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EDDYGLEDSON SOUZA GAMA

PROBLEMS ABOUT MEAN CURVATURE

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Tese apresentada ao Programa de Pós-graduação em Matemática do Departamento de Matemática da Universidade Federal do Ceará, como parte dos requisitos necessários para a obtenção do título de Doutor em Matemática.
Área de concentração: Geometria Diferencial.

Orientador: Prof. Dr. Luquésio Petrola de Melo Jorge
Coorientador: Prof. Dr. Francisco Martín Serrano
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I dedicate this work to my father José Maria, my mother Simone Conceição, my brothers Eddygeferson, José Henrique and José Maria filho, my grandmother Maria do Carmo, my grandfather Francisco Souza, my great grandmother Maria Souza da Conceição, my cousin Jessica Melo, my sisters in law Paula Mikaelly and Francisca Wemelisa, and finally I dedicate this work to my lovely future wife Francisca Edcarla.

In memory of Luiza Lourenço de Sousa.

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Até um pingo de água em uma folha pode se tornar algo incrível quando visto de uma perspectiva diferente.
Even a trickle of water on a leaf can become something incredible when viewed from a different perspective.

RESUMO

Essa tese está dividida em três capítulos. No primeiro capítulo faz-se uma breve introdução das ferramentas necessárias para o desenvolvimento do trabalho. Por sua vez, no segundo capítulo desenvolve-se a teoria de Jenkins-Serrin para os casos vertical e horizontal. No tocante o caso vertical, prove-se apenas a existência de solução do problema de Jenkins-Serrin do tipo I, quando M é rotacionalmente simétrico e tem curvaturas sectional não-positiva. No entanto, com respeito ao caso horizontal, prova-se a existência e unicidade global, naturalmente admitindo que o espaço base M tem uma particular estrutura. A terceira, e última parte dessa tese é dedicada a prova de um resultado de caracterização de translating solitons em $\mathbb{R}^{n+1}$. Mais precisamente, prova-se que os únicos exemplos C^1-assintóticos a dois meio-hiperplanos fora de um cilindro são os hiperplanos paralelos ao vetor $\mathbf{e}_{n+1}$ e os elementos da família associada ao cilindro grim reaper inclinado.

Palavras-chaves: Solitons de translação. Problema de Jenkins-Serrin. Cilindro grim reaper inclinado.

ABSTRACT

This thesis is divided into three chapters. In the first chapter, it is done a brief introduction of the main tools necessary for the development of this work. In turn, in the second chapter it develops the Jenkins-Serrin theory for vertical and horizontal cases. Regarding the vertical case, it only proves the existence of the solution of Jenkins-Serrin problem for the type I, when M is rotationally symmetric and has non-positive sectional curvatures. However, with respect to the horizontal case, the existence and the uniqueness is proved in a general way, namely assuming that the base space M has a particular structure. The third and last chapter of this thesis is devoted to proving a result of the characterization of translating solitons in $\mathbb{R}^{n+1}$. More precisely, it is proved that the unique examples C^1-asymptotic to two half-hyperplanes outside a cylinder are the hyperplanes parallel to $\mathbf{e}_{n+1}$ and the elements of the family associated with the tilted grim reaper cylinder in $\mathbb{R}^{n+1}$.

Keywords: Translating solitons. Jenkins-Serrin problem. Tilted grim reaper cylinder.

LIST OF FIGURES

LIST OF SYMBOLS

$\mathrm{T}\Sigma$	tangent bundle of Σ
$\mathrm{T}^{\top}\Sigma$	normal bundle of Σ
Ric_{Σ}	Ricci tensor of Σ
$C^k(\Sigma)$	space of all functions with derivative up to order k on Σ
$C^k_c(\Sigma)$	space of all functions with derivative up to order k and compact support on Σ
$\mathcal{L}_Y X$	Lie derivative of X with respect to Y
∇u	gradient of u
$\nabla^2 u$	Hessian of u
$\mathrm{Tr}_k u$	the sum of the k lowest eigenvalues of $\nabla^2 u$
Δu	Laplacian of u
$\mathrm{div}\, X$	divergence of the vector field X
$\mathcal{A}_g[\Sigma]$	Area of Σ with respect to the metric endowed by g
L_g	Jacobi operator associated with the metric endowed by g
$\mathcal{H}^k$	k-dimensional Hausdorff measure
$G^k(\Sigma)$	Grassman manifold of all unorientated k-dimensional subspace of $\mathrm{T}\Sigma$
$\mathcal{V}^k(\Sigma)$	space of all k-dimensional varifolds in Σ
$\mathcal{RV}^k(\Sigma)$	space of all k-dimensional rectifiable varifolds in Σ
$\mathcal{IV}^k(\Sigma)$	space of all k-dimensional integral varifolds in Σ
$\mathrm{spt}\ V$	support of k-dimensional varifold V

CONTENTS

1 INTRODUCTION

The techniques from the geometric flow has shown their power in the proof of Thurston's conjecture by PERELMAN (2008b,a) using Ricci flow and Penrose's inequality by HUISKEN and ILMANEN (2001) using the inverse of the mean curvature flow. After that, the techniques coming from geometric flow have been turned one of the most important tools of the differential geometry. Besides of the massive applications into geometric problems, these techniques have shown their power into other areas as physic, computation etc.

Under this optical of geometric flow, here we are interested to study a particular type of extrinsic geometric flow, namely the mean curvature flow (or flow of the mean curvature vector field). So before proceeding, let us define what means a hypersurface flow by their mean curvature vector field. Let N be a Riemannian manifold and $F_0 : \Sigma \to N$ be an immersion of Σ into N. Suppose that $F : \Sigma \times [0, T) \to N$ is an one-parameter family of immersions of Σ into N. Then we say that the family $F : \Sigma \times [0, T) \to N$ flow by their mean curvature vector field with initial data F_0 provided that

$$\begin{cases} \partial_t F(p,t) &= \vec{\mathbf{H}}(F(p,t)) & p \in \Sigma \\ F(p,0) &= F_0(p) & p \in \Sigma \end{cases}$$

where $\vec{\mathbf{H}}(F(p,t))$ denotes the mean curvature vector field of the hypersurface $F_t(\Sigma) := F(\Sigma, t)$ at $F(p,t)$. Here and after the mean curvature is the trace of the second fundamental form. When Σ is a compact hypersurface in N, the existence and the uniqueness for short times can be seen in (ECKER, 2004), (HUISKEN and POLDEN, 1999) and (MANTEGAZZA, 2010). It is important we point out here that HUISKEN and POLDEN (1999) proved the existence and uniqueness in a large class of geometric flows that contains the mean curvature flow.

Although we have defined the mean curvature flow in a general setting, actually here we are interested to study the mean curvature flow in a Riemannian product $M \times \mathbb{R}$, where M is a complete Riemannian manifold with a Riemannian metric σ. Indeed to be honest here we are interested in a particular solution of the mean curvature flow called the translating soliton (or translator) in $M \times \mathbb{R}$.

We say that an oriented hypersurface Σ in $M \times \mathbb{R}$ is a translating soliton with speed c(> 0) provided that

$$\vec{\mathbf{H}} = c\partial_t^{\perp},$$

where $\perp$ denotes the normal projection over the normal bundle of Σ.

The first fact about translating solitons are that they are eternal solution for the mean curvature flow. Indeed, let $F : M \times \mathbb{R} \times \mathbb{R} \to M \times \mathbb{R}$ be the flux of vector field $c\partial_t$, then the restriction of F to $\Sigma \times \mathbb{R}$ flow by their the mean curvature vector field, up to intrinsic diffeomorphism on Σ given by the tangent vector field $c\partial_t^{\top}$. Thus, up to change of coordinate on Σ, one has

$$\partial_t F_(p,t) = c\partial_t^{\perp}.$$

The second, in fact the most important fact about translating solitons, they are as the blow up limit near the singularity of the mean curvature flow. At this moment we will focus in $\mathbb{R}^{n+1}$ to define what means a singularity. Under this supposition, the singularity come out naturally when we envelop a compact hypersurface in $\mathbb{R}^{n+1}$ by mean curvature. This means that there exists a maximum time, $T_{\max} > 0$, so that the flow cannot be extended to a flow until $T_{\max} + \epsilon$ for all $\epsilon > 0$, see (ECKER, 2004), (HUISKEN, 1984, 1990), (MANTEGAZZA, 2010) and (WHITE, 2015). Indeed, as was proved by HUISKEN (1984)(see also HUISKEN (1990)) the behaviour of the flow near the singularity is described by the form that the second fundamental form blow-up. More precisely, he showed what follows: let Σ be an embedded and compact hypersurface in $\mathbb{R}^{n+1}$ and $F : \Sigma \times [0, T_{\max}) \to \mathbb{R}^{n+1}$ be the maximal flow (which means that $T_{\max}$ is the maximal time that the flow exists), then if A_t denotes the second fundamental form of $\Sigma_t := F_t(\Sigma)$, then $\max_{\Sigma_t} |A_t|^2$ is unbounded as $t \to T_{\max}$. Actually, he proved that

$$\max_{\Sigma_t} |A_t|^2 \geq \frac{1}{2(T-t)}.$$

Thus we classified the singularities according to the rate at which $\max_{p\in\Sigma_t} |A_t|$ blows up as follows: if there is a constant $C > 1$ such that

$$\max_{\Sigma_t} |A_t| \sqrt{2(T-t)} \leq C,$$

then we say that the flow develops a *Type I singularity* at instant T. Otherwise, that is, if

$$\limsup_{t\to T} \max_{\Sigma_t} |A_t| \sqrt{(T-t)} = +\infty,$$

we say that is a *Type II singularity.*

Once defined what a singularity means we can come back to talk how translating solitons appear the blow up near the singularity of the mean curvature flow. In dimension two, ANGENENT (1991) proved, in the case of self-intersect convex (in a certain sense) planar curves, that singularities of the *shortening flow* (the mean curvature flow in $\mathbb{R}^2$) are asymptotic (after a subtle rescaling) to the grim reaper curve $y = -\log(\cos x)$, $x \in (-\pi/2, \pi/2)$, which is a translation curve with respect to translation along of the flow of $\mathbf{e}_2$. In higher dimension $n(\geq 2)$, HUISKEN and SINESTRARI (1999) proved that if M has non-negative mean curvature and if the flow develops a *type II of singularity*, then after a particular rescaling the limit flow is the evolution of a convex translating soliton in $\mathbb{R}^{n+1}$ along of a flow of a vector $v \in \mathbb{R}^{n+1}$.

A remarkable property of translating solitons was obtained by ILMANEN (1994). He proved that translating solitons are minimal hypersurfaces in $M \times \mathbb{R}$ with respect to a conformal metric called Ilmanen's metric. In particular, we can use all local tools from the theory of minimal hypersurfaces in this setting.

Under this perspective of seeing translating solitons as minimal hypersurfaces, here we use this parallel in order to study of translating solitons into two different terms. The Jenkins-Serrin problem in $M \times \mathbb{R}$ and classification of translating solitons in $\mathbb{R}^{n+1}$.

In the Chapter 3 we study the Jenkins-Serrin problem in $M \times \mathbb{R}$. We start this chapter by giving a brief digression of the problem. This is done turn our exposition more didactic and for localize our problem in an optical of the theory already known. After this digression, in the remaining part of this chapter we develop the results obtained in (GAMA *et al.*, 2019a) and (GAMA *et al.*, 2019b) in collaboration with Esko Heinonen, Jorge H. Lira and Francisco Martín. In these two works, we obtained results of existence of Jenkins-Serrin problem in the vertical case (graph along of the flow of ∂_t) and in the horizontal case (graph Killing along of the flow of a Killing vector field in M).

In turn in the Chapter 4 we study translating solitons in $\mathbb{R}^{n+1}$. Similar what we did in Chapter 3, we start this chapter by localizing our study in the perspective what already known. After that, in the remaining part of the chapter, we develop the results obtained in (GAMA and MARTÍN, 2018) and (GAMA, 2019). In these two works, been the first one in collaboration with Francisco Martín, we obtained a result of characterization of two important families of translating solitons in $\mathbb{R}^{n+1}$, the hyperplanes parallel to $\mathbf{e}_{n+1}$ and the family associated with the tilted grim reaper cylinder. The ideas here are detected the shape of the hypersurface knowing its behaviour at the "wings" of the hypersurface. Doing a parallel with the minimal case in $\mathbb{R}^{n+1}$, our results could be seen as the "analogous" to the result of characterization due to SCHOEN (1983b). They proved that the catenoids and the hyperplanes are the unique examples of minimal hypersurfaces with finite total curvature and two embedded ends in $\mathbb{R}^{n+1}$. Although this little comparative, the technique used here differ from the method used by SCHOEN (1983b), because here we essentially use the theory of varifolds to obtain our results, however SCHOEN (1983b) used in a clever way the Alexandrov's method to detect the shape of the hypersurface.

2 BACKGROUND

In this section we shall give a brief revision of the most important results that we are going to use throughout the thesis. This review will be divided into three parts. In the first one, we shall do a little revision about differential geometry that will be useful later. In turn in the second one, we shall talk about Geometric measure theory, more precisely, we are going to collect subject from the theory of varifolds that will be used throughout the thesis. Finally, in the third and last part, we shall obtain the remaining matter that we are going to need to develop the thesis about translating solitons.

2.1 Differential geometry

This section is devoted to give a little review of two important facts from the differential geometry. The first is the classical analytic tool, the maximum principle. The second fact is the existence of the Plateau problem in the piecewise convex Riemannian manifold.

2.1.1 *Minimal Hypersurfaces*

Let N be a Riemannian manifold with a Riemannian metric g without boundary and $\Sigma \hookrightarrow N$ be a hypersurface in N. Take any point $q \in \Sigma$, we denote by

$$D_r(p) = \{v \in \mathrm{T}_p\Sigma \ : \ |v| < r\}$$

the tangent ball around p of radius r. Consider $\mathrm{T}_p\Sigma$ as a vector subspace of T_pN and let ν be an unit normal vector to $\mathrm{T}_p\Sigma$ in T_pN. Fix a sufficiently small $\epsilon > 0$ and denote by $W_{r,\epsilon}(p)$ the solid cylinder around p, i. e.

$$W_{r,\epsilon}(p) := \left\{\exp_p(q + t\nu) \ : \ q \in D_r(p) \ \text{and} \ |t| < \epsilon\right\},$$

where exp is the exponential map of N at p. Given a smooth function $u : D_r(p) \to \mathbb{R}$, the graph of u over $D_r(p)$ is the set given by

$$\mathrm{Graph}[u] := \left\{\exp_p(q + u(q)\nu) \ : \ q \in D_r(p)\right\}.$$

Coming back to Σ now, it is known that if we take r and ϵ small enough, then $\Sigma \cap W_{r,\epsilon}(p)$ is a graph of a smooth function u defined over $D_r(p)$. Endowing T_pN with the metric pull-back from g via exponential map $\exp_p$ we can see $\mathrm{Graph}[u]$ as the hypersurface in T_pN given by $\mathrm{Graph}[u] = \{q + u(q)\nu \in \mathrm{T}_pN \ : \ q \in D_r(p)\}$. We assume this identification in the rest of the section.

Suppose now Σ' is another hypersurface in N. We say that Σ' lies locally *one* *side* of Σ if either $\Sigma' \cap \Sigma = \varnothing$ or for all $p \in \Sigma' \cap \Sigma$ we have $N_\Sigma(p) = N_{\Sigma'}(p)$, and if $\Sigma \cap W_{r,\epsilon}(p) = \text{Graph}[u]$ on $D_r(p)$ and $\Sigma' \cap W_{r,\epsilon}(p) = \text{Graph}[v]$ on $D_r(p)$, then either $u \geq v$ or $v \geq u$ on $D_r(p)$, where here N_Σ (respectively, $N_{\Sigma'}$) denotes a local unit normal along Σ (respectively, Σ').

The local description for Σ as a graph shows its power when Σ is a minimal hypersurface in N. To be precise, consider the notation of the first paragraph, consider the local coordinate $\{x_1, \ldots, x_{n-1}\}$ for $D_r(p) \subset \text{T}_p\Sigma \subset \text{T}_pN$, and the coordinate $\{x_1, \ldots, x_{n-1}, x_n = \nu\}$ for T_pN. Arguing as in (COLDING and MINICOZZI II, 2011) we may conclude that the function u satisfies an uniformly elliptic quasilinear equation. Besides of this, if $v : D_r(p) \to \mathbb{R}$ is another smooth function so that Graph$[v]$ is a minimal hypersurface in T_pN too, then the function $w = u - v$ satisfies an uniformly elliptic differential equation with smooth coefficients

$$a_{ij}\partial^2_{ij}w + b_i\partial_i w + cw = 0.$$

This last fact is the key point that we would want to mention here, because of this PDE and the theory developed in (GILBARG and TRUDINGER, 2001) we can conclude the following two versions of the maximum principle.

Theorem 2.1. *Suppose that Σ_1 and Σ_2 are minimal hypersurfaces in N. If $\Sigma_1 \cap \Sigma_2 \neq \varnothing$ and Σ_1 lies locally one side of Σ_2, then $\Sigma_1 = \Sigma_2$.*

When the hypersurfaces and N have boundary, we have the following version of the maximum principle.

Theorem 2.2. *Let N be a Riemmannian manifold with non-empty boundary. Suppose that Σ_1 and Σ_2 are minimal hypersurfaces in N with boundary so that $\partial\Sigma_1$ and $\partial\Sigma_2$ lie on ∂N. If $\partial\Sigma_1 \cap \partial\Sigma_2 \neq \varnothing$ and Σ_1 lies locally one side of Σ_2, then $\Sigma_1 = \Sigma_2$.*

Remark 2.1. *Note that we can omit the hypothesis that N has boundary above by assuming Σ_1 and $\Sigma_2 \subset \Omega$ and $\partial\Sigma_1$ and $\partial\Sigma_2 \subset \partial\Omega$, where $\Omega \subset N$ is a smooth closed domain in N.*

In addition of these theorems, the local description as a graph also allows to conclude the following result provided that N has dimension three, see (COLDING and MINICOZZI II, 2011) for the proof of this result.

Theorem 2.3. *Suppose that Σ_1 and Σ_2 are two minimal surfaces in N^3 which have non-empty intersection and do not coincide on an open set. Then Σ_1 and Σ_2 intersect transversely except at an isolated set of points $\mathcal{A}$. Moreover, given any point $p \in \mathcal{A}$ there exists a integer $k \geq 2$ and a neighbourhood $U \ni p$ where the intersection consists of $2k$ embedded arc meeting at p.*

Remark 2.2. *This theorem tells that if two minimal surfaces are not equal and have non-empty intersection, then the set of intersection of these surfaces has a particular structure depending if Σ_1 and Σ_2 are transversal or not. More precisely, If the tangent planes of*

Σ_1 and Σ_2 at p are transversal, then $\mathcal{A}$ is locally a smooth curve across p. On the other hand, if Σ_1 and Σ_2 are not transversal at p, then $\mathcal{A}$ is locally an even set of smooth curves meeting at p. Geometrically, this last fact tells that the graphs of Σ_1 and Σ_2 change of side near p, and the smooth curves are exactly the arcs where the graphs change of side.

2.1.2 *Schoen's Estimate*

In this part we are going to state Schoen's theorem about the estimates of the second fundamental form of stable surfaces. So before proceeding, let us define what means hypersurface be stable in general setting. Let N be an n-dimensional oriented Riemannian manifold with a Riemannian metric g and Σ be a minimal hypersurfaces in N.

Definition 2.1. *We say that Σ is stable provided that if X is a vector field in N with compact support in N so that $X(p)\perp \mathrm{T}_p\Sigma$ for all $p \in \Sigma$, and $\phi : (-\epsilon, \epsilon) \times N \to N$ is an one-parameter family of diffeomorphisms so that $\phi_s(p) := \phi(s,p) = p$ for all $p \notin \mathrm{supp}X$, $\phi_0(p) = p$ and $X = (\phi_s)_* \left(\frac{\mathrm{d}}{\mathrm{d}s}\right)_{|s=0}$, if $\partial\Sigma \neq \varnothing$ we also assume $X|_{\partial\Sigma} = 0$, then it holds*

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}_{|s=0} \mathcal{A}_g[\phi_s(\Sigma)] \geq 0.$$

Here $\mathcal{A}_g[\Sigma]$ indicates the area of Σ in N with the metric induced by g.

Next, we state the following result due to SCHOEN (1983a) about estimates of the second fundamental form of stable surface.

Theorem 2.4. *Let Σ be an immersed stable surface in an $3-$dimensional Riemannian manifold N. Given $r \in (0,1]$, and a point $p \in \Sigma$ such that the geodesic ball $B_r(p)$ in Σ has compact closure in Σ. Then, there exists a constant c which depends only on the curvature of N in $B_r(p)$ so that*

$$|A|^2(p) \leq \frac{c}{r^2}.$$

Furthermore, if $B_r^3(p)$ denotes a geodesic ball in N and $B_r^3(p) \cap \Sigma$ has compact support on Σ, then there exists a constant $\epsilon > 0$ depending on the curvature of N in $B_r^3(p)$ and the injectivity radius of N at p in such a way $B_{\epsilon r}^3(p) \cap \Sigma$ is a union of embedded discs having the square of the norm of the second fundamental form bounded by $\tilde{c}/r^2$ for a constant $\tilde{c}$ depending on the curvatures of N in $B_r^3(p)$.

Remark 2.3. *Before proceeding we must say something about the previous theorem. The previous theorem say that if we have a sequence of stable surfaces in $\{\Sigma_n\}$ so that $B_r^3(p) \cap \Sigma_n$ has compact support on Σ_n for all n. Then, there exists a $\epsilon > 0$ depending on the curvature of N in $B_r^3(p)$ so that $|A_n| \leq \tilde{c}/r^2$ for a constant $\tilde{c}$ depending on the curvatures of N in $B_r^3(p)$, where A_n indicates the norm of the second fundamental form of $B_r^3(p) \cap \Sigma_n$.*

This result and the following result we will be useful later. The proof of this result follows a similar argument as in Lemma 2.4 in (COLDING and MINICOZZI II, 2011), see also SPRUCK and XIAO (2018).

Lemma 2.1. *Let Σ be an immersed stable surface in an $3-$dimensional Riemannian manifold N. Suppose that for same constant $c > 0$ we have*

$$\sup_{\Sigma} |A|^2 \leq \frac{c}{r^2}.$$

Then, there exists a constant $\epsilon > 0$ depending on the curvature of N so that if $\mathrm{dist}_{\Sigma}(p, \partial\Sigma) > 2\epsilon$, then $B_{2\epsilon}(p)$ is a graph of a function u over $\mathrm{T}_p\Sigma$ with gradient and Hessian bounded by a constant only depends on N and ϵ.

2.1.3 *Existence of minimal embedded disk*

The most important problems in differential geometry in the last century was to prove the existence of solutions of the Plateau's problem. This problem asks the following: given a finite family of simple closed curves in a three-dimensional Riemannian manifold N^3, then does it possible to find a minimal immersion in N with boundary this family?

Although this problem was already known for Euler and Lagrange in the eighteenth century, and by Plateau in the nineteenth century, the proof of the existence of such solutions was given by RADÓ (1930) and DOUGLAS (1931) at the beginning of the last century, when M is either $\mathbb{R}^3$ (RADÓ, 1930) or $\mathbb{R}^{n+1}$ (DOUGLAS, 1931). After that, in a deeply work, MORREY (1948) extended the existence when N is now a homogeneous manifold. A hard extension of the Morrey's work was carried out by MEEKS III and YAU (1982a,b). They proved the existence of solutions of Plateau's problem when now N is a piecewise convex manifold. So before we state this theorem, we need some notation.

Definition 2.2. *A manifold N is called be piecewise convex, if N is a precompact domain with boundary of a large Riemannian manifold $\hat{\mathrm{N}}$ and ∂N is formed by a finite family $\{N_i\}$ of convex (with respect to the unit inward pointing normal) smooth hypersurface with boundary in $\hat{\mathrm{N}}$ with boundary in ∂N, each N_i is a compact domain of a smooth surface $\hat{N}_i$ in $\hat{N}$, $N_i = \hat{N}_i \cap N$ and each $\partial\hat{N}_i \subset \partial\hat{N}$.*

The main theorem can now be stated as follows.

Theorem 2.5 (Existence of minimal disk)**.** *Let N be a piecewise convex manifold and γ be a Jordan's curve on ∂N null-homotopic. Then there exists a minimal embedded disk into N with boundary γ.*

Besides of this theorem, later we will need of the following variation when we have two Jordan's curves in ∂N. This result is also due to MEEKS III and YAU (1982a,b)

Theorem 2.6 (Existence of least area cylinder)**.** *Let N be a piecewise convex $3-$manifold and γ_1 and γ_2 two disjoint Jordan's curves on ∂N. Assume that exist a bounded cylinder $\mathcal{C}$ with boundary γ_1 and γ_2 and $d\{\gamma_1, \gamma_2\} < d\{\gamma_1\} + d\{\gamma_2\}$. Then there exists an immersed* *connected least area cylinder in N with boundary γ_1 and γ_2.*

Remark 2.4. *Above $d\{\gamma_i\}$ denotes the minimum of the area of all disk in N with boundary γ_i and $d\{\gamma_1, \gamma_2\}$ denotes the minimum of the area of all cylinder in N with boundary γ_1 and γ_2.*

Remark 2.5. *Notice that the solutions of the Plateau problem are stable surfaces in N.*

2.2 Theory of varifolds

In this part we are going to follow the exposition given by ALLARD (1972), SIMON (1983) and FEDERER (1996) for theory of varifolds, see also appendix in (WHITE, 2009) for a brief explanation or the recent reference MAGGI (2012). Throughout this part N denote an $(n+1)$-dimensional Riemannian manifold with a Rimennian metric g, d indicates the distance function in N and $U \subset N$ is an open subset.

2.2.1 *Hausdorff measure of dimension* k

Let $\alpha(k)$ be the area of the unit ball in $\mathbb{R}^k$. Whenever $0 < \delta \leq +\infty$ and $A \subset U$, we define

$$\mathcal{H}^k_{N\delta}(A) = \inf_C \left\{ \sum_{S \in C} \alpha(k) \left(\frac{\text{diam } S}{2} \right)^k \right\},$$

where C denotes a countable family of subset of U so that

$$A \subset \bigcup_{S \in C} S \text{ and diam } S := \sup_{x,y \in S} d(x,y) \leq \delta.$$

Each $\mathcal{H}^k_{N\delta}$ is an outer measure which satisfies $\mathcal{H}^k_{N\delta_1}(A) \leq \mathcal{H}^k_{N\delta_2}(A)$ whenever $\delta_2 \leq \delta_1$ and $A \subset U$. Moreover, if $\text{dist}(A,B) > \delta$ one has $\mathcal{H}^k_{N\delta}(A \cup B) = \mathcal{H}^k_{N\delta}(A) + \mathcal{H}^k_{N\delta}(B)$ by definition. These properties together tells that the limit

$$\mathcal{H}^k_N(A) = \sup_{\delta > 0} \mathcal{H}^k_{N\delta}(A) = \lim_{\delta \to 0} \mathcal{H}^k_{N\delta}(A)$$

there exists in $[0, \infty]$ for all $A \subset U$ and by Caratheodory's criterion $\mathcal{H}^k_N$ is a outer measure on the algebra of Borel set of U.

Definition 2.3 (k-dimensional Hausdorff measure)**.** *The measure $\mathcal{H}^k_N$ is called the k-dimensional Hausdorff measure in U.*

We have placed here the index N at $\mathcal{H}^k_N$, differentiating of the usual notation $\mathcal{H}^k$, to indicate that the measure $\mathcal{H}^k_N$ depends on the geometry of N. The next result clarifies what we are trying to say.

Theorem 2.7. *The measure $\mathcal{H}^{n+1}_N$ coincides with the Riemannian measure associated* *with the metric g. That is, we have*

$$\mathcal{H}_N^{n+1}(A) = \int_A \mathrm{d}\mu_N \text{ for all } A \subset U \text{ Borel set},$$

where $\mathrm{d}\mu_N$ *denotes the element of volume Riemannian associated to* g. *Actually, for all* $k \in \{1, \ldots, n\}$, *the measure* $\mathcal{H}_N^k$ *restricts to an k-dimensional submanifold* Σ *in* U *coincides with the Riemannian measure associated with the metric* g *restricts on* Σ.

The proof of this fact can be seen in SIMON (1983) and MAGGI (2012) for $\mathbb{R}^{n+1}$ and FEDERER (1996) for any Riemannian metric in the chapter three. For simplicity from now on, we will denote $\mathcal{H}_N^k$ by $\mathcal{H}^k$ whenever this does not generate problems.

2.2.2 *Rectifiable set*

We start by defining the main concept of this part.

Definition 2.4 (k-dimensional rectifiable set)**.** *We say that a set* E *is an* $k-$*dimensional rectifiable set provided* $E = M_0 \cup (\bigcup_{i=1}^{\infty} M_i)$, *where each* M_i *is a* $k-$*dimensional submanifold of class* C^1 *and* $\mathcal{H}^k(M_0) = 0$.

Example 2.1. *All submanifolds of dimension* k *in* U *are examples of* $k-$*dimensional rectifiable sets.*

Assume for instant that E is an k-dimensional rectifiable set in such a way $\mathcal{H}^k(E \cap K) < +\infty$ for all $K \subset U$ compact. Thus $\mathcal{H}^k \llcorner E$ is a Radon measure in U defined by $\mathcal{H}^k \llcorner E(A) = \mathcal{H}^k(E \cap A)$. Another example of Radon measure can be obtained as follows: let θ be a non-negative locally finite $\mathcal{H}^k$-integrable function on E, then $\mathcal{H}^k \llcorner (\theta, E)$ defined by

$$\mathcal{H}^k \llcorner (\theta, E)(A) := \int_{A \cap E} \theta(p) \mathrm{d}\mathcal{H}^k(p)$$

is a Radon measure in U.

Next, we would like to define the tangent space to an $k-$dimensional rectifiable set. Let E be an k-dimensional rectifiable set in U and $p \in U$. Take r smaller than the injectivity radius of p in N, hence $\exp_p \colon B_r(0) \to B_r(p)$ is a diffeomorphim, and thus $\exp_p^{-1}(E \cap B_r(p))$ is an k-dimensional rectifiable set in $B_r(0) \subset T_pN (= \mathbb{R}^{n+1})$, here $B_r(0)$ denotes the open ball of radius r in T_pN and $B_r(p)$ is the geodesic ball of radius r and center p. Whenever $\lambda > 0$ we define the map $\eta_{p,\lambda} : \mathrm{T}_pN \to \mathrm{T}_pN$ by setting $\eta_{p,\lambda}(q) = \frac{q}{\lambda}$. With these notations we can state the next result about the existence of tangent spaces. The proof of this result when $N = \mathbb{R}^{n+1}$ can be seen in MAGGI (2012).

Proposition 2.1. *Let* E *be an k-dimensional rectifiable set in* U *and* $\theta : E \to R$ *be a locally finite* $\mathcal{H}^k$*-integrable function. Then for* $\mathcal{H}^k$ *almost all* $p \in E$ *there exists an unique* $k-$*dimensional subspace* T_pE *of* T_pN *such that as* $r \to 0$ *it holds*

$$(\eta_{p,\lambda})_{\#}(\exp_p)_{\#}^{-1}\mathcal{H}^k \llcorner (\theta, E) \overset{*}{\rightharpoonup} \theta(p)\mathcal{H}^k \llcorner (T_pE), \tag{1}$$

in T_pN*, here* $(\eta_{p,\lambda})_\#(\exp_p)^{-1}_\#\mathcal{H}^k\llcorner(\theta,E)(A) := \mathcal{H}^k\llcorner(\theta,E)((\exp_p)(\lambda A))$ *and*

$$(\eta_{p,\lambda})_\#(\exp_p)^{-1}_\#\mathcal{H}^k\llcorner(\theta,E) \overset{*}{\rightharpoonup} \theta(p)\mathcal{H}^k\llcorner(T_pE)$$

means

$$\lim_{\lambda\to 0}\int_{T_pN}\phi(q)\mathrm{d}(\eta_{p,\lambda})_\#(\exp_p)^{-1}_\#\mathcal{H}^k_U\llcorner(\theta,E)(q) = \theta(p)\int_{T_pN}\phi(q)\mathrm{d}\mathcal{H}^k\llcorner(T_pE)(q),$$

for all $\phi \in C^0_c(T_pN)$*.*

Remark 2.6. *Notice that the measure which we are adopting in* $T_pN(=\mathbb{R}^{n+1})$ *is the standard Hausdorff measure* $\mathcal{H}^k$ *for* $\mathbb{R}^{n+1}$*. Moreover, the measure* $(\eta_{p,\lambda})_\#(\exp_p)^{-1}_\#\mathcal{H}^k_U\llcorner(\theta,E)$ *in* $B_{\lambda^{-1}r}(0)\subset T_pN$ *is defined by*

$$(\eta_{p,\lambda})_\#(\exp_p)^{-1}_\#\mathcal{H}^k_U\llcorner(\theta,E)(A) = \mathcal{H}^k_U\llcorner(\theta,E)((\exp_p)(\lambda A)) = \int_{(\exp_p)(\lambda A)\cap E}\theta(d)\mathrm{d}\mathcal{H}^k_U(d).$$

Definition 2.5 (Tangent space)**.** *The unique* $k-$*dimensional subspace* T_pE *given by the last proposition is called the tangent space to* E *at* p*.*

Remark 2.7. *It is important to point out here that the definition above coincides with the classical definition of tangent space when* E *is a smooth submanifold in* N*.*

2.2.2.1 *Co-Area Formulae*

We would like to finish this part by reminding the Co-Area formulae. This formula gives a simple way to compute the integral of an $\mathcal{H}^k$-integrable function in term of the level-set of an C^1 function. More precisely, let $\rho : U \to \mathbb{R}$ be a proper locally Lipschtiz function and E be an $k-$dimensional rectifiable set in E, then for all $\mathcal{H}^k$-integrable function u one has

$$\int_{E_r} u(p)\cdot\sqrt{g(\nabla\rho,\nabla\rho)}\mathrm{d}\mathcal{H}^k(p) = \int_{-\infty}^{r}\int_{\rho^{-1}(s)\cap E} u(p)\mathrm{d}\mathcal{H}^{k-1}(p)\mathrm{d}s, \tag{2}$$

where $E_r := E\cap\rho^{-1}(-\infty,r])$.

Remark 2.8. *The proof and extension of this formula can be seen in FEDERER (1996), SIMON (1983) and MAGGI (2012).*

2.2.3 ***Varifolds***

For all $k\in\{1,\dots,n+1\}$ let

$$G_k(U) := \{(p,\Pi)\ :\ p\in M \text{ and } \Pi \text{ is an } k-\text{dimensional subspace of } T_pM\}$$

be the Grassmann manifold of all unorientated $k-$dimensional subspace of TN on U.

Definition 2.6 (Varifold)**.** *We say that V is an k-dimensional varifold in U, if V is a non-negative Radon measure in $G_k(U)$.*

The space of all $k-$dimensional varifolds in U will be denoted by $\mathcal{V}_k(U)$. The topology that we will consider in $\mathcal{V}_k(U)$ is the weak*-topology which is defined as follows: we say that V_i converges weakly* to V on $\mathcal{V}_k(U)$ if for all $\phi \in C_c(G_n(U))$ it holds

$$\lim_{i\to\infty} \int_{G_k(U)} \phi(x,\Pi)\mathrm{d}V_i(x,\Pi) = \int_{G_k(U)} \phi(x,\Pi)\mathrm{d}V(x,\Pi).$$

As $\{V_i\}$ converges weakly to V, then we will write $V_i \overset{*}{\rightharpoonup} V$.

Denote by $\pi : G_k(U) \to U$ the projection map of $G_k(U)$ onto U defined by $\pi(p,\Pi) = p$. Let V be an k-dimensional varifold in U. From V we obtain a Radon measure μ_V in U called the weight measure associated to V by setting $\mu_V(A) = V(\pi^{-1}(A))$.

Definition 2.7. *We say that an k-dimensional varifold V has locally bounded area provided that $\mu_V(F) < \infty$ for all compact set F in U. More generally, we say that a sequence of $k-$dimensional varifolds $\{V_i\}$ has locally bounded area provided that for all compact subset F in U there exists a constant c $(= c(F))$ so that*

$$\mu_{V_i}(F) \le c(F) \text{ for all } i.$$

Next we define the support of an $k-$dimensional varifold V denoted by $\operatorname{spt} V$ as the smallest closed set F so that $\mu_V(U \setminus F) = 0$. In particular, the support of an k-dimensional varifold V is a subset of U. Notice this is not the support of V seen as measure in $G_n(U)$.

Definition 2.8. *An k-dimensional varifold V is called connected provided* $\operatorname{spt} V$ *is a connected subset in U.*

Turn out that if W is an open subset of U, then we can get a natural k-dimensional varifold on $G_k(W)$ from V denoted by $V\llcorner_{G_k(W)}$ by putting $V\llcorner_{G_k(W)}(A) := V(A \cap G_k(W))$. Sometimes later in the proofs we are going to use the same notation for V and its restriction to $G_k(W)$ without any comment. This omission of notation could generate problems later, but whenever we were made to use of this convection, we shall specify the sets.

