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RENATO OLIVEIRA TARGINO

LIPSCHITZ GEOMETRY OF COMPLEX PLANE ALGEBRAIC CURVES

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LIPSCHITZ GEOMETRY OF COMPLEX PLANE ALGEBRAIC CURVES:

Thesis submitted to the Graduate Program of the Mathematical Departament of Universidade Federal do Ceará in partial fulfillment of the necessary requirements for the degree of Ph.D in Mathematics. Area of expertise: Singularities

Advisors: Prof. Dr. Alexandre César Gurgel Fernandes and Prof. Dr. Anne Pichon

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I dedicate this work to my wife and my daughter.

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RESUMO

Apresentamos a classificação completa de curvas algébricas planas e complexas, com métrica euclidiana induzida, a menos de homeomorfismo bilipschitz. Em particular, provamos um teorema que fornece a classificação completa da geometria Lipschitz no infinito de curvas planas algébricas complexas. Sintetizamos objetos combinatórios que codificam tanto a geometria Lipschitz como a geometria Lipschitz no infinito de curvas algébricas planas e complexas.
Palavras-chave: Curvas Algébricas planas e complexas. Geometria Lipschitz.

ABSTRACT

We present the complete classification of complex plane algebraic curves, equipped with the induced Euclidean metric, up to global bilipschitz homeomorphism. In particular, we prove a theorem giving a complete classification of the Lipschitz geometry at infinity of complex algebraic plane curves. We synthesize combinatorial objects that encode both Lipschitz geometry and Lipschitz geometry at infinity of complex algebraic plane curves.

Keywords: Plane algebraic curves. Lipschitz geometry.

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1 INTRODUCTION

One of the most natural questions in the investigation of a class of mathematical objects is the problem of classification of these objects. Here the classification problem is treated from the outer metric viewpoint: all the subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$ are considered equipped with the induced Euclidean metric. The problem of classification of germs of complex analytic sets up to bilipschitz homeomorphism has been intensively studied in the last years. One of the recent works on this subject, Neumann and Pichon (2014) proved that two germs of plane analytic curve are bilipschitz homeomorphic if only if they have the same embedded topological type. Neumann and Pichon (2014) named an equivalence class of germ of an analytic curve $(C, 0)$ up to bilipschitz homeomorphism of the Lipschitz geometry of $(C, 0)$. Previous contributions were made by Fernandes (2003) and, Pham Teissier (1969).

From another way, looking to scrutinize global Lipschitz geometry of algebraic sets in some sense, Fernandes and Sampaio (2020) arrived to the notion of bilipschitz equivalence at infinity of subsets in the Euclidean space, that means, two subsets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are bilipschitz equivalent at infinity if there exist compact subsets $K \subset \mathbb{R}^n$ and $K' \subset \mathbb{R}^m$ and a bilipschitz homeomorphism $\phi : X \backslash K \to Y \backslash K'$. Following Neumann and Pichon's (2014) vocabulary, the equivalence class of X in this relation is called the Lipschitz geometry at infinity of X. Fernandes and Sampaio (2020) proved among other things that a pure dimensional complex algebraic subset of $\mathbb{C}^n$ with the same Lipschitz geometry at infinity as a Euclidean space must be an affine linear space of $\mathbb{C}^n$.

Our class of mathematical objects are complex plane algebraic curves and we give a complete classification of their Lipschitz geometry (see Theorem 3.9). We say that two plane algebraic curves C and Γ have the same Lipschitz geometry if there exists a bilipschitz map $\psi : C \to \Gamma$ (see Definition 3.1). In particular, we prove a theorem (see Theorem 3.8) giving a complete classification of the Lipschitz geometry at infinity (see Definition 3.5) of complex algebraic plane curves. We synthesize combinatorial objects that encode the Lipschitz geometry of complex algebraic plane curves (see Definition 3.16).

2 TOPOLOGY OF ALGEBRAIC CURVES

In this section we present classical material on topological aspects of complex algebraic plane curves. The central result of this section is the classification of topological type of complex curve germs. We introduce Eggers-Wall tree and carousel tree which are combinatorial objects that encode the topological type of germs of analytic curves.

2.1 Algebraic and projective curves

We start introducing notations and definitions commonly used in the study of complex algebraic plane curves.

Definition 2.1. *A* ***complex algebraic plane curve*** *C is the zero locus of a non-constant complex polynomial f in two variables, i.e., $C = \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\}$ where $f \in \mathbb{C}[x, y]$. Since $\mathbb{C}[x, y]$ is factorial, f can be written as a product $f_1^{\alpha_1} \dots f_k^{\alpha_k}$, with $f_1, \dots, f_k$ irreducible and f_i, f_j are coprime, and the α_j's are positive integers. The curve C_j defined by $f_j = 0$ is called an* ***irreducible component*** *of C. The curve C is said to be* ***reduced*** *if each $\alpha_j = 1$.*

We deal only with plane curves (curves in the complex plane $\mathbb{C}^2$ or the projective plane $\mathbb{P}^2$) and real curves are not treat here, thus we usually omit the terms "complex" and "plane" and just say algebraic curve.

Since we are only interested in the Lipschitz geometry (see Definition 3.1) of algebraic curves, we shall confine ourselves entirely to the case of reduced curves. With this restriction, Proposition 2.2 (BRIESKORN and KNÖRRER, 1986) shows that there is a reasonable connection between polynomials in $\mathbb{C}[x, y]$ and algebraic curves.

Proposition 2.2 (Study's lemma)**.** *Let C and C' be complex plane algebraic curves defined by f and g, respectively. Suppose that $C \subset C'$, then f is a divisor of g.*

Proof. Suppose first that f is irreducible. If $f \in \mathbb{C}[x]$ then $f = (x - a_1) \cdots (x - a_d)$ and $g(a_i, y) = 0$ in $\mathbb{C}[y]$, that is a_i is a root of $g \in \mathbb{C}[y][x]$. By the Division Theorem, $x - a_i$ is a divisor of g for all i, so it is f. We can assume $f \notin \mathbb{C}[x]$ and write

$$f = a_0(x)y^d + \cdots + a_d(x), \text{ with } a_0 \neq 0 \text{ in } \mathbb{C}[x].$$

If $g \in \mathbb{C}[x]$ then there is $\lambda \in \mathbb{C}$ such that $g \cdot a_0(\lambda) \neq 0$ for all $y \in \mathbb{C}$. But this contradicts the fact that $f(\lambda, y)$ has roots. We can assume that $g \notin \mathbb{C}[x]$. Thus we can write

$$(*) \quad \mathbb{C}[x] \ni R(f, g) = fu + gv,$$

where $R(f, g)$ denotes the resultant of f and g. There exists a zero of f for each $x_0 \in$

$\mathbb{C}$ such that $a_0(x_0) \neq 0$, and by hypothesis this is also a zero of g. By $(*)$, we have $R(f,g)(x_0) = 0$. Consequently, the function $a_0 R(f,g)$ vanish in $\mathbb{C}$. Since $a_0 \neq 0$, it follows that $R(f,g) = 0$ and this means that f and g have a non-constant common divisor, and by irreducibility of f it must be f itself.

Now, for the general case we decompose f in irreducible factors, $f_1, \ldots, f_r$. Since $\{f_i = 0\} \subset C \subset C'$, by the first part we have f_i is a divisor of g for all $i = 1, \cdots, r$, so it is f since f is square-free (i.e. f does not have multiple factors). $\square$

Remark 2.3. *Up to linear isomorphism we may suppose that an algebraic curves is defined by monic polynomial in* $\mathbb{C}[x][y]$. *Indeed, let* C *be an algebraic curve defined by* $f \in \mathbb{C}[x,y]$ *with degree* d*, then we can write* f *as* $f(x,y) = a_0(x)y^d + a_1(x)y^{d-1} + \cdots + a_d(x)$, *where* $a_j(x) = \alpha_j x^j + \cdots \in \mathbb{C}[x]$. *The coefficient of* y^d *in* $f(x+\lambda y, y)$ *is* $\sum_j \alpha_j \lambda^j$ *since*

$$a_j(x+\lambda y)y^{d-j} = (\alpha_j(x+\lambda y)^j + \cdots)y^{d-j} = \alpha_j \lambda^j y^d + \cdots .$$

There is $j' > 0$ *such that* $\alpha_{j'} \neq 0$ *(*f *has degree* d*), so we can choose* $\lambda \in \mathbb{C}$ *such that* $\sum_j \alpha_j \lambda^j = 1$. *The map* $T : \mathbb{C}^2 \to \mathbb{C}^2$ *defined by* $A(x,y) = (x+\lambda y, y)$ *is an linear isomorphism such that* $P \circ A$ *is a monic polynomial in* $(\mathbb{C}[x])[y]$ *and we have* $C = A^{-1}(\{P \circ A = 0\})$.

Study's lemma allows us to associate each algebraic curve C with a polynomial which is unique up to a constant factor and therefore the degree of curve is well-defined.

Definition 2.4. *The* ***degree*** *of the curve* C*, denoted by* $\deg C$*, defined by a polynomial* f *is the degree of* f.

The degree will be very important for us since it is related with the behavior of the curve at infinity. To understand this behavior at infinity we consider the following compactification of the plane $\mathbb{C}^2$.

Definition 2.5. *The* ***projective plane****, denoted by* $\mathbb{P}^2$*, is defined as the set of one-dimensional linear subspaces of* $\mathbb{C}^3$*, with the quotient topology determined by the natural projection* $\pi : \mathbb{C}^3 \backslash \{0\} \to \mathbb{P}^2$ *sending each point* (x,y,z) *to the subspace spanned by* (x,y,z). *Let* $[x : y : z] = \pi(x,y,z) \in \mathbb{P}^2$ *denote the line spanned by* (x,y,z) *and* $\iota : \mathbb{C}^2 \hookrightarrow \mathbb{P}^3$ *be the parametrization given by* $\iota(x,y) = [x : y : 1]$. *The complement of* $\iota(\mathbb{C}^2)$ *in* $\mathbb{P}^2$ *is called the* ***line at infinity*** *and we denote it by* L_∞.

Using the coordinates $[x : y : z]$ of the previous definition, the line at infinity is defined by the equation $z = 0$. The line at infinity is a simple example of what we call projective curve. If F is a homogeneous polynomial in $\mathbb{C}^3$ and if $(x,y,z) = \lambda(x',y',z')$ for some $\lambda \in \mathbb{C}\backslash\{0\}$, then $F(x,y,z) = 0$ if and only if $F(x',y',z') = 0$. Therefore, for points $[x : y : z] \in \mathbb{P}^2$ the property $F(x,y,z) = 0$ only depends on the class $[x : y : z]$.

Definition 2.6. *Let* F *be a homogeneous polynomial in* $\mathbb{C}[x,y,z]$. *The set* $\{[x : y : z] \in$

$\mathbb{P}^2 : F(x, y, z) = 0\}$ *is called a* ***projective curve****.*

We also consider the following compactification of an algebraic curve in the projective plane.

Definition 2.7. *Let $f \in \mathbb{C}[x, y]$ be a polynomial of degree d. The* ***homogenization*** *of f is the homogeneous polynomial $\widetilde{f} \in \mathbb{C}[x, y, z]$ defined by*

$$\widetilde{f}(x, y, z) = z^d f\left(\frac{x}{z}, \frac{y}{z}\right).$$

Let C be the algebraic curve with equation $f(x, y) = 0$. The projective curve $\widetilde{C} = \{[x : y : z] \in \mathbb{P}^2 : \widetilde{f}(x, y, z) = 0\}$ is called the ***homogenization*** *of C. The* ***points at infinity*** *of C are the elements of the intersection $\widetilde{C} \cap L_\infty$. Let $[a : b : 0]$ be a point at infinity of C, the subspace spanned by (a, b) in $\mathbb{C}^2$ is called* ***tangent line to*** *C* ***at infinity*** *associated to the point $[a : b : 0]$.*

The points at infinity of C are the points $[x : y : 0] \in \mathbb{P}^2$ satisfying $f_d(x, y) = 0$, where f_d denotes the homogeneous polynomial composed by the monomials in f of degree d. Thus there exist at almost d points at infinity. The zero set of f_d in $\mathbb{C}^2$ is the union of tangent lines to C at infinity.

Example 2.8. *Consider the polynomial $f(x, y) = y^2x - y$, and let C_λ be the complex algebraic plane curve with equation $f(x, y) + \lambda = 0$ for $\lambda \in \mathbb{C}$. One has $\widetilde{f}(x, y, z) = y^2x - yz^2 + \lambda z^3$, and the points at infinity of C_λ are $[1 : 0 : 0]$ and $[0 : 1 : 0]$ and the tangent lines at infinity to C_λ are the coordinates axis.*

Remark 2.9. *We recall that $\mathbb{P}^2$ is covered by three coordinate neighborhoods $U_i, i = 1, 2, 3$, where*

$$U_1 = \{[x : y : z] \in \mathbb{P}^2 : x \neq 0\}, U_2 = \{y \neq 0\} \text{ and } U_3 = \mathbb{P}^2 \backslash L_\infty.$$

One has coordinate charts $\varphi_i : U_i \to \mathbb{C}^2$ defined by $\varphi_1[x : y : z] = (z/x, y/x)$, $\varphi_2[x : y : z] = (z/y, x/y)$ and $\varphi_3[x : y : z] = (x/z, y/z)$. Now, when $\mathcal{C}$ is a projective curve defined by a homogeneous polynomial F, the part of $\mathcal{C}$ lying in the coordinate neighborhood $U_3 \subset \mathbb{P}^2$, namely $\mathcal{C} \cap U_3$, is obviously just

$$\mathcal{C} \cap U_3 = \{[x : y : 1] : F(x, y, 1) = 0\}.$$

Thus, if we set $f(x, y) := F(x, y, 1)$, the curve C defined by $f = 0$ satisfies $C = \varphi_3^{-1}(U_3 \cap \mathcal{C})$ and $\widetilde{C} = \mathcal{C}$.

To understand better an algebraic or a projective curve we compare them with a complex manifold. In order to do this, we conveniently divide the points of the curve into two classes.

Definition 2.10. *Let $\mathcal{S}$ be a smooth complex surface and $A \subset \mathcal{S}$ a subset. A point p of*

A is called ***regular*** *if there is a neighborhood U of p in $\mathcal{S}$ such that $A \cap U$ is a complex submanifold of $\mathcal{S}$. A non-regular point is called* ***singular*** *point.*

Example 2.11. *We will consider typical examples of singularities (singular points) of algebraic curves. First of all, such can be the points of intersection of several complex manifolds, e.g., $xy = 0$ or $y(y - x^2) = 0$.*

Another example of singularity is the cusp $C : y^2 = x^3$. The differential of $f = y^2 - x^3$ vanishes only at the origin, hence $C \backslash \{0\}$ is a one-dimensional complex manifold. The set C is the image of the complex plane $\mathbb{C}$ under the holomorphic map $\phi(z) = (z^2, z^3)$. Since $\phi : \mathbb{C} \to X$ is bijective, there exists a topological structure such that C is a topological submanifold. However, if g is a holomorphic function and equal to zero in a neighborhood of 0 on C, then $g(\phi(z)) \equiv 0$ in a neighborhood of 0 in $\mathbb{C}$. Therefore, by Taylor's formula,

$$\frac{\partial g}{\partial x}(0)2z + \frac{\partial g}{\partial y}(0)3z^2 + \cdots \equiv 0$$

and hence $\partial g/\partial x(0) = \partial g/\partial y(0) = 0$. Thus, there exists no neighborhood of the origin in which C is a complex submanifold. This can also be seen geometrically: if C would be a complex submanifold, then the plane x-axis would be tangent to C at the origin (the distance from $(x, y) \in C$ to x-axis is equal to $|y| = |x|^{\frac{3}{2}} = o(|(x, y)|)$. Therefore, in a small neighborhood of the origin the orthogonal projection of C on the x-axis would be bijective; however, it is not: the two distinct points $(x, \pm\sqrt{x^3}) \in C$ are projected into $(x, 0), x \neq 0$.

Let C be an algebraic curve given by the equation $f = 0$. By the implicit function theorem, the set of singular points is contained in the intersection of the curves C, $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$. But the intersection of curves without common irreducible components is finite. Let $f_1, \ldots, f_r$ be the irreducible factors of f. Then

$$\frac{\partial f}{\partial x} = \sum_i f_1 \ldots \frac{\partial f_i}{\partial x} \ldots f_r.$$

Thus C and $\partial f/\partial x = 0$ do not have common irreducible component and the singular points are finite.

Proposition 2.12. *Two complex algebraic plane curves C and Γ without a common component have at most finite intersection points.*

Proof. We can assume that C and Γ are defined by monics polynomials f resp. g in $\mathbb{C}[x][y]$. Thus the resultant $R(f, g)$ is a non-constant polynomial in $\mathbb{C}[x]$ and $R(f, g)(x_0) = 0$ if and only if $f(x_0, y) \in \mathbb{C}[y]$ and $g(x_0, y) \in \mathbb{C}[y]$ have a common root. The result follows. $\square$

Actually, the singular points of curve are easy to determine.

Proposition 2.13. *Let C be an algebraic (resp. a projective) curve given by the equation $f = 0$ (resp. $F = 0$). The singular points of C are the elements of the intersection of the* *curves C, $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$ (resp. C, $\partial F/\partial x = 0, \partial F/\partial y = 0$ and $\partial F/\partial z = 0$).*

Proof. For algebraic curves see (EBELING, 2007, Proposition 2.36). Let $\mathcal{C}$ be a projective curve defined by F and let p be a point in $\mathcal{C}$. Suppose first that $p \in U_3$ where $U_3 = \{[x : y : z] \in \mathbb{P}^2 : z \neq 0\}$ and consider the coordinate chart $\varphi_3[x : y : z] = (x/z, y/z)$. Then p is a singular point of $\mathcal{C}$ if and only if $\varphi_3(p)$ is a singular point of the algebraic curve C defined by $f = 0$ where $f(x, y) := F(x, y, 1)$.

