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Statistical model for a quantum noiseless subsystem

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ABSTRACT

One of the most promising physical properties for implementing quantum technology is light polarization. However, since light polarization is fragile, it is crucial to use quantum error correction in order to make quantum information over optical networks feasible. This paper performs a statistical analysis of a noiseless subsystem technique to correct errors on quantum information sent through light polarization. We discuss the performance of the noiseless subsystem scheme in a noisy channel using a two-dimensional random walk to represent the channel variation. Finally, we propose an expression to measure the efficiency of the analyzed setup using the degree of depolarization of the light.

1. Introduction

Quantum communication and computation are new areas of information processing that use quantum mechanics to implement new ways of communicating and computing without counterpart in the classical world, such as quantum key distribution [1–3], quantum teleportation [4,5], and quantum searching [6].

One of the most promising physical properties in experimental realization of quantum technologies is light polarization [7]. However, it is well known that light polarization is fragile and changes in an unpredictable way when propagating in standard single-mode fibers. Thus, in order to make quantum technology based on light polarization feasible, quantum error correction (QEC) schemes must be employed, that is, the unpredictable changes in light polarization must be controlled. QEC can be achieved by using quantum codes [8–10].

The QEC statistically analyzed follows the approach presented in [11]. The basic idea is that pulses can travel through short or long paths, and the quantum states $|S\rangle$ and $|L\rangle$ represent the pulses traveling through the short and the long paths, respectively, or similarly, the pulses arriving in the early and late time slot. We assume that any state of the two-dimensional Hilbert space spanned by the basic states $|S\rangle$ and $|L\rangle$ can be prepared (any state $\alpha|S\rangle+\beta|L\rangle$ can be prepared, where α and β are complex numbers such that $|\alpha|^2+|\beta|^2=1$).

Our paper analyzes a scheme for quantum error correction using noiseless subsystems to photons. The first scheme without entanglement was proposed in [12] and other schemes were proposed later [13,14]. Schemes with entanglement were proposed, without experimental verification, in [15] and their experimental realization was described in [16]. The essential idea in these schemes is to separate the components of polarization-encoded qubit in time slots as in [11]. The slow variation of the channel transformation in the time does not change the information when the quantum state travels through it. In this work, a statistical analysis is done considering probabilistic variations between the components of the time-bin quantum state. An expression to measure the efficiency of the quantum error correction setup is proposed taking into account a statistical model, purposed by us, of the discussed channel. We conclude that the efficiency decreases exponentially with channel length. The statistical analysis purposed requires the knowledge of the degree of depolarization dynamics, which can be estimated [17,18]. Other approaches for performing statistical analysis require knowing the variance of the channel parameters (rotation angle and phase shift angle) in function of the channel length. However, these values are very hard to obtain experimentally for different channel lengths, which fully justify our approach.

The paper is outlined as follows. In Section 2, the quantum error correction system based on noiseless subsystem is presented. In Section 3, the correction presented in Section 2 is analyzed when statistical variations between the components of the time-bin state occur. Finally, we draw some conclusions in Section 4. The required analytical equalities for Section 3 are presented in the Appendix.

2. Quantum noiseless subsystem

We start by discussing the single-photon linear–optical scheme for quantum error correction proposed in [13]. The scheme is depicted in Fig. 1.

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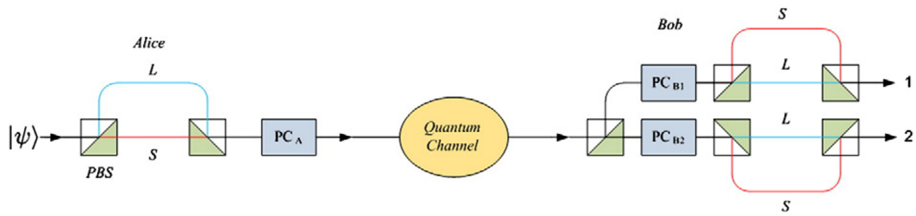

Fig. 1. Optical scheme for single-photon quantum error correction: PBS (polarization beam splitter) and PC (Pockels cell).

The transmitter, Alice, has a single photon in an unknown polarization state $|\psi\rangle=\alpha|H\rangle+\beta|V\rangle$, where $|H\rangle$ ($|V\rangle$) represents the horizontal (vertical) state of polarization. After the photon passes the unbalanced polarization interferometer the state is $\alpha|H,S\rangle+\beta|V,L\rangle$, since the horizontal component takes the short path, S, while the vertical component takes the long path, L. Alice turns on her Pockels cell only when the L-path component is present, effecting the transformation $|V,L\rangle\rightarrow|H,L\rangle$. Hence, the state that Alice sends to Bob is $\alpha|H,S\rangle+\beta|H,L\rangle$.

