Implementation of a Viscoelastic Constitutive Model Using the Object-Oriented Programming Approach

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Summary

The numerical implementation of the time-dependent linear viscoelastic constitutive model using object-oriented programming is discussed here. This paper is divided in two main parts: (i) a brief discussion of the numerical integration of linear viscoelastic constitutive equations, including the use of Prony series for characterization of viscoelastic materials (e.g. asphalt pavement surfaces) and (ii) the class organization and methods implemented in viscoelastic related classes of an object-oriented framework. The implementation is part of an ongoing development Finite Element (FE) computer system for pavement analysis and research.

Introduction

There is a still a current need for new methodologies for pavement analysis and design that can more accurately predict pavement distresses. The evolution of numerical methods such as the Finite Element Method (FEM) and the Boundary Element Method (BEM) has made the mechanistic-empirical methods more popular for pavement analysis applications. Nevertheless, it is generally assumed the pavement system to have a linear elastic behavior and subjected to static loads. Efforts to implement more advanced material models as well as dynamic loading in pavement analysis are found in [1] and [2], respectively.

Asphalt mixtures are known to present a viscoelastic behavior and many moduli can be used to describe their viscoelastic behavior, in both time and frequency domain [3, 4, 5, 6, 7, 8]. Christensen [3] mentions that for isotropic materials it is convenient to divide the material characterization in two parts: simple shear and dilatation. A reasonable simplification is to assume a constant Poisson's ratio ν [9], therefore requiring a single material characterization. In fact, it is convenient to perform a simple shear test in frequency domain and then interconvert to a dilatation expression in time domain [6], for proper use in a Finite Element code. The latter modulus, named relaxation modulus $E(t)$, is experimentally obtained for constant strain, that is $\varepsilon(t) = H(t-\tau_0)\varepsilon_0$, where H is the unit step function and ε_0 is a prescribed strain [3, 4]. A proper

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mathematical representation of the relaxation modulus $E(t)$ is an exponential series (1), also known as Prony series, fitted with the collocation method [5].

$$E(t) = E_\infty + \sum_{i=1}^{p} E_i e^{-t/\rho_i} \tag{1}$$

The number of terms p is usually between 5 and 11 terms and depends on the time range of material behavior. The use of a Prony series (1) is due its semi-group property which enables a computationally efficient recursive expression to update the stress field [10].

Numerical Implementation of Linear Viscoelasticity

The numerical implementation of the viscoelastic model can be performed by direct integration of the convolution integral (2) in some set of discrete time [3].

$$\sigma = \int_{t_0=0}^{t} E(t-\tau)\frac{\partial \varepsilon}{\partial \tau} d\tau \tag{2}$$

Although this is the most direct manner [11], it is computationally inefficient. In fact, using such approach to evaluate the response of a given discrete time $t+\Delta t$, it is mandatory to store the response of *all* previous discrete times from initial time of loading $t_0 = 0$ to immediately previous discrete time t [12, 13]. It requires significant storage memory and computational effort, even in the case of isotropic materials. To deal with such computational drawbacks, alternative formulations have been proposed by many authors [13, 14, 15, 16, 17, 18].

In this paper we present an incremental numeric method for the isotropic linear viscoelastic constitutive model characterized by a Prony series (1), which is simple to implement in a Finite Element program [15, 19]. This method was first proposed by Taylor *et al.* [18] and it enables a computationally efficient recursive expression to update the stress field. For the sake of convenience, the expressions that follow are proper for the one-dimensional case, but this method was also implemented for two-dimensional (plane stress, plane strain), axisymmetric and three-dimensional isotropic cases.

According to Eq. (3), we assume stress σ_t is known at a given discrete time t and the current stress $\sigma_{t+\Delta t}$ is updated by an incremental quantity $\Delta\sigma$. Applying the convolution integral (2), the amount of stress increment $\Delta\sigma$ can be divided into two basic parts (4). Applying Prony series (1) in the first part of the r.h.s. of Eq. (4) we can find a closed form for modulus $\bar{E}$ indicated in Eq. (5).

$$\sigma_{t+\Delta t} = \sigma_t + \Delta\sigma \tag{3}$$

$$\Delta\sigma = \overline{E}\Delta\varepsilon + \Delta\hat{\sigma} \tag{4}$$

$$\overline{E} = E_\infty + \frac{1}{\Delta t}\sum_{i=1}^{p} E_i \rho_i (1 - e^{-\Delta t/\rho_i}) \tag{5}$$