In order get others important examples varifolds, let E be an k-dimensional rectifiable set in U and θ a non-negative locally finite $\mathcal{H}^k$-integrable function, we define an k-dimensional varifold $V(\theta, E)$ by setting

$$V(\theta,E)(A) := \int_{\{x\in E:\ (x,T_xE)\in A\}} \theta(x)\mathrm{d}\mathcal{H}^k(x) = \int_{E\cap\pi(A)} \theta(x)\mathrm{d}\mathcal{H}^k(x) \text{ for all } A \in G_n(U).$$

Remark 2.9. *Notice that $k-$dimensional submanifolds in U are examples of k-dimensional* *varifolds.*

Before we define two more important subspaces in $\mathcal{V}^k(U)$, we need some notation. Let V be an k-dimensional varifold, we say that V is an k-dimensional rectifiable varifold if there exist k-rectifiable set E in U and non-negative locally finite $\mathcal{H}^k$-integrable function θ so that

$$V = V(\theta, E).$$

If θ is an integer value function, thus we say that V is an k-dimensional integral varifold. With these new notations, we denote by $\mathcal{RV}_k(U)$ the space of all k-dimensional rectifiable varifolds and $\mathcal{IV}_k(U) \subset \mathcal{RV}_k(U)$ the space of all k-dimensional integral varifolds. These two spaces, in fact the second ones, play an important role in this thesis.

2.2.4 *First Variation*

Let us begin this part by reminding the first variation formula for smooth submanifolds. Let $\phi : U \times (-\epsilon, \epsilon) \to U$ be an one-parameter family of proper diffeomorphims in U associated to the C^1 vector field X with compact support on U, if Σ has boundary we suppose that $X|_{\partial\Sigma} \equiv 0$. We know from the differential geometry, see for example (LAWSON, 1980) or (LI, 2012), that the following expression holds

$$\frac{d}{dt}_{t=0}\text{Area}[\phi_t(\Sigma)] = \frac{d}{dt}_{t=0}\mathcal{H}^k[\phi_t(\Sigma)] = \int_\Sigma \text{div}_\Sigma X(p) \mathrm{d}\mathcal{H}^k(p).$$

On the other words, this tells that the function

$$\delta\Sigma(X) := \frac{d}{dt}_{t=0}\text{Area}[\phi_t(\Sigma)] = \int_\Sigma \text{div}_\Sigma X(p) d\mathcal{H}^k(p)$$

is a linear functional on the space of all C^1 vector field in U with compact support in U so that $X|_{\partial\Sigma} = 0$. This is the key point that we would want to comment here, this expression motives we define the first variation of a $k-$dimensional varifold as a linear functional on the set of all C^1 vector field in U with compact support.

Let V be an k-dimensional varifold on U and X be a C^1 vector field in U with compact support. We define the first variation formulae on the space of C^1 vector fields in N with compact support as the linear functional δV defined by

$$\delta V(X) = \int_{G_n(U)} \text{div}_\Pi X(p) \mathrm{d}V(p, \Pi),$$

where here

$$\text{div}_\Pi X(p) = \sum_{i=1}^{k} g\left(\nabla_{E_i} X, E_i\right)$$

and $\{E_i\}$ is an orthonormal basis for Π and ∇ denotes the Levi-Civita connection of U

associated with the Riemannian metric g.

Definition 2.9. *We say that an k-dimensional varifold V is stationary provided that* $\delta V \equiv 0$.

Example 2.2. *k-dimensional minimal hypersurfaces in U are examples of k-dimensional stationary varifolds in U.*

The importance of the k-dimensional stationary varifolds for us here is that they satisfy the monotonicity formula. More precisely, one important consequence of $\delta V \equiv 0$ is that the function

$$r \mapsto e^{\Lambda r} \frac{\mu_V(B_r(p))}{\alpha(k) r^k}$$

is non-decreasing for all $p \in U$ when r is small enough, here the constant Λ depends on the geometry of the ambient space N, see (SIMON, 1983). In particular, for any $k-$dimensional stationary varifolds the density $\Theta^k(V, p)$ always exists at each point $p \in U$.

Definition 2.10. *The s-dimensional density of V at p, denoted by $\Theta^s(V, p)$, is defined by*

$$\Theta^s(V, p) := \lim_{r \to 0} \frac{\mu_V(B_r(p))}{\alpha(s) r^s},$$

here $B_r(p)$ indicates the geodesic ball in U with center p.

We finish this part of our study by stating the following result about the density of stationary varifolds whose proof can be seen in (SIMON, 1983).

Proposition 2.2 (Upper semicontinuous of density)**.** *Let $\{V_i\}$ be a sequence of k-dimensional stationary varifold which converges weakly* to V and $\{p_i\}$ be a sequence of point in U that converges to $p \in U$, then one has*

$$\Theta^k(V, p) \geq \limsup \Theta^k(V_i, p_i).$$

2.2.5 *Compactness theorems*

At this moment, we obtain the compactness theorems for varifolds setting. These theorems will be one of the most important tools of this thesis. Before we state these theorems, we need to introduce some notations.

Definition 2.11. *Let V be an n-dimensional varifold in U. We say that $p \in \operatorname{spt} V$ is a regular point of V provided there exists an open neighbourhood W of p in U so that* $\operatorname{spt} V \cap W$ *is a smooth hypersurface without boundary in W. The set of all regular points of V will be denoted by* $\operatorname{reg} V$. *The set* $\operatorname{sing} V := (\operatorname{spt} V \setminus \operatorname{reg} V) \cap U$ *is called the singular set of V.*

Definition 2.12. *An n-dimensional integral varifold V is called stable provided it is stationary and* $\operatorname{reg} V$ *is stable in the sense of Definition 2.1 in U.*

Example 2.3. *All stable minimal hypersurfaces in N are examples of n-dimensional stable integral varifolds.*

Next we need to come back to use the notations from Section 2.1 for defining what means convergence in an $C^{\infty}-$topology.

Definition 2.13. *Let $\{\Sigma_i\}$ be a sequence of hypersurfaces in U. We say that $\{\Sigma_i\}$ converges in $C^{\infty}-$ topology with finite multiplicity to a smooth embedded hypersurface Σ if*

i. *Σ consists of accumulations points of $\{\Sigma_i\}$, that is, for each $p \in \Sigma$ there exists a sequence $\{p_i\}$ such that $p_i \in \Sigma_i$, for each $i \in \mathbb{N}$, and $p = \lim_i p_i$;*

ii. *For every $p \in \Sigma$ there exists $r, \epsilon > 0$ such that $\Sigma \cap W_{r,\epsilon}(p)$ can be represented as the graph of a function u over $B_r(p)$;*

iii. *For i large enough, the set $\Sigma_i \cap W_{r,\epsilon}(p)$ consists of a finite number, k, independent of i, of graphs of functions $u_i^1, \ldots, u_i^k$ over $D_r(p)$ in such a way for all $l \in \{1, \ldots, k\}$ u_i^l and any of its derivatives converges uniformly to u.*

The multiplicity of a given point $p \in \Sigma$ is defined by k. As $\{\Sigma_i\}$ converges smoothly to Σ, then we will write $\Sigma_i \to \Sigma$.

Once defines this last ingredient, we finally can talk about one of the main tools of this thesis, the compactness theorems. We begin the statement of these results with the compactness theorem for stationary integral varifold due to ALLARD (1972) (see also SIMON (1983)).

Theorem 2.8 (Compactness Theorem for Stationary Integral Varifold)**.** *Let $\{V_i\}$ be a sequence of $n-$dimensional stationary integral varifolds in U whose area is locally bounded in U, then a subsequence of $\{V_i\}$ converge weakly* to an $n-$dimensional stationary integral varifold V in U.*

Remark 2.10. *Notice that in the previous theorem we may have $V = \varnothing$. Indeed, if $V_i := \{x \in \mathbb{R}^{n+1} : \langle x, \mathbf{e}_{n+1}\rangle = i\}$, then $\{V_i\}$ is a sequence of n-dimensional stationary integral varifold in $\mathbb{R}^{n+1}$, endowed with the Euclidean metric, whose area is locally bounded and $V_i \overset{*}{\rightharpoonup} \varnothing$.*

Turn out that when we know that each varifold V_i is stable too, and if the singular set of each V_i satisfies a subtle condition, then we can conclude that the convergence above is stronger than weakly* convergence. This theorem is due to SCHOEN and SIMON (1981) when the singular set has "Hausdorff dimension" at most $n-2$. The strong version that we are going to start in a moment is due to WICKRAMASEKERA (2014)(see also BELLETTINI and WICKRAMASEKRA (2018) and BELLETTINI and WICKRAMASEKRA (2019) for another extension when the varifold has "prescribed mean curvature"). Before we state this theorem, we need one more definition.

Definition 2.14 ($\alpha-$structure hypothesis)**.** *We say that an $n-$dimensional integral varifold V satisfies the $\alpha-$structure hypothesis provided that for all $p \in \mathrm{sing} V$ there exist no $r > 0$ in such a way that $\mathrm{spt}\, V \cap B_r(p)$ is the union of a finite number of embedded $C^{1,\alpha}$ hypersurfaces with boundary in $B_r(p)$, all having a common $C^{1,\alpha}$ boundary in $B_r(p)$ containing p and no two intersecting except along of their boundary, here $B_r(p)$ denotes the geodesic ball in U.*

Theorem 2.9 (Strong Compactness Theorem)**.** *Let $\{V_i\}$ be a sequence of $n-$dimensional stable integral varifolds in U with locally bounded area. Suppose that if $n \geq 7$ then $\mathcal{H}^{n-7+\beta}(\text{sing } V_i \cap U) = 0$ for all $\beta > 0$, if $n = 7$ then* sing $V_i \cap U$ *and if $1 \leq n \leq 6$ then* sing $V_i \cap U = \varnothing$ *for all i and each V_i satisfies the $\alpha-$structure hypothesis. Then there exist a subsequence $\{V_{i_k}\} \subset \{V_k\}$ and an $n-$dimensional stable integral varifold V in U so that:*

i. $V_{i_k} \overset{*}{\rightharpoonup} V$ *in U;*

ii. sing V *satisfies $\mathcal{H}^{n-7+\beta}(\text{sing } V \cap U) = 0$ for all $\beta > 0$ if $n \geq 7$,* sing $V \cap U$ *is discrete if $n = 7$ and* sing $V \cap U = \varnothing$ *if $1 \leq n \leq 6$;*

iii. $\text{spt } V_{i_k} \to \text{spt } V$ *in $U \setminus \text{sing} V$.*

2.2.5.1 *Area blow-up set*

In general, it is not so easy task to prove that a sequence of $n-$dimensional varifolds has locally bounded area. So we need to look for a criterion that ensures over certain conditions the sequence must have locally bounded area. This criterion is due to WHITE (2016), and it will be our focus of study now.

We begin our study with the following definition.

Definition 2.15. *Let Z be a closed set in U. We say that Z is an $(k, 0)$ subset of U provided the following property holds: if $u : U \to \mathbb{R}$ is a C^2 function so that $u|Z$ has a local maximum at p, then*

$$\text{Tr}_k \nabla^2 u \leq 0,$$

where $\nabla^2 u$ denotes the Hessian of u and $\text{Tr}_k \nabla^2$ denote the sum of the k lowest eigenvalues of $\nabla^2 u$ with respect to the Riemannian metric g.

The subtlety of this definition is that $(n, 0)$ sets satisfy a kind of type of the barrier principle, see WHITE (2016) for another extension for $(k, 0)$ sets as $k \leq n - 1$.

Theorem 2.10 (Strong Barrier Principle)**.** *Let Z be an $(n, 0)$ set in U and K be a closed region of U with smooth, connected boundary ∂K such that $Z \subset K$ and so that*

$$g(\vec{\mathbf{H}}_{\partial K}, \nu) \geq 0$$

everywhere on ∂K, where $\vec{\mathbf{H}}_{\partial K}(p)$ denotes the mean curvature vector field at p and ν denotes the unit normal at p to ∂K which point into K. If Z contains any point of ∂K, then it must contain all of ∂K.

The next result tells us that the $(n, 0)$ set comes out naturally as the area blow-up set from the sequence of $n-$dimensional minimal hypersurface. It is important we point out here that this theorem is true when each Σ_i is in fact an $n-$dimensional stationary varifold with "boundary". We only enunciate this version because it is sufficient for our future application, see WHITE (2016) for such extension.

Theorem 2.11 (Blow-up set structure)**.** *Let $\{V_i\}$ be a sequence of $n-$dimensional stationary integral varifold in U without "boundary" and define*

$$\mathcal{B} := \{p \in U\colon \limsup_i \mathcal{H}^n(\Sigma_i \cap B_r(p)) = +\infty \text{ for every } r > 0\},$$

then $\mathcal{B}$ is an $(n,0)$ set in U.
Definition 2.16. *The set $\mathcal{B}$ above is called the area blow-up set of the sequence $\{V_i\}$.*
Remark 2.11. *Although we have not defined what the boundary of an $n-$dimensional varifold means, this will not cause problems for us, because all varifolds that we will use here have no boundary. For example, we can prove that the weakly limit of a sequence of varifolds without boundary has not boundary too.*

2.2.5.2 *Regularity type Allard*

We finish this part by stating a regularity theorem type Allard due to WHITE (2016). Before we state it, we need to introduce some notation.
Definition 2.17 (Converges of sets)**.** *We say that a sequence of subsets $\{S_i\}$ in U converges as set to $S \subset U$ if it holds*

$$S := \{p \in U\colon \limsup_i \operatorname{dist}\{p, S_i\} = 0\} \text{ and so } S = \{p \in U\colon \liminf_i \operatorname{dist}\{p, S_i\} = 0\}.$$

Now the regularity theorem promised can be stated as follows.
Theorem 2.12 (Regularity type Allard)**.** *Let $\{\Sigma_i\}$ be a sequence of $n-$dimensional, properly embedded minimal hypersurface without boundary in U. Suppose that Σ_i converges as sets to a subset of an $n-$dimensional, connected, properly embedded hypersurface without boundary Σ in U. Assume also there exists a point $p \in \Sigma$ and a neighbourhood W of p in U so that $\Sigma_i \cap W$ converges weakly to $\Sigma \cap W$ with multiplicity one. Then $\{\Sigma_i\}$ converges smoothly to Σ and with multiplicity one everywhere.*
Remark 2.12. *Here $\Sigma_i \cap W$ converges weakly to $\Sigma \cap W$ with multiplicity one means*

$$V(1, \Sigma_i \cap W) \overset{*}{\rightharpoonup} V(1, \Sigma \cap W).$$

2.2.6 *Maximum Principle for Varifolds*

In this part we shall obtain the last ingredient that we will need later. Here we are going to obtain the versions of the Theorem 2.1 and Theorem 2.2 and principle of barrier for varifolds setting. These results for varifolds and the compactness theorems from the last section are the two most important tools of this thesis.

Before we go on, we need to define a subtle variation of a varifold be stationary.
Definition 2.18. *Let K be a closed domain of U with smooth boundary ∂K. We say that*

an $n-$dimensional varifold V in K minimizes the area to the first order in K if one holds

$$\delta V(X) \geq 0,$$

for all C^1 vector field X with compact support in U so that $g(X, \nu_{\partial K}) \geq 0$ everywhere on ∂K, where $\nu_{\partial K}$ denotes the unit normal to ∂K which point into K.

Once define this, we can state the next barrier principle due to SOLOMON and WHITE (1989) and WHITE (2010).

Theorem 2.13. *Let K be a closed domain of U with smooth, connected boundary ∂K so that*

$$g(\vec{\mathbf{H}}_{\partial K}, \nu_{\partial K}) \geq 0$$

everywhere on ∂K, where $\vec{\mathbf{H}}_{\partial K}(p)$ denotes the mean curvature vector field at p and ν denotes the unit normal at p to ∂K which point into K. Let V be an n-dimensional varifold that minimizes to the first order in K. Thus

i. If spt *V contains any point of ∂K, then it must contain all of ∂K and the mean curvature of ∂K must be vanish everywhere on ∂K;*

ii. If V is a stationary integral varifold, then V can be written as $W + W'$, where W and W' are stationary integral varifolds, the support of W is ∂K and the support of W' is disjoint from ∂K.

Actually, later we will be interested in the following consequence of this theorem.

Corollary 2.1. *Let K be a closed domain of U with smooth, connected boundary ∂K so that*

$$g(\vec{\mathbf{H}}_{\partial K}, \nu_{\partial K}) \geq 0$$

everywhere on ∂K, where $\vec{\mathbf{H}}_{\partial K}(p)$ denotes the mean curvature vector field at p and ν denotes the unit normal at p to ∂K which point into K. Let V be an n-dimensional connected varifold that minimizes to the first order in K. Thus if $\operatorname{spt} V$ *contains any point of ∂K, then* $\operatorname{spt} V = \partial K$.

We finish our exposition about varifolds setting with the following two results. The first ones was proved by ILMANEN (1996). This result tells when the regular set of an $n-$dimensional stationary varifold is connected.

Theorem 2.14 (Connectedness of the regular set)**.** *Suppose that V an $n-$dimensional stationary varifold in U so that* $\operatorname{spt} V$ *is connected in U and $\mathcal{H}^{n-2}(\operatorname{sing} V) = 0$. Then* $\operatorname{reg} V$ *is connected in U.*

The second result is a sharp generalization of Theorem 2.1 to the varifolds setting due to WICKRAMASEKERA (2014).

Theorem 2.15 (Sharp Maximum Principle for Integral Varifold)**.** *Suppose that V_1 and*

V_2 are $n-$dimensional stationary integral varifolds in U so that

$$\mathcal{H}^{n-1}(\operatorname{spt} V_1 \cap \operatorname{spt} V_2) = 0,$$

then $\operatorname{spt} V_1 \cap \operatorname{spt} V_2 = \varnothing$.

2.3 Translating solitons in $M \times \mathbb{R}$

In this part we obtain the remaining matter that we shall need later to develop this thesis. Here we follow the exposition given by ALÍAS, LIRA, and RIGOLI (2017) and LIRA and MARTÍN (2019) to this subject. For the sake of simplicity throughout this section M denotes a complete oriented Riemannian manifold with a Riemannian metric σ and $c > 0$ is a constant positive.

2.3.1 *Translating Solitons*

Let Σ be a oriented hypersurface in $M \times \mathbb{R}$, we say that a hypersurface Σ is a translating soliton with respect to the parallel vector field ∂_t with translation speed $c \in \mathbb{R}$ provided that

$$\vec{\mathbf{H}} = c\,\partial_t^{\perp},$$

where $\vec{\mathbf{H}}$ is the mean curvature vector field of Σ and $\perp$ indicates the projection onto the normal bundle of Σ. Hence, if N is the unit normal vector field along Σ, then the mean curvature of Σ satisfies

$$H = cg_0(\partial_t, N), \tag{3}$$

where $g_0 = \sigma + dt^2$ denotes the Riemannian product metric in $M \times \mathbb{R}$.

Remark 2.13. *For us the mean curvature H of Σ is the trace of the second fundamental form of Σ.*

Before we go on, let us give one natural example of translating soliton in $M \times \mathbb{R}$.

Example 2.4. *If Σ is a minimal hypersurface on M, then $\Sigma \times \mathbb{R}$ is a translating soliton.*

Remark 2.14. *Notice that the hypersurface $M \times \{t\}$ is not a translating soliton in $M \times \mathbb{R}$.*

As it was proven by ILMANEN (1994) translating solitons are minimal hypersurfaces with respect to the so-called Ilmanen's metric

$$g_c := e^{\frac{2c}{m}t}(\sigma + \mathrm{d}t^2), \tag{4}$$

where $m = dimM$.

Lemma 2.2 (T. Ilmanen)**.** *Translating solitons with translation speed $c \in \mathbb{R}$ are minimal hypersurfaces in the product $M \times \mathbb{R}$ with respect to the Ilmanen's metric $g_c = e^{\frac{2c}{m}t}(\sigma + \mathrm{d}t^2)$.*

Proof. Indeed, let $\{E_i\}$ be an orthonormal frame and N be an unit normal for Σ seen as hypersurface in $M \times \mathbb{R}$ endowed with the Riemannian metric g_0, and consider the orthonormal frame $\{e^{-\frac{c}{m}t}E_i\}$ and unit normal $e^{-\frac{c}{m}t}N$ for Σ seen now as hypersurface in $M\times\mathbb{R}$ endowed with the Riemannian metric g_c. Since g_c is conformal to g_0, then we have the following relationship between the connections ∇^c associated to g_c and ∇^0 associated to g_0

$$\nabla^c_Y X = \nabla^0_Y X + \frac{c}{m}\left\{g_0(X,\partial_t)Y + g_0(Y,\partial_t)X - g_0(X,Y)\partial_t\right\}.$$

From this equality follows

$$\nabla^c_{e^{-\frac{c}{m}t}E_i}(e^{-\frac{c}{m}t}E_i) = e^{-\frac{2c}{m}t}\left[\nabla^0_{E_i}E_i + \frac{c}{m}\left\{g_0(E_i,\partial_t)E_i - \partial_t\right]\right\}.$$

Hence, one holds

$$\begin{aligned}
-H_c &= \operatorname{div}_c(e^{-\frac{c}{m}t}N) = g_c\left(\nabla^c_{e^{-\frac{c}{m}t}E_i}(e^{-\frac{c}{m}t}N), e^{-\frac{c}{m}t}E_i\right)\\
&= -g_c\left(\nabla^c_{e^{-\frac{c}{m}t}E_i}(e^{-\frac{c}{m}t}E_i), e^{-\frac{c}{m}t}N\right)\\
&= -g_c\left(e^{-\frac{2c}{m}t}\left[\nabla_{E_i}E_i + \frac{c}{m}\left\{g_0(E_i,\partial_t)E_i - \partial_t\right]\right\}, e^{-\frac{c}{m}t}N\right)\\
&= -e^{-\frac{c}{m}t}g_0\left(\left[\nabla_{E_i}E_i + \frac{c}{m}\left\{g_0(E_i,\partial_t)E_i - \partial_t\right]\right\}, N\right)\\
&= -e^{-\frac{c}{m}t}\left[g_0\left(\nabla_{E_i}E_i, N\right) - cg_0(\partial_t, N)\right]\\
&= -e^{-\frac{c}{m}t}\left[H - cg_0(\partial_t, N)\right],
\end{aligned}$$

where H_c indicates the mean curvature of Σ as hypersurface of $M \times \mathbb{R}$ with respect to the Riemannian metric g_c and H denotes the mean curvature of Σ as hypersurface of $M \times \mathbb{R}$ with respect to the Riemannian metric g_0. In particular, we have $H_c = e^{-\frac{c}{m}t}\left[H - cg_0(\partial_t, N)\right]$. This complete the proof of the lemma. □

Remark 2.15. *Notice that Ilmanen's metric is not a complete metric in $M\times\mathbb{R}$, however we need that $(M \times \mathbb{R}, g_0)$ be complete.*

Actually, this lemma is not the original viewpoint of ILMANEN (1994), in this moment we come back to endow $M\times\mathbb{R}$ with the metric g_0. Doing this, ILMANEN (1994) saw translating solitons are critical points of the area functional

$$\mathcal{A}_{g_c}[\Sigma] = \int_\Sigma \mathrm{d}\mu^c_\Sigma = \int_\Sigma e^{c\eta}\mathrm{d}\mu_\Sigma$$

where $\eta = t|_\Sigma$ and $\mathrm{d}\mu^c_\Sigma = e^{c\eta}\,\mathrm{d}\mu_\Sigma$ is the area element of Σ induced by g_c. A straightforward calculation shows that the Euler-Lagrange equation associated with this variational problem is

$$H - cg_0(\partial_t, N) = 0.$$

Namely, assume that $\phi : \Sigma \times (-\epsilon, \epsilon) \to M \times \mathbb{R}$ is a normal variation of Σ with support compact on Σ. Suppose that $\phi_*\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)_{|s=0} = vN$, where $v \in C^\infty_c(\Sigma)$ and define $\Sigma_s = \phi_s(\Sigma)$.

From the first variation formulae (see Lemma 5 in (LAWSON, 1980) or (LI, 2012)) we know that

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{d}\mu_{\Sigma_s} = -vH_{\Sigma_s}\mathrm{d}\mu_{\Sigma_s}.$$

Hence, one has

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{A}_{g_c}[\Sigma_s] \;=\; \frac{\mathrm{d}}{\mathrm{d}s}\int_{\Sigma_s}\mathrm{d}\mu^c_{\Sigma_s} = \frac{\mathrm{d}}{\mathrm{d}s}\int_{\Sigma_s}e^{c\eta}\mathrm{d}\mu_{\Sigma_s} = -\int_{\Sigma_s}(H_{\Sigma_s} - cg_0(\partial_t, N_{\Sigma_s}))e^{c\eta}v\mathrm{d}\mu_{\Sigma_s}, \quad (5)$$

where $\mathrm{d}\mu_{\Sigma_s}$ indicates the Riemannian element of area of Σ_s seen as hypersurface of $M\times\mathbb{R}$ with the metric g_0. This proof our claim.

Next we would like to figure out the second variation formulae at $s = 0$. We can compute this as follows: differentiating (5) at $s = 0$, then after simplification one gets

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}_{|s=0}\mathcal{A}_{g_c}[\Sigma_s] \;=\; -\int_{\Sigma}e^{c\eta}v\frac{\mathrm{d}}{\mathrm{d}s}_{|s=0}(H_{\Sigma_s} - cg_0(\partial_t, N_{\Sigma_s}))\mathrm{d}\mu_{\Sigma}.$$

Turn out that if Z is a vector field on Σ, then we have

$$\begin{aligned}
0 \;&=\; \frac{\mathrm{d}}{\mathrm{d}s}g_0((\phi_s)_*(Z), N_{\Sigma_s}) = g_0\left(\nabla_{(\phi_s)_*(\frac{\mathrm{d}}{\mathrm{d}s})}(\phi_s)_*(Z), N_{\Sigma_s}\right) + g_0\left(Z, \frac{\mathrm{d}}{\mathrm{d}s}N_{\Sigma_s}\right)\\
&=\; g_0\left((\phi_s)_*(\nabla_{\partial_s}Z), N_{\Sigma_s}\right) + g_0\left(Z, \frac{\mathrm{d}}{\mathrm{d}s}N_{\Sigma_s}\right)\\
&=\; g_0\left((\phi_s)_*(\nabla_Z\partial_s), N_{\Sigma_s}\right) + g_0\left(Z, \frac{\mathrm{d}}{\mathrm{d}s}N_{\Sigma_s}\right)\\
&=\; g_0\left(\nabla_{(\phi_s)_*(Z)}(\phi_s)_*\left(\frac{\mathrm{d}}{\mathrm{d}s}\right), N_{\Sigma_s}\right) + g_0\left(Z, \frac{\mathrm{d}}{\mathrm{d}s}N_{\Sigma_s}\right).
\end{aligned}$$

Thus, using that $(\phi_s)_*\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)_{|s=0} = vN$, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}s}_{|s=0}N_{\Sigma_s} = -\nabla v.$$

Turn out that this implies that

$$\frac{\mathrm{d}}{\mathrm{d}s}_{|s=0}g_0(\partial_t, N_{\Sigma_s}) = g_0\left(\partial_t, \frac{\mathrm{d}}{\mathrm{d}s}N_{\Sigma_s}\right) = g_0\left(\partial_t, \nabla v\right).$$

On the other hand, using that (see Theorem 32 in (LAWSON, 1980) or (LI, 2012))

$$\frac{\mathrm{d}}{\mathrm{d}s}_{|s=0}H_{\Sigma_s} = \Delta v + (|A|^2 + \mathrm{Ric}(N,N))v,$$

we conclude that the second variation of the area is given by

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}_{|s=0}\mathcal{A}_{g_c}[\Sigma_s] := -\int_{\Sigma}e^{c\eta}v\mathrm{L}_{g_c}[v]\mathrm{d}\mu_{\Sigma},$$

where the Jacobi operator L_{g_c} is defined by

$$\mathrm{L}_{g_c}[v] = \Delta v + c g_0(\partial_t, \nabla v) + (|A|^2 + \mathrm{Ric}_{M\times\mathbb{R}}(N,N))v, \quad v \in C^2(\Sigma),$$

where $|A|$ is the norm of the second fundamental form of Σ and $\mathrm{Ric}_{M\times\mathbb{R}}$ is the Ricci curvature of $M \times \mathbb{R}$, both calculated with respect to the Riemannian metric g_0, and the gradient and the divergent are computed with respect to the metric induces by g_0 on Σ.

Definition 2.19. *We say that a translating soliton Σ in $M \times \mathbb{R}$ is stable provided that*

$$-\int_\Sigma v \mathrm{L}_{g_c}[v] e^{c\eta} \mathrm{d}\mu_\Sigma \geq 0 \text{ for all } v \in C_c^2(\Sigma).$$

Remark 2.16. *Perhaps the better notation for stability above should be g_c-stable to indicate the dependence of the metric g_c, but whenever we use this notion of stability we shall specify what is the metric that the stability is being taken.*

Remark 2.17. *It is important to point out here that this definition of stability coincides with that given in Definition 2.12.*

2.3.1.1 *Jacobi fields*

Next, we would like to study a criterion for deciding when a certain translation solution is stable. Motivated by what happens in the minimal case, this question of deciding when this hypersurface is stable can be obtained by proving that a particular function is a positive Jacobi field. So let us start by finding a particular Jacobi field.

Proposition 2.3. *Let Σ be a translating soliton in $M \times \mathbb{R}$ and Z be a Killing vector field in $M \times \mathbb{R}$ endowed with the metric g_0 in such a way that $g_0(Z, \partial_t)$ is constant on $M \times \mathbb{R}$. Define $u := g_0(Z, N)$ on Σ, then u is a Jacobi field for L_{g_c}, i. e. u satisfies*

$$\mathrm{L}_{g_c}[u] = \Delta u + c g_0(\partial_t, \nabla u) + (|A|^2 + \mathrm{Ric}_{M\times\mathbb{R}}(N,N))u = 0$$

Proof. Indeed, from Proposition 1 in (FORNARI and RIPOLL, 2004) we know that u satisfies

$$\Delta u + g_0(Z, \nabla H) + (\mathrm{Ric}_{M\times\mathbb{R}}(N,N) + |A|^2)u = 0.$$

Next notice that $\nabla u = -2AZ^\top - \nabla_N Z$. Indeed,

$$\begin{aligned} g_0(X, \nabla u) &= X(u) = g_0(\nabla_X Z, N) + g_0(Z, \nabla_X N) = -g_0(\nabla_N Z, X) - g_0(Z, AX) \\ &= -g_0(\nabla_N Z, X) - g_0(AZ^\top, X). \end{aligned}$$

On the other hand, using that $g_0(Z, \partial_t)$ is constant on Σ and so $g_0(\nabla_N Z, \partial_t) = 0$, one obtains that

$$\begin{aligned} g_0(\nabla H, Z) &= Z(H) = cZg_0(\partial_t, N) = cg_0\left(\partial_t, \nabla_Z N\right) = -cg_0\left(\partial_t, AZ^\top\right) \\ &= cg_0\left(\partial_t, \nabla u\right). \end{aligned}$$

Thus, it holds $\mathrm{L}_{g_c}[u] = 0$. □

Next we will see how we can get the stability from the previous result.

Proposition 2.4. *Let Σ be a translating soliton in $M \times \mathbb{R}$ so that there exists Z a Killing vector field in $M \times \mathbb{R}$ endowed with the metric g_0 in such a way that $g_0(Z, N)_{|\Sigma} > 0$ and $g_0(Z, \partial_t)$ is constant on $M \times \mathbb{R}$.. Then Σ is stable.*

Proof. Firstly note that for all $\rho \in C_c^2(\Sigma)$ we have

$$\begin{aligned} \mathrm{L}_{g_c}[\rho u] &= \Delta(\rho u) + cg_0(\partial_t, \nabla(\rho u)) + (\mathrm{Ric}(N, N) + |A|^2)(\rho u) \\ &= \rho \mathrm{L}_{g_c}[u] + u\Delta\rho + cug_0(\partial_t, \nabla\rho) + 2g_0(\nabla\rho, \nabla u) \\ &= u\left[\Delta\rho + cg_0(\partial_t, \nabla\rho)\right] + 2g_0(\nabla\rho, \nabla u). \end{aligned}$$

In turn, this equality and the divergence theorem imply

$$\begin{aligned} 0 &= \int_\Sigma \operatorname{div}\left(\frac{1}{2}u^2 e^{c\eta}\nabla\rho^2\right) \mathrm{d}\mu_\Sigma = \frac{1}{2}\int_\Sigma \left[g_0(\nabla u^2, \nabla\rho^2) + cu^2 g_0(\partial_t, \nabla\rho^2) + u^2\Delta\rho^2\right] e^{c\eta}\mathrm{d}\mu_\Sigma \\ &= \int_\Sigma \left\{2u\rho g_0(\nabla u, \nabla\rho) + cu^2\rho g_0(\partial_t, \nabla\rho) + u^2[\rho\Delta\rho + g_0(\nabla\rho, \nabla\rho)]\right\} e^{c\eta}\mathrm{d}\mu_\Sigma \\ &= \int_\Sigma \left[(\rho u)\mathrm{L}_{g_c}[\rho u] + u^2 g_0(\nabla\rho, \nabla\rho)\right] e^{c\eta}\mathrm{d}\mu_\Sigma. \end{aligned}$$

Therefore

$$-\int_\Sigma (\rho u)\mathrm{L}_{g_c}[\rho u] e^{c\eta}\mathrm{d}\mu_\Sigma = \int_\Sigma u^2 g_0(\nabla\rho, \nabla\rho) e^{c\eta}\mathrm{d}\mu_\Sigma \geq 0 \text{ for all } \rho \in C_c^2(\Sigma). \tag{6}$$

Finally, whenever $\phi \in C_c^2(\Sigma)$ we also have $\rho = \phi/u \in C_c^2(\Sigma)$, so putting this choice of ρ onto (6) one gets

$$-\int_\Sigma \phi \mathrm{L}_{g_c}[\phi] e^{c\eta}\mathrm{d}\mu_\Sigma \geq 0.$$

This completes the proof. □

Remark 2.18. *This Proposition also was proved by SHARIYARI (2015) and ZHOU (2018) when M has dimension two.*

2.3.2 *Vertical translating graphs*

Let Σ be a translating soliton in $M \times \mathbb{R}$. Suppose that Σ is a *vertical graph*, that is

$$\Sigma = \{(x, u(x)) \in M \times \mathbb{R} \ : \ x \in \Omega\}$$

of a smooth function u defined in a domain $\Omega \subset M$ with boundary (possibly empty.) In this case, we denote $\Sigma = \mathrm{Graph}^v[u]$ and we refer to those solitons as *vertical translating graphs*.

As consequence of (3), we would like to conclude that u must satisfy the following partial differential equation

$$\mathrm{div}_M \left(\frac{\nabla u}{W} \right) = \frac{c}{W}, \tag{7}$$

where $W := \sqrt{1 + |\nabla u|^2}$, and the gradient and divergence operators are taken with respect to the metric σ of M. This can be done noting firstly that Σ can be oriented by the unit upward pointing normal vector field

$$N = \frac{1}{W}(\partial_t - \nabla u)$$

with ∇u translated from $x \in \Omega$ to the point $(x, u(x)) \in \Sigma$. Now consider an orthonormal frame $\{E_i\}$ to Σ and a orthonormal frame $\{e_i\}$ to M we compute

$$\begin{aligned} H &= -\mathrm{div}_\Sigma N = -g_0\left(\nabla_{E_i} N, E_i\right) = -g_0\left(\nabla_{E_i} N, E_i\right) - g_0\left(\nabla_N N, N\right) = -\mathrm{div}_{M\times\mathbb{R}} N \\ &= -\mathrm{div}_{M\times\mathbb{R}} \left(\frac{1}{W}(\partial_t - \nabla u) \right) = -\mathrm{div}_{M\times\mathbb{R}} \left(\frac{1}{W}\partial_t \right) + \mathrm{div}_{M\times\mathbb{R}} \left(\frac{1}{W}\nabla u \right) \\ &= -g_0\left(\nabla \left(\frac{1}{W} \right), \partial_t \right) + g_0\left(\nabla_{e_i} \left(\frac{1}{W}\nabla u \right), e_i \right) + g_0\left(\nabla_{\partial_t} \left(\frac{1}{W}\nabla u \right), \partial_t \right) \\ &= g_0\left(\nabla_{e_i} \left(\frac{1}{W}\nabla u \right), e_i \right) = \mathrm{div}_M \left(\frac{\nabla u}{W} \right). \end{aligned}$$

On the other hand, since $H = cg_0\left(\partial_t, N\right) = c/W$ we get the claim.

Remark 2.19. *The equation*

$$\mathrm{div}_M \left(\frac{\nabla u}{W} \right) = H($$

is called the equation of the graphs with prescribed mean curvature H.

2.3.2.1 *Homology inequality for vertical graphs*

Let us continue assuming that $\Sigma := \mathrm{Graph}^v[u]$ is a *vertical translating graph* and N indicates the upward unit normal to Σ, where $u : \Omega \to \mathbb{R}$ is a smooth function. Notice that since ∂_t is a Killing vector field in $M \times \mathbb{R}$ endowed with the Riemannian metric

g_0, and it holds $g_0(N, \partial_t) = 1/W > 0$, since $|\partial_t|^2 = 1$ then we can apply Proposition 2.4 to conclude that Σ is stable. As we will use of this fact many times throughout the thesis, we enunciate it as a lemma.