In theory, quantum communication protocols work well for fully polarized light. However, when fully polarized light propagates in standard optical fiber, the light polarization suffers several random rotations due to random birefringence in the fiber that can be produced, for example, by mechanical stress like bending, impurities during the fabrication process, and noncircularity of the core. This effect can be expressed by a unitary transformation $U(\phi,\chi)$ such that $U(\phi,\ \chi)|H\rangle=cos\phi|H\rangle+e^{i\chi}sin\phi|V\rangle$ and $U(\phi,\ \chi)|V\rangle=-sin\phi|H\rangle+e^{i\chi}cos\phi|V\rangle$ which describes a general qubit transformation (excluding a global phase which does not have physical significance), where ϕ denotes the angle rotation and χ is a shift phase between the two polarization components. So, we define the channel noise, which we abusively refer just as channel, as a product of two matrices where each of them has one random variable, χ and ϕ

$$U(\phi,\chi)=\begin{pmatrix}1 & 0\\ 0 & e^{i\chi}\end{pmatrix}\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}. \tag{1}$$

In this section we only consider the (not so realistic) scenario where both components, S and L, have the same channel (noise), that is, the noise in both components is modeled by the same unitary transformation U. From the general expression (1) of U we have that

$$\begin{aligned}U(\phi,\chi)\alpha|H,S\rangle+U(\phi,\chi)\beta|H,L\rangle&=\alpha\left(cos\phi|H,S\rangle+e^{i\chi}sin\phi|V,S\rangle\right)\\&+\beta\left(cos\phi|H,L\rangle+e^{i\chi}sin\phi|V,L\rangle\right).\end{aligned} \tag{2}$$

As it can be seen in Fig. 1, the Pockels cell PC_{B1} is activated only when the S-path component is present; similarly PC_{B2} is activated only when the L-path component is present. At each mode, 1 (upper arm) and 2 (lower arm), there exists an unbalanced polarization interferometer. In these interferometers, the horizontal component propagates through the long path while the vertical component propagates through the short path. When the corrupted state, described in Eq. (2), arrives at Bob's place it has suffered two transformations. First, after passing through the first PBS and Pockels cells, it is transformed into

$$\alpha\left(cos\phi|H,S\rangle^{2}+e^{i\chi}sin\phi|H,S\rangle^{1}\right)+\beta\left(cos\phi|V,L\rangle^{2}+e^{i\chi}sin\phi|V,L\rangle^{1}\right). \tag{3}$$

At last, after passing through the unbalanced polarization interferometers, the final state becomes

$$cos\phi\left(\alpha|H\rangle^{1}+\beta|V\rangle^{1}\right)+e^{i\chi}sin\phi\left(\alpha|H\rangle^{2}+\beta|V\rangle^{2}\right). \tag{4}$$

In Eqs. (3) and (4), the superscripts 1 and 2 denote the paths in the direction of the output modes 1 and 2, respectively. The qubit emerges randomly in either one of the two output modes (1 or 2) according to a distribution that depends on the channel parameter ϕ. Thus Bob obtains the uncorrupted state $\alpha|H\rangle+\beta|V\rangle$ either in mode 1, with probability $\cos^2\phi$, or in mode 2 with probability $\sin^2\phi$. When the channel is approximately an ideal channel ($\phi\approx 0$) Bob receives the uncorrupted state most likely in mode 1. On the other hand, when ϕ is allowed to vary over its range according to a uniform distribution, the probability of obtaining the uncorrupted state in either mode tends to 1/2. However, Bob will always receive the uncorrupted state. Thus, following the previous analysis, by using an optical delay and an electro-optic switch to form time multiplexing, it is possible to avoid the channel noise, and send the original state uncorrupted (but it emerges in different times). As we shall see in the next section, this is not the case when we model differently the noise in the S- and L-path component.