We apply the result for C. Thus the point $\varphi_3(p)$ is a singular point of C if only if the point $\varphi_3(p)$ belongs to the intersection of the curves C, $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$. But, of course, $\partial f/\partial x = \partial F/\partial x$ and $\partial f/\partial x = \partial F/\partial y$. Summarizing p is singular point of $\mathcal{C}$ if and only if

$$F(p) = \frac{\partial F}{\partial x}(p) = \frac{\partial F}{\partial y}(p) = 0.$$

Finally, by Euler formula:

$$(\deg F)F = x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial x} + z\frac{\partial F}{\partial z},$$

and then p is singular point of $\mathcal{C}$ if and only if $F(p) = \frac{\partial F}{\partial x}(p) = \frac{\partial F}{\partial y}(p) = \frac{\partial F}{\partial z}(p) = 0$. Similarly, for $p \in U_2$ or $p \in U_1$.

□

A regular point $p = (x_0, y_0)$ of an algebraic curve C, defined by f, appears as those points at which the tangent problem has a simple, unique solution. Namely, the tangent line at p is

$$\frac{\partial f}{\partial x}(p)(x - x_0) + \frac{\partial f}{\partial y}(p)(y - y_0) = 0.$$

Notice that the right side of the above equation is the first term of the Taylor expansion of f at p. Similarly, let $\mathcal{C}$ be a projective curve and p at regular point of $\mathcal{C}$. We choose $[x : y : z]$ for $\mathbb{P}^2$ such that p has coordinates $[a : b : 1]$. The **tangent line** of $\mathcal{C}$ at p is the homogenization of the tangent line for the algebraic curve C defined by $f = 0$ where $f(x, y) := F(x, y, 1)$. Thus it has equation

$$\frac{\partial F}{\partial x}(p)x + \frac{\partial F}{\partial y}(p)y + \frac{\partial F}{\partial z}(p)z = 0.$$

This motivates the following definition:

Definition 2.14. *Let C be an algebraic (resp. a projective) curve defined by $f = 0$ and let p be a point of C.*

1. *The **multiplicity** $\upsilon_p(C)$ of C at p is the order of the lowest non-vanishing term in the Taylor expansion of f at p.*
2. *The **tangents** to C at p are the lines through p which cut C with multiplicity bigger*

than $\upsilon_p(C)$ at p. Counting multiplicities, C has exactly $\upsilon_p(C)$ tangents at p.

By Proposition 2.13, a point p of the curve C is regular if and only if the multiplicity of C at p is one. To conclude this section let us present a result about global topology of algebraic curve.

Proposition 2.15. *Irreducible algebraic curves are connected.*

See (e.g. LEFSCHETZ, 1953, p. 97) for a proof of Proposition 2.15.

2.2 Topological type of germs of analytic curves

Let C be an algebraic or a projective curve and p a point at C. How does C look like in a small neighbourhood of p? For certain purposes it can cause trouble to choose a particular neighborhood U of p. We are interested only in the curve in arbitrarily small neighborhood of p. Then it is more convenient to consider the system of all neighborhoods U of p and this leads us to the definition of germs of sets.

Definition 2.16. *Let T be a topological space and $p \in T$. We define an equivalence relation on the set of all subsets of T, by setting $X \sim Y$ if $X \cap U = Y \cap U$ for some neighborhood U of p. An equivalence class of a set X in this relation is called the* ***germ*** *of X at p and we denote the germ of X at p by (X, p).*

In addition, we write $(X, p) \subset (Y, p)$ when there is a neighborhood U of p such that $X \cap U \subset Y \cap U$. Clearly, $(X, p) \subset (Y, p)$ and $(Y, p) \subset (X, p)$ if and only if $(X, p) = (Y, p)$. Similarly, if we are interested only in the behaviour of a map in arbitrarily neighborhood of a point, there is a notion of germs of maps.

Definition 2.17. *Let T be a topological space, $p \in T$ and N any set. We define an equivalence relation on the set of all maps $f : T \to N$, by setting $f \sim_p g$ if $f|_U = g|_U$ for some neighborhood U of p. An equivalence class of a map $f : T \to N$ in this relation is called the* ***germ*** *of $f : T \to N$ at p and we denote it by $f : (T, p) \to N$.*

Note that for any representative g of the class $f : (T, p) \to N$ we have $g(p) = f(p)$. When we want to emphasise the image of p by f, say $f(p) = q$, we may write $f : (T, p) \to (N, q)$. The usual adjectives that we use to a map naturally extend to a map germ.

Definition 2.18. *We say that a map germ $f : (T, p) \to N$ is continuous if there exists a representative of it that is continuous. More generally, we say that $f : (T, p) \to N$ has some property if one of its representatives has that property.*

We return to our question: how (C, p) looks like? There are multiple answers to this vague question depending on the category we work in, i.e., on the equivalence relation between germs we choose. Here we consider the topological equivalence relations.

Definition 2.19. *Let T and T' be topological spaces. Two germs $(X,p) \subset (T,p)$ and $(Y,q) \subset (T',p)$ are* ***topologically equivalent*** *if there exists a germ of homeomorphism $\psi : (X,p) \to (Y,q)$.*

We say that (X,p) and (Y,q) are ***embedded topologically equivalent*** *if there exists a germ of homeomorphism $\phi : (T,p) \to (T',q)$ such that $\psi(X) = Y$. The equivalence class of (X,p) in this relation is called* ***topological type*** *of (X,p).*

To study the topological type of germ (C,p) of an algebraic curve we will intersect C with a small sphere S_ϵ centered at p.

Proposition 2.20. *Let C be an algebraic curve and $p \in C$. Every sufficiently small sphere S_ϵ centered at p intersect C smooth manifold.*

For a proof, see (MILNOR, 1968, Corollary 2.9). The intersection of C with a small sphere S_ϵ centered at p is a compact smooth 1-manifold. Therefore each connected component of the intersection is diffeomorphic to the circle $\mathbb{S}^1$. As an example, if the intersection has only one component, then C must be a topological manifold near p by Conical Structure Theorem 2.23. On the other hand, the embedded of the intersection in the sphere can be quite interesting even for simple singularities like intersection of three lines going through the origin.

Example 2.21. *Let C be the union of the lines $y = 0, y = x$ and $y = -x$. We use the correspondence*

$$(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 \leftrightarrow (x, y) \in \mathbb{C}^2$$

to identify $\mathbb{R}^4$ with $\mathbb{C}^2$. The set $\mathbb{S}^3 \backslash \{(0,0,0,1)\}$ is a neighborhood of $\mathbb{S}^3 \cap C$ so we can view the embedded into $\mathbb{R}^3$ using the stereographic projection from the north pole $\sigma : \mathbb{S}^3 \backslash \{(0,0,0,1)\} \to \mathbb{R}^3$ defined by

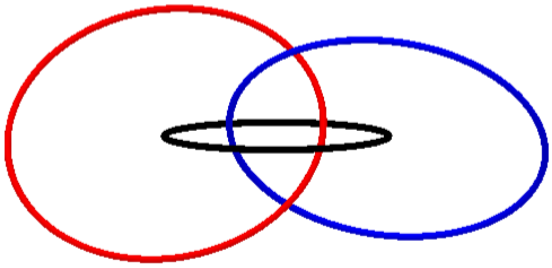
$$\sigma(x_1, x_2, y_1, y_2) = \frac{(x_1, x_2, y_1)}{1 - y_2}.$$

The image of the intersection $C \cap \mathbb{S}^3$ by the stereographic projection is the union of the circles $\{(u,v,w) \in \mathbb{R}^3 : u^2 + v^2 = 1, w = 0\}, \{u^2 + (v-1)^2 + w^2 = 2, u = v\}$ and $\{u^2 + (v+1)^2 + w^2 = 2, w = -u\}$ (see Figure 1.)

For a smooth point p of an algebraic curve C nothing interesting happens in the embedded of the intersection $S_\epsilon(p) \cap C$ in the sphere $S_\epsilon(p)$ centred at p of radius $\epsilon > 0$ for ϵ sufficiently small.

Proposition 2.22. *Let C be an algebraic curve and let $p \in C$ be a regular point. For all sufficiently small $\epsilon > 0$ the pair $(S_\epsilon(p), S_\epsilon(p) \cap C)$ is diffeomorphic to the pair $(\mathbb{S}^3, \mathbb{S}^3 \cap P)$ where P is plane going through the origin (a 2-dimensional linear subspace of $\mathbb{R}^4$).*

Figure 1 – Linked circles of Example 2.21.



Source: Elaborated by the author.

Proof. See (MILNOR, 1968, Lemma 2.12). □

The homeomorphism class of the germ (C, p) of an algebraic curve C is completely determined by the number of connected components of the link of a curve as stated in the classical Conical Structure Theorem, see (MILNOR, 1968, Theorem 2.10).

Let $B_\epsilon(p)$ be the open ball with radius $\epsilon > 0$ centered at the point p of $\mathbb{C}^2$ and let $S_\epsilon(p)$ be its boundary.

Theorem 2.23 (Conical Structure Theorem)**.** *Let C be an algebraic curve and $p \in C$. For $\epsilon > 0$, set $K_\epsilon = S_\epsilon(p) \cap C$. There exists $\epsilon_0 > 0$ such that for every ϵ with $0 < \epsilon \leq \epsilon_0$, the pair $(B_\epsilon(p), B_\epsilon(p) \cap C)$ is homeomorphic to the pair $(B_\epsilon(p), \mathrm{Cone}(K_\epsilon))$ where $\mathrm{Cone}(K_\epsilon)$ is the union of all line segments $tq + (1-t)p, 0 \leq t \leq 1$ and $q \in K_\epsilon$.*

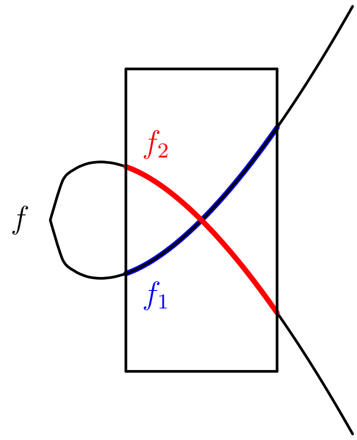
In other words, the homeomorphism class of the pair $(B_\epsilon(p), B_\epsilon(p) \cap C)$ does not depend on ϵ when ϵ is sufficiently small. In particular, the homeomorphism class of the intersection $C \cap S_\epsilon(p)$ is well defined for the germ (C, p).

Definition 2.24. *When $0 < \epsilon \leq \epsilon_0$, the intersection $C \cap S_\epsilon$ is called the* ***link*** *of (C, p).*

As we said the number of connected components of the link completely determines the homeomorphism class of (C, p). But the number of irreducible components of the curve C does not determine the connected components of the link.

Example 2.25. *Consider the curve α defined by the irreducible polynomial $f = y^2 - x^2(1+x)$. Notice that the curve α in a small neighborhood U of the origin is the union of zero locus of holomorphic functions $f_1(x, y) = y - x\sqrt{1+x}$, $f_2(x, y) = y + x\sqrt{1+x}$ (analytic curves). The analytic curves intersect only at the origin and intersect small spheres centered at the origin transversely, since they are graphs (see Fig. 2). The decomposition of the link of α in connected components is therefore not reflected by the decomposition of the polynomial f into the product of polynomials. Rightly so, because the curve α indeed does not decompose into two pieces over the whole plane, but only in a suitably chosen, sufficiently small, neighborhood.*

Figure 2 – Graphs of f_1 and f_2 in a neighborhood of the origin.



Source: Elaborated by the author.

Definition 2.26. *Let $\mathcal{S}$ be a smooth complex surface (e.g. $\mathbb{C}^2$ or $\mathbb{P}^2$). A set $C \subset S$ is an analytic curve if for each point $p \in \mathcal{S}$ there are a neighborhood U of p and a non-constant holomorphic function $f : U \to \mathbb{C}$ such that*

$$C \cap \mathcal{S} = \{z \in U : f(z) = 0\}.$$

In other words, locally C is the zero locus of holomorphic functions. The locality of the definition is important not only for generality. With this definition we have plenty of analytic curves in compact surfaces!

All over this section, $\mathcal{S}$ denotes a complex manifold of dimension two. We fix a point $p \in \mathcal{S}$. All coordinate charts of this section are defined in a neighborhood of p, moreover, the charts are centred at p, that means, the point p always has coordinates $(0, 0) \in \mathbb{C}^2$. The set of all germs of holomorphic functions $h : (\mathcal{S}, p) \to \mathbb{C}$ obviously constitutes a ring: one defines the sum and product of germs representative wise. We denote this ring by $\mathcal{O}_{p,\mathcal{S}}$. When the surface is understood or not important we usually omit mention of it and just write $\mathcal{O}_p$. The ring $\mathcal{O}_{p,\mathcal{S}}$ share all algebraic properties with the ring $\mathbb{C}\{x, y\}$, see (EBELING, 2007, Proposition 2.15).

Proposition 2.27. *Any chart φ of $\mathcal{S}$ centred at p induces an isomorphism between the ring $\mathcal{O}_{p,\mathcal{S}}$ and the ring $\mathbb{C}\{x, y\}$ of convergent power series at the origin.*

We now introduce a fundamental theorem in the further study of local rings of holomorphic function germs. This is the Weierstrass's Preparation Theorem, which will play a central role in the study of germs of analytic curves.

Theorem 2.28 (Weierstrass's Preparation Theorem)**.** *Let $f(x, y)$ be a convergent power series in $\mathbb{C}\{x, y\}$ with $f(0, y) \not\equiv 0$. Then f can be represented uniquely in the form*

$$f(x, y) = (y^k + c_1(x)y^{k-1} + \cdots + c_k(x))u(x, y)$$

where the functions $c_i(x) \in \mathbb{C}\{x\}$ *and* $u(x,y) \in \mathbb{C}\{x,y\}$ *is a unit (that is* $u(0,0) \neq 0$*).*

Proofs of Theorem 2.28 can be found in many accessible books, for instance, see (CHIRKA, 1989, Theorem 1.1.1) or (EBELING, 2007, Theorem 2.1).

Thus, the zero set of holomorphic function in two variables coincides with the zero set of a polynomial in one variable y, with coefficients that holomorphically depend on the remaining variable x and with leading coefficient 1. This polynomial is called **Weierstrass polynomial**. This result implies that the only difference between algebraic curves and analytic curves is that the latter case we allow coefficients in the ring $\mathbb{C}\{x\}$ that contains the ring $\mathbb{C}[x]$.

Remark 2.29. *The condition that* $f(0,y) \not\equiv 0$ *is not an essential restriction. Up to linear isomorphism we may suppose that a non-zero power series is regular of order* k *in* y*, that is* $f(0,y) = a_{0k}y^k + \cdots, a_{0k} \neq 0$*. This follows from the same kind of argument as in Remark 2.3.*

A useful consequence of the Weierstrass's Preparation Theorem is that $\mathbb{C}\{x,y\}$ is, like $\mathbb{C}[x,y]$, factorial, see (EBELING, 2007, Proposition 2.20).

Proposition 2.30. *The ring* $\mathbb{C}\{x,y\}$ *is factorial.*

Definition 2.31. *Let* $(C,p) \subset (\mathcal{S},p)$ *be a germ of analytic curve defined by* $f \in \mathcal{O}_p$*. We can write* f *as a product* $f_1^{\alpha_1} \cdots f_k^{\alpha_k}$*, with* $f_1, \ldots, f_k$ *irreducible and* f_i*,* f_j *are coprime for* $i \neq j$*, and the* α_j*'s are positive integers. The zero set of* f_j*'s are the* ***branches*** *of* C*. When* $k = 1$*, we say that* C *is* ***irreducible****. The holomorphic function* f *is* ***reduced*** *if each* $\alpha_j = 1$*.*

We will always suppose all equations for curves are reduced. Let $(C,p) \subset (\mathcal{S},p)$ be a germ of analytic curve defined by $f = 0, f \in \mathcal{O}_p$. Since we want to carry out only purely local investigations we can suppose that $p = 0 \in \mathbb{C}^2$ and $f \in \mathbb{C}\{x,y\}$. We want to describe the zero set of f, i.e. the solution set of the equation $f(x,y) = 0$, in a suitable neighborhood of the origin. We shall see that the solution set has a parametrization. This parametrization in a certain sense can be obtained quite explicitly. To do this one uses a method which goes back to Newton.

Theorem 2.32 (The Newton-Puiseux theorem)**.** *Let* $f \in \mathbb{C}\{x,y\}$ *be an irreducible power series regular of order* n *in* y *and let* C *be the curve defined by* $f = 0$*. Then there exist a neighborhood* B *of* 0 *in* $\mathbb{C}$*, a neighborhood* V *of the origin in* $\mathbb{C}^2$ *and a homeomorphism map* $\gamma : B \to C \cap V$ *of the form* $\gamma(w) = (w^n, \eta(w))$ *with* $\eta(w) \in \mathbb{C}\{w\}$*. Moreover, the restriction* $\gamma : B\backslash\{0\} \to C \cap V\backslash\{(0,0)\}$ *is biholomorphic.*

Proof. See in (BRIESKORN and KNÖRRER, 1986, Theorem 8.3.1). □

In other words, any equation $f(x,y) = 0$ where $f \in \mathbb{C}\{x,y\}$ with $f(0,0) = 0, f(0,y) \not\equiv 0$ admits a solution of the form $y = \eta(x^{1/n})$ with $\eta \in \mathbb{C}\{t\}$ and n is a positive integer. Note that the branch $f = 0$ is smooth, if and only, $n = 1$ or $\eta'(0) \neq 0$.

We recall some definitions and conventions about power series with positive rational exponents. Let n be a positive integer, the ring $\mathbb{C}[[x^{1/n}]]$ consists of sequence $(A_k)_{k\in\mathbb{N}}$ of elements of $\mathbb{C}$. Let $\eta = (A_k)_{k\in\mathbb{N}} \in \mathbb{C}[[x^{1/n}]]$, we denote this element by

$$\eta = \sum_{k=0}^{\infty} A_k x^{k/n}.$$

The **exponents** of η are the numbers k/n such that $A_k \neq 0$. We denote the set of exponents of η by $\mathcal{E}(\eta)$. The **order** of $\eta \neq 0$, denoted by $\operatorname{ord}_x \eta$, is the smallest exponent of η. For technical reasons it is convenient to define the order of the zero to be $+\infty$. The subgroup of n-th roots of 1 acts on $\mathbb{C}[[x^{1/n}]]$ by the rule

$$(\rho, \eta) \to \eta(\rho \cdot x^{1/n}) := \sum_{k=0}^{\infty} A_k \rho^k x^{k/n}, \text{ where } \rho \text{ is a } n\text{-th root of } 1.$$

The next definitions of this section depend on the choice of a smooth curve L in $\mathcal{S}$ passing through p. Assume that a coordinate system (x, y) such that $L = \{x = 0\}$ is fixed. Let C be a curve on $\mathcal{S}$ and assume that A is a branch of C different from the curve L. Relative to the system (x, y), the branch A may be defined by a Weierstrass polynomial $f_A \in \mathbb{C}\{x\}[y]$, which is monic, and of degree d_A in y. Note that the degree d_A does not depend on the coordinate system that sends L to y-axis.