Lemma 2.3. *All vertical translating graphs are stable.*

Indeed, as we shall prove now using ideas come from SOLOMON (1986), *vertical translating graphs* are in fact strictly area-minimizing inside the cylinder $\Omega \times \mathbb{R}$.

Proposition 2.5. *Let* $u : \overline{\Omega} \to \mathbb{R}$ *a smooth function over a smooth domain* $\Omega \subset M$ *so that* $\mathrm{Graph}^v[u]$ *is a vertical translating graph in* $M \times \mathbb{R}$. *Assume that* Σ *is any other hypersurface in solid cylinder* $\overline{\Omega} \times \mathbb{R}$ *so that* $\partial\Sigma = \partial\mathrm{Graph}^v[u]$. *Then, it holds*

$$\mathcal{A}_{g_c}[\mathrm{Graph}^v[u]] \leq \mathcal{A}_{g_c}[\Sigma].$$

Moreover, the equality is arrived if, and only if $\Sigma = \mathrm{Graph}^v[u]$.

Proof. Suppose first that Σ lies one-side of $\mathrm{Graph}^v[u]$ and let U be the domain in $\overline{\Omega} \times \mathbb{R}$ limited by Σ and $\mathrm{Graph}^v[u]$. Next, consider the vector field X in $\overline{\Omega} \times \mathbb{R}$ obtained from the unit upward pointing normal $N_{\mathrm{Graph}^v[u]}$ to $\mathrm{Graph}^v[u]$ by parallel transport along the flux of ∂_t. That is, X is given by

$$X(p,t) = e^{ct}\left(\frac{\partial_t}{W} - \frac{\nabla u}{W}\right) \text{ for all } (p,t) \in \overline{\Omega} \times \mathbb{R}.$$

We have

$$\begin{aligned}
\mathrm{div}_{M\times\mathbb{R}} X &= \mathrm{div}_{M\times\mathbb{R}}\left[e^{ct}\left(\frac{\partial_t}{W} - \frac{\nabla u}{W}\right)\right] \\
&= cg_0\left(\partial_t, \left(\frac{\partial_t}{W} - \frac{\nabla u}{W}\right)\right)e^{ct} + e^{ct}\,\mathrm{div}_{M\times\mathbb{R}}\left(\frac{\partial_t}{W} - \frac{\nabla u}{W}\right) \\
&= \left[c\frac{1}{W} - \mathrm{div}_M\left(\frac{\nabla u}{W}\right)\right]e^{ct} = 0.
\end{aligned}$$

Thus $\mathrm{div}_{M\times\mathbb{R}} X = 0$, and the divergence theorem applying to U and X implies, up to a sign, that

$$\begin{aligned}
0 &= \int_{\mathrm{Graph}^v[u]} g_0(X, N_{\mathrm{Graph}^v[u]})\mathrm{d}\mathcal{H}^n - \int_\Sigma g_0(X, N_\Sigma)\mathrm{d}\mathcal{H}^n \geq \int_{\mathrm{Graph}^v[u]} e^{ct}\mathrm{d}\mathcal{H}^n - \int_\Sigma e^{ct}\mathrm{d}\mathcal{H}^n \\
&= \mathcal{A}_{g_c}[\mathrm{Graph}^v[u]] - \mathcal{A}_{g_c}[\Sigma].
\end{aligned}$$

This completes the proof when Σ lies oneside of $\mathrm{Graph}^v[u]$. The general case can be obtained by breaking the hypersurface Σ into many parts so that each part lies one-side of $\mathrm{Graph}^v[u]$. Finally, the fact about the equality follows remarking that we can not have the equality if any part of Σ does not lie in $\mathrm{Graph}^v[u]$. ∎

Remark 2.20. *This proposition also was proven by XIN (2015) when* $M = \mathbb{R}^n$.

2.3.3 *Horizontal translating graphs*

Next we would like to define what means a horizontal in $M \times \mathbb{R}$. More precisely, in the context of DAJCZER, HINOJOSA, and LIRA (2008), we would like to define what means a graph over Killing. At this time we will suppose that M is the warped product $S^{n-1} \times_\rho \mathbb{R}$, where the factor S^{n-1} is complete hypersurface endowed with a Riemannian metric ϱ and ρ is any positive smooth function in S^{n-1}. With these notations the Riemannian metric that we are assuming at M is

$$h_0 = \varrho + \rho^2(x)ds^2. \tag{8}$$

In particular the Riemannian metric in $M \times \mathbb{R}$ is

$$g_0 = \varrho + \rho^2(x)ds^2 + dt^2. \tag{9}$$

Remark 2.21. *We know from (DAJCZER, HINOJOSA, and LIRA, 2008) that this structure of the warped product in M is always obtained when M admits a complete non-singular Killing field with integrable orthogonal distribution.*

With this convention for M we define $\mathbb{P} = S^{n-1} \times \mathbb{R}$, with the Riemannian metric $h_0 = \varrho + dt^2$ and write $M^2 \times \mathbb{R} = \mathbb{P} \times_\rho \mathbb{R}$. By a *horizontal graph* in $M \times \mathbb{R} (= \mathbb{P} \times_\rho \mathbb{R})$ over a domain $\Omega \subset \mathbb{P}$ means a hypersurface $\Sigma \subset M \times \mathbb{R}$ given by

$$\Sigma = \{(p, u(p)) \in \mathbb{P} \times_\rho \mathbb{R} \,(= M \times \mathbb{R})\colon p \in \Omega\},$$

where $u\colon \Omega \to \mathbb{R}$ is a smooth function. Sometimes, to simplify the notation, we will write also $\mathrm{Graph}^h[u]$ to mean the horizontal graph of u.

Remark 2.22. *The horizontal graphs, that we are considering in this paper, are graphs in the direction of the Killing field ∂_s. However, we are representing them as "vertical" graphs since they are graphs in $\mathbb{P} \times_\rho \mathbb{R}$ "over" a domain in $\mathbb{P}$. Therefore the last coordinate is the coordinate associated to the flow lines of ∂_s. Moreover, for us a horizontal line means a flow line of the vectorfield ∂_s, i.e. $\{q\} \times \mathbb{R} = \{(q, s) \in \mathbb{P} \times_\rho \mathbb{R} \,(= M \times \mathbb{R})\colon s \in \mathbb{R}\}$.*

We have just seen at Lemma 2.2 that translating solitons in $M \times \mathbb{R}$ are minimal hypersurface in $M \times \mathbb{R}$ endowed the metric $g_c := e^{t\frac{2c}{m}} g_0$. In particular, since we are considering the Riemannian metric $g_0 = h_0 + \rho^2 ds^2 = \varrho + dt^2 + \rho^2(x)ds^2$ in $M \times \mathbb{R}$, the Ilmanen's metric can be written as

$$g_c = e^{t\frac{2c}{m}}(\varrho + dt^2 + \rho^2(x)ds^2) = h_c + e^{t\frac{2c}{m}}\rho^2(x)ds^2,$$

where h_c denotes the restriction of Ilmanen's metric g_c to $\mathbb{P}$. Note that g_c is still a warped metric. Differentiating of the vertical case, in the remain part of this subsection we will always consider the metric h_c in $\mathbb{P}$ and the metric g_c in $\mathbb{P} \times_{e^{t\frac{c}{m}}\rho(x)} \mathbb{R} (= M \times \mathbb{R})$. Also, to simplify the notation we will denote by $f\colon M\times\mathbb{R}\to\mathbb{R}$ the function $f(x,t)=e^{\frac{c}{m}t}\rho(x)$.

Suppose $\Sigma=\mathrm{Graph}^h[u]$ that is a *horizontal translating graph* in $M\times\mathbb{R}$, where $u\colon\Omega\subset\mathbb{P}\to\mathbb{R}$ is a smooth function. Thus Σ can be oriented by the unit upward pointing normal vector field

$$N=\frac{1}{f}\frac{\partial_s}{W}-f\frac{\nabla u}{W},$$

where $W=\sqrt{1+f^2h_c(\nabla u,\nabla u)}$ and to simplify the notation, we continue denoting by ∇u the translation ∇u from $x\in\Omega$ to the point $(x,u(x))\in\Sigma$. Next notice that from (3) we can check that u satisfies the partial differential equation

$$\mathrm{div}_{\mathbb{P}}\left(f^2\frac{\nabla u}{W}\right)=0\quad\text{in }\Omega,\tag{10}$$

where the gradient and divergence are taken with respect to the metric h_c in $\mathbb{P}$. Indeed, observe that

$$\mathrm{N}=\frac{1}{f}\frac{\partial_s}{W}-f\frac{\nabla u}{W}=\frac{\partial_s}{f^2W_f}-\frac{\nabla u}{W_f},$$

where $fW_f=W$. Since Graph$[u]$ is a minimal hypersurface in $(M\times\mathbb{R},g_c)$ we have

$$\begin{aligned}
0&=\mathrm{div}_\Sigma\left(\mathrm{N}\right)=\mathrm{div}_{M\times\mathbb{R}}\left(\mathrm{N}\right)=\mathrm{div}_{M\times\mathbb{R}}\left(\frac{\partial_s}{f^2W_f}-\frac{\bar\nabla u}{W_f}\right)\\
&=\mathrm{div}_{M\times\mathbb{R}}\left(\frac{\partial_s}{f^2W_f}\right)-\mathrm{div}_{M\times\mathbb{R}}\left(\frac{\bar\nabla u}{W_f}\right)\\
&=\frac{1}{f^2W_f}\mathrm{div}_{M\times\mathbb{R}}\left(\partial_s\right)+g_c\left(\nabla\left(\frac{1}{f^2W_f}\right),\partial_s\right)-\mathrm{div}_{M\times\mathbb{R}}\left(\frac{\bar\nabla u}{W_f}\right)\\
&=-\mathrm{div}_{M\times\mathbb{R}}\left(\frac{\bar\nabla u}{W_f}\right)=-\frac{1}{f^2}g_c\left(\bar\nabla_{\partial_s}\left(\frac{\bar\nabla u}{W_f}\right),\partial_s\right)-\mathrm{div}_{\mathbb{P}}\left(\frac{\nabla u}{W_f}\right)\\
&=\frac{1}{f^2}g_c\left(\bar\nabla_{\partial_s}\partial_s,\frac{\bar\nabla u}{W_f}\right)-\mathrm{div}_{\mathbb{P}}\left(\frac{\nabla u}{W_f}\right),
\end{aligned}$$

where ∇u denotes the gradient of u on $(\mathbb{P},h_c=g_c|\mathbb{P})$, $\bar\nabla u$ indicates the gradient of u in $(M\times\mathbb{R},g_c)$ and we have used the fact that $\mathbb{P}$ is totally geodesic in $M\times\mathbb{R}$. As $g_c(\partial_s,\partial_s)=f^2$, one obtains

$$2fg_c(X,\bar\nabla f)=X(f^2)=X(g_c(\partial_s,\partial_s))=2g_c(\bar\nabla_X\partial_s,\partial_s)=-2g_c(\bar\nabla_{\partial_s}\partial_s,X),$$

for all $X\in\mathfrak{X}(M\times\mathbb{R})$. Consequently,

$$\bar\nabla_{\partial_s}\partial_s=-f\bar\nabla f\tag{11}$$

and therefore, using again that $\mathbb{P}$ is totally geodesic, we conclude that

$$0 = h_c\left(\frac{\nabla f}{f}, \frac{\nabla u}{W_f}\right) + \operatorname{div}_{\mathbb{P}}\left(\frac{\nabla u}{W_f}\right) = \frac{1}{f}\operatorname{div}_{\mathbb{P}}\left(f\frac{\nabla u}{W_f}\right) = \frac{1}{f}\operatorname{div}_{\mathbb{P}}\left(f^2\frac{\nabla u}{W}.\right)$$

Remark 2.23. *Different of the vertical case, in the horizontal case seems more simple to work with Ilmanen's metric than the product metric in $M \times \mathbb{R}$. There are two facts that motives this, the first is that we are seeing translating solitons are minimal hypersurfaces in $M \times \mathbb{R}$ and so we can apply the local theory of minimal hypersurfaces in this setting. The second facts is for simplicity even. For example, in Section 3.1 we will need to define other metric in $M \times \mathbb{R}$ conformal to g_c, and we will work with a so-called $f-$geodesic throughout this section and this could generate confusion.*

2.3.3.1 *Homology inequality for horizontal graphs*

Here we will adapt the computations done in subsection 2.3.2.1 above for *horizontal translating graphs* setting. So suppose that $\Sigma := \mathrm{Graph}^h[u]$ is a *horizontal translating graph* and N indicates the upward unit normal along of Σ, where $u : \Omega \to \mathbb{R}$ is a smooth function. Since ∂_s is a Killing vector field in $\mathbb{P} \times \mathbb{R}(= M \times \mathbb{R})$ endowed with the metric g_0, and it satisfies $g_0(N, \partial_s) = f/W > 0$, then from Proposition 2.4 we can conclude the next result.

Lemma 2.4. *All horizontal translating graphs are stable.*

The analogous of Proposition 2.5 for *horizontal translating graphs* setting can be stated as follows.

Proposition 2.6. *Let $u : \overline{\Omega} \to \mathbb{R}$ a smooth function over a domain $\Omega \subset \mathbb{P}$ so that* $\mathrm{Graph}^h[u]$ *is a horizontal translating graph in $\mathbb{P} \times \mathbb{R}(= M \times \mathbb{R})$. Assume that Σ is any other hypersurface in the Killing cylinder $\overline{\Omega} \times \mathbb{R}$ such way that $\partial\Sigma = \partial\mathrm{Graph}^h[u]$. Then, one has*

$$\mathcal{A}_{g_c}[\mathrm{Graph}^h[u]] \leq \mathcal{A}_{g_c}[\Sigma].$$

Moreover, the equality is true provided that $\Sigma = \mathrm{Graph}^h[u]$.

Proof. Essentially the proof of this case follows a similar strategy of the proof of Proposition 2.5, but here we are using the metric g_c in $M \times \mathbb{R}$. Suppose first that Σ lies oneside of $\mathrm{Graph}^h[u]$ and let U be the domain in $\overline{\Omega} \times \mathbb{R}$ limited by Σ and $\mathrm{Graph}^h[u]$. Consider the vector field X in $\overline{\Omega} \times \mathbb{R}$ obtained from the unit upward pointing normal $N_{\mathrm{Graph}^h[u]}$ of $\mathrm{Graph}^h[u]$ by parallel transport across the along line of the flow of ∂_s. That is, X is given by

$$X(p,s) = \frac{1}{f}\frac{\partial_s}{W} - f\frac{\nabla u}{W} \text{ for all } (p,s) \in \overline{\Omega} \times \mathbb{R}.$$

Using that $\mathrm{Graph}^h[u]$ is minimal one gets

$$\mathrm{div}_{\mathbb{P}\times\mathbb{R}}\, X = 0.$$

Thus the divergence theorem applying to U and X implies, up to a sign, that

$$\begin{aligned} 0 &= \int_{\mathrm{Graph}^h[u]} g_c(X, N_{\mathrm{Graph}^h[u]})\mathrm{d}\mu_{\mathrm{Graph}^h[u]} - \int_\Sigma g_c(X, N_\Sigma)\mathrm{d}\mu_H \\ &\geq \int_{\mathrm{Graph}^h[u]} \mathrm{d}\mu_{\mathrm{Graph}^h[u]} - \int_\Sigma \mathrm{d}\mu_\Sigma = \mathcal{A}_{g_c}[\mathrm{Graph}^h[u]] - \mathcal{A}_{g_c}[\Sigma]. \end{aligned}$$

This completes the proof when Σ lies oneside of $\mathrm{Graph}^h[u]$. The general case can be obtained by breaking the hypersurface Σ into many parts so that each part lies one-side of $\mathrm{Graph}^h[u]$. Finally, the fact about the equality follows remarking that we can not have the equality if any part of Σ does not lie in $\mathrm{Graph}^h[u]$. □

3 JENKINS-SERRIN THEORY FOR TRANSLATING GRAPHS

Let M^n be a complete Riemannian manifold and $\Omega \subset M$ be a domain (not necessarily bounded) with piecewise smooth boundary. Assume that the boundary can be composed as $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, where the sets Γ_1 and Γ_2 are disconnected so that any smooth connected component of Γ_i does not intersect any other smooth connected component of Γ_i for $i \in \{1, 2\}$. A classical problem in differential geometry is to find sufficient and necessary conditions for the existence of prescribed mean curvature surfaces with possibly infinite boundary data. More precisely, we want to solve the Dirichlet problem

$$\begin{cases} \operatorname{div}\left(f^2 \frac{\nabla u}{\sqrt{1+f^2|\nabla u|^2}}\right) = H(x, u, \nabla u), & \text{in } \Omega; \\ u = \varsigma, & \text{on } \Gamma_0; \\ u = +\infty, & \text{on } \Gamma_1; \\ u = -\infty, & \text{on } \Gamma_2, \end{cases} \tag{12}$$

where $H\colon M \times C^{2,\alpha}(M) \times TM \to \mathbb{R}$ is a locally Lipschitz function, $f\colon M \to \mathbb{R}$ is a known smooth function and $\varsigma\colon \Gamma_0 \to \mathbb{R}$ is a given continuous function called the continuous data.

The most famous and most important example of solutions of the equation (12) in $M = \mathbb{R}^2$ with $\Omega = [-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$ and $f \equiv 1$ was given by H. Scherk in 1834. Namely, he showed that the function $u = \log(\cos x/\cos y)$ is a solution of (12) with $\Gamma_0 = \varnothing$ and $H \equiv 0$. After this graph becomes known as Scherk's minimal surface.

Passing a hundred years, JENKINS and SERRIN (1966) associated the existence of solutions of (12) when $H \equiv 0$ and $f \equiv 1$ in $M = \mathbb{R}^2$ over bounded domain with algebraic conditions involving the length of "admissible polygons" in the domain. The central idea in (JENKINS and SERRIN, 1966) was using part of Scherk's surface as barrier to study the divergence set associated with a monotone sequence of solution of (12). As consequence of this local analysis over the divergence set and the algebraic conditions over the "admissible polygons" they guaranteed that the divergence set is empty. That way they ensured that a subsequence of a sequence of solution of (12) must converge to a function which is a solution of (12) with prescribed data on Γ_0. After that, the Dirichlet problem (12) becomes known as the Jenkins-Serrin problem.

An important extension of Jenkins and Serrin ideas was carried by SPRUCK (1972) when H is constant. He extended the results of Jenkins and Serrin in $\mathbb{R}^3$, when $f \equiv 1$, $M = \mathbb{R}^2$ and over bounded domains. Besides this, he gave local existence for general domain in $\mathbb{R}^n$. Unfortunately, his approach does not work well for domains with complicated topology in $\mathbb{R}^n$.

Using a very different method, MASSARI (1977) and TOMAINI (1986) studied the case of prescribed mean curvature when $f \equiv 1$, but now H is not constant. They extended the results of (JENKINS and SERRIN, 1966) for solutions of (12) when H satisfies some "structural conditions". Their idea was to replace the algebraic conditions involving the length of "admissible polygons" found out by JENKINS and SERRIN (1966) for conditions on certain functional defined on Caccioppoli sets. An elegant exposition of Massari's ideas in the case $H \equiv 0$ and $f \equiv 1$ can be found in GIUSTI (1984).

More recently the Jenkins-Serrin problem has been studied in many different settings and we mention here the works that we have found. Beginning with NELLI and ROSENBERG (2002) who studied the existence of solution of (12) in $\mathbb{H}^2 \times \mathbb{R}$, when $\Omega \subset \mathbb{H}^2$ is a bounded domain, $H \equiv 0$ and $f \equiv 1$. They results was extended firstly by ROSENBERG (2002) for $\mathbb{S}^2 \times \mathbb{R}$, and it also was extended by PINHEIRO (2009) in a general way for $M^2 \times \mathbb{R}$, when now M^2 denotes a complete Riemannian surface and $\Omega \subset M^2$ is a geodesically convex and bounded domain. MAZET, RODRÍGUEZ, and ROSENBERG (2011) remarked that the results obtained by PINHEIRO (2009) can be extended to more general domains than geodesicaly convex by using the Perron's method. Furthermore, they proved the existence of solutions of (12) when $H \equiv 0$, $f \equiv 1$ and Ω could be an unbounded domain in M.

Using ideas close to the approaching of PINHEIRO (2009), NGUYEN (2014) extended results further into the case of Sol_3 when $H \equiv 0$ and f is a subtle known function. Her idea was to see Sol_3 as the warped space $\mathbb{H}^2 \times_y \mathbb{R}$. After that, she proved that is possible to carry out the Nelli, Rosenberg and Pinheiro ideas into this new ambient. Another interesting extension of now the original viewpoint of JENKINS and SERRIN (1966) ideas was given by YOUNES (2010). They proved the existence of minimal sections of the Riemannian bundle $\pi : \widetilde{PSL}_2(\mathbb{R}) \to \mathbb{H}^2$ over any "admissible domain" in $\mathbb{H}^2$.

A very interesting application of Jenkins and Serrin ideas was obtained by COLLIN and ROSENBERG (2010). They proved the existence of solution of (12) with $f \equiv 1$ and $H \equiv 0$, and now Ω is an "ideal polygon" in $\mathbb{H}^2$. After that, as application of them results they constructed a harmonic diffeomorphism of $\mathbb{H}^2$ into the complex plane $\mathbb{C}$. Later these results was generalized by GÁLVEZ and ROSENBERG (2010) for any Hadamard surface. In fact, almost all results that we have mentioned until now have a natural extension when $f \equiv 1$ and H is a constant, but in this setting the domain must satisfy some conditions over the "reflection" of the edges. As an example of these extensions, we can quote the results obtained by HAUSWIRTH, ROSENBERG, and SPRUCK (2009); FOLHA and MELO (2011); FOLHA and ROSENBERG (2012) and KLASER and MENEZES (2019).

Using a different approach, EICHMAIR and METZGER (2016) studied the existence of Scherk type solutions for the Jang's equation in Riemannian manifolds with dimension at most 7. Moreover, as application their techniques, they proved the existence solution of (12) with $\Gamma_0 = \varnothing$, H constant and $f \equiv 1$ when now M could be a Riemannian manifold of dimension at most 7.

Once we have done this brief digression about the Jenkins-Serrin problem, we can finally say what was our contribution in this setting. In this chapter, we study the Jenkins-Serrin problem for translating graphs in $M \times \mathbb{R}$ in the vertical direction and the horizontal direction. As we have seen at the subsection 2.3.2 and the subsection 2.3.3, the equation (12) also describe the equation of vertical translating graphs, with $f \equiv 1$ and $H = c/\sqrt{1+|\nabla u|^2}$ by (7), and the equation of the horizontal translating graphs, with f a known function which depends on the horizontal (Killing) vector field and $H \equiv 0$ by (10).

Here we will divide our studied into two parts. Firstly, we work in the horizontal case. The problem in this setting comes out because of no completeness of Ilmanen's metric. However, we overcome this difficulty by using ideas which were developed by EICHMAIR and METZGER (2016) and by HOFFMAN, ILMANEN, MARTÍN, and WHITE (2019). In the second part of this chapter we study the equation of vertical translating graphs. This can be carried out by using minimal graphs as a barrier. Unfortunately, this procurement only allows to prove the existence of Jenkins-Serrin solution of type I. Essentially, the problem when we try to execute the whole Jenkins and Serrin ideas in this setting lies in the fact that we must understand which means $H = 1/\sqrt{1+|\nabla u|^2}$ on the equation (12). This term comes out because the vector field ∂_t is only conformal in $M \times \mathbb{R}$ endowed with Ilmanen's metric. However, we would like to point out here that HOFFMAN, MARTÍN, and WHITE (2019) have proven the existence of Scherk *vertical translating graphs* in $\mathbb{R}^3$ when Ω is a *rhombus* domain in $\mathbb{R}^2$.

This chapter is structured into two parts. In the first part, we develop the Jenkins-Serrin theory for *horizontal translating graph*. Besides this, we finish this first part by giving some special examples of "admissible domains" in $\mathbb{R}^3$ and $\mathbb{H}^2 \times \mathbb{R}$. In turn, in the second part we carry out the Jenkins-Serrin theory for *vertical translating graph* setting.

3.1 Horizontal case

Let us remember some notations from Section 2.3.3. In what follows, we will fix $c > 0$. Notice that since we are working at dimension two, then $M = S \times_\rho \mathbb{R}$, where S is either $\mathbb{S}^1$ or $\mathbb{R}$ and $\rho : S \to \mathbb{R}$ is a positive smooth function. Moreover, as we are seeing $M \times \mathbb{R} = \mathbb{P} \times_\rho \mathbb{R}$, where $\mathbb{P} = S \times \mathbb{R}$ with the metric $h_0 = \varphi(x)^2 dx^2 + dt^2$. Thus the Ilmanen's metric can be written as $g_c = h_c + f^2 ds^2$, where $h_c = e^{ct}(\varphi(x)^2 dx^2 + dt^2)$ is the metric induces on $\mathbb{P}$ by g_c and $f^2 = e^{ct}\rho^2(x)$. From now on we always adopt the Riemannian metric g_c in $M \times \mathbb{R}$ and the Riemannian metric h_c in $\mathbb{P}$. Moreover ∇ will denote the Riemannian connection associated to g_c.

We know from (10) that the graph of a function $u : \Omega \to \mathbb{R}$ is a *horizontal* *translating graph* provided that

$$\operatorname{div}_{\mathbb{P}}\left(f^2\frac{\nabla u}{W}\right) = 0 \quad \text{in } \Omega,$$

where $W = \sqrt{1 + f^2 h_c(\nabla u, \nabla u)}$, and the gradient and divergence are taken with respect to the metric h_c in $\mathbb{P}$, and Ω is a domain in $\mathbb{P}$. Moreover, we orient $\mathrm{Graph}^h[u]$ by the unit normal

$$\mathrm{N} = \frac{1}{f}\frac{\partial_s}{W} - f\frac{\nabla u}{W}.$$

3.1.1 *A conformal geometry in* $M \times \mathbb{R}$

Our interest here is to collect some computations from a conformal geometry of $(M \times \mathbb{R}, g_c)$ that we will be used later.

Let $\gamma\colon [0,1] \to M \times \mathbb{R}$ be a parametrized curve in $M \times \mathbb{R}$. We define the f-length of γ, denoted by $\mathfrak{L}_f[\gamma]$, as the length of γ with respect to the conformal metric $f^2 g_c$. That is

$$\mathfrak{L}_f[\gamma] = \int_0^1 f(\gamma(r))\sqrt{g_c(\gamma'(r), \gamma'(r))_{\gamma(r)}}\, dr. \tag{13}$$

We will work with a special type of curves that will play the role of geodesic in this horizontal case.

Definition 3.1. *Let γ be a curve in $M \times \mathbb{R}$. We say that γ is an f-geodesic provided γ is a geodesic in $M \times \mathbb{R}$ with respect to the metric $f^2 g_c$.*

By differential geometry we know that

Proposition 3.1. *Let γ be a curve in $M \times \mathbb{R}$. Then γ is an $f-$geodesic, if and only if,*

$$\bar{\nabla}_{\gamma'}\gamma' = g_c(\gamma', \gamma')\frac{\bar{\nabla} f}{f} - 2g_c\left(\frac{\bar{\nabla} f}{f}, \gamma'\right)\gamma', \tag{14}$$

where $\bar{\nabla}_r\gamma'$ denotes the covariant derivative of γ' along γ with respect to g_c.

Proof. We just need to use the following relationship between the connections associated to the metric g_c and $f^2 g_c$

$$\tilde{\nabla}_Y X = \bar{\nabla}_Y X + g_c\left(X, \frac{\bar{\nabla} f}{f}\right) Y + g_c\left(Y, \frac{\bar{\nabla} f}{f}\right) X - g_c\left(X, Y\right)\frac{\bar{\nabla} f}{f},$$

where $\tilde{\nabla}$ (respectively, ∇) denotes the Levi-Civita connection in $M \times \mathbb{R}$ with the metric $\sigma_c = f^2 g_c = e^{2\log f} g_c$ (respectively, g_c) and the definition of $f-$geodesic. ☐

Definition 3.2 (f-curvature)**.** *Let γ be a curve in $\mathbb{P}$. The (scalar) f-curvature of γ is*

$$\mathrm{k}_f[\gamma] := \mathrm{k}_{h_c}[\gamma] - h_c\left(\frac{\nabla f}{f}, N\right), \tag{15}$$

where $\mathrm{k}_{h_c}[\gamma]$ *denotes the geodesic curvature of* γ *in* $(\mathbb{P}, h_c)$ *and* $N \in T\mathbb{P}$ *denotes the unit normal along* γ*.*

Remark 3.1. *Using again the formulae*

$$\tilde{\nabla}_Y X = \bar{\nabla}_Y X + g_c\left(X, \frac{\bar{\nabla} f}{f}\right) Y + g_c\left(Y, \frac{\bar{\nabla} f}{f}\right) X - g_c(X,Y)\frac{\bar{\nabla} f}{f},$$

where $\tilde{\nabla}$ *(respectively,* ∇*) denotes the Levi-Civita connection in* $M \times \mathbb{R}$ *with the metric* $\sigma_c = f^2 g_c = e^{2\log f} g_c$ *(respectively,* g_c*), we see that the definition above is exactly the definition of geodesic curvature in* $M \times \mathbb{R}$ *with the metric* σ_c*. Notice also that* $\mathbb{P}$ *continues been totally geodesic in* $\mathbb{P} \times \mathbb{R} (= M \times \mathbb{R})$ *with the metric* σ_c*.*

Before proceeding, we will remark some properties of f-geodesics that will be used later.

Proposition 3.2. *We have the following properties*

(*i*) *Let* γ *be a curve in* $\mathbb{P}$*. If* $\gamma \times \mathbb{R} := \{(p,s) \in \mathbb{P} \times_\rho \mathbb{R}(= M \times \mathbb{R}) \colon p \in \gamma,\ s \in \mathbb{R}\}$ *denotes the cylinder over* γ*, then*

$$\mathrm{k}_f[\gamma] = \mathrm{H}_{\gamma \times \mathbb{R}},$$

where $\mathrm{H}_{\gamma\times\mathbb{R}}$ *denotes the mean curvature of* $\gamma \times \mathbb{R}$ *in* $(M \times \mathbb{R}, g_c)$*.*

(*ii*) *A curve* γ *on* $\mathbb{P}$ *is an* f*-geodesic in* $\mathbb{P}$*, if and only if,* γ *is an* f*-geodesic in* $M \times \mathbb{R}$*.*

(*iii*) *Let* γ *be a curve in* $\mathbb{P}$ *and consider the Killing rectangle over* γ*, with height* h*, defined by* $\gamma \times [0,h] = \{(p,s) \in \mathbb{P} \times_\rho \mathbb{R} \colon p \in \gamma,\ s \in [0,h]\}$*, where* $h > 0$*. Then we have*

$$\mathcal{A}_{g_c}[\gamma \times [0,h]] = \int_0^1 \int_0^h f(\gamma(r))\sqrt{h_c(\gamma'(r), \gamma'(r))}\, dr dz = h\mathfrak{L}_f[\gamma],$$

where $\mathcal{A}_{g_c}[\gamma \times [0,h]]$ *denotes the area of* $\gamma \times [0,h]$ *with respect to the metric* g_c*. Note that the length of a segment* $\{((x,t),s) \colon s \in [0,h]\}$ *of a flow line through the point* $(x,t) \in \mathbb{P}$ *is given by* $hf(x,t)$*.*

Proof. Regarding (i) notice that $\{\gamma', \partial_s/f\}$ is an orthonormal frame for $\gamma \times \mathbb{R}$, so one has

$$\begin{aligned}\bar{H}_{\gamma\times\mathbb{R}} &= \left(\bar{\nabla}_{\gamma'}\gamma' + \bar{\nabla}_{\partial_s/f}(\partial_s/f)\right)^{\perp} \\ &= \left(\bar{\nabla}_{\gamma'}\gamma' - \frac{\bar{\nabla} f}{f}\right)^{\perp},\end{aligned}$$

here we are using the fact that $\bar{\nabla}_{\partial_s}\partial_s = -f\bar{\nabla} f$ which was proved in (11). Now if $N \in T\mathbb{P}$ denotes the unit normal to γ, then the horizontal left of N defined by $N(p,s) := N(p)$ is an unit normal vector field along $\gamma \times \mathbb{R}$. Therefore, the scalar mean curvature of $\gamma \times \mathbb{R}$ is given by

$$\begin{aligned} \mathrm{H}_{\gamma\times\mathbb{R}} &= g_c\left(\bar{\nabla}_{\gamma'}\gamma', N\right) - g_c\left(\frac{\bar{\nabla} f}{f}, N\right) \\ &= g_c(\nabla_{\gamma'}\gamma', N) - g_c\left(\frac{\nabla f}{f}, N\right) = \mathrm{k}_f[\gamma]. \end{aligned}$$

This concludes the proof of item i. About (ii), we can see it from (14), since $\mathbb{P}$ is totally geodesic in $M \times \mathbb{R}$. Finally, (iii) can be checking by computing the metric induced by g_c in $\gamma \times \mathbb{R}$ and the definition of area. □

Remark 3.2. *From (ii) above, we see that there is a correspondence between f-geodesics and minimal cylinders over $\mathbb{P}$ in $M \times \mathbb{R}$.*

We finish this part by recalling some properties from the classical theory about the existence of geodesics and exponential mapping that will be used later, see for example DO CARMO (2011) or PETERSEN (2006) for more information about this subject.

Proposition 3.3. *The f-geodesics are critical points of the f-length with respect to proper variations. Moreover, the f-geodesics are local minimizers of the f-length.*

and

Proposition 3.4. *Given any point $p \in \mathbb{P}$, then there exists a neighbourhood $U \ni p$ such that given any $q_1, q_2 \in \overline{U}$ then there is a unique f-geodesic joining q_1 and q_2, and the interior of this f-geodesic lies in U.*

Remark 3.3. *The neighbourhood given by Proposition 3.4 will be called geodesically f-convex neighbourhood.*

3.1.2 *Local existence*

Here we would like to prove the local existence for the equation

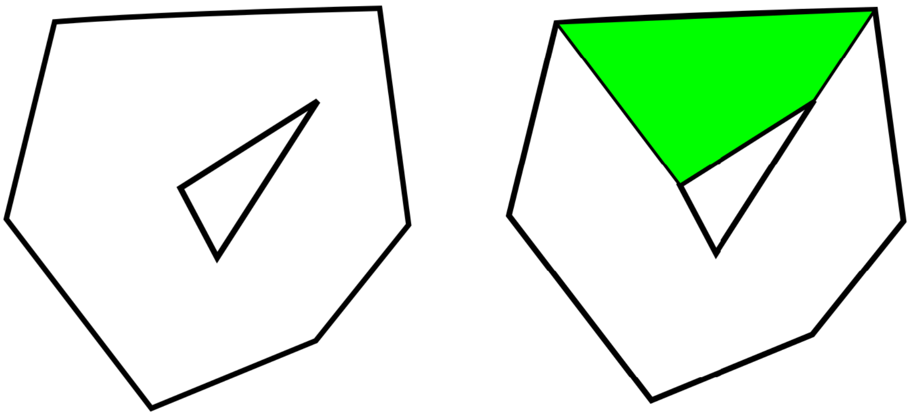
$$\operatorname{div}\left(f^2 \frac{\nabla u}{W}\right) = 0 \tag{16}$$

over admissible domains that are geodesically $f-$convex too. So we start by defining what means a domain be admissible.

Definition 3.3 (Admissible domain)**.** *Let $\Omega \subset \mathbb{P}$ be a precompact domain. We say that Ω is an admissible domain if $\partial\Omega$ is a union of f-geodesic arcs $A_1, \dots, A_s$, $B_1 \dots, B_r$, f-convex arcs $C_1, \dots, C_t$, and the end points of these arcs and no two arcs A_i and no two arcs B_i have a common endpoint, see Figure 1*

Definition 3.4 (Admissible polygon)**.** *Let Ω be an admissible domain. We say that $\mathcal{P}$ is an admissible polygon if $\mathcal{P} \subset \overline{\Omega}$, the boundary of $\mathcal{P}$ is formed by edges of $\partial\Omega$ and f-geodesic arcs on Ω, and the vertices of $\mathcal{P}$ are chosen among the vertices of Ω, see Figure 1*

Figure 1 – Admissible domain (left) and an admissible domain with an admissible polygon (right).



Suppose now that $\Omega \subset \mathbb{P}$ is an admissible domain with $\partial\Omega = \cup_i J_i$, where the family $\{J_i\} \subset \partial\Omega$ is a closed cover of $\partial\Omega$ and satisfies $J_i \cap J_{i+1} = \alpha_i$ for all $i \in \{1, \ldots, v-1\}$, and $J_v \cap J_1 = \alpha_v$, where $\{\alpha_i\}$ denotes the set of endpoints of the arcs J_i. Let $c = \{c_i \colon J_i \to \mathbb{R}\}$ be a family of bounded continuous functions. Consider the curve $\gamma_c \subset \partial\Omega \times \mathbb{R}$ given by $\gamma_c(x) = (x, c_i(x))$ if $x \in \operatorname{int} J_i$ and γ_c is a (horizontal) line joining $(\alpha_i, c_i(\alpha_i))$ and $(\alpha_i, c_{i+1}(\alpha_i))$ if $x = \alpha_i$. As we shall see now it is always possible to get a solution of (16) with boundary data γ_c over a geodesically f-convex domain. Here bounded data γ_c means that the solution equals to c_i on int J_i.