The main task in this method is to provide an efficient expression to evaluate the second term of the r.h.s of Eq. (4). Again, using the convolution integral (2) and Prony series (1), we can analytically develop Eq. (6) containing internal recursive variables $S_i(t)$ (7), which depends only on the respective internal variables of the second previous time of interest $t - \Delta t$. The internal variable at initial time $S_i(t_0)$ can be considered zero without incurring in much error. This formulation express an efficient recursive way to update the stress, avoiding direct numerical integration of the convolution integral and consequently this method saves computational memory and effort.

$$\Delta\hat{\sigma} = -\sum_{i=1}^{p}\left(1 - e^{-\Delta t/\rho_i}\right)S_i(t) \tag{6}$$

$$S_i(t) = \left[e^{-\Delta t/\rho_i} S_i(t - \Delta t) + E_i \rho_i (1 - e^{-\Delta t/\rho_i})\right]\frac{\Delta\varepsilon}{\Delta t}, \qquad i = 1,\ldots,p \tag{7}$$

When the formulation previously described is considered in a generalized two or three dimensional Finite Element problem, the basic local equilibrium equation assumes the vector form in Eq. (8) [13], where $\mathbf{B}$ is the stress-strain matrix and $\mathbf{f}_{t+\Delta t}$ is the current external force vector. For the isotropic formulation the modulus $\overline{E}$ (5) is still a scalar, but the stresses $\boldsymbol{\sigma}_t$ (first part of r.h.s of Eq. (5)), $\Delta\hat{\boldsymbol{\sigma}}$ of Eq. (6) and the incremental displacement $\Delta\mathbf{u}$ are vectors. The latter substitutes the incremental strain.

$$\left(\int_V \mathbf{B}^T \overline{E}\mathbf{B}dV\right)\Delta\mathbf{u} = \mathbf{f}_{t+\Delta t} - \int_V \mathbf{B}^T(\boldsymbol{\sigma}_t + \Delta\hat{\boldsymbol{\sigma}})dV \tag{8}$$

OOP Approach for Viscoelastic Constitutive Model Implementation

In the development of large and complex computational systems based on Finite Element Method (FEM) it has been commonly adopted the paradigm of object-oriented programming (OOP) [20, 21, 22]. The central principles of OOP as class definitions, inheritance and polimorfism improve the efficiency, reusability, data management and increases maintainability of a computational system [21, 22, 23, 24]. The viscoelastic constitutive formulation described in the previous section was numerically implemented in a FE computer system under development that use OOP techniques and whose hierarchy of the main classes is described in Figure 1. The relationships among these main classes if of type “has a”. For generality, these main classes are abstraction of essential components of the Finite Element Method [23]. This paper concern just in the classes related to the viscoelastic constitutive material. For more details of the description of all classes the reader is referred to [20, 25].

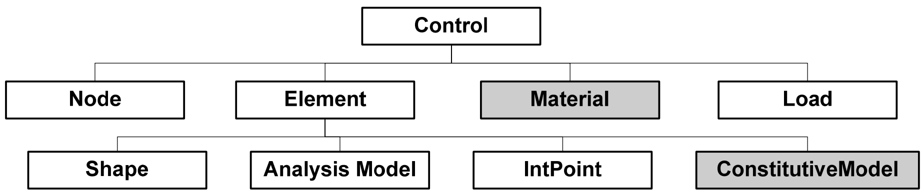


Figure 1: Classes hierarchy of FE computer system

For the viscoelastic constitutive model implemented, a derived class *MatViscoElastic* from *Material* abstract class storages material parameters – Poisson’s ratio v and coefficients E_∞, E_i, ρ_i of Prony series (1) – and creates query methods to be used by other classes. On the other hand, the *ConstitutiveModel* abstract class is responsible for the computation of the current stress vector $\boldsymbol{\sigma}_{t+\Delta t}$ for a given strain vector $\boldsymbol{\varepsilon}_{t+\Delta t}$. The current development of the *ConstitutiveModel* base class enables the classical linear elastic, two resilient models for soils and the linear viscoelastic analysis, as depicted in the derived classes presented in Figure 2.