By Newton-Puiseux Theorem 2.32, there exists a parametrization of A of the form $\gamma_A(w) = (w^{d_A}, \eta_A(w))$ where $\eta_A(w) = \sum_{k>0} a_k w^k \in \mathbb{C}\{w\}$. Let n be the product of the degrees of the Weierstrass polynomials of the branches of C different from L. We consider the formal power series $\sum_{k>0} A_k x^{k/n} \in \mathbb{C}[[x^{1/n}]]$ where

$$A_k = \begin{cases} a_{\frac{kd_A}{n}}, & \text{if } n \text{ divides } kd_A \\ 0, & \text{otherwise.} \end{cases}$$

We denote by $\eta_A(x^{1/n})$ the formal power series $\sum_{k>0} A_k x^{k/n}$.

Definition 2.33. *The **Newton-Puiseux roots relative to** L of the branch A are the formal power series $\eta_A(\rho \cdot x^{1/n}) \in \mathbb{C}[[x^{1/n}]]$, for ρ running through the n-th roots of 1.*

Let $\rho \in \mathbb{C}$ be a primitive n-root of unity, notice that there are only d_A Newton-Puiseux roots relative to L of the branch A, namely

$$\eta_A(\rho \cdot x^{1/n}), \ldots, \eta_A(\rho^{d_A} \cdot x^{1/n}).$$

All the Newton-Puiseux roots relative to L of the curve A have the same exponents. Some of those exponents may be distinguished by looking at the differences of roots:

Definition 2.34. *The* ***characteristic exponents relative to*** L *of the curve* A *are the* x*-orders* $\text{ord}_x(\eta_A - \eta'_A)$ *of the differences between distinct Newton-Puiseux roots relative to* L *of* A.

The fact that in the previous definition we mentioned only the dependency on L, and not the whole coordinate system (x, y), comes from Proposition 2.43.

The characteristic exponents relative to L of A consist of exponents of η_A which, when written as a quotient of integers, need a denominator strictly bigger than the lowest common denominator of the previous exponents. That is: $\frac{l}{n}$ is characteristic exponent relative to L of A if and only if $N_l \frac{l}{n} \notin \mathbb{Z}$ where $N_l = \min\{N \in \mathbb{Z}\,; \mathcal{E}(\eta_A) \cap [0, \frac{l}{n}) \in \frac{1}{N}\mathbb{Z}\}$.

Example 2.35. *Let* C *be the projective curve defined by* $(zx - y^2)(xy^2 - z^3) = 0$, $p = [1 : 0 : 0]$ *and* L *be the line at infinity. We consider the chart* $\varphi_1 : U_1 \to \mathbb{C}^2$ *defined by* $\varphi_1[x : y : z] = (z/x, y/x) = (u, v)$*, in this coordinate* L *is given by* $u = 0$ *and* C *by* $(u - v^2)(v^2 - u^3) = 0$. *Thus the curve* C *has two branches* $A : v^2 = u^3$ *and* $B : v^2 = u$ *parametrized by*

$$\gamma_A(w) = (w^2, w^3),\ \gamma_B(w) = (w^2, w),$$

respectively. The Newton-Puiseux roots relative to L *of* A *are*

$$\eta_A(u^{1/4}) = u^{6/4}, \qquad \eta_A(\rho \cdot u^{1/4}) = \rho^2 u^{6/4},$$

where ρ *is a primitive 4-th root of unity. While the Newton-Puiseux roots relative to* L *of* B *are*

$$\eta_B(u^{1/4}) = u^{2/4}, \qquad \eta_B(\rho u^{1/4}) = \rho^2 u^{2/4}.$$

The characteristic exponent relative to the line at infinity of A *is* $3/2$. *The characteristic exponent of* B *relative to* L *is* $1/2$. *Notice that if we have choose* L *to be* $y = 0$ *in* $\mathbb{P}^2$*, we would end up with different characteristic exponents for the branch* A*, namely,* $2/3$ *.*

As the above example shows the characteristic exponents of a branch relative to a smooth curve L varies with L. But only for special choice of L.

Proposition 2.36. *The characteristic exponents of a branch relative to a smooth curve* L *are independent of* L *once* L *is transversal to the branch.*

Proof. See (WALL, 2004, Corollary 4.1.4). □

From the above proposition together with Proposition 2.43 we conclude that the characteristic exponents of a branch A relative to a smooth curve L is independent of the system of coordinate (x, y) once $L = \{x = 0\}$ and L is transversal to the branch, that is, once its tangent does not coincide with the tangent line of A. One speaks then of **the generic characteristic exponents of** A. Its main property is that it is a complete invariant of the topological type of the branch.

Theorem 2.37. *Two branch germs (A, p) and (B, q) have the same topological type if and only if A and B have the same generic characteristic exponents.*

Proof. See (WALL, 2004, Theorem 5.5.8). □

Most computations of other topological invariants of the pair $(\mathcal{S}, A)$ are done in terms of its generic characteristic exponents sequence.

Let us consider now the case of a curve with several branches. In order to have a complete invariant for the topological type of the curve, one needs to know not only the characteristic exponents of its branches, but also the exponent of coincidence of its pairs of branches:

Definition 2.38. *If A and B are two distinct branches of C, then their **exponent of coincidence relative** to L is defined by:*

$$k_L(A, B) := \max\{\text{ord}_x(\eta_A - \eta_B)\},$$

where $\eta_A, \eta_B \in \mathbb{C}[[x^{1/n}]]$ vary among the Newton-Puiseux roots of A and B, respectively.

The above notions appears to depend on coordinates, but we see at once that is not the case for L transversal to the curve. On the other hand, Proposition 2.43 tells us that the exponent of coincidence relative to L between branches of C only depends on L.

Proposition 2.39. *The exponent of coincidence relative to L of the branch of a curve C is independent of the system of coordinate (x, y) once $L = \{x = 0\}$ and L is transversal to the curve.*

Proof. See (WALL, 2004, Lemma 4.1.1 and Lemma 4.1.2). □

One speaks then of **the generic exponents of coincidence of** C. We finish this section with the complete classification of the topological type for analytic curves.

Theorem 2.40. *Two analytic curves germs (C, p) and (Γ, q) have the same topological type if and only if there is a bijection between the irreducible components of C and Γ such that*

1. *the generic characteristic exponents of corresponding components are the same*
2. *the generic exponent of coincidence of corresponding components coincide.*

2.3 Combinatorics of topological type of germs of analytic curves

There are several objects that encode the topological type of germ of a complex curve (C, p), here we present two of them: carousel tree and Eggers-Wall tree. We define the Eggers-Wall tree $\Theta_L(C)$ of such a germ relative to a smooth branch L going through point p. The definition of Eggers-Wall tree which are given in this thesis are the same present in (BARROSO, PÉREZ, and POPESCU-PAMPU, 2019). Then we explain how to pass from $\Theta_L(C)$ to the Eggers-Wall tree $\Theta_{L'}(C)$ relative to another smooth branch L' transversal to C and L at p. Finally, we describe how one gets the Eggers-Wall tree from the carousel tree. A similar process is also described in (NEUMANN and PICHON, 2014).

We keep using convention of previous section, that is: $\mathcal{S}$ denotes a complex manifold of dimension two. We fix a point $p \in \mathcal{S}$, a smooth curve L at p and a coordinate system (x, y) such that $p = (0, 0)$ and $L = \{x = 0\}$.

Let (C, p) be a germ of complex curve on $(\mathcal{S}, p)$ and assume that A is a branch of C different from the curve L. The Eggers-Wall tree of A relative to L is a geometrical way of encoding the set of characteristic exponents, as well as the sequence of their successive common denominators.

Definition 2.41. *The Eggers-Wall tree $\Theta_L(A)$ of the curve A relative to L is a compact oriented segment endowed with the following supplementary structures:*

- *an increasing homeomorphism $\mathbf{e}_{L,A} : \Theta_L(A) \to [0, \infty]$, **the exponent function**;*
- ***marked points**, which are by definition the points whose values by the exponent function are the characteristic exponents of A, as well as the smallest end of $\Theta_L(A)$, labeled by L, and the greatest end, labeled by A.*
- *an **index function** $\mathbf{i}_{L,A} : \Theta_L(A) \to \mathbb{N}$, which associates to each point $P \in \Theta_L(A)$ the smallest common denominator of the exponents of a Newton-Puiseux root of A which are strictly less than $\mathbf{e}_{L,A}(P)$.*

Let us consider now the case of a curve with several branches. In order to construct the Eggers-Wall tree in this case, one needs to know not only the characteristic exponents of its branches, but also the exponent of coincidence of its pairs of branches.

Definition 2.42. *Let C be a germ of curve on (S, p). Let us denote by $\mathcal{I}_C$ the set of branches of C which are different from L. The Eggers-Wall tree $\Theta_L(C)$ of C relative to L is the rooted tree obtained as the quotient of the disjoint union of the individual Eggers-Wall trees $\Theta_L(A), A \in \mathcal{I}_C$, by the following equivalence relation. If $A, B \in \mathcal{I}_C$, then we glue $\Theta_L(A)$ with $\Theta_L(B)$ along the initial segments $\mathbf{e}_{L,A}^{-1}([0, k_L(A, B)])$ and $\mathbf{e}_{L,B}^{-1}([0, k_L(A, B)])$ by:*

$$\mathbf{e}_{L,A}^{-1}(\alpha) \sim \mathbf{e}_{L,B}^{-1}(\alpha), \text{ for all } \alpha \in [0, k_L(A, B)].$$

*One endows $\Theta_L(C)$ with the **exponent function** $\mathbf{e}_L : \Theta_L(C) \to [0, \infty]$ and the index* *function* $\mathbf{i}_L : \Theta_L(C) \to \mathbb{N}$ *induced by the initial exponent functions* $\mathbf{e}_{L,A}$ *and* $\mathbf{i}_{L,A}$ *respectively, for* A *varying among the irreducible components of* C *different from* L*. The tree* $\Theta_L(L)$ *is the trivial tree with vertex set a singleton whose element is labelled by* L*. If* L *is an irreducible component of* C*, then the marked point* $L \in \Theta_L(L)$ *is identified with the root of* $\Theta_L(L)$ *for any* $A \in \mathcal{I}_C$*. The set of marked points of* $\Theta_L(C)$ *is the union of the set of marked points of the Eggers-Wall tree of the branches of* C *and of the set of ramification points of* $\Theta_L(C)$*.*

The fact that in the previous notations $\Theta_L(C), \mathbf{e}_L, \mathbf{i}_L$ we mentioned only the dependency on L, and not the whole coordinate system (x, y), comes from the following fact :

Proposition 2.43. *The Eggers-Wall tree* $\Theta_L(C)$*, seen as a rooted tree endowed with the exponent function* $\mathbf{e}_L$ *and the index function* $\mathbf{i}_L$*, depends only on the pair* (C, L)*, where* L *is defined by* $x = 0$*.*

Proof. See (BARROSO, PÉREZ, and POPESCU-PAMPU, 2019, Proposition 3.9). □

The Eggers-Wall tree $\Theta_L(C)$ is explicitly defined in terms of characteristic exponents of branches relative L and the exponent of coincidence between branches relative to L. Now, if L is transversal to C, Theorem 2.40 says that these data determines and is determined by the topology type of (C, p).

Proposition 2.44. *For two curves germs* (C, p) *and* (Γ, q)*, the following are equivalent:*

1. (C, p) *and* (Γ, q) *have the same topological type;*
2. *There is an isomorphism of Eggers-Wall trees preserving the exponent and index functions* $\Theta_L(C) \to \Theta_{L'}(C')$ *where* L *and* L' *are a smooth curve transversal to* C *and* C' *at* p *and* q*, respectively.*

Proof. See (WALL, 2004, Proposition 4.3.9) and Theorem 2.40. □

Let L be a smooth curve going through p not transversal to C. Is still true that $\Theta_L(C)$ determines the topological type of (C, p)?. In view of Proposition 2.44, it is enough to answer the following question: let L' be a smooth branch transversal to C, does the tree $\Theta_L(C)$ determine $\Theta_{L'}(C)$?. The answer is affirmative and the algorithm to pass from one tree to another is given in the Inversion Theorem for Eggers-Wall tree.

Theorem 2.45. *Let* L *and* L' *be two transversal smooth branches at* p *which are components of the reduced curve germ* (C, p)*. Let us denote by* $\pi_{[L,L']} : \Theta_L(C) \to \Theta_L(C)$ *the projection on the segment* $[L, L']$ *defined by:* $\pi_{[L,L']}(P)$ *is the vertex in the intersection* $[L, P] \cap [P, L'] \cap [L, L']$*. Then the trees associated with* $\Theta_{L'}(C)$ *and* $\Theta_L(C)$ *coincide and the functions* $\mathbf{e}_{L'}$*,* $\mathbf{i}_{L'}$ *are determined by:*

$$\mathbf{e}_{L'} + 1 = \frac{\mathbf{e}_L + 1}{\mathbf{e}_L \circ \pi_{[L,L']}}, \qquad \mathbf{i}_{L'} = \begin{cases} 1, & \text{on } [L, L'], \\ (\mathbf{e}_L \circ \pi_{[L,L']}) \cdot \mathbf{i}_L, & \text{otherwise.} \end{cases}$$

Proof. See (BARROSO, PÉREZ, and POPESCU-PAMPU, 2019, Theorem 4.5). □

When L or L' is not a branch of C, we determine the Eggers-Wall tree $\Theta_{L'}(C)$ from $\Theta_L(C)$ by constructing first $\Theta_L(C \cup L \cup L')$, by applying then Theorem 2.45 to it in order to get $\Theta_{L'}(C \cup L \cup L')$, and by passing finally to the subtree $\Theta_{L'}(C)$.

To construct $\Theta_L(C \cup L \cup L')$ from $\Theta_L(C)$ one needs to know only the coincidence exponent between L' and the branches of C relative to L. Since L' is transversal to both curves C and L, we can choose to work in a coordinate system (x, y) such that $L = \{x = 0\}$ and $L' = \{y = 0\}$, and for any branch A of C, its Newton-Puiseux roots η_A relative to L satisfy $\text{ord}_x(\eta_A) \in (0, 1]$. But one has that $\text{ord}_x(\eta_A) = \text{ord}_x(\eta_A - 0) = k_L(A, L')$. Let $\Theta_L(C)_1$ be the subtree of $\Theta_L(C)$ consisting of points in $\Theta_L(C)$ of index 1. Immediately, one has that the tree $\Theta_L(C \cup L \cup L')$ is obtained from $\Theta_L(C)$ by adding an edge $[U, L']$ where U is

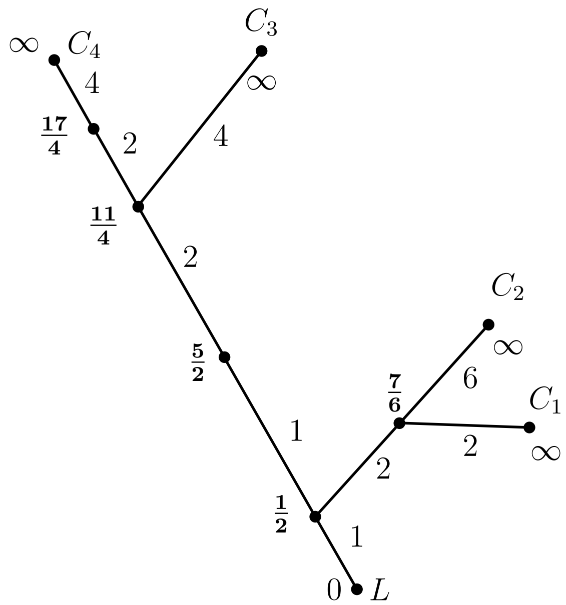
- the highest end of $\Theta_L(C)_1$, when the exponent function takes only values < 1 in restriction to $\Theta_L(C)_1$ (case in which $\Theta_L(C)_1$ is a segment);
- the unique point of $\Theta_L(C)$ of exponent 1, otherwise.

Example 2.46. *Consider a plane curve singularity $C = \bigcup_{i=1}^{4} C_i$ whose branches C_i are defined by the Newton-Puiseux series η_i, where:*

$$\eta_1 = x^{1/2}, \quad \eta_2 = -x^{1/2} + x^{7/6}, \quad \eta_3 = x - x^{5/2} + x^{11/4}, \quad \eta_4 = x + x^{5/2} + x^{17/4}.$$

Let L be the y-axis and L' be the x-axis. One has $k_L(C_1, C_2) = \frac{7}{6}$, $k_L(C_1, C_3) = k_L(C_1, C_4) = k_L(C_2, C_3) = k_L(C_2, C_4) = \frac{1}{2}$, $k_L(C_3, C_4) = 11/4$, and the Eggers-Wall tree of C relative to L is drawn Figure 3.

