additional boundary condition

$$L'(t) = -y_x(L(t),t), \quad t \in (0,T). \tag{47}$$

Here, $y = y(x,t)$ is the state and $v = v(x,t)$ is a control; it acts on the system at any time through the nonempty open set $\omega = (a,b)$ with $0 < a < b < L_*$; 1_ω denotes the characteristic function of the set ω; we will assume that $y^0 \in H_0^1(0, L_0)$ and $L(0) = L_0$, see Fig. (1).

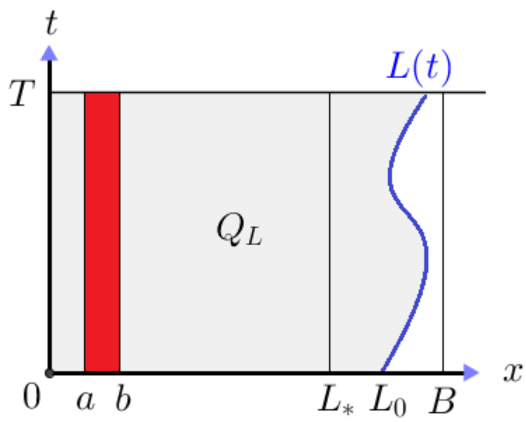

Figure 1: The situation in (46)

The main goal of this paper is to analyze the null controllability of (46). It will be said that (46) is null-controllable at time T if, for each $y^0 \in H_0^1(0,T)$, there exists

$v \in L^2(\omega \times (0,T)$, a function $L \in C^1([0,T])$ and an associated solution $y = y(x,t)$ satisfying (46), (47) and

$$y(x,T) = 0, \;\; x \in (0, L(T)). \tag{48}$$

On the other hand, it will be said that (46) is approximately controllable in $L^2(0, L(T))$ at time T if, for any $y^0 \in H_0^1(0, L_0)$ and any $\varepsilon > 0$, there exists a control $v \in L^2(\omega \times (0,T))$, a function $L \in C^1([0,T])$ and an associated state $y = y(x,t)$ satisfying (46), (47) and

$$\|y(\cdot, T)\|_{L^2(0,L(T))} \leq \varepsilon. \tag{49}$$

The controllability of linear and semilinear parabolic systems has been analyzed in several papers. Among them, let us mention (FURSIKOV and IMANUVILOV, 1996a), (BARBU, 2000), (FERNÁNDEZ-CARA and ZUAZUA, 2000) and (DOUBOVA *et al.*, 2002).

On the other hand, free-boundary problems similar to (46)-(47) have been motivated by different applications such as:

- Tumor growth and other problems from mathematical biology; see (FRIEDMAN and Aguda, 2006) and (2012).
- Fluid-solid interaction; see (DOUBOVA and FERNÁNDEZ-CARA, 2005), (VÁZQUEZ and ZUAZUA, 2003) and (LIU, TAKAHASHI, and TUCSNAK, 2013).
- Gas flow through porous media; see (ARONSON, 1983), (FASANO, 2005) and (VÁZQUEZ, 2007).
- Solidification and related Stefan problems; see (FRIEDMAN, 2010).
- The analysis and computation of free surface flows; see (HERMANS, 2010), (STOKER, 1957) and (WROBEL and BREBBIA, 1991).

Denote by y^* the extension of y by 0. The main result in this paper is the following:

Theorem 2.1. *Assume that f is globally Lipschitz continuous, $f(0) = 0$, $T > 0$ and $B > 0$. Also, assume that $0 < a < b < L_* < L_0 < B$. Then* (46) *is locally null-controllable. More precisely, there exists $\varepsilon > 0$ such that, if $\|y^0\|_{H_0^1(0,L_0)} \leq \varepsilon$ there exists triplets (L, v, y) with*

$$\begin{cases} L \in C^1([0,T]), \quad L_* \leq L(t) \leq B, \\ v \in L^2(\omega \times (0,T)), \;\; y^* \in C^0([0,T]; H_0^1(0,B)), \end{cases} \tag{50}$$

satisfying (46), (47) *and* (48).

Remark 2.1. Theorem 2.1 is still true when we consider, instead of (46), a boundary controlled system with the control acting just at $x = 0$. This can be deduced in a simple way as follows:

1. Take $\delta > 0$ and solve the following control problem

$$
\begin{cases}
\widetilde{y}_t - \widetilde{y}_{xx} + f(\widetilde{y}) = v1_{(-\delta/2,0)}, & (x,t) \in \widetilde{Q}_L, \\
\widetilde{y}(0,t) = 0, \;\; \widetilde{y}(L(t),t) = 0, & t \in (0,T), \\
\widetilde{y}(x,0) = \widetilde{y}^0(x), & x \in (-\delta, L_0), \\
L(0) = L_0,
\end{cases}
\tag{51}
$$

$$
L'(t) = -\widetilde{y}_x(L(t),t), \;\; t \in (0,T), \tag{52}
$$

$$
\widetilde{y}(x,T) = 0, \;\; x \in (-\delta, L(T)). \tag{53}
$$

Here, $\widetilde{y}^0$ is the extension of y^0 by 0, v is a distributed control with support in the cylinder $(-\delta/2, 0) \times (0,T)$ and $\widetilde{Q}_L = \{(x,t) : x \in (-\delta, L(t)), \;\; t \in (0,T)\}$.

2. Denote by y the restriction to Q_L of the function $\widetilde{y}$ and set $h(t) = \widetilde{y}(0,t)$. The triplet (L, h, y) is the solution of the boundary null controllability problem. □

Remark 2.2. An even more interesting case is found when the control acts on the free boundary:

$$
\begin{cases}
y_t - y_{xx} + f(y) = 0, & (x,t) \in Q_L, \\
y(0,t) = 0, \;\; y(L(t),t) = h(t), & t \in (0,T), \\
y(x,0) = y^0(x), & x \in (0, L_0), \\
L(0) = L_0,
\end{cases}
$$

together with (47) and (48). This control problem needs a deeper analysis. □

The rest of this paper is organized as follows. In Subsection 2.2, we prove a global Carleman inequality, whence we deduce an observability inequality needed to prove the null controllability of a linear variant of (46)-(47). We also establish a regularity property for $y_x(L(t),t)$. In Subsection 2.3, we give the proof of Theorem 2.1. Subsection 2.4 deals with some additional comments.

2.2 A controllability result for the linear heat equation in a non-cylindrical domain

2.2.1 *The problem and the result*

Our final goal is to prove Theorem 2.1. We will use a fixed point argument and, for this purpose, we must first study the null controllability problem for the linear system:

$$
\begin{cases}
y_t - y_{xx} + a(x,t)y = v1_\omega, & (x,t) \in Q_L, \\
y(0,t) = 0, \;\; y(L(t),t) = 0, & t \in (0,T), \\
y(x,0) = y^0(x), & x \in (0, L_0),
\end{cases}
\tag{54}
$$

where $a \in L^\infty((0,B) \times (0,T))$ and the function $L \in C^1([0,T])$ is given and satisfies, $0 < a < b < L_* < L(t) < B$.

After an appropriate change of variable, (54) can be rewritten in the form

$$\begin{cases} w_s - w_{\xi\xi} + B(\xi,s)w_\xi + C(\xi,s)w = h, & (\xi,s) \in (0,L_0)\times(0,S), \\ w(0,s) = 0, \;\; w(L_0,s) = 0, & s \in (0,S), \\ w(\xi,0) = y^0(\xi), & \xi \in (0,L_0), \end{cases} \tag{55}$$

with $B, C \in L^\infty((0,L_0)\times(0,S))$ and $h \in L^2((0,L_0)\times(0,S))$.

We can easily verify that there exists a unique solution y to (54), with $y^* \in L^2(0,T;H^2(0,B))$ and $y^*_t \in L^2(0,T;L^2(0,B)$. Consequently,

$$y^* \in C^0([0,T];H^1_0(0,B)).$$

Theorem 2.2. *For any $y^0 \in H^1_0(0,L_0)$ and $\varepsilon > 0$, there exist pairs $(v_\varepsilon, y_\varepsilon)$ with*

$$v_\varepsilon \in L^2(\omega\times(0,T)), \quad y^*_\varepsilon \in C^0([0,T];H^1_0(0,B))$$

satisfying (54) and

$$\|y_\varepsilon(\cdot,T)\|_{L^2(0,L(T))} \le \varepsilon. \tag{56}$$

Furthermore, the control v_ε can be found such that

$$\|v_\varepsilon\|_{L^2(\omega\times(0,T))} \le C_1\|y^0\|_{L^2(0,L_0)}, \tag{57}$$

where $C_1 > 0$ only depends on $L_*, B, \omega, \|L'\|_\infty, \|a\|_{L^\infty(Q_0)}$ and T.

The proof follows rather standard arguments. The main tool is a global Carleman estimate for the solution to the *adjoint system* of (54), that is given by

$$\begin{cases} -\varphi_t - \varphi_{xx} + a(x,t)\varphi = u, & (x,t) \in Q_L, \\ \varphi(0,t) = 0, \;\; \varphi(L(t),t) = 0, & t \in (x,T), \\ \varphi(x,T) = \varphi^0(x), & x \in (0,L(T)), \end{cases} \tag{58}$$

where $u \in L^2(Q_L)$ and $\varphi^0 \in L^2(0,L(T))$.

An immediate consequence of Theorem 2.2 is the following:

Corollary 2.1. *For any $y^0 \in H^1_0(0,L_0)$, there exists pairs (v,y), with*

$$v \in L^2(\omega\times(0,T)), \;\; y^* \in C^0([0,T];H^1_0(0,B)),$$

satisfying (54) and (48). Furthermore, v can be found such that

$$\|v\|_{L^2(\omega\times(0,T))} \le C_2\|y^0\|_{H^1_0(0,L_0)}$$

where C_2 only depends on $L_*, B, \omega, \|L'\|_\infty, \|a\|_{L^\infty(Q_0)}$ and T.

This will be recalled in the next section.

2.2.2 *A global Carleman inequality for the linear heat equation and its consequences*

Let us first introduce some weight functions. Let us denote the lateral boundary of Q_L by

$$\Sigma_L = \{(x,t) : x = 0 \text{ or } x = L(t),\ 0 < t < T\}.$$

Lemma 2.1. *Let ω_0 be a non-empty open set with $\overline{\omega_0} \subset (a,b)$. There exists a function $\eta_0 \in C^1(\overline{Q_L})$ with $\eta_{0,xx} \in C^0(\overline{Q_L})$ such that*

$$\begin{cases} \eta_0(x,t) = 0, & (x,t) \in \Sigma_L, \\ |\eta_{0,x}| > 0, & (x,t) \in \overline{Q_L} \setminus (\omega_0 \times (0,T)), \\ \eta_0(x,t) = 1 - \dfrac{x-b}{l(t)-b}, & (x,t) \in (b, L(t)) \times (0,T). \end{cases}$$

The proof of this Lemma can be found in (FERNÁNDEZ-CARA, LIMACO, and MENEZES, 2016), Lemma 2.1. We introduce now the weight functions

$$\alpha(x,t) := \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x,t)}}{\beta(t)},$$

$$\xi(x,t) := \frac{e^{\lambda\eta(x,t)}}{\beta(t)},$$

where $\beta(t) = t(T-t)$, $\eta(x,t) = \eta_0(x,t) + 1$ and $\lambda > 0$.

The following result contains a Carleman estimate for the solutions to the adjoint systems (58); it is inspired by the ideas in Fursikov-Imonolov (1996a) and the proof is identical to the proof of Theorem 2.2 in (FERNÁNDEZ-CARA, LIMACO, and MENEZES, 2016).

Theorem 2.3. *Let η, α, β and ξ be the functions defined above. There exist positive constants λ_0, s_0 and C_0, only depending on $L_*, B, \omega, \|L'\|_\infty, \|z\|_{L^\infty(Q_0)}$ and T, such that, for any $s \geq s_0$ and any $\lambda \geq \lambda_0$, we have*

$$
\begin{aligned}
&\iint_{Q_L} e^{-2s\alpha}\left(\frac{1}{s\xi}(|\varphi_t|^2+|\varphi_{xx}|^2)+\lambda^2 s\xi|\varphi_x|^2+\lambda^4 s^3\xi^3|\varphi|^2\right)dx\,dt\\
&+\int_0^T e^{-2s\alpha(L(t),t)}\lambda s\xi(L(t),t)|\varphi_x(L(t),t)|^2\,dt\\
&+\int_0^T e^{-2s\alpha(0,t)}\lambda s\xi(0,t)|\varphi_x(0,t)|^2\,dt\\
&\le C_0\left(\iint_{Q_L} e^{-2s\alpha}|u|^2\,dx\,dt+\iint_{\omega\times(0,T)} e^{-2s\alpha}\lambda^4 s^3\xi^3|\varphi|^2\,dx\,dt\right)
\end{aligned}
\tag{59}
$$

In a second step, we will prove an observability inequality for the solutions to the adjoint systems. This is a consequence of the previous Carleman inequality.

Proposition 2.1. *There exists a constant $C>0$, only depending on L_*, B, ω, $\|L'\|_\infty$, $\|z\|_{L^\infty(Q_0)}$ and T, such that for any $\varphi^0\in L^2(0,L(T))$, the associated solution to (58) with $u=0$ satisfies*

$$
\int_0^{L_0}|\varphi(x,0)|^2\,dx\le C\iint_{\omega\times(0,T)}|\varphi|^2\,dx\,dt \tag{60}
$$

Proof. Let us take $\lambda=\lambda_0$ and $s=s_0$ in (3.4). Then

$$
\iint_{Q_L} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt\le C\iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt
$$

and, consequently,

$$
\begin{aligned}
\int_{T/4}^{3T/4}\int_0^{L(t)}|\varphi|^2\,dx\,dt &\le C\int_{T/4}^{3T/4}\int_0^{L(t)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt\\
&\le C\iint_{\omega\times(0,T)}|\varphi|^2\,dx\,dt.
\end{aligned}
\tag{61}
$$

On the other hand, if we introduce the auxiliary function $\psi=e^{t\|a\|_\infty}\varphi$, we find that

$$
-\frac{1}{2}\frac{d}{dt}\left(\int_0^{L(t)}|\psi|^2\,dx\right)+\int_0^{L(t)}|\psi_x|^2\,dx+\int_0^{L(t)}(\|a\|_\infty+a)|\psi|^2\,dx=0,
$$

for all $t\in(0,T)$, whence

$$\frac{d}{dt}\left(\int_0^{L(t)} |\psi|^2\,dx\right) \geq 0.$$

This implies

$$\int_0^{L(0)} |\varphi(x,0)|^2\,dx \leq e^{T\|a\|_\infty}\int_0^{L(t)} |\varphi(x,t)|^2\,dx \quad \forall t\in(0,T)$$

and

$$\frac{T}{2}\int_0^{L(0)} |\varphi(x,0)|^2\,dx \leq e^{T\|a\|_\infty}\int_{T/4}^{3T/4}\int_0^{L(t)} |\varphi(x,t)|^2\,dx\,dt. \tag{62}$$

From (61) and (62), we conclude the proof. □

The observability inequality (60) leads to the approximate controllability result in Theorem 2.2. The argument is well known; see (FABRE, PUEL, and ZUAZUA, 1995) for more details. Thus, let $y^0 \in L^2(0,L_0)$ and $\varepsilon > 0$ be given and let us introduce the functional $J_\varepsilon(\cdot,a,L)$, with

$$J_\varepsilon(\varphi^0;a,L) = \frac{1}{2}\iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt + \varepsilon\|\varphi^0\|_{L^2(0,L(T))} + (\varphi(\cdot,0),y^0)_{L^2(0,L_0)}$$

for all $\varphi^0 \in L^2(0,L(T))$.

Here, φ is the solution to (58). Using (60), it is relatively easy to check that $J_\varepsilon(\cdot;a,L)$ is strictly convex, continuous, and coercive in $L^2(0,L(T))$, so it possesses a unique minimum $\varphi_\varepsilon^0 \in L^2(0,L(T))$, whose associated solution is denoted by φ_ε. Let us now introduce the control $v_\varepsilon = \varphi_\varepsilon 1_\omega$, and let us denote by y_ε the solution to (54) associated to v_ε. Then, either $\varphi_\varepsilon^0 = 0$, or we can differentiate the functional at φ_ε^0 and obtain a necessary condition to reach a minimum at φ_ε^0:

$$\begin{cases} \displaystyle\iint_{\omega\times(0,T)} \varphi_\varepsilon\varphi\,dx\,dt & + \quad \varepsilon\left(\dfrac{\varphi_\varepsilon^0}{\|\varphi_\varepsilon^0\|_{L^2(0,L(T))}},\varphi^0\right)_{L^2(0,L(T))} \\ & + \quad (\varphi(\cdot,0),y^0)_{L^2(0,L_0)} = 0 \\ \forall\varphi^0 \in L^2(0,L(T)), & \end{cases} \tag{63}$$

From this and (60) for $\varphi^0 = \varphi_\varepsilon^0$, we get the estimate (57). On the other hand, since the systems (54) and (58) are in duality, we also have

$$\iint_{\omega\times(0,T)} \varphi_\varepsilon \varphi \, dx\, dt = (\varphi^0, y_\varepsilon(T))_{L^2(0,L(T))} - (\varphi(0), y^0)_{L^2(0,L_0)} \tag{64}$$

which, combined with (63), yields (56).

2.2.3 *The uniform Hölder-continuity of* y_x

We introduce here a class of functions of standard use in the regularity theory of parabolic equations, see (LADYZHENSKAIA, SOLONNIKOV, and URALCEVA, 1988).

Let us fix an integer $m \geq 0$ and $\alpha \in (0,1)$. Let us set $Q_0 = (0,B)\times(0,T)$, let $G \subset Q_0$ be a non-empty open set and let us assume the $D_t^r D_x^s u$ is continuous in $\overline{G}$ for $2r+s < m+\alpha$. Then, we set

$$\langle u\rangle_{x,G}^{(\alpha)} = \sup_{(x,t),(x',t)\in\overline{G}} \frac{|u(x,t)-u(x',t)|}{|x-x'|^\alpha}, \quad \langle u\rangle_{x,G}^{(m+\alpha)} = \sum_{2r+s=m} \langle D_t^r D_x^s u\rangle_{x,G}^{(\alpha)},$$

$$\langle u\rangle_{t,G}^{(\alpha/2)} = \sup_{(x,t),(x,t')\in\overline{G}} \frac{|u(x,t)-u(x,t')|}{|t-t'|^{\alpha/2}}, \quad \langle u\rangle_{t,G}^{(\frac{m+\alpha}{2})} = \sum_{2r+s=m} \langle D_t^r D_x^s u\rangle_{t,G}^{(\frac{\alpha}{2})}.$$

$$|u|_G^{(m+\alpha)} = \sum_{2r+s\leq m} \|D_t^r D_x^s u\|_{L^\infty(G)} + \langle u\rangle_{x,G}^{(m+\alpha)} + \langle u\rangle_{t,G}^{(\frac{m+\alpha}{2})},$$

The space of the functions $u = u(x,t)$, such that $|u|_G^{(m+\alpha)} < +\infty$ will be denoted by

$$K^{m,\alpha}(\overline{G}).$$

This is a separable Banach space for $|\cdot|_G^{m,\alpha}$. Furthermore, it is easy to check that $K^{m,0}(\overline{G}) = C^m(\overline{G})$ and, if $m+\alpha < m'+\alpha'$, the embedding $K^{m',\alpha'}(\overline{G}) \hookrightarrow K^{m,\alpha}(\overline{G})$ is compact.