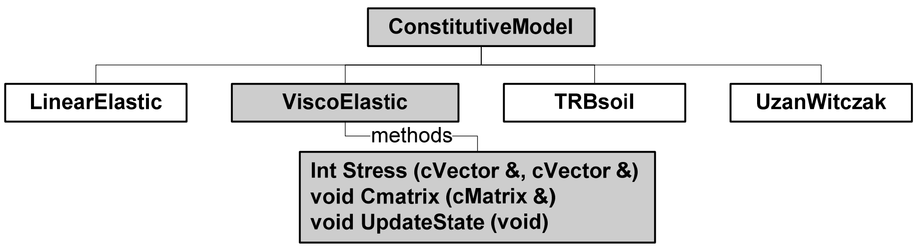


Figure 2: Derived classes of *ConstitutiveModel* base class

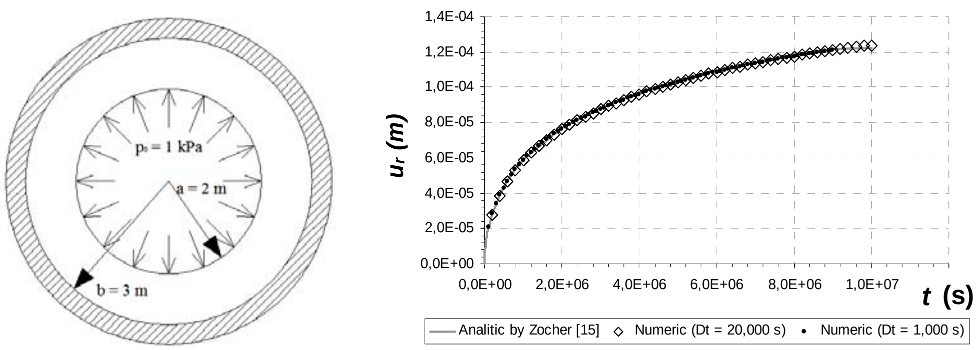
The *Constitutive Model* has access to the *Material* data and *AnalysisModel* methods through the associated element, so duplication of *Material* data is avoided and minimizes the amount of computer memory required by the system [20]. It is important to say that each region of the domain has a different stress/strain history, so an object of the *ConstitutiveModel* class is defined for each integration point of the finite element mesh [2, 13, 20]. Specifically looking at the viscoelastic problem, a derived class *ViscoElastic* is created. The method *Stress* computes the stress field according to proper corresponding vector expressions of Eqs. (3), (4), (5), (6) and (7). For a general two/three-dimensional problem it was implemented the method *Cmatrix* to compute the tangent constitutive matrix $\mathbf{C}$. Finally, it was implemented the *UpdateState* method to update the internal variables of the problem, such as $S_i(t)$ of Eq. (5).

Example Problem

In order to validate the implementation the viscoelastic constitutive model discussed in this paper, this section shows a simple example of the radial displacement u_r of a thick-walled viscoelastic cylinder encased in a shell of infinite stiffness [15]. The internal radius a, external radius b and constant internal pressure p_0 are indicated in Figure 3a. The Poisson's ration used was $\nu =$ 0.30. The analysis was performed in axisymmetric mode with the viscoelastic properties extracted from [6] and presented in Table 1. The numerical responses were compared with analytical solution of Eq. (9) in which *D(t)* is the creep compliance interconverted from relaxation modulus *E(t)* also extracted from [6].

Table 1: Prony series of viscoelastic material

E_∞ (MPa)	i	1	2	3	4	5	6	7	8	9	10	11
	ρ_i (s)	2.E-2	2.E-1	2.E0	2.E1	2.E2	2.E3	2.E4	2.E5	2.E6	2.E7	2.E8
2.24	E_i (MPa)	194	283	554	602	388	156	41.0	13.8	3.68	0.790	0.960



(a) geometry and load (b) radial displacement (*r* = 2.22m)

Figure 3: viscoelastic cylinder encased in a shell of infinite stiffness

$$u_r(r,t) = \frac{p_0 a^2 b(1+\nu)(1-2\nu)}{a^2 + (1-2\nu)b^2}\left(\frac{b}{r} - \frac{r}{b}\right)D(t) \quad (9)$$

The results of Figure 3b indicate the time function radial displacement $u_r(r,t)$ at radial position r = 2.22m. For FE numeric analysis it was evaluated two time steps Δt = 20,000s and Δt = 1,000s, which are very small according to response of viscoelastic material analyzed and constant load condition.

A good agreement between the analytic and FE results were found for both time steps, which indicate a correct implementation of the viscoelastic constitutive model on the proposed system for pavement analysis and research.

Conclusion

A formulation has been briefly reported for modeling the response of linear viscoelastic materials, as asphalt pavement surfaces. This formulation was correct implemented in a Finite Element computer system for pavement analysis and research using comprehensively Object-Oriented techniques, which enables more efficient data management and simpler expansion of the code under development in order to include more accurate models for mechanistic pavement analysis.

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