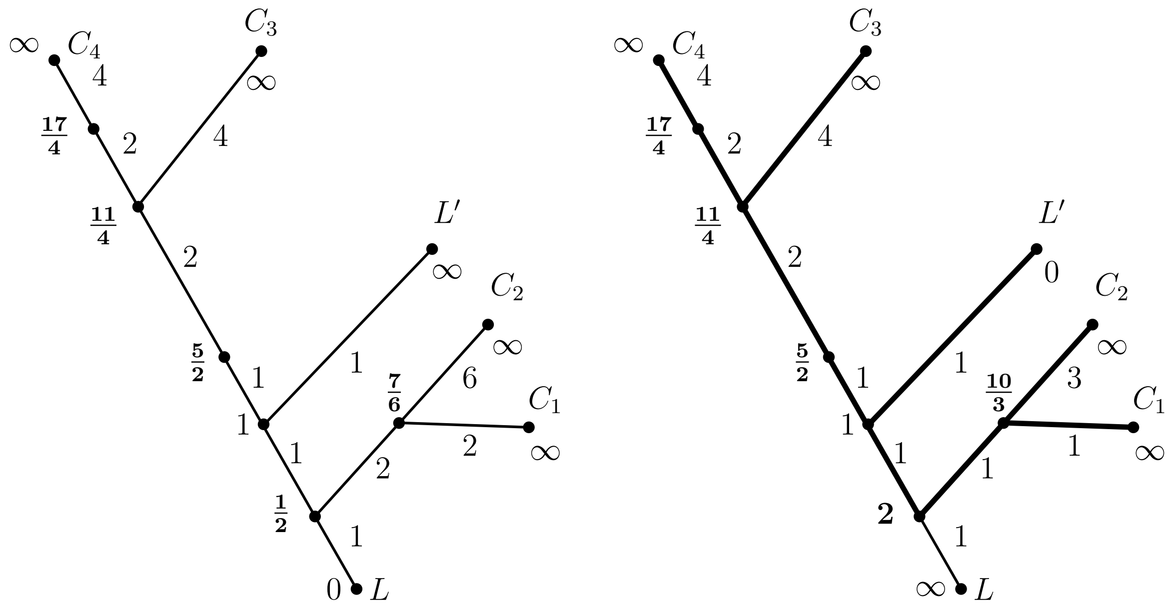
Figure 3 – Eggers-Wall tree $\Theta_L(C)$ of Example 2.46



Source: Elaborated by the author.

In Figure 4 are represented the Eggers-Wall trees $\Theta_L(C \cup L \cup L')$ and $\Theta_{L'}(C \cup L \cup L')$

Figure 4 – The Eggers-Wall tree $\Theta_L(C \cup L \cup L')$ of Example 2.46 on the left, compared with $\Theta_{L'}(C \cup L \cup L')$ on the right. The subtree $\Theta_{L'}(C)$ is in heavier lines.



Source: Elaborated by the author.

The carousel tree is a variant of the Eggers-Wall tree, but using all the Newton-Puiseux roots of C, not only one root for each branch. The name was introduced in (NEUMANN and PICHON, 2014) and it is inspired by the carousel geometrical model for the link of the curve C described in (WALL, 2004, Section 5.3).

Definition 2.47. *Let C be a germ of curve on $\mathcal{S}$. Let us denote by $[d_C]$ the set $\{1, \ldots, d_C\}$ and let $\eta_j, j \in [d_C]$ be the Newton-Puiseux roots relative to L of C. Consider the map* $\operatorname{ord}_x\colon [d_C] \times [d_C] \to \mathbb{Q} \cup \{\infty\}$, $(j,k) \mapsto \operatorname{ord}_x(\eta_j - \eta_k)$. *The map* ord_x *has the property that* $\operatorname{ord}_x(j,l) \geq \min\{\operatorname{ord}_x(j,k), \operatorname{ord}_x(k,l)\}$ *for any triple j,k,l. So for any $q \in \mathbb{Q} \cup \{\infty\}$, the relation on the set $[d_C]$ given by $j \sim_q k \Leftrightarrow \operatorname{ord}_x(j,k) \geq q$ is an equivalence relation. Name the elements of the set* $\operatorname{ord}_x([d_C] \times [d_C]) \cup \{0\}$ *in ascending order: $0 = q_0 < q_1 < \cdots < q_r = \infty$. For each $i = 0, \ldots, r$ let $G_{i,1}, \ldots, G_{i,\mu_i}$ be the equivalence classes for the relation $\sim_{q_i}$. So $\mu_r = d_C$ and the sets $G_{r,j}$ are singletons while $\mu_0 = \mu_1 = 1$ and $G_{0,1} = G_{1,1} = [d_C]$. We form a tree with these equivalence classes $G_{i,j}$ as vertices and edges given by inclusion relations: there is an edge between $G_{i,j}$ and $G_{i+1,k}$ if $G_{i+1,k} \subseteq G_{i,j}$. The vertex $G_{0,1}$ is the root of this tree and the singleton sets $G_{r,j}$ are the leaves. We weight each vertex with its corresponding q_i. The* ***carousel tree relative*** *to L is the tree obtained from this tree by suppressingvalency 2 vertices: we remove each such vertex and amalgamate its two adjacent edges into one edge.*

We will describe how one gets the Eggers-Wall tree from the carousel tree.

This process is essentially the same process described in (NEUMANN and PICHON, 2014, Lemma 3.1). At any vertex v of the carousel tree we have a weight q_v which is one of the q_i's. Let d_v be the denominator of the q_v when q_v is written as a quotient of coprime integers.

The process of obtaining the Eggers-Wall tree from the carousel tree is an induction process in i. First, we label the edge between $G_{0,1}$ and $G_{1,1}$ by 1. The subtrees cut off above $G_{1,1}$ consist of groups of $d_{G_{1,1}}$ isomorphic trees, with possibly one additional tree. We label the edge connecting $G_{1,1}$ to this additional tree, if it exists, with 1, and then delete all but one from each group of $d_{G_{1,1}}$ isomorphic trees. Finally, we label the remaining edges contain $G_{1,1}$ with $\mathrm{lcm}\{d_{G_{1,1}}, 1\}$.

Inductively, let v vertex with weight q_i. Let v' be the adjacent vertex below v along the path from v up to the root vertex and let $l_{vv'}$ be the label of the edge between v and v'. The subtrees cut off above v consist of groups of $\frac{\mathrm{lcm}\{d_v, l_{vv'}\}}{l_{vv'}}$ isomorphic trees, with possibly one additional tree. We label the edge connecting v to this additional tree, if it exists, with $l_{vv'}$, and then delete all but one from each group of $\frac{\mathrm{lcm}\{d_v, l_{vv'}\}}{l_{vv'}}$ isomorphic trees below v. Finally, we label the remain edges contain v with $\mathrm{lcm}\{d_v, l_{vv'}\}$.

The resulting tree, with the q_v labels at vertices and the extra label on the edges is easily recognized as the Eggers-Wall tree relative to L of C.

Example 2.48. *Let L be the y-axis. Consider a plane curve C whose branches A and B are parametrized by*

$$\gamma_A(w) = (w^4, w^6 + w^7),\ \gamma_B(w) = (w^2, w),$$

respectively. The Newton-Puiseux roots relative to L of A are

$$\eta_A(x^{1/8}) = x^{12/8} + x^{14/8}, \qquad \eta_A(\rho x^{1/8}) = \rho^4 x^{12/8} + \rho^6 x^{14/8},$$
$$\eta_A(\rho^2 x^{1/8}) = x^{12/8} + \rho^4 x^{14/8}, \qquad \eta_A(\rho^3 x^{1/8}) = \rho^4 x^{12/8} + \rho^2 x^{14/8},$$

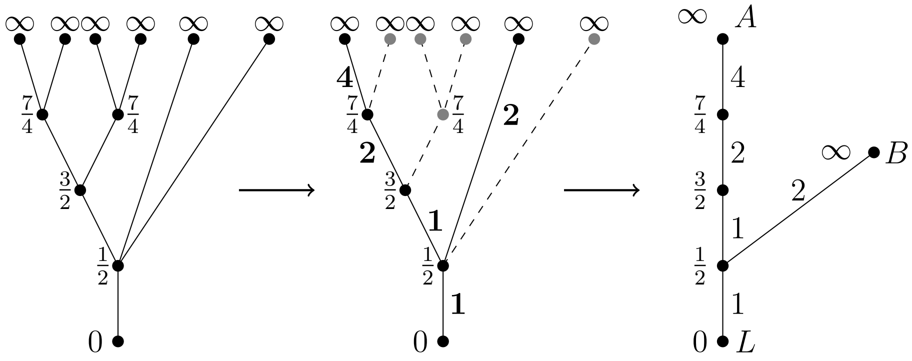
where ρ is a primitive 8-th root of unity. While the Newton-Puiseux roots relative to L of B are

$$\eta_B(x^{1/8}) = x^{4/8}, \qquad \eta_B(\rho x^{1/8}) = \rho^4 x^{4/8}.$$

The characteristic exponents relative to y-axis of A are $3/2, 7/4$. The characteristic exponent of B relative to y-axis is $1/2$.

Figure 5 illustrates the above process for the curve C.

Figure 5 – From the carousel tree to the Eggers-Wall tree.



Source: Elaborated by the author.

3 LIPSCHITZ GEOMETRY

Here we present our main result: a complete classification of complex plane algebraic curves, equipped with the induced Euclidean, up to global bilipschitz homeomorphism. In particular, we prove a theorem giving the complete classification of the Lipschitz geometry at infinity of complex algebraic plane curves. We synthesize combinatorial objects that encode both Lipschitz geometry and Lipschitz geometry at the infinity of complex algebraic plane curves.

3.1 Main result

Every subset X of the Euclidean space $\mathbb{R}^n$ can naturally be considered as a metric space: just consider the restriction of the Euclidean metric d to X, that is, use between the points of X the same distance they had as a point of $\mathbb{R}^n$. The induced metric on X by the Euclidean metric are called **outer metric** by some authors contrasting with another natural metric: the inner metric. The **inner metric** induced by the Euclidean metric is the function $d_i : X \times X \to [0, \infty]$ defined by

$$d_i(x, y) := \inf \operatorname{length}(\gamma),$$

where the infimum of the length is taken over all rectifiable curves $\gamma : [0, 1] \to X$ from x to y, i.e. $\gamma(0) = x, \gamma(1) = y$. The metric d_i is finite if and only if every pair of points in X can be joined by a rectifiable curve. Note that always $d(x, y) \leq d_i(x, y)$. Unless we explicitly specify otherwise, we always use the outer metric on subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$. For classification results for Lipschitz geometry using the inner metric see (BIRBRAIR, 1999; BIRBRAIR and MOSTOWSKI, 2000; BIRBRAIR, NEUMANN, and PICHON, 2014).

We are concerned with bilipschitz equivalence between metric spaces.

Definition 3.1. *Let* (M, d) *and* (M', d') *be two metric spaces. A map* $f : M \to M'$ *is*

Lipschitz *if there exists a real constant* $c > 0$ *such that*

$$d'(f(x), f(y)) \leq cd(x, y) \text{ for all } x, y \in M.$$

A Lipschitz map $f : M \to M'$ *is called* ***bilipschitz*** *if its inverse exists and it is Lipschitz. We say that* M *and* M' *are* ***bilipschitz equivalent*** *if there exists a bilipschitz map* $f : M \to M'$ *between them. The equivalence class of* M *in this relation is called the* ***Lipschitz geometry*** *of* M*.*

When $M = M'$ *and the identity* $\mathrm{id} : (M, d) \to (M, d')$ *is a bilipschitz map we say that the metrics* d *and* d' *are* ***equivalent****.*

Proposition 3.2. *Let* M *be a set endowed with two metrics* d *and* d'*. If there exists a bilipschitz map* $f : (M, d) \to (M, d')$*,* d *and* d' *are equivalent.*

Proof. There exists $k > 0$ such that

$$\frac{1}{k}d(x, y) \leq d'(f(x), f(y)) \leq kd(x, y),$$
$$\frac{1}{k}d'(x, y) \leq d(f^{-1}(x), f^{-1}(y)) \leq kd'(x, y).$$

Then we have

$$\frac{1}{k^2}d(x, y) \leq \frac{1}{k}d'(f(x), f(y)) \leq d(f^{-1}(f(x)), f^{-1}(f(y))) \leq kd'(f(x), f(y)) \leq k^2 d(x, y).$$

□

To study Lipschitz geometry of subsets of the Euclidean space $\mathbb{R}^n$ endowed with the induced Euclidean metric one can use any metric on $\mathbb{R}^n$ equivalent to the Euclidean metric. When we do not explicitly say which metric we are considering in $\mathbb{R}^n$, it is understood that we are using the Euclidean metric. There are two metrics in $\mathbb{R}^n$ that are of simpler formal handling, which we will use in $\mathbb{R}^n$, when convenient. They are

$$d_1(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|$$
$$d_\infty(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

For any $x, y \in \mathbb{R}^n$, the inequalities apply:

$$d_\infty(x, y) \leq d(x, y) \leq d_1(x, y) \leq nd_\infty(x, y).$$

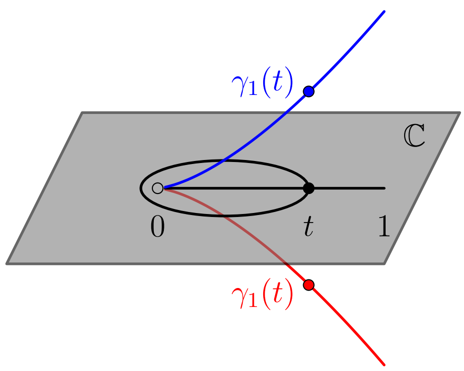
The bilipschitz equivalence is much coarser than isometric equivalence. For instance, germs of analytic curve at regular points, obviously, have the same Lipschitz geometry as $(\mathbb{C}, 0) \subset (\mathbb{C}^2, 0)$ while germs of analytic curves at smooth points are rarely isometric to $(\mathbb{C}, 0)$ since curvature can be interpreted as obstruction to flatness (locally isometric to the Euclidean space), see for instance (LEE, 1997, Theorem 7.3).

Also, the bilipschitz equivalence for germs of analytic curves is strictly finer than the equivalence up to homeomorphism.

Example 3.3. *The germ $(C, 0)$ of the cusp $C : y^2 = x^3$ is homeomorphic to $(\mathbb{C}, 0)$ (see Theorem 2.23), but it does not have the same Lipschitz geometry as the plane $(\mathbb{C}, 0)$. We prove the last claim.*

Suppose there exists a bilipschitz map $\psi : (C, 0) \to (\mathbb{C}, 0)$. Then ψ is bilipschitz for the inner metric and since the inner metric for $(\mathbb{C}, 0)$ is the same as the outer metric, this implies that $((C, 0), d_i)$ has the same Lipschitz geometry as $((C, 0), d)$. By Proposition 3.2, d_i and d are equivalent in $(C, 0)$. Consider the orthogonal projection $\pi : C \to \mathbb{C}, \pi(x, y) = x$. We see that π is a ramified cover with degree 2 and with critical set containing only the origin. We have two different liftings $\gamma_1(t) = (t, t^{3/2})$ and $\gamma_2(t) = (t, -t^{3/2})$ of the segment $r(t) = t, t > 0$. By monodromy theorem (see e.g. LEE, 2013, Proposition A.77(c)), any path in $C\backslash\{(0, 0)\}$ connecting $\gamma_1(t)$ and $\gamma_2(t)$ is the lifting of a loop, based at the point t which is not contractible in $\mathbb{C}\backslash\{0\}$ (see Fig.6).

Figure 6 – Liftings of $r(t)$.



Source: Elaborated by the author.

Thus, the length of such a path must be at least $2t$. Any path in C connecting $\gamma_1(t)$ and $\gamma_2(t)$ either goes through the origin or is a path in $C\backslash\{(0, 0)\}$. In either case, the length of such a path must be at least $2t$. It implies that the inner distance, $d_i(\gamma_1(t), \gamma_2(t))$, in C, between $\gamma_1(t)$ and $\gamma_2(t)$, is at least $2t$. It follows that

$$\lim_{t\to 0} \frac{||\gamma_1(t) - \gamma_2(t)||}{d_i(\gamma_1(t), \gamma_2(t))} \leq \lim_{t\to 0} \frac{t^{3/2}}{2t} = 0.$$

This contradicts the assumption that d_i is equivalent to d in $(C, 0)$.

In general, Neumann and Pichon (2014) proved for germs of analytic curve that the Lipschitz geometry is the same as the topological type.

Theorem 3.4. *Let $(C_1, 0) \subset (\mathbb{C}^2, 0)$ and $(C_2, 0) \subset (\mathbb{C}^2, 0)$ be two germs of analytic curves. The following are equivalent:*

1. *$(C_1, 0)$ and $(C_2, 0)$ have the same Lipschitz geometry;*
2. *there is a homeomorphism of germs $\phi\colon (C_1, 0) \to (C_2, 0)$, holomorphic except at 0, which is bilipschitz for the outer metric;*
3. *$(C_1, 0)$ and $(C_2, 0)$ have the same embedded topology;*
4. *there is a bilipschitz homeomorphism of germs $h\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ with $h(C_1) = C_2$.*

Since this result deals with germs of analytic curves, we only have a classification of the Lipschitz geometry for points with small distance between them. It is natural to ask what happens with points with big distance between them. To answer this, in some sense, Fernandes and Sampaio (2020) arrived to the notion of bilipschitz equivalence at infinity.

Definition 3.5. *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be two subsets. We say that X and Y are* **bilipschitz equivalent at infinity** *if there exist compact subsets $K \subset \mathbb{R}^n$ and $\widetilde{K} \subset \mathbb{R}^m$, and a bilipschitz map $\Phi\colon X\backslash K \to Y\backslash\widetilde{K}$. The equivalence class of X in this relation is called the **Lipschitz geometry at infinity** of X.*

Clearly, bilipschitz equivalence is more refined than bilipschitz equivalence at infinity and bilipschitz equivalence for germs. In fact, they are strictly coarser than the bilipschitz equivalence as the following examples show.

Example 3.6. *Since the parabola $P : x = y^2$ is smooth, (P, p) and $(\mathbb{C}, 0)$ have the same Lipschitz geometry for every point $p \in P$. The map $h : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $h(x, y) = (x-y^2, y)$ is a homeomorphism such that $h(P) = y$-axis. With similar arguments of example 3.3 one can prove that the parabola does not have the same Lipschitz geometry at infinity as a line.*

The following example shows that the number of irreducible components of an algebraic curve is not an invariant for the Lipschitz geometry at infinity, but it is, as expected, for the Lipschitz geometry, see Theorem 3.9.