Let us denote by N_0 the norm of y^0 in $L^2(0,L_0)$ and let (v,y) be a control-state pair furnished by Theorem 2.2. Let b' be given with $b < b' < L_0$ and let us set

$$R_L = Q_L \cap \{(x,t) : x > b'\}.$$

From Theorems 10.1 and 11.1 in (LADYZHENSKAIA, SOLONNIKOV, and URALCEVA, 1988, pp. 204 and 211), we can affirm that $y \in K^{1,\alpha}$ for all $\alpha \in [0,1/2)$, the function V_L with $V_L(t) := y_x(L(t),t)$ satisfies $V_L \in C^{0,\alpha}([0,T])$ and, furthermore,

$$\|V_L\|_{C^{0,\alpha}([0,T])} \leq C\|y\|_{L^\infty(Q_L)} \tag{65}$$

where the constant $C > 0$ only depends on C_1 and N_0 and α only depends on L_* and B.

Let us write $y = \widehat{y} + \widetilde{y}$ where $\widehat{y}$ is the solution to (54) with $y^0 \equiv 0$ and $\widetilde{y}$ is the solution to (54) with $v \equiv 0$.

Using Gronwall's Lemma, one can easily prove that

$$\|\widehat{y}\|_{L^\infty(Q_L)} \leq C(\|a\|_{L^\infty(Q_0)}, \|L'\|_\infty, B, T)\|v\|_{L^2(\omega\times(0,T))}. \tag{66}$$

On the other hand, from the Maximum Principle, we have

$$\|\widetilde{y}\|_{L^\infty(Q_L)} \leq C(\|a\|_{L^\infty(Q_0)}, \|L'\|_\infty, T)\|y^0\|_{L^\infty(0,L_0)}. \tag{67}$$

Consequently, we see that

$$\|V_L\|_{C^{0,\alpha}([0,T])} \leq C_3\|y^0\|_{L^\infty(0,B)} \tag{68}$$

where the constant $C_3 > 0$ only depending on $\|a\|_{L^\infty(Q_0)}, \|L'\|_\infty, B, T, L_*, \omega$ and N_0.

The estimate (68) will be crucial in the proof of Theorem 2.1 in the next section.

2.3 Proof of Theorem 2.1

In a first step, let us assume that $f \in C^1(\mathbb{R})$ and $|f'|$ is uniformly bounded.

We define the function $g : \mathbb{R} \to \mathbb{R}$ as follows:

$$g(s) = \frac{f(s)}{s} \text{ for } s \neq 0 \text{ and } g(0) = f'(0).$$

For any $(z, \ell) \in L^\infty(Q_0) \times C^1([0,T])$ with $L_* \leq \ell \leq B$ and any $y^0 \in H_0^1(0, L_0)$, we consider the following controllability problem

$$\begin{cases} y_t - y_{xx} + g(z)y = v1_\omega, & (x,t) \in Q_\ell, \\ y(0,t) = 0, \ \ y(\ell(t),t) = 0, & t \in (x,T), \\ y(x,0) = y^0(x), & x \in (0, L_0), \end{cases} \tag{69}$$

$$\|y(\cdot,T)\|_{L^2(0,\ell(T))} \leq \varepsilon \tag{70}$$

.

Let us introduce the set

$$\mathcal{N} = \{z \in L^\infty(Q_0) : \|z\|_{L^\infty(Q_0)} \leq R\},$$

where $R > 0$ will be defined later. Let $R_1 > 0$ be given and let us set

$$\mathcal{M} = \{\ell \in C^1([0,T]) : L_* \leq \ell \leq B,\ \ell(0) = L_0,\ \|\ell'\|_\infty \leq R_1\}.$$

We will consider the mapping $\Lambda_\varepsilon : \mathcal{N} \times \mathcal{M} \mapsto L^\infty(Q_0) \times C^1([0,T])$ defined by $\Lambda_\varepsilon = (y_\varepsilon^*, L_\varepsilon)$, where y_ε satisfies (69) and (70) for $v = \varphi_\varepsilon|_{\omega\times(0,T)}$, φ_ε is the unique minimum of $J_\varepsilon(\cdot; g(z), \ell)$ and

$$L_\varepsilon(t) = L_0 - \int_0^t y_{\varepsilon,x}(\ell(s), s)\, ds \tag{71}$$

We will apply a fixed point technique to prove Theorem 2.1. First, note that in view of the results in Section 2, Λ_ε is well defined. Moreover, one has

$$\|y_\varepsilon^*\|_{L^\infty(Q_0)} \leq C_4\|y^0\|_{L^\infty(0,L_0)},$$

where C_4 only depends on L_*, B, ω, R_1 and T,

$$\|L_\varepsilon'\| \leq C_3\|y^0\|_{H_0^1(0,L_0)}$$

and, consequently,

$$|L_\varepsilon(t) - L_0| \leq C_3 T\|y^0\|_{H_0^1(0,L_0)}, \quad \forall t \in [0,T].$$

Therefore, if we take

$$R = C_4\|y^0\|_{L^\infty(0,L_0)}$$

and, we assume that

$$\|y^0\|_{H_0^1(0,L_0)} \leq \min\left(\frac{R_1}{C_3}, \frac{B - L_*}{C_3 T}, \frac{L_0 - L_*}{C_3 T}\right),$$

we find that Λ maps $\mathcal{N} \times \mathcal{M}$ into itself.

Let us now prove that, for some $\alpha \in (0,1)$, Λ_ε maps the bounded sets of $L^\infty(Q_0) \times C^1([0,T])$ into bounded sets in $K^{0,\alpha}(\overline{Q}_0) \times C^{1,\alpha}([0,T])$. We will use the results from (LADYZHENSKAIA, SOLONNIKOV, and URALCEVA, 1988, see Theorems 7.1 and 10.1, Ch. III). Thus, there exists $\alpha \in (0,1/2)$ (only depending on L_*, B and T) such that $y_\varepsilon \in K^{0,\alpha}(\overline{Q}_0)$ and there exists a constant $C > 0$, only depending on L_*, B, T, α and $\|y^0\|_{H_0^1(0,L_0)}$ such that

$$|y_\varepsilon|_{Q_0}^{(0+\alpha)} \leq C;$$

more details can be found in (FERNÁNDEZ-CARA, LIMACO, and MENEZES, 2016).

On the other hand, from (93) we already knew that

$$\|L_\varepsilon\|_{C^{1,\alpha}} \leq C, \tag{72}$$

where $C > 0$ only depends on the previous data and N_0. As a consequence, Λ_ε maps $\mathcal{N} \times \mathcal{M}$ into a compact set of $L^\infty(Q_0) \times C^1([0,T])$.

Now, we will show that $(z,\ell) \mapsto \Lambda(z,\ell)$ is a continuous mapping. Thus, let the (z_n,ℓ_n) be such that

$$(z_n,\ell_n) \to (z,\ell), \text{ in } L^\infty(Q_0) \times C^1([0,T]) \tag{73}$$

and let us set $(y^*_{\varepsilon,n}, L_{\varepsilon,n}) = \Lambda_\varepsilon(z_n,\ell_n)$.

Obviously, $\Lambda_\varepsilon(z_n,\ell_n)$ converge strongly to some $(y^*_\varepsilon, L_\varepsilon)$. We must prove that $(y^*_\varepsilon, L_\varepsilon) = \Lambda_\varepsilon(z,\ell)$.

To this purpose, the following result will be used:

Proposition 2.2. *Let M be the mapping*

$$M : \mathcal{N} \times \mathcal{M} \mapsto L^2(0,1),$$

where $M(z,\ell) = \psi^0_\varepsilon$, $\psi^0_\varepsilon(\zeta) \equiv \varphi^0_\varepsilon(\zeta\ell(T))$ and φ^0_ε is the minimizer of $J_\varepsilon(\cdot; g(z), \ell)$.

If $z_n \to z \in L^\infty(Q_0)$ and $\ell_n \to \ell$ strongly in $C^1([0,T])$, then $\psi^0_{\varepsilon,n}$ converges strongly in $L^2(0,1)$ to ψ^0_ε.

The proof can be obtained as in (FERNÁNDEZ-CARA, LIMACO, and MENEZES, 2016).

A direct use of Proposition 2.2 shows that the controls $v_{\varepsilon,n}$ associated to the (z_n,ℓ_n) converge strongly in $L^2(\omega \times (0,T))$ to the control v_ε associated to (y,ℓ):

$$v_{\varepsilon,n} \to v_\varepsilon \text{ strongly in } L^2(\omega \times (0,T)).$$

Thus, it is straightforward to check that the $(y^*_{\varepsilon,n}, L_{\varepsilon,n})$ converge to $\Lambda_\varepsilon(z,\ell)$ and consequently is continuous.

A consequence of these propositions of Λ_ε is that there exists $\delta > 0$ (independent of ε) such that, if $\|y^0\|_{H^1_0(0,L_))} \leq \delta$, Schauder's Theorem can be applied to the fixed point equation $(y,L) = \Lambda_\varepsilon(y,L)$.

Let $(y_\varepsilon, L_\varepsilon)$ be a fixed point of Λ for each $\varepsilon > 0$. Then it is clear that $(y_\varepsilon, L_\varepsilon)$ satisfies, together with v_ε, (46), (47), (56) and (57). Moreover, L_ε and v_ε are uniformly bounded in $C^{1+\alpha}([0,T])$ and $L^2(\omega \times (0,T))$, respectively. Consequently, our assertion is proved.

Now, at least for a subsequence, one has

$$L_\varepsilon \to L \text{ strongly in } C^1([0,T]) \text{ and } v_\varepsilon \to v, \text{ weakly in } L^2(\omega \times (0,T))$$

as $\varepsilon \to 0$. Obviously, (y, L, v) satisfies (2.1) and (47). Also, it is clear that y satisfies (48).

This proves the result when f is of class C^1.

The general case can be easily obtained through a simple approximation process. Hence, The proof of Theorem 2.1 is completed.

2.4 Additional comments and questions

The global null controllability of (46)–(47) is an open question. As noticed in (FERNÁNDEZ-CARA, LIMACO, and MENEZES, 2016), it is open even in the case $f \equiv 0$. It s not clear at all how the existence of a fixed point of Λ_ε can be obtained for large y^0.

On the other hand, for higher spatial dimensions, the local null controllability is also open. In view of the previous results and arguments, a natural strategy would be to introduce a mapping of the form

$$(z, \ell) \in \mathcal{L} \mapsto \Lambda(z, \ell) = (y^*, L) \in \mathcal{L},$$

where v is a minimal L^2-norm control that produces a state satisfying

$$\|y(T)\|_{L^2(\Omega(T))} \leq \varepsilon, \ \ x \in \Omega(T).$$

and $\{\Omega(t)\}_{t\in[0,T]}$ is a family of sets whose boundaries are parameterized by ℓ and try to prove the existence of a fixed point. But, again, this does not seem easy.

On the other hand, it is not difficult to prove a result similar to Theorem 2.1 under spherical symmetry hypotheses. Indeed, it suffices to adapt the assumptions on the data ω and y^0 and define the weights appropriately.

3 LOCAL NULL CONTROLLABILITY OF A FREE-BOUNDARY PROBLEM FOR THE VISCOUS BURGERS EQUATION

3.1 Introduction

We will consider a 1D nonlinear system which models the interaction between a fluid and its boundary. We will assume that the velocity of the fluid is governed by the viscous Burgers equation and, for simplicity, that the fluid density is constant. Thus, the proposed system is the following:

$$\begin{cases} y_t + yy_x - y_{xx} = v1_\omega, & (x,t) \in Q_\ell, \\ y(0,t) = 0, \;\; y(\ell(t),t) = \ell'(t), & t \in (0,T), \\ y_x(\ell(t),t) = -\ell''(t) & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,\ell_0), \\ \ell(0) = \ell_0, \;\; \ell'(0) = \ell_1, \end{cases} \tag{74}$$

Here, $T > 0$, $0 < a < b < \ell_* < \ell_0 < B$ and $Q_\ell = \{(x,t) : x \in (0,\ell(t)), \;\; t \in (0,T)\}$. Also, 1_ω is the characteristic of $\omega = (a,b)$, $y(x,t)$ is the velocity of fluid particle located at x at time t, v is a distributed control with support in the cylinder $\omega \times (0,T)$ which can be interpreted as an external force field acting on the fluid, ℓ is a function in the set

$$X := \{\ell \in C^2([0,T]) : 0 < \ell_* \leq \ell(t) \leq B, \forall t \in (0,T)\},$$

and the initial data satisfy

$$y^0 \in H^1(0,\ell_0), \;\; 0 < \ell_* < \ell_0 < B \text{ and } y^0(\ell_0) = \ell_1.$$

Note that two conditions are required. Since y is the velocity of the fluid, we have to assume $y(\ell(t),t) = \ell'(t)$ and each boundary particle at $\ell(t)$ is accelerated by $y_x(\ell(t),t)$.

A model similar to (74) for the interaction between a fluid and a solid represented by a point mass is considered and analyzed in (VÁZQUEZ and ZUAZUA, 2003) and (VÁZQUEZ and ZUAZUA, 2006) and investigated from the viewpoint of the null controllability in (DOUBOVA and FERNÁNDEZ-CARA, 2005) and (LIU, 2011).

The modeling and analysis of fluid-solid interaction have attracted a lot of attention in recent years. In particular, in the case of two- and three-dimensional Navier-Stokes fluids in contact with one or more rigid or elastic bodies, this has been the goal for instance of (DESJARDINS, 2000), (TAKAHASHI, 2003) and

TAKAHASHI and TUCSNAK (2004).

We define $Q = (0, B) \times (0, T)$. Let $\widehat{y}$ be the extension of y defined below:

$$\widehat{y}(x,t) = \begin{cases} y(x,t), & \text{in } Q_\ell \\ \\ \ell'(t), & \text{in } Q \setminus Q_\ell \end{cases}$$

It will be said that (74) is null controllable at time T if, for every $(y^0, \ell_0, \ell_1) \in H^1(0, \ell_0) \times (\ell_*, B) \times \mathbb{R}$, there exist $v \in L^2(\omega \times (0,T))$ and an associated solution $(\widehat{y}, l) \in C^0([0,T]; L^2(0,B)) \times C^2([0,T])$ satisfying

$$\widehat{y}(x,T) = 0, \quad x \in (0,B).$$

The controllability of PDE has also been the object of extensive research during the last years. Since the pioneering papers such as (LIONS, 1988b) and (1988a), where systems governed by linear wave and heat equations were considered, a lot of works has been done in this area, such as (FABRE, PUEL, and ZUAZUA, 1995), (FURSIKOV and IMANUVILOV, 1996a), (ZUAZUA, 1991) and (2007) for the approximate, exact and null controllability of semilinear parabolic and hyperbolic PDEs.

The first main result of this paper is the following:

Theorem 3.1. *Let us assume $0 < a < b < \ell_* < \ell_0 < B$. Then (74) is locally null-controllable. More precisely, there exists $\varepsilon > 0$ such that, if $(y^0, \ell_1) \in H^1(0,\ell_0) \times \mathbb{R}$ and $\|(y^0, \ell_1)\|_{H^1(0,\ell_0)\times\mathbb{R}} \leq \varepsilon$, we can find controls v and associated solutions (y, ℓ) satisfying*

$$v \in L^2(\omega \times (0,T)), \quad \widehat{y} \in C^0([0,T]; H^1(0,B)), \quad \ell \in X$$

and

$$\widehat{y}(x,T) = 0, \quad x \in (0,B). \tag{75}$$

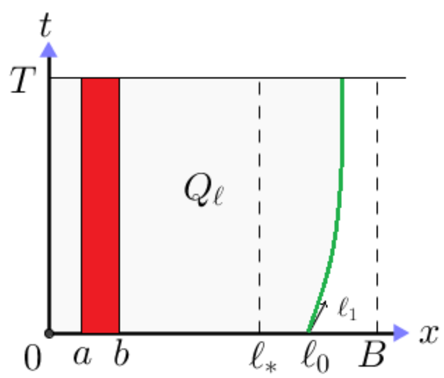

Figure 2: The situation in Theorem 3.1

For the proof, in a first step, we must consider a linearized system:

$$\begin{cases} y_t + \overline{y} y_x - y_{xx} = v 1_\omega, & (x,t) \in Q_{\overline{\ell}}, \\ y(0,t) = 0, \;\; y(\overline{\ell}(t), t) = \ell'(t), & t \in (0,T), \\ y_x(\overline{\ell}(t), t) = -\ell''(t) & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0, \ell_0), \\ \ell(0) = \ell_0, \;\; \ell'(0) = \ell_1, \end{cases} \tag{76}$$

with potential $\overline{y} \in L^\infty(Q)$ and $\overline{\ell} \in X$ such that $\overline{\ell}(0) = \ell_0, \;\; \overline{\ell}'(0) = \ell_1$ and $|\overline{\ell}'| < 1$ given.

The following result holds:

Theorem 3.2. *For any $v \in L^2(\omega \times (0,T))$ and any $(y^0, \ell_1) \in H^1(0,\ell_0) \times \mathbb{R}$, there exists a unique solution (y, ℓ) to (76) satisfying*

$$\widehat{y} \in C^0([0,T]; L^2(0,B)) \cap L^2(0,T; H^1(0,B)), \;\; \widehat{y}_t \in L^2(0,T; H^{-1}(0,B)) \tag{77}$$

and

$$\ell \in H^2(0,T). \tag{78}$$

Furthermore, there exists a positive constant C, only depending on $\ell_*, B, T, \|\overline{y}\|_{L^\infty(Q)}$ and ℓ_0, such that

$$\begin{aligned} &\|\widehat{y}\|_{L^2(0,T;H^1(0,B))} + \|\widehat{y}_t\|_{L^2(0,T;H^{-1}(0,B))} + \|\ell'\|_{H^1(0,T)} \\ &\quad \le C\big(\|(y^0, \ell_1)\|_{L^2(0,\ell_0)\times\mathbb{R}} + \|v\|_{L^2(\omega\times(0,T))}\big). \end{aligned} \tag{79}$$

Remark 3.1. It is interesting to note that Theorem 3.1 is still true for similar boundary controllability problem, with the control $h \in L^\infty(0,T)$ acting at $x = 0$. This can be deduced from the case shown in this paper by simple extension process. □

This Section is organized as follows. In Subsection 3.2, we state and prove a uniform approximate controllability result. This will rely on useful global Carleman estimates and some related observability inequalities. We will also establish a regularity property for the states. In Subsection 3.3, we give the proof of Theorem 3.1. Subsection 3.4 deals with some additional comments and the Appendix contains the proof of the Carleman estimates.