3. Correction when there is a fast variation between pulses

In this section we consider a much more realistic case than that presented in Section 2. We analyze the performance of the noiseless subsystem setup of Fig. 1 when the channel is very noisy, that is, the fiber birefringence has local fast variations and it changes during the time interval between the short and long pulses. In other words, we analyze the case when there are variations in the channel after the passage of the short pulse and before the passage of the long pulse. In this scenario, the short pulse and the long pulse have different unitary matrices to represent the channel evolution, i.e., the matrix $U_S(\phi_S,\chi_S)$ is applied in the states $|H,S\rangle$ and the matrix $U_L(\phi_L,\chi_L)$ is applied in the state $|H,L\rangle$. Thus, the channel evolution is written as

$$\alpha\left(cos\phi_L|H\rangle+e^{i\chi_L}sin\phi_L|V\rangle\right)+\beta\left(cos\phi_S|H\rangle+e^{i\chi_S}sin\phi_S|V\rangle\right). \tag{5}$$

The efficiency of the noiseless subsystem is measured by the fidelity between the input state of the channel and the output state of the decoder. By definition, the value of the fidelity is:

$$F^2=|\alpha|^4+|\beta|^4+2|\alpha|^2|\beta|^2f \tag{6}$$

$$f=cos(\phi_S)cos(\phi_L)+sin(\phi_S)sin(\phi_L)cos(\chi_L-\chi_s). \tag{7}$$

Now, by taking $\phi_S=\phi$, $\phi_L=\phi+\Phi$, $\chi_S=\chi$ and $\chi_L=\chi+X$ we are able to rewrite f as follows:

$$\begin{aligned}f&=cos(\phi)cos(\phi+\Phi)+sin(\phi)sin(\phi+\Phi)cos(X)\\&=\frac{1}{2}\{cos\Phi(1+cosX)+cos(2\phi+\Phi)(1-cosX)\}\end{aligned}. \tag{8}$$

We stress that $\Phi=\phi_L-\phi_S$ and $X=\chi_L-\chi_S$ are the differences between the rotation angles and the phase shift angles of the long and short pulses, respectively. Observe that f is a parameter depending on the channel variations. Moreover, if $X=\Phi=0$ there is no error and $F^2=1$. Furthermore, note that the expected value of F^2 depends of the expected value of f, since $E(F^2)=|\alpha|^4+|\beta|^4+2|\alpha|^2|\beta|^2E(f)$.

We consider that the random variable ϕ is uniformly distributed, since, without any extra information, it is natural to assume that ϕ can take any possible value with the same probability. As the analyzed function is $\cos(2\phi+\Phi)$, we assume that ϕ is a uniform random variable ranging over $\left[\frac{-\pi-\Phi}{2}, \frac{\pi-\Phi}{2}\right]$. In this way, we can write the expected value of f for ϕ by

$$E_\phi(f) = \frac{1}{2}\left\{\cos\Phi(1+\cos X) + (1-\cos X)\underbrace{\frac{1}{\pi}\int_{\frac{-\pi-\Phi}{2}}^{\frac{\pi-\Phi}{2}}\cos(2\phi+\Phi)d\phi}_{0}\right\}$$
$$= \frac{1}{2}\{\cos\Phi(1+\cos X)\}. \tag{9}$$

In order to describe the evolution of the variables Φ and X through the channel position we consider discrete steps and, therefore, we obtain a two-dimensional random walk, one-dimension for the variable Φ and one-dimension for the variable$X$. The discrete random walk describes how the values of Φ and Xchange while the short and long pulses cross the channel. Discrete time is obtained by dividing the length $z$of the channel in n equal parts in such way that $n=z/\Delta z$ and Δz is the (constant) difference between the positions of the short and the long pulse. Thus, we consider that discrete time ranges from $t_0, t_1, \ldots, t_n$ where t_0 is the initial time and t_n is the time that the long pulse arrives the end of the channel. See Fig. 2 to understand the time evolution of the system.

As depicted in Fig. 2, at the instant t_0, the later pulse $|H, L\rangle$, is entering the channel, whereas the earlier pulse $U_{S_0}|H, S\rangle$, is already at the point z_1. After, at instant t_1, the later pulse is at point z_1 and the earlier pulse is at point z_2, moreover, the later pulse develops into $U_{L_1}|H, L\rangle$ whereas the earlier pulse develops into $U_{S_1}U_{S_0}|H, S\rangle$. At each ith section of the channel, the earlier pulse and the later pulse develop according with unitary evolutions U_{S_i} and U_{L_i} respectively. Therefore, we can write the global channel transformation as

$$\alpha\prod_{i=0}^{n} U_{L_i}|H,L\rangle + \beta\prod_{i=0}^{n} U_{S_i}|H,S\rangle. \tag{10}$$

Note that the values of Φ and X evolve from Φ_0 and X_0 at time t_0 to Φ_n and X_n at time t_n. The evolution will be described as an unbiased random walk suffering, at each step, small variations $\delta\Phi>0$ and $\delta X>0$. Clearly, $\delta\Phi$ and δX depend on the interval Δz. Using the proposed random walk model for Φ and X it is possible to calculate the expected value of the parameter$f$.