Example 3.7. *The hyperbola $H : xy = 1$ and the axes $A : xy = 0$ does not have the same Lipschitz geometry since they are not even homeomorphic: $A\backslash\{(0, 0)\}$ has two connected components while $H\backslash\{p\}$ has only one for any point $p \in H$. But the hyperbole H has the same Lipschitz geometry at infinity as the axes A.*

Indeed, let $\Delta_2(0) = \{(x, y) \in \mathbb{C}^2 : |x| \leq 2, |y| \leq 2\}$ and consider the map $\psi : H\backslash\Delta_2(0) \to A\backslash\Delta_2(0)$ given by

$$\psi(x, y) = \begin{cases} (x, 0) & \text{if } |x| > 2, \\ (0, y) & \text{if } |x| \leq 2. \end{cases}$$

Let $(x_i, 1/x_i) \in H$ with $|x_i| > 2$ for $i = 1, 2$. We have

$$d_1\Big(\Big(x_1, \frac{1}{x_1}\Big), \Big(x_2, \frac{1}{x_2}\Big)\Big) = |x_1 - x_2| + \Big|\frac{1}{x_1} - \frac{1}{x_2}\Big| = |x_1 - x_2|\Big(1 + \frac{1}{|x_1 x_2|}\Big).$$

And thus

$$\frac{4}{5} d_1\Big(\Big(x_1, \frac{1}{x_1}\Big), \Big(x_2, \frac{1}{x_2}\Big)\Big) \le d_1\Big(\psi\Big(x_1, \frac{1}{x_1}\Big), \psi\Big(x_2, \frac{1}{x_2}\Big)\Big) \le d_1\Big(\Big(x_1, \frac{1}{x_1}\Big), \Big(x_2, \frac{1}{x_2}\Big)\Big).$$

Similarly, for $(1/y_i, y_i) \in H$ with $|y_i| > 2$ for $i = 1, 2$. Now, let $(x, 1/x), (1/y, y) \in H$ with $|x|, |y| > 2$. We have

$$d_1\Big(\Big(x, \frac{1}{x}\Big), \Big(\frac{1}{y}, y\Big)\Big) = \Big|x - \frac{1}{y}\Big| + \Big|\frac{1}{x} - y\Big| = (|x| + |y|)\Big(1 - \frac{1}{|xy|}\Big).$$

And thus

$$\frac{3}{4} d_1\Big(\Big(x_1, \frac{1}{x_1}\Big), \Big(x_2, \frac{1}{x_2}\Big)\Big) \le d_1\Big(\psi\Big(x_1, \frac{1}{x_1}\Big), \psi\Big(x_2, \frac{1}{x_2}\Big)\Big) \le d_1\Big(\Big(x_1, \frac{1}{x_1}\Big), \Big(x_2, \frac{1}{x_2}\Big)\Big).$$

Notice that H is irreducible while A is not. This means that the Lipschitz geometry at infinity does not tell us the number of irreducible components.

We will prove that the Lipschitz geometry at infinity of a complex algebraic plane curve C determines and is determined by the topological type of the germ of the curve $\widetilde{C} \cup L_\infty$ at each point at infinity of C. Since the topological types of germs of complex plane curves are encoded in dual resolution graphs of minimal good resolutions we also encode the Lipschitz geometry at infinity in a tree obtained as a quotient of dual resolution graphs as follows.

Theorem 3.8. *Let C and C' be two complex algebraic plane curves. The following are equivalent:*

1. *C and C' have the same Lipschitz geometry at infinity;*
2. *there is a bijection ψ between the set of points at infinity of C and the set of points at infinity of C' such that $(\widetilde{C} \cup L_\infty, p)$ has the same topological type as $(\widetilde{C'} \cup L_\infty, \psi(p))$;*
3. *there is an isomorphism between the Lipschitz tree at infinity of C and C' (see definition 3.10).*

Armed with the classification of the Lipschitz geometry of germs and of the Lipschitz geometry at infinity of complex algebraic plane curves we obtain our main result.

Theorem 3.9. *Let C and Γ be two complex plane algebraic curves with irreducible components $C = \bigcup_{i \in I} C_i$ and $\Gamma = \bigcup_{j \in J} \Gamma_j$. The following are equivalent:*

1. *C and Γ have the same Lipschitz geometry;*
2. *there are bijections $\sigma : I \to J$ and φ between the set of singular points of $\widetilde{C} \cup L_\infty$*

and the set of singular points of $\widetilde{\Gamma} \cup L_\infty$ *such that* $p \in L_\infty$ *if only if* $\varphi(p) \in L_\infty$*,* $(\widetilde{C} \cup L_\infty, p)$ *has the same topological type as* $(\widetilde{\Gamma} \cup L_\infty, \varphi(p))$*, and each* $(\widetilde{C}_i \cup L_\infty, p)$ *has the same topological type as* $(\widetilde{\Gamma}_{\sigma(i)} \cup L_\infty, \varphi(p))$*;*

3. *there is an isomorphism between the Lipschitz graph of* C *and* Γ *(see definition 3.16).*

3.2 Lipschitz geometry at infinity determines topological type

In this section, we define the Lipschitz tree at infinity of a complex algebraic plane curve. Then we prove the equivalence of (2) and (3) and that (1) implies (2) of Theorem 3.8.

To define the Lipschitz tree at infinity of a complex algebraic plane curve we recall the basic vocabulary of resolution of singularities. Let $(C, p) \subset (\mathcal{S}, p)$ be a germ of a singular complex curve. We remember that the blowing up of $\mathcal{S}$ with centre p produces a smooth surface $\mathcal{S}_1$, a holomorphic map $\pi_1 : \mathcal{S}_1 \to \mathcal{S}$ such that $\pi_1 : \mathcal{S}_1 \backslash \pi_1^{-1}(p) \to \mathcal{S} \backslash \{p\}$ is biholomorphic, the **exceptional curve** $E_1 = \pi_1^{-1}(p)$, and the **strict transform** C_1 which is the topological closure $\overline{\pi_1^{-1}(C \backslash \{p\})}$. The map π_1 is called the **blowing up** of $\mathcal{S}$ with centre p. A **good minimal resolution** of C is a map $\pi : \mathcal{S}_n \to \mathcal{S}$ which is a composite of finite and minimal sequence of blowing ups $\pi_i : \mathcal{S}_i \to \mathcal{S}_{i-1}$ such that the strict transform $C_n = \overline{\pi^{-1}(C \backslash \{p\})}$ is smooth and meets the exceptional curves $\pi^{-1}(p) = E_1 \cup E_2 \cup \cdots \cup E_n$ transversely at a regular point.

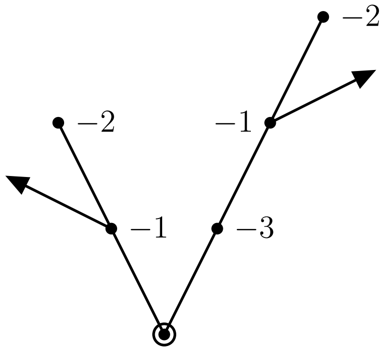
Definition 3.10. *Let* C *be a complex algebraic plane curve,* $p_1, \ldots, p_m$ *its points at infinity and let* $B_1^{(j)}, \ldots, B_{k_j}^{(j)}$ *be the branches of* $(\widetilde{C}, p_j)$*. A* ***good minimal resolution*** *of* $(\widetilde{C} \cup L_\infty, p_1)$ *produces a smooth surface* $S_{(1)}$*, a projection* $\pi_{(1)} : S_{(1)} \to \mathbb{P}$*, a sequence of exceptional curves* $E_1^{(1)}, \ldots, E_{r_1}^{(1)}$ *and strict transform curves* $\mathcal{B}_1^{(1)}, \ldots, \mathcal{B}_{k_1}^{(1)}$ *of the branches* $B_1^{(1)}, \ldots, B_{k_1}^{(1)}$ *and the strict transform* $\mathcal{L}_\infty$ *of the line at infinity* L_∞*. Then, we resolve the strict transform* $C^{(1)} = \pi_{(1)}^{-1}(\widetilde{C})$ *at the singular point* $\pi_{(1)}^{-1}(p_2)$*. We repeat this process for all points at infinity of* C*.*

The ***Lipschitz tree at infinity*** *of* C *is a rooted tree with vertices* $V_k^{(j)}$ *corresponding to the curves* $E_k^{(j)}$ *labeled with its self-intersection number, arrow vertices* $W_i^{(j)}$ *corresponding to the branches* $\mathcal{B}_i^{(j)}$ *not labeled and a root corresponding to the strict transform* $\mathcal{L}_\infty$ *of the line at infinity. We put an edge joining vertices if and only if the corresponding curves intersect.*

Example 3.11. *The Lipschitz tree at infinity of the complex algebraic plane curve defined by* $(y - x^2)(y^3 - x) = 0$ *is drawn in Figure 7.*

We point out that the Lipschitz tree at infinity of C is obtained as the quotient of the disjoint union of the individual dual resolution graph of minimal good resolutions of $(\widetilde{C} \cup L_\infty, p_i)$, by identifying all vertices corresponding to the strict transform of the line at infinity and put it as the root. We recall that by (WALL, 2004, Theorem 8.1.7), the

Figure 7 – Lipschitz tree at infinity of Example 3.11.



Source: Elaborated by the author.

isomorphism class of the dual resolution graph of a minimal good resolutions of germ of complex curve at singular point determines and it is determined by its topological type. This explain the equivalence (2) $\Leftrightarrow$ (3) of Theorem 3.8.

For the implication (1) $\Rightarrow$ (2) of Theorem 3.8, we introduce the asymptotic notations of Bachmann-Landau which are convenient for study of Lipschitz geometry. See (KNUTH, 1976) for a historical survey about these notation.

Definition 3.12. *Let $f, g : (0, +\infty) \to (0, +\infty)$ be positive functions. We say*

1. *f **is big-Theta of** g, and we write $f(t) = \Theta(g(t))$, if there exists $R_0 > 0$ and a constant $c > 0$ such that $\frac{1}{c}g(t) \leq f(t) \leq cg(t)$ for all $t > R_0$.*
2. *f **is small-o of** g, and we write $f(t) = o(g(t))$, if $\limsup_{t\to\infty} \frac{f(t)}{g(t)} = 0$.*

Lemma 3.13. *Let C be a complex algebraic plane curve, and let $P : \mathbb{C}^2 \to \mathbb{C}$ be a linear projection whose kernel does not contain any tangent line at infinity to C. Then there exist a compact set K and a constant $M > 1$ such that for each $u, u' \in C\backslash K$, there is an arc $\tilde{\alpha}$ in $C\backslash K$ joining u to a point u'' with $P(u'') = P(u')$ and*

$$d(u, u') \leq \text{length}(\tilde{\alpha}) + d(u'', u') \leq Md(u, u').$$

Proof. After a linear change of coordinates if necessary, we may assume that P is the projection on the first coordinate and that the y-axis is not a tangent line at infinity to C. Let $[1 : a_1 : 0], \ldots, [1 : a_m : 0]$ be the points at infinity of C. For each i, let $B_{i1}, \ldots, B_{ik_i}$ be the branches of $(\widetilde{C}, [1 : a_i : 0])$.

The open set $U = \{[x : y : z] \in \mathbb{P}^2 : x \neq 0\}$ contains all the points at infinity of C, so we can use the coordinate chart $\varphi : U \to \mathbb{C}^2$ defined by $\varphi([x : y : z]) = (z/x, y/x)$ to obtain Newton-Puiseux parametrization of the branch $\varphi(B_{ij})$ for each i. Let $\epsilon > 0$ sufficiently small such that there exists Newton-Puiseux parametrization $\gamma_{ij} : D_\epsilon \to \mathbb{C}^2$

of $\varphi(B_{ij})$ given by

$$\gamma_{ij}(w) = (w^{d_{ij}}, a_i + v_{ij}(w)),$$

where D_ϵ is the open disk of radius ϵ centered at the origin and $v_{ij} \in \mathbb{C}\{w\}, v_{ij}(0) = 0$. Let $\Gamma_{ij} : D_\epsilon \backslash \{0\} \to \mathbb{C}^2$ given by

$$\Gamma_{ij}(w) = (\iota^{-1} \circ \varphi^{-1} \circ \gamma_{ij})(w) = \left(\frac{1}{w^{d_{ij}}}, \frac{a_i + v_{ij}(w)}{w^{d_{ij}}} \right).$$

We will prove that the compact $K = C \backslash \bigcup_{ij} \Gamma_{ij}(D_\epsilon \backslash \{0\})$ satisfies the desired conditions.

We claim that there exists a constant $c > 0$ such that $C \backslash K$ is a subset of the cone $\{(x, y) \in \mathbb{C}^2; |y| \leq c|x|\}$. Moreover, c may be chosen such that the tangent space of $C \backslash K$ at a point p, denoted by T_pC, is also a subset of the same cone.

The first part of this statement is easy to check. In particular, it follows that $P|_{\Gamma_{ij}(D_\epsilon \backslash \{0\})}$ is a covering map for all i, j. Differentiating Γ_{ij} gives

$$\Gamma'_{ij}(w) = \left(-\frac{d_{ij}}{w^{d_{ij}+1}}, \frac{wv'_{ij}(w) - d_{ij}v_{ij}(w)}{w^{d_{ij}+1}} - a_i \frac{d_{ij}}{w^{d_{ij}+1}} \right).$$

Thus the points $(x, y) \in T_{\Gamma_{ij}(w)}C$ satisfies $|y - a_ix| \leq \eta_{ij}|x| \Rightarrow |y| \leq (\eta_{ij} + |a_i|)|x|$ where $\eta_{ij} = \sup \left| \frac{wv'_{ij}(w) - d_{ij}v_{ij}(w)}{d_{ij}} \right|$. Now, putting $c = \max_{ij}\{\eta_{ij} + |a_i|\}$ we have

$$T_pC \subset \{(x, y) \in \mathbb{C}^2; |y| \leq c|x|\} \text{ for all } p \in C \backslash K,$$

as claimed.

Suppose $u, u' \in C \backslash K$ are arbitrary. Let i_0, j_0, i'_0, j'_0 such that $u \in \Gamma_{i_0j_0}(D_\epsilon \backslash \{0\})$ and $u' \in \Gamma_{i'_0j'_0}(D_\epsilon \backslash \{0\})$ and suppose that $1/\epsilon^{d_{i_0j_0}} \leq 1/\epsilon^{d_{i'_0j'_0}}$. Let $R = 1/\epsilon^{d_{i_0j_0}}$ and choose a path $\alpha : [0, 1] \to \mathbb{C} \backslash D_R$ such that $\alpha(0) = P(u), \alpha(1) = P(u')$ and $\text{length}(\alpha) \leq \pi R|P(u) - P(u')|$. Consider the lifting $\tilde{\alpha}$ of α by $P|_{\Gamma_{i_0j_0}(D_\epsilon \backslash \{0\})}$ with origin u and let u'' be its end. We obviously have

$$d(u, u') \leq \text{length}(\tilde{\alpha}) + d(u', u'').$$

On the other hand, since P is linear, $dP_p = P|_{T_pC}$. Thus

$$\frac{1}{\sqrt{1 + c^2}} \leq ||dP_p|| \leq 1 \text{ for all } p \in C \backslash K.$$

In particular, $\text{length}(\tilde{\alpha}) \leq \sqrt{1 + c^2}\,\text{length}(\alpha) \leq \pi R\sqrt{1 + c^2}|P(u) - P(u')|$, as $|P(u) - P(u')| \leq d(u, u')$, we obtain

$$\text{length}(\tilde{\alpha}) \leq \pi R\sqrt{1 + c^2}d(u, u').$$

If we join the segment $[u,u']$ to $\tilde{\alpha}$ at u, we have a curve from u' to u'', so $d(u',u'') \leq (1+\pi R\sqrt{1+c^2})d(u,u')$. Finally,

$$\operatorname{length}(\tilde{\alpha}) + d(u',u'') \leq (1+2\pi R\sqrt{1+c^2})d(u,u'),$$

and the constant $M = 1+2\pi R\sqrt{1+c^2}$ satisfies the desired conditions. □

Remark 3.14. *In the above lemma, we prove that $P|_{C\backslash K} : C\backslash K \to \mathbb{C}\backslash P(K)$ is a covering map. Moreover, $P|_{C\backslash K}$ has derivative bounded above and below by positive constants. In particular, for a non-constant arc α the quotient*

$$\operatorname{length}(\tilde{\alpha})/\operatorname{length}(\alpha)$$

is bounded above and below by positive constants.

The demonstration technique of (1) ⇒ (2) the Theorem 3.8 is similar to the case of germ of complex curves in (NEUMANN and PICHON, 2014). In particular, it is based on a so-called "bubble trick" argument.

Proof of (1) ⇒ (2) *of Theorem 3.8.* We first prove that the Lipschitz geometry at infinity gives us the number of points at infinity. Let $f \in \mathbb{C}[x,y]$ be a polynomial that defines C which does not have multiple factors. Let $n = \deg f$, then by a linear change of coordinates if necessary, we can assume that the monomial y^n has coefficient equal to 1 in f.

The points at infinity of C are the points $[x:y:0] \in \mathbb{P}^2$ satisfying $f_n(x,y) = 0$, where f_n denotes the homogeneous polynomial composed by the monomials in f of degree n, so $[0:1:0]$ is not a point at infinity of C.

We claim that there are constant $c > 0$ and an open Euclidean ball $B_{R_0}(0)$ of radius R_0 centered at the origin such that $|y| \leq c|x|$ for all $(x,y) \in C\backslash B_{R_0}(0)$. Indeed, otherwise, there exists a sequence $\{z_k = (x_k,y_k)\} \subset C$ such that $\lim\limits_{k\to+\infty} \|z_k\| = +\infty$ and $|y_k| > k|x_k|$. Thus, taking a subsequence, one can suppose that $\lim\limits_{k\to+\infty} \frac{y_k}{|y_k|} = y_0$ for some y_0 such that $|y_0| = 1$. Since $\frac{|x_k|}{|y_k|} < \frac{1}{k}$, $\lim\limits_{k\to+\infty} \frac{z_k}{\|z_k\|} = (0,y_0)$. On the other hand,

$$0 = f(z_k) = f\left(\|z_k\|\frac{z_k}{\|z_k\|}\right) = \|z_k\|^n \sum_{i=0}^{n} \frac{1}{\|z_k\|^{n-i}} f_i\left(\frac{z_k}{\|z_k\|}\right),$$

where f_i denotes the homogeneous polynomial composed by the monomials in f of degree i. This implies that

$$0 = f(z_k) = \sum_{i=0}^{n} \frac{1}{\|z_k\|^{n-i}} f_i\left(\frac{z_k}{\|z_k\|}\right),$$

Letting $k \to \infty$ yields $f_n(0, y_0) = 0$, which implies that $[0 : 1 : 0]$ is a point at infinity of C, this is a contradiction. Therefore, the claim is true.