Theorem 3.1 (Local Existence)**.** *Let Ω be a geodesically f-convex domain which is also an admissible domain in $\mathbb{P}$ as above. Let $c = \{c_i \colon J_i \to \mathbb{R}\}$ be a family of bounded continuous functions and γ_c the curve associated to c. Then there exists an unique solution of* (16) *with boundary data c_i on J_i.*

Proof. By Proposition 3.2 (i) the domain bounded formed by the part of the solid cylinder over Ω between $\mathbb{P} + \inf_i c_i$ and $\mathbb{P} + \sup_i c_i$ is piecewise convex in the sense of Definition 2.2 with respect to the metric g_c. Therefore, we can solve the Plateau problem with boundary data γ_c by Theorem 2.5. So it remains to prove that this Plateau's solution is a graph over the domain $\Omega \subset \mathbb{P}$.

Firstly, let us prove that the tangent space at any point of Σ does not contain ∂_s. In fact, suppose that there exists a point $(p_0, r) = p \in \Sigma$ $(p_0 \in \Omega)$ so that $\partial_s \in T_p\Sigma$. Admit $\{\partial_s, v\}$ is an orthonormal basis for $T_p\Sigma$, where $v \in T_p\mathbb{P}_r$, where $\mathbb{P}_r := \{(p, r) \in \mathbb{P} \times \mathbb{R} \ : \ p \in \mathbb{P}\}$. By Proposition 3.4 and Proposition 3.3 there exists an $f-$geodesic α through p at 0 and $\alpha'(0) = v$. Moreover, since Ω is geodesicaly $f-$convex, then α does not accumulate inside Ω and goes out to Ω. This means that α must go out Ω, and clearly α intersects $\partial\Omega$ just at two points. Consider now $\Lambda := \alpha \times \mathbb{R}$ the cylinder over α which is minimal in $M \times \mathbb{R}$ by Remark 3.2 (i). By our assumption Λ and Σ have the same tangent space at p. Therefore, near p, $I = \Sigma \cap \Lambda$ contains at least two curves that intersect transversely at p by Theorem 2.3. If there exists a closed curve β in $I \setminus \partial\Sigma$, then β is the boundary of a minimal disk D in Σ. Thus we could choose a geodesic curve ζ in D so that the totally geodesic surface $\zeta \times \mathbb{R}$ touches D at an interior point. But this is impossible by Theorem 2.1.

Since I does not contain a closed curve, each of the branches leaving p must go to $\partial\Sigma$. Moreover, γ intersects $\partial\Omega$ at two points so at least two of these branches must go to the same point or horizontal segment on $\partial\Sigma$. However, this fact yields again closed curve that bounds a minimal surfaces and we get a contradiction with Theorem 2.1. Therefore $T_p\Sigma$ does not contain ∂_s.

With this information in mind we would like to show that intΣ is a graph. We can work out this as follows: suppose there exist two consecutive points p and q in Σ which lie in the same horizontal line passing through a point of Ω. We know that Σ divides $\Omega \times \mathbb{R}$ at two connected components. So by our hypothesis, we can orient Σ so that $g_c(N, \partial_s) > 0$, where N is the unit normal along Σ. On the other hand, since p and q are consecutive we must have either $g_c(N(p), \partial_s) > 0$ and $g_c(N(q), \partial_s) < 0$ or $g_c(N(p), \partial_s) < 0$ and $g_c(N(q), \partial_s) > 0$ which is impossible. Therefore, any vertical line over any point of Ω intersects Σ just in an unique point. In particular, Σ must be a graph over Ω of a smooth function $u : \Omega \to \mathbb{R}$.

The uniqueness of $\text{Graph}^h[u]$ can be obtained as follows: consider the foliation $\{\text{Graph}^h[u+s]\}_s$ of $\Omega \times \mathbb{R} (\subset \mathbb{P} \times \mathbb{R})$. If there exists other $v : \Omega \to \mathbb{R}$ solution of (16) so that $v|_{J_i} = c_i$ different of u, then $\text{Graph}^h[v]$ must intersect some $\text{Graph}^h[u+s]$ at an interior point which is impossible by Theorem 2.1. □

Remark 3.4. *It is important to point out here that the Theorem 3.1 (and then Theorem 3.2) is not in contradiction with Proposition 30 of (CHINI and MØLLER, 2018) because the cylinder over the domain considered by them is neither f-convex nor an admissible domain in the sense of Definition 3.4. They proved that in $\mathbb{R}^3 = [\mathbf{e}_3]^\perp \times \mathbb{R}$ there are convex domains with respect to the Euclidean metric on $[\mathbf{e}_3]^\perp$ which does not admit horizontal translating graph solution.*

3.1.3 *Interior gradient estimate*

The next step to study the Jenkins-Serrin problem is to understand how we can get the solutions of (16) for more general domains, once that the Theorem 3.1 is only local. As it is classical, this can be done by using Perron's method. However, for use it we need to get a compactness theorem for solutions of (16). In turn, this can be obtained by getting an interior gradient estimate. So allow us to begin by getting the interior gradient estimate.

Proposition 3.5 (Interior gradient estimate)**.** *Let $\{u_n\}$ be a sequence of solutions of (16) on a domain $\Omega \subset \mathbb{P}$, not necessarily admissible neither geodesically $f-$convex. Let*

$p \in \Omega$ and $r > 0$ be small enough so that the g_0-geodesic ball $B_{2r}(p) \subset\subset \Omega$. Assume that $|u_n(q)| \leq K$ for all $n \in \mathbb{N}$ and $q \in B_{2r}(p)$. Then there exists a constant $c > 0$ such that

$$\sup_{q \in B_r(p)} h_c(\nabla u_n(q), \nabla u_n(q)) \leq c \text{ for all } n \in \mathbb{N}.$$

Proof. The proof will be done by contradiction. Assume that

$$\sup_{q \in B_r(p)} h_c(\nabla u_n(q), \nabla u_n(q)) \to +\infty.$$

Thus, up to extracting a subsequence, we would find a sequence $\{x_n\} \subset B_r(p)$ such that

$$h_c(\nabla u_n(x_n), \nabla u_n(x_n)) \to \infty$$

as $n \to \infty$. Since $\overline{B}_r(p)$ is compact in $(M \times \mathbb{R}, g_0)$ (see Remark 2.15) we could assume that $x_n \to x_\infty$ in $(M \times \mathbb{R}, g_c)$. On the other hand $\{u_n(x_n)\}$ is a bounded sequence, so we could also assume $u_n(x_n) \to \alpha$ as $x_n \to x_\infty$.

Let $\Sigma_n = \{(x, u_n(x)) \in \mathbb{P} \times \mathbb{R}(= M \times \mathbb{R}) \colon x \in B_{2r}(p)\}$ be the *horizontal translating graph* of u_n over the ball $B_{2r}(p)$. Then $\{\Sigma_n\}$ is a sequence of stable g_c-minimal surfaces, by Proposition 2.4, with locally bounded area in $\{(x, s) \in \mathbb{P} \times \mathbb{R}(= M \times \mathbb{R}) \colon x \in B_{2r}(p)$ and $s \in \mathbb{R}\}$ since we have

$$\mathcal{A}_c[\Sigma_n] \leq \frac{1}{2}\mathcal{A}_c[K],$$

for all compact subset K of $B_{2r}(p)$ so that ∂K is C^1 by Proposition 2.6. Therefore all conditions of Theorem 2.9 are satisfy, so we could assume, up to a subsequence, that $\Sigma_n \to \Sigma_\infty$, where Σ_∞ is a smooth stable minimal surface inside of the cylinder $B_{2r} \times \mathbb{R}(\subset \mathbb{P} \times \mathbb{R})$, since the singular set is empty at dimension 2. Note that Σ_∞ is not empty because $(x_\infty, \alpha) \in \Sigma_\infty$.

Claim 3.1. *Each connected component of Σ_∞ is a smooth horizontal graph.*

Proof of the Claim 3.1. If the contrary of this is true, then we could suppose that there exists a connected component $S \subset \Sigma_\infty$ that is not a graph over a subset of $B_{2r}(p)$. Because each Σ_n is a graph over $B_{2r}(p)$, and $\Sigma_n \to \Sigma_\infty$ smoothly, we obtain that any horizontal line $(q, \mathbb{R})$, $q \in B_{2r}(p)$, intersects S in a connected subset on S. Since we are assuming that S is not a graph, there exists a horizontal line $(q, \mathbb{R})$, $q \in B_{2r}(p)$, such that $(q, [a, a+\epsilon]) \subset S$ for some ϵ small.

Let $S(\theta) = \{((x, t), s+\theta) \in \mathbb{P} \times \mathbb{R} \colon ((x, t), s) \in S\}$ be a translation of S by θ in the direction of ∂_s. Since $(q, [a, a+\epsilon]) \subset S$, Theorem 2.1 would imply that $S(\theta) = S$ for all $\theta \in (0, \epsilon)$ and it would follow that S is a cylinder $S' \times \mathbb{R} \subset \mathbb{P} \times \mathbb{R}$, where S' is a curve in $B_{2r}(p)$. But this is impossible since each $\Sigma_n \subset \{(x, [-K, K]) \colon x \in B_{2r}(p)\}$. Therefore

S is a horizontal graph of a continuous function u_∞.

To conclude that S is a graph of a smooth function, we will use a Radó-Alexandrov type argument. For this, we denote by

$$\Lambda_\beta = \{((x,t),\beta)\colon (x,t) \in \mathbb{P},\ \beta \in \mathbb{R}\}$$

a foliation of $M \times \mathbb{R}$ by surfaces. Define

$$S_+(\beta) = \{((x,t),s) \in S\colon s \leq \beta\} \text{ and } S_-(\beta) = \{((x,t),s) \in S\colon s \geq \beta\}$$

to be the parts of S that lies on different sides of Λ_β, and

$$S_+^*(\beta) = \{((x,t),\beta - s)\colon ((x,t),s) \in S_+\}$$

the reflection of S_+ with respect to Λ_β. Since S is a graph of a continuous function, $S_+^*(\beta)$ and $S_-(\beta)$ can intersect only along the boundary lying on the plane Λ_β.

Now assume that there exists a point $q = ((x,t),u_\infty(x,t)) \in S$ so that the normal to S at q is perpendicular to ∂_s. Then, reflecting with respect to the plane $\Lambda_{u_\infty(x,t)}$ through q, we would obtain that $S_+^*(u_\infty(x,t))$ and $S_-(u_\infty(x,t))$ would intersect along the plane $\Lambda_{u_\infty(x,t)}$, and they would have a common tangent plane at q so that locally they lie on different sides of this tangent plane. So Theorem 2.2 implies that $S_+^*(u_\infty(x,t)) = S_-(u_\infty(x,t))$ but this is a contradiction since S is a graph. Therefore S is a graph of a smooth function. □

Claim 3.2. *Σ_∞ is connected.*

Proof of the Claim 3.2. Indeed, notice that the projection of Σ_∞ over $B_{2r}(p)$ is onto, because each horizontal line across the point of $B_{2r}(p)$ intersects Σ_n. Now if Σ_∞ was not connected, then we could find a simple closed curve α in $B_{2r}(p) \times \mathbb{R}$ that intersects Σ_∞ at an unique point, since each connected component of Σ_∞ is a horizontal graph. So this curve would intersect Σ_n at a unique point for all n large enough, but this arrives at a contradiction because each simple closed curve in $B_{2r}(p) \times \mathbb{R}$ must intersect α at an even number of points counting the multiplicity. This proves that Σ_∞ is connected. □

Now the assumption $h_c(\nabla u_n(x_n), \nabla u_n(x_n)) \to \infty$ as $n \to \infty$ implies that the normal to Σ_∞ at (x_∞, α) is perpendicular to ∂_s. But this is a contradiction with Σ_∞ being a graph of a smooth function over an $B_{2r}(p)$. Therefore, there exists a constant c so that

$$\sup_{q \in B_r(p)} h_c(\nabla u_n(q), \nabla u_n(q)) \leq c \text{ for all } n \in \mathbb{N}.$$

□

We finish this part by showing how the compactness theorem follows from the interior gradient estimate. Here Ω continues to be a domain, not necessarily admissible domain neither geodesically $f-$convex.

Proposition 3.6 (Compactness Theorem)**.** *Let $\{u_n\}$ be a sequence of solutions of* (16) *on a domain $\Omega \subset \mathbb{P}$. Suppose that $\{u_n\}$ is locally bounded on compact subsets of Ω. Then there exists a subsequence of $\{u_n\}$ that converges smoothly on compact subsets of Ω to a solution u of* (16).

Proof. First of all we have to observe that the Proposition 3.5 tells that for all compact subset $K \subset\subset \Omega$, there exists a constant $c(K) > 0$ (depending on K) so that

$$h_c(\nabla u_n, \nabla u_n) \leq c(K) \text{ on } K \text{ for all } n.$$

Now the Di Giorgi-Nash-Moser estimate implies that for all compact subset $K \subset \Omega$ the $C^{1,\alpha}-$norm of $\{u_n\}$ is bounded by a constant that depends only K. In turn, Schauder's estimates implies that the C^k-norm of $\{u_n\}$ on compact subset $K \subset \Omega$ is bounded by a constant that depends only K.

Now Arzelá-Ascoli's Theorem and the Diagonal argument show that there exist a function $u : \Omega \to \mathbb{R}$ so that a subsequence of $\{u_n\}$ converges uniformly on compact subsets of Ω to u and u is a solution of (16). Here we are using that the restriction of the metric g_c to $\overline{\Omega} \times \mathbb{R}$ is complete, so we can use Arzelá-Ascoli's theorem □

3.1.4 *Perron's method*

As we have mentioned earlier in this part we want to extend Theorem 3.1 over more general domains. Here we will follow the elegant exposition given by GILBARG and TRUDINGER (2001) for Perron's method.

Given $u \in C^0(\Omega)$, we say that u is a *subsolution* in $\Omega \subset \mathbb{P}$ if for all $A \subset\subset \Omega$ and every solution v of (16) in A such that $u \leq v$ on ∂A, we have $u \leq v$ in A. A *supersolution* is defined in a similar way but with opposite inequality.

As we will see now this flexible version of *subsolution* (respectively, *supersolution*) for (16) enjoys of the following useful properties.

(i) A function $u \in C^2(\Omega)$ is a *subsolution* (respectively, *supersolution*) if and only if

$$\mathrm{div}_{\mathbb{P}}\left(f^2\frac{\nabla u}{W}\right) \geq 0 \ \left(\mathrm{div}_{\mathbb{P}}\left(f^2\frac{\nabla u}{W}\right) \leq 0\right);$$

Proof. Namely the maximum principle implies that if $\mathrm{div}_{\mathbb{P}}\left(f^2\frac{\nabla u}{W}\right) \geq 0$, then u is a *subsolution*. Suppose now that u is a *subsolution* and $\mathrm{div}_{\mathbb{P}}\left(f^2\frac{\nabla u}{W}\right) < 0$ at $p \in \Omega$. Take a geodesic ball $B_r(p) \subset\subset \Omega$ which is geodesicaly $f-$convex and $\mathrm{div}_{\mathbb{P}}\left(f^2\frac{\nabla u}{W}\right) < 0$ on $B_r(p)$. By Theorem 3.1, there exists a function $v : B_r(p) \to \mathbb{R}$ solution of (16) so that $v \equiv u$ on $\partial B_r(p)$. In turn the maximum principle implies that $v < u$ in $B_r(p)$

which is impossible since u is a *subsolution*. Therefore, it holds $\mathrm{div}_{\mathbb{P}}\left(f^2 \frac{\nabla u}{W}\right) \geq 0$. $\square$

(ii) Suppose that Ω is a bounded domain. Let $u \in C^0(\Omega)$ be a *subsolution* and $v \in C^0(\Omega)$ be a *supersolution* such that $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω;

Proof. Admit there exists $p \in \Omega$ so that $\sup_\Omega(u - v) = u(p) - v(p) > 0$ and call $M = u(p) - v(p) > 0$. Let $B_r(p) \subset\subset \Omega$ be a geodesic ball in a such that way $u \neq v$ on $\partial B_r(p)$. By Theorem 3.1 there exist functions $\bar{u}, \bar{v} : B_r(p) \to \mathbb{R}$ solutions of (16) so that $\bar{u} = u$ on $\partial B_r(p)$ and $\bar{v} = v$ on $\partial B_r(p)$. By our hypothesis over u and v we must have

$$u \leq \bar{u} \text{ and } \bar{v} \leq v \text{ in } B_r(p).$$

In particular,

$$M \geq \sup_{\partial B_r(p)} (\bar{u} - \bar{v}) \geq (\bar{u}(p) - \bar{v}(p)) \geq (u(p) - v(p)) = M.$$

Hence, by the maximum principle, one has $\bar{u} - \bar{v} \equiv M$ in $B_r(p)$, consequently we also must have $u - v = \bar{u} - \bar{v} = M$ on $\partial B_r(p)$ which is impossible. $\square$

(iii) Let u be a *subsolution* in Ω and A be a subset strictly contained in Ω. Assume that $v \in C^2(A)$ is a solution of (16) with $v = u$ on ∂A. Define a function $U \in C^0(\Omega)$ (called lifting of u in A by v) given by

$$U(p) = \begin{cases} v(p), & p \in A \\ u(p), & p \in \Omega \setminus A. \end{cases}$$

Then U is a *subsolution* in Ω. Similar result holds also for *supersolutions*;

Proof. Notice first of all that $U \geq u$ in Ω, since $v \geq u$ in A by definition of *subsolution*. Now let $B \subset \Omega$ be a domain and w be a solution of (16) in B such that $w \geq U$ on ∂B. This implies that $w \geq U \geq u$ in ∂B. Consequently, since u is *subsolution* it holds $w \geq u$ in B. Therefore, one has $w \geq U$ in $B \setminus A$. In turn, as we also have $w \geq U$ in $\partial B \cap A$ and $U = v$ is a solution of (16) in $B \cap A$, then we must have $w \geq U$ in $B \cap A$ too, by the maximum principle. $\square$

(iv) If $u_1, \ldots, u_r$ are *subsolutions* in Ω, then $u := \max\{u_1, \ldots, u_r\}$ is a *subsolution* in Ω. On the other hand, If $u_1, \ldots, u_r$ are *supersolution* in Ω, then $u := \min\{u_1, \ldots, u_r\}$ is a *supersolution* in Ω.

Proof. In fact, take any domain $B \subset \Omega$ and any solution v of (16) in B so that $v \geq \max\{u_1, \ldots, u_r\}$ on ∂B. Then one has $v \geq u_i$ for all i on ∂B, consequently it holds $v \geq u_i$ for all i in B. In particular, $v \geq \max\{u_1, \ldots, u_r\}$ in B. About the second statement, if we put $v_i = -u_i$, then the first part implies the second part. $\square$

Now suppose that Ω is a bounded domain (not necessarily admissible domain) and let $c : \partial\Omega \to \mathbb{R}$ be a bounded function. We say that a function $u \in C^0(\Omega)$ is a *subfunction* (*superfunction*) relative to c if u is a *subsolution* (*supersolution*) in Ω and $u \leq c$ ($u \geq c$) on $\partial\Omega$. Observe that by (iii) $\inf c$ is a *subfunction* relative to c and $\sup c$ is a *superfunction* relative to c. Denote by $\mathcal{S}_c$ the set of all *subfuctions* relative to c. Essentially, as we shall see now, the Perron's method gives a constructive way to construct solutions of (16) from $\mathcal{S}_c$.

Theorem 3.2 (Perron's method)**.** *The function $u(p) = \sup_{v\in\mathcal{S}_c} v(p)$ is a smooth solution of* (16) *on* Ω.

Proof. The proof follows the same strategy as in (GILBARG and TRUDINGER, 2001). Firstly, notice that $\inf c \leq u \leq \sup c$ by (ii). Secondly, take any point $p \in \Omega$ and let $\{u_n\}$ be a sequence in $\mathcal{S}_c$ such that $u_n(p) \to u(p)$. If replacing u_n by $\max\{u_n, \inf c\}$ we can suppose that u_n is bounded. Thirdly, choose a geodesic ball $B_r(p) \subset\subset \Omega$ so that $\partial B_r(p)$ is $f-$convex. This $f-$convexity allows to get for all n a smooth function $v_n : B_r(p) \to \mathbb{R}$ so that $v_n = u_n$ on $\partial B_r(p)$, by Theorem 3.1. In turn, by (i) and (ii) we must have $u_n \leq v_n$. In particular, if V_n denotes the lifting of u_n in $B_r(p)$ by v_n then $V_n(p) \to u(p)$.

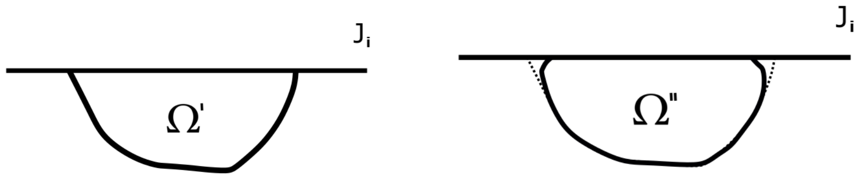
On the other hand, by Proposition 3.6 up to a subsequence $\{V_n\}$ converges on compact subset of $B_r(p)$ to a solution V of (16) in $B_r(p)$, observe that $V(p) = u(p)$. So if we could conclude that $V = u$ in $B_r(p)$ we finish the proof. Namely, we already have $V \leq u$ in $B_r(p)$, so we need to prove that $V \geq u$ in $B_r(p)$. Suppose, then there exists $q \in B_r(p)$ so that $V(q) < u(q)$, therefore there exists $\bar{u} \in \mathcal{S}_c$ so that $V(q) < \bar{u}(q)$. If we define $w_n = \max\{V_n, \bar{u}\}$ then $w_n \in \mathcal{S}_c$ by (iv). Now let W_n be the unique solution of (16) with $W_n = w_n$ on $\partial B_r(p)$ and call $\bar{W}_n$ the lifting of w_n in $B_r(p)$ by W_n. By Proposition 3.6 we can suppose $\{\bar{W}_n\}$ converges to a solution $\bar{W}$ of (16) in $B_r(p)$ and such function satisfies $V \leq \bar{W} \leq u$ in $B_r(p)$. Moreover, as $V(q) < \bar{u}(q)$, we have $V(q) < \bar{W}(q)$ and $\bar{W}(p) = V(p)$, since $V(p) = u(p)$ Hence, by the maximum principle we arrive at a contraction. □

Suppose that Ω is a bounded admissible domain (not necessarily geodesicaly $f-$convex) with $\partial\Omega = \cup J_i$, where J_i's are connected f-convex arcs on $\partial\Omega$ so that $J_i \cap J_k$ is either an endpoint of both arcs or is empty, and let $c = \{c_i : J_i \to \mathbb{R}\}$ be a family of bounded continuous functions. Then, as we will see now the Perron's solution has the specific boundary behaviour.

Theorem 3.3 (Perron's method-boundary data)**.** *Suppose that u is the solution given by Theorem 3.2. Then u satisfies $u = c_i$ on* $\operatorname{int} J_i$.

Proof. Fix an $f-$convex arc J_i. Take any point $p \in \operatorname{int} J_i$ and let $B_r(p)$ a geodesic ball which is $f-$convex. If we call $\Omega' := B_r(p) \cap \bar{\Omega}$, then for r small enough we conclude that Ω' is geodesicaly $f-$convex and Ω' does not intersect any vertices of Ω, see Figure 2. Moreover, by "smoothly" the corner of Ω', we get an C^2 domain $\Omega'' \subset \overline{\Omega}$ which is also geodesicaly $f-$convex such that a part of J_i centred at p lies in $\partial\Omega''$, see Figure 2. In turn

Figure 2 – Representation of Ω'(left) and Ω''(right).



using this domain and by Theorem 3.1 there exist $w^+, w^- : \Omega'' \to \mathbb{R}$ such that $w^\pm = c_i$ in $\partial\Omega'' \cap J_i$, $w^+ \geq \max_i\{\sup_{J_i} c_i\}$ and $w^- \leq \min_i\{\inf_{J_i} c_i\}$. Thus we shall have by the maximum principle and (ii) that

$$w^- \leq u \leq w^+ \text{ in } \Omega''.$$

In particular, we must have $u(p) = c_i(p)$ and u continuous at p. □

3.1.5 *Maximum principle*

This part of the section is devoted to obtain a particular variation of the maximum principle. Here the admissible domains are these according to the definition 3.4.

Theorem 3.4 (Maximum principle). *Let $\Omega \subset \mathbb{P}$ be a bounded admissible domain. Suppose that u_1 and u_2 are solutions of* (16) *such that*

$$\liminf_{x \to \partial\Omega}(u_2(x) - u_1(x)) \geq 0$$

with possible exception of finite number of points $\{q_1, \dots, q_r\} = E \subset \partial\Omega$. Then $u_2 \geq u_1$ in Ω with strict inequality unless $u_2 = u_1$.

Proof. The proof follows a similar strategy of the proof given by (SPRUCK, 1972). We start by defining a function $\varphi\colon \Omega \to \mathbb{R}$ given by

$$\varphi = \begin{cases} K - \epsilon, & \text{if } u_1 - u_2 \geq K; \\ u_1 - u_2 - \epsilon, & \text{if } \epsilon < u_1 - u_2 \leq K; \\ 0, & \text{if } u_1 - u_2 \leq \epsilon, \end{cases}$$

where $K, \epsilon > 0$ are constants, K large and ϵ small. We have that φ is a locally Lipschitz function with $0 \leq \varphi \leq K$, $\nabla\varphi = \nabla u_1 - \nabla u_2$ in the set $\{x \in \Omega\colon \epsilon < u_1(x) - u_2(x) < K\}$ and $\nabla\varphi = 0$ almost everywhere in the complement of this set.

For each point $q_i \in E$, let $B_\epsilon(q_i)$ be an open geodesic disk with center q_i and radius ϵ. Denote $\Omega_\epsilon = \Omega \setminus \cup_i B_\epsilon(q_i)$ and suppose that $\partial\Omega_\epsilon = \tau_\epsilon \cup \rho_\epsilon$, where $\rho_\epsilon =$

$\cup_i(\partial B_\epsilon(q_i)\cap\Omega)$ and $\tau_\epsilon=\partial\Omega_\epsilon\cap\partial\Omega$. Since $\liminf(u_2-u_1)\geq 0$ in $\partial\Omega\setminus E$, we have $\varphi\equiv 0$ in a neighbourhood of τ_ϵ. Define

$$J:=\int_{\rho_\epsilon}\varphi\left[h_c\left(f^2\frac{\nabla u_1}{W_1},\nu\right)-h_c\left(f^2\frac{\nabla u_2}{W_2},\nu\right)\right], \tag{17}$$

where ν is the unit outer normal to Ω_ϵ and $W_i=\sqrt{1+f^2h_c(\nabla u_i,\nabla u_i)}$. From (17), and $0\leq\varphi\leq K$, we obtain from (13) that

$$J\leq 2K\sum_{i=1}^{r}\mathfrak{L}_f[\partial B_\epsilon(q_i)]. \tag{18}$$

On the other hand, since φ is a locally Lipschitz function, we have

$$\begin{aligned}\operatorname{div}_{\mathbb{P}}\left[\varphi\left(f^2\frac{\nabla u_1}{W_1}-f^2\frac{\nabla u_2}{W_2}\right)\right]&=h_c\left(\nabla\varphi,f^2\frac{\nabla u_1}{W_1}-f^2\frac{\nabla u_2}{W_2}\right)\\&\quad+\varphi\left[\operatorname{div}_{\mathbb{P}}\left(f^2\frac{\nabla u_1}{W_1}\right)-\operatorname{div}_{\mathbb{P}}\left(f^2\frac{\nabla u_2}{W_2}\right)\right],\end{aligned}$$

almost everywhere in Ω. Therefore, by the divergence theorem, one obtains

$$\begin{aligned}J&=\int_{\Omega_\epsilon}\left[h_c\left(\nabla\varphi,f^2\frac{\nabla u_1}{W_1}-f^2\frac{\nabla u_2}{W_2}\right)+\varphi\left(\operatorname{div}_{\mathbb{P}}\left(f^2\frac{\nabla u_1}{W_1}\right)-\operatorname{div}_{\mathbb{P}}\left(f^2\frac{\nabla u_2}{W_2}\right)\right)\right]\\&\geq\int_{\Omega_\epsilon}h_c\left(\nabla\varphi,f^2\frac{\nabla u_1}{W_1}-f^2\frac{\nabla u_2}{W_2}\right).\end{aligned} \tag{19}$$

Now if $N_i=\frac{\partial_s}{fW_i}-f\frac{\nabla u_i}{W_i}$, then

$$\begin{aligned}h_c\left(\nabla u_1-\nabla u_2,f^2\frac{\nabla u_1}{W_1}-f^2\frac{\nabla u_2}{W_2}\right)&=g_c\left(N_1-N_2,W_1N_1-W_2N_2\right)\\&=W_1-(W_1+W_2)g_c(N_1,N_2)+W_2\\&=\frac{1}{2}(W_1+W_2)g_c(N_1-N_2,N_1-N_2).\end{aligned} \tag{20}$$

From (18), (19) and (20) we get

$$2K\sum_{i=1}^{r}\mathfrak{L}_f[\partial B_\epsilon(q_i)]\geq\frac{1}{2}\int_{\Omega_\epsilon\cap\{0<u_1-u_2<K\}}(W_1+W_2)g_c(N_1-N_2,N_1-N_2)\geq 0,$$

and in particular, letting $\epsilon\to 0$ we arrive that

$$\int_{\{0<u_1-u_2<K\}}(W_1+W_2)g_c(N_1-N_2,N_1-N_2)=0.$$

Therefore $N_1=N_2$ in $\{x\in\Omega\colon 0<u_1-u_2<K\}$, and consequently also $\nabla u_1=\nabla u_2$ in

the same set. As K was arbitrary, we conclude that $\nabla u_1 = \nabla u_2$ whenever $u_1 > u_2$.

To finish the proof, assume that $\{0 < u_1 - u_2\}$ contains a connected component with non-empty interior. Then, by the previous argument, $u_1 = u_2 + c$, where c is a positive constant, and consequently by maximum principle we have $u_1 = u_2 + c$ in Ω. On the other hand, as $\liminf(u_2 - u_1) \geq 0$ for any approach of $\partial\Omega \setminus E$, c must be a non-positive constant, which is impossible, and therefore $u_2 \geq u_1$. □

3.1.6 *Scherk's translator barrier*

The next step to extend the Jenkins-Serrin theory to the horizontal translating setting it is to construct a specific solution that looks like a part of Scherk's surface. This is the natural generalization of the barriers used by JENKINS and SERRIN (1966) to get information about monotony sequences of solutions of (16). Our proof follows a similar strategy as in (NELLI and ROSENBERG, 2002), (PINHEIRO, 2005, 2009) and (NGUYEN, 2014).

Proposition 3.7 (Scherk's surface). *Let $\Omega \subset \mathbb{P}$ be a geodesically f-convex and admissible domain whose boundary $\partial\Omega$ is an union of four f-geodesic arcs A_1, A_2, C_1 and C_2 so that A_1 and A_2 do not have common endpoints. Assume also that*

$$\mathfrak{L}_f[A_1] + \mathfrak{L}_f[A_2] < \mathfrak{L}_f[C_1] + \mathfrak{L}_f[C_2].$$

Then, given any bounded continuous data $c_i \colon C_i \to \mathbb{R}$, there exists a solution u of (16) *such that $u = c_i$ on C_i and $u \to \infty$ along $A_1 \cup A_2$.*

Proof. The proof will be divided into two cases depending on the continuous boundary data c_i.

Case $c_1 = c_2 \equiv 0$.

Consider the sequence of curves $\{\gamma_n\} \subset \partial\Omega \times \mathbb{R}$, where $\gamma_n(x) = (x, 0)$ for all $x \in \mathrm{C}_1 \cup \mathrm{C}_2$, $\gamma_n(x) = (x, n)$ for all $x \in A_1 \cup A_2$ and γ_n is a "horizontal" segment joining the vertices $(x, 0)$ and (x, n) when x is a vertex of $\partial\Omega$. By Theorem 3.1 there exists a solution $u_n \colon \Omega \to \mathbb{R}$ of (16) with the continuous curve γ_n as the boundary. Moreover, by Theorem 3.4 the sequence $\{u_n\}$ is monotone increasing. So we need to prove that $\{u_n\}$ is locally bounded on compact subsets of Ω and hence, by Theorem 3.6, we can obtain a subsequence of $\{u_n\}$ converging smoothly on compact subsets of Ω to a solution u of (16) satisfying the required properties.

In order of control the sequence on compact subsets of Ω, we construct a minimal cylinder, and for this, consider the minimal disk $D_i^h = C_i \times [0, h]$, that is the rectangle over C_i with height h. Then D_i^h is an area-minimizing, that means that it has least area. Indeed, suppose that Σ is any minimal disk with boundary ∂D_i^h.

As we are considering the metric g_c in $M \times \mathbb{R}$, and so we equip Σ with the Riemannian metric that is the restrictions of g_c onto Σ. If we write $h_\Sigma = s|_\Sigma$ as the "height function" of Σ, we see that

$$\nabla h_\Sigma = (\bar{\nabla} s)^\top = \frac{\partial_s}{f^2} - g_c\left(N_\Sigma, \frac{\partial_s}{f^2}\right) N_\Sigma. \tag{21}$$

Taking the divergence we can conclude that

$$\begin{aligned}
\Delta^\Sigma h_\Sigma &= \mathrm{div}_\Sigma(\nabla h_\Sigma) = \mathrm{div}_\Sigma\left(\frac{\partial_s}{f^2} - g_c\left(N_\Sigma, \frac{\partial_s}{f^2}\right) N_\Sigma\right) \\
&= \mathrm{div}_\Sigma\left(\frac{\partial_s}{f^2}\right) - g_c\left(N_\Sigma, \frac{\partial_s}{f^2}\right) \mathrm{div}_\Sigma\left(N_\Sigma\right) = g_c\left(\partial_s, \nabla_\Sigma\left(\frac{1}{f^2}\right)\right) \\
&= -2g_c\left(\nabla^\Sigma \log f, \nabla^\Sigma h_\Sigma\right),
\end{aligned}$$

in the second line we have used that Σ is minimal and ∂_s is a Killing vector field. Thus, one has $\Delta^\Sigma h_\Sigma + 2g_c\left(\nabla^\Sigma \log f, \nabla^\Sigma h_\Sigma\right) = 0$, and hence h_Σ is harmonic with respect to the weighted Laplacian, so the maximum principle implies that the maximum and the minimum of h_Σ are attained at the boundary of Σ. Therefore, the co-area formulae (2) gives

$$\mathcal{A}_{g_c}[\Sigma] = \int_\Sigma d\mu_\Sigma = \int_0^h \int_{h_\Sigma^{-1}(t)} \frac{1}{\sqrt{g_c(\nabla h_\Sigma, \nabla h_\Sigma)}} ds_t dt.$$

From (21) one obtains

$$\begin{aligned}
g_c(\nabla h_\Sigma, \nabla h_\Sigma) &= g_c\left(\frac{\partial_s}{f^2} - g_c\left(N_\Sigma, \frac{\partial_s}{f^2}\right) N_\Sigma, \frac{\partial_s}{f^2} - g_c\left(N_\Sigma, \frac{\partial_s}{f^2}\right) N_\Sigma\right) \\
&= \frac{1}{f^2} g_c\left(\frac{\partial_s}{f} - g_c\left(N_\Sigma, \frac{\partial_s}{f}\right) N_\Sigma, \frac{\partial_s}{f} - g_c\left(N_\Sigma, \frac{\partial_s}{f}\right) N_\Sigma\right) \leq \frac{1}{f^2}.
\end{aligned}$$

Consequently, by Proposition 3.3 and Remark 3.2 (iii) we have

$$\begin{aligned}
\mathcal{A}_{g_c}[\Sigma] &\geq \int_0^h \int_{h_\Sigma^{-1}(t)} f ds_t dt = \int_0^h \mathfrak{L}_f[h_\Sigma^{-1}(t)] dt \\
&\geq \int_0^h \mathfrak{L}_f[C_i] dt = \mathcal{A}_{g_c}[C_i \times [0,h]] = \mathcal{A}_{g_c}[D_i^h]
\end{aligned}$$

and D_i^h is an area-minimizing with respect to the area functional.