These variations result from the channel transitions $U(\phi(z), \chi(z))$ to $U(\phi(z+\Delta z), \chi(z+\Delta z))$. Each step happens with probability $p_{\delta\Phi}$ for $+\delta\Phi$ and probability $(1-p_{\delta\Phi})$ for $-\delta\Phi$. The same happens to δX, that is, probability $p_{\delta X}$ for $+\delta X$ and probability $(1-p_{\delta X})$ for $-\delta X$. Under the unbiased assumption for the random walk, we have that $p_{\delta\Phi}=p_{\delta X}=1/2$ and so

$$E(E_\Phi(f)) = \frac{1}{2}M_n(\delta\Phi)\{1+M_n(\delta X)\}$$
$$= \frac{1}{2}\cos^n(\delta\Phi)\{1+\cos^n(\delta X)\} \tag{11}$$

Where $M_n(\cdot)$ is the expected value of the one-dimensional random walk after n steps (see Eq. (25) in the Appendix). Since $\delta\Phi$ and δX depend on the interval Δz, the expected value of f also depends on Δz (the difference between the long and short arms in the interferometer) and of the channel length z (since $n=z/\Delta z$). By making the usual approximation to $\cos(x)\approx 1-x^2/2$, then, when δ and δX are close to 0, we can write

$$E(E_\Phi(f)) = \frac{1}{2}e^{-\frac{(\delta\Phi(\Delta z))^2}{2\Delta z}z}\left\{1+e^{-\frac{(\delta X(\Delta z))^2}{2\Delta z}z}\right\} \tag{12}$$

Observe that Eqs. (11) and (12) would be a good outcome if the variations $\delta\Phi$ and δX were known. However, it is very hard to obtain these values experimentally. So, the only possible conclusion from those two equations is the exponential decrease of $E(E_\Phi(f))$ with z.

Since the small variations $\delta\Phi$ and $\delta X$of the channel are unknown there is not knowledge about the performance of system studied. To solve this problem we need to devise a statistical model based on parameters of the channel whose dynamics are known. A natural question arises: which parameters can be used to express the channel variation between the components $|H, S\rangle$ and $|H, L\rangle$? For the next analysis, we choose the degree of depolarization, since it is a quantity that has been used in many works to describe the polarization channel dynamics and can be inferred via the Stokes parameters [17,18].

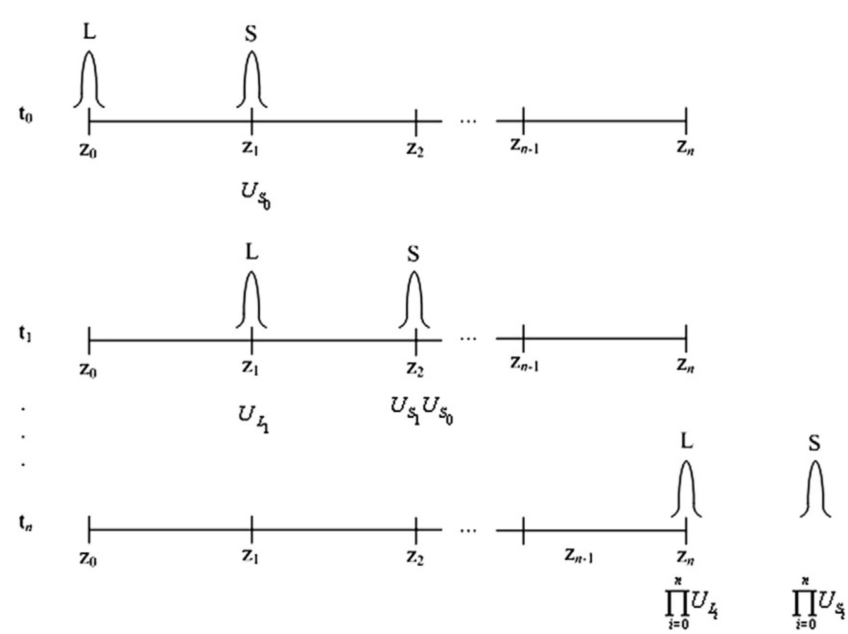

Fig. 2. The time evolution of the long and short pulses in the channel.