Now, let $[1 : a_j : 0], j = 1, \ldots, m \leq n$ be the points at infinity of C. We define cones

$$V_j := \{(x, y) \in \mathbb{C}^2 : |y - a_j x| \leq \epsilon |x|\}$$

where $\epsilon > 0$ is small enough that the cones are disjoint except at 0. Then increasing $R_0 > 0$, if necessary,

$$C \backslash B_{R_0}(0) \subset \bigcup_{j=1}^{m} V_j.$$

Indeed, otherwise, there exists a sequence $\{z_k = (x_k, y_k)\} \subset C$ such that $\lim_{k \to +\infty} \|z_k\| = +\infty$ and $|y_k - a_j x_k| > \epsilon |x_k|$ for all $j = 1, \ldots, m$. Again, since $\|z_k\| \to +\infty$ as $k \to \infty$, we have

$$\lim_{k \to \infty} f_n\left(\frac{z_k}{\|z_k\|}\right) = 0.$$

On the other hand, writing $f_n(x, y) = \prod_{j=1}^{m} (y - a_j x)^{d_j}$, where d_j is a positive integer such that $n = \sum_{1 \leq j \leq m} d_j$, we have

$$\left\| f_n\left(\frac{z_k}{\|z_k\|}\right) \right\| = \frac{\prod_{j=1}^{m} |y_k - a_j x_k|^{d_j}}{\|z_k\|^n} \geq \left(\frac{\epsilon |x_k|}{\|z_k\|}\right)^n.$$

But, because of the first claim, we have

$$\frac{|x_k|}{\|z_k\|} = \frac{1}{\sqrt{1 + \left|\frac{y_k}{x_k}\right|^2}} \geq \frac{1}{\sqrt{1 + c^2}},$$

which derives a contradiction.

We denote by $\mathcal{C}_j$ the part of $C \backslash B_{R_0}(0)$ inside V_j. Now, let $K, K' \subset \mathbb{C}^2$ be compact sets such that there is a bilipschitz map $\Phi : C \backslash K \to C' \backslash K'$. Let $[1 : a'_j : 0], j = 1, \ldots, m'$ be the points at infinity of C'. We repeat the above arguments for C', then increasing $R_0 > 0$, if necessary,

$$C' \backslash B_{R_0}(0) \subset \bigcup_{j=1}^{m'} V'_j, \text{ where } V'_j := \{(x, y) \in \mathbb{C}^2 : |y - a'_j x| \leq \epsilon |x|\}.$$

Likewise, denote by $\mathcal{C}'_j$ the set $(C' \backslash B_{R_0}(0)) \cap V'_j$. We have $\Phi(C \backslash B_R(0)) \subset C' \backslash B_{h(R)}(0)$ with $h(R) = \Theta(R)$. Since $\operatorname{dist}(\mathcal{C}_j \backslash B_R(0), \mathcal{C}_k \backslash B_R(0)) = \Theta(R)$ we have

$$\operatorname{dist}(\Phi(\mathcal{C}_j \backslash B_R(0)), \Phi(\mathcal{C}_k \backslash B_R(0))) = \Theta(R).$$

Notice that the sets $\mathcal{C}'_l, l = 1, \ldots, m'$ have the following property: the distance between any two connected component of $\mathcal{C}'_l$ outside a ball of radius $h(R)$ around 0 is $o(R)$. Then, we cannot have

$$\Phi(\mathcal{C}_j \backslash B_R(0)) \subset \mathcal{C}'_l \backslash B_{h(R)}(0) \text{ and } \Phi(\mathcal{C}_k \backslash B_R(0)) \subset \mathcal{C}'_l \backslash B_{h(R)}(0)$$

for $k \neq j$ then $m \leq m'$ and using the inverse Φ^{-1} we get $m = m'$.

Now, we obtain the topological type of $\widetilde{C} \cup L_\infty$ at the points at infinity. Without loss of generality, we can suppose that $[1 : a_1 : 0] = [1 : 0 : 0]$ is a point at infinity for C. We extract the characteristic and the coincidence exponents relative to L_∞ of the curve $(\widetilde{C} \cup L_\infty, [1 : 0 : 0])$ using the coordinate system and the induced Euclidean metric d on $\mathcal{C}_1$. Next, we prove that these data determine the topology type of $(\widetilde{C} \cup L_\infty, [1 : 0 : 0])$. Finally, we prove that these data can be obtained without using the chosen coordinate system and even using the equivalent metric d' induced by Φ, for this we operate the "bubble trick".

Let $U = \{[x : y : z] \in \mathbb{P}^2 : x \neq 0\}$ and consider the coordinate chart $\varphi : U \to \mathbb{C}^2$ defined by $\varphi([x : y : z]) = (z/x, y/x) = (u, v)$. In this local coordinates, $\varphi([1 : 0 : 0])$ is the origin and we have $\mathrm{ord}_v(\widetilde{f} \circ \varphi^{-1})(0, v) = d_1$. Let $B_1, \ldots, B_{k_1}$ be the branches of $(\varphi(\widetilde{C} \cap U), 0)$. Every branch of the curve $(\varphi(\widetilde{C} \cap U), 0)$ has a Newton-Puiseux parametrization of the form

$$\gamma_s(w) = \left(w^{d_{1s}}, \sum_{k>0} a_{sk} w^k \right),$$

where d_{1s} are positive integers such that $\sum_{s=1}^{k_1} d_{1s} = d_1$. Then, increasing $R_0 > 0$ if necessary, the images of the maps

$$\Gamma_s(w) = (\iota^{-1} \circ \varphi^{-1} \circ \gamma)(w) = \left(\frac{1}{w^{d_{1s}}}, \frac{1}{w^{d_{1s}}} \sum_{k>0} a_{sk} w^k \right), s = 1, \ldots, k_1$$

cover $\mathcal{C}_1$. Therefore, the lines $x = t$ for $t \in (R_0, \infty)$ intersect $\mathcal{C}_1$ in d_1 points $p_1(t), \ldots, p_{d_1}(t)$ which depend continuously on t. Denote by $[d_1]$ the set $\{1, \ldots, d_1\}$. For each $j, k \in [d_1]$ with $j < k$, the distance $d(p_j(t), p_k(t))$ has the form $\Theta(t^{1-q(j,k)})$, where $q(j, k) = q(k, j)$ is either a characteristic Puiseux exponent relative to L_∞ for a branch of the plane curve $(\widetilde{C} \cup L_\infty, [1 : 0 : 0])$ or a coincidence exponent relative to L_∞ between two branches of $(\widetilde{C} \cup L_\infty, [1 : 0 : 0])$. For $j \in [d_1]$ define $q(j, j) = \infty$.

Lemma 3.15. *The map* $q \colon [d_1] \times [d_1] \to \mathbb{Q} \cup \{\infty\}$, $(j, k) \mapsto q(j, k)$, *determines the topological type of* $(\widetilde{C} \cup L_\infty, [1 : 0 : 0])$.

Proof. The topological type of $(\widetilde{C} \cup L_\infty, [1 : 0 : 0])$ is encoded by its Eggers-Wall tree relative to a smooth branch $\mathcal{L}$ transversal to $(\widetilde{C} \cup L_\infty, [1 : 0 : 0])$ (see Proposition 2.44).

To prove the lemma we notice that the function q is the same as the function ord_x of Definition 2.47. By the process described in Section 2.3, one obtains the Eggers-Wall tree relative to L_∞ of $(\widetilde{C} \cup L_\infty, [1:0:0])$. By applying the Inversion Theorem for Eggers-Wall tree 2.45 to $\Theta_{L_\infty}(\widetilde{C} \cup L_\infty \cup \mathcal{L}, [1:0:0])$, one gets the Eggers-Wall tree $\Theta_{\mathcal{L}}(\widetilde{C} \cup L_\infty, [1:0:0])$.

□

As already noted, this discovery of the topology type involved the chosen coordinate system and the metric d. We must show we can discover it using d' and without using the chosen coordinate system.

The points $p_1(t), \dots, p_{d_1}(t)$ that we used to find the numbers $q(j,k)$ were obtained by intersecting $\mathcal{C}_1$ with the line $x = t$. The arc $t \in (R_0, \infty) \mapsto p_1(t)$ satisfies

$$d(0, p_1(t)) = \Theta(t). \tag{1}$$

Moreover, the other points $p_2(t), \dots, p_{d_1}(t)$ are in the disk of radius ηt centered at $p_1(t)$ in the plane $x = t$. Here, $\eta > 0$ can be as small as we like, so long as R_0 is then chosen sufficiently big.

Instead of a disk of radius ηt, we can use a ball $B(p_1(t), \eta t)$ of radius ηt centered at $p_1(t)$. This ball $B(p_1(t), \eta t)$ intersects $\mathcal{C}_1$ in d_1 topological disks $D_1(\eta t), \dots, D_{d_1}(\eta t)$, named such that $p_l(t) \in D_l(\eta t), l = 1, \dots, d_1$ and thus $\mathrm{dist}(D_j(\eta t), D_k(\eta t)) \leq d(p_j(t), p_k(t))$. On the other hand, let $\widetilde{p}_l(t) \in D_l(\eta t), l = 1, \dots, d_1$ such that

$$\mathrm{dist}(D_j(\eta t), D_k(\eta t)) = d(\widetilde{p}_j(t), \widetilde{p}_k(t)).$$

Consider the projection $P \colon \mathbb{C}^2 \to \mathbb{C}$ given by $P(x, y) = x$ and let α_t be the segment in $\mathbb{C}$ joining $P(\widetilde{p}_j(t))$ to $P(\widetilde{p}_k(t))$ and let $\tilde{\alpha}_t$ be the lifting of α_t by the restriction $P|_{C \setminus B_{R_0}(0)}$ with origin $\widetilde{p}_k(t)$. Applying Lemma 3.13 to P with $u = \widetilde{p}_k(t)$ and $u' = \widetilde{p}_j(t)$, we then obtain

$$d(\widetilde{p}_j(t), \widetilde{p}_k(t)) \geq \frac{1}{M}(\mathrm{length}(\tilde{\alpha}_t) + d(\widetilde{p}_j(t), \tilde{\alpha}_t(1))) \geq \frac{1}{M} d(\widetilde{p}_j(t), \tilde{\alpha}_t(1)).$$

But $d(\widetilde{p}_j(t), \tilde{\alpha}_t(1)) = \Theta(t^{1-q(j,k)})$ since $P(\widetilde{p}_j(t)) = P(\tilde{\alpha}_t(1))$ and $|P(\widetilde{p}_j(t))| = \Theta(t)$.

We now replace the arc p_1 by any continuous arc on $\mathcal{C}_1$ satisfying (1) and we still denote this new arc by p_1. If η is sufficiently small it is still true that $B_{\mathcal{C}_1}(p_1(t), \eta t) := \mathcal{C}_1 \cap B(p_1(t), \eta t)$ consists of d_1 disks $D_1(\eta t), \dots, D_{d_1}(\eta t)$ with $\mathrm{dist}\big(D_j(\eta t), D_k(\eta t)\big) = \Theta(t^{1-q(j,k)})$. So at this point, we have gotten rid of the dependence on the choice of coordinate system in discovering the topology, but not yet of the dependence on the metric d.

A L-bilipschitz change to the metric may make the components of $B_{\mathcal{C}_1}(p_1(t), \eta t)$ disintegrate into many pieces, so we can no longer simply use distance between all pieces.

To resolve this, we consider $B_{\mathcal{C}_1}(p_1(t), \eta t/L)$ and $B_{\mathcal{C}_1}(p_1(t), \eta Lt)$. Note that

$$B_{\mathcal{C}_1}(p_1(t), \eta t/L) \subset B'_{\mathcal{C}_1}(p_1(t), \eta t) \subset B_{\mathcal{C}_1}(p_1(t), \eta Lt),$$

where B' means we are using the modified metric d'.

Denote by $D_j(\eta t/L)$ and $D_j(\eta Lt), j = 1, \ldots, d_1$ the disk of $B_{\mathcal{C}_1}(p_1(r), \eta t/L)$ and $B_{\mathcal{C}_1}(p_1(r), \eta Lt)$, respectively, so that $D_j(\eta t/L) \subset D_j(\eta Lt)$ for $j = 1, \ldots, d_1$. Thus $B'_{\mathcal{C}_1}(p_1(t), \eta t)$ has d_1 components such that each one contains at most one component of $B_{\mathcal{C}_1}(p_1(r), \eta t/L)$. Therefore, exactly d_1 components of $B'_{\mathcal{C}_1}(p_1(t), \eta t)$ intersect $B_{\mathcal{C}_1}(p_1(t), \eta t/L)$. Naming these components $D'_1(\eta t), \ldots, D'_{d_1}(\eta t)$, such that $D_j(\eta t/L) \subset D'_j(\eta t) \subset D_j(\eta Lt), j = 1, \ldots, d_1$, we still have $\text{dist}(D'_j(\eta t), D'_k(\eta t)) = \Theta(t^{1-q(j,k)})$ since

$$\text{dist}(D_j(\eta Lt), D_k(\eta Lt)) \leq \text{dist}(D'_j(\eta t), D'_k(\eta t)) \leq \text{dist}(D_j(\eta t/L), D_k(\eta t/L)).$$

So the $q(j, k)$ are determined by the distance between $D'_j(\eta t), j = 1, \ldots, d_1$.

Up to now, we have used the metric d to select the components $D'_j(\eta t), j = 1, \ldots, d_1$ of $B'_{\mathcal{C}_1}(p_1(t), \eta t)$. To avoid using the metric d, consider $B'_{\mathcal{C}_1}(p_1(t), \eta t/L^2)$. We have

$$B_{\mathcal{C}_1}(p_1(t), \eta t/L^3) \subset B'_{\mathcal{C}_1}(p_1(t), \eta t/L^2) \subset B_{\mathcal{C}_1}(p_1(t), \eta t/L) \subset D'_1(\eta t) \cup \cdots \cup D'_{d_1}(\eta t).$$

This implies that $B'_{\mathcal{C}_1}(p_1(t), \eta t/L^2)$ intersects exactly the components $D'_j(\eta t), j = 1, \ldots, d_1$ of $B'_{\mathcal{C}_1}(p_1(t), \eta t)$. So we can only use the metric d' to select these components and we are done.

□

3.3 Topological type determines Lipschitz geometry at infinity

In this subsection, we prove that (2) implies (1) of Theorem 3.8. For this, we will construct a bilipschitz map between complex algebraic plane curves with the same data in (2).

Proof of the implication (2) ⇒ (1) *of Theorem 3.8.* Let C_1 and C_2 be complex algebraic plane curves with the same data described by (2) of of Theorem 3.8. Choose (x, y) coordinates in such a way that none of the curves have the point $[0 : 1 : 0]$ as a point at infinity.

Let $[1 : a^l_1 : 0], \ldots, [1 : a^l_{m_l} : 0]$ be the points at infinity of $C_l, l = 1, 2$, denoted in such a way that $(\widetilde{C}_1 \cup L_\infty, [1 : a^1_i : 0])$ has the same topological type as $(\widetilde{C}_2 \cup L_\infty, [1 : a^2_i : 0])$. Then, by 2.44, for any smooth branch L_1 (resp. L_2) through $[1 : a^1_i : 0]$ (resp. $[1 : a^2_i : 0]$) transversal to $(C_1 \cup L_\infty, [1 : a^1_i : 0])$ (resp. $(C_1 \cup L_\infty, [1 : a^2_i : 0])$) the Eggers-Wall trees $\Theta_{L_1}(\widetilde{C}_1 \cup L_\infty, [1 : a^1_i : 0])$ and $\Theta_{L_2}(\widetilde{C}_2 \cup L_\infty, [1 : a^2_i : 0])$ are isomorphic. Then, we apply the

Inversion Theorem for Eggers-Wall tree 2.45 to both and we get that $\Theta_{L_\infty}(\widetilde{C}_1, [1 : a_i^1 : 0])$ and $\Theta_{L_\infty}(\widetilde{C}_2, [1 : a_i^2 : 0])$ are isomorphic.

For each i, let $B_{i1}^l, \ldots, B_{ik_i}^l$ be the branches of $(\widetilde{C}_l, [1 : a_i^l : 0]), l = 1, 2$. Again, we denoted in such a way that $(B_{ij}^1, [1 : a_i^1 : 0])$ has the same topological type as $(B_{ij}^2, [1 : a_i^2 : 0])$. From what has been said above, we have that B_{ij}^1 and B_{ij}^2 have the same characteristic exponents relative to L_∞ and $k_{L_\infty}(B_{ij}^1, B_{ij'}^1) = k_{L_\infty}(B_{ij}^2, B_{ij'}^2)$.

The open set $U = \{[x : y : z] \in \mathbb{P}^2 : x \neq 0\}$ contains all the points at infinity of $C_l, l = 1, 2$. We can use the coordinate chart $\varphi : U \to \mathbb{C}^2$ defined by $\varphi([x : y : z]) = (z/x, y/x)$ to obtain a Newton-Puiseux parametrization of the branches $\varphi(B_{ij}^l)$. Let D_{ϵ_0} be the open disk of radius $\epsilon_0 > 0$ centered at the origin with ϵ_0 sufficiently small such that there exist Newton-Puiseux parametrization $\gamma_{ij}^l : D_{\epsilon_0} \to \mathbb{C}^2$ of $\varphi(B_{ij}^l)$ given by

$$\gamma_{ij}^l(w) = \left(w^{d_{ij}}, a_i^l + \sum_{k>0} a_{ijk}^l w^k \right).$$

Let $\Gamma_{ij}^l : D_{\epsilon_0} \backslash \{0\} \to \mathbb{C}^2$ given by

$$\Gamma_{ij}^l(w) = (\iota^{-1} \circ \varphi^{-1} \circ \gamma_{ij}^l)(w) = \left(\frac{1}{w^{d_{ij}}}, \frac{a_i^l + \sum_{k>0} a_{ijk}^l w^k}{w^{d_{ij}}} \right), l = 1, 2.$$

Consider the compact set $K_\epsilon^l = C_l \backslash \bigcup_{ij} \Gamma_{ij}^l(D_\epsilon \backslash \{0\}), l = 1, 2$. We will prove that there exists $\epsilon > 0$ that the map

$$\begin{array}{rcl} \Phi : C_1 \backslash K_\epsilon^1 & \longrightarrow & C_2 \backslash K_\epsilon^2 \\ \Gamma_{ij}^1(w) & \longmapsto & \Gamma_{ij}^2(w) \end{array}$$

is bilipschitz.