3.2 A controllability result and a regularity property

In the section, we assume that $\ell_0 > 0, T > 0$ and $0 < a < b < \ell_* < \ell_0 < B$ are given. We fix $(y^0, \ell_1) \in H^1(0,\ell_0) \times \mathbb{R}$ and $\overline{\ell} \in X$ with $\overline{\ell}(0) = \ell_0$ and $|\overline{\ell}'| < 1$.

3.2.1 *A uniform approximate*

We will need an estimate for the velocity of the fluid at time T and the control of the system (76), such that the control keeps an explicit dependence with respect to (y^0, ℓ_1). The precise result is the following.

Theorem 3.3. *For any $(y^0, \ell_1) \in H^1(0, \ell_0) \times \mathbb{R}$ and any $\varepsilon > 0$, there exist a control $v_\varepsilon \in L^2(\omega \times (0,T))$ such that the corresponding solution $(y_\varepsilon, \ell_\varepsilon)$ of* (76) *verifies*

$$\|(y_\varepsilon(T), \ell_\varepsilon'(T))\|_{L^2(0,\overline{\ell}(T))\times\mathbb{R}} \leq \varepsilon, \ \ \forall x \in (0, \overline{\ell}(T)), \tag{80}$$

Moreover, v_ε can be chosen satisfying the estimate

$$\|v_\varepsilon\|_{L^2(\omega\times(0,T))} \leq C_1 \|(y^0, \ell_1)\|_{L^2(0,\ell_0)\times\mathbb{R}}, \tag{81}$$

where C_1 is a positive constant, only depending on $\ell_*, B, \omega, \|\overline{y}\|_{L^\infty(Q)}$ and T.

The proof follows rather standard arguments. The main tool is a global Carleman estimate for the solution to the *adjoint system* of (76), that is given by

$$\begin{cases} -\varphi_t - (\overline{y}\varphi)_x - \varphi_{xx} = g(x,t), & (x,t) \in Q_{\overline{\ell}}, \\ \varphi(0,t) = 0, \ \ \varphi(\overline{\ell}(t), t) = m'(t), & t \in (0,T), \\ \varphi_x(\overline{\ell}(t), t) = m''(t) - \overline{y}(\overline{\ell}(t), t)m'(t), & t \in (0,T), \\ \varphi(x, 0) = \varphi^T(x), & x \in (0, \overline{\ell}(T)), \\ m(T) = \overline{\ell}(T), \ \ m'(T) = m^1, \end{cases} \tag{82}$$

where $g \in L^2(Q_{\overline{\ell}})$ and $(\varphi^T, m^1) \in L^2(0, \overline{\ell}(T)) \times \mathbb{R}$.

This will be established in the next subsection.

3.2.2 *An observability inequality*

In this subsection, we will establish a technical result needed in this section. More precisely, we will present and prove the required Carleman estimates for the systems (82).

In this technique it is fundamental to use some weight functions.

Lemma 3.1. *Let ω_0 be a non-empty open set with $\overline{\omega_0} \subset (a,b)$. There exists a function $\eta_0 \in C^1(\overline{Q_{\overline{\ell}}})$ with $\eta_{0,xx} \in C^0(\overline{Q_{\overline{\ell}}})$ such that*

$$\begin{cases} |\eta_{0,x}| > 0, \ \ (x,t) \in \overline{Q_{\overline{\ell}}} \setminus (\omega_0 \times (0,T)), \\ \eta_{0,x}(0,t) = 0, \\ \eta_0(x,t) = \dfrac{\overline{\ell}(t) - x}{\overline{\ell}(t) - b}, \ \ (x,t) \in (b, \ell(t)) \times (0,T). \end{cases}$$

For the proof, it suffices to take (for instance)

$$\eta_0(x,t) = \begin{cases} \dfrac{x^2}{a}, & \text{if } \ 0 \le x < a \\ a + p\left(\frac{2(x-a)}{b-a}, \frac{b-a}{2a}\right), & \text{if } \ a \le x < \frac{a+b}{2} \\ a + p\left(\frac{2(b-x)}{b-a}, \frac{b-a}{2(\overline{\ell}(t)-b)}\right), & \text{if } \ \frac{a+b}{2} \le x < b \\ 1 - \dfrac{x-b}{\overline{\ell}(t)-b}, & \text{if } \ b \le x < \overline{\ell}(t) \end{cases}$$

where

$$p(x,y) = xy + (10-6y)x^3 + (8y-15)x^4 + (6-3y)x^5.$$

Let $\lambda > 1$ large enough and $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(Q_{\overline{\ell}})}$. Let us introduce the weights

$$\xi(x,t) = \frac{e^{\lambda\eta(x,t)}}{t(T-t)} \ \text{ and } \ \alpha(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x,t)}}{t(T-t)},$$

where $\eta(x,t) = \eta_0(x,t) + 1$. Then we have the following Carleman estimate:
Theorem 3.4. *Let η, ξ and α be the functions defined above. There exist positive constants λ_0, s_0 and C_0 only depending on $\|\overline{y}\|_{W^{1,\infty}(Q_{\overline{\ell}})}, \ell_*, \omega, B$ and T such that for any $s \ge s_0$, $\lambda \ge \lambda_0$ any $g \in L^2(Q_{\overline{l}})$ and any $\varphi^T \in L^2(0, \overline{\ell}(T))$, one has*

$$\begin{aligned} &\iint_{Q\overline{\ell}} e^{-2s\alpha}\left(\frac{1}{s\xi}(\varphi_t + \varphi_{xx}) + s\lambda^2\xi|\varphi_x|^2 + s^3\lambda^4\xi^3|\varphi|^2\right) dx\,dt \\ &+ s\lambda \int_0^T e^{-2s\alpha(\overline{\ell}(t),t)}\xi(\overline{\ell}(t),t)|\varphi_x(\overline{\ell}(t),t)|^2\,dt \\ &+ s^3\lambda^3 \int_0^T e^{-2s\alpha(\overline{\ell}(t),t)}\xi(\overline{\ell}(t),t)^3|m'(t)|^2\,dt \\ &\le C_0\left(\|e^{-s\alpha}g\|_2^2 + s^3\lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt\right), \end{aligned} \tag{83}$$

where φ is the corresponding solution of (82).

The prof of this Theorem is given in the Appendix.

We now prove the observability inequality for the solutions to the adjoint system. Observe that it is a consequence of the Carleman Inequality presented above.
Proposition 3.1. *There exists $C > 0$, only depending on $\ell_*, B, \omega, \|\overline{y}\|_\infty$ and T, such that for any $\varphi^T \in L^2(0, \overline{\ell}(T))$, the associated solution to (82) with $g = 0$ satisfies*

$$\int_0^{\ell_0} |\varphi(x,0)|^2\,dx + |m'(0)|^2 \le C \iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt \tag{84}$$

Proof. Let us take $\lambda = \lambda_0$ and $s = s_0$ in Theorem 3.4. Then

$$\begin{aligned} &\iint_{Q\overline{\ell}} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt + \int_0^T e^{-2s\alpha(\overline{l}(t),t)}\xi(\overline{\ell}(t),t)^3|m'(t)|^2\,dt \\ &\quad\le C\iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \end{aligned} \tag{85}$$

and consequently

$$\begin{aligned} &\int_{T/2}^{3T/2}\int_0^{\overline{\ell}(t)} |\varphi|^2\,dx\,dt + \int_{T/2}^{3T/2}|m'(t)|^2\,dt \\ &\quad\le C\iint_{\omega\times(0,T)}|\varphi|^2\,dx\,dt. \end{aligned} \tag{86}$$

Let us multiply the equation (82) by φ and let us integrate with respect to x in $(0,\overline{\ell}(t))$. Then we see that

$$-\frac{1}{2}\frac{d}{dt}\left(\int_0^{\overline{\ell}(t)}|\varphi|^2\,dx + |m'|^2\right) + \int_0^{\overline{\ell}(t)}|\varphi_x|^2\,dx + \int_0^{\overline{\ell}(t)}\overline{y}\varphi\varphi_x\,dx = 0.$$

Consequently,

$$-\frac{d}{dt}\left(\int_0^{\overline{\ell}(t)}|\varphi|^2\,dx + |m'|^2\right) \le \|\overline{y}\|^2_{L^\infty(Q_{\overline{\ell}})}\int_0^{\overline{\ell}(t)}|\varphi|^2\,dx$$

We deduce that

$$\frac{d}{dt}\left(e^{t\|\overline{y}\|^2_{L^\infty(Q_{\overline{\ell}})}}\int_0^{\overline{\ell}(t)}|\varphi|^2\,dx + |m'|^2\right) \ge 0 \ \ \forall t\in(0,T)$$

and, consequently,

$$\int_0^{\ell_0}|\varphi(x,0)|^2\,dx + |m'(0)|^2 \le e^{t\|\overline{y}\|^2_{L^\infty(Q_{\overline{\ell}})}}\left(\int_0^{\overline{\ell}(t)}|\varphi(x,t)|^2\,dx + |m'(t)|\right),$$

forall $t\in(0,T)$ and

$$\begin{aligned} &\frac{T}{2}\left(\int_0^{\ell_0}|\varphi(x,0)|^2\,dx + |m'(0)|^2\right) \\ &\quad\le e^{T\|\overline{y}\|^2_{L^\infty(Q_{\overline{l}})}}\left(\int_{T/2}^{3T/2}\int_0^{\overline{l}(t)}|\varphi(x,t)|^2\,dx + \int_{T/2}^{3T/2}|m'(t)|\right). \end{aligned} \tag{87}$$

From (86) and (87), the proof of Proposition 3.1 is done. □

3.2.3 *Proof of Theorem 3.3*

This controllability result, Theorem 3.3, is implied by the observability estimate (84). Let $(y^0, \ell_1) \in L^2(0, \ell_0) \times \mathbb{R}$ and $\varepsilon > 0$ be given. Let us introduce the functional J_ε, with

$$\begin{aligned} J_\varepsilon(\varphi^T, m^1) &= \frac{1}{2}\iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt + \varepsilon\|(\varphi^T, m^1)\|_{L^2(0,\bar{\ell}(T))\times\mathbb{R}} \\ &\quad +((\varphi(0), m'(0)), (y^0, \ell_1))_{L^2(0,\ell_0)\times\mathbb{R}} \end{aligned}$$

for $\varphi^T \in L^2(0, \bar{\ell}(T))$. Here, φ is solution of (82) with $\varphi(T) = \varphi^T$. Using (84) it is not difficult to check that J_ε is strictly convex, continuous, and

$$\liminf_{\|(\varphi^T,m^1)\|_{L^2(0,\bar{\ell}(T))\times\mathbb{R}}\to\infty} \frac{J_\varepsilon(\varphi^T, m^1)}{\|(\varphi^T, m^1)\|_{L^2(0,\bar{\ell}(T))\times\mathbb{R}}} \geq \varepsilon. \tag{88}$$

This implies that the functional J_ε is coercive.

Consequently, J_ε achieves its minimum at a unique point

$$(\varphi_\varepsilon^T, m_\varepsilon^1) \in L^2(0, \bar{\ell}(T)) \times \mathbb{R}$$

.

Let $(\varphi_\varepsilon, m'_\varepsilon)$ be the solution of (82) associated to $(\varphi_\varepsilon^T, m_\varepsilon^1)$. Let us now introduce the control $v_\varepsilon = \varphi_\varepsilon 1_\omega$, and let us denote by $(y_\varepsilon, \ell'_\varepsilon)$ the solution of (76) associated to v_ε. Let us remark that the unique interesting case to be studied turns out to be when $\varphi_\varepsilon^T \neq 0$; see (FABRE, PUEL, and ZUAZUA, 1995) for more details. Under this assumption, we can differentiate the functional J_ε at $(\varphi_\varepsilon^T, m_\varepsilon^1)$ and obtain a necessary condition for J_ε to reach a minimum at $(\varphi_\varepsilon^T, m_\varepsilon^1)$

$$\begin{aligned} J'_\varepsilon(\varphi_\varepsilon^T, m_\varepsilon^1)(\varphi^T, m^1) &= \iint_{\omega\times(0,T)} \varphi_\varepsilon\varphi\,dx\,dt \\ &\quad +\varepsilon\frac{((\varphi_\varepsilon^T, m_\varepsilon^1), (\varphi^T, m^1))_{L^2(0,\ell_0)\times\mathbb{R}}}{\|(\varphi_\varepsilon^T, m_\varepsilon^1)\|_{L^2(0,\bar{\ell}(T))\times\mathbb{R}}} \\ &\quad +((\varphi(0), m'(0)), (y^0, \ell_1))_{L^2(0,\ell_0)\times\mathbb{R}} \\ &= 0, \end{aligned} \tag{89}$$

for every $(\varphi^T, m^1) \in L^2(0, \overline{\ell}(T)) \times \mathbb{R}$. From this equality and (84) written for $(\varphi^T, m^1) = (\varphi^T_\varepsilon, m^1_\varepsilon)$, we obtain

$$\|v_\varepsilon\|_{L^2(\omega\times(0,T))} \leq C_1 \|(y^0, \ell_1)\|_{L^2(0,\overline{\ell}(T))\times\mathbb{R}}, \tag{90}$$

where $C_1 > 0$ only depends on $\ell_*, B, \omega, \|\overline{y}\|_\infty$ and T. Since systems (76) and (82) are in duality, we have

$$\begin{aligned} \iint_{\omega\times(0,T)} \varphi_\varepsilon \varphi \, dx\, dt &= ((\varphi^T, m^1), (y_\varepsilon(T), \ell'_\varepsilon(T)))_{L^2(0,\overline{\ell}(T))\times\mathbb{R}} \\ &\quad -((\varphi(0), m'(0)), (y^0, \ell_1))_{L^2(0,\ell_0)\times\mathbb{R}} \end{aligned} \tag{91}$$

which, combined with (89), yields

$$\|(y_\varepsilon(T), \ell'_\varepsilon(T))\|_{L^2(0,\overline{\ell}(T))\times\mathbb{R}} \leq \varepsilon, \;\; \forall x \in (0, \overline{\ell}(T)).$$

3.2.4 *A regularity property*

We introduce here a class of functions of standard use in the regularity theory of parabolic equations, see (LADYZHENSKAIA, SOLONNIKOV, and URALCEVA, 1988)).

Let us fix an integer $m \geq 0$ and $\alpha \in (0,1)$. Let us set $Q = \Omega \times (0,T)$, let $G \subset Q$ be a non-empty open set and let us assume the $D^r_t D^s_x u$ is continuous in $\overline{G}$ for $2r + s < m + \alpha$. Then, we set

$$\langle u \rangle^{(\alpha)}_{x,G} = \sup_{(x,t),(x',t)\in\overline{G}} \frac{|u(x,t) - u(x',t)|}{|x - x'|^\alpha}, \;\; \langle u \rangle^{(m+\alpha)}_{x,G} = \sum_{2r+s=m} \langle D^r_t D^s_x u \rangle^{(\alpha)}_{x,G},$$

$$\langle u \rangle^{(\alpha/2)}_{t,G} = \sup_{(x,t),(x,t')\in\overline{G}} \frac{|u(x,t) - u(x,t')|}{|t - t'|^{\alpha/2}}, \;\; \langle u \rangle^{(\frac{m+\alpha}{2})}_{t,G} = \sum_{2r+s=m} \langle D^r_t D^s_x u \rangle^{(\frac{\alpha}{2})}_{t,G}.$$

$$|u|^{(m+\alpha)}_G = \sum_{2r+s\leq m} \|D^r_t D^s_x u\|_{L^\infty(G)} + \langle u \rangle^{(m+\alpha)}_{x,G} + \langle u \rangle^{(\frac{m+\alpha}{2})}_{t,G},$$

The space of the functions $u = u(x,t)$, such that $|u|^{(m+\alpha)}_G < \infty$ will be denoted by

$$K^{m,\alpha}(\overline{G}).$$

This is a separable Banach space for $|\cdot|^{m,\alpha}_G$. Furthermore, it is easy to check that $K^{m,0}(\overline{G}) = C^m(\overline{G})$ and, if $m+\alpha < m'+\alpha'$ the embedding $K^{m',\alpha'}(\overline{G}) \hookrightarrow K^{m,\alpha}(\overline{G})$ is compact.

Let us denote by N_0 the norm of y^0 in $L^2(0,L_0)$ and let (v,y) be a control-state pair furnished by Theorem 3.3. Let b' be given with $b < b' < L_0$ and let us set

$$R_{\overline{\ell}} = Q_{\overline{\ell}} \cap \{(x,t) : x > b'\}.$$

Let $\delta > 0$ be sufficiently small to have $b < b' - \delta < b' + \delta < \ell_*$. From Theorem 10.1 in (LADYZHENSKAIA, SOLONNIKOV, and URALCEVA, 1988, p. 204), we can say that

$$y_\varepsilon \in K^{1,\alpha}([b'-\delta, b'+\delta] \times (0,T)),$$

where $\alpha \in [0,1/2)$.

Let us now introduce the following change of variables in $R_{\overline{l}}$:

$$\xi(x,t) = \frac{b'(\overline{\ell}(t)-x) + \ell_0(x-b')}{\overline{\ell}(t)-b'}, \quad s(t) = \int_0^t \left(\frac{\ell_0 - b'}{\overline{\ell}(\tau)-b'}\right)^2 d\tau$$

and let us set $z(\xi,s) = y(x,t)$. Then, we have

$$\begin{cases} -z_s - \overline{z}z_\xi - z_{\xi\xi} = 0, \ \text{ in } R, \\ z(0,s) = 0, \ \ z(\ell_0,s) = \ell'(t(s)), \ \ s \in (0,S), \\ z_\xi(\ell_0,s) = -\ell''(t(s)), \ \ s \in (0,S), \\ z(\xi,0) = y^0(\xi), \ \ \xi \in (b',\ell_0), \end{cases} \tag{92}$$

where $R = (b',\ell_0)\times(0,S)$ with $S = s(T)$ and $\overline{z} \in L^\infty((b',\ell_0)\times(0,S))$ and $\|\overline{z}\|_{L^\infty((b',\ell_0)\times(0,S))}$ is bounded a constant only depending on $\|\overline{y}\|_\infty, \ell_*, \ell_0$ and b'. Taking again into account Theorems 10.1 and 11.1 of (LADYZHENSKAIA, SOLONNIKOV, and URALCEVA, 1988, p.211), we can verify that the function z satisfies

$$z \in K^{1,\alpha}(\overline{R}).$$

Consequently, we can verify that the function $\ell''(t) := -y_{\varepsilon,x}(\ell(t),t)$ satisfies

$$\|\ell''\|_{C^{0,\alpha/2}([0,T])} \leq C_2 \|(y^0,\ell_1)\|_{H^1(0,\ell_0)\times\mathbb{R}}, \tag{93}$$

where the constant $C_2 > 0$ only depending on $\|\overline{y}\|_\infty, \ell_*, \ell_0, B, \omega$ and T.