The degree of polarization is a quantity used to describe the portion of an electromagnetic wave that is polarized. Note that the degree of polarization $1-\xi(z)$ is a function of the channel length z. A perfectly polarized wave has $1-\xi(z)=1$, whereas a totally depolarized wave has $\xi(z)=1$. A wave that is partially polarized, and therefore can be represented by a superposition of a polarized and depolarized component, has degree of polarization ranging between 0 and 1. For a single polarized photon the same idea is applied to describe if a quantum state is totally mixed, denoted by $I/2$, or totally pure, denoted byρ. A pure quantum state has degree of purity equal to 1, whereas a totally mixed state has degree of purity equal to 0. So, a quantum state can be also represented as superposition of a pure quantum state and a totally mixed state

$$\xi(z)\frac{I}{2} + (1-\xi(z))\rho. \tag{13}$$

There is a correlation between the degree of depolarization of light $\xi(z)$ and the changes in the parameters that represents the channel $\phi(z)$ and $\chi(z)$. The variations $\Phi(\Delta z)$ and $X(\Delta z)$ are unknown, but $\xi(\Delta z)\in[0,\ 1]$ can be calculated [17]. We know that there is a correlation between the degree of depolarization $\xi(\Delta z)$ and the variation of the unknown channel parameters, in detail, there exist unknown values $\Delta\Phi>0$ and $\Delta X>0$ such that $\xi(\Delta z)$ denotes the probability that $\Phi(\Delta z)\geq\Delta\Phi$ and $X(\Delta z)\geq\Delta X$. In other words, there are some unknown values $\Delta\Phi>0$ and $\Delta X>0$ for which $\xi(\Delta z)$ is the probability that, in step Δz, the variation of Φ surpasses $\Delta\Phi$ and the variation of X surpassesΔX. For instance, consider a photon propagating a distance Δz, we may denote that $\xi(\Delta z)$ represents the probability associated with the event such that $\Phi(\Delta z)\geq\pi=\Delta\Phi$ and $X(\Delta z)\geq1.5\pi=\Delta X$. If Δz represents a short distance of propagation, then the event is rare, because it is unlikely that the pulse suffers this variation in such little distance. Therefore, the degree of depolarization can express this reality, because $\xi(\Delta z)\approx0$. In [18] practical results were presented that can be used to determine $\xi(\Delta z)$.

We note that the unknown values $\Delta\Phi$ and $\Delta X$are far larger than the values $\delta\Phi$ and δX used in Eqs. (11) and (12) (since in the latter we consider very small variations). Therefore, the values Δ and ΔX will be added as steps to represent the channel variation only when the event $(\Delta z)\geq\Delta\Phi$ and $X(\Delta z)\geq\Delta X$ occurs. More precisely, let $[z_{i-1}, z_i]$ be the ith interval of the channel, where $\Delta z=z_i-z_{i-1}$. If $\Phi(\Delta z)\geq\Delta\Phi$ and $X(\Delta z)\geq\Delta X$, we assume that $U(\phi_{i-1},\ \chi_{i-1})\neq U(\phi_i,\ \chi_i)$, such that $\phi_i=\phi_{i-1}+\Delta$ and $\chi_i=\chi_{i-1}+\Delta X$. Otherwise, we consider that $U(\phi_{i-1},\chi_{i-1})=U(\phi_i,\chi_i)$.

Recall that we are dividing a channel with length z in n equal parts in such way that $n=z/\Delta z$. As depicted in Fig. 2, at each ith section of the channel, the earlier pulse and the later pulse develop according with unitary evolutions U_{S_i} and U_{L_i} respectively. And so, the global channel transformation is

$$\alpha\prod_{i=0}^{n} U_{L_i}|H,L\rangle + \beta\prod_{i=0}^{n} U_{S_i}|H,S\rangle. \tag{14}$$

In each section of channel (with length Δz) the probability of $U_{L_i}\neq U_{S_i}$ is $\xi(\Delta z)$. Though $U_{L_i}\neq U_{S_i}$ is a rare event, i.e. $\xi(\Delta z)\approx0$, the number of trials n is very big, and so, the hypothesis that variations occur in some part of the channel must be tested. What determines the number of trials is the channel length (and also Δz).

We assume that variations in and X are rare events (they occur with probability $\xi(\Delta z)$) and that, when they occur, Φ and X are modified by $\pm\Delta\Phi$ and $\pm\Delta X$, respectively. Thus, we derive a two-dimensional random walk which starts at point 0 and at each step moves by $\pm\Delta\Phi$ ($\pm\Delta X$). Each step $+\Delta\Phi$ happens with probability $p_{\Delta\Phi}$ and with probability $(1-p_{\Delta\Phi})$ for $-\Delta\Phi$ (and similar to ΔX, that is, probability $p_{\Delta X}$ for $+\Delta X$ and probability $(1-p_{\Delta X})$ for $-\Delta X$). Thus, the discrete random variation in the channel can be expressed by $=t\Delta$ and $X=s\Delta X$. Moreover, we have

$$P_n(k) = \binom{n}{k}\xi(\Delta z)^k(1-\xi(\Delta z))^{(n-k)} \tag{15}$$

$$P_k(p_{\Delta X},s) = \frac{k!}{\left(\frac{k+s}{2}\right)!\left(\frac{k-s}{2}\right)!}p_{\Delta X}^s(1-p_{\Delta X})^{(k-s)} \tag{16}$$

$$P_k(p_{\Delta\Phi},t) = \frac{k!}{\left(\frac{k+t}{2}\right)!\left(\frac{k-t}{2}\right)!}p_{\Delta\Phi}^t(1-p_{\Delta\Phi})^{(k-t)}. \tag{17}$$