Now, to construct the cylinder, consider first the piecewise cylinder

$$\mathcal{C}_h := \Omega \cup \Omega_h \cup (A_1 \times [0,h]) \cup (A_2 \times [0,h]),$$

where $\Omega_h = \{(p,h) \in \mathbb{P} \times \mathbb{R}(= M \times \mathbb{R})\colon p \in \Omega\}$. As

$$\mathcal{A}_{g_c}(\mathcal{C}_h) = 2\mathcal{A}_{g_c}[\Omega] + \mathcal{A}_{g_c}[A_1 \times [0,h]] + \mathcal{A}_{g_c}[A_2 \times [0,h]],$$

it holds

$$\mathcal{A}_{g_c}[\mathcal{C}_h] - \mathcal{A}_{g_c}[D_1^h] - \mathcal{A}_{g_c}[D_2^h] = 2\mathcal{A}_c[\Omega] + h(\mathfrak{L}_f[A_1] + \mathfrak{L}_f[A_2] - \mathfrak{L}_f[C_1] - \mathfrak{L}_f[C_2]) < 0,$$

provided that $h \geq h_0$ for some h_0 large enough. Fix some $h \geq h_0$, then by Theorem 2.6 there exists a stable minimal cylinder Θ_h inside $\Omega \times \mathbb{R}$ with boundary ∂D_1^h and ∂D_2^h.

Observe that Θ_h is above $\mathrm{Graph}^h[u_n]$ for all n. In fact, if we translate Θ_h to height n we see by Theorem 2.1 that Θ_h does not intersect $\mathrm{Graph}^h[u_n]$. Furthermore, if we translate Θ_h comes back we see that Θ_h does not intersect $\mathrm{Graph}^h[u_n]$ until we arrive in the original position of Θ_h by Theorem 2.1. Consequently, Θ_h is above $\mathrm{Graph}^h[u_n]$ for all n.

Next, denote by Υ the connected component of $\Omega \times \mathbb{R} \setminus \Theta_h$ which is non-compact. Notice that the set $\Upsilon_\kappa = \{(p,s) \in \Upsilon\colon |s| \leq \kappa\}$ is piecewise convex for all $\kappa \geq h$ in the sense of Definition 2.2. So for all $\kappa \geq h$ there exists a stable minimal cylinder $\mathcal{C}_\kappa$ in Υ_κ with boundary $\partial(C_1 \times [0,\kappa]) \cup \partial(C_2 \times [0,\kappa])$ by Theorem 2.6. Notice also that the family $\{\mathcal{C}_\kappa\}$ has locally bounded area in $\Omega \times \mathbb{R}$ since each solution of the Plateau's problem is also a minimum of the area functional amount our hypersurface with the same boundary.

Fix $\varsigma > \kappa$. Translating the cylinder $\mathcal{C}_\kappa$ to height $\varsigma - \kappa$ and coming back to the original position we see that $\mathcal{C}_\kappa$ and $\mathcal{C}_\varsigma$ do not have point contact in $\Omega \times \mathbb{R}$. Moreover, along of the horizontal segment across the endpoints of the arc C_i, we see by Theorem 2.2 that $\mathcal{C}_\kappa$ and $\mathcal{C}_\varsigma$ cannot have the same tangent plane. So the tangent plane of $\mathcal{C}_\varsigma$ on the common part of the horizontal segment across the endpoints of C_i is controlled by the tangent plane of $\mathcal{C}_h$.

Now for all $n > h (n \in \mathbb{N})$ let Σ_n the cylinder obtained by translating $\mathcal{C}_{2n}$ down by height $-n$. Then $\{\Sigma_n\}$ is a sequence of stable hypersurfaces with locally bounded area. At that time we cannot use Theorem 2.9 to conclude that $\Sigma_n \to \Sigma_\infty$, because Σ_n has boundary. However, from Theorem 2.4 and Lemma 2.1, we may conclude what follows.

Claim 3.3. *After passing to a subsequence we have* $\Sigma_n \to (A_1 \times \mathbb{R}) \cup (A_2 \times \mathbb{R})$.

Proof of the Claim 3.3. The proof of this fact can be done as follows. Let $B_r(p)$ be a geodesic ball in $\Omega \times \mathbb{R}$ what does not intersect $\partial(\Omega \times \mathbb{R})$. If we take r small enough we can ensure that each connected component of $\Sigma_n \cap B_r(p)$ is a graph of its tangent plane, by Theorem 2.4 and Lemma 2.1. If there is one component, then a similar argument as in Proposition 3.6 proves that, after passing to a subsequence, we may assume that this sequence converges as graphs to a function defined over a open subset of the plane tangent plane limit of the sequence of tangent plane. If there are more than one component, then we apply this argument to each component.

On the other hand, if $B_r(p)$ intersects $\partial(\Omega \times \mathbb{R})$, then as the tangent plane of $\mathcal{C}_h$. at the boundary control the range of the tangent plane of Σ_n, this implies that we have uniformly estimates at the boundary. Consequently, we can apply the previous argument, with the boundary now, to conclude that, after passing to a subsequence, we can suppose that the sequence $\{\Sigma_n \cap B_r(p)\}$ converges. Now, the diagonal argument joint with a covering of $\overline{\Omega \times \mathbb{R}}$ imply that $\{\Sigma_n\}$ must converges in $\overline{\Omega \times \mathbb{R}}$ to a smooth surfaces Σ_∞ with boundary $\partial(A_1 \times \mathbb{R}) \cup \partial(A_2 \times \mathbb{R})$.

To conclude that $\Sigma_\infty = (A_1 \times \mathbb{R}) \cup (A_2 \times \mathbb{R})$, we parametrize C_i by $\zeta_i : [0,1] \to C_i$. Taking the correct orientation in C_i, we can find a foliation of Ω by $f-$geodesic satisfies what follows: if $t \in [0,1]$ then we denotes by λ_t the unique $f-$geodesic in Ω joint $\zeta_1(t)$ and $\zeta_2(t)$, $\lambda_0 = A_1$ and $\lambda_1 = A_2$. Next, we consider the family of minimal surfaces $\{\lambda_t \times \mathbb{R}\}_{t\in[0,1]}$ in $\Omega \times \mathbb{R}$. If Σ_∞ is not $(A_1 \times \mathbb{R}) \cup (A_2 \times \mathbb{R})$, then we can find a $t \in (0,1)$ so that $\lambda_t \times \mathbb{R}$ touches Σ_∞ either at a finite point or at a infinite point.

Turn out that Theorem 2.1 implies that the first case is impossible. Regarding the second case, it implies that $\mathrm{dist}(\Sigma_\infty, \lambda_t \times \mathbb{R}) = 0$, consequently there exists a sequence of point $\{p_n\}$ in Σ_∞ so that $\lim_n \mathrm{dist}(p_n, \lambda_t \times \mathbb{R}) = 0$, notice that $\{p_n\}$ is away from the boundary of Σ_∞. Let Λ_n be the surface obtained from the translation of Σ_∞ by $-x_n$. Then by the previous argument $\{\Lambda_n\}$, after passing to a subsequence, must converges to smooth surface with boundary Σ' in $\overline{\Omega \times \mathbb{R}}$. Furthermore, Σ' touches $\lambda_t \times \mathbb{R}$ at a finite point, so by Theorem 2.2 we must have $\Sigma' = \lambda_t \times \mathbb{R}$ which is impossible since the boundary of Λ_n is away from $\lambda_t \times \mathbb{R}$. □

In particular, this claim says that the sequence of $\{\pi(\mathcal{C}_{2n})\}$ is an exhaustion of Ω, where π denotes the projection over $\mathbb{P}$. Finally, in order of finish the proof, we must observe that the same argument of the proof of Theorem 3.3 allows us to conclude that $u|_{c_1 \cup c_2} \equiv 0$.

General case (c_1 and c_2 are a bounded function).

Suppose that $|c_i| \leq K$ and let $v\colon \Omega \cup C_1 \cup C_2 \to \mathbb{R}$ be the function of the first case. Let $\{\gamma_n \subset \partial\Omega \times \mathbb{R}\}$ be the sequence of curves, where $\gamma_n(x) = (x, \min\{n, c_i(x)\})$ for all $x \in \mathrm{C}_i$, $\gamma_n(x) = (x, n)$ for all $x \in A_1 \cup A_2$ and γ_n is a horizontal segment joining the vertices $(x, 0)$ and (x, n) when x is a vertex of $\partial\Omega$. By Theorem 3.1 there exists a solution $u_n\colon \Omega \to \mathbb{R}$ of (16) with continuous boundary curve γ_n. Moreover, by Theorem 3.4 the sequence $\{u_n\}$ is monotone non-decreasing and $-K \leq u_n \leq v + K$ in Ω. Hence, by Theorem 3.6 we obtain that $\{u_n\}$ converges smoothly on compact subsets of Ω to a solution u of (16) with the required properties. As it was mentioned earlier, this last claim about the continuous data can be obtained by using the same strategy of the proof of Theorem 3.3. □

Now let us do some applications of the previous result.

Proposition 3.8. *Let $\Omega \subset \mathbb{P}$ be a bounded domain such that $\partial\Omega$ is a union of an f-geodesic arc A and an f-convex arc C with their endpoints. Assume there exists a geodesically f-convex domain $\Omega' \subset \mathbb{P}$ so that $\Omega \subset \Omega'$ and its boundary $\partial\Omega'$ is a union of four f-geodesic arcs A_1, A_2, C_1 and C_2 so that A_1 and A_2 do not have common endpoints and $A \subset A_1$. Moreover assume that*

$$\mathfrak{L}_f[A_1] + \mathfrak{L}_f[A_2] < \mathfrak{L}_f[C_1] + \mathfrak{L}_f[C_2].$$

Then, given any bounded continuous function $\zeta\colon C \to \mathbb{R}$, there exists a solution of (16) *in Ω such that $u \to \infty$ on A and has the continuous boundary data ζ on C.*

Proof. Let $\{\gamma_n\} \subset \partial\Omega \times \mathbb{R}$ be a sequence of curves, where $\gamma_n(x) = (x, \min\{\zeta(x), n\})$ for all $x \in C$, $\gamma_n(x) = (x, n)$ for all $x \in A$ and γ_n is a horizontal segment joining the vertices $(x, 0)$ and $(x, \min\{\zeta(x), n\})$ when x is a vertex of $\partial\Omega$, then by Theorem 3.1 there exists a solution $u_n\colon \Omega \to \mathbb{R}$ of (16) with continuous boundary curve γ_n. Moreover, by Theorem 3.4 the sequence $\{u_n\}$ is an increasing. Now, if v denotes the function over Ω' given by the previous result with continuous data 0, then we must have

$$\inf_C \zeta \leq u_n \leq \sup_C \zeta + v \ \text{ in } \ \Omega$$

by Theorem 3.4. Thus, Theorem 3.6 yields that u_n converges on compact subsets of Ω to a solution u of (16). Finally, if we argue as in the proof of Theorem 3.3 we can conclude that u has the specific continuous data. □

Until now we have proven that the continuous data of the limit of a convergent sequence of solution of (16) can be controlled, if the sequence is defined over specific geodesicaly $f-$convex domains. Now we will extend this fact to encompass the general domains.

Proposition 3.9. *Let $\Omega \subset \mathbb{P}$ be a domain. Suppose that γ is an f-convex arc in $\partial\Omega$. Let $\{u_n\}$ be a sequence of solutions of* (16) *that converges uniformly to a solution u of* (16) *on compact subsets of Ω. Suppose that $u_n \in C^0(\Omega \cup \gamma)$ and $u_n|_\gamma$ converge uniformly on compact subsets of γ to a function $\zeta\colon \gamma \to \mathbb{R}$ that is continuous or $\zeta \equiv \pm\infty$. Then u is continuous in $\Omega \cup \gamma$ and $u|_\gamma = \zeta$.*

Proof. Given $p \in \gamma$, assume that $\zeta(p) > K$, where K is a fixed constant. After of all, let us observe that if we prove that there exists a neighbourhood U of p in $\Omega \cup \gamma$ so that $u > K$ in U to conclude that u is continuous at p. The same argument works if we want to prove the existence of this neighbourhood when $\zeta(p) < K$. In particular, these claims tells that we can argue as in the proof of Theorem 3.3 to conclude that u has the specific continuous data.

In order of proof the previous claim, fix a constant $\bar{K} \in (K, \zeta(p))$. Since $u_n|\gamma$ converge uniformly to ζ on compact subsets of γ, there exists a subarc $\lambda \subset \gamma$ containing p in its interior so that $u_n > \bar{K}$ for all $n \geq n_0$ on λ, for some n_0 large enough. Moreover, we can assume that λ lies in a neighbourhood of p which is geodesically f-convex by taking λ small enough. Notice also that, if λ is small enough, we have two cases to analyse:

(i) λ is an f-geodesic;

(ii) there exists a sequence $\{p_n\} \subset \lambda \setminus \{p\}$ so that $p_n \to p$ and $\mathrm{k}_f[\lambda](p_n) > 0$.

Suppose λ is an f-geodesic, then we can construct an admissible domain $\Delta \subset \Omega$ with four edges A_1, A_2, λ' and λ so that A_1 and A_2 do not have common endpoints, and $\mathfrak{L}_f[A_1] + \mathfrak{L}_f[A_2] < \mathfrak{L}_f[\lambda'] + \mathfrak{L}_f[\lambda]$. By Proposition 3.7 there exists a solution v of (16) so that $v \to \infty$ along $A_1 \cup A_2$, $v = \bar{K}$ on λ and $v = \tilde{K}$ on λ', where $\tilde{K} = \inf_{\lambda'} u_n > -\infty$, since u_n converge on compact subset to u. Now by Theorem 3.4 we conclude $v < u$ in Δ.

On the other hand, if there exists a sequence $\{p_n\} \subset \lambda \setminus \{p\}$ so that $p_n \to p$ and $\mathrm{k}_f[\lambda](p_n) > 0$, then we can get a domain $\Delta \subset \overline{\Omega}$ so that $\partial\Delta = \eta \cup \lambda'$, where λ' is a subarc of λ which contain p in its interior and η is a $f-$geodesic arc joining the endpoints of λ'. By Proposition 3.8 there exists a solution $v : \Delta \to \mathbb{R}$ of (16) so that $v \to \infty$ along η and $v = \bar{K}$ on λ'. Again by Theorem 3.4 we must have $v < u$ in Δ. In particular, in both cases there exists a small neighbourhood U of p in $\Omega \cup \gamma$ so that $u \geq \bar{K} > K$ in U. □

Notice that the previous proof motives the proof of the following proposition.

Proposition 3.10. *Let $\Omega \subset \mathbb{P}$ be a bounded domain and $\gamma \subset \partial\Omega$ be a strictly f-convex curve with respect to inner unit normal to $\partial\Omega$. Suppose that $\{u_n\}$ is a sequence of solutions of* (16) *in Ω such that $u_n \geq \kappa$ (respectively $u_n \leq \kappa$) on γ, where K is a constant. Then given any compact subarc $\lambda \subset \gamma$ there exists a neighbourhood $U(\gamma)$ (depending of γ) in $\overline{\Omega}$ and a constant $K(\gamma) > 0$ (depending of γ) such that $u_n \geq \kappa - K(\gamma)$ (respectively $u_n \leq \kappa + K(\gamma)$) for all n in $U(\gamma)$.*

Proof. The proof can be done as follows: since γ is strictly $f-$convex we can break up γ into some small subarcs $\{\gamma_1, \ldots, \gamma_i\}$ so that $\gamma = \cup\gamma_j$, $\gamma_{j-1} \cap \gamma_j$ is a small not empty subarc on γ and each γ_j lies in a geodesicaly $f-$convex neighbourhood of $\mathbb{P}$. For each γ_j let η_j the $f-$geodesic arc in Ω joining the endpoints of γ_j and call Δ_j the subdomain in $\overline{\Omega}$ with boundary $\gamma_j \cup \eta_j$. Turn out that if we are careful, we can assume that every domain Δ_j satisfies the condition of Proposition 3.8, so for every j there exists a solution $v_j : \Delta_j \to \mathbb{R}$ so that $v_j = \kappa$ on γ_j and $v_j \to -\infty$ (respectively $v_j \to +\infty$) on η_j. Using these functions are barrier we construct the neighbourhood $U(\gamma)$ and find the constant $K(\gamma)$. □

3.1.7 *Straight line lemma*

This section is devoted to give a geometric proof of the straight line lemma by using tools get from geometric measure theory. This lemma says that the unique possibility of a solution of (16) blow-up (respectively down) along of γ is if γ is an $f-$geodesic. The ideas that we will develop here are inspired on the argument of (EICHMAIR and METZGER, 2016).

Lemma 3.1 (Straight line lemma)**.** *Let $\Omega \subset \mathbb{P}$ be a domain such that $\gamma \subset \partial\Omega$ is an open arc and suppose that $u\colon \Omega \to \mathbb{R}$ is a solution of* (16)*. If $u(x) \to \pm\infty$ when $x \to \gamma$, then γ is an f-geodesic.*

Proof. Let us suppose that $u \to +\infty$ along γ. Fix any $p \in \gamma$ and let $B_r((p,0))$ a geodesic ball in $\mathbb{P} \times \mathbb{R}$ $(= M \times \mathbb{R})$ with center $(p,0)$ and radius small so that $B_r((p,0)) \cap (\partial(\Omega \times \mathbb{R}) \setminus (\gamma \times \mathbb{R})) = \varnothing$. Take any sequence $\{p_n\} \subset \Omega$ with $p_n \to p$ and $p_n \in B_r((p,0))$, we also will suppose that $p_n \neq p_m$ if $n \neq m$. Consider the sequence of surface in $\mathbb{P} \times \mathbb{R}$ $(= M \times \mathbb{R})$ given by $\{\Sigma_n = \mathrm{Graph}[u - u(p_n)]\}$ Our hypothesis says that $\Sigma_n \cap B_r((p,0))$ is not empty for all n. Let S_n be the connected component which contains p_n. We know by Proposition 2.4 and Proposition 2.6 that $\{S_n\}$ is a sequence of stable surfaces with bounded area in $B_r((p,0))$, so by Theorem 2.9 we may assume, after passing to a subsequence, $S_n \to S_\infty$, where S_∞ is a smooth not empty surface in $B_r((p,0))$ because $(p,0) \in S_\infty$. In order of conclude the proof, we would like to prove that S_∞ would lie in $\gamma \times \mathbb{R}$. In particular, this last claim implies that γ is smooth on a subarc centred at p.

If there exists any point $q \in S_\infty \setminus \gamma \times \mathbb{R} \subset \overline{\Omega} \times \mathbb{R}$, then by definition of C^∞ convergence there exists a sequence of point $\{q_n\}$ so that $q_n \in S_n$ and $q_n \to q$. Bringing back this information to u, this says that $q_n = (\hat{q}_n, u(\hat{q}_n) - u(p_n))$, consequently it holds $u(\hat{q}_n) \to +\infty$ too. Hence, since we are assuming $B_r((p,0)) \cap \partial(\gamma \times \mathbb{R}) = \varnothing$, then $\{\hat{q}_n\}$ can not accumulate neither in any other part of $\partial\Omega$ unless γ nor inside Ω too. Thus $\hat{q}_n \to \gamma$ which arrives at a contradiction since $q \notin \gamma \times \mathbb{R}$. Therefore $S_\infty \subset \gamma \times \mathbb{R}$. In particular γ must be smooth too. Furthermore, the condition $S_\infty \subset \gamma \times \mathbb{R}$ also implies that γ is a $f-$geodesic by Remark 3.2 (i). □

Before proceeding to prove the next proposition we need to observe the following consequence of the previous proof.

Escólio 3.1. *Let $\Omega \subset \mathbb{P}$ be a domain and $\gamma \subset \partial\Omega$ be an $f-$geodesic. Assume that $u\colon \Omega \to \mathbb{R}$ is a solution of* (16) *so that $u(x) \to \pm\infty$ when $x \to \gamma$. Thus, if $B_r((p,0))$ is a sufficiently small geodesic ball in $\mathbb{P} \times \mathbb{R}$ $(= M \times \mathbb{R})$ with center $(p,0) \in \gamma \times \mathbb{R}$ so that $B_r((p,0)) \cap (\partial(\Omega \times \mathbb{R}) \setminus (\gamma \times \mathbb{R})) = \varnothing$, then there exists n_1 so that if $n > n_1$ then the number of connected component of $\Sigma_n \cap B_r((p,0))$ is exactly one, where $\Sigma_n = \mathrm{Graph}^h[u - u(p_n)]$.*

Proof. We start by defining $A = B_r((p,0)) \cap (\gamma \times \mathbb{R})$ and for $\epsilon << r$ we define $U_\epsilon := \{x \in B_r((p,0))\colon \operatorname{dist}\{x, A\} < \epsilon\}$, $\Upsilon_\epsilon := B_r((p,0)) \setminus U_\epsilon$ and $A_\epsilon := \partial U_\epsilon \setminus A$ (the other connected component of ∂U_ϵ inside $B_r((p,0))$).

Claim 3.4. *Given $\epsilon << r$ there exists n_0 so that if $n > n_0$, then $\Sigma_n \cap B_r((p,0))$ does not intersect Υ_ϵ.*

Proof of the Claim 3.4. In fact, notice that the projection K_ϵ of $\overline{\Upsilon_\epsilon}$ over $\mathbb{P}$ is a compact subset of Ω. Hence, the portion of corresponded of the image K_ϵ in Graph$[u]$ is a compact subset of $\mathbb{P} \times \mathbb{R}$. Therefore, since we are assuming $u(p_n) \to +\infty$, there exists n_0 so that $n > n_0$ implies Σ_n does not intersect intersects Υ_ϵ. □

Now we would like to conclude that the number of connected component is exactly one for n large enough. Otherwise, we could form two sequences $\{\Phi_k\}$ and $\{\Psi_k\}$ so that Φ_k and Ψ_k lie in $\Sigma_{n_k} \cap B_r((p,0))$. By Claim 3.4 for all $\epsilon << r$ there exists k_0 so that $k > k_0$ implies Φ_k and Ψ_k must lie in U_ϵ. This fact joint with the condition of $\{\Phi_k\}$ and $\{\Psi_k\}$ are sequences of stable surfaces with locally bounded area implies that, up to a subsequence, Φ_k and Ψ_k converge smoothly to A, by Theorem 2.9 perhaps with multiplicity.

Now take $\epsilon <<$ Area$[A \cap B_{\frac{r}{2}}((p,0))]$ so that the cylinder C_ϵ in $\partial B_{\frac{r}{2}}((p,0))$ with boundary $\partial\{A \cap B_{\frac{r}{2}}((p,0))\} \cup \partial\{A_\epsilon \cap B_{\frac{r}{2}}((p,0))\}$ satisfies $\mathcal{A}_{g_c}[C_\epsilon] < \mathcal{A}_{g_c}[\cap B_{\frac{r}{2}}((p,0))]$. Using the previous information about the convergence of Φ_k and Ψ_k we may conclude that there exists k_0 so that if $k > k_0$ then Φ_k and Ψ_k lie in U_ϵ and

$$\left|\mathcal{A}_{g_c}[A \cap B_{\frac{r}{2}}((p,0))] - \mathcal{A}_{g_c}[\Phi_k \cap B_{\frac{r}{2}}((p,0))]\right| \;<\; \frac{\epsilon}{2}$$

and

$$\left|\mathcal{A}_{g_c}[A \cap B_{\frac{r}{2}}((p,0))] - \mathcal{A}_{g_c}[\Psi_k \cap B_{\frac{r}{2}}((p,0))]\right| \;<\; \frac{\epsilon}{2}.$$

In particular, if $k > k_0$ the cylinder B_k in $\partial B_r((p,0))$ with boundary $\partial\Phi_k$ and $\partial\Psi_k$ satisfies

$$\begin{aligned} \mathcal{A}_{g_c}[B_k] \;&\leq\; \mathcal{A}_{g_c}[C_\epsilon] < \frac{1}{2}\mathcal{A}_{g_c}[A \cap B_{\frac{r}{2}}((p,0))] \\ &<\; \frac{1}{2}\left\{\mathcal{A}_{g_c}[\Phi_k \cap B_{\frac{r}{2}}((p,0))] + \mathcal{A}_{g_c}[\Psi_k \cap B_{\frac{r}{2}}((p,0))]\right\} + \frac{\epsilon}{2} \end{aligned} \tag{22}$$

On the other hand, by Proposition 2.6 it holds

$$\mathcal{A}_{g_c}[B_k] > \mathcal{A}_{g_c}[\Phi_k \cap B_{\frac{r}{2}}((p,0))] + \mathcal{A}_{g_c}[\Psi_k \cap B_{\frac{r}{2}}((p,0))]$$

for all k. So from (22), if $k > k_0$ one has

$$\epsilon > \mathcal{A}_{g_c}[\Phi_k \cap B_{\frac{r}{2}}((p,0))] + \mathcal{A}_{g_c}[\Psi_k \cap B_{\frac{r}{2}}((p,0))],$$

which is impossible. Therefore, there exists $n_1 > n_0$ so that if $n > n_1$ then the number of connected component on $\Sigma_n \cap B_r((p,0))$ is exactly one. It is important we point out here that the strategy above also proves that the multiplicity of the convergence

$\Sigma_n \cap B_r((p,0)) \to A$ is one. $\square$

Before we state the next proposition we need some notation. Let $\gamma \subset \partial\Omega$ be a smooth open arc. We know that in a small neighbourhood U of γ in $\overline{\Omega}$ the distance function $\mathrm{dist}(\gamma,\cdot)$ is smooth function and $(r,q) \in [0,\epsilon) \times \gamma \to \exp_q(q - r\nu(q))$ is a local coordinate to U, where $\exp_q$ denotes the exponential map at q and ν denotes the unit outer normal to γ with respect to Ω.

Proposition 3.11. *Suppose that $u\colon \Omega \to \mathbb{R}$ is a solution of (16) and $\gamma \subset \partial\Omega$ is an f-geodesic. Then for every $\delta \in (0,1)$ and every compact arc $\lambda \subset \gamma$ there exists $\eta(\delta,\lambda) > 0$ so that if $\mathrm{dist}(p,\lambda) < \eta$, then*

$$1 \geq h_c\left(f\frac{\nabla u}{W},\nu\right)(p) \geq 1-\delta\ ,\ \ \textit{if } u \to +\infty \ \textit{along}\ \gamma$$

and

$$-1 \leq h_c\left(f\frac{\nabla u}{W},\nu\right)(p) \leq -1+\delta\ ,\ \ \textit{if } u \to -\infty \ \textit{along}\ \gamma,$$

where $W^2 = 1 + f^2 g_c(\nabla u,\nabla u)$ and ν denotes the unit outer normal along $\partial\Omega_\epsilon := \{x \in \Omega\ :\ \mathrm{dist}(x,\partial\Omega) = \epsilon\}$ for all $\epsilon \in (0,\eta(\lambda,\delta))$

Proof. Assume that $u \to +\infty$, the same proof works when $u \to -\infty$. If our claim is not true, then there exist $\delta \in (0,1)$ and a sequence $\{p_n\} \subset U \subset \Omega$ so that $\mathrm{dist}(\lambda,p_n) \to 0$ but

$$-\,h_c(N(p_n,u(p_n)),\nu) = h_c\left(f\frac{\nabla u}{W},\nu(p_n)\right)(p_n) \leq 1-\delta, \tag{23}$$

where N denotes the unit upward normal to Graph$[u]$. Thus, up to a subsequence, we may assume that $p_n \to p \in \gamma$. Now we are going to use the argument of the last proof to conclude the proof. Let $B_r((p,0))$ be a geodesic ball in $\mathbb{P}\times\mathbb{R}$ $(= M\times\mathbb{R})$ with center $(p,0)$ and radius small so that $B_r((p,0)) \cap \{\partial(\Omega\times\mathbb{R}) \setminus (\gamma\times\mathbb{R})\} = \varnothing$. Consider the sequence of surface $\{\Sigma_n = \text{Graph}[u-u(p_n)]\} \subset \mathbb{P}\times\mathbb{R}$ $(= M\times\mathbb{R})$. By the previous argument, we may assume that Σ_n has one unique connected component S_n inside $B_r((p,0))$, and therefore $(p_n,0) \in S_n$. So $\{S_n\}$ is a sequence of stable surfaces with locally bounded area in $B_r((p,0))$, and consequently it holds $S_n \to S_\infty$, where $S_\infty = (\gamma\times\mathbb{R}) \cap B_r((p,0))$, but this contraction (23) unless $h_c(N(p_n,u(p_n)),\nu(p_n)) \to 1$. However, this is not the case, because N is the unit upward normal to Graph$[u]$. $\square$

3.1.8 *Flux formula*

This subsection is dedicated to the study of the flow formula. So first of all, let us start by defining what means that. Let $\Omega \subset \mathbb{P}$ be a domain such that $\partial\Omega$ is C^1 smooth. Suppose $u : \Omega \to \mathbb{R}$ is a solution of (16), then by the divergence theorem, one has

$$\int_{\partial\Omega} \frac{f^2}{W} h_c(\nabla u, \nu) = 0,$$

where ν is unit outer normal to $\partial\Omega$. This motivates us to define a flux

$$\mathrm{F}_u[\gamma] = \int_\gamma \frac{f^2}{W} h_c(\nabla u, \nu). \tag{24}$$

We would conclude that (24) makes sense ever when u is not smooth on γ. Namely, let α be a curve in Ω with the same endpoint of γ and call D the domain in Ω with boundary $\gamma \cup \alpha$, motivated by the divergence theorem applied to D, we define

$$\int_\gamma \frac{f^2}{W} h_c(\nabla u, \nu) := -\int_\alpha \frac{f^2}{W} h_c(\nabla u, \nu),$$

where ν denotes the unit outer normal to D. It remains to conclude that the previous definition independent of the α. In fact, let β be another curve with the same endpoints of γ and D the 2−chair in Ω with boundary $\alpha \cup \beta$. The divergence theorem applied to D allows us to conclude

$$\int_\beta \frac{f^2}{W} h_c(\nabla u, \nu) := -\int_\alpha \frac{f^2}{W} h_c(\nabla u, \nu),$$

where ν denotes here the unit outer normal to D. In particular, regarding the orientation on α and β endowed by γ one gets

$$\int_\beta \frac{f^2}{W} h_c(\nabla u, \nu) := \int_\alpha \frac{f^2}{W} h_c(\nabla u, \nu),$$

Now we will collect some properties of the flux formula that it will be useful later.

Lemma 3.2. *Let u be a solution of* (16) *in an admissible domain Ω.*

(*i*) *Then, for all piecewise smooth polygon $\mathcal{P}$ (not necessary admissible) in Ω we have*

$$\mathrm{F}_u[\partial\mathcal{P}] = 0,$$

(*ii*) *Then for every curve γ in Ω we have*

$$|\mathrm{F}_u[\gamma]| \leq \mathfrak{L}_f[\gamma],$$

(*iii*) *Then, if $\gamma \subset \partial\Omega$ is an f-geodesic such that u tends to $+\infty$ on γ, we have*

$$\mathrm{F}_u[\gamma] = \mathfrak{L}_f[\gamma],$$

(*iv*) *Then, if* $\gamma \subset \partial\Omega$ *is an* f*-geodesic such that* u *tends to* $-\infty$*, we have*

$$\mathrm{F}_u[\gamma] = -\mathfrak{L}_f[\gamma],$$

(*v*) *Then, if* $\gamma \subset \partial\Omega$ *is an* f*-convex curve , i.e.* $\mathrm{k}_f[\gamma] \geq 0$ *along* γ*, such that* u *is continuous and finite on* γ*, then*

$$|\mathrm{F}_u[\gamma]| < \mathfrak{L}_f[\gamma].$$

Proof. The divergence theorem and (16) imply (i). Moreover, since $\left|\frac{f}{W}h_c(\nabla u, \nu)\right| = |h_c(N, \nu)| \leq 1$, where N denotes the unit upward normal to Graph$[u]$ it holds $|\mathrm{F}_u[\gamma]| \leq \mathfrak{L}_f[\gamma]$. Thus, we have (ii).

Regarding (iii), let η be an arc of γ and η_ϵ be a curve in Ω which distant ϵ of η. Call α_1^ϵ and α_2^ϵ the curves that connected the endpoints of η and η_ϵ. If we denote by $\mathcal{P}$ the domain with boundary $\eta \cup \eta_\epsilon \cup \alpha_1^\epsilon \cup \alpha_2^\epsilon$, then by (i) and (ii) one holds

$$\mathrm{F}_u[\eta] = -\mathrm{F}_u[\eta_\epsilon] - \mathrm{F}_u[\alpha_1^\epsilon] - \mathrm{F}_u[\alpha_2^\epsilon] \geq -\mathrm{F}_u[\eta_\epsilon] - \mathfrak{L}_f[\alpha_1^\epsilon] - \mathfrak{L}_f[\alpha_1^\epsilon], \tag{25}$$

where ν denotes the unit outer normal to $\partial\mathcal{P}$. On the other hand, given $\delta \in (0,1)$ if we take ϵ is small enough ($\epsilon < \delta$) then by Proposition 3.11 one has

$$-\mathrm{F}_u[\eta_\epsilon] = -\int_{\eta_\epsilon} \frac{f^2}{W} h_c(\nabla u, \nu) > \int_{\eta_\epsilon} f(1-\delta) = (1-\delta)\mathfrak{L}_f[\eta_\epsilon], \tag{26}$$

since the unit outer normal to γ is minus the unit outer normal to η_ϵ in the orientation considered by Proposition 3.11. Since $\mathfrak{L}_f[\alpha_1^\epsilon] \to 0$ as $\epsilon \to 0$, then from (25) and (26) it holds

$$\mathrm{F}_u[\eta] > (1-\delta)\mathfrak{L}_f[\eta_\epsilon] - \mathfrak{L}_f[\alpha_1^\epsilon] - \mathfrak{L}_f[\alpha_1^\epsilon]$$

Letting $\delta \to 0$ we obtain $\mathrm{F}_u[\eta] \geq \mathfrak{L}_f[\eta]$. Therefore $\mathrm{F}_u[\eta] = \mathfrak{L}_f[\eta]$. Since η was arbitrary, then the same conclusion is true by γ. Notice that essentially the same argument proves (iv) up to a sign.

It remains to prove (v). In order to prove that, fix any $p \in \gamma$ and let $B_r(p)$ be a geodesic ball in $\mathcal{P}$ so that $B_r(p) \cap (\partial\Omega \setminus \gamma) = \varnothing$ and $B_r(p)$ lies in a geodesicaly $f-$convex neighbourhood. By Theorem 3.1 there exists a solution $v : B_r(p) \cap \Omega \to \mathbb{R}$ of (16) so that $v = u$ on $\partial(B_r(p) \cap \Omega) \setminus (\gamma \cap B_r(p))$ and $v = u + 1$ on $\gamma \cap B_r(p)$. Using that

$$h_c\left(\nabla v - \nabla u, f^2\frac{\nabla v}{W_v} - f^2\frac{\nabla u}{W}\right) = \frac{1}{2}(W_v + W_u) g_c(N_v - N_u, N_v - N_u),$$

where $W_u := \sqrt{1 + f^2 g_c(\nabla u, \nabla u)}$ and $N_u = \frac{\partial_s}{fW_u} - f\frac{\nabla u}{W_u}$, by the proof of Theorem 3.4 and

v and u are solutions of (16) one has

$$\int_{B_r(p)\cap\Omega} \operatorname{div}\left(\{v-u\}\left\{f^2\frac{\nabla v}{W_v}-f^2\frac{\nabla u}{W_u}\right\}\right)=\int_{B_r(p)\cap\Omega} h_c\left(\nabla v-\nabla u, f^2\frac{\nabla v}{W_v}-f^2\frac{\nabla u}{W_u}\right)>0.$$

In turn, by the divergence theorem, we have

$$\begin{aligned} 0 &< \int_{\partial(B_r(p)\cap\Omega)} h_c\left(\{v-u\}\left\{f^2\frac{\nabla v}{W_v}-f^2\frac{\nabla u}{W_u}\right\},\nu\right)\\ &= \int_{B_r(p)\cap\gamma} h_c\left(f^2\frac{\nabla v}{W_v}-f^2\frac{\nabla u}{W_u},\nu\right)=\mathrm{F}_v[\gamma\cap B_r(p)]-\mathrm{F}_u[\gamma\cap B_r(p)]. \end{aligned}$$

Thus, $\mathfrak{L}_f[\gamma\cap B_r(p)]\geq \mathrm{F}_v[\gamma\cap B_r(p)]>\mathrm{F}_u[\gamma\cap B_r(p)]$. In turn, if $w:\Omega\cap B_r(p)\to\mathbb{R}$ is a solution of (16) so that $w=u$ on $\partial(B_r(p)\cap\Omega)\setminus(\gamma\cap B_r(p))$ and $w=u-1$ on $\gamma\cap B_r(p)$ one obtains $-\mathfrak{L}_f[\gamma\cap B_r(p)]<\mathrm{F}_u[\gamma\cap B_r(p)]$. Therefore it holds $\mathfrak{L}_f[\gamma\cap B_r(p)]]>|\mathrm{F}_u[B_r(p)\cap\gamma]|$, and consequently $\mathfrak{L}_f[\gamma]>|\mathrm{F}_u[\gamma]|$. □

We finish this subsection with the following variation of the items (iii) and (iv).

Lemma 3.3. *Let $\{u_n\}$ be a sequence of solutions of* (16) *on a domain $\Omega\subset\mathbb{P}$ so that u_n's are continuous up to γ, where γ is an $f-$geodesic on $\partial\Omega$. Then*

(i) If $\{u_n\}$ diverges uniformly to $+\infty$ on compact subset of γ, while remaining uniformly bounded on compact subset of Ω, then

$$\lim_{n\to\infty}\mathrm{F}_{u_n}[\gamma]=\mathfrak{L}_f[\gamma],$$

(ii) If $\{u_n\}$ diverges uniformly to $-\infty$ on compact subset of Ω, while remaining uniformly bounded on compact subset of γ, then

$$\lim_{n\to\infty}\mathrm{F}_{u_n}[\gamma]=\mathfrak{L}_f[\gamma],$$

Proof. We will prove (i) firstly. The proof of this item follows of the following claim joint with the argument used for prove item (iii) in Lemma 3.2.