Claim. *Consider the projection $P \colon \mathbb{C}^2 \to \mathbb{C}$ given by $P(x, y) = x$. In order to check that Φ is a Lipschitz map it is enough to consider points in $C_1 \backslash K_\epsilon^1$ with the same x coordinate. That is, there exists a constant $c > 0$ such that*

$$d\big(\Gamma_{ij}^2(w'), \Gamma_{i'j'}^2(w'')\big) \leq c d\big(\Gamma_{ij}^1(w'), \Gamma_{i'j'}^1(w'')\big),$$

for all w', w'' such that $P(\Gamma_{ij}^1(w')) = P(\Gamma_{i'j'}^1(w''))$.

Indeed, let $\Gamma_{ij}^1(w)$ and $\Gamma_{i'j'}^1(w')$ be any two elements of $C_1 \backslash K_\epsilon^1$ and suppose that $1/\epsilon^{d_{ij}} \leq 1/\epsilon^{d_{i'j'}}$. Let α be a curve in $\mathbb{C} \backslash D_{1/\epsilon^{d_{ij}}}$ joining $P(\Gamma_{ij}^1(w))$ to $P(\Gamma_{i'j'}^1(w'))$ as in Lemma 3.13. Let $\tilde{\alpha}_1$ (resp. $\tilde{\alpha}_2$) be the lifting of α by the restriction $P|_{\Gamma_{ij}^1(D_\epsilon \backslash \{0\})}$ (resp. $P|_{\Gamma_{ij}^2(D_\epsilon \backslash \{0\})}$) with origin $\Gamma_{ij}^1(w)$ (resp. $\Gamma_{ij}^2(w)$). Consider the unique $w'' \in D_\epsilon$ such that $\Gamma_{ij}^1(w'')$ is the end of $\tilde{\alpha}_1$. Notice that $P \circ \Gamma_{ij}^1 = P \circ \Gamma_{ij}^2$ and by uniqueness of lifts $\tilde{\alpha}_2 = \Gamma_{ij}^2 \circ (\Gamma_{ij}^1)^{-1} \circ \tilde{\alpha}_1$ which implies that $\Gamma_{ij}^2(w'')$ is the end of $\tilde{\alpha}_2$.

We have

$$d\big(\Gamma^2_{ij}(w),\Gamma^2_{i'j'}(w')\big)\le \mathrm{length}(\tilde\alpha_2)+d\big(\Gamma^2_{ij}(w''),\Gamma^2_{ij}(w')\big).$$

According to the Remark 3.14, there are constant, say c_1 and c_2 such that $\mathrm{length}(\tilde\alpha_2)\le c_1\,\mathrm{length}(\alpha)\le c_1c_2\,\mathrm{length}(\tilde\alpha_1)$. By hypothesis, there exists a constant $c>0$ such that

$$d\big(\Gamma^2_{ij}(w''),\Gamma^2_{ij}(w')\big)\le cd\big(\Gamma^1_{ij}(w''),\Gamma^1_{ij}(w')\big).$$

Therefore setting $C=\max\{c_1c_2,c\}$, we obtain

$$d\big(\Gamma^2_{ij}(w),\Gamma^2_{i'j'}(w')\big)\le C\Big(\mathrm{length}(\tilde\alpha_1)+d\big(\Gamma^1_{ij}(w''),\Gamma^1_{ij}(w')\big)\Big).$$

Applying Lemma 3.13 to C_1 with $u=\Gamma^1_{ij}(w)$ and $u'=\Gamma^1_{i'j'}(w')$, we then have

$$d\big(\Gamma^2_{ij}(w),\Gamma^2_{i'j'}(w')\big)\le CMd\big(\Gamma^1_{ij}(w),\Gamma^1_{i'j'}(w')\big).$$

This proves Φ is Lipschitz and the claim.

Now, let B^1_{ij} and $B^2_{i'j'}$ be branches of $\widetilde{C}_1$ and $\widetilde{C}_2$, respectively, with $i\ne i'$. Let $s\in(0,1]\to\Gamma^1_{ij}(ws^{1/d_{ij}})$ and $s\in(0,1]\to\Gamma^1_{i'j'}(w's^{1/d_{i'j'}})$ be the two real arcs with $w^{d_{ij}}=(w')^{d_{i'j'}}$. Then we have

$$d\big(\Gamma^1_{ij}(ws^{1/d_{ij}}),\Gamma^1_{i'j'}(w's^{1/d_{i'j'}})\big)=\frac{1}{s|w^{d_{ij}}|}\Big|a^1_{ij}-a^1_{i'j'}+\sum_{k>0}a^1_{ijk}w^ks^{k/d_{ij}}-\sum_{k>0}a^1_{i'j'k}(w')^ks^{k/d_{i'j'}}\Big|.$$

and

$$d\big(\Phi(\Gamma^1_{ij}(ws^{1/d_{ij}})),\Phi(\Gamma^1_{i'j'}(w's^{1/d_{i'j'}})\big)=\frac{1}{s|w^{d_{ij}}|}\Big|a^2_{ij}-a^2_{i'j'}+\sum_{k>0}a^2_{ijk}w^ks^{k/d_{ij}}-\sum_{k>0}a^2_{i'j'k}(w')^ks^{k/d_{i'j'}}\Big|$$

Hence the ratio

$$d\big(\Gamma^1_{ij}(ws^{1/d_{ij}}),\Gamma^1_{i'j'}(w's^{1/d_{i'j'}})\big)\Big/ d\Big(\Phi(\Gamma^1_{ij}(ws^{1/d_{ij}})),\Phi(\Gamma^1_{i'j'}(w's^{1/d_{i'j'}}))\Big) \tag{2}$$

tends to the non-zero constant $\frac{|a^1_{ij}-a^1_{i'j'}|}{|a^2_{ij}-a^2_{i'j'}|}$ as s tends to 0 for every such pairs (w,w'). So there exists $\epsilon>0$ such that for each such (w,w') with $|w|=1$ and each $s<\epsilon$, the quotient (2) belongs to $[1/c,c]$ where $c>0$.

Now, consider the branches B^1_{ij} and B^2_{ij}. Let $s\in(0,1]\to\Gamma^1_{ij}(ws)$ and $s\in$

$(0,1] \to \Gamma^1_{i'j'}(w's)$ be the two real arcs with $w^{d_{ij}} = (w')^{d_{ij}}$. Then we have

$$d\big(\Gamma^1_{ij}(ws), \Gamma^1_{ij}(w's)\big) = \frac{1}{s^{d_{ij}}|w^{d_{ij}}|}\left|\sum_{k>0} a^1_{ijk}(w^k - (w')^k)s^k\right|$$

and

$$d\big(\Phi(\Gamma^1_{ij}(ws)), \Phi(\Gamma^1_{ij}(w's)\big) = \frac{1}{s^{d_{ij}}|w^{d_{ij}}|}\left|\sum_{k>0} a^2_{ijk}(w^k - (w')^k)s^k\right|$$

Let k_0 be the minimal element of $\{k; a^1_{ijk} \neq 0 \text{ and } w^k \neq (w')^k\}$. Then k_0/d_{ij} is a characteristic exponent for B^1_{ij} relative to L_∞, so $a^1_{ijk_0}$ and $a^2_{ijk_0}$ are non-zero. Hence the ratio

$$d\big(\Gamma^1_{ij}(ws), \Gamma^1_{ij}(w's)\big) \Big/ d\Big(\Phi(\Gamma^1_{ij}(ws)), \Phi(\Gamma^1_{ij}(w's)\big)\Big) \tag{3}$$

tends to the non-zero constant $c_{ijk_0} = \frac{|a^1_{ijk_0}|}{|a^2_{ijk_0}|}$ as s tends to 0.

Notice that the integer k_0 depends on the pair of points (w, w'). But k_0/d_{ij} is a characteristic exponent relative to L_∞ of B^1_{ij}. Therefore there is a finite number of values for k_0 and c_{ijk_0}. Moreover, the set of pairs (w, w') such that $w \neq w'$ and $w^{d_{ij}} = (w')^{d_{ij}}$ consists of a disjoint union of $d_{ij} - 1$ lines, say $L_l = \{(w, \exp(2\pi l/d_{ij})w), w \in \mathbb{C}^*\}, l = 1, \ldots, d_{ij} - 1$. Observe that for any $(w, w') \in L_l$ the quotient (3) tends to positive constant as $s \to 0$ which does not depend on the pair (w, w'). So there exists $\epsilon_1 > 0$ such that for each such (w, w') with $|w| = 1$ and each $s \leq \epsilon_1$, the quotient (3) belongs to $[1/c, c]$ where $c > 0$, as claimed.

For the case of branches B^1_{ij} and $B^2_{ij'}$ with $j \neq j'$, the same arguments work taking into account their coincidence exponent relative to L_∞.

□

3.4 Lipschitz geometry of complex algebraic plane curves

In this subsection, we present the complete classification of the Lipschitz geometry of complex algebraic plane curves. We define the Lipschitz graph of a complex algebraic plane curves which is a combinatorial object that encode its Lipschitz geometry.

Let C be a complex algebraic plane curve. A sequence of **good minimal resolution** of $\widetilde{C}$ produces a smooth curve $\widetilde{\mathcal{C}}$. By (BRIESKORN and KNÖRRER, 1986, Lemma 9.2.3), the connected components of $\widetilde{\mathcal{C}}$ correspond bijectively to the irreducible components of C.

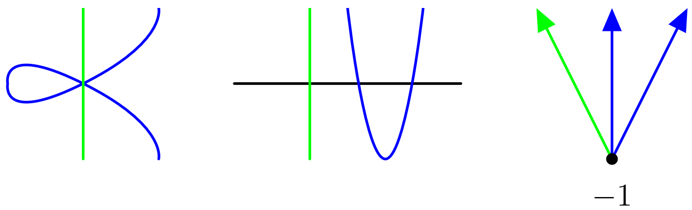
Definition 3.16. *Let C be a complex algebraic plane curve with irreducible components $C_1, \ldots, C_n$. A sequence of* ***good minimal resolution*** *of $\widetilde{C} \cup L_\infty$ produces a sequence of exceptional curves $E_1, \ldots, E_r$ and strict transform curves $\mathcal{C}_1, \ldots, \mathcal{C}_n$ of the curves $C_1, \ldots, C_n$ and the strict transform $\mathcal{L}_\infty$ of the line at infinity L_∞.*

The ***Lipschitz graph of*** *C is rooted graph with vertices V_k corresponding to* *the curves E_k labeled with its self-intersection number, vertices W_i corresponding to the curves $\mathcal{C}_i$ not labeled and a root corresponding to the $\mathcal{L}_\infty$. We put one edge joining vertices for each intersection point of the corresponding curves.*

Remark 3.17. *Let $p_1, \ldots, p_m$ be the singular points of $\widetilde{C} \cup L_\infty$. We point out that the Lipschitz graph C is obtained as the quotient of the disjoint union of the individual dual resolution graph of minimal good resolutions of $(\widetilde{C} \cup L_\infty, p_i)$, by identifying all vertices corresponding to the branch of an irreducible component C_j for all $j = 1, \ldots, n$. Then it is clear that the Lipschitz graph of C is determined by the topological type of the germs $(\widetilde{C} \cup L_\infty, p_i)$ and $(\widetilde{C}_j \cup L_\infty, p_i)$.*

Example 3.18. *Let C be a complex algebraic plane curve defined by $x[y^2 - x^2(1+x)] = 0$. We have two singular points for $\widetilde{C} \cup L_\infty$, namely $[0:0:1]$ and $[0:1:0]$. Dual resolution graph of a good minimal resolution at the singular point $[0:0:1]$ is drawn in Figure 8.*

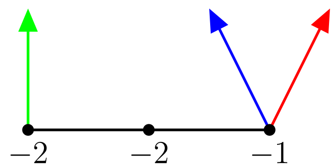
Figure 8 – The blue arrow vertices correspond to the branches of $C_1 : y^2 - x^2(1+x) = 0$ and the green arrow vertex corresponds to the branch of $C_2 : x = 0$.



Source: Elaborated by the author.

Dual resolution graph of a good minimal resolution for the singular point $[0:1:0]$ is drawn in Figure 9.

Figure 9 – The blue arrow vertex corresponds to the branch of $C_1 : y^2 - x^2(1+x) = 0$ and the green arrow vertex corresponds to the branch of $C_2 : x = 0$. The red arrow vertex corresponds to the branch of the line at infinity.

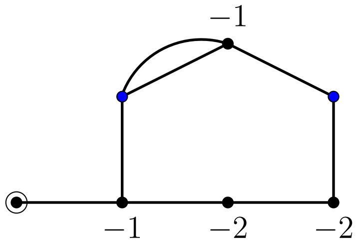


Source: Elaborated by the author.

Connecting these graphs in the way described above we obtain the Lipschitz graph for C (see Fig. 10).

We can do the inverse process: start from a Lipschitz graph of a complex algebraic plane curve C and obtain the individuals dual resolution graph of minimal good resolutions of $(\widetilde{C} \cup L_\infty, p_i)$. Then by (WALL, 2004, Theorem 8.1.7) we extract the following data: the topological type of the germ of the curve $\widetilde{C} \cup L_\infty$ at each of its singular points.

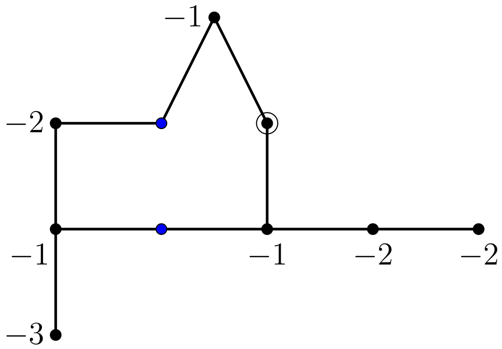
Figure 10 – Lipschitz graph of Example 3.18. The vertices that correspond to irreducible components of the curve are distinguished from the other vertices by the fact that they are not labeled. But to improve the visualization of the graph we put a distinct color to such vertices.



Source: Elaborated by the author.

Example 3.19. *Suppose that Figure 11 is a Lipschitz graph of a complex algebraic plane curve C. If we erase the vertices corresponding to the irreducible components of $\widetilde{C} \cup L_\infty$,*

Figure 11 – A given Lipschitz graph of a complex algebraic plane curve C.



Source: Elaborated by the author.

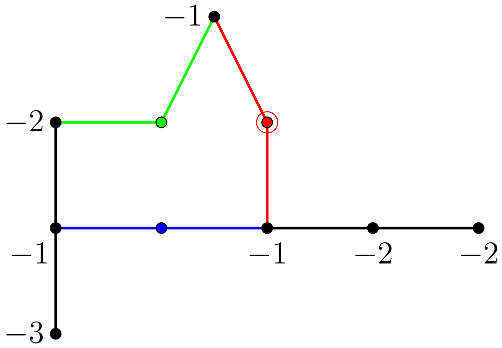
we get three graphs with some no end edges. We put an arrow vertex in each no end edges. But before doing all that we distinguish by colors the vertices corresponding to irreducible components and the edges connected to them to discern which branches belongs to an irreducible component (see Fig. 12).

Now, we delete the vertex corresponding to irreducible components and put the arrows vertices in the no end edges with the same color as the edge (see Fig.13).

The colors tell us the relation between branches and irreducible components. This and the dual resolution graphs $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ (see Fig. 13) are sufficient to determine the topological type of the germ of the irreducible components at each of its singular points. To see this we subject the graphs $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ repeatedly to a contraction operation which corresponds to blowing down of a curve. We call a vertex in a dual resolution graph (not associate to a minimal resolution) ***contractible*** *when it has label -1 and valency less than tree. Contraction of one of these vertices consists:*

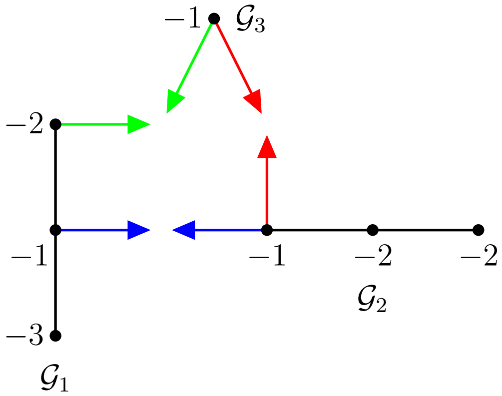
- *if this vertex has valency 2, in adding one to the intersection numbers of its labeled*

Figure 12 – There are three irreducible components, say, R (red) corresponding to the line at infinity, G (green) and B (blue) corresponding to the irreducible components of C.



Source: Elaborated by the author.

Figure 13 – There are three singular points of $\widetilde{C} \cup L_\infty$, say p_1, p_2 and p_3 with dual resolution graphs $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$, respectively.



Source: Elaborated by the author.

adjacent vertices, removing the vertex, and amalgamating its two adjacent edges into one edge.

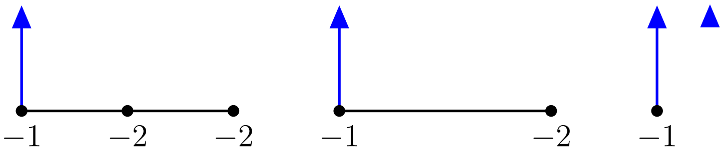
- *if this vertex has valency 1, in adding one to the intersection numbers of its labeled adjacent vertex and removing the vertex and its adjacent edge.*

Definition 3.20. *The **contraction process** of a dual resolution graph of a germ of a complex algebraic plane curve* (Γ, p) *with respect to one of its irreducible components* (Γ', p) *consists in removing the arrow vertices and the edges connected to it except the ones which corresponds to the branches of* (Γ', p)*. In the resulting graph one repeatedly applies all possible contractions. The non-contractible graph finally obtained is the dual resolution graph of* (Γ', p)*.*

Notice we get only arrow vertex at the end of contraction process if and only if (Γ', p) *is smooth. For instance, to determine the dual resolution graph of* (B, p_2) *one removes the red and arrow vertex and applies three contractions (see Fig. 14).*

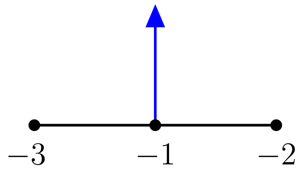
To determine the dual resolution graph of (B, p_1) *one removes the green edge and its arrow vertex: there are no contractible vertices (see Fig. 15).*

Figure 14 – Contraction process of the germ (B, p_2).