3.3 The local null controllability of the nonlinear system

To prove that (74) is null controllable, we proceed as follows. Let us set

$$Z = L^2(Q) \times X_0, \;\; X_0 := \{\ell \in X : \ell(0) = \ell_0,\; \ell'(0) = \ell_1,\; \|\ell'\|_\infty < 1\}.$$

Let Z_R be the closed convex set

$$Z_R = \{(z,\ell) \in Z : \|\overline{z}\|_{L^\infty(Q)} < R,\; \|\ell'\|_\infty < 1\}$$

(R will be chosen below).

We will denote by $\|\cdot\|_Z$ the usual norm in Z:

$$\|(z,\ell)\|_Z = \|(z,\ell)\|_{L^2(Q)\times C^0([0,T])}.$$

Let $\varepsilon > 0$ be given. For any $(z,\ell) \in Z$, let us denote by v_ε the minimal L^2 norm control satisfying, together with the state $(y_\varepsilon, k_\varepsilon)$, the following:

$$\begin{cases} y_{\varepsilon,t} + z y_{\varepsilon,x} - y_{\varepsilon,xx} = v_\varepsilon 1_\omega, & (x,t) \in Q_\ell \\ y_\varepsilon(0,t) = 0, \;\; y_\varepsilon(\ell(t),t) = k_\varepsilon(t), & t \in (0,T) \\ y_{\varepsilon,x}(\ell(t),t) = -k_\varepsilon'(t), & t \in (0,T) \\ y_\varepsilon(x,0) = y^0(x), & x \in (0,\ell_0) \\ k_\varepsilon(0) = \ell_0, \;\; k_\varepsilon(0) = \ell_1, & \end{cases}$$

$$\|(y_\varepsilon(T), k_\varepsilon(T))\|_{L^2(0,\ell(T))\times\mathbb{R}} \le \varepsilon.$$

We will set

$$\Theta_\varepsilon(z,\ell) := v_\varepsilon$$

and

$$\Lambda_\varepsilon(z,\ell) := (\widehat{y}_\varepsilon, L_\varepsilon),$$

where

$$\widehat{y}_\varepsilon(x,t) = \begin{cases} y_\varepsilon(x,t), \;\text{ in } \; Q_\ell \\ k_\varepsilon(t), \;\text{ in } \; Q \setminus Q_\ell \end{cases}$$

and

$$L_\varepsilon(t) = \ell_0 + \int_0^t k_\varepsilon(s)\, ds.$$

We will try to apply Schauder's Fixed-Point Theorem to the mapping Λ_ε in Z_R. To this end, we note for the moment that, in view of Theorems 3.2 and 3.3, the following estimates hold:

$$\|v_\varepsilon\|_{L^2(\omega\times(0,T))} \le C_1 \|(y^0, \ell_1)\|_{L^2(0,\ell_0)\times\mathbb{R}}$$

and

$$\|(\widehat{y}_\varepsilon, k_\varepsilon)\|_Z \leq C_3\|(y^0, \ell_1)\|_{L^2(0,\ell_0)\times\mathbb{R}},$$

where $C_3 > 0$ only depend ons ℓ_*, B, $\|z\|_{L^\infty(Q)}$, ω and T.

It is clear that $(z, l) \mapsto \Lambda_\varepsilon(z, l)$ is a well defined mapping from Z into a bounded set of $L^\infty(Q) \times C^0([0, T])$. On other hand, we have by construction $L_\varepsilon \in C^2([0, T])$, $L'_\varepsilon(t) \equiv k_\varepsilon(t)$ and $L_\varepsilon(0) = \ell_0$, whence

$$|L'_\varepsilon(t)| \leq C_3\|(y^0, \ell_1)\|_{H^1(0,\ell_0)\times\mathbb{R}} \text{ and } |L_\varepsilon(t) - \ell_0| \leq TC_3\|(y^0, \ell_1)\|_{H^1(0,\ell_0)\times\mathbb{R}}.$$

Therefore, if we choose $\|(y^0, \ell_1)\|_{H^1(0,\ell_0)\times\mathbb{R}}$ such that

$$\|(y^0, \ell_1)\|_{H^1(0,\ell_0)\times\mathbb{R}} \leq \min\left(1, \frac{R}{C_3}, \frac{B-\ell_0}{TC_3}, \frac{\ell_0-\ell_*}{TC_3}\right),$$

the property $\Lambda_\varepsilon(Z_R) \subset Z_R$ holds.

We denote by $X^2(Q)$ the following Banach space:

$$X^2(Q) := L^2(0, T; H^1(0, B)) \cap W^{1,2}(0, T; L^2(0, B)).$$

Let us prove that Λ_ε maps bounded sets in Z into bounded sets in $X^2(Q) \times C^{2,\alpha}([0, T])$, for some $\alpha \in (0, 1/2)$. This will suffice to our purpose, since this space is compactly embedded in $L^2(Q) \times C^2([0, T])$. But this is clear: indeed, combining (79), (81) and (93), we deduce that

$$\|\widehat{y}_\varepsilon\|_{X^2(Q)} + \|k_\varepsilon\|_{C^{1,\alpha/2}([0,T])} \leq C_4\|(y^0, \ell_1)\|_{H^1(0,\ell_0)\times\mathbb{R}},$$

where C_4 is a positive constant, only depending on ℓ_*, B, $\|z\|_{L^\infty(Q)}$, ω and T.

Now, we will show that $(z, \ell) \mapsto \Lambda_\varepsilon(z, \ell)$ is a continuous mapping on Z. Thus, let the (z_n, ℓ_n) be such that

$$(z_n, \ell_n) \to (z, \ell) \text{ in } Z, \tag{94}$$

and let us set $(y_{\varepsilon,n}, L_{\varepsilon,n}) = \Lambda_\varepsilon(z_n, l_n)$ for all n. We must prove that $(y_\varepsilon, L_\varepsilon) = \Lambda_\varepsilon(z, \ell)$.

Proposition 3.2. *Let us consider the mapping $M : Z \to L^2(0, \ell_0)\times C^0([0, T])$, with $M(z, \ell) = (\varphi^T_\varepsilon, m^1_\varepsilon)$, where $(\varphi^T_\varepsilon, m^1_\varepsilon)$ is the minimizer of J_ε and*

$$J_\varepsilon(\varphi^T, m^1) = \frac{1}{2}\iint_{\omega\times(0,T)} |\varphi|^2\, dx\, dt + \varepsilon\|(\varphi^T, m^1)\|_{L^2(0,l(T))\times\mathbb{R}}$$

$$+((\varphi(0), m'(0)), (y^0, \ell_1))_{L^2(0,\ell_0)\times\mathbb{R}}$$

for all $(\varphi^T, m^1) \in L^2(0,\ell(T)) \times \mathbb{R}$ *(here,* (φ, m') *is solution of (82) with potential* z *and initial condition* (φ^T, m^1)*).*

One has the following:

i *If* $\{z_n\}$ *is a bounded sequence in* $L^\infty(Q)$ *and the* $\ell_n \in X_0$*, there exists a constant* $C > 0$ *independent of* n *such that*

$$\|\varphi^T_{\varepsilon,n}\|_{L^2(0,L_0)} \le C, \ \ \forall n,$$

where $(\varphi^T_{\varepsilon,n}, m^1_{\varepsilon,n})$ *is the minimizer of the functional* $J_{\varepsilon,n}$ *corresponding to* z_n *and* Q_{l_n}*.*

ii *If* $z_n \to z \in L^\infty(Q)$ *weak-*$*$ *and* $\ell_n \to \ell \in C^0([0,T])$*, then* $\varphi^T_{\varepsilon,n}$ *converges strongly in* $L^2(0,l(T))$ *to* φ^T_ε*.*

After this proposition, it is not difficult to check that, indeed,

$$(\widehat{y}_{\varepsilon,n}, L_{\varepsilon,n}) \to (\widehat{y}_\varepsilon, L_\varepsilon) \ \text{ strongly in } Z$$

and, consequently, Λ_ε is continuous.

We can apply Schauder's Theorem to Λ_ε. Let $(y_\varepsilon, L_\varepsilon)$ be a fixed point of Λ_ε for each $\varepsilon > 0$. Then, the v_ε are uniformly bounded in $L^2(\omega \times (0,T))$ and it is clear that, at least for a subsequence,

$$v_\varepsilon \to v, \ \text{ weakly in } \ L^2(\omega \times (0,T)) \ \text{ as } \ \varepsilon \to 0,$$

where v satisfies, together with some (y, L), the following:

$$\begin{cases} y_t + yy_x - y_{xx} = v1_\omega, & (x,t) \in Q_L, \\ y(0,t) = 0, \ \ y(L(t),t) = L'(t), & t \in (0,T), \\ y_x(L(t),t) = -L''(t), & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,\ell_0), \\ L(0) = \ell_0, \ \ L'(0) = \ell_1 \end{cases} \tag{95}$$

and

$$y(x,T) = 0, \ \ x \in (0,\ell_0), \ \ L'(T) = 0.$$

3.4 Additional comments and questions

- A global null control result is, to our knowledge, unknown.
- It is also interesting to analyze a similar model with two fluids and an external force field only acting on one of the fluids. The objective would be in this case to see whether the system can be driven exactly to zero with only one control starting from an arbitrary state, see (LIU, 2011) for a local result.

- Consider the following inviscid Burgers free-boundary problem:

$$\begin{cases} y_t + yy_x = v1_\omega, & (x,t) \in Q_\ell, \\ y(0,t) = 0, \;\; y(\ell(t),t) = \ell'(t), & t \in (0,T), \\ y_x(\ell(t),t) = -\ell''(t), & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,\ell_0), \\ \ell(0) = \ell_0, \;\; \ell'(0) = \ell_1. \end{cases} \tag{96}$$

On the other hand, for each $\nu > 0$, consider the following system, similar to (46):

$$\begin{cases} y_t + yy_x - \nu y_{xx} = v1_\omega, & (x,t) \in Q_\ell, \\ y(0,t) = 0, \;\; y(l(t),t) = l'(t), & t \in (0,T), \\ y_x(\ell(t),t) = -\ell''(t), & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,\ell_0), \\ \ell(0) = \ell_0, \;\; \ell'(0) = \ell_1. \end{cases} \tag{97}$$

Assume that $(y^0, \ell_1) \in H^1(0,\ell_0) \times \mathbb{R}$ and $0 < a < b < \ell_* < \ell_0 < B$ and let v_ν be a null control for (97) and (y_ν, ℓ'_ν) an associated state satisfying

$$y_\nu(x,T) = 0, \;\; x \in (0,\ell_0), \;\; \ell'_\nu(T) = 0.$$

Is it possible to prove convergence result, at least for a subsequence, of the v_ν and the (y_ν, ℓ_ν) respectively to v and (y,ℓ) ?

- As a system in higher spatial dimension, consider the Navier-Stokes free-boundary problem

$$\begin{cases} y_t - \Delta y + (y,\nabla)y + \nabla p = v1_\omega, & (x,t) \in \Omega(t) \times (0,T), \\ \nabla \cdot y = 0, & (x,t) \in \Omega(t) \times (0,T), \\ y(l(t,s),t) = l_t(t,s), & t \in (0,T), \;\; s \in (0,1), \\ y(x,0) = y^0(x), & x \in \Omega_0, \end{cases}$$

together with

$$\begin{cases} \dfrac{\partial y}{\partial \eta}(l(t,s),t) = -l_{tt}(t,s) \cdot \eta, \;\; t \in (0,T), \;\; s \in (0,1) \\ l(t,0) = l(t,1), \;\; l_s(t,0) = l_s(t,1), \;\; t \in (0,T) \end{cases}$$

where $\Omega(t) \subset \mathbb{R}^n, n = 2$ or 3 and $\partial\Omega(t) = l(t,s), \;\; t \in (0,T), \;\; s \in (0,1)$.
The local null controllability is open in this case (in fact, even for the Stoles system, the answer is unknown).

4 REMARKS CONCERNING THE APPROXIMATE CONTROLLABILITY OF THE BOUSSINESQ AND NAVIER-STOKES SYSTEMS

In this Section, we deal with the 3D Navier-Stokes and Boussinesq systems in a cube. We prove some extensions and variants of a result by Guerrero, Imanuvilov and Puel that concerns the (global) approximate boundary controllability.

4.1 Introduction

Let $T > 0$ and let Ω be the open set

$$\Omega = \{x \in \mathbb{R}^3 : x_1, x_2, x_3 \in (0,1)\},$$

whose boundary is denoted by $\partial\Omega$. We will use the notation $Q := \Omega \times (0,T)$ and $\Sigma := \partial\Omega \times (0,T)$.

Let us introduce the Hilbert spaces

$$H(\Omega) = \{w \in L^2(\Omega)^3 : \nabla \cdot w = 0 \ \text{ in } \ \Omega, \ \ w \cdot n = 0 \ \text{ on } \ \partial\Omega\}$$

(where $n = n(x)$ is the outward unit normal vector at $x \in \partial\Omega$) and

$$V_0(\Omega) = \{w \in H_0^1(\Omega)^3 : \nabla \cdot w = 0 \ \text{ in } \ \Omega\}.$$

We consider the three-dimensional Navier-Stokes and Boussinesq systems

$$\begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = f, \ \ \nabla \cdot u = 0 & \text{in} \ \ Q \\ u(0, x_2, x_3, t) = 0, & \text{in} \ \ (0,1)^2 \times (0,T) \\ u(x,0) = u_0(x) & \text{in} \ \ \Omega \end{cases} \tag{98}$$

and

$$\begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = \theta e_N + f, \ \ \nabla \cdot u = 0 & \text{in} \ \ Q \\ \theta_t - \Delta\theta + u \cdot \nabla\theta = g & \text{in} \ \ Q \\ u(0, x_2, x_3, t) = 0, \ \ \theta(0, x_2, x_3, t) = 0 & \text{in} \ \ (0,1)^2 \times (0,T) \\ (u(x,0), \theta(x,0)) = (u_0(x), \theta_0(x)) & \text{in} \ \ \Omega. \end{cases} \tag{99}$$

Here, $f \in L^2(0,T; L^2(\Omega)^3)$, $g \in L^2(0,T; L^2(\Omega))$ are given source terms, $u_0 \in H(\Omega)$ and $\theta_0 \in L^2(\Omega)$.

In a recent work, Guerrero, Imanuvilov and Puel (GUERRERO, IMANUVILOV, and PUEL, 2012) have established a result concerning the approximate controllability of (98). Specifically, they have proved that, for any u_0 and f, there exists a sequence

$\{f_n\}$ in $L^2(0,T;L^2(\Omega)^3)$ such that $f_n \to f$ in an appropriate sense and, for each n, the corresponding system (98) is null-controllable, with controls supported by the faces on the boundary where $x_1 \neq 0$.

This paper is devoted to present some extensions and variants that include in particular a result of the same kind for the Boussinesq system (99).

Note that, in view of the time irreversibility of (98) and (99), we cannot expect the exact controllability to hold to an arbitrary target function. On the other hand, recall that the global approximate controllability is an open question for these systems, due to the presence of a Dirichlet condition at $x_1 = 0$.

Let us now recall some (partial) results concerning the controllability of (98) and (99).

Global controllability results can be proved using the arguments in (FURSIKOV and IMANUVILOV, 1999) if the control is exerted on the whole boundary. On the other hand, the local exact controllability to bounded trajectories with distributed controls was first established in (FERNÁNDEZ-CARA, GUERRERO, IMANUVILOV, and PUEL, 2004) and GUERRERO (2006), respectively for the Navier-Stokes and Boussinesq systems. This has been revisited and improved in a set of papers, where it was shown that $N-1$ or even less scalar controls suffice; see (FERNÁNDEZ-CARA, GUERRERO, IMANUVILOV, and PUEL, 2006; CARREÑO, 2012; CARREÑO and GUERRERO, 2013; CORON and LISSY, 2014). In (CORON, 1996), the global approximate controllability of the 2D Navier-Stokes equations completed with Navier slip boundary conditions was proved. Then, in (CORON and FURSIKOV, 1996), a global exact controllability result was established for the same system in a 2D manifold without boundary.

The first main result in this paper is the following:

Theorem 4.1. *Assume that* $(u_0, \theta_0) \in V_0(\Omega) \times H^1(\Omega)$ *and* $(f, g) \in L^2(Q)^3 \times L^2(Q)$. *Then, there exists a sequence* $\{(f_\varepsilon, g_\varepsilon)\}_{\varepsilon>0}$ *in* $L^2(Q)^3 \times L^2(Q)$ *such that*

$$(f_\varepsilon, g_\varepsilon) \to (f, g) \ \ \text{in} \ \ L^r(0,T;H^{-1}(\Omega)^3) \times L^r(0,T;H^{-1}(\Omega))$$

for all $r \in (1, 4/3)$ *and there exist solutions* $(u_\varepsilon, p_\varepsilon, \theta_\varepsilon)$ *to the null controllability problems*

$$\begin{cases} u_{\varepsilon,t} - \Delta u_\varepsilon + (u_\varepsilon, \nabla) u_\varepsilon + \nabla p_\varepsilon = \theta_\varepsilon e_N + f_\varepsilon, \ \ \nabla \cdot u_\varepsilon = 0 & \text{in } Q \\ \theta_{\varepsilon,t} - \Delta \theta_\varepsilon + u_\varepsilon \cdot \nabla \theta = g_\varepsilon & \text{in } Q \\ u_\varepsilon(0, x_2, x_3, t) = 0, \ \ \theta_\varepsilon(0, x_2, x_3, t) = 0 & \text{in } (0,1)^2 \times (0,T) \\ (u_\varepsilon(x,0), \theta_\varepsilon(x,0)) = (u_0(x), \theta_0(x)) & \text{in } \Omega \\ (u_\varepsilon(x,T), \theta_\varepsilon(x,T)) = (0,0) & \text{in } \Omega, \end{cases}$$

with

$$u_\varepsilon \in L^2(0,T;V(\Omega)) \cap L^\infty([0,T];H(\Omega))$$

and

$$\theta_\varepsilon \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty([0,T];L^2(\Omega)).$$

As in (GUERRERO, IMANUVILOV, and PUEL, 2012), the proof of Theorem 4.1 will take four steps. Thus, we divide our time interval $(0,T)$ in four subintervals, where different strategies are used:

- In the first interval $(0,T_1)$ no control is needed, so we let the Boussinesq system evolve from our initial condition (u_0,θ_0) to same $(u_\varepsilon,\theta_\varepsilon)$ with zero Dirichlet boundary conditions.
- In the second time interval, we explicitly give our solution $(u_\varepsilon,\theta_\varepsilon)$. This way, we drive (u,θ) to some compactly supported state $(u_{1,\varepsilon_1},\theta_{1,\varepsilon_1})$ at a time T_2.
- In the third time interval, we construct our solution $(u_\varepsilon,\theta_\varepsilon)$ in a much more intrinsic way. Indeed, we write $(u_\varepsilon,\theta_\varepsilon)$ as the sum of three and two functions: a very particular solution (U,Θ) to the Boussinesq system constructed multiplied by a large parameter plus a solution (y,h) to a transport equation plus acouple of the form $(W,0)$, where W solves a linear Stokes system.
  This allows to drive the $(u_\varepsilon,\theta_\varepsilon)$ to a solution to a heat equation.
- In the last time interval, we reduce the question to drive $(u_\varepsilon,\theta_\varepsilon)$ to zero, that is, a null controllability problem for a system composed of two coupled 1D parabolic equations. In view of well known results, this is easy to achieve and allows to conclude.