Claim 3.5. *Given $\epsilon\in(0,1)$ there exists a $\delta>0$ depends only on ϵ so that if* $\operatorname{dist}(p,\gamma)<\delta$, *then*

$$h_c\left(f(p)\frac{\nabla u_n}{W_n}(p),\nu(p)\right)>1-\epsilon \text{ for all } n,$$

where $W_n^2=1+f^2|\nabla u_n|^2$ and $\nu(p)$ indicates the outer unit normal to $\Omega\setminus\{q\in\Omega\colon$ $\operatorname{dist}(q,\gamma)\geq\operatorname{dist}(p,\gamma)\}$ *at p.*

Proof of the Claim 3.5. To prove this fact we need of Theorem 2.4 and Lemma 2.1. Here we will use the fact that horizontal graphs are stable by Lemma 2.4.

Suppose that this claim is not true, then there could $\epsilon\in(0,1)$ so that the claim is not true for all n. This means that we could find a sequence of point $\{p_k\} \in \Omega$ so that $p_k \to p \in \text{int}\gamma$ and a sequence of index $\{n_k\}$ in such that way that

$$-h_c\left(N_{\text{Graph}^h[u_{n_k}]}(p_k), \nu(p_k)\right) = h_c\left(f(p_n)\frac{\nabla u_{n_k}}{W_{n_k}}(p_k), \nu(p_k)\right) \leq 1-\epsilon \text{ for all } k, \quad (27)$$

where $N_{\text{Graph}^h[u_{n_k}]}(p_k)$ indicates the unit upward normal to $\text{Graph}^h[u_{n_k}]$.

In turn, using that $\{u_{n_k}\}$ is unbounded on γ and bounded on compact subset of Ω, then we could find a $r > 0$ small enough so that the intrinsic geodesic ball $B_k = B_r(p_k)$ in $\text{Graph}^h[u_{n_k}]$ belong to $\text{Graph}^h[u_{n_k}] \setminus \partial\text{Graph}^h[u_{n_k}]$ for all k and this ball is a graph over the tangent plane $\text{T}_{(p_k, u_{n_k}(p_k))}\text{Graph}^h[u_{n_k}]$. Now, up to a subsequence, we could assume that the sequence of geodesic ball $\{B_k\}$ converges to a graph B_∞ over the tangent plane $\pi = \lim_k \text{T}_{(p_k, u_{n_k}(p_k))}\text{Graph}^h[u_{n_k}]$. However, (27) implies that $-h_c\left(N_{B_\infty}(p), \nu(p)\right) \geq 1-\epsilon$. In particular, there are points in the projection of B_∞ over $\mathbb{P}$ outside Ω, consequently it also there are points in the projection of B_k over $\mathbb{P}$ outside Ω for all k large enough which is impossible. This proves the claim □

Regarding the item (ii). The proof of it follows from the following claim, which the proof is exactly the same of the previous proof, joint with the argument used for prove item (iii) in Lemma 3.2.

Claim 3.6. *Given $\epsilon \in (0,1)$ there exists a $\delta > 0$ depends only on ϵ so that if* $\text{dist}(p,\gamma) < \delta$, *then*

$$h_c\left(f(p)\frac{\nabla u_n}{W_n}(p), \nu(p)\right) > 1-\epsilon \text{ for all } n.$$

□

3.1.9 *Divergence and convergence sets*

The next step to extend the Jenkins-Serrin theory to our setting is to know which are the structure of the divergence and convergence sets of a monotonic sequence of solutions of (16). This study we will be done in this subsection. We will begin by establishing the next result about the structure of convergence set.

Proposition 3.12 (Structure of convergence set)**.** *Let $\{u_n\}$ be an increasing (respectively decreasing) sequence of solutions of* (16) *over a domain $\Omega \subset \mathbb{P}$. Then there exists an open set $\mathcal{C} \subset \Omega$, called the convergence set, such that $\{u_n\}$ converges on compact subsets of $\mathcal{C}$ to a solution of* (16) *and diverges uniformly to $+\infty$ (respectively $-\infty$) on compact subsets of $\mathcal{D} = \Omega \setminus \mathcal{C}$. The set $\mathcal{D}$ will be called the divergence set of $\{u_n\}$. Moreover, if $\{u_n\}$ is bounded at a point $p \in \Omega$, then the convergence set $\mathcal{C}$ is non-empty.*

Proof. Suppose that $\{u_n\}$ is an increasing sequence. In fact, up to a reflection in $\mathbb{P}\times\mathbb{R}$ ($= M\times\mathbb{R}$), we can always suppose this without loss of generality. Given any point $p \in \Omega\cap\mathcal{C}$

and suppose that $u_n(p) \to \alpha \in \mathbb{R}$. Take ϵ small enough so that $\partial B_\epsilon(p)$ is a strictly f-convex curve, i.e. $\mathrm{k}_f[\partial B_\epsilon(p)] > 0$, where $B_\epsilon(p)$ denotes the geodesic ball with center p and radius ϵ on $\mathbb{P}$. Consider the sequence of surfaces $\{\Sigma_n = \mathrm{Graph}[u_n|_{B_\epsilon(p)}]\}$ in the solid cylinder $B_\epsilon(p) \times \mathbb{R}$. As $\{\Sigma_n\}$ is a sequence of stable surfaces with locally bounded area, by Proposition 2.4 and Proposition 2.6, then after passing to a subsequence, we may suppose that $\{\Sigma_n\}$ converges smoothly to Σ_∞ in $B_\epsilon(p) \times \mathbb{R}$, here we are using the fact that we are working in a 3−dimensional manifold $M \times \mathbb{R}$ and so we have regularity at Theorem 2.9, i. e. there is not singular set.

As $u_1 \leq u_n$ for all n, we can use the approaching of Proposition 3.5 to obtain that Σ_∞ is a smooth graph u_∞ over $B_r(p)$. Hence $u_n|_{B_r(p)}$ converges on compact subsets to u_∞, here is the whole sequence $\{u_n\}$ since it is increasing. Therefore $B_r(p) \subset \mathcal{C}$ and this completes the proof that $\mathcal{C}$ is open and non-empty if there exists a point $p \in \Omega$ such that $\{u_n(p)\}$ is a bounded sequence. □

Now we are going to see how we can determine the structure of divergence set by using what we have just developed until now.

Proposition 3.13 (Structure of divergence set)**.** *Let $\Omega \subset \mathbb{P}$ be an admissible domain whose boundary is a union of f-convex arcs C_i. Let $\{u_n\}$ be either an increasing or a decreasing sequence of solutions of* (16) *over Ω such that for all open arcs C_i the functions u_n extend continuously to C_i and either $u_n|_{C_i}$ converge uniformly to a continuous function or $+\infty$ or $-\infty$, respectively. If $\mathcal{D}$ denotes the divergence set of $\{u_n\}$, then $\mathcal{D}$ satisfies the following properties.*

(*i*) *$\partial\mathcal{D}$ consists of a union of a set of non-intersecting interior f-geodesics in Ω, joining two points of $\partial\Omega$, and arcs on $\partial\Omega$. These arcs will be called chords. Moreover, a component of $\mathcal{D}$ cannot be an isolated point;*

(*ii*) *No two interior chords in $\partial\mathcal{D}$ can have a common endpoint at a convex corner of $\mathcal{D}$;*

(*iii*) *A component of $\mathcal{D}$ cannot be an interior chord;*

(*iv*) *The endpoints of interior f-geodesic chords are among the vertices of $\partial\Omega$.*

Proof. Let us assume that $\{u_n\}$ is an increasing sequence. If $\mathcal{D} = \Omega$ there is nothing to prove, so we can suppose that $\mathcal{D} \neq \Omega$. Under this hypothesis, Lemma 3.1 implies that $\partial\mathcal{D}$ consists of interior f-geodesics in Ω and arcs of $\partial\Omega$. We will prove initially that $\mathcal{D}$ cannot have isolated points. Indeed, if p is an isolated point of $\mathcal{D}$, then we can construct a quadrilateral domain $\Omega' \subset \Omega$ satisfying the condition of Proposition 3.7 so that $p \in \mathrm{int}\,\Omega'$. Moreover, we can suppose that $\overline{\Omega'}$ does not intersect $\mathcal{D}\backslash\{p\}$. Now consider $M = \sup_{C_1 \cup C_2} |u_n| < \infty$, where C_1 and C_2 denotes the large edges of Ω'. If v denotes the function given by Proposition 3.7, then by Theorem 3.4 one gets $-M - v \leq u_n \leq M + v$ in Ω' which is impossible since $u_n(p) \to +\infty$. This contradiction shows that $\mathcal{D}$ cannot have isolated points. Note that this argument proves also that a chord of $\partial\mathcal{D}$ cannot have an endpoint in the interior of Ω, since we can get a domain Ω' satisfying the conditions of Proposition 3.7 so that the endpoint of this chord lies in Ω' and a part of this chord lies in Ω'.

Next we prove that the interior f-geodesics are non-intersecting. In fact, if the contrary of this was true, then we can construct a triangle Δ with edges a_1, a_2 and a_3 so that $a_1, a_2 \subset \partial\mathcal{D}$ and Δ lies either in $\mathcal{C}$ or in $\mathcal{D}$. Assume first that Δ lies in $\mathcal{C}$. Then by Lemma 3.2 (i) we have

$$0 = \mathrm{F}_{u_n}[\partial\Delta] = \mathrm{F}_{u_n}[a_1] + \mathrm{F}_{u_n}[a_2] + \mathrm{F}_{u_n}[a_3]. \tag{28}$$

Since a_1 and a_2 lies on $\partial\mathcal{D}$ we have $\lim_n \mathrm{F}_{u_n}[a_i] = -\mathfrak{L}_f[a_i]$ for $i = \{1, 2\}$, by Lemma 3.3. On the other hand, again by Lemma 3.2 we have $|\mathrm{F}_{u_n}[a_3]| \leq \mathfrak{L}_f[a_3]$, so we get a contradiction with (28). Therefore we must have $\Delta \subset \mathcal{D}$. In turn, we must have $\lim_n \mathrm{F}_{u_n}[a_i] = \mathfrak{L}_f[a_i]$ for $i = \{1, 2\}$. To see this, note that since $a_i \subset \partial\mathcal{C}$ for $i = \{1, 2\}$, then by Lemma 3.3, we must have $\lim_n \mathrm{F}_{u_n}[a_i] = -\mathfrak{L}_f[a_i]$ for $i = \{1, 2\}$ in $\mathcal{C}$. Now using the previous argument we arrive again to a contradiction, and this proves (i).

In order to get (ii), assume that there exist two interior chords γ_1 and γ_2 with a common endpoint $p \in \partial\Omega$. Again, we can construct a triangle Δ with edges a_1, a_2 and a_2 so that $a_1, a_2 \subset \partial\mathcal{D}$ and Δ lies either in $\mathcal{C}$ or in $\mathcal{D}$. Then the same argument as above proves (ii).

To prove the assertion (iii), suppose that γ is an interior chord that is a connected component of $\mathcal{D}$. Fix any point $p \in \gamma$ which lies in $\operatorname{int}\Omega$. Clearly we can construct a quadrilateral domain Ω' such that it satisfies the properties of Proposition 3.7. If $\partial\Omega' = A_1 \cup A_2 \cup C_1 \cup C_2$, then γ only intersects A_1 and A_2 at an unique interior point on these arcs and $\overline{\Omega'}$ does not intersect $\mathcal{D} \setminus \gamma$. Consider $M = \sup_{C_1 \cup C_2} |u_n| < \infty$ and let $v\colon \Omega' \to \mathbb{R}$ be the function given by Proposition 3.7. Using Theorem 3.4 one obtains $-M - v \leq u_n \leq M + v$ in Ω' which is impossible since an arc of γ lies in Ω'. This concludes the proof of the (iii).

Finally, assume that there exists a chord γ with endpoint $p \in \operatorname{int} C_i$ for some C_i. If $\mathrm{k}_f[C_i](p) > 0$ then Lemma 3.10 gives us a contradiction. On the other hand, if $\mathrm{k}_f[C_i](p) = 0$, then we have two cases to check: either there is a sequence $\{p_n\} \subset C_i$ so that $p_n \to p$ and $\mathrm{k}_f[C_i](p_n) > 0$ or there is a subarc η of C_i so that $\mathrm{k}_f[\eta] \equiv 0$ and p lies in the interior of η. The first case would imply that it is possible to find a domain Δ satisfying the condition of Proposition 3.8 so that p lies in the interior of the arc of $\partial\Delta$ which is not an f-geodesic and $\Delta \subset \overline{\Omega}$. Suppose first that $\{u_n\}$ is unbounded on C_i and let $v\colon \Delta \to \mathbb{R}$ be the function given by Proposition 3.8 with continuous data 0 satisfying $v \to -\infty$ along α, where α is the part of $\partial\Delta$ which is $f-$geodesic. If K is an arbitrary fixed constant, then by Theorem 3.4 one has

$$-v + K < u_n \ \text{ in } \ \Delta \ \text{ for all } \ n \, \text{large enough}\,.$$

Thus, since K was arbitrary, this implies that a small neighbourhood of p lies in $\mathcal{D}$, but this is impossible because $\gamma \subset \partial\mathcal{D}$. On the other hand, if $\{u_n\}$ is bounded on C_i and $v\colon \Delta \to \mathbb{R}$ is the function given by Proposition 3.8 with continuous data K, where $K = \sup_C |u_n|$, then by Theorem 3.4 one obtains $u_n \leq v$ in Δ, which again leads at a contradiction.

Hence, there exists a subarc η of C_i so that $\mathrm{k}_f[C_i] \equiv 0$ on η and p lies in the interior of η. Again, we have two cases to check: either $\{u_n\}$ is unbounded or $\{u_n\}$ is bounded on C_i. If $\{u_n\}$ is unbounded on C_i, then we can find a triangle Δ with edges a_1, a_2 and a_2 so that $a_1 \subset \gamma$, $a_2 \subset C_i$ and a_3 lies in $\mathcal{C}$, and a similar argument as in the proof of i. would lead at a contradiction. In turn, if $\{u_n\}$ is bounded on γ, we can find a triangle Δ with edges a_1, a_2 and a_3 so that $a_1 \subset \gamma$, $a_2 \subset \mathcal{D} \cap C_i$ and a_3 lies in $\mathcal{D}$ which is impossible. This finish the proof of (iv). □

The next proposition summarizes what we shall need about the structure of divergence set later.

Proposition 3.14. *Let $\Omega \subset \mathbb{P}$ be an admissible domain whose boundary is the union of f-convex arcs C_i. Let $\{u_n\}$ be either an increasing or a decreasing sequence of solutions to* (16) *over Ω such that for every open arc C_i, u_n extends continuously to C_i and either $u_n|_{C_i}$ converge uniformly to a continuous function or $+\infty$ or $-\infty$, respectively. Let $\mathcal{D}$ be the divergence set of $\{u_n\}$. Then each connected component of $\mathcal{D}$ is an admissible polygon in Ω.*

3.1.10 *Existence of Jenkins-Serrin graphs*

Finally, in this subsection we are going to prove the existence and uniqueness of Jenkins-Serrin solution of (16). Before stating the main result, we need some notations. Henceforth Ω will denote an admissible domain in $\mathbb{P}$ so that

$$\partial\Omega = \left(\bigcup_{i=1}^{l} A_i\right) \bigcup \left(\bigcup_{j=1}^{t} B_j\right) \bigcup \left(\bigcup_{k=1}^{z} C_k\right),$$

where the arcs A_i and B_j are f-geodesics and the arcs C_k are f-convex.

Definition 3.5. *A function $u\colon \Omega \to \mathbb{R}$ is called a Jenkins-Serrin solution of* (16) *over Ω with continuous boundary data $c_k\colon C_k \to \mathbb{R}$ if u is a solution of* (16) *such that $u = c_k$ on C_k for all k, $u \to +\infty$ on A_i for all i, and $u \to -\infty$ on B_j for all j. If $\{C_k\} = \varnothing$, then we only require that $u \to +\infty$ on A_i for all i and $u \to -\infty$ on B_j for all j.*

Moreover, we will need of the following notation. Let $\mathcal{P}$ be an admissible polygon in Ω, we define

$$\alpha_f(\mathcal{P}) = \sum_{A_i \subset \partial\mathcal{P}} \mathfrak{L}_f[A_i] \quad \text{and} \quad \beta_f(\mathcal{P}) = \sum_{B_i \subset \partial\mathcal{P}} \mathfrak{L}_f[B_i].$$

Theorem 3.5 (Existence of Jenkins-Serrin graph)**.** *Let $\Omega \subset \mathbb{P}$ be an admissible domain such that for any admissible polygon $\mathcal{P} \subset \overline{\Omega}$ we have*

$$2\alpha_f(\mathcal{P}) < \mathfrak{L}_f[\partial\mathcal{P}] \quad \text{and} \quad 2\beta_f(\mathcal{P}) < \mathfrak{L}_f[\partial\mathcal{P}]. \tag{29}$$

Then

(a) If $\{C_k\} \neq \varnothing$ and $c_k\colon C_k \to \mathbb{R}$ are given continuous functions, then there exists a Jenkins-Serrin solution of (16) *with continuous boundary data c_k.*

(b) If $\{C_k\} = \varnothing$ and $\alpha_f(\Omega) = \beta_f(\Omega)$, then there exists a Jenkins-Serrin solution of (16)*.*

Furthermore, if u is a Jenkins-Serrin solution of (16) *with continuous boundary data*

$$c_k\colon C_k \to \mathbb{R}$$

and if $\{C_k\} \neq \varnothing$, then inequalities (29) *hold for all admissible polygon $\mathcal{P}$ in Ω, and if $\{C_k\} = \varnothing$ then we also have $\alpha_f(\Omega) = \beta_f(\Omega)$.*

Proof. The proof will be divided into three cases depending on the structure of $\partial\Omega$.

1st Case: Assume that $\{B_j\} = \varnothing$ and each function c_k is continuous and bounded from below.

By Theorem 3.2 and Theorem 3.3 there exists a solution u_n of (16) satisfying $u_n|A_i = n$ and $u_n|C_k = \min\{n, c_k\}$. Moreover, by Theorem 3.4 the sequence $\{u_n\}$ is increasing. Let $\mathcal{D}$ be the divergence set of $\{u_n\}$. If $\mathcal{D} \neq \varnothing$, then by Proposition 3.14 each connected component of $\mathcal{D}$ is an admissible polygon to Ω. Taking any connected component $\mathcal{P} \subset \mathcal{D}$ and using Lemma 3.2 and Lemma 3.3 we conclude that

$$0 = \mathrm{F}_{u_n}[\partial\mathcal{P}] = \sum_{A_i \subset \partial\mathcal{P}} F_{u_n}[A_i] + F_{u_n}\left[\partial\mathcal{P} \setminus \bigcup_{A_i \subset \partial\mathcal{P}} A_i\right],$$

$$\left|\sum_{A_i \subset \partial\mathcal{P}} F_{u_n}[A_i]\right| \leq \alpha_f(\mathcal{P})$$

and

$$\lim_n \mathrm{F}_{u_n}\left[\partial\mathcal{P} \setminus \bigcup_{A_i \subset \partial\mathcal{P}} A_i\right] = -\mathfrak{L}_f\left[\partial\mathcal{P} \setminus \bigcup_{A_i \subset \partial\mathcal{P}} A_i\right] = -\mathfrak{L}_f[\partial\mathcal{P}] + \alpha_f(\mathcal{P}), \tag{30}$$

where the first equality in (30) holds due to the argument that we used to prove the assertion (i) in Proposition 3.13. This would imply $\mathfrak{L}_f[\partial\mathcal{P}] \leq 2\alpha_f(\mathcal{P})$, which is a contradiction, and therefore we must have $\mathcal{D} = \varnothing$. Now by Proposition 3.6 a subsequence of $\{u_n\}$ (in fact, all sequence since it is increasing) converges uniformly on compact subsets of Ω to a solution u of (16). Furthermore, Proposition 3.9 says that u has the required properties.

Now we prove that the existence of a solution implies the structural conditions (29). For this, suppose that $u\colon \Omega \to \mathbb{R}$ is a Jenkins-Serrin solution of (16) with boundary data $c_k\colon C_k \to \mathbb{R}$, where c_k is continuous and bounded from below. Take any admissible polygon $\mathcal{P}$ in Ω. By Lemma 3.2 we have

$$\begin{aligned}\alpha_f(\mathcal{P}) = \mathrm{F}_u\left[\bigcup_{A_i\subset\partial\mathcal{P}} A_i\right] &= -\mathrm{F}_u\left[\partial\mathcal{P}\setminus\bigcup_{A_i\subset\partial\mathcal{P}} A_i\right]\\ &< \mathfrak{L}_f\left[\partial\mathcal{P}\setminus\bigcup_{A_i\subset\partial\mathcal{P}} A_i\right] = \mathfrak{L}_f[\partial\mathcal{P}] - \alpha_f(\mathcal{P}),\end{aligned}$$

since there exists at least one arc η of $\partial\mathcal{P}$ so that either η lies in Ω or η coincides with an arc C_k. Therefore $2\alpha_f(\mathcal{P}) < \mathfrak{L}_f[\partial\mathcal{P}]$ for each admissible polygon $\mathcal{P}$ in Ω.

2nd Case: Assume that $\{A_i\} \neq \varnothing$, $\{B_j\} \neq \varnothing$ and $\{C_k\} \neq \varnothing$.

By the first case there exist solutions u^+ and u^- of (16) so that

$$u^+ \equiv 0 \text{ on } \{B_j\},\ u^+|C_k = \max\{0, c_k\} \text{ and } u^+ \to +\infty \text{ on } \{A_i\}$$

and

$$u^- \equiv 0 \text{ on } \{A_i\},\ u^-|C_k = \min\{0, c_k\} \text{ and } u^- \to -\infty \text{ on } \{B_j\}.$$

Moreover, by Proposition 3.2 and Proposition 3.3 for each n there exists a solution u_n of (16) so that

$$u_n \equiv n \text{ on } \{A_i\},\ u_n|C_k = \tilde{c}_k \text{ and } u_n \equiv -n \text{ on } \{B_j\},$$

where

$$\tilde{c}_k = \begin{cases} n, & \text{if } c_k \geq n;\\ c_k, & \text{if } -n \leq c_k \leq n;\\ -n, & \text{if } c_k \leq -n.\end{cases}$$

Since $u^- \leq u_n \leq u^+$, by Theorem 3.4, then by Proposition 3.6 and Proposition 3.9 a subsequence of $\{u_n\}$ must converge uniformly on compact subsets of Ω to a solution u of (16) with the required boundary data.

To conclude this case, we prove that the existence of a solution implies the structural conditions (29). Suppose that $u\colon \Omega \to \mathbb{R}$ is a Jenkins-Serrin solution with continuous boundary data $c_k\colon C_k \to \mathbb{R}$. Take any admissible polygon $\mathcal{P}$ in Ω. If $\mathcal{P} \neq \Omega$, then there exists an edge of $\partial\mathcal{P}$ which lies in Ω, and from Lemma 3.2 we obtain

$$\begin{aligned}\beta_f(\mathcal{P}) = -\mathrm{F}_u\left[\bigcup_{B_j\subset\partial\mathcal{P}} B_j\right] &= \mathrm{F}_u\left[\partial\mathcal{P}\setminus\bigcup_{B_j\subset\partial\mathcal{P}} B_j\right]\\ < \mathfrak{L}_f\left[\partial\mathcal{P}\setminus\bigcup_{B_j\subset\partial\mathcal{P}} B_j\right] &= \mathfrak{L}_f[\partial\mathcal{P}] - \beta_f(\mathcal{P}).\end{aligned}$$

Therefore $2\beta_f(\mathcal{P}) < \mathfrak{L}_f[\partial\mathcal{P}]$, and by the first case, we have also

$$2\alpha_f(\mathcal{P}) < \mathfrak{L}_f[\partial\mathcal{P}]$$

for each admissible polygon $\mathcal{P} \neq \Omega$. As these conditions are satisfied also when $\mathcal{P} = \Omega$, by Lemma 3.2 (v) we finish the proof the second case.

3rd Case: Assume that $\{C_k\} = \varnothing$.

Firstly, notice that the hypothesis on Ω implies that $l = t$, i.e. there are equal number of arcs A_i and B_j. For each n let v_n be the solution of (16) satisfying $v_n|A_i = n$ and $v_n|B_j = 0$. Clearly by Theorem 3.4 we must have $0 \le v_n \le n$. Given any $c \in (0, n)$, we denote

$$E_c = \{p \in \Omega \colon v_n(p) > c\} \quad \text{and} \quad F_c = \{p \in \Omega \colon v_n(p) < c\}.$$

Let E_c^i be the connected component of E_c whose closure contains A_i, and similarly let F_c^j be connected component of F_c whose closure contains B_j. Notice that if $E_c \neq \bigcup_i E_c^i$, then v_n is a constant by maximum principle. Hence $E_c = \bigcup_i E_c^i$, and similarly we conclude that $F_c = \bigcup_j F_c^j$.

Let now c be so close to n that $\{E_c^i\}$'s are pairwise disjoint. This is possible by our assumption on Ω and u_n. Define

$$\mu(n) = \inf\{c \in (0,n) \colon E_c^i \cap E_c^j = \varnothing \ \text{ for all } \ i \neq j\}.$$

Since $\overline{\Omega}$ is compact, there exists at least one pair i and j so that

$$\overline{E}^i_{\mu(n)} \cap \overline{E}^j_{\mu(n)} \neq \varnothing.$$

Moreover, for each i there exists j so that

$$F^i_{\mu(n)} \cap F^j_{\mu(n)} = \varnothing,$$

because if this was not the case, then $\cup_i F^i_{\mu(n)}$ would be connected, and consequently $\overline{E}^i_{\mu(n)} \cap \overline{E}^j_{\mu(n)} = \varnothing$.

Now, for every n, we define the function $u_n = v_n - \mu(n)$. We would like to prove that $\{u_n\}$ is locally bounded on compact subsets of Ω. To do this, we note that by the first case there exist auxiliary functions u_i^+ and u_i^- that satisfy

$$u_i^+ \equiv 0 \text{ on } \partial\Omega \setminus A_i, \quad u_i^+|_{A_i} = +\infty$$

and

$$u_i^-|_{B_j} \equiv -\infty \text{ for } j \neq i, \text{ and } u_i^- = 0 \text{ on } \partial\Omega \setminus \bigcup_{j\neq i} B_j.$$

Then, given any $p \in \Omega$, we define the functions

$$u^+(p) = \max_i \{u_i^+(p)\} \text{ and } u^-(p) = \max_i \{u_i^-(p)\},$$

and claim that

$$u^- \leq u_n \leq u^+$$

holds in Ω.

Let $p \in \Omega$, and note first that if $u_n(p) = 0$, then we have the claim. Therefore, we suppose that $u_n(p) > 0$, which implies that $v_n(p) > \mu(n)$, and consequently we must have $p \in E^i_{\mu(n)}$. Since $u_n \leq u_i^+$ on $\partial E^i_{\mu(n)}$, then by Theorem 3.4 we must have $u_n \leq u_i^+ \leq u^+$ in $E^i_{\mu(n)}$. As u^- is negative, we have the desired inequality $u_n(p) > 0$. Finally, if $u_n(p) < 0$ we can apply the same argument replacing $E^i_{\mu(n)}$ by $F^i_{\mu(n)}$. Therefore $\{u_n\}$ is locally bounded on compact subsets of Ω.

By construction

$$u_n|A_i = n - \mu(n) \quad \text{and} \quad u_n|B_j = -\mu(n),$$

and to finish the proof, we show that $\{n - \mu(n)\}$ and $\{\mu(n)\}$ are diverging to infinity. Then we would have that a subsequence of $\{u_n\}$ converges uniformly on compact subsets of Ω to a solution u of (16) with the desired properties. We show that $\{n - \mu(n)\}$ diverges, and similar argument proves the claim also for $\{\mu(n)\}$. On the contrary, suppose that there exists a subsequence of $\{n - \mu(n)\}$ converging to a finite limit τ. This implies that $\mu(n) \to +\infty$ and hence

$$u_n = n - \mu(n) \to \tau \text{ on } A_i \text{ and } u_n = -\mu(n) \to -\infty \text{ on } B_j.$$

Let u be the solution obtained from a convergent subsequence of $\{u_n\}$ so that

$$u \to \tau \text{ on } A_i \quad \text{and} \quad u_n \to -\infty \text{ on } B_j.$$

From Lemma 3.2 one has

$$0 = F_u[\partial\Omega] = F_u\left[\bigcup_i A_i\right] + F_u\left[\bigcup_j B_j\right],$$

but other the hand Lemma 3.2 also gives

$$\left|F_u\left[\bigcup_i A_i\right]\right| < \alpha_f(\Omega) \quad \text{and} \quad F_u\left[\bigcup_j B_j\right] = -\beta_f(\Omega),$$

which is a contradiction with our hypothesis on Ω. Consequently $\{n - \mu(n)\}$ is diverging to infinity.

Finally, let us prove that the existence implies the structural conditions (29) in Ω. Really, recall that we proved in the previous case that the existence of Jenkins-Serrin solution implies the structural conditions (29) for each admissible polygon $\mathcal{P} \neq \Omega$. Therefore, it remains to prove the last structural condition when $\mathcal{P} = \Omega$. But the last condition follows now by Lemma 3.2, since

$$\beta_f(\Omega) = -\mathrm{F}_u\left[\bigcup_j B_j\right] = \mathrm{F}_u\left[\partial\Omega \setminus \bigcup_j B_j\right] = \mathrm{F}_u\left[\bigcup_i A_i\right] = \alpha_f(\Omega).$$

□

The uniqueness of Jenkins-Serrin solution will follow from a little variation of the ideas of the proof of Theorem 3.4.

Theorem 3.6 (Uniqueness of Jenkins-Serrin graph)**.** *Let $\Omega \subset \mathbb{P}$ be a bounded admissible domain and suppose that u_1 and u_2 are solutions of* (16)*. Then, if $\{C_k\} \neq \varnothing$ and $u_1 = u_2$ on $\{C_k\}$, we have $u_1 = u_2$ in Ω. In turn, if $\{C_k\} = \varnothing$, then $u_2 - u_1$ is a constant.*

Proof. Consider

$$\varphi = \begin{cases} K, & \text{if } u_1 - u_2 \geq K; \\ u_1 - u_2, & \text{if } -K < u_1 - u_2 \leq K; \\ -K, & \text{if } u_1 - u_2 \leq -K, \end{cases}$$

where K is a large constant. Then φ is a Lipschitz function such that $-K \leq \varphi \leq K$, $\nabla\varphi = \nabla u_1 - \nabla u_2$ in the set $\{x \in \Omega\colon -K < u_1(x) - u_2(x) < K\}$ and $\nabla\varphi = 0$ almost everywhere is the complement of $\{x \in \Omega\colon -K < u_1(x) - u_2(x) < K\}$. Let

$$\Omega_{\epsilon,\delta} = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) \geq \epsilon\} \setminus \bigcup_{p \in \Upsilon} B_\delta(p),$$

where $\epsilon, \delta > 0$ are small constants with $\delta > \epsilon$ and Υ denotes the set of endpoints of A_i

and B_j. Define also a function

$$J = \int_{\partial\Omega_{\epsilon,\delta}} \varphi \left[h_c\left(f^2 \frac{\nabla u_1}{W_1}, \nu \right) - h_c\left(f^2 \frac{\nabla u_2}{W_2}, \nu \right) \right], \tag{31}$$

where ν denotes the outer unit normal to $\partial\Omega_{\epsilon,\delta}$. Since φ is a Lipschitz function, the divergence theorem and (20) give

$$\begin{aligned} J &= \int_{\Omega_{\epsilon,\delta}} h_c\left(\nabla\varphi, f^2 \frac{\nabla u_1}{W_1} - f^2 \frac{\nabla u_2}{W_2} \right) \\ &= \int_{\Omega_{\epsilon,\delta}} \frac{1}{2}(W_1 + W_2) g_c(N_1 - N_2, N_1 - N_2), \end{aligned} \tag{32}$$

where $N_i = \frac{\partial_s}{fW_i} - f\frac{\nabla u_i}{W_i}$.

On the other hand, observe that the boundary $\partial\Omega_{\epsilon,\delta}$ is formed by arcs A_i', B_j', C_k' and parts of $\partial B_\delta(p)$ when p moves along Υ. Here $A_i' = \partial\Omega_{\epsilon,\delta} \cap \{x \in \Omega\colon \operatorname{dist}(x, A_i) \le \epsilon\}$ and similarly for B_j' and C_k'.

Next we define

$$\Gamma = \partial\Omega_{\epsilon,\delta} \setminus \bigcup_i A_i' \bigcup_j B_j' \bigcup_k C_k'.$$

With this notation we obtain

$$\begin{aligned} J = &\int_{\Gamma} \varphi \left[h_c\left(f^2 \frac{\nabla u_1}{W_1}, \nu \right) - h_c\left(f^2 \frac{\nabla u_2}{W_2}, \nu \right) \right] \\ &+ \int_{\bigcup_i A_i'} \varphi \left[h_c\left(f^2 \frac{\nabla u_1}{W_1}, \nu \right) - h_c\left(f^2 \frac{\nabla u_2}{W_2}, \nu \right) \right] \\ &+ \int_{\bigcup_j B_j'} \varphi \left[h_c\left(f^2 \frac{\nabla u_1}{W_1}, \nu \right) - h_c\left(f^2 \frac{\nabla u_2}{W_2}, \nu \right) \right] \\ &+ \int_{\bigcup_k C_k'} \varphi \left[h_c\left(f^2 \frac{\nabla u_1}{W_1}, \nu \right) - h_c\left(f^2 \frac{\nabla u_2}{W_2}, \nu \right) \right]. \end{aligned} \tag{33}$$

Since $\varphi = 0$ in $\{C_i\}$ if δ is small enough, the first and the last terms of (33) can be estimated by

$$\left| \int_{\Gamma} \varphi \left[h_c\left(f^2 \frac{\nabla u_1}{W_1}, \nu \right) - h_c\left(f^2 \frac{\nabla u_2}{W_2}, \nu \right) \right] \right| \le 2K \sum_{p \in \Upsilon} \mathfrak{L}_f[\partial B_\delta(p)] \tag{34}$$

and

$$\left| \int_{\bigcup_k C_k'} \varphi \left[h_c\left(f^2 \frac{\nabla u_1}{W_1}, \nu \right) - h_c\left(f^2 \frac{\nabla u_2}{W_2}, \nu \right) \right] \right| \le 2\epsilon \sum_k \mathfrak{L}_f[C_k]. \tag{35}$$

Regarding the second and third term of (33), note that the arcs A_i' and B_j' are ϵ-close to

A_i and B_j, respectively. By Proposition 3.11, if ϵ is small enough,

$$1 \geq h_c\left(f\frac{\nabla u_i}{W_i}, \nu\right) \geq 1-\delta \quad \text{on} \quad \gamma, \quad \text{if} \quad u \to +\infty \quad \text{along } \gamma' \text{ and } \operatorname{dist}_H(\gamma, \gamma') < \epsilon$$

and

$$-1 \leq h_c\left(f\frac{\nabla u_i}{W_i}, \nu\right) \leq -1+\delta \quad \text{on} \quad \gamma, \quad \text{if} \quad u \to -\infty \quad \text{along } \gamma' \text{ and } \operatorname{dist}_H(\gamma, \gamma') < \epsilon,$$

where γ' is an arc of $\partial\Omega$ and dist_H denotes the Hausdorff distance. In particular, these inequalities yield

$$\left|\int_{A_i'} \varphi\left[h_c\left(f^2\frac{\nabla u_1}{W_1}, \nu\right) - h_c\left(f^2\frac{\nabla u_2}{W_2}, \nu\right)\right]\right| \leq K\delta\mathfrak{L}_f[A_i'] \tag{36}$$

and

$$\left|\int_{B_j'} \varphi\left[h_c\left(f^2\frac{\nabla u_1}{W_1}, \nu\right) - h_c\left(f^2\frac{\nabla u_2}{W_2}, \nu\right)\right]\right| \leq K\delta\mathfrak{L}_f[B_j']. \tag{37}$$

Finally from (32), (33), (34), (35), (36) and (37) one has

$$\begin{aligned}\int_{\Omega_{\epsilon,\delta}} \frac{1}{2}(W_1+W_2)g_c(N_1-N_2, N_1-N_2) \leq 2\epsilon\sum_i \mathfrak{L}_f[C_i] + 2K\sum_{p\in\Upsilon}\mathfrak{L}_f[\partial B_\delta(p)] \\ + \sum_i K\delta\mathfrak{L}_f[A_i'] + \sum_j K\delta\mathfrak{L}_f[B_j'].\end{aligned}$$

Letting $\delta \to 0$ above, we conclude that $N_1 = N_2$ in $\{-K < u_1 - u_2 < K\}$. Thus $\nabla u_1 = \nabla u_2$ in $\{-K < u_1 - u_2 < K\}$, but since K was arbitrary constant, then we shall have $u_1 = u_2 + c$ in Ω, where c is a constant. In turn, if $\{C_i\} \neq \varnothing$ we must have $c = 0$. □

3.1.11 *Examples of admissible domains in* $\mathbb{R}^3$ *and* $\mathbb{H}^2 \times \mathbb{R}$

We finish this part of our work by giving some examples of domains that satisfy (29) in $\mathbb{R}^3$ and in $\mathbb{H}^2 \times \mathbb{R}$.