Source: Elaborated by the author.

Figure 15 – Minimal resolution graph of the germ (B, p_1).



Source: Elaborated by the author.

Thus, we extract from the Lipschitz graph the following data:

- *the number of irreducible components.*
- *there are three singular points, say p_1, p_2 and p_3 with dual resolution graphs $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$, respectively. By (WALL, 2004, Theorem 8.1.7), this is equivalent to know the topological type of the germs $(\widetilde{C} \cup L_\infty, p_1), (\widetilde{C} \cup L_\infty, p_2)$ and $(\widetilde{C} \cup L_\infty, p_3)$.*
- *the dual resolution graphs of the germs $(G, p_1), (B, p_1), (B, p_2)$ and (G, p_3), obtained by the contraction process. By (WALL, 2004, Theorem 8.1.7), this is equivalent to know the topological type of these germs.*

From the Example 3.19 it is easy to see that the equivalence $(2) \Leftrightarrow (3)$ of Theorem 3.9 holds. Now, we deal with the equivalence between (1) and (2).

Proof of $(1) \Leftrightarrow (2)$ *of Theorem 3.9.* We start assuming that there exists a bilipschitz map $\phi : C \to \Gamma$. By Theorem 3.4, for each singular point $p \in C$ the topological type of the germ (C, p) is the same as the topological type of $(\Gamma, \phi(p))$. By item (2) of Theorem 3.8, there is a bijection ψ between the set of points at infinity of C and the set of points at infinity of Γ such that $(\widetilde{C} \cup L_\infty, p)$ has the same topological type as $(\widetilde{\Gamma} \cup L_\infty, \psi(p))$.

Restricting ϕ to smooth points of C we get a homeomorphism between $C \backslash \Sigma(C) = \bigcup_{i \in I} C_i \backslash \Sigma(C)$ and $\Gamma \backslash \Sigma(\Gamma) = \bigcup_{j \in J} \Gamma_j \backslash \Sigma(\Gamma)$, where $\Sigma(C)$ and $\Sigma(\Gamma)$ denote the singular points of C and Γ, respectively. Since C_i, Γ_j are irreducible and $\Sigma(C)$ and $\Sigma(\Gamma)$ are finite, $C_i \backslash \Sigma(C)$ and $\Gamma_j \backslash \Sigma(\Gamma)$ are connected. Then the map $\sigma : I \to J$, defined by $\sigma(i) = j \in J$ if and only if $\phi(C_i \backslash \Sigma(C)) = \Gamma_j \backslash \Sigma(\Gamma)$, is a bijection.

We extend the application $\phi|_{C_i \backslash \Sigma(C)}$ to topological closure of $C_i \backslash \Sigma(C)$ and we get the bilipschitz map $\phi_i : C_i \to \Gamma_{\sigma(i)}, \phi_i = \phi|_{C_i}$. Applying Theorem 3.4 to $\phi_i : C_i \to \Gamma_{\sigma(i)}$, we obtain that for each singular point $p \in C_i$ the topological type of the germ (C_i, p) is the same as the topological type of $(\Gamma_{\sigma(i)}, \phi(p))$.

By item (2) of Theorem 3.8, there is a bijection ψ_i between the set of points at infinity of C_i and the set of points at infinity of $\Gamma_{\sigma(i)}$ such that $(\widetilde{C}_i \cup L_\infty, p)$ has the same topological type as $(\widetilde{\Gamma}_{\sigma(i)} \cup L_\infty, \psi_i(p))$. Moreover, ψ_i can be chosen to be the restriction of ψ to the points at infinity of C_i.

Recall the parametrization $\iota : \mathbb{C}^2 \to \mathbb{P}^2$ of $\mathbb{P}^2$ given by $\iota(x, y) = [x : y : 1]$. Then the bijection $\varphi : \Sigma(\widetilde{C} \cup L_\infty) \to \Sigma(\widetilde{\Gamma} \cup L_\infty)$ defined by

$$\varphi(p) = \begin{cases} \psi(p) & \text{if } p \in L_\infty, \\ \iota(\phi(\iota^{-1}(p))) & \text{otherwise,} \end{cases}$$

give us the bijection of item (2) of Theorem 3.9.

Now, the reciprocal, i.e., that (2) implies (1) of Theorem 3.9. We can assume that $I = J = \{1, \ldots, m\}$ and $\sigma = \text{id}$. The item (2) of Theorem 3.9 implies that both curves C_i and Γ_i have the same number of points at infinity and singular points for $i = 0, \ldots, m$ where $C_0 = C$ and $\Gamma_0 = \Gamma$.

Let $p_1, \ldots, p_s$ be the singular points of $\widetilde{C}$ and let $q_1, \ldots, q_s$ be the singular points of $\widetilde{\Gamma}$ which are not point at infinity of C and Γ, respectively. We denote in such a way that $(\widetilde{C}_i, p_l)$ has the same topological type as $(\widetilde{\Gamma}_i, q_l)$ for $l = 1, \ldots, s$ and $i = 0, 1, \ldots, m$.

Similarly, let $p_{s+1}, \ldots, p_m$ be the points at infinity of C and let $q_{s+1}, \ldots, q_m$ be the points at infinity of Γ denoted in such a way that $(\widetilde{C}_i \cup L_\infty, p_l)$ has the same topological type as $(\widetilde{\Gamma}_i \cup L_\infty, q_l)$ for $l = s+1, \ldots, m$ and $i = 0, 1, \ldots, m$.

Let $B(p_l) \subset \mathbb{P}^2$ be a regular coordinate ball, that is, there exist a smooth coordinate ball $B'(p_l) \supseteq \overline{B(p_l)}$. Shrinking $B'(p_l)$ if necessary, we can assume that $B'(p_l) \cap B'(p_j) = \emptyset$ for $l \neq j$ and we can apply Theorem 3.4 , i.e., for $l = 1, \ldots, s$ there exists a bilipschitz map

$$\phi_l : C \cap \iota^{-1}(B'(p_l)) \to \phi_l(C \cap \iota^{-1}(B'(p_l))) \subset \Gamma$$

which is biholomorphic except at $\iota^{-1}(p_l)$ and $\phi_l(\iota^{-1}(p_l)) = \iota^{-1}(q_l)$.

Similarly, by Theorem 3.8, there exist a bilipschitz

$$\Phi : C \cap \Big(\bigcup_{l=s+1}^{m} \iota^{-1}(B'(p_l))\Big) \to \Phi\Big(C \cap \Big(\bigcup_{l=s+1}^{m} \iota^{-1}(B'(p_l))\Big)\Big) \subset \Gamma$$

which is biholomorphic. Then the curve C is almost covered with domains of bilipschitz maps. The part that is missing is inside of $C \backslash \Big(\bigcup_{l=1}^{m} \iota^{-1}(B(p_l)\Big)$ which is a union of connected compact orientable surfaces K_i with boundary. More precisely, let $K_i = C_i \backslash \Big(\bigcup_{l=1}^{m} \iota^{-1}(B(p_l))\Big)$. Recall that connected compact orientable surfaces with boundary are classified up to diffeomorphism by the Euler characteristic and the number of connected components of boundary see, for instance, (HIRSCH, 1976, Chapter 9, Theorem 3.11).

Let us calculate the Euler characteristic of K_i. Shrinking $B'(p_l)$ if necessary, we assume that $B'(p_l)$ intersect $\widetilde{C}_i$ if and only if p_l is a singular point of $\widetilde{C}_i \cup L_\infty$. By additive property of the Euler characteristic we have:

$$\begin{aligned}\chi(\widetilde{C}_i) &= \chi\Big(\Big(\widetilde{C}_i \backslash \bigcup_l B(p_l)\Big) \cup \Big(\bigcup_l \overline{B(p_l)} \cap \widetilde{C}_i\Big)\Big)\\ &= \chi\Big(\widetilde{C}_i \backslash \bigcup_l B(p_l)\Big) + \chi\Big(\bigcup_l \overline{B(p_l)} \cap \widetilde{C}_i\Big) - \chi\Big(\Big(\widetilde{C}_i \backslash \bigcup_l B(p_l)\Big) \cap \Big(\bigcup_l \overline{B(p_l)} \cap \widetilde{C}_i\Big)\Big),\end{aligned}$$

Note that all spaces appearing in the above equation are compact and triangulable. By the Conical Structure Theorem and the additive property we know that

$$\chi\Big(\bigcup_l \overline{B(p_l)} \cap \widetilde{C}_i\Big) = \sum_l \chi\big(\overline{B(p_l)} \cap \widetilde{C}_i\big) = m_i,$$

where m_i is the number of singular points of $\widetilde{C}_i \cup L_\infty$. Shrinking each $B(p_l)$ if necessary, by Proposition 2.20 we may assume that the boundary of $B(p_l)$ intersect $\widetilde{C}_i$ transversally. Thus there are two possibilities: $B(p_l) \cap \widetilde{C}_i = \emptyset$ or $B(p_l) \cap \widetilde{C}_i$ is a smooth compact 1-manifold. In the latter case, by the classification theorem of smooth 1-manifold, the intersection is diffeomorphic to $\mathbb{S}^1$. In both cases we have

$$\chi\Big(\Big(\widetilde{C}_i \backslash \bigcup_l B(p_l)\Big) \cap \Big(\bigcup_l \overline{B(p_l)} \cap \widetilde{C}_i\Big)\Big) = 0.$$

On the other hand, (WALL, 2004, Theorem 7.1.1) tells us a formula for the Euler characteristic of a curve in terms of its degree and its singularities. More precisely,

$$\chi(\widetilde{C}_i) = 3d_i - d_i^2 + \sum_{p \in \widetilde{C}_i} \mu_p(\widetilde{C}_i)$$

where d_i denotes the degree of $\widetilde{C}_i$ and $\mu_p(\widetilde{C}_i)$ denotes the Milnor number of $\widetilde{C}_i$ at p.

Recall that the Milnor number is an invariant of the topological type, for instance, see (WALL, 2004, Theorem 6.5.9). The degree of the curve in its turn is an invariant of the topological type of the germs $(\widetilde{C}_i \cup L_\infty, p_l)$ since

$$\deg \widetilde{C}_i = \sum_{p \in \widetilde{C}_i \cap L_\infty} (\widetilde{C}_i \cdot L_\infty)_p,$$

where $(\widetilde{C}_i \cdot L_\infty)_p$ denotes the intersection number between $\widetilde{C}_i$ and L_∞. Also, the invariance of degree for the Lipschitz geometry at infinity of complex algebraic curves is given in (BOBADILLA, FERNANDES, and SAMPAIO, 2018, Corollary 3.2). Having said that, we have $\chi(\widetilde{C}_i) = \chi(\widetilde{\Gamma}_i)$ and for K_i,

$$\chi(K_i) = \chi(\widetilde{C}_i \backslash \cup_l B(p_l)) = \chi(\widetilde{C}_i) - m_i. \tag{4}$$

Let $\mathcal{B}(q_l) = \iota\Big(\Phi\big(C \cap \iota^{-1}(B(p_l))\big)\Big) \cup \{p_l\}$ for $l = s+1, \ldots, m$ and $\mathcal{B}(q_l) = \iota\Big(\phi_l\big(C \cap \iota^{-1}(B(p_l))\big)\Big)$ for $l = 1, \ldots, s$ and

$$\mathcal{K}_i = \Gamma_i \backslash \bigcup_l \phi_l(C \cap \iota^{-1}(B(p_l))) \cup \Phi(C \cap (\bigcup_{l=s+1}^{m} \iota^{-1}(B(p_l))).$$

To calculate the Euler characteristic of $\mathcal{K}_i$ we notice that $\mathcal{K}_i = \Gamma_i \backslash \Big(\bigcup_{l=1}^{m} \iota^{-1}(\mathcal{B}(q_l))\Big)$. And by similar arguments as above one has

$$\chi(\mathcal{K}_i) = \chi\Big(\widetilde{\Gamma}_i \backslash \bigcup_l \mathcal{B}(q_l)\Big) = \chi(\widetilde{\Gamma}_i) - m_i. \tag{5}$$

It follows from equation (4) and (5) that K_i and $\mathcal{K}_i$ have the same Euler characteristic. The map $f_i : \partial K_i \to \partial \mathcal{K}_i$ defined by the restrictions

$$\phi_l|_{C_i \cap \iota^{-1}(\partial B(p_l))} \text{ for } l = 1, \ldots, s \text{ and } \Phi|_{C_i \cap (\iota^{-1}(\partial B(p_l)))} \text{ for } i = s+1, \ldots, m$$

is bihomorphic. Now, we use a slight generalization of the classification of smooth compact surface with boundary:

Lemma 3.21. *Let K_i and $\mathcal{K}_i$ be connected compact orientable smooth surfaces with boundary and let $f_i : \partial K_i \to \partial \mathcal{K}_i$ be an orientation-preserving diffeomorphism. Then f_i extends to a diffeomorphism $F_i : K_i \to \mathcal{K}_i$ if only if K_i and $\mathcal{K}_i$ have the same Euler characteristic.*

Proof. The boundaries $\partial K_i, \partial \mathcal{K}_i$ are smooth compact 1-manifolds and thus its connected components are diffeomorphic to $\mathbb{S}^1$. Since f_i is a diffeomorphism between ∂K_i and $\partial \mathcal{K}_i$, they have the same number of connected components. Let $g_i : K_i \to \mathcal{K}_i$ be the diffeomorphism given by the classification of smooth compact surface theorem (HIRSCH, 1976, Chapter 9, Theorem 3.11). Up to isotopy every orientation-preserving diffeomorphism of $\mathbb{S}^1$ is the identity (HIRSCH, 1976, Chapter 8, Theorem 3.3), then we know that the restriction map $g_i|_{\partial K_i}$, and f_i are isotopic, say by $H_i : [0,1] \times \partial K_i \to \partial \mathcal{K}_i, H_i(0, \cdot) = f_i, H_i(1, \cdot) = g_i|_{\partial K_i}$.

The collar neighborhood theorem (LEE, 2013, Theorem 9.26) shows that ∂K_i has a collar neighborhood $\mathcal{C}$ in K_i; which is the image of a smooth embedding $E : [0,1) \times \partial K_i \to K_i$ satisfying $E(0, x) = x$ for all $x \in \partial K_i$. To simplify notation, we use this embedding to identify $\mathcal{C}$ with $[0,1) \times \partial K_i$ and denote a point in $\mathcal{C}$ as an ordered pair (s, x) with $s \in [0,1)$ and $x \in \partial K_i$; thus $(s, x) \in \partial K_i$ if and only if $s = 0$. For any $a \in (0,1)$, let $\mathcal{C}(a) = \{(s, x) \in \mathcal{C} : 0 \leq s < a\}$ and $K_i(a) = K_i \backslash \mathcal{C}(a)$.

Let $\gamma : [0,1] \to [0,1]$ be a smooth map that satisfies $\gamma(0) = 0$ and $\gamma(s) = 1$ for $\frac{1}{2} \leq s \leq 1$. Define $F : K_i \to \mathcal{K}_i$ by

$$F_i(p) = \begin{cases} g_i(p), & \text{if } p \in \text{Int}K_i\left(\frac{1}{2}\right), \\ (s, H_i(x, \gamma(s)), & p = (s,x) \in \mathcal{C}. \end{cases}$$

These definitions both give the map g_i on the set $\mathcal{C}\backslash\overline{\mathcal{C}\left(\frac{1}{2}\right)}$ where they overlap, so F_i is a diffeomorphism extension of f_i. □

The map $F : K_i \to \mathcal{K}_i$ is a diffeomorphism between compact sets, so it is a bilipschtz map. The maps F, Φ, ϕ_l agree on the components of the boundary of K_i. It follows that there exist a bilipschitz map $\Psi : C \to \Gamma$ such that $\Psi|_{K_i} = F_i$, $\Psi|_{\iota^{-1}(B(p_l))\cap C} = \phi_l$ and $\Psi|_{C\cap\left(\bigcup_{l=s+1}^{m}\iota^{-1}(B(p_l))\right)} = \Phi$. □

4 CONCLUSION

In this thesis, we present a complete classification of complex plane algebraic curves, equipped with the induced Euclidean metric, up to global bilipschitz homeomorphism.

We start from the local classification of these objects by following results and ideas of Pham and Teissier (1969), Fernandes (2003) and Neumann and Pichon (2014). The novelty of this thesis is to bring a complete description of the Lipschitz geometry at infinity of those curves. Finally, we gather all these information to get at a complete classification of global Lipschitz geometry of those curves and we encoded it in a discrete invariant so-called Lipschitz graph.

To better understand the implications of our results on the behavior of complex polynomial in two variables future studies one could address the problem of bilipschitz contact equivalence between them. More precisely, we say that two complex polynomials $f, g : \mathbb{C}^2 \to \mathbb{C}$ are bi-Lipschitz contact equivalent if there exists $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ a bilipschitz homeomorphism and there exist positive constants A and B such that

$$A|f(p)| \leq |(g \circ \Phi)(p)| \leq B|f(p)|.$$

Birbrair, Fernandes and Grandjean (2015) consider the problem of bilipschitz contact equivalence of complex analytic function-germs of two variables. Their main result, Theorem 4.2, states that the bilipschitz contact equivalence class of a plane complex analytic function-germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ determines and is determined by purely numerical data, namely: the Puiseux pairs of each branch of its zero locus, the multiplicities of its irreducible factors and the intersection numbers of pairs of branches of its zero locus.

Putting it differently, the bilipschitz contact equivalence of f is described by the Lipschitz geometry of its zero locus and the multiplicities of its irreducible factors.

We hope that a similar result holds true for the global case, in other words, we think the bilipschitz contact class of a polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ is described by the global Lipschitz geometry of finitely many fibers of f.

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