The following two results concern generalizations of Theorem 1 in (GUERRERO, IMANUVILOV, and PUEL, 2012). In the first one, we prove that the approximate boundary controllability can also be obtained with controls acting only on three faces of the unit cube. In the second one, we show that Ω can be a much more general set, namely a bounded domain of $\mathbb{R}^3$ whose boundary contains a piece of a plane entirely located inside one of the associated semispaces, see Fig. (3).

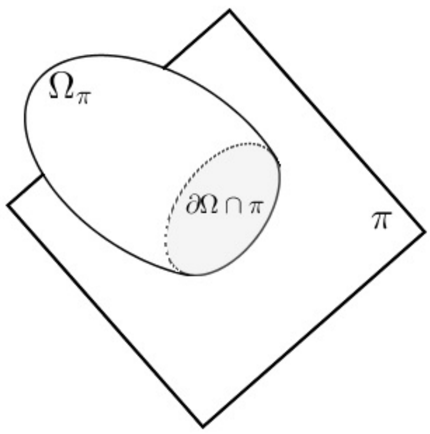

Figure 3: The situation in Theorem 4.3

Theorem 4.2. *Assume that $u_0 \in H(\Omega)$ and $f \in L^2(0,T;L^2(\Omega)^3)$ are given. Then, there exists* *a sequence* $\{(f_\varepsilon)\}_{\varepsilon>0}$ *in* $L^2(0,T;L^2(\Omega)^3)$ *such that*

$$f_\varepsilon \to f \ \text{ in } \ L^r(0,T;H^{-1}(\Omega)^3)$$

for all $r \in (1,4/3)$ *and there exist solutions* $(u_\varepsilon, p_\varepsilon)$ *to the null controllability problems*

$$\begin{cases} u_{\varepsilon,t} - \Delta u_\varepsilon + (u_\varepsilon,\nabla)u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon & \text{in} \quad \Omega\times(0,T) \\ \nabla\cdot u_\varepsilon = 0 & \text{in} \quad \Omega\times(0,T) \\ u_\varepsilon(0,x_2,x_3,t) = u_\varepsilon(1,x_2,x_3,t) = u_\varepsilon(x_1,x_2,0,t) = 0 & \text{in} \quad (0,1)^2\times(0,T) \\ u_\varepsilon(x,0) = u_0(x), \ u_\varepsilon(x,T) = 0 & \text{in} \quad \Omega. \end{cases}$$

Now, let Π be a plane in $\mathbb{R}^3$, let Π^+ be one of the semispaces determined by Π and let $\Omega_\Pi \subset \mathbb{R}^3$ be a bounded domain satisfying

$$\Omega_\Pi \subset \Pi^+, \ \ \Omega_\Pi \cap \Pi \ \text{ is a non-empty open set}$$

and let us consider the Navier-Stokes system

$$\begin{cases} u_t - \Delta u + (u,\nabla)u + \nabla p = f & \text{in} \quad \Omega_\Pi\times(0,T) \\ \nabla\cdot u = 0 & \text{in} \quad \Omega_\Pi\times(0,T) \\ u(x,t) = 0 & \text{in} \quad (\partial\Omega_\Pi\cap\Pi)\times(0,T) \\ u(x,0) = u_0(x) & \text{in} \quad \Omega_\Pi. \end{cases}$$

Theorem 4.3. *Assume that* $u_0 \in H(\Omega_\Pi)$ *and* $f \in L^2(\Omega_\Pi\times(0,T))$. *Then, there exists a sequence* $\{(f_\varepsilon)\}_{\varepsilon>0}$ *in* $L^2(\Omega_\Pi\times(0,T))^3$ *such that*

$$f_\varepsilon \to f \ \text{ in } \ L^r(0,T;H^{-1}(\Omega_\Pi)^3)$$

for all $r \in (1,4/3)$ *and there exist solutions* $(u_\varepsilon, p_\varepsilon)$ *to the null controllability problems*

$$\begin{cases} u_{\varepsilon,t} - \Delta u_\varepsilon + (u_\varepsilon,\nabla)u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon & \text{in} \quad \Omega_\Pi\times(0,T) \\ \nabla\cdot u_\varepsilon = 0 & \text{in} \quad \Omega_\Pi\times(0,T) \\ u_\varepsilon(x,t) = 0 & \text{on} \quad (\partial\Omega_\Pi\cap\Pi)\times(0,t) \\ u_\varepsilon(x,0) = u_0(x), \ u_\varepsilon(x,T) = 0 & \text{in} \quad \Omega_\Pi. \end{cases}$$

A similar result can be deduced for the Boussinesq system in $\Omega_\Pi\times(0,T)$. For brevity, we leave the details to the reader.

This section is organized as follows. In the next subsection, we construct some intermediate functions and we prove some crucial estimates.
In the Subsection 4.3, the proof of Theorem 4.1 is given, following the ideas in (GUERRERO, IMANUVILOV, and PUEL, 2012). Subsection 4.4 deals with the proofs of Theo-

rems 4.2 and 4.3. Finally, in Subsection 4.5, we present some additional comments and questions.

4.2 Some auxiliary problems and estimates

In this subsection, we will construct a specific solution (U, P, Θ) to the Boussinesq system with boundary conditions, with $(U, \nabla)U \equiv 0$.

4.2.1 *The Navier-Stokes system with a boundary control acting on three faces*

Let $z = z(x_1, x_3, t)$ be solution to the following system for the 2D heat PDE:

$$\begin{cases} z_t - (z_{x_1x_1} + z_{x_3x_3}) = c(t), \ (x_1, x_3, t) \in (0,1)^2 \times (0,T) \\ z(0, x_3, t) = z(1, x_3, t) = z(x_1, 0, t) = 0, \ x_1, x_3 \in (0,1), \, t \in (0,T) \\ z(x_1, 1, t) = w(t), \ (x_1, t) \in (0,1) \times (0,T) \\ z(x_1, x_3, 0) = 0, \ (x_1, x_3) \in (0,1)^2. \end{cases}$$

Here, $c \in C^2([0,T])$ is a positive function with positive $c(0)$ (as large as needed) and w is a nonnegative function satisfying

$$w(t) \in C^\infty([0,T]), \ \ w(0) = 0, \ \ w'(0) = c(0), \ \ w''(0) = c'(0). \tag{100}$$

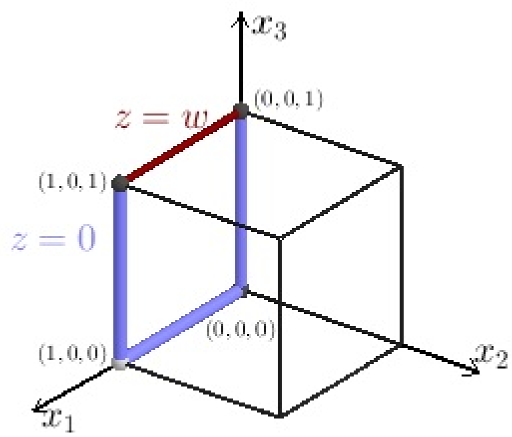

Figure 4: The situation in Theorem 4.2

Thanks to the compatibility condition (100), we can argue as in (GUERRERO, IMANUVILOV, and PUEL, 2012) and check that

$$z \in C^2([\delta, 1-\delta]^2 \times [0,T]) \ \ \forall \delta > 0.$$

On the other hand, thanks to Taylor's formula, we can obtain functions β_δ, γ_δ^i, λ_δ and μ_δ^{ij}

in $C^0([\delta, 1-\delta]^2 \times [0,T])$ such that

$$\begin{aligned} &z(x_1,x_3,t) = c(0)t + \beta_\delta(x_1,x_3,t)t^2 \\ &z_{x_i}(x_1,x_3,t) = \gamma_\delta^i(x_1,x_3,t)t^2, \;\; i \in \{1,3\} \\ &z_t(x_1,x_3,t) = c(0) + \lambda_\delta(x_1,x_3,t)t \\ &z_{x_i,x_j}(x_1,x_3,t) = \mu_\delta^{ij}(x_1,x_3,t)t, \;\; i,j \in \{1,3\}. \end{aligned} \tag{101}$$

Let $\mathcal{G}$ and $\mathcal{I}$ be given by

$$\mathcal{G} = \{\, (x_1,x_2,x_3) : x_2 \in \mathbb{R}, \;\; (x_1,x_3) \in (0,1)^2 \,\},$$

$$\mathcal{I} = (\{0,1\} \times \mathbb{R} \times (0,1)) \cup ((0,1) \times \mathbb{R} \times \{0\}).$$

Now, we introduce the functions U and q, with $U(x,t) := (0, z(x_1,x_3,t), 0)$ and $q := -c(t)x_2$. Note that the couple (U,q) satisfies

$$\begin{cases} U_t - \Delta U + \nabla q = 0, & \text{in} \quad \mathcal{G} \times (0,T) \\ \nabla \cdot U = 0, & \text{in} \quad \mathcal{G} \times (0,T) \\ U(x,t) = 0, & \text{on} \quad \mathcal{I} \times (0,T) \\ U(x,0) = 0, & \text{in} \quad \mathcal{G}. \end{cases}$$

Later, we will look for a solution to the Navier-Stokes system of the form

$$u = N^2 U + y + \xi(t) W,$$

where N is a large constant, y is the solution to a transport equation, W solves a Stokes system and $\xi \in C^2[0,2/N]$ is a cut-off function.

4.2.1.1 *Transport equation*

For an arbitrary initial condition $y_0 \in V_0(\Omega) \cap C_0^1(\Omega)$ extended by zero on $\mathcal{G}$ we consider the system

$$\begin{cases} y_t + N^2(U,\nabla)y + N^2(y,\nabla)U = 0, & (x,t) \in Q_{2/N}, \\ y(x,t) = 0, & (x,t) \in \Sigma_{2/N}, \\ y(x,0) = y_0(x), & x \in \Omega, \end{cases} \tag{102}$$

Here, we have used the notation

$$Q_{2/N} = \mathcal{G} \times (0,2/N), \;\; \Sigma_{2/N} = \mathcal{I} \times (0,2/N).$$

Let us introduce the constant

$$C_\delta := \sup\{\|g\|_{C^0((\delta,1-\delta)^2\times[0,T])} : g \in \{\beta_\delta, \gamma^i_\delta, \mu^{ij}_\delta\}\}.$$

We will look for a particular estimate for y, with an explicit dependence on N_δ that is satisfied when it is large enough. It is given in the following lemma:

Lemma 4.1. *Let $y_0 \in C^1_0(\Omega) \cap V_0(\Omega)$. Then, there exists $N_\delta = N(\delta)$ such that, for any $N > N_\delta$, there exist a solution y to* (102) *and a positive constant K_δ (independent of N), with the following properties:*

$$\|y\|_{C^1(\overline{Q}_{2/N})} \leq K_\delta \|y_0\|_{C^1_0(\overline{\Omega})} \tag{103}$$

and

$$y(x,t) = 0, \;\; t \in [1/N, 2/N], \;\; x \in \Omega.$$

Proof. Let us consider the Banach space

$$Y = \{y \in C^1(Q_{2/N}) : y(x,0) = y_0(x)\}.$$

Let us assume that $\operatorname{supp} y_0 \subset (\delta, 1-\delta)^3$.

There exists N_δ such that, for any $N > N_\delta$, the existence of y can be established by applying Banach's Fixed-Point Theorem to the mapping Λ, where

$$\Lambda(y)(x,t) = y_0(x - N^2 Z(x,t)) - N^2 \int_0^t (y, \nabla)U(x - N^2 Z(x,s), s)\, ds,$$

$$Z(x,t) = \left(0, \int_0^t z(x_1, x_3, s) ds, 0\right), \;\; (x,t) \in Q_{2/N}$$

Let us denote $y_0 = (y_{0,1}, y_{0,2}, y_{0,3})$ and $U = (U_1, U_2, U_3)$. Then, we have

$$y_1(x,t) = y_{0,1}(x - N^2 Z(x,t)),$$

$$y_2(x,t) = y_{0,2}(x - N^2 Z(x,t)) - N^2 \int_0^t y \cdot \nabla U_2(x - N^2 Z(s,x), s)\, ds,$$

$$y_3(x,t) = y_{0,3}(x - N^2 Z(x,t)).$$

From these formulae, it is easy to check that, for N large enough, one has:

$$\begin{aligned} \|y\|_{C^0(Q_{2/N})} &\leq C\|y_0\|_{C^0(\overline{\Omega})} \\ \|\nabla y_1\|_{C^0(Q_{2/N})} &\leq C\|\nabla y_{0,1}\|_{C^0(\overline{\Omega})} \\ \|\nabla y_3\|_{C^0(Q_{2/N})} &\leq C\|\nabla y_{0,3}\|_{C^0(\overline{\Omega})}. \end{aligned} \tag{104}$$

On the other hand, we also have

$$
\begin{aligned}
\|\nabla y_2\|_{C^0(Q_{2/N})} &\leq 7\|\nabla y_{0,2}\|_{C^0(\overline{\Omega})} + C_\delta \|y_0\|_{C^0(\overline{\Omega})} \\
&+ 3\frac{C_\delta}{N}\|\nabla y\|_{C^0(Q_{2/N})} + 15\frac{C_\delta^2}{N^2}\|\nabla y\|_{C^0(Q_{2/N})}.
\end{aligned} \tag{105}
$$

From (104) and (105), the inequality (103) holds (for N large enough). This ends the proof. □

4.2.1.2 *The solution W to a Stokes-like system with $\nabla \cdot W = -\nabla \cdot y$*

Consider the following Stokes problem:

$$
\begin{cases}
W_t - \Delta W + \nabla r = 0, & (x,t) \in Q_{2/N}, \\
\nabla \cdot W = -\nabla \cdot y, & (x,t) \in Q_{2/N}, \\
W(x,t) = 0, & (x,t) \in \Sigma_{2/N}, \\
W(x,0) = 0, & x \in \Omega, \\
W(x,t) \to 0 \text{ as } |x_2| \to +\infty.
\end{cases} \tag{106}
$$

The following result holds:

Proposition 4.1. *Let W be the solution to problem* (106). *Then, for any $p \in (1, \infty)$,*

$$
\|W\|_{L^p(Q_{2/N})} \leq \frac{C(p)}{N^{1/p}}\|y_0\|_{C^3(\overline{\Omega})}. \tag{107}
$$

Furthermore, there exists a positive constant $C > 0$ independent of N such that

$$
\|W\|_{C^0([0,2/N];L^2(\mathcal{G}))} + \|\partial_{x_2} W\|_{C^0([0,2/N];L^2(\mathcal{G}))} + \|\partial_{x_3} W\|_{C^0([0,2/N];L^2(\mathcal{G}))} \leq \frac{C}{N^{1/4}}. \tag{108}
$$

The proof can easily be obtained arguing as in (GUERRERO, IMANUVILOV, and PUEL, 2012) (see the proof of Proposition 1).

4.2.2 *Boussinesq system*

We will construct a specific solution (U, Θ) to the Boussinesq system. Let us first introduce the functions $z_2 = z_2(x_1,t)$, $z_3 = z_3(x_1,t)$ and $\Theta = \Theta(x_1,t)$: z_2 is the solution to the system

$$
\begin{cases}
\partial_t z_2 - \partial^2_{x_1 x_1} z_2 = c(t), & (x_1,t) \in (0,1) \times (0,T), \\
z_2(0,t) = 0, \quad z_2(1,t) = w_2(t), & t \in (0,T), \\
z_2(x_1,0) = 0 & x_1 \in (0,1),
\end{cases} \tag{109}
$$

where $c \in C^2([0,T])$ is a positive function and w_2 is a nonnegative function satisfying

$$w_2(t) \in C^\infty[0,T], \quad w_2(0) = 0, \quad w_2'(0) = c(0), \quad w_2''(0) = c'(0);$$

on the other hand, (z_3, Θ) solves

$$\begin{cases} \partial_t z_3 - \partial^2_{x_1x_1} z_3 = c(t) + \Theta(x_1,t), & (x_1,t) \in (0,1)\times(0,T) \\ \partial_t \Theta - \partial^2_{x_1x_1}\Theta = 0, & (x_1,t) \in (0,1)\times(0,T) \\ z_3(0,t) = 0, \quad z_3(1,t) = w_3(t), & t \in (0,T) \\ \Theta(0,t) = 0, \quad \Theta(1,t) = w(t) & t \in (0,T) \\ z_3(x_1,0) = 0, \quad \Theta(x_1,0) = 0 & x_1 \in (0,1). \end{cases} \tag{110}$$

with

$$\begin{gathered} w_3 \in C^\infty[0,T],\ w_3(0) = 0,\ w_3'(0) = c(0),\ w_3''(0) = c'(0), \\ w \in C^\infty[0,T],\ w(0) = w'(0) = w''(0) = 0. \end{gathered}$$

Proposition 4.2. *Under the above assumptions on w_2 and c, there exist a unique solution to* (109) *with*

$$z_2 \in L^2(0,T;H^1(0,1)) \cap L^\infty((0,1)\times(0,T)), \quad z_{2,t} \in L^2(0,T;H^{-1}(0,1)).$$

Furthermore, for all small $\delta > 0$, we have that $z_2 \in C^2([\delta,1]\times[0,T])$ and there exist functions $\beta_2,\ \gamma_2,\ \mu_2$ and λ_2 such that

(i) $z_2(x_1,t) = c(0)t + \beta_2(x_1,t)t^2$, $|\beta_2| \le C_\delta$,

(ii) $\partial_{x_1} z_2(x_1,t) = \gamma_2(x_1,t)t^2$, $|\gamma_2| \le C_\delta$,

(iii) $\partial_t z_2(x_1,t) = c(0) + \mu_2(x_1,t)t$, $|\mu_2| \le C_\delta$,

(iv) $\partial^2_{x_1x_1} z_2(x_1,t) = \lambda_2(x_1,t)t$, $|\lambda_2| \le C_\delta$.

The proof is not difficult. For instance, let us see how *(i)* can be proved.
We simply write that

$$\begin{aligned} z_2(x,t) &= z_2(x_1,0) + \int_0^t z_{2,t}(x_1,s)\,ds \\ &= z_{2,t}(x,0)t + \left(\int_0^t z_{2,t}(x_1,s)\,ds - t z_{2,t}(x_1,0)\right) \\ &= c(0)t + \beta_2(x_1,t)t^2 \end{aligned}$$

with $0 < \tilde{t} < t$, where we have used the notation $\beta_2(x_1,t) := \left(z_{2,t}(x,\tilde{t}) - z_{2,t}(x_1,0)\right)t^{-1}$.