Where Eq. (15) is the probability of occurring k variations in n trials, Eq. (16) represents the probability of $X=s\Delta X$ when k variations $(s\leq k)$ occur and Eq. (17) represents the probability of $\Phi=t\Delta\Phi$ when k variations $(t\leq k)$ occur. Thus, after n trials the probability of $X=s\Delta X$ and $\Phi=t\Delta\Phi$ is

$$P_n(s,t) = \sum_{k=1}^{n} P_n(k)P_k(p_{\Delta X},s)P_k(p_{\Delta\Phi},t). \tag{18}$$

Where $s,t\in S=\{-k,-k+2,\ldots,k-2,k\}$ such that $\sum_{s,t\in S}P_n(s,t)=1$. What can we say about the expected values of the positions and X of the walk after k steps? It is not hard to see that $E(\Phi)=E(X)=0$ when $p_{\Delta\Phi}=p_{\Delta X}=1/2$.

Recall we have $\xi(\Delta z)\approx0$ since in a short length of propagation we expect the pulse to have small depolarization. Note that if and $X$are modified by $\pm\Delta$ and $\pm\Delta X$ (which occurs with probability $\xi(\Delta z)$), at the ith section of the channel, then U_{L_i} is different from U_{S_i}. So, the probability, in the limit $n\to\infty$, that we have $k$of such variations is given by the limit of the binomial distribution

$$\begin{aligned}\lim_{n\to\infty}P_n(\xi,k) &= \lim_{n\to\infty}\binom{n}{k}\xi(\Delta z)^k(1-\xi(\Delta z))^{(n-k)}\\ &= \lim_{n\to\infty}\binom{n}{k}\left(\frac{\xi(\Delta z)z}{n\Delta z}\right)^k\left(1-\frac{\xi(\Delta z)z}{n\Delta z}\right)^{(n-k)}\\ &= \lim_{n\to\infty}\underbrace{\frac{n!}{n^k(n-k)!}}_{1}\frac{\left(\frac{\xi(\Delta)}{\Delta z}z\right)^k}{k!}\underbrace{\left(1-\frac{\xi(\Delta z)z}{n\Delta z}\right)^n}_{e^{-\frac{\xi(\Delta z)}{\Delta z}z}}\overbrace{(1-\xi(\Delta z))^{-k}}^{1}\\ &= \frac{\left(\frac{\xi(\Delta z)}{\Delta z}z\right)^k}{k!}e^{-\frac{\xi(\Delta z)}{\Delta z}z}.\end{aligned} \tag{19}$$

The result, as expected, is the Poisson distribution, which is the discrete probability distribution that expresses the probability that k events (local birefringence variation between the earlier pulse and later pulse) occur in a fixed channel length, provided that these events occur with an average rate, $z\xi(\Delta z)/\Delta z$, and independently of the time since the last event occurred. In other words, k is the number of occurrences of an event and $z\xi(\Delta z)/\Delta z$ is a positive real number, equal to the expected number of channel variations between the later pulse and earlier pulse that occur during a given channel length z. This limit describes the law of rare events, since each of the individual Bernoulli events is rarely triggered. The name may be misleading because the total count of success events in a Poisson process needs not to be rare if the channel length is very long, on the other hand, the parameter $z\xi(\Delta z)/\Delta z$ is not small. Though a channel variation between the earlier pulse and the later pulse is a very rare event, the number of trials grows significantly with the channel length. Therefore, the expected value of f is calculated by

$$E(f) = \frac{e^{-\frac{\xi(\Delta z)}{\Delta z}z}}{2}\sum_{k=1}^{\infty}\frac{\left(\frac{\xi(\Delta z)}{\Delta z}z\right)^k}{k!}\sum_{s,t=-k,-k+2,\ldots}^{k}E_{\phi}(f)P_k(p_{\Delta X},s)P_k(p_{\Delta},t). \tag{20}$$