3.1.11.1 *Examples in* $\mathbb{R}^3$

In this case $\mathbb{P}$ is a vertical plane ($\mathbb{R}^2$) containing the vector $\mathbf{e}_3$ in $\mathbb{R}^3$, so after to rotation, we can suppose that $\mathbb{P} = \mathbb{R}^2 := \{(0, x_2, x_3) \ : \ x_2 \text{ and } x_3 \in \mathbb{R}\}$. Moreover, the Ilmanen's metric is given by $g_c = e^{cx_3}\langle\cdot,\cdot\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the Euclidean metric of $\mathbb{R}^3$, and consequently the function f is given by $f = e^{c\frac{x_3}{2}}$.

Next we are going to obtain the expression of the $f-$geodesic equation in term of the Euclidean metric in $\mathbb{P}=\mathbb{R}^2$. To do this, recall that as we are assuming the metric $h_c = g_c|_{\mathbb{P}}$ in $\mathbb{P}$ and the equation of $f-$geodesic is

$$\mathrm{k}_{h_c}[\gamma] - h_c\left(\frac{\tilde{\nabla} f}{f}, \tilde{N}\right) = 0,$$

here $\tilde{N}$ denotes the unit normal to γ in $\mathbb{P}$ and the gradient $\tilde{\nabla} f$ is taken with respect to h_c. As the metric h_c is conformal to the Euclidean metric in $\langle\cdot,\cdot\rangle$, then for all vector field X we have

$$f^2\langle\tilde{\nabla} f, X\rangle = h_c\left(\tilde{\nabla} f, X\right) = X(f) = \langle\tilde{\nabla} f, X\rangle,$$

where ∇f indicates the gradient of f with respect to the Euclidean metric $\langle\cdot,\cdot\rangle$. Therefore

$$\tilde{\nabla} f = \frac{\nabla f}{f^2} = \frac{c}{2}\frac{\mathbf{e}_3}{f}. \tag{38}$$

In turn, it is known that we have the following relationship between the metric h_c in $\mathbb{P}$ and the Euclidean metric $\langle\cdot,\cdot\rangle$

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{c}{2}\left\{\langle X, \mathbf{e}_3\rangle Y + \langle Y, \mathbf{e}_3\rangle X - \langle X, Y\rangle \mathbf{e}_3\right\},$$

where $\tilde{\nabla}$ denotes the Levi-Civita's connection associated to h_c and ∇ denotes the Levi-Civita's connection associated to $\langle\cdot,\cdot\rangle$. Hence,

$$\begin{aligned}
\mathrm{k}_{h_c}[\gamma] &= \frac{h_c\left(\tilde{\nabla}_r\gamma', \tilde{N}\right)}{h_c\left(\gamma',\gamma'\right)} = \frac{\left\langle\tilde{\nabla}_r\gamma', \tilde{N}\right\rangle}{\langle\gamma',\gamma'\rangle} \\
&= \frac{\left\langle\nabla_r\gamma' + \frac{c}{2}\left\{\langle\gamma',\mathbf{e}_3\rangle\gamma' + \langle\gamma',\mathbf{e}_3\rangle\gamma' - \langle\gamma',\gamma'\rangle\mathbf{e}_3\right\}, \frac{N}{f}\right\rangle}{\langle\gamma',\gamma'\rangle} \\
&= \frac{1}{f}\frac{\langle\nabla_r\gamma', N\rangle}{\langle\gamma',\gamma'\rangle} - \frac{c}{2f}\langle\mathbf{e}_3, N\rangle,
\end{aligned} \tag{39}$$

where $\tilde{\nabla}_r\gamma$ (respectively $\nabla_r\gamma$) denotes the covariant derivative of γ' with respect to h_c (respectively $\langle\cdot,\cdot\rangle$) and N denotes the unit normal to γ, notice that $\tilde{N} = N/f$.

From (38) and (39) one obtains

$$\begin{aligned}
0 &= \mathrm{k}_{h_c}[\gamma] - h_c\left(\frac{\tilde{\nabla} f}{f}, \tilde{N}\right) = \frac{1}{f}\frac{\langle\nabla_r\gamma', N\rangle}{\langle\gamma',\gamma'\rangle} - \frac{c}{2f}\langle\mathbf{e}_3, N\rangle - \frac{c}{2f}\langle\mathbf{e}_3, N\rangle \\
&= \frac{1}{f}\left\{\frac{\langle\nabla_r\gamma', N\rangle}{\langle\gamma',\gamma'\rangle} - c\langle\mathbf{e}_3, N\rangle\right\}.
\end{aligned}$$

Thus, it holds

$$\mathrm{k}[\gamma] = c\langle N, \mathbf{e}_3\rangle, \tag{40}$$

where $\mathrm{k}[\gamma]$ denotes the scalar curvature of γ in $\mathbb{P}$, N denotes the unit normal to γ and $\langle\cdot,\cdot\rangle$ is the Euclidean metric of $\mathbb{P} = \mathbb{R}^2$. In particular, f-geodesic are translating curves in $\mathbb{R}^2$.

It remains to compute all translating curves in $\mathbb{R}^2$. Let us assume now that $c > 0$ and notice that if γ is a line in $\mathbb{P}$ parallel to $\mathbf{e}_3$, then γ is a translating curve in $\mathbb{P}$ by (40). In turn, if we suppose that $\gamma = \{(0, x, \phi(x)) \in \mathbb{P} \ : \ x \in (a, b)\}$, where $a < b$, then $\gamma' = (0, 1, \phi'(x))$ and $N = \frac{(0,-\phi'(x),1)}{\sqrt{1+(\phi')^2}}$, so one has

$$\mathrm{k}[\gamma] = \frac{\phi''}{(1+(\phi')^2)^{\frac{3}{2}}} \text{ and } c\langle N, \mathbf{e}_3\rangle = \frac{c}{\sqrt{1+(\phi')^2}}.$$

Thus, ϕ satisfies the ODE

$$\frac{\phi''}{1+(\phi')^2} = c. \tag{41}$$

However, $x \in (-\pi/(2c), \pi/(2c)) \to \phi(x) = -\frac{1}{c}\log\cos(cx)$ is a solution of (41) and $\phi' = \tan(cx) \to +\infty$ as $x \to \pm\pi/(2c)$. These conditions say that the lines parallel to $\mathbf{e}_3$ and the grim reaper curve $\mathcal{G}_c = (0, x, -\frac{1}{c}\log\cos(cx))$ $(x \in (-\pi/(2c), \pi/(2c)))$ are the unique translating curves in $\mathbb{P}$, up to translation in $\mathbb{P}$, since they are geodesics with respect to a conformal metric.

Now we are going to see how we can produce admissible domains $\Omega \subset \mathbb{P}$ that are bounded by vertical line segments and parts of the grim reaper curves, see Figure 3. If we assign boundary data $+\infty$ on the parts of the grim reaper curve (corresponding to the edges A_1, A_2 in Theorem 3.5) and continuous data (0 in Fig. 3) on the vertical segments (corresponding to the edges C_1, C_2), the condition for the existence of solutions becomes

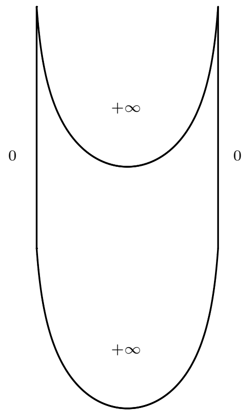
$$\mathfrak{L}_f[A_1] + \mathfrak{L}_f[A_2] < \mathfrak{L}_f[C_1] + \mathfrak{L}_f[C_2].$$

Consider the following parametrizations

$$\begin{aligned}
A_1 = \alpha_1 &= \left\{ \left(0, x, a - \frac{1}{c}\log\cos(cx)\right) : x \in (r, s) \right\}, \\
A_2 = \alpha_2 &= \left\{ \left(0, x, b - \frac{1}{c}\log\cos(cx)\right) : x \in (r, s) \right\}, \\
C_1 = \zeta_1 &= \left\{ (0, r, x) \colon x \in \left(a - \frac{1}{c}\log\cos(cs), b - \frac{1}{c}\log\cos(cs)\right) \right\}, \quad \text{and} \\
C_2 = \zeta_2 &= \left\{ (0, s, x) \colon x \in \left(a - \frac{1}{c}\log\cos(cr), b - \frac{1}{c}\log\cos(cr)\right) \right\}
\end{aligned}$$

for the edges of Ω in the plane $\mathbb{P} \subset \mathbb{R}^3$, where $-\pi/(2c) < s < r < \pi/(2c)$, $a, b \in \mathbb{R}$ and

Figure 3 – Basic solution.



$a < b$. Then from (13) we have

$$\begin{aligned}\mathfrak{L}_f[A_1] &= \mathfrak{L}_f[\alpha_1] = \int_r^s f(\alpha_1)\sqrt{g_c(\alpha_1', \alpha_1')_{\alpha_1}}dx = \int_r^s e^{c\left(a - \frac{1}{c}\log\cos(cx)\right)}\sqrt{1+\tan^2(cx)}dx \\ &= e^{ca}\int_r^s \sec^2(cx)dx = c^{-1}e^{ca}(\tan(cs) - \tan(cr)).\end{aligned}$$

Analogously, we conclude

$$\begin{aligned}\mathfrak{L}_f[A_2] &= c^{-1}e^{cb}(\tan(cs) - \tan(cr)) \\ \mathfrak{L}_f[C_1] &= c^{-1}\sec(cs)(e^{cb} - e^{ca}) \\ \mathfrak{L}_f[C_2] &= c^{-1}\sec(cr)(e^{cb} - e^{ca})\end{aligned}$$

In particular, it holds

$$\begin{aligned}\mathfrak{L}_f[A_1] + \mathfrak{L}_f[A_2] = c^{-1}(e^{cb} + e^{ca})(\tan(cr) - \tan(cs)) \\ \mathfrak{L}_f[C_1] + \mathfrak{L}_f[C_2] = c^{-1}(e^{cb} - e^{ca})(\sec(cr) + \sec(cs)).\end{aligned} \tag{42}$$

If we fix $a < b$, then choosing $r - s > 0$ small enough, we ensure that $\mathfrak{L}_f[A_1] + \mathfrak{L}_f[A_2] < \mathfrak{L}_f[C_1] + \mathfrak{L}_f[C_2]$.

On the other hand, if $r > s$ are fixed, then choosing $b - a > 0$ small enough in (42), we can guarantee that $\mathfrak{L}_f[A_1] + \mathfrak{L}_f[A_2] > \mathfrak{L}_f[C_1] + \mathfrak{L}_f[C_2]$. In particular, if we rename C_i by B_i, there are $b > a$ and $r > s$ so that $\mathfrak{L}_f[A_1] + \mathfrak{L}_f[A_2] = \mathfrak{L}_f[B_1] + \mathfrak{L}_f[B_2]$, and we obtain the structural condition of the case (b) in Theorem 3.5.

3.1.11.2 *Examples in* $\mathbb{H}^2 \times \mathbb{R}$

At this time we are going to consider the hyperbolic plane $\mathbb{H}^2$ as a warped product $\mathbb{H}^2 = \mathbb{R} \times_{e^x} \mathbb{R}$ with the metric

$$dx^2 + e^{2x} ds^2. \tag{43}$$

Then the vector field ∂_s is a Killing field with norm $|\partial_s|_{(x,s)} = e^x$, and the x-axis is an integral curve of the distribution orthogonal to ∂_s. In this case we can take the vertical plane $\mathbb{P}$ in $\mathbb{H}^2 \times \mathbb{R}$ to be the vertical plane over x-axis

$$\mathbb{P} = \{(x, t, s) \colon x, t \in \mathbb{R},\ s = 0\},$$

and with this choice we have $f = e^{c\frac{t}{2}} e^x$. Recall that we are endowing $\mathbb{P}$ with metric $h_c = e^{ct}(dx^2 + dt^2)$. Furthermore, by (15) we have that σ is a f-geodesic provided that

$$\mathrm{k}_{h_c}[\sigma] = h_c \left(\frac{\tilde{\nabla} f}{f}, \tilde{N} \right),$$

where $\tilde{N}$ is the unit normal along σ and $\tilde{\nabla} f$ is taken with respect to the metric h_c in $\mathbb{P}$. Using the metric h_c is conformal to the Euclidean metric $h_0 = dx^2 + dt^2$, we conclude

$$\frac{\tilde{\nabla} f}{f} = e^{-ct} \frac{\nabla f}{f} = e^{-ct} \left(\frac{c}{2} \partial_t + \partial_x \right).$$

On the other hand, it also holds

$$\begin{aligned}
\mathrm{k}_{h_c}[\sigma] &= \frac{h_c\left(\tilde{\nabla}_r \sigma', \tilde{N}\right)}{h_c(\sigma', \sigma')} = \frac{h_0\left(\tilde{\nabla}_r \sigma', \tilde{N}\right)}{h_0(\sigma', \sigma')} \\
&= \frac{1}{h_0(\sigma', \sigma')} h_0 \left(\nabla_r \sigma' + \frac{c}{2} \left\{ h_0(\sigma', \partial_t)\sigma' + h_0(\sigma', \partial_t)\sigma' - h_0(\sigma', \sigma')\partial_t \right\}, \frac{N}{e^{\frac{c}{2}t}} \right) \\
&= \frac{1}{e^{\frac{c}{2}t}} \frac{h_0(\nabla_r \sigma', N)}{h_0(\sigma', \sigma')} - \frac{c}{2e^{\frac{c}{2}t}} h_0(\partial_t, N) = e^{-ct/2} \mathrm{k}_{h_0}[\sigma] - e^{-ct/2} h_0\left(\frac{c}{2}\partial_t, N\right),
\end{aligned}$$

where $\tilde{\nabla}_r \sigma$ (respectively $\nabla_r \sigma$) denotes the covariant derivative of σ' with respect to h_c (respectively h_0) and N denotes the unit normal to σ and $\tilde{N} = e^{-ct/2} N$, where N denotes the unit normal along σ with respect to $h_0 = dx^2 + dt^2$ and $\mathrm{k}_{h_0}[\sigma]$ denotes the scalar geodesic curvature of σ with respect to h_0. Therefore, we have

$$\mathrm{k}_{h_0}[\sigma] = h_0(c\partial_t + \partial_x, N), \tag{44}$$

From this equality, we may conclude that lines in the direction $c\partial_t + \partial_x$ are f-geodesics in $\mathbb{P}$.

To compute the other f-geodesics, let us denote $\vec{\tau} = \partial_x + c\partial_t$ and $\vec{\varsigma} = c\partial_x - \partial_t$ and notice that $\{\vec{\varsigma}, \vec{\tau}\}$ is a positive frame of $\mathbb{P}$. As the curve cannot be tangent to τ, write $\sigma(x) = x\vec{\varsigma} + \varphi(x)\vec{\tau}$, where $x \in \mathbb{R}$ and φ is a smooth function. As $\sigma' = \vec{\varsigma} + \varphi'(x)\vec{\tau}$, then $N = \frac{1}{|\vec{\tau}|\sqrt{1+(\varphi'(x))^2}}(-\varphi'(x)\vec{\varsigma} + \tau)$ and $\nabla_r\sigma' = \sigma'' = \varphi''(x)\vec{\tau}$. From (44) we can conclude that

$$\begin{aligned} 0 &= \mathrm{k}_{h_0}[\sigma] - h_0\left(c\partial_t + \partial_x, N\right) \\ &= \frac{1}{h_0(\sigma', \sigma')} h_0(\nabla_r\sigma', N) - h_0(\tau, N) \\ &= \frac{1}{|\vec{\tau}|^2(1 + (\varphi'(x))^2)} \frac{\varphi''(x)}{|\vec{\tau}|\sqrt{1 + (\varphi'(x))^2}} - \frac{1}{|\vec{\tau}|\sqrt{1 + (\varphi'(x))^2}}. \end{aligned}$$

Therefore, one holds

$$\frac{\varphi''}{1 + (\varphi')^2} = |\vec{\tau}|^2.$$

Consequently $\varphi(x) = -|\vec{\tau}|^{-2} \log\cos(|\vec{\tau}|^2 x)$ for $x \in (-\pi/(2|\vec{\tau}|^2), \pi/(2|\vec{\tau}|^2))$. Using translation of σ we can conclude that f-geodesics of $\mathbb{P}$ are either lines in the direction of $\vec{\tau}$ or translating the curve σ above, which is the grim reaper curve in the direction of $\vec{\tau}$. Finally, the argument of the subsection 3.1.11.1 allows us to conclude the existence of similar basic domains.

3.1.11.2.1 *A new example of translating soliton in* $\mathbb{H}^{n+1} \times \mathbb{R}$

Notice that since we are assuming $\mathbb{H}^2 = \mathbb{R} \times_{e^x} \mathbb{R}$, then by Remark (3.2) one concludes what follows.

Proposition 3.15. *The hypersurface $\sigma \times \mathbb{R}$ is a complete, properly embedded translating soliton in $\mathbb{H}^2 \times \mathbb{R}$ with respect to ∂_t with speed c. Moreover, $\alpha \times \mathbb{R}$ is a complete, properly embedded translating soliton in $\mathbb{H}^2 \times \mathbb{R}$, where α is any line parallel to $\vec{\tau} = \partial_x + c\partial_t$.*

Remark 3.5. *We say that a translating soliton Σ in $M \times \mathbb{R}$ is complete provided that it is complete as hypersurface in $M \times \mathbb{R}$ with the product metric.*

Actually, this ingenuous trick of seeing translating solitons as Killing cylinder it does not a punctual fact for surfaces. We shall see now that it is possible to get one example of translating soliton that looks like the grim reaper cylinder in $\mathbb{H}^{n+1} \times \mathbb{R}$ by seeing it as a Killing cylinder over on a specific curve.

Consider the following model for the hyperbolic space as a warped product in $\mathbb{H}^{n+1} = \mathbb{R}^{n+1}$ endowed with the metric

$$e^{2x_{n+1}}(dx_1^2 + \cdots + dx_n^2) + dx_{n+1}^2$$

and in $\mathbb{H}^{n+1}\times\mathbb{R}$ we adopt the Riemannian metric

$$g := e^{2x_{n+1}}(dx_1^2+\cdots+dx_n^2)+dx_{n+1}^2+dt^2.$$

Assuming these, we choose $\mathbb{P}^2 := \mathbb{R}^2 = \{\underbrace{0,\dots,0}_{n}\}\times\mathbb{R}^2$ endowed with the Riemannian metric $h := dx_{n+1}^2+dt^2$, notice that $\mathbb{P}$ is totally geodesic in $\mathbb{H}^{n+1}\times\mathbb{R}$. Next, consider the family of Killing vector fields $\{\partial_1,\partial_2,\dots,\partial_n\}$, observe that $\mathbb{P}$ is a leaf of the normal distribution associated to this family.

Now let σ_n be a curve on $\mathbb{P}^2$ so that the Killing cylinder $\mathbb{R}^n\times\sigma_n$ is a translating soliton in $\mathbb{R}^n\times\mathbb{P}^2$ ($=\mathbb{H}^{n+1}\times\mathbb{R}$) with respect to ∂_t and speed c, and N denotes the unit normal vector field along σ_n. In particular, we can get a unit normal vector field N in $\mathbb{R}^n\times\sigma_n$ by defining $N(x,p) := N(p)$, for all $(x,p)\in\mathbb{R}^n\times\Sigma$.

Remark 3.6. *Notice that we are considering the coordinate $\{x_1,\dots,x_n,p\}$ in $\mathbb{R}^n\times\sigma_n$, because of this, we wrote $\mathbb{R}^n\times\sigma_n$ in the place of $\sigma_n\times\mathbb{R}^n$ to denote the Killing cylinder over σ_n.*

Assume that σ_n is a parametrization by arclength of σ_n, and consider the local orthonormal frame $\{\sigma_n',\partial_1/e^{x_{n+1}},\dots,\partial_n/e^{x_{n+1}}\}$ for $\mathbb{R}^n\times\sigma_n$. Thus, one has

$$\begin{aligned}
-ch(N,\partial_t) &= -cg(N,\partial_t) = \operatorname{div}_{\mathbb{R}\times\sigma_n} N\\
&= g\left(\nabla_{\sigma_n'}N,\sigma_n'\right)+\sum_{i=1}^n g\left(\nabla_{\frac{\partial_i}{e^{x_{n+1}}}}N,\frac{\partial_i}{e^{x_{n+1}}}\right)\\
&= -h\left(N,\nabla_r\sigma_n'\right)+\sum_{i=1}^n g\left(\nabla_{\frac{\partial_i}{e^{x_{n+1}}}}N-\nabla_N\left(\frac{\partial_i}{e^{x_{n+1}}}\right),\frac{\partial_i}{e^{x_{n+1}}}\right)\\
&= -\mathrm{k}_{\mathbb{P}^2}[\sigma_n]+\sum_{i=1}^n g\left(\left[N,\frac{\partial_i}{e^{x_{n+1}}}\right],\frac{\partial_i}{e^{x_{n+1}}}\right)\\
&= -\mathrm{k}_{\mathbb{P}^2}[\sigma_n]-\sum_{i=1}^n g\left(\left[\frac{\partial_i}{e^{x_{n+1}}},N\right],\frac{\partial_i}{e^{x_{n+1}}}\right)\\
&= -\mathrm{k}_{\mathbb{P}^2}[\sigma_n]-\sum_{i=1}^n g\left(\mathcal{L}_N\left(\frac{\partial_i}{e^{x_{n+1}}}\right),\frac{\partial_i}{e^{x_{n+1}}}\right)\\
&= -\mathrm{k}_{\mathbb{P}^2}[\sigma_n]-\sum_{i=1}^n g\left(N\left(\frac{1}{e^{x_{n+1}}}\right)\partial_i,\frac{\partial_i}{e^{x_{n+1}}}\right)\\
&= -\mathrm{k}_{\mathbb{P}^2}[\sigma_n]+ng(N,\partial_{n+1})\\
&= -\mathrm{k}_{\mathbb{P}^2}[\sigma_n]+n\cdot h(N,\partial_{n+1})
\end{aligned}$$

Therefore

$$\mathrm{k}_{\mathbb{P}^2}[\sigma_n] = h(N,c\partial_t+n\partial_{n+1}). \tag{45}$$

In particular, σ_n must be a translating curve in $\mathbb{P}^2$ with respect to the vector $\vec{\tau}_n := c\partial_t+n\partial_{n+1}$.

Now we would like to compute all translating curves with respect to $\vec{\tau}_n$. Naturally the lines parallel to $\vec{\tau}_n$ are translating curves in $\mathbb{P}^2$ with respect to $\vec{\tau}_n$. To compute the remains translating curves we argue as early: define $\vec{\varsigma}_n = -n\partial_t + c\partial_{n+1}$ and suppose that $\sigma_n = x\vec{\varsigma}_n + \varphi_n(x)\vec{\tau}_n$. Arguing as early we shall conclude that

$$\varphi_n(x) = -|\vec{\tau}_n|^{-2} \log\cos(|\vec{\tau}_n|^2 x) \text{ for } x \in \left(-\pi/(2|\vec{\tau}_n|^2), \pi/(2|\vec{\tau}_n|^2)\right).$$

In particular, using that the translating curves are geodesics with respect to a conformal metric in $\mathbb{P}^2$, we may conclude that all translating curves with respect to $\vec{\tau}_n$ in $\mathbb{P}$ are the lines parallel to $\vec{\tau}_n$ and $\beta_n(x) := x\vec{\varsigma}_n - |\vec{\tau}_n|^{-2}\log\cos(|\vec{\tau}_n|^2 x)\vec{\tau}_n$, for $x \in (-\pi/(2|\vec{\tau}_n|^2), \pi/(2|\vec{\tau}_n|^2))$, up to translation. This fact and (45) imply the next result.

Proposition 3.16. *The hypersurface $\mathbb{R}^n \times \beta_n$ is a complete, properly embedded translating soliton in $\mathbb{H}^{n+1} \times \mathbb{R}$ with respect to ∂_t with speed c. Moreover, $\mathbb{R}^n \times \alpha_n$ is also a complete, properly embedded translating soliton in $\mathbb{H}^n \times \mathbb{R}$, where α_n is any line parallel to $\vec{\tau}_n = n\partial_{n+1} + c\partial_t$.*

Remark 3.7. *The example $\mathbb{R}^n \times \alpha_n$ already appeared in (LIRA and MARTÍN, 2019), however the first one $\mathbb{R}^n \times \beta_n$ is a new example of a complete, properly embedded translating soliton in $\mathbb{H}^{n+1} \times \mathbb{R}$. For $n = 1$, the example $\mathbb{R}^1 \times \beta_1$ already has appeared in (GAMA et al., 2019b).*

3.2 Vertical case

We start this part by fixing some notation and recalling some notation from the subsection 2.3.2. Henceforth in this part M will be a complete Riemannian surface endowed with a rotationally symmetric metric σ whose sectional curvatures are nonpositive. Let Ω be a domain in M and $u : \Omega \to \mathbb{R}$ be a smooth function.

We know from subsection 2.3.2 that $\text{Graph}^v[u]$ is a translating soliton provided that

$$\text{div}_M \left(\frac{\nabla u}{W}\right) = \frac{c}{W}, \tag{46}$$

where $W := \sqrt{1 + |\nabla u|^2}$, and the gradient and divergence operators are taken with respect to the metric σ on M. Besides this, we orient $\text{Graph}^v[u]$ by the unit normal vector field

$$N = \frac{1}{W}(\partial_t - \nabla u).$$

We finish this introduction we the following lemma.

Lemma 3.4. *Suppose that γ is a curve in M. Then the mean curvature $\widetilde{H}_{\gamma\times\mathbb{R}}$ of $\gamma \times \mathbb{R}$ in $(M \times \mathbb{R}, g_c)$ is given by*

$$\widetilde{H}_{\gamma\times\mathbb{R}}(x,t) = e^{-\frac{c}{2}t}\mathrm{k}[\gamma](x) \tag{47}$$

up to a sign, for all $(x,t) \in \gamma \times \mathbb{R}$. Here $\mathrm{k}[\gamma]$ is the scalar curvature of γ in (M,σ).

Proof. Indeed, in Lemma 3.3 we have proved that the mean curvature of a hypersurface Σ in $M\times\mathbb{R}$ with the product metric and the Ilmanen's metric has the following relationship

$$H_c = e^{-\frac{c}{m}t}\left[H - cg_0(\partial_t, N)\right],$$

here H_c (respectively H) denotes the mean curvature of Σ in $M\times\mathbb{R}$ with the metric g_c (respectively $\sigma + dt^2$). From this equality, using that the mean curvature of the cylinder is equal to the scalar curvature of the curve, we conclude the proof of the lemma. □

3.2.1 *Local Existence*

Following what we have done in the horizontal case, in this part we shall prove the local existence of solution of (46) over admissible domains. So, before proceeding, we will define what is an admissible domain in the vertical case.

Definition 3.6 (Admissible domain)**.** *Let Ω be a connected domain in M. We say that Ω is an admissible domain provided that it is geodesicaly convex and bounded domain, and $\partial\Omega$ is a union of geodesic arcs $A_1,\ldots,A_s, B_1\ldots,B_r$, convex arcs $C_1,\ldots,C_t$, the end points of these arcs and that no two arcs A_i and no two arcs B_i have a common endpoint.*

Remark 3.8. *Here the geodesic and convexity are taken with respect to the metric σ in M.*

Definition 3.7 (Admissible polygon)**.** *Let Ω be an admissible domain. We say that $\mathcal{P}$ is an admissible polygon provided that $\mathcal{P}\subset\Omega$ and the vertices of $\mathcal{P}$ are chosen among the vertices of Ω.*

Remark 3.9. *Recall that a domain Ω is called to be a geodesicaly convex domain, if two any points in Ω can be joined by a geodesic segment contained in Ω.*

Now suppose that $\Omega\subset M$ is an admissible domain with $\partial\Omega = \cup_i J_i$, where the family $\{J_i\}\subset\partial\Omega$ is a closed cover of $\partial\Omega$ which satisfies $J_i\cap J_{i+1} = \alpha_i$ for all $i\in\{1,\ldots,v-1\}$, and $J_v\cap J_1 = \alpha_v$, where $\{\alpha_i\}$ denotes the set of endpoints of the arcs J_i. Let $c = \{c_i\colon J_i\to\mathbb{R}\}$ be a family of bounded continuous functions. Consider the curve $\gamma_c\subset\partial\Omega\times\mathbb{R}$ given by $\gamma_c(x) = (x, c_i(x))$ if $x\in\operatorname{int} J_i$ and γ_c is a vertical line joining $(\alpha_i, c_i(\alpha_i))$ and $(\alpha_i, c_{i+1}(\alpha_i))$ if $x = \alpha_i$. Using the classical results about the solvability of the Plateau problem, we can conclude that it is always possible to get a solution of (46) with boundary data c_i over an admissible domain.

Theorem 3.7 (Local existence)**.** *Suppose that Ω is an admissible connected domain as above which is also geodesicaly convex. Let γ_c be the curve in $\partial\Omega\times\mathbb{R}$ associated to the family $c = \{c_i\colon J_i\to\mathbb{R}\}$. Then there exists an unique solution of* (46) *with data c_i on* $\operatorname{int} J_i$.

Proof. The proof is similar to that given in Theorem 3.1. Firstly, note that the domain in $\Omega\times\mathbb{R}$ limited by $\text{Graph}^v[\varphi - t]$ and $\text{Graph}^v[\varphi + t]$ $(t > 0)$ is piecewise convex in the sense of Definition 2.2, where $\varphi : M \to \mathbb{R}$ indicates the rotationally symmetric function given by LIRA and MARTÍN (2019) at Theorem 5. Namely by Lemma 3.4 $\partial\Omega \times \mathbb{R}$ is mean convex in $M \times \mathbb{R}$ with the metric g_c. Thus, there exists an embedded minimal disk Σ in $\Omega \times \mathbb{R}$ endowed with metric g_c with boundary γ_c by Theorem 2.5. Therefore, it remains to prove that $\text{int}(\Sigma)$ is a *vertical graph* over Ω.

Firstly, let us show that for all $p \in \text{int}(\Sigma)$ the tangent space $T_p\Sigma$ does not contain ∂_t. Otherwise there exists a point $p \in \text{int}(\Sigma)$ such that $p \in M \times \{c\}$ for some $c \in \mathbb{R}$ and that the tangent space $T_p\Sigma$ contains ∂_t. Take an orthonormal basis $\{\partial_t,\ v\}$ of $T_p\Sigma$, where v is tangent to $M \times \{c\}$. Let γ be the unique geodesic in $M \times \{c\}$ with respect to g_0 such that $\gamma(0) = p$ and $\gamma'(0) = v$. Note that γ intersects $\partial(\Omega \times \mathbb{R})$ exactly in two points, since γ cannot accumulate inside Ω because it is geodesicaly convex.

Now we know from Lemma 3.4 (or Remark 2.4) that $\gamma \times \mathbb{R}$ is minimal in $M \times \mathbb{R}$ endowed with the metric g_c and $T_p(\gamma \times \mathbb{R}) = T_p\Sigma$. So, near p the set $I = \Sigma \cap (\gamma \times \mathbb{R})$ contains at least two curves that intersect transversely at p, by Theorem 2.3. Turn out that if there exists a closed curve α in $I \setminus \partial\Sigma$, then α is the boundary of a minimal disk D in Σ. Thus we could choose a geodesic curve β in D so that the totally geodesic surface $\beta \times \mathbb{R}$ touches D at an interior point. But this is impossible by Theorem 2.1.

Finally, using a similar strategy as at the end of Proposition 3.7 we shall conclude that Σ is a *vertical graph* and it is unique. □

3.2.2 *Maximum principle*

As the last step to prove the main theorem of this part, we will need to obtain a version of the maximum principle that is applicable in our setting, so we will get this now. In this part, the norm, the gradient and the divergent are taken with respect to the metric σ in M.

Proposition 3.17 (Maximum principle)**.** *Let $\Omega \subset M$ be an admissible domain. Suppose that u_1 and u_2 satisfy*

$$\text{div}\left(\frac{\nabla u_1}{\sqrt{1+|\nabla u_1|^2}}\right) \geq \text{div}\left(\frac{\nabla u_2}{\sqrt{1+|\nabla u_2|^2}}\right),$$

and $\liminf(u_2 - u_1) \geq 0$ *for any approach of $\partial\Omega$, with possible exception of finite numbers of points $\{q_1, \ldots, q_r\} = E \subset \partial\Omega$. Then $u_2 \geq u_1$ on $\partial\Omega \setminus E$ with strict inequality unless $u_2 = u_1$.*

Proof. Let K and ε be positive constants, with K large enough and ε small enough to be defined. Define a function

$$\varphi = \begin{cases} K-\varepsilon, & \text{if } u_1-u_2 \geq K; \\ u_1-u_2-\varepsilon, & \text{if } \varepsilon < u_1-u_2 \leq K; \\ 0, & \text{if } u_1-u_2 \leq \varepsilon. \end{cases}$$

Notice that φ is Lipschitz with $0 \leq \varphi \leq K$. In fact, we have $\nabla\varphi = \nabla u_1 - \nabla u_2$ in the set $\{\varepsilon < u_1 - u_2 < K\}$ and $\nabla\varphi = 0$ almost everywhere in the complement of $\{\varepsilon < u_1 - u_2 < K\}$. In particular, we have a control of $\nabla\varphi$ in the compact set $\overline{\{\varepsilon < u_1 - u_2 < K\}}$, and therefore in whole Ω. Around any point $q_i \in E$, consider an open geodesic disk $B_\varepsilon(q_i)$ of radius ε and center q_i. Let $\Omega_\varepsilon := \Omega \setminus \cup B_\varepsilon(q_i)$, and suppose that $\partial\Omega_\varepsilon = \tau_\varepsilon \cup \rho_\varepsilon$, where $\rho_\varepsilon = \cup(\partial B_\varepsilon(q_i) \cap \Omega)$ and $\tau_\varepsilon = \partial\Omega_\varepsilon \cap \partial\Omega$. Since $\liminf(u_2 - u_1) \geq 0$ in $\partial\Omega \setminus E$, we have $\varphi \equiv 0$ in a neighbourhood of τ_ε.

Next we would like to study the quantity

$$J := \int_{\rho_\varepsilon} \varphi \left\{ \sigma\left(\frac{\nabla u_1}{W_1}, \nu\right) - \sigma\left(\frac{\nabla u_2}{W_2}, \nu\right) \right\}, \tag{48}$$

where ν is the unit outer conormal to Ω_ε and $W_i = \sqrt{1 + |\nabla u_i|^2}$.

Naturally the condition $0 \leq \varphi \leq K$ implies from (48) that

$$J \leq 2K \sum_{i=1}^{r} ||\partial B_\varepsilon(q_i)||, \tag{49}$$

where $||\partial B_\varepsilon(q_i)||$ denotes the length of $\partial B_\varepsilon(q_i)$ with respect to the Riemannian metric σ. On the other hand, using that φ is a Lipschitz functions one concludes

$$\operatorname{div}\left(\varphi\left\{\frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}\right\}\right) = \nabla\varphi\left\{\frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}\right\} + \varphi\left\{\operatorname{div}\left(\frac{\nabla u_1}{W_1}\right) - \operatorname{div}\left(\frac{\nabla u_2}{W_2}\right)\right\},$$

almost everywhere in Ω. Thus, by divergence theorem ones gets

$$\begin{aligned} J &= \int_{\Omega_\varepsilon} \left\{ \sigma\left(\nabla\varphi, \left(\frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}\right)\right) + \varphi\left(\operatorname{div}\left(\frac{\nabla u_1}{W_1}\right) - \operatorname{div}\left(\frac{\nabla u_2}{W_2}\right)\right) \right\} \\ &\geq \int_{\Omega_\varepsilon} \sigma\left(\nabla\varphi, \left(\frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}\right)\right). \end{aligned} \tag{50}$$

Now if $N_i := \frac{\partial_t}{W_i} - \frac{\nabla u_i}{W_i}$, then

$$\begin{aligned}\sigma\left(\nabla u_1 - \nabla u_2, \left(\frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}\right)\right) &= g_0\left(N_1 - N_2, W_1 N_1 - W_2 N_2\right) \\ &= W_1 - (W_1 + W_2) g_0(N_1, N_2) + W_2 \\ &= \frac{1}{2}(W_1 + W_2) g_0(N_1 - N_2, N_1 - N_2). \qquad (51)\end{aligned}$$

From (49), (50) and (51) we get

$$2K \sum_{i=1}^{r} ||\partial B_\varepsilon(q_i)|| \geq \frac{1}{2} \int_{\Omega_\varepsilon \cap \{0 < u_1 - u_2 < K\}} (W_1 + W_2) g_0(N_1 - N_2, N_1 - N_2) \geq 0.$$

Letting $\varepsilon \to 0$ we obtain

$$\int_{\{0 < u_1 - u_2 < K\}} (W_1 + W_2) g_0(N_1 - N_2, N_1 - N_2) = 0.$$

Hence $N_1 = N_2$ in $\{0 < u_1 - u_2 < K\}$ which implies that we must have $\nabla u_1 = \nabla u_2$ in $\{0 < u_1 - u_2 < K\}$. In turn, as K was arbitrary constant, then $\nabla u_1 = \nabla u_2$ in the set $\{0 < u_1 - u_2\}$. Finally, to complete the proof, let us suppose now that $\{0 < u_1 - u_2\}$ contains a connected component with non-empty interior. By the previous argument $u_1 = u_2 + c$, where c is a positive constant, so by the maximum principle $u_1 = u_2 + c$ in Ω. On the other hand, as $\liminf(u_2 - u_1) \geq 0$ for any approach of $\partial\Omega \setminus E$, then c must be non-positive, which is impossible. This finishes the proof. □

Remark 3.10. *Different what happen in the horizontal case, Proposition 3.17 is a comparison principle for divergence form operators. This fact deserves be pointed out here because the comparison principle and the maximum principle are not equivalent in general setting.*

3.2.3 *Existence of Jenkins-Serrin graphs type I*

Before we star the main result of existence, let us fix some notations. From now on Ω will be an admissible domain in M so that

$$\partial\Omega = \left(\bigcup_{i=1}^{l} A_i\right) \bigcup \left(\bigcup_{k=1}^{z} C_k\right),$$

where the arcs A_i are geodesics and the arcs C_k are convex in M with the metric σ.