The proof of *(ii)*, *(iii)* and *(iv)* follows through analogous computations.

A similar result can be established for the solution (z_3,Θ) to the system (110):

Proposition 4.3. *Under the above assumptions on $w_3,\ w_2$ and c, there exists a unique solution*

(z_3, Θ) *to* (110) *with*

$$z_3 \in L^2(0,T;H^1(0,1)) \cap L^\infty((0,1)\times(0,T)),\ z_{3,t} \in L^2(0,T;H^{-1}(0,1)),$$

$$\Theta \in L^2(0,T;H^1(0,1)) \cap L^\infty((0,1)\times(0,T)),\ \Theta_t \in L^2(0,T;H^{-1}(0,1)).$$

Furthermore,, for all small $\delta > 0$, *we have that* $z_3, \Theta \in C^2([\delta,1]\times[0,T])^2$ *and there exist functions* β_3, γ_3, μ_3, λ_3, β, γ *and* μ *such that*

(i) $z_3(x_1,t) = c(0)t + \beta_3(x_1,t)t^2$ *and* $\Theta(x_1,t) = \beta(x_1,t)t^2$, *with* $|\beta|, |\beta_3| \leq C_\delta$,

(ii) $\partial_{x_1} z_2(x_1,t) = \gamma_3(x_1,t)t^2$ *and* $\partial_{x_1}\Theta(x_1,t) = \gamma(x_1,t)t^2$, *with* $|\gamma|, |\gamma_3| \leq C_\delta$,

(iii) $\partial_t z_2(x_1,t) = c(0)+\mu_3(x_1,t)t$ *and* $\partial_t\Theta(x_1,t) = \partial_{x_1x_1}\Theta(x_1,t) = \mu(x_1,t)t$, *with* $|\mu|, |\mu_3| \leq C_\delta$,

(iv) $\partial^2_{x_1x_1} z_2(x_1,t) = \lambda_3(x_1,t)t$, *with* $|\lambda_3| \leq C_\delta$.

Now, consider the functions $U(x,t) = (0, z_2(x_1,t), z_3(x_1,t))$, $\Theta = \Theta(x,t)$ as before and $q(x,t) = -(x_2+x_3)c(t)$. Observe that (U,q,Θ) solves the following Boussinesq problem:

$$\begin{cases} U_t - \Delta U + (U,\nabla)U + \nabla q = \Theta e_3, & \text{in } \mathcal{G}\times(0,T) \\ \nabla\cdot U = 0 & \text{in } \mathcal{G}\times(0,T) \\ \Theta_t - \Delta\Theta + U\cdot\nabla\Theta = 0 & \text{in } \mathcal{G}\times(0,T) \\ U(0,x_2,x_3,t) = 0, \ \ \Theta(0,x_2,x_3,t) = 0 & \text{in } \mathcal{G}\times(0,T) \\ U(x,0) = 0, \ \ \Theta(x,0) = 0 & \text{in } \mathcal{G}. \end{cases} \tag{111}$$

In the proof of Theorem 4.1, the construction of the solution to (99) is divided into four steps. In one of them, (u,θ,p) is written in the form

$$\begin{aligned} u(x,t) &= N^2U(x,t) + y(x,t) - W(x,t), & (x,t) \in \Omega\times(T_1,T_2) \\ \theta(x,t) &= N^2\Theta(x,t) + h(x,t), & (x,t) \in \Omega\times(T_1,T_2) \\ p(x,t) &= N^2q(x,t) + r(x,t), & (x,t) \in \Omega\times(T_1,T_2) \end{aligned}$$

where (y,h) is the solution to a transport equation and W is the solution to a linear Stokes system. In the next two paragraphs, we construct (y,h) and W and we prove some estimates.

For any $\delta > 0$, we define

$$C^0_\delta(\bar{\mathcal{G}}\times[0,2/N])^4 := \{(y,h) \in C^0(\mathcal{G}\times[0,2/N])^4;\ y = 0,\ h = 0 \text{ for } x_1 \in [0,\delta]\}.$$

4.2.2.1 *Transport equation*

For an arbitrary initial condition extended by zero on $\mathcal{G}$ and for some $N \in \mathbb{N}$ large enough, which will be defined precisely later, we solve the following null controllability problem for the transport equation

$$\begin{cases} y_t + N^2(U,\nabla)y + N^2(y,\nabla)U = he_3, & \text{in} \quad Q_{2/N}, \\ h_t + N^2 U\cdot\nabla h + N^2 y\cdot\nabla\Theta = 0 & \text{in} \quad Q_{2/N}, \\ y(0,x_2,x_3,t) = 0, \quad h(0,x_2,x_3,t) = 0 & \text{in} \quad \mathbb{R}^2\times(0,2/N), \\ y(x,0) = y_0, \quad h(x,0) = h_0 & \text{in} \quad x\in\mathcal{G}, \end{cases} \tag{112}$$

where $Q_{2/N} = \mathcal{G}\times(0,2/N)$.

Lemma 4.2. *Let us assume that* $(y_0,h_0)\in(C_0^1(\Omega)\cap V_0(\Omega))\times C_0^1(\Omega)$. *Then there exists* $N_0(\delta)$ *such that, for any* $N\geq N_0(\delta)$, *there exists a solution* (y,h) *to problem* (112) *and a positive constant* $C(\delta)$ *independent of* N, *such that* $(y,h)\in C_\delta^0(\bar{\mathcal{G}}\times[0,2/N])$,

$$\|y\|_{C_\delta^1(\bar{Q}_{2/N})} + \|h\|_{C_\delta^1(\bar{Q}_{2/N})} \leq C(\delta), \tag{113}$$

$$\|y_t\|_{C_\delta^1(\bar{Q}_{2/N})} + \|h_t\|_{C_\delta^1(\bar{Q}_{2/N})} \leq C(\delta) \tag{114}$$

and

$$y(x,t) = 0, \quad h(x,t) = 0 \quad (x,t)\in\Omega\times[1/N,2/N].$$

Proof. Since $(v_0,h_0)\in(C_0^1(\Omega)\cap V_0(\Omega))\times C_0^1(\Omega)$, then there exist $\delta>0$ such that

$$\operatorname{supp} v_0 \cup \operatorname{supp} h_0 \subset [\delta,1]\times[0,1]^2.$$

The existence of the solutions is established by the Banach's Fixed Point Theorem. Let us consider the Banach space

$$Y = \{(y,h)\in C_\delta^0(\bar{G}\times[0,2/N])^4;\ y(x,0) = y_0(x),\ h(x,0) = h_0(x)\}$$

and let us introduce $Z = (0,Z_2,Z_3)$, with

$$Z_2(x_1,t) = \int_0^t z_2(x_1,s)\,ds,\ Z_3(x_1,t) = \int_0^t z_3(x_1,s)\,ds,\ (x_1,t)\in(0,1)\times(0,T).$$

We will define the mapping $\Lambda : Y\to Y$ as follows: for each $(\tilde{y},\tilde{h})\in Y$, $\Lambda(\tilde{y},\tilde{h}) = (y,h)$ if and only if

$$\begin{aligned} y(x,t) &= y_0(x - N^2 Z(x_1,t)) + \left(\int_0^t \tilde{h}(x - N^2 Z(x_1,s),s)\,ds\right)e_3 \\ &- N^2\int_0^t ((\tilde{y},\nabla)U)(x - N^2 Z(x_1,s),s)\,ds, \end{aligned} \tag{115}$$

$$h(x,t) = h_0(x - N^2 Z(x_1,t)) - N^2 \int_0^t (\tilde{y}\cdot\nabla\Theta)(x - N^2 Z(x_1,s),s)\,ds. \tag{116}$$

Let us see that, if $N_0(\delta)$ is large enough, we can apply Banach's Fixed Point Theorem to Λ and deduce the existence of a fixed point. For $(y_1,h_1),(y_2,h_2)\in Y$, by propositions (4.2) and (110) we have

$$\begin{aligned} N^2 \int_0^t ((y_2,\nabla)U)(x - N^2 Z(x_1,s),s)\,ds \\ -N^2 \int_0^t ((y_1,\nabla)U)(x - N^2 Z(x_1,s),s)\,ds \\ \leq \frac{C_\delta}{N}\|y_2 - y_1\|_{C^0_\delta(\bar{\mathcal{G}}\times[0,2/N])^3} \end{aligned}$$

and

$$\begin{aligned} N^2 \int_0^t (y_2\cdot\nabla\Theta)(x - N^2 Z(x_1,s),s)\,ds \\ -N^2 \int_0^t (y_1\cdot\nabla\Theta)(x - N^2 Z(x_1,s),s)\,ds \\ \leq \frac{C_\delta}{N}\|y_2 - y_1\|_{C^0_\delta(\bar{\mathcal{G}}\times[0,2/N])^3}. \end{aligned}$$

Then, we obtain

$$\|\Lambda(y_2,h_2) - \Lambda(y_1,h_2)\|_Y \leq \frac{C_\delta}{N}\|(y_2,h_2) - (y_1,h_1)\|_{C^0_\delta(\bar{\mathcal{G}}\times[0,2/N])^4}.$$

Therefore, for N_0 large enough Λ possesses at least one fixed point (y,h). Obviously, $\Lambda(y,h) = (y,h)$ is a solution to (112).

Now let us verify the estimates (113) and (114).

First, note that

$$\begin{aligned} |N^2\nabla U| &\leq CN^2t^2, \quad \text{in} \quad [\delta,1]\times[0,2/N], \\ |N^2\nabla\Theta| &\leq CN^2t^2, \quad \text{in} \quad [\delta,1]\times[0,2/N]. \end{aligned}$$

We can verify that

$$|y| \leq C + \int_0^t |h(x - N^2 Z(x_1,s),s)|ds + C\int_0^t |y(x - N^2 Z(x_1,s),s)|\,ds,$$

$$|h| \leq C + C\int_0^t |y(x - N^2 Z(x_1,s),s)|\,ds.$$

Now as $t\in[0,2/N]$, we immediately obtain

$$\|y\|_{C^0_\delta(\bar{Q}_{2/N})} + \|h\|_{C^0_\delta(\bar{Q}_{2/N})} - \frac{C_1}{N}(\|y\|_{C^0_\delta(\bar{Q}_{2/N})} + \|h\|_{C^0_\delta(\bar{Q}_{2/N})}) \leq C_2.$$

Therefore, choosing N large enough, we have

$$\|y\|_{C^0_\delta(\bar{Q}_{2/N})} + \|h\|_{C^0_\delta(\bar{Q}_{2/N})} \leq 2C_2. \tag{117}$$

Taking the derivatives of (115) and (116) with respect to x_1 and using Propositions 4.2 and 110, after some computations, we deduce that

$$\begin{gathered}|\partial_{x_1} y| \leq C + \frac{C}{N} + \frac{C}{N}\|\partial_{x_1} h\|_{C^0_\delta(\bar{Q}_{2/N})} + \frac{C}{N^2}\|\partial_{x_2} h\|_{C^0_\delta(\bar{Q}_{2/N})} \\ + \frac{C}{N^2}\|\partial_{x_3} h\|_{C^0_\delta(\bar{Q}_{2/N})} + \frac{C}{N}\|\partial_{x_1} y\|_{C^0_\delta(\bar{Q}_{2/N})} + \frac{C}{N^2}\|\partial_{x_2} y\|_{C^0_\delta(\bar{Q}_{2/N})} \\ + \frac{C}{N^2}\|\partial_{x_3} y\|_{C^0_\delta(\bar{Q}_{2/N})}\end{gathered}$$

and

$$|\partial_{x_1} h| \leq C + \frac{C}{N} + \frac{C}{N}\|\partial_{x_1} y\|_{C^0_\delta(\bar{Q}_{2/N})} + \frac{C}{N^2}\|\partial_{x_2} y\|_{C^0_\delta(\bar{Q}_{2/N})} + \frac{C}{N^2}\|\partial_{x_3} y\|_{C^0_\delta(\bar{Q}_{2/N})}.$$

With a similar argument, we obtain estimates of the same kind for $\partial_{x_2} y$, $\partial_{x_3} y$, $\partial_{x_2} h$ and $\partial_{x_3} h$. Then, we can write

$$\|\nabla y\|_{C^0_\delta(\bar{Q}_{2/N})} + \|\nabla h\|_{C^0_\delta(\bar{Q}_{2/N})} - \frac{C_1}{N}\|\nabla y\|_{C^0_\delta(\bar{Q}_{2/N})} - \frac{C_1}{N}\|\nabla h\|_{C^0_\delta(\bar{Q}_{2/N})} \leq C_2$$

and, for N large enough, we have

$$\|\nabla y\|_{C^0_\delta(\bar{Q}_{2/N})} + \|\nabla h\|_{C^0_\delta(\bar{Q}_{2/N})} \leq 2C_2. \tag{118}$$

Taking the derivative of (115) and (116) in time, thanks the properties of (y, h) in (117) and (118), we get:

$$\|y_t\|_{C^1_\delta(\bar{Q}_{2/N})} + \|h_t\|_{C^1_\delta(\bar{Q}_{2/N})} \leq C.$$

Now, our objective is to elect a N_0 such that

$$y_1(t, x) = 0, \quad (t, x) \in [1/N, 2/N] \times \Omega. \tag{119}$$

Notice that

$$y_1(x,t) = y_{0,1}(x - N^2 Z(x_1,t))$$

$$y_2(t,x) = y_{0,2}(x - N^2 Z(x_1,t)) - N^2 \int_0^t \partial_{x_1} z_2(x_1,s) y_1(x - N^2 Z(x_1,s),s)\, ds$$

$$y_3(t,x) = y_{0,3}(x - N^2 Z(x_1,t)) - N^2 \int_0^t \partial_{x_1} z_3(x_1,s) y_1(x - N^2 Z(x_1,s),s)\, ds + \int_0^t h(x - N^2 Z(x_1,s),s)\, ds$$

$$h(t,x) = h_0(x - N^2 Z(x_1,t)) - N^2 \int_0^t y_1(x - N^2 Z(x_1,s),s) \partial_{x_1} \Theta(x_1,s)\, ds.$$

For $x_1 \in [0,\delta/2]$, we have that $y_1 = 0$ and $h = 0$. Recall that $c(t)$ is a positive function. Without loss of generality we way assume that

$$c(0) > 3.$$

Observe that by Proposition 4.2 we have

$$-N^2 \int_0^t z_2(x_1,s) ds = -N^2 \int_0^t (sc(0) + \beta_2(x_1,s)s^2)\, ds \leq -\frac{c(0)}{2} + \frac{8}{3N} C_\delta,$$

for N enough large we can assume that

$$x_2 - N^2 Z_2(x_1,t) < 0,\ (x_1,t) \in [\delta/2,1] \times [1/N, 2/N], \tag{120}$$

consequently, we obtain (119).

Let check that

$$y_2(t,x) = y_3(t,x) = h(t,x) = 0, \ \ t \in [1/N, 2/N],\ x \in \Omega.$$

To this purpose, we consider the curve

$$(\tilde{x}_2(t), \tilde{x}_3(t)) = (N^2 \int_0^t z_2(x_1,s)\, ds + \alpha_1, N^2 \int_0^t z_3(x_1,s)\, ds + \alpha_2),$$

with α_1 and α_2 constant such that $(\tilde{x}_2(t), \tilde{x}_3(t)) \in (0,1)^2$. Denoting

$$c_{x_1}(t) = y_2(x_1, \tilde{x}_2(t), \tilde{x}_3(t), t),$$

we have that

$$c'_{x_1}(t) = \partial_t y_2 + N^2 z_2 \partial_{x_2} y_2 + N^2 z_3 \partial_{x_3} y_2 = -N^2 \partial_{x_1} z_2(x_1) v_{0,1}(x_1, \alpha_1, \alpha_2).$$

Then, for any fixed $x_1 \in [0, \delta/2]$ we have that $c'_{x_1}(t) = 0$, wich implies that $c_{x_1}(t) = c_{x_1}(0) = y_2(x_1, \alpha_1, \alpha_2, 0) = v_{0,2}(x_1, \alpha_1, \alpha_2)$ and therefore $y_2 = 0,\ x_1 \in [0, \delta/2]$.

Now if $(x_1, x_2, x_3) \in (\delta/2, 1) \times (0, 1)^2$, by (120)

$$N^2 \int_0^t z_2(x_1, s) ds > 1,$$

which implies that $\alpha_1 < 0$. Then, $c'_{x_1}(t) = 0$, which gives $c_{x_1}(t) = c_{x_1}(0) = y_2(x_1, \alpha_1, \alpha_2, 0) = v_{0,2}(x_1, \alpha_1, \alpha_2) = 0$ and, therefore, $y_2 = 0,\ (x_1, x_2, x_3) \in \left(\frac{\delta}{2}, 1\right) \times (0, 1)^2$.

We can get similar properties for h, consequently for y_3. This ends the proof.

□

Finally, consider the Stokes problem

$$\begin{cases} \partial_t W - \Delta W + \nabla r = 0 & \text{in} \quad Q_{2/N} \\ W(0, x_2, x_3, t) = W(1, x_2, x_3, t) = 0 & \text{in} \quad \mathbb{R}^2 \times (0, 2/N) \\ W(x_1, x_2, x_3, t) \to 0 & \text{as} \quad |x_2| + |x_3| \to \infty \\ \nabla \cdot W = \nabla \cdot y & \text{in} \quad Q_{2/N} \\ W(x, 0) = 0 & \text{in} \quad \mathcal{G}, \end{cases} \tag{121}$$

where y is the function furnished by Lemma 4.2.

Proposition 4.4. *Let W be the solution to problem* (121)*. Then, for any $p \in (1, \infty)$, one has* (107)*. Furthermore, there exists a positive constant $C > 0$ independent of N such that* (108) *is satisfied.*

The proof is given in (GUERRERO, IMANUVILOV, and PUEL, 2012) (see Proposition 1).

4.3 Proof of Theorem 4.1

As mentioned above, the proof of Theorem 4.1 closely follows Theorem 1 in (GUERRERO, IMANUVILOV, and PUEL, 2012) and, as there, is divided in several steps, each them related to a time subinterval.