When the channel grows the number of trials also grows. We consider the parameter $z\xi(\Delta z)/\Delta z$ to be very big, in such way that the number of trials is approximately infinity. In addition, and to simplify the analysis, we consider that each step $\Delta(\Delta X)$ is shifted left or right with probability 1/2, and so we consider a unbiased random walk. Using the analysis presented in the Appendix, we can write the expected value to f as follows:

$$E(f) = \frac{1}{2}\left(e^{-(1-cos(\Delta))\frac{\xi(\Delta z)}{\Delta z}z} + e^{-(1-cos(\Delta)cos(\Delta X))\frac{\xi(\Delta z)}{\Delta z}z}\right). \tag{21}$$

When the earlier and later pulses do not have different local variations between them in any sections of the channel ($\Delta = \Delta X = 0$), the expected value of f is always 1 for any channel length z. However, when these variations exist, even small ones, the expected value of f decreases exponentially with the channel length z. Hence, the expected value of the fidelity (see Eq. (6) for F^2) has a minimum value of $|\alpha|^4 + |\beta|^4$ to a sufficiently long channel.

The expected value of f is illustrated in the Fig. 3. The parameters Δ and ΔX are considered equals and belong to the interval $[0, 0.1\pi]$. Another parameter in the expected value of f is $\xi(\Delta z)z/\Delta z$ which belongs to the interval [0,1000] in the Fig. 3. In [17], a simple quantum model for light depolarization shows that $\xi(z)$ increases exponentially with z. In [19] some results were simulated for the behavior of the degree of polarization in classical light. In general, the degree of polarization as function of propagation distance along optical fiber is the unity for initial light (totally polarized) and decreases to zero when the distance of propagation increases. However, the shape of the function changes with the variation of some parameters (coupling coefficient between the orthogonal fields and spectral width source).

The values Δ and ΔXwere considered unknown. Then, we choose the values $\Delta = \pi$ and $\Delta X = 0$, since these values minimize $E(f)$. Thus, we have the following limit:

$$e^{-\frac{2\xi(\Delta z)}{\Delta z}z} \leq E(f) \leq 1. \tag{22}$$

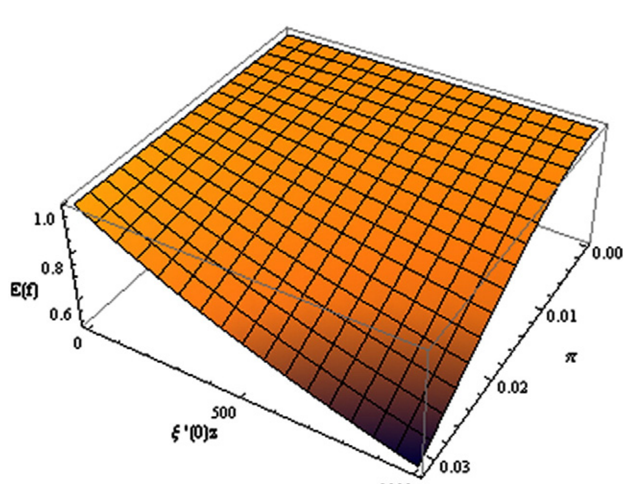

Fig. 3. The value $E(f)$ in function of $\xi(\Delta z)z/\Delta z$ and $\Delta = \Delta X$.

Finally, by picking the worst case, the equation to estimate the fidelity between the quantum state sent by Alice, $\alpha|H\rangle + \beta|V\rangle$ and the quantum state corrected by Bob is

$$F_e^2 = |\alpha|^4 + |\beta|^4 + 2|\alpha|^2|\beta|^2 e^{-\frac{2\xi(\Delta z)}{\Delta z}z}. \tag{23}$$

All parameters in Eq. (23), the channel length z, the sub-channel length $\Delta z \neq 0$, the degree o depolarization $\xi(\Delta z)$ and the quantum state sent by Alice are known in the design. Now, we have an expression to estimate the performance of the quantum error correction setup based in noiseless subsystem assuming more realistic conditions for the channel.

4. Conclusion

We analyzed the performance of a quantum noiseless subsystem considering variations in the channel when the birefringence has a local variation after the passing of the short pulse and before the passing of the long pulse. By employing two different approaches, we conclude that the expected value of the fidelity between the input quantum state in the encoder and the corrected quantum state decreases exponentially when the length of the channel increases. However, the second statistical analysis requires the knowledge of the degree of depolarization dynamics while in the first analysis the required parameters are the variations in the rotation angle and phase shift angle of the state polarization. The second approach is more advantageous because we are able to know the dynamics of the degree of depolarization and, therefore, it can be used to estimate the efficiency of the quantum error correction based in noiseless subsystem.