• FIRST STEP:

We know there exists at least one weak solution (u, p, θ) to the problem

$$\begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = \theta e_3 + f, \quad \nabla \cdot u = 0 & \text{in} \quad Q, \\ \theta_t - \Delta\theta + u \cdot \nabla\theta = g & \text{in} \quad Q, \\ u = 0, \quad \theta = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ (u(x,0), \theta(x,0)) = (u_0, \theta_0) & \text{in} \quad \Omega, \end{cases}$$

with

$$u \in L^2(0,T;V_0(\Omega)) \cap L^\infty([0,T];H(\Omega)), \ \theta \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty([0,T];L^2(\Omega)).$$

Let $\tilde{T}_1 \in (T - \delta_0, T)$ be such that $(\tilde{u}_1, \tilde{\theta}_1) := (u(\tilde{T}_1), \theta(\tilde{T}_1)) \in V_0(\Omega) \times H_0^1(\Omega)$ and

$$\|f\|_{L^2(T-\delta,T;V_0'(\Omega))} + \|g\|_{L^2(T-\delta,T;H^{-1}(\Omega))} \leq \frac{\varepsilon}{5}.$$

For a small interval $(\tilde{T}_1, \tilde{T}_1 + \eta)$ with $\tilde{T}_1 + \eta < T$, there exists a unique strong solution (u, p, θ) to the Boussinesq problem, such that $(u(\tilde{T}_1), \theta(\tilde{T}_1)) = (\tilde{u}_1, \tilde{\theta}_1)$ (see for instance VISKIK and FURSIKOV (1988)) and there exists $T_1 \in (\tilde{T}_1, \tilde{T}_1 + \eta)$ with

$$(u(T_1), \theta(T_1)) \in ((H^2(\Omega)^3 \cap V_0(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)).$$

On the interval $(0, T_1)$ we do not exert any control and take

$$u_\varepsilon := u, \quad p_\varepsilon := p, \quad f_\varepsilon := f, \quad \theta_\varepsilon := \theta, \quad g_\varepsilon := g.$$

• SECOND STEP:

Write $(u_1, \theta_1) := (u(T_1), \theta(T_1))$ and take $u_{1,\alpha} \in V_0(\Omega) \cap C_0^\infty(\Omega)^3$ and $\theta_{1,\alpha} \in C_0^\infty(\Omega)$ such that

$$(u_{1,\alpha}, \theta_{1,\alpha}) \to (u_1, \theta_1) \text{ in } V_0(\Omega) \times H_0^1(\Omega) \text{ as } \alpha \to 0^+$$

and

$$\|u_{1,\alpha}\|_{V_0(\Omega)} + \|\theta_{1,\alpha}\|_{H_0^1(\Omega)} \leq 2\,(\|u_1\|_{V_0(\Omega)} + \|\theta_1\|_{H_0^1(\Omega)}).$$

Let $T_2 \in (T_1, T)$ be a time; its precise value will be given below. We introduce now $(u_\varepsilon, p_\varepsilon, \theta_\varepsilon)$ in (T_1, T_2), with

$$u_\varepsilon = \frac{(t - T_1)}{(T_2 - T_1)} u_{1,\alpha} + \frac{(T_2 - t)}{(T_2 - T_1)} u_1, \quad p_\varepsilon = 0, \quad f_\varepsilon = \mathcal{L} u_\varepsilon - \theta_\varepsilon e_3,$$

$$\theta_\varepsilon = \frac{(t - T_1)}{(T_2 - T_1)} \theta_{1,\alpha} + \frac{(T_2 - t)}{(T_2 - T_1)} \theta_1, \quad g_\varepsilon = \mathcal{M}_\varepsilon \theta_\varepsilon$$

where

$$\begin{aligned}\mathcal{L}u_\varepsilon &= \partial_t u_\varepsilon - \Delta u_\varepsilon + (u_\varepsilon, \nabla)u_\varepsilon \text{ and}\\ \mathcal{M}_\varepsilon\theta_\varepsilon &= \partial_t\theta_\varepsilon - \Delta\theta_\varepsilon + u_\varepsilon \cdot \nabla\theta_\varepsilon.\end{aligned}$$

Then, it is clear that

$$(u_\varepsilon(T_1), \theta_\varepsilon(T_1)) = (u_1, \theta_1), \quad (u_\varepsilon(T_2), \theta_\varepsilon(T_2)) = (u_{1,\alpha}, \theta_{1,\alpha}), \quad \nabla \cdot u_\varepsilon = 0$$

and the couple $(f_\varepsilon, g_\varepsilon)$ satisfies

$$(f_\varepsilon, g_\varepsilon) \in L^2(0,T; L^2(\Omega))^3 \times L^2(0,T; L^2(\Omega)),$$

$$\begin{aligned}\|f_\varepsilon\|_{L^2(T_1,T_2;V_0'(\Omega))} \leq\ & \frac{C}{\sqrt{T_2 - T_1}}\|u_{1,\alpha} - u_1\|_{V_0(\Omega)} +\\ & C\sqrt{T_2 - T_1}\left(\|u_1\|_{H_0^1(\Omega)^3} + \|u_1\|^2_{H_0^1(\Omega)^3} + \|\theta_1\|_{H_0^1(\Omega)}\right)\end{aligned}$$

and

$$\begin{aligned}\|g_\varepsilon\|_{L^2(T_1,T_2;H^{-1}(\Omega))} \leq\ & \frac{C}{\sqrt{T_2 - T_1}}\|\theta_{1,\alpha} - \theta_1\|_{H_0^1(\Omega)} +\\ & C\sqrt{T_2 - T_1}\left(\|\theta_1\|_{H_0^1(\Omega)} + \|\theta_1\|_{H_0^1(\Omega)}\|u_1\|_{H_0^1(\Omega)^3}\right).\end{aligned}$$

Accordingly, we can choose first T_2 close enough of T_1 and then α small enough to have

$$\|f_\varepsilon\|_{L^2(T_1,T_2;V_0'(\Omega))} + \|g_\varepsilon\|_{L^2(T_1,T_2;H^{-1}(\Omega))} \leq \frac{\varepsilon}{5}.$$

• THIRD STEP:

Now, we will work in the interval $[T_2, T_2 + 2/N]$, where $N > N(\delta)$ and $N(\delta)$ is furnished by Lemma 4.2. The initial data are:

$$u_2 := u_\varepsilon(T_2) \in V_0(\Omega) \cap C_0^\infty(\Omega)^3, \;\; \theta_2 := \theta_\varepsilon(T_2) \in C_0^\infty(\Omega).$$

In this step, we will take

$$\begin{aligned}u_\varepsilon(x,t) &= N^2\tilde{U}(x,t) + \tilde{y}(x,t) - \xi(t - T_2)\tilde{W}(x,t),\\ p_\varepsilon(x,t) &= -N^2(x_2 + x_3)c(t - T_2) + \tilde{r}(x,t),\\ \theta_\varepsilon(x,t) &= N^2\tilde{\Theta}(x,t) + \tilde{h}(x,t)\end{aligned}$$

and

$$\begin{aligned}f_\varepsilon &= -\Delta\tilde{y} + ((\tilde{y} - \tilde{W}), \nabla)(\tilde{y} - \tilde{W}) - N^2(\tilde{U}, \nabla)\tilde{W} - N^2(\tilde{W}, \nabla)\tilde{U} - \alpha_t\tilde{W},\\ g_\varepsilon &= -\Delta\tilde{h} + (\tilde{y} - \tilde{W}) \cdot \nabla\tilde{h} - N^2\tilde{W} \cdot \nabla\tilde{\Theta},\end{aligned}$$

where $\tilde{U}$, $\tilde{\Theta}$, etc. are respectively U, Θ, etc. written at time $t - T_2$, (U, θ) is the solution to (111) , (y, h) is the solution to (112) with initial data $(y_0, h_0) = (u_2, \theta_2)$, (W, r) is the solution to (121) and $\xi \in C^2([0, 2/N])$ is a cut-off function satisfying

$$\xi(t) = 1 \text{ in } t \in [0, 1/N] \text{ and } \xi(t) = 0 \text{ in a neighborhood of } 2/N.$$

From the properties of (y, h) deduced in Lemma 4.2 and the definitions of U and W, we have the following:

$$(u_\varepsilon, \theta_\varepsilon)(T_2 + 2/N) = N^2(U, \Theta)(2/N), \quad \nabla \cdot u_\varepsilon = 0$$

and

$$u_\varepsilon(0, x_2, x_3, t) = 0, \quad \theta_\varepsilon(0, x_2, x_3, t) = 0 \text{ in } (0, 1)^2 \times (T_2, T_2 + 2/N).$$

Our goal is to verify that, for N large enough, we get

$$\|f_\varepsilon\|_{L^2(T_2, T_2+2/N; V_0'(\Omega))} + \|g_\varepsilon\|_{L^2(T_2, T_2+2/N; H^{-1}(\Omega))} \leq \frac{\varepsilon}{5}.$$

First, note that Lemma 4.2 yields

$$\|\Delta \tilde{y}\|_{L^2(T_2, T_2+2/N; V_0'(\Omega))} + \|\Delta \tilde{h}\|_{L^2(T_2, T_2+2/N; H^{-1}(\Omega))} \leq \frac{C}{N^{1/2}}.$$

Let us decompose Ω in two parts

$$\Omega_1 := (0, \delta/2) \times (0, 1)^2 \text{ and } \Omega_2 := (\delta/2, 1) \times (0, 1)^2.$$

Recall that $\nabla \cdot y = \nabla \cdot W$ in $Q_{2/N}$ and $y = 0$ in Ω_1. Consequently,

$$\begin{aligned} \|N^2(\tilde{W} \cdot \nabla)\tilde{U}\|_{V_0'(\Omega)} &= \sup_{b \in V_0(\Omega), \|b\|_{V_0(\Omega)} = 1} \int_{\Omega_1} N^2(\tilde{W}, \nabla)\tilde{U} b \, dx \\ &+ \sup_{b \in V_0(\Omega), \|b\|_{V_0(\Omega)} = 1} \int_{\Omega_2} N^2(\tilde{W}, \nabla)\tilde{U} b \, dx \\ &= - \sup_{b \in V_0(\Omega), \|b\|_{V_0(\Omega)} = 1} \int_{\Omega_1} N^2 \tilde{W} \cdot \nabla b \, \tilde{U} \, dx \\ &+ \sup_{b \in V_0(\Omega), \|b\|_{V_0(\Omega)} = 1} \int_{\Omega_2} N^2(\tilde{W}, \nabla)\tilde{U} b \, dx. \end{aligned}$$

The first term is bounded by $C\|N\tilde{W}\|_{L^2(\Omega)}\|N\tilde{U}\|_{L^\infty(\Omega)}$. On the other hand,

$$\int_{\Omega_2} N^2(\tilde{W}, \nabla)\tilde{U} b \, dx \leq \|N\nabla\tilde{U}\|_{L^\infty((\delta/2) \times \mathbb{R}^2)}\|N\tilde{W}\|_{L^2(\Omega)}.$$

Thanks to Propositions 4.2 and 4.3, there exists $C(\delta) > 0$ such that

$$\|N\nabla\tilde{U}\|_{L^\infty(T_2,T_2+2/N;L^\infty((\delta/2,1)\times\mathbb{R}^2)} \leq C.$$

Therefore, we see from (108) that

$$\begin{aligned}\|N^2(\tilde{W},\nabla)\tilde{U}\|_{L^r(T_2,T_2+2/N;V_0'(\Omega))} &\leq C\left(\int_{T_2}^{T_2+2/N}\|NW\|_{L^2(\Omega)}^r\,dt\right)^{1/r}\\ &\leq CN^{3/4-1/r}.\end{aligned}$$

Similarly, the following estimate can be obtained:

$$\begin{aligned}\|N^2\tilde{W}\cdot\nabla\Theta\|_{L^r(T_2,T_2+2/N;H^{-1}(\Omega))} + \|N^2(\tilde{U}\cdot\nabla)\tilde{W}\|_{L^r(T_2,T_2+2/N;V_0'(\Omega))}\\ \leq CN^{3/4-1/r}.\end{aligned}$$

Next, using (107), we deduce that

$$\|\alpha_t W\|_{L^r(T_2,T_2+2/N;L^2(\Omega)^3)} \leq \|W\|_{L^r(T_2,T_2+2/N;L^2(\Omega))} \leq C(r)N^{-1/r}\|u_2\|_{C^1(\Omega)}.$$

From Lemma 4.2 and (107), the following is found:

$$\|((\tilde{y}-\tilde{W})\cdot\nabla)(\tilde{y}-\tilde{W})\|_{L^2(T_2,T_2+2/N;V_0')} \leq C\|\tilde{y}-\tilde{W}\|^2_{L^4(T_2,T_2+2/N;L^4(\Omega)^3)} \to 0$$

and

$$\begin{aligned}&\|(\tilde{y}-\tilde{W})\cdot\nabla h\|_{L^2(T_2,T_2+2/N;H^{-1})}\\ &\quad\leq C\|\tilde{h}\|_{L^4(T_2,T_2+2/N;L^\infty(\Omega))}\|\tilde{y}-\tilde{W}\|_{L^4(T_2,T_2+2/N;L^2(\Omega)^3)} \to 0\end{aligned}$$

as $N \to +\infty$. This concludes the step.

• FOURTH STEP:

Finally, we set $T_3 := T_2 + 2/N$ and we work in the interval $[T_3, T]$. Note that $(u_\varepsilon, p_\varepsilon, \theta_\varepsilon)$ arrives to time T_3 with the structure

$$\begin{aligned} u_\varepsilon(x,T_3) &= (0, N^2 z_2(x_1,2/N), N^2 z_3(x_1,2/N)),\\ \theta_\varepsilon(x,T_3) &= N^2(\partial_t z_3 + \Delta z_3)(x_1,2/N),\end{aligned}$$

which leads to a heat equation and a system of two coupled one-dimensional parabolic equations. The second component of u_ε can be driven to zero at time $t = T$ by solving a standard null controllability problem for a linear heat equation. On the other hand, the third component and θ_ε can be driven to zero by solving a (less standard) null controllability problem for a system of two coupled 1D parabolic PDEs.

Indeed, in the interval $[T_3, T]$, we take $f_\varepsilon = 0$ and $g_\varepsilon = 0$. It is well-known that the boundary null controllability holds for the linear equation, see (IMANUVILOV, 1995). Hence, there exists $\rho = \rho(t) \in L^\infty(0, T - T_3)$ such that the solution to

$$\begin{cases} \partial_t \overline{z} - \partial^2_{x_1x_1}\overline{z} = 0, & \text{in } (0,1)\times(0,T-T_3) \\ \overline{z}(0,t) = 0, \quad \overline{z}(1,t) = \rho(t), & \text{in } (0,T-T_3) \\ \overline{z}(x_1,0) = N^2 z_2(x_1, 2/N) & \text{in } (0,1) \end{cases}$$

satisfies

$$\overline{z}(x_1, T - T_3) = 0 \text{ in } (0,1).$$

On the other hand, it is proved in (FERNÁNDEZ-CARA, GONZÁLEZ-BURGOS, and de TERESA, 2010) that, if $A \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$, μ_1 and μ_2 are the eigenvalues of A, $\text{rank}[B|AB] = 2$, $(T/\pi)(\mu_1 - \mu_2)$ is not a integer of the form $4(m+1)$ or $2m+1$ with $m \geq 1$ and $\overline{y}_0 \in H^{-1}(0,1)^2$, there exists a control $v = v(t) \in L^2(0, T - T_3)$ such that the associated solution to the system

$$\begin{cases} \partial_t \overline{y} - \partial^2_{x_1x_1}\overline{y} = A\overline{y}, & \text{in } (0,1)\times(0,T-T_3) \\ \overline{y}(0,t) = 0, \quad \overline{y}(1,t) = Bv, & \text{in } (0,T-T_3) \\ \overline{y}(x_1,0) = \overline{y}_0(x_1) & \text{in } (0,1) \end{cases} \tag{122}$$

satisfies

$$\overline{y}(x_1, T - T_3) = 0 \text{ in } (0,1). \tag{123}$$

Then, it suffices to define u_ε and θ_ε in (T_3, T) as follows:

$$\begin{cases} u_\varepsilon(x,t) = (0, \overline{z}(x_1, t - T_3), \overline{y}_1(x_1, t - T_3)) \\ \theta_\varepsilon(x,t) = \overline{y}_2(x_1, t - T_3), \end{cases}$$

where $(\overline{y}_1, \overline{y}_2)$ is, together with some v, a solution to the problem (122)–(123) with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \overline{y}_0 = N^2(z_3, \partial_t z_3 + \Delta z_3)(x_1, 2/N).$$

Finally,

$$u_\varepsilon(\cdot, T) = 0, \quad \theta_\varepsilon(\cdot, T) = 0$$

and we clearly have

$$\|f - f_\varepsilon\|_{L^r(0,T;V_0'(\Omega))} + \|g - g_\varepsilon\|_{L^r(0,T;H^{-1}(\Omega))} \leq \varepsilon.$$

4.4 Proofs of Theorems 4.2 and 4.3

The proof of Theorem 4.2 is similar. Again, it is composed of several steps, each of them related to a time subinterval and, for brevity, we will only give an idea of what is actually different from the proof of Theorem 4.1.

The first and second steps are almost identical (of course, there is no θ_ε now). In the third step, we take again $T_3 = T_2 + 2/N$ and we introduce

$$u_\varepsilon(x,t) = N^2 U(x, t - T_2) + y(x, t - T_2) + \theta(t - T_2) W(x, t - T_2),$$

$$p_\varepsilon(x,t) = N^2 x_2 c(x, t - T_2) - r(x, t - T_2),$$

where the functions U, y, θ, W, r and c are as in Subsection 4.2.

It is easy to check that $(u_\varepsilon, p_\varepsilon)$ solves

$$\begin{cases} u_{\varepsilon,t} - \Delta u_\varepsilon + (u_\varepsilon, \nabla) u_\varepsilon = f_\varepsilon, & \text{in } \Omega \times (T_2, T_3) \\ \nabla \cdot u_\varepsilon = 0, & \text{in } \Omega \times (T_2, T_3) \\ u_\varepsilon(0, x_2, x_3, t) = u_\varepsilon(1, x_2, x_3, t) = u_\varepsilon(x_1, x_2, 0, t) = 0 & \text{on } (0,1)^2 \times (T_2, T_3) \\ u_\varepsilon(x, 0) = u_0(T_2), & \text{in } \Omega, \end{cases}$$

where we have set $T_3 = T_2 + 2/N$, with

$$\begin{aligned} f_\varepsilon(x,t) &= (-\Delta y + N^2 (U, \nabla)\theta W + N^2 (\theta W, \nabla) U + \theta_t W)(x, t - T_2) \\ &+ ((y + \theta W), \nabla)(y + \theta W))(x, t - T_2). \end{aligned}$$

From (101), Lemma 4.1 and Proposition 4.1, we can verify that, for N large enough, we have

$$\|f_\varepsilon\|_{L^2(T_2, T_2 + 2/N; V'(\Omega))} \le \frac{\varepsilon}{5}$$

and

$$u_\varepsilon(x, T_2 + 2/N) = N^2 U(x, 2/N).$$

In the fourth step, we take $T_3 := T_2 + 2/N$ and we note that u_ε possesses at time T_3 the structure

$$u_\varepsilon(x, T_2 + 2/N) = (0, N^2 z(x_1, x_3, 2/N), 0).$$

The second coordinate of u_ε can be driven to zero at time $t = T$ by solving a null controllability problem for a linear 2D heat equation. More precisely, let us take $f_\varepsilon = 0$ in $[T_3, T]$. It is well-known that there exist controls $\rho = \rho(x_1, t)$ in $L^\infty((0,1) \times (0, T - T_3))$

such that the associated solution to

$$\begin{cases} \overline{z}_t - (\overline{z}_{x_1x_1} + \overline{z}_{x_3x_3}) = c(t), \ (x_1, x_3, t) \in (0,1)^2 \times (0, T - T_3) \\ \overline{z}(0, x_3, t) = \overline{z}(1, x_3, t) = \overline{z}(x_1, 0, t) = 0, \ t, \in (0, T - T_3), \ x_1, x_3 \in (0,1) \\ \overline{z}(x_1, 1, t) = \rho(x_1, t), \ (x_1, t) \in (0,1) \times (0, T - T_3) \\ \overline{z}(x_1, x_3, 0) = N^2 z(x_1, x_3, 2/N), \ (x_1, x_3) \in (0,1)^2 \end{cases}$$

satisfies

$$\overline{z}(x_1, T - T_3) = 0 \ \text{ in } \ (0,1)$$

(see IMANUVILOV (1995)). Then, it is sufficient to take in (T_3, T)

$$u_\varepsilon(x,t) = (0, \overline{z}(x_1, x_3, t - T_3), 0).$$

This way, we get

$$u_\varepsilon(\cdot, T) = 0$$

and

$$\|f - f_\varepsilon\|_{L^r(0,T;V_0'(\Omega))} \leq \varepsilon.$$

We now give the proof of Theorem 4.3.