The expected value of the fidelity (measurement used to quantify the efficiency of the error correction scheme based in quantum noiseless subsystem) is a function of the channel length and the distance between the long and the short arm in the unbalanced interferometer Δz. Thus, the efficiency of the setup discussed in this work can be estimated by the distance between the encoder and decoder. Furthermore, the distance Δz can be tuned to avoid interference between long and short pulses caused by dispersion effects (polarization-mode dispersion, chromatic dispersion, etc).

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Appendix

Details about Eq. (21)

This appendix presents the details about the development of Eq. (20) into Eq. (21). Firstly, let us develop the following expression:

$$\begin{aligned}&\sum_{s,t=-k,-k+2,\ldots}^{k}E_{\phi}(f)P_k(p_{\Delta X},s)P_k(p_{\Delta\Phi},t)\\ &= \frac{1}{2}\sum_{s,t=-k,-k+2,\ldots}^{k}cos(t\Delta\Phi)\{1+cos(s\Delta\Phi)\}P_k(p_{\Delta X},s)P_k(p_{\Delta},t)\\ &= \frac{1}{2}\left\{\sum_{t=-k,-k+2,\ldots}^{k}cos(t\Delta)P_k(p_{\Delta},t)\left(1+\sum_{s=-k,-k+2,\ldots}^{k}cos(s\Delta X)P_k(p_{\Delta X},s)\right)\right\}\end{aligned} \tag{24}$$

Considering $p_\Delta = p_{\Delta X} = 1/2$, consider:

$$\begin{aligned}
M_k(\delta) &= \sum_{r=-k,-k+2,\dots}^{k} cos(r\delta)P_k(1/2,r) \\
&= \sum_{r=-k,-k+2,\dots}^{k} \frac{k!}{\left(\frac{k+r}{2}\right)!\left(\frac{k-r}{2}\right)!}\frac{cos(r\delta)}{2^k} \\
&= \sum_{r=-k,-k+2,\dots}^{k} \frac{k!}{\left(\frac{k+r}{2}\right)!\left(\frac{k-r}{2}\right)!}\frac{e^{ir\delta}+e^{-ir\delta}}{2^{k+1}} \\
&= \frac{1}{2}\left(\sum_{r=-k,-k+2,\dots}^{k} \frac{k!}{\left(\frac{k+r}{2}\right)!\left(\frac{k-r}{2}\right)!}\frac{e^{ir\delta}}{2^k}\right. \\
&\quad \left. + \sum_{r=-k,-k+2,\dots}^{k} \frac{k!}{\left(\frac{k+r}{2}\right)!\left(\frac{k-r}{2}\right)!}\frac{e^{-ir\delta}}{2^k}\right) \\
&= \frac{1}{2}\left(\sum_{r=-k,-k+2,\dots}^{k} \frac{k!}{\left(\frac{k+r}{2}\right)!\left(\frac{k-r}{2}\right)!}\frac{e^{i\delta\left(\frac{k+r}{2}\right)}e^{-i\delta\left(\frac{k-r}{2}\right)}}{2^k}\right) \\
&\quad + \frac{1}{2}\left(\sum_{r=-k,-k+2,\dots}^{k} \frac{k!}{\left(\frac{k+r}{2}\right)!\left(\frac{k-r}{2}\right)!}\frac{e^{-i\delta\left(\frac{k+r}{2}\right)}e^{i\delta\left(\frac{k-r}{2}\right)}}{2^k}\right) \\
&= \frac{1}{2}\left(\left(\frac{e^{i\delta}+e^{-i\delta}}{2}\right)^k + \left(\frac{e^{-i\delta}+e^{i\delta}}{2}\right)^k\right) \\
&= (cos(\delta))^k.
\end{aligned} \tag{25}$$

Now, we write $\lambda = \xi(\Delta z)z/\Delta z$ to simplify notation. Thus, the expression (24) can be written by

$$\begin{aligned}
E(f) &= \frac{e^{-\lambda}}{2}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}M_k(\Delta)(1+M_k(\Delta X)) \\
&= \frac{1}{2}e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}cos^k(\Delta)\left(1+cos^k(\Delta X)\right) \\
&= \frac{1}{2}e^{-\lambda}\left(\sum_{k=0}^{\infty}\frac{(\lambda cos(\Delta))^k}{k!} + \sum_{k=0}^{\infty}\frac{(\lambda cos(\Delta)cos(\Delta X))^k}{k!}\right) \\
&= \frac{1}{2}\left(e^{-(1-cos(\Delta))\lambda} + e^{-(1-cos(\Delta)cos(\Delta X))\lambda}\right).
\end{aligned} \tag{26}$$

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