In fact, Theorem 4.3 can be viewed as a Corollary of Theorem 1 in GUERRERO, IMANUVILOV, and PUEL (2012). Indeed, let $R \in \mathbb{R}^3$ be a cube, with edges not necessarily parallel to the axes and let us denote by Γ_0 one of its faces. It is clear that, after appropriate rotation and tranlation, we can construct right hand sides $f_\varepsilon \in L^2(R \times (0,T))$ satisfying

$$f_\varepsilon \to f \ \text{ in } L^r(0, T; H^{-1}(R)),$$

for all $r \in (1, 4/3)$ and solutions $(v_\varepsilon, p_\varepsilon)$ to the corresponding Navier-Stokes systems

$$\begin{cases} v_{\varepsilon,t} - \Delta v_\varepsilon + (v_\varepsilon, \nabla) v_\varepsilon + \nabla p_\varepsilon = f_\varepsilon, & \text{in} \quad R \times (0,T) \\ \nabla \cdot v_\varepsilon = 0, & \text{in} \quad R \times (0,T) \\ v_\varepsilon = 0, & \text{on} \quad \Gamma_0 \times (0,T) \\ v_\varepsilon(x, 0) = u_0(x) & \text{in} \quad R \end{cases}$$

that satisfy

$$u_\varepsilon(x, T) = 0 \ \text{ in } \ R.$$

Let R be such that $\Omega_\Pi \subset R$. Then, we just take

$$u_\varepsilon := v_\varepsilon\big|_{\Omega_\Pi \times (0,T)}$$

and we immediately conclude.

Remark 4.1. Let us set $\Gamma_1 = \partial\Omega \setminus (\{0\} \times (0,1)^2)$ and let $\mathcal{O}$ be a neighborhood of Γ_1 in Ω. It is not difficult to obtain from Theorem 1 in (GUERRERO, IMANUVILOV, and PUEL, 2012) a global controllability result of the same kind for the Navier-Stokes system with distributed controls, supported by $\mathcal{O} \times (0,T)$. However, a similar result for the Boussinesq system is, to our knowledge, unknown. □

4.5 Final comments and questions

The previous results do not imply actually global approximate controllability, since the right hand sides f and g to be modified slightly (the same can be said on the results in (GUERRERO, IMANUVILOV, and PUEL, 2012)). What we would need is a uniform bound of the controls in some Banach space B allowing to take limits as ε to 0. But, at present, this is missing.

Thus, it would be interesting to be able to modify the constructions of u_ε and θ_ε paying special attention to the behavior of their traces.

Another possible approach relies on the following idea:

1. Solve the exremal problems

$$\begin{cases} \text{Minimize } \; J_\varepsilon(h) = \|h\|_B \\ \text{Subject to } h \in \mathcal{B} \end{cases}$$

   where $\mathcal{B}$ is the family of boundary null controls for the Boussinesq system with f replaced by f_ε that belong to B.
2. Then, prove that the solutions satisfy

$$\|\tilde{h}_\varepsilon\|_B \leq C.$$

Observe that, with a suitable choice of B, all these problems are solvable. Therefore, one can probably use an optimality characterization to get some information.

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APPENDIX

For completeness, let us sketch the proof of Theorem 3.4. We will closely follow the arguments in (DOUBOVA and FERNÁNDEZ-CARA, 2005)

Let $(\varphi, m') \in C^2(Q_{\bar{l}}) \times C^1([0,T])$ be such that

$$\begin{cases} -\varphi_t - \varphi_{xx} = g(x,t), \;\; (x,t) \in Q_{\bar{\ell}} \\ \varphi(0,t) = 0, \;\; \varphi(\bar{\ell}(t),t) = m'(t), \;\; t \in (0,T) \\ \varphi_x(\bar{\ell}(t),t) = m''(t) - \overline{y}(\bar{\ell}(t),t)m'(t) \;\; t \in (0,T) \end{cases} \tag{124}$$

and $s > 1$. Let us introduce

$$\psi = e^{-s\alpha}\varphi.$$

Notice that

$$\psi(x,0) = \psi(x,T) \equiv 0, \;\; \psi(0,t) \equiv 0 \tag{125}$$

and

$$\psi(\bar{l}(t),t) = e^{-s\alpha(\bar{\ell}(t),t)}m'(t). \tag{126}$$

We have the following equality:

$$\psi_t + \psi_{xx} - 2s\lambda\eta_x\xi\psi_x + s^2\lambda^2\eta_x^2\xi^2\psi - s\lambda(\eta_x\xi)_x\psi + s\alpha_t\psi = e^{-s\alpha}g.$$

We can rewrite in the form

$$M_1\psi + M_2\psi = g_s\psi \tag{127}$$

where

$$\begin{cases} M_1\psi = \psi_{xx} + s^2\lambda^2\eta_x^2\xi^2\psi + s\alpha_t\psi \\ M_2\psi = \psi_t - 2s\lambda\eta_x\xi\psi_x \\ g_s\psi = e^{-s\alpha}g + s\lambda(\eta_x\xi)_x\psi \end{cases}$$

We have fron (127) that

$$\|M_1\psi\|_2^2 + \|M_2\psi\|_2^2 + 2(M_1\psi, M_2\psi)_2 = \|g_s\psi\|_2^2, \tag{128}$$

where $\|\cdot\|_2$ and $((\cdot,\cdot))_2$ denote the usual norm and scalar product in $L^2(Q_{\bar{l}})$, respectively.

Let us compute the scalar product on the left-hand side of (128). We can write

$$(M_1\psi, M_2\psi)_2 = I_{11} + I_{12} + I_{21} + I_{22} + I_{31} + I_{32},$$

where I_{ij} denote the scalar products of the terms of $M_1\psi$ and $M_2\psi$.

After some manipulation, we get

$$\begin{aligned}
&\|M_1\psi\|_2^2 + \|M_2\psi\|_2^2 + 2s\lambda^2 \iint_{Q\bar{\ell}} \eta_x^2 \xi |\psi_x|^2 \, dx \, dt \\
&+6s^3\lambda^4 \iint_{Q\bar{\ell}} \eta_x^4 \xi^3 |\psi|^2 \, dx \, dt \\
&+2s\lambda \int_0^T \frac{1}{\bar{\ell}(t) - b} \xi(\bar{\ell}(t), t) |\psi_x(\bar{\ell}(t), t)|^2 \, dt \\
&+2s^3\lambda^3 \int_0^T \frac{1}{(\bar{\ell}(t) - b)^3} e^{-2s\alpha(\bar{\ell}(t),t)} \xi(\bar{\ell}(t), t)^3 |m'(t)|^2 \, dt \\
&+2 \int_0^T \psi_x(\bar{\ell}(t), t) \psi_t(\bar{\ell}(t), t) \, dt \\
&= -2s^2\lambda \int_0^T \frac{1}{\bar{\ell}(t) - b} e^{-2s\alpha(\bar{\ell}(t),t)} \xi(\bar{\ell}(t), t) \alpha_t(\bar{\ell}(t), t) |m'(t)|^2 \, dt \qquad (129) \\
&+2s^2\lambda^2 \iint_{Q\bar{\ell}} \eta_x \xi (\eta_x \xi)_t |\psi|^2 \, dx \, dt - 2s^2\lambda^2 \iint_{Q\bar{\ell}} \eta_x^2 \xi \alpha_t |\psi|^2 \, dx \, dt \\
&-2s^2\lambda \iint_{Q\bar{\ell}} \eta_x \xi \alpha_{xt} |\psi|^2 \, dx \, dt + s \iint_{Q\bar{\ell}} \alpha_{tt} |\psi|^2 \, dx \, dt \\
&-2s\lambda \iint_{Q\bar{\ell}} \eta_{xx} \xi |\psi_x|^2 \, dx \, dt + \|g_s\psi\|_2^2 \\
&-2s^2\lambda \iint_{Q\bar{\ell}} \eta_{xx} \xi \alpha_t |\psi|^2 \, dx \, dt \\
&-6s^3\lambda^3 \iint_{Q\bar{\ell}} \eta_x^2 \eta_{xx} \xi^3 |\psi|^2 \, dx \, dt
\end{aligned}$$

On the other hand, it is easy to verify that

$$|\xi_t| \leq \lambda C(|\bar{\ell}'|_\infty, \ell_*, \omega, T)\xi^2, \qquad (130)$$

$$\begin{aligned} |\alpha_t| &\leq \lambda\eta_t\xi + T\xi^2 \\ &\leq \lambda C(|\overline{\ell}'|_\infty, \ell_*, \omega, T)\xi^2, \end{aligned} \tag{131}$$

$$|\alpha_t(\overline{\ell}(t), t)| \leq \lambda C(|\overline{\ell}'|_\infty, \ell_*, T)\xi(\overline{\ell}(t), t)^2, \tag{132}$$

$$|\alpha_{tt}| \leq \lambda^2 C(|\overline{\ell}'|_\infty, \ell_*, \omega, T)\xi^3 \tag{133}$$

and

$$|\alpha_{xt}| \leq \lambda^2 C(|\overline{\ell}'|_\infty, \ell_*, \omega, T)\xi^2. \tag{134}$$

Remember that $g_s\psi = e^{-s\alpha}g + s\lambda(\eta_x\xi)_x\psi$. Then, we can obtain:

$$\begin{aligned} \|g_s\psi\|_2^2 &\leq 2\|e^{-s\alpha}g\|_2^2 + 2s^2\lambda^2 \iint_{Q_{\overline{\ell}}} |(\eta_x\xi)_x|^2|\psi|^2 \, dx \, dt \\ &\leq 2\|e^{-s\alpha}g\|_2^2 + Cs^2\lambda^4 \iint_{Q_{\overline{\ell}}} |\xi|^2|\psi|^2 \, dx \, dt, \end{aligned} \tag{135}$$

where $C = C(\ell_*, \omega, B)$.

From the previous estimates and (129), the following is found:

$$
\begin{aligned}
&\|M_1\psi\|_2^2 + \|M_2\psi\|_2^2 + 2s\lambda^2 \iint_{Q\bar{\ell}} \eta_x^2 \xi |\psi_x|^2 \, dx\, dt \\
&+6s^3\lambda^4 \iint_{Q\bar{\ell}} \eta_x^4 \xi^3 |\psi|^2 dxdt \\
&+2s\lambda \int_0^T \frac{1}{\bar{\ell}(t)-b} \xi(\bar{\ell}(t),t) |\psi_x(\bar{\ell}(t),t)|^2 \, dt \\
&+2s^3\lambda^3 \int_0^T \frac{1}{(\bar{\ell}(t)-b)^3} e^{-2s\alpha(\bar{\ell}(t),t)} \xi(\bar{\ell}(t),t)^3 |m'(t)|^2 \, dt \\
&+2\int_0^T \psi_x(\bar{\ell}(t),t)\psi_t(\bar{\ell}(t),t)\, dt \\
&\leq 2C(|\bar{\ell}'|_\infty, \ell_*, T) s^2\lambda^2 \int_0^T e^{-2s\alpha(\bar{\ell}(t),t)} \xi(\bar{\ell}(t),t)^3 |m'(t)|^2 \, dt \\
&-2s\lambda \iint_{Q\bar{\ell}} \eta_{xx} \xi |\psi_x|^2 \, dx\, dt + 2\|e^{-s\alpha} g\|_2^2 \\
&+Cs^3\lambda^4 \iint_{Q\bar{\ell}} \xi^3 |\psi|^2 \, dx\, dt
\end{aligned}
\tag{136}
$$

with $C = C(\|\bar{\ell}'\|_\infty, \ell_*, \omega, T, B)$.

Notice that $|\eta_x| \geq \gamma$ in $\overline{Q_{\bar{\ell}}} \setminus (\omega_0 \times (0,T))$, where

$$
\frac{1}{\gamma} = \max_{0\leq t\leq T} \{a, \bar{\ell}(t) - b\}.
$$

With this in mind, we get from (136) that

$$
\begin{aligned}
&\|M_1\psi\|_2^2+\|M_2\psi\|_2^2+s^3\lambda^4\iint_{Q\overline{\ell}}\xi^3|\psi|^2\,dx\,dt\\
&+\frac{s\lambda}{B-b}\int_0^T\xi(\overline{\ell}(t),t)|\psi_x(\overline{\ell}(t),t)|^2\,dt\\
&+s^3\lambda^3\int_0^T e^{-2s\alpha(\overline{\ell}(t),t)}\xi(\overline{\ell}(t),t)^3|m'(t)|^2\,dt\\
&+2\int_0^T\psi_x(\overline{\ell}(t),t)\psi_t(\overline{\ell}(t),t)\,dt\\
&\leq 2\|e^{-s\alpha}g\|_2^2-2s\lambda\iint_{Q\overline{\ell}}\eta_{xx}\xi|\psi_x|^2\,dx\,dt\\
&+Cs^3\lambda^4\iint_{\omega\times(0,T)}\xi^3|\psi|^2\,dx\,dt,
\end{aligned}
\tag{137}
$$

with $C=C(\|\overline{\ell}'\|_\infty,\ell_*,\omega,T,B)$, s and λ sufficiently large.

Using the fact that $M_1\psi=\psi_{xx}+s^2\lambda^2\eta_x^2\xi^2\psi+s\alpha_t\psi$, we obtain

$$
\iint_{Q_{\overline{\ell}}}\frac{1}{s\xi}|\psi_{xx}|^2\,dx\,dt\leq 4\|M_1\psi\|_2^2+Cs^3\lambda^3\iint_{Q_{\overline{\ell}}}\xi^3|\psi|^2\,dx\,dt \tag{138}
$$

On the other hand, by integrating by parts, we see that

$$
\begin{aligned}
s\lambda^2\iint_{Q_{\overline{\ell}}}\xi|\psi_x|^2\,dx\,dt=&-s\lambda^2\iint_{Q_{\overline{\ell}}}\xi\psi_{xx}\psi\,dx\,dt-s\lambda^3\iint_{Q_{\overline{\ell}}}\eta_x\xi\psi_x\psi\,dx\,dt\\
&+s\lambda^2\iint_{Q_{\overline{\ell}}}\xi(\overline{\ell}(t),t)\psi_x(\overline{\ell}(t),t)\psi(\overline{\ell}(t),t)\,dt.
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
s\lambda^2 \iint_{Q_{\overline{\ell}}} \xi|\psi_x|^2\,dx\,dt &\leq \iint_{Q_{\overline{\ell}}} \frac{1}{s\xi}|\psi_{xx}|^2\,dx\,dt + s^3\lambda^4 \iint_{Q_{\overline{\ell}}} \xi^2|\psi|^2\,dx\,dt \\
&+s\lambda^4 \iint_{Q_{\overline{\ell}}} \eta_x^2\xi|\psi|^2\,dx\,dt \\
&+s\lambda \iint_{Q_{\overline{\ell}}} \xi(\overline{\ell}(t),t)|\psi_x(\overline{\ell}(t),t)|^2\,dt \\
&+s\lambda^3 \iint_{Q_{\overline{\ell}}} \xi(\overline{\ell}(t),t)|\psi(\overline{\ell}(t),t)|^2\,dt
\end{aligned}
\tag{139}
$$

and

$$
\begin{aligned}
&\|M_2\psi\|_2^2 + \iint_{Q\overline{\ell}} \left(\frac{1}{s\xi}\psi_{xx} + s\lambda^2\xi|\psi_x|^2 + s^3\lambda^4\xi^3|\psi|^2\right)\,dx\,dt \\
&+s\lambda \int_0^T \xi(\overline{\ell}(t),t)|\psi_x(\overline{\ell}(t),t)|^2\,dt \\
&+s^3\lambda^3 \int_0^T e^{-2s\alpha(\overline{\ell}(t),t)}\xi(\overline{\ell}(t),t)^3|m'(t)|^2\,dt \\
&+2\int_0^T \psi_x(\overline{\ell}(t),t)\psi_t(\overline{\ell}(t),t)\,dt \\
&\leq C\left(\|e^{-s\alpha}g\|_2^2 + s^3\lambda^4 \iint_{\omega\times(0,T)} \xi^3|\psi|^2\,dx\,dt\right),
\end{aligned}
\tag{140}
$$

with $C = C(\|\overline{\ell}'\|_\infty, \ell_*, \omega, T, B)$.

Using the fact that $M_2\psi = \psi_t - 2s\lambda\eta_x\xi\psi_x$, we find

$$
\iint_{Q_{\overline{\ell}}} \frac{1}{s\xi}|\psi_t|^2\,dx\,dt \leq 2\|M_2\psi\|_2^2 + Cs\lambda^2 \iint_{Q_{\overline{\ell}}} \xi|\psi_x|^2\,dx\,dt. \tag{141}
$$

Consequently,

$$
\begin{aligned}
&\iint_{Q\bar{\ell}} \left(\frac{1}{s\xi}(\psi_t + \psi_{xx}) + s\lambda^2 \xi |\psi_x|^2 + s^3\lambda^4\xi^3|\psi|^2 \right) \, dx \, dt \\
&+ s\lambda \int_0^T \xi(\bar{\ell}(t), t) |\psi_x(\bar{\ell}(t), t)|^2 \, dt \\
&+ s^3\lambda^3 \int_0^T e^{-2s\alpha(\bar{\ell}(t),t)} \xi(\bar{\ell}(t), t)^3 |m'(t)|^2 \, dt \\
&+ 2 \int_0^T \psi_x(\bar{\ell}(t), t) \psi_t(\bar{\ell}(t), t) \, dt \\
&\leq C \left(\|e^{-s\alpha} g\|_2^2 + s^3\lambda^4 \iint_{\omega \times (0,T)} \xi^3 |\psi|^2 \, dx \, dt \right)
\end{aligned}
\tag{142}
$$

fo some $C = C(\|\bar{l}'\|_\infty, \ell_*, \omega, T, B)$.

Returning to the original variables, it is not difficult to deduce (83) and conclude.