Definition 3.8. *A function $u\colon \Omega \to \mathbb{R}$ is called a Jenkins-Serrin solution of* (46) *type I over Ω with continuous boundary data $c_k\colon C_k \to \mathbb{R}$ if u is a solution of* (46) *such that $u = c_k$ on C_k for all k, $u \to +\infty$ on A_i for all i.*

Moreover, we will need of the following notation. Let $\mathcal{P}$ be an admissible polygon in Ω. Then with the notations above, we define

$$\alpha(\mathcal{P}) = \sum_{A_i \subset \partial\mathcal{P}} \mathfrak{L}_\sigma[A_i],$$

where $\mathfrak{L}_\sigma[\eta]$ denotes the length of η is taken with respect to the metric σ.

Theorem 3.8 (Existence of Jenkins-Serrin graph type I)**.** *Let $\Omega \subset M$ be an admissible domain with $\{B_i\} = \varnothing$. Given any continuous data $c_k\colon C_k \to \mathbb{R}$, there exists a Jenkins-Serrin solution $u\colon \Omega \to \mathbb{R}$ for the translating soliton equation with continuous data $u|_{C_k} = c_k$, if for any admissible polygon $\mathcal{P}$ we have*

$$2\alpha(\mathcal{P}) < \mathfrak{L}_\sigma(\mathcal{P}). \tag{52}$$

Proof. Define a family of curves $\{\gamma_n\}$ by setting $\gamma_n(x) = (x, n)$ for all $x \in A_i$, $\gamma_n(x) = (x, \min\{c_k, n\})$ for all $x \in \operatorname{int} C_k$ for all j and γ_n is the vertical segment joint (x, n) to $(x, \min\{c_k(x), n\})$ when x is a vertices of Ω. By Theorem 3.8, for all $n \in \mathbb{N}$, there exists $u_n\colon \Omega \to \mathbb{R}$ so that Graph$[u_n]$ is a *vertical translating graph* in $\Omega \times \mathbb{R}$ with boundary γ_n. Notice that if $n > m$ we have $u_n \geq u_m$ on $\partial\Omega$, so $u_n > u_m$ in Ω by Proposition 3.17. Hence $\{u_n\}$ is a monotone sequence. Next, taking into account results of PINHEIRO (2009) or MAZET, RODRÍGUEZ, and ROSENBERG (2011), the structural conditions (52) guarantees that there exists a Jenkins-Serrin solution $v\colon \Omega \to \mathbb{R}$ for the minimal graph equation with continuous data c_k. Since

$$\operatorname{div}\left(\frac{v}{\sqrt{1+|v|^2}}\right) = 0 < \frac{1}{\sqrt{1+|u_n|^2}} = \operatorname{div}\left(\frac{u_n}{\sqrt{1+|u_n|^2}}\right)$$

and $\liminf(v - u_n) \geq 0$ on $\partial\Omega \setminus E$, where E is the set of vertices of Ω, the Proposition 3.17 implies $v > u_n$ for all n. Therefore $\lim u_n = u$ exists and satisfies

$$\operatorname{div}\left(\frac{u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{1}{\sqrt{1+|u|^2}}$$

in Ω. Clearly $u|_{C_k} = c_k$ and $u \to +\infty$ as we approach A_i for all i. $\square$

4 CHARACTERIZATION OF THE FAMILY ASSOCIATED TO THE TILTED GRIM REAPER CYLINDER

Differentiating what we have done in Section 3 which we have proven the existence of Jenkins-Serrin graphs in the vertical direction and the horizontal (Killing) direction, in this section we are interested to obtain a characterization of a particular family of complete translating solitons in $\mathbb{R}^{n+1}$, naturally the family associated to the tilted grim reaper cylinders.

We just have seen in Subsection 3.1.11.1 that the grim reaper curve and the line parallel to $\mathbf{e}_2$ are the unique examples of translating soliton in $\mathbb{R}^2$ with respect to the vector $\mathbf{e}_2$, up to a translation. From this curve grim reaper we can create new examples of soliton by taking the product of this curve with $\mathbb{R}^{n-1}$, the resultant hypersurface is called the grim reaper cylinder. This hypersurface has the following parametrization

$$F_0 : \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \times \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n+1}$$

given by

$$F_0(x_1, \ldots, x_n) = (x_1, \ldots, x_n, -\log \cos x_1).$$

On the other hand, from this hypersurface we also can produce other examples of translating solitons just by subtle scaling and rotating F_0 in such a way that keeps the translating velocity $\mathbf{e}_{n+1}$. In this way, we obtain an one-parameter family of translating solitons parametrized by

$$F_\theta : \left(-\tfrac{\pi}{2\cos(\theta)}, \tfrac{\pi}{2\cos(\theta)}\right) \times \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n+1}$$

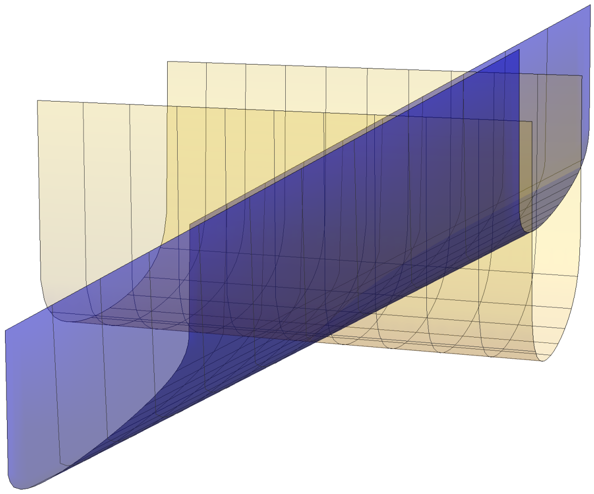
defined by

$$F_\theta(x_1, \ldots, x_n) = (x_1, \ldots, x_n, -\sec^2(\theta) \log \cos(x_1 \cos(\theta)) + \tan(\theta) x_n), \tag{53}$$

where $\theta \in [0, \pi/2)$. Notice that the limit of the family F_θ, as θ tends to $\pi/2$, is a hyperplane parallel to $\mathbf{e}_{n+1}$ (see Figure 4). The family $\left\{F_\theta\left(\left(-\tfrac{\pi}{2\cos(\theta)}, \tfrac{\pi}{2\cos(\theta)}\right) \times \mathbb{R}^{n-1}\right)\right\}_{\theta \in [0,\pi/2)}$ is called the family associated to the tilted grim reaper cylinder.

Another interesting example of translating solitons in $\mathbb{R}^{n+1}$ $(n \geq 2)$ was given by CLUTTERBUCK, SCHNÜRER, and SCHULZE (2007), they proved the existence of an entire graphical translator in $\mathbb{R}^{n+1}$ over $\mathbb{R}^n (= \mathbb{R}^n \times \{0\})$ that is rotationally symmetric, strictly convex with translating velocity $\mathbf{e}_{n+1}$. This example becomes known as the translating paraboloid soliton or bowl soliton. Moreover, they found an one-parameter family $\{W^n_\lambda\}_{\lambda > 0}$ of rotationally invariant cylinders called translating catenoids soliton (see Figure 5). The parameter λ control the size of the neck of each translating soliton. The limit,

Figure 4 – The regular grim reaper cylinder in $\mathbb{R}^3$ and the tilted grim reaper for $\theta = \pi/4$ and the translating catenoid W_2^2.



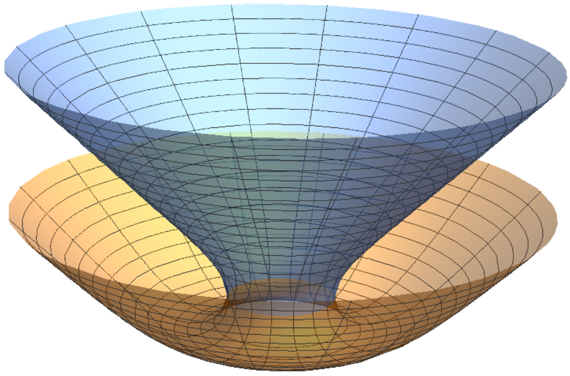
as $\lambda \to 0$, of W_λ^n consists of two copies of the bowl soliton with a singular point at the axis of symmetry. Furthermore, they classified all the translating solitons of revolution by proving that the family of translating catenoids and the bowl soliton are the unique examples of rotationally symmetric translating solitons in $\mathbb{R}^{n+1}$ with translating velocity $\mathbf{e}_{n+1}$, up to a translation.

Until here all the examples that we have mentioned have a trivial topology which means that they could be seen as the sphere $\mathbb{S}^n$ without either one or two points or a simply connected region on $\mathbb{S}^n$. Examples with no trivial topology in $\mathbb{R}^3$ were obtained by using Kapouleas's techniques. These examples were given by DÁVILA, DEL PINO, and NGUYEN (2017), NGUYEN (2009), NGUYEN (2013), NGUYEN (2015) and SMITH (2017). It is important we point out here that the examples obtained by Nguyen have infinite topology.

More recently, HOFFMAN, ILMANEN, MARTÍN, and WHITE (2019) (see also BOURNI, LANGFORD, and TINAGLIA (2018)) proved the existence one-parameter family of strictly convex *vertical translating graphs* in $\mathbb{R}^{n+1}$ called $\Delta-$wing of width w. Furthermore, they classified all complete *vertical translating graphs* in $\mathbb{R}^3$. More precisely, they showed that the family of the grim reaper cylinder, the bowl soliton and the family of $\Delta-$wings are the unique examples of complete *vertical translating graphs* in $\mathbb{R}^3$ (see Figure 6). Using this result of classification, HOFFMAN, MARTÍN, and WHITE (2019) proved the existence and uniqueness of example of translating soliton like Scherk in $\mathbb{R}^3$. Moreover, taking subtle limit on the domain they got a two-parameter family of new examples of translating solitons, one like the helicoid, other doubly periodic like Scherk and two another new examples, without analogous with the minimal case, the Scherknoid and Pitchfork (see Figure 6 and Figure 7).

Another result of classification for bowl soliton was given by WANG (2011).

Figure 5 – The catenoid translator W_2^2.



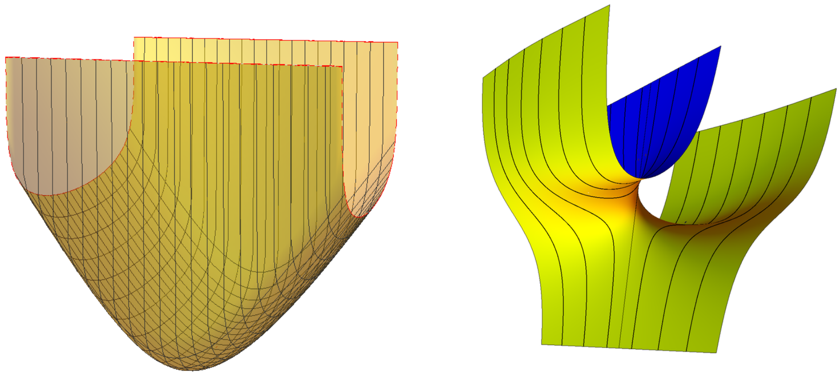
He characterized the bowl soliton as the only convex translating soliton which is an entire graph. Very recently, SPRUCK and XIAO (2018) have proved that a complete *vertical translating graph* must be convex. In particular, complete entire *vertical translating graph* must be the bowl soliton by Wang's theorem.

Using a little different approaching, HASLHOFER (2015) showed that any strictly convex, uniformly two-convex translator which is non-collapsing is necessarily rotationally symmetric. In this line of work, BOURNI and LANGFORD (2016) proved that a translator which arises as a proper blow-up limit of a two-convex mean curvature flow of immersed hypersurfaces is rotationally symmetric. Some interesting classification results for the grim reaper cylinders was found by TASAYCO and ZHOU (2017). They proved the uniqueness of grim reaper cylinders in $\mathbb{R}^{n+1}$ when $n \in \{2, 3\}$ in function of the range of the second fundamental form.

Using the Alexandrov's method of moving hyperplanes, MARTÍN, SAVAS-HALILAJ, and SMOCZYK (2015) get the first characterization of the bowl soliton in term of its asymptotic behaviour. More precisely, they proved that if a translating soliton is C^∞-asymptotic to a bowl soliton, then it must be the Bowl soliton. Besides that, these authors obtained one of the first characterizations of the family of tilted grim reaper cylinders, as the only connected translation solitons in $\mathbb{R}^{n+1}$, $n \geq 2$, such that the function $|A|^2 H^{-2}$ has a local maximum in $M \setminus H^{-1}(0)$.

Another characterization of the grim reaper cylinder in $\mathbb{R}^3$, in terms of its asymptotic behaviour, was given by MARTÍN, PÉREZ-GARCÍA, SAVAS-HALILAJ, and SMOCZYK (2016) (see also PÉREZ-GARCÍA (2016)). They proved that the grim reaper cylinder is the only connected, properly embedded, translating soliton of dimension 2, with locally bounded genus and being C^1-asymptotic to two different half-planes. Their clever ideas was to use the maximum principle combined with a compactness theorem for minimal surfaces in 3-manifolds due to WHITE (2016) to determine the asymptotic shape of the surface. Finally, the authors applied the maximum principle to prove that if a translating soliton is smoothly asymptotic to a grim reaper cylinder, then it must

Figure 6 – $\Delta-$wing translator(left) and the pitchfork translator(right).



coincide with the grim reaper cylinder.

Unfortunately, as it is not known whether White's compactness theorem has an extension for higher dimensions and, even in dimension 3, it does not work without the hypothesis of locally bounded genus, then the proof in MARTÍN *et al.* (2016) fails for higher dimensions and without the hypothesis of locally bounded genus. Moreover, the tilted grim reaper cylinder given by (53) is C^1-asymptotic to two half-hyperplanes outside a non-horizontal cylinder (see Remark 4.2 below). Hence, it is natural to ask if it is possible to generalize the theorem for arbitrary dimensions $n \geq 2$, without any further assumptions about the topology of the soliton or the axis of the cylinder.

These questions were what motivated our works in (GAMA and MARTÍN, 2018) and (GAMA, 2019). As we shall see afterwards the variation of the maximum principle from Subsection 2.2.6 and the compactness theorems from Subsection 2.2.5 allow to give a positive answer to these questions.

This chapter is structured as follows. In the Section 4.1 we fix some notations that we going to use after and refine the Proposition 2.6 for any Killing vector field in $\mathbb{R}^{n+1}$. In turn in the Section 4.2, we obtain a lemma which shows that every complete, properly embedded translating soliton in $\mathbb{R}^n$ which is C^1-asymptotic to two half-hyperplanes has a surprising amount of internal dynamical periodicity in the space $\mathcal{IV}_n(\mathbb{R}^{n+1})$. Finally, in the Section 4.3 we prove our main theorems.

4.1 Translating solitons in $\mathbb{R}^{n+1}$

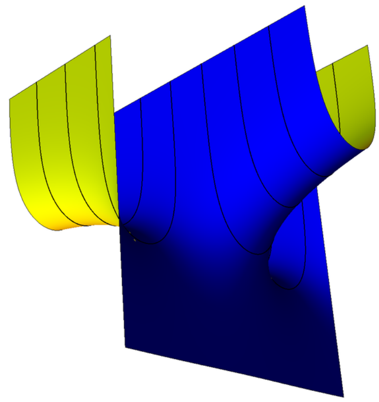
We remind the notations that we use throughout this chapter. Recall that an oriented hypersurface $M \subset \mathbb{R}^{n+1}$ is called to be a *translating soliton* provided that it satisfies

$$\vec{\mathbf{H}} = v^{\perp},$$

where v is a fixed vector and $\vec{\mathbf{H}}$ denotes the mean curvature vector field. In particular, one has

$$H = \langle v, N \rangle, \tag{54}$$

Figure 7 – The scherkenoid translator.



where N denotes the unit normal along M. To make our study more simple, we will always suppose from now on that $v = \mathbf{e}_{n+1}$, where $B = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n+1}\}$ is the canonical basis of $\mathbb{R}^{n+1}$. Moreover, as translating solitons are minimal hypersurfaces is $\mathbb{R}^{n+1}$ with Ilmanen's metric $g = e^{\frac{2}{n}x_{n+1}}\langle\cdot,\cdot\rangle$, where $\langle\cdot,\cdot\rangle$ is the Euclidean metric, then from now on we always adopt in $\mathbb{R}^{n+1}$ the metric g, unless otherwise stated. Thus we are seeing translating solitons are minimal hypersurfaces. Recall that for us a complete translating soliton M in $\mathbb{R}^{n+1}$ means that the hypersurface M is complete in $\mathbb{R}^{n+1}$ with the Euclidean metric.

Next, we need to define what means a hypersurface be asymptotic to half-hyperplanes outside a cylinder.

Definition 4.1. *Let $\mathcal{H}$ a open half-hyperplane in $\mathbb{R}^{n+1}$ and w the unit inward pointing normal of $\partial\mathcal{H}$. For a fixed positive number δ, denote by $\mathcal{H}(\delta)$ the set given by*

$$\mathcal{H}(\delta) := \{p + tw : p \in \partial\mathcal{H} \text{ and } t > \delta\}.$$

We say that a smooth hypersurface M is C^k-asymptotic to the open half-hyperplane $\mathcal{H}$ if M can be represented as the graph of a C^k- function $\varphi : \mathcal{H} \longrightarrow \mathbb{R}$ such that for every $\epsilon > 0$, there a (big) $\delta > 0$, so that for any $j \in \{1, 2, \ldots, k\}$ it holds

$$\sup_{\mathcal{H}(\delta)} |\varphi| < \epsilon \text{ and } \sup_{\mathcal{H}(\delta)} |D^j\varphi| < \epsilon.$$

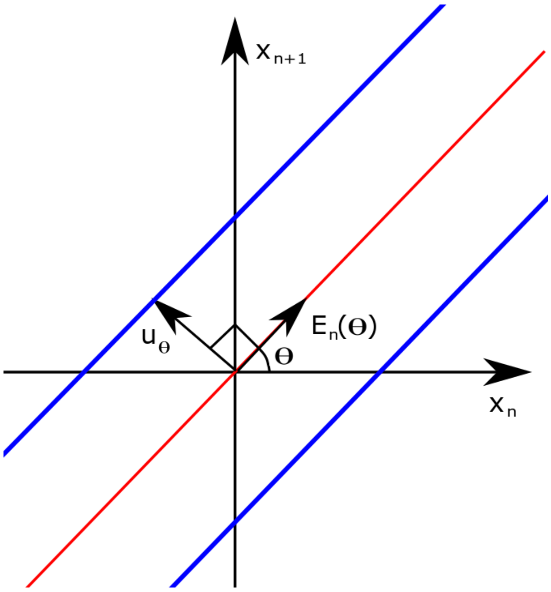
We say that a smooth hypersurface M is C^k-asymptotic outside a cylinder to two half-hyperplanes $\mathcal{H}_1$ and $\mathcal{H}_2$ provided there exists a solid cylinder $\mathcal{C}$ such that:

i. The solid cylinder $\mathcal{C}$ contains the boundaries of the half-hyperplane $\mathcal{H}_1$ and $\mathcal{H}_2$,

ii. $M \setminus \mathcal{C}$ consists of two connected components M_1 and M_2 that are C^k-asymptotic to $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Remark 4.1. *The solid cylinders in $\mathbb{R}^{n+1}$ with the Euclidean metric that we are considering are those that are isometric to $\mathbb{D}(r) \times \mathbb{R}^{n-1}$, where $\mathbb{D}(r)$ is the disk of radius r in*

Figure 8 – Intersection of the cylinder $\mathcal{C}_\theta(1)$(blue) with the plane $[\mathbf{e}_n, \mathbf{e}_{n+1}]$.



$\mathbb{R}^2$.

Let us give some examples that are C^1-asymptotic to two hyperplanes outside a cylinder.

Example 4.1. *The hyperplanes parallel to* $\mathbf{e}_{n+1}$ *are* $C^\infty-$*asymptotic outside a cylinder to two half-hyperplanes.*

Example 4.2. *Each element of the family of the grim reaper cylinders is* $C^\infty-$*asymptotic to two half-hyperplanes outside a particular tilted cylinder.*

Proof. To see this, observe that the map F_θ defined early is a parametrization of $\mathrm{Graph}^v[f_\theta]$, where

$$\begin{array}{rcl} f_\theta : \left(-\frac{\pi}{2\cos(\theta)}, \frac{\pi}{2\cos(\theta)}\right) \times \mathbb{R}^{n-1} & \longrightarrow & \mathbb{R} \\ x = (x_1, \dots, x_n) & \longmapsto & -\sec^2(\theta)\log\cos(x_1\cos(\theta)) + \tan(\theta)x_n \end{array}.$$

Remarked this, we consider the vectors $u_\theta := -\sin(\theta)\mathbf{e}_n + \cos(\theta)\mathbf{e}_{n+1}$ and $E_n(\theta) = \cos(\theta)\mathbf{e}_n + \sin(\theta)\mathbf{e}_{n+1}$ (see Figure 8). Next we define the solid cylinder $\mathcal{C}_\theta(s) = \{x \in \mathbb{R}^{n+1} : \langle x, \mathbf{e}_1\rangle^2 + \langle u_\theta, x\rangle^2 \leq s^2\}$ and the half-hyperplanes

$$\mathcal{H}_- := \left\{x \in \mathbb{R}^{n+1} : \langle x, \mathbf{e}_1\rangle = -\frac{\pi}{2\cos(\theta)} \text{ and } \langle x, u_\theta\rangle \geq 0\right\}$$

and

$$\mathcal{H}_+ := \left\{x \in \mathbb{R}^{n+1} : \langle x, \mathbf{e}_1\rangle = \frac{\pi}{2\cos(\theta)} \text{ and } \langle x, u_\theta\rangle \geq 0\right\}.$$

We want to conclude that $\mathrm{Graph}^v[f_\theta]$ is $C^\infty-$asymptotic to $\mathcal{H}_-$ and $\mathcal{H}_+$ outside $C_\theta(s)$ for some subtle choose of s. To do this, first observe that if $x = (x_1, \dots, x_n, x_{n+1}) \in$

$\mathrm{Graph}^v[f_\theta]$, then it holds $\langle x, u_\theta\rangle \geq 0$, since $x_1 \in \left(-\frac{\pi}{2\cos(\theta)}, \frac{\pi}{2\cos(\theta)}\right)$. Thus, if $r(x) = \langle x, u_\theta\rangle$ denotes the height function in $\mathbb{R}^{n+1}$ with respect to the vector u_θ, then $r \geq 0$ on $\mathrm{Graph}^v[f_\theta]$.

Now if $(x_1, \ldots, x_n, -\sec(\theta)\log\cos(x_1\cos(\theta)) + \tan(\theta)x_n) = (y_1, \ldots, y_n, y_{n+1})$, then

$$y_1 = \frac{1}{\cos(\theta)} \arctan\left(\pm\sqrt{e^{2r(x)\cos(\theta)} - 1}\right). \tag{55}$$

In particular, this equality implies that $\mathrm{Graph}^v[f_\theta]$ can be seen as the union of two horizontal graphs defined over $\mathcal{H}_-$ and $\mathcal{H}_+$, respectively. Indeed, considering the orthonormal basis $\{\mathbf{e}_2, \ldots, \mathbf{e}_{n-1}, E_n(\theta), u_\theta\}$ in $\mathcal{H}_\pm$ one has

$$\mathcal{H}_- = \left\{ x = -\frac{\pi}{2\cos(\theta)} e_1 + \sum_{j=2}^{n-1} \alpha_j \mathbf{e}_j + \alpha_n E_n(\theta) + r u_\theta \ : \ \alpha_i \in \mathbb{R} \text{ and } r \geq 0 \right\}.$$

and

$$\mathcal{H}_+ = \left\{ x = \frac{\pi}{2\cos(\theta)} e_1 + \sum_{j=2}^{n-1} \alpha_j \mathbf{e}_j + \alpha_n E_n(\theta) + r u_\theta \ : \ \alpha_i \in \mathbb{R} \text{ and } r \geq 0 \right\}.$$

Therefore, $\mathrm{Graph}^v[f_\theta] = \mathrm{Graph}^h[f_-] \cup \mathrm{Graph}^h[f_+]$, where $f_\pm : \mathcal{H}_\pm \to \mathbb{R}$ are defined by

$$f_\pm = \frac{1}{\cos(\theta)} \arctan\left(\pm\sqrt{e^{2r\cos(\theta)} - 1}\right).$$

Finally, we fix some $\delta > 0$, and we take $s(\delta)$ so that

$$\frac{1}{\cos^2(\theta)} \arctan^2\left(\sqrt{e^{2\delta\cos(\theta) - 1}}\right) + \delta^2 = s(\delta)^2.$$

With this choice one has

$$\mathrm{Graph}^v[f_\theta] \setminus \mathcal{C}_\theta(s(\delta)) = \mathrm{Graph}^h[f_-|_{\mathcal{H}_-(\delta)}] \cup \mathrm{Graph}^h[f_+|_{\mathcal{H}_+(\delta)}],$$

where $\mathcal{H}_\pm(\delta) = \{x \in \mathcal{H}_\pm \ : \ \langle x, u_\theta\rangle \geq \delta\}$. Using this equality, we can conclude that $\mathrm{Graph}^v[f_\theta]$ is C^∞−asymptotic to $\mathcal{H}_-(\delta)$ and $\mathcal{H}_+(\delta)$ outside $\mathcal{C}_\theta(s(\delta))$. □

We would like to finish this part with the next general results whose proof is similar to that given at Lemma 2.3 and Proposition 2.5. The expression $\mathcal{A}_g[\Sigma]$ indicates the area of the hypersurface Σ as a hypersurface in $\mathbb{R}^{n+1}$ with the metric g and $\mathrm{Graph}^\Pi[f] := \{x + f(x)\nu \colon x \in \overline{\Omega}\}$, where ν is an unit normal vector to Π. Notice that $\langle v, e_{n+1}\rangle$ is constant.

Proposition 4.1. *Suppose that* $\mathrm{Graph}^\Pi[f]$ *is a translating soliton in* $M \times \mathbb{R}$. *Then*

$\mathrm{Graph}^{\Pi}[f]$ *is stable in* $\mathbb{R}^{n+1}$ *with the metric* g.

Proposition 4.2. *Let* $f : \overline{\Omega} \to \mathbb{R}$ *a smooth function over a domain* $\overline{\Omega} \subset \Pi$, *where* Π *is a hyperplane in* $\mathbb{R}^{n+1}$. *Suppose that* $\mathrm{Graph}^{\Pi}[f]$ *is a translating graph in* $\mathbb{R}^{n+1}$. *Assume that* Σ *is any other hypersurface inside the cylinder* $\{x + s\nu \colon x \in \overline{\Omega} \text{ and } s \in \mathbb{R}\}$ *so that* $\partial\Sigma = \partial\mathrm{Graph}^{\Pi}[f]$, *thus we have*

$$\mathcal{A}_g[\mathrm{Graph}^{\Pi}[f]] \leq \mathcal{A}_g[\Sigma].$$

Moreover, the equality is true provided that $\Sigma = \mathrm{Graph}^{\Pi}[u]$.

4.2 Dynamic lemma and applications

Throughout this subsection we are fixing $\theta \in [0, \pi/2)$, and we continue to call $u_\theta := -\sin(\theta) \cdot \mathbf{e}_n + \cos(\theta) \cdot \mathbf{e}_{n+1}$. Furthermore, given any $r > 0$ we consider the cylinder

$$\mathcal{C}_\theta(r) := \{x \in \mathbb{R}^{n+1} \ : \ \langle x, \mathbf{e}_1 \rangle^2 + \langle u_\theta, x \rangle^2 \leq r^2\}.$$

Henceforth, M^n denotes a complete, connected, properly embedded translating soliton in $\mathbb{R}^{n+1}$ such that, outside $\mathcal{C}_\theta(r)$, M is C^1-asymptotic to two half-hyperplanes $\mathcal{H}_1$ and $\mathcal{H}_2$. Our main lemma can be stated as follows.

Lemma 4.1 (Dynamics Lemma). *Let* M *be a hypersurface as above. Suppose that* $\{b_i\}_{i\in\mathbb{N}}$ *is a sequence in* $[\mathbf{e}_1, u_\theta]^{\perp}$ *and let* $\{M_i\}_{i\in\mathbb{N}}$ *be a sequence of hypersurfaces given by* $M_i := M + b_i$. *Then there exist a connected* $n-$*dimensional stationary integral varifold* M_∞ *and a subsequence* $\{M_{i_k}\} \subset \{M_i\}$ *so that*

(i) $M_{i_k} \overset{*}{\rightharpoonup} M_\infty$ *in* $\mathbb{R}^{n+1}$;

(ii) $\mathrm{sing}\ M_\infty$ *satisfies* $\mathcal{H}^{n-7+\beta}(\mathrm{sing}\ M_\infty \cap (\mathbb{R}^{n+1} \setminus \mathcal{C}_\theta(r))) = 0$ *for all* $\beta > 0$ *if* $n \geq 7$, $\mathrm{sing}\ M_\infty \cap (\mathbb{R}^{n+1} \setminus \mathcal{C}_\theta(r))$ *is discrete if* $n = 7$ *and* $\mathrm{sing}\ M_\infty \cap (\mathbb{R}^{n+1} \setminus \mathcal{C}_\theta(r)) = \varnothing$ *if* $1 \leq n \leq 6$;

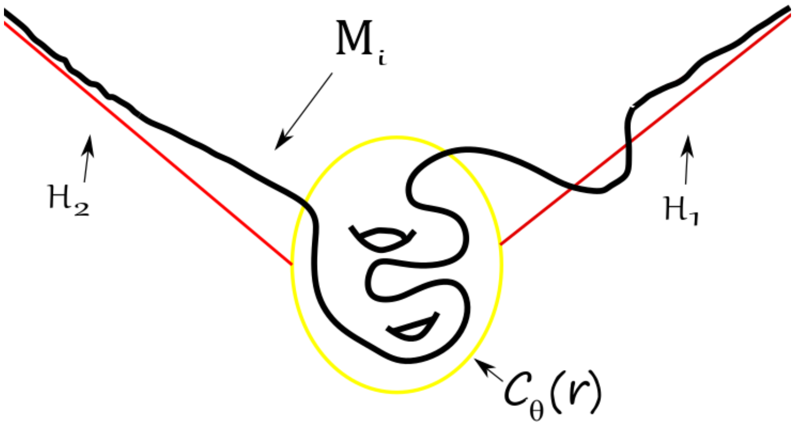
(iii) $M_{i_k} \to \mathrm{spt}\, M_\infty$ *in* $\mathbb{R}^{n+1} \setminus (\mathcal{C}_\theta(r) \cup \mathrm{sing} M_\infty)$.

Remark 4.2. *Above we are using the same notation for the varifold associated to* M_{i_k} *and for itself. So at i. we are seeing* M_{i_k} *as an n-dimensional varifold, however, at iii. we are seeing* M_{i_k} *as a hypersurface in* $\mathbb{R}^{n+1}$.

Proof. The strategy of the proof follows a similar argument as in (MARTÍN *et al.*, 2016),(PÉREZ-GARCÍA, 2016) and (GAMA and MARTÍN, 2018). However, this proof is different of those proofs because we use Proposition 4.2 to conclude that the sequence has locally bounded area.

From our assumption on M, there exist smooth functions $\varphi_1 : \mathcal{H}_1 \to \mathbb{R}$ and $\varphi_2 : \mathcal{H}_2 \to \mathbb{R}$ such way $M \setminus \mathcal{C}_\theta(r) = \mathrm{Graph}^{\Pi_1}[\varphi_1] \cup \mathrm{Graph}^{\Pi_2}[\varphi_2]$, where Π_i denotes the hyperplane in $\mathbb{R}^{n+1}$ which contains $\mathcal{H}_i$. Notice that $M_i \setminus \mathcal{C}_\theta(r) = \mathrm{Graph}^{\Pi_1}[\varphi_1^i] \cup \mathrm{Graph}^{\Pi_2}[\varphi_2^i]$, where $\varphi_j^i(x) = \varphi_j(x - b_i)$

Figure 9 – Transversal section of the behaviour of M_i.



Claim 4.1. *$\{M_i \setminus \mathcal{C}_\theta(r)\}$ has locally bounded area.*

Proof of the Claim 4.1. Indeed, fix any point $p \in \mathbb{R}^{n+1} \setminus \mathcal{C}_\theta(r)$ and take $\epsilon(> 0)$ small enough so that $B_\epsilon(p)$ does not intersect $\mathcal{C}_\theta(r)$, where $B_\epsilon(p)$ denotes the geodesic ball in $\mathbb{R}^{n+1}$ with center p and radius ϵ (see Figure 9). With this notation, whenever $B_\epsilon(p)$ intersects any connected component of $M_i \setminus \mathcal{C}_\theta(r)$ Proposition 4.2 implies that

$$\mathcal{A}_g[B_\epsilon(p) \bigcap (M_i \setminus \mathcal{C}_\theta(r))] \leq \frac{1}{2}\mathcal{A}_g[\partial B_\epsilon(p)].$$

This completes the proof. □

Now the claim above implies that the area blow-up set

$$\mathcal{B} := \{p \in \mathbb{R}^{n+1} \ : \limsup \mathcal{A}_g(M_i \cap B_r(p)) = \infty \text{ for every } r > 0\} \tag{56}$$

lies inside the cylinder $\mathcal{C}_\theta(r)$ and is an $(n, 0)$ set by Theorem 2.11. We would like to conclude that $\mathcal{B} = \varnothing$, so the sequence $\{M_i\}$ has locally bounded area. Arguing by contradiction, let us suppose that $\mathcal{B} \neq \varnothing$. In this case, we could take a tilted grim reaper cylinder whose axis is perpendicular to $\mathcal{C}_\theta(r)$ and it does not intersect $\mathcal{B}$. Now we could move the tilted grim reaper cylinder until we get a first point of contact with $\mathcal{B}$, but Theorem 2.10 implies that $\mathcal{B}$ must contain the tilted grim reaper cylinder, which is absurd.

Therefore the sequence $\{M_i\}$ has locally bounded area, by Theorem 2.8 there exists a subsequence of $\{M_{i_k}\}$ that converges weakly∗ to the stationary integral varifold M_∞. Furthermore, as outside $\mathcal{C}_\theta(r)$ both connected components of M_i are graphs, so stable by Proposition 4.1 and satisfies the $\alpha-$structure hypothesis (Definition 2.14), then we can apply Theorem 2.9 to conclude that $M_{i_k} \to \text{spt } M_\infty$ in $\mathbb{R}^{n+1} \setminus (\mathcal{C}_\theta(r) \cup \text{sing } M_\infty)$ and the set singular sing M_∞ satisfies (ii), notice that the $\alpha-$structure hypothesis is automatically satisfied in this case, since it is the limit of varifolds that satisfies it. In particular, this implies that spt M_∞ is smooth outside $\mathcal{C}_\theta(r)$ and away from sing M_∞.

Using this last fact, we can conclude the connectedness of spt M_∞ as follows. Taking into account that any loop in $\mathbb{R}^{n+1}$ intersects spt M_∞ in an even numbers of points (counting multiplicity), since each M_i is an embedded, then both wings of M_∞ must lie in the same connected component. Indeed, if this was not true, then we could choose the above mentioned loop intersecting spt M_∞ at one unique point (because spt M_∞ is smooth outside $\mathcal{C}_\theta(r) \cup$ sing M_∞) which is absurd. This implies that if spt M_∞ is not connected, there would be a connected component inside the cylinder. In this case we can consider a suitable tilted grim reaper (whose axis is perpendicular to u_θ) of sufficiently large coordinates in the direction of u_θ so that it does not intersect the solid cylinder. Now, if we move it in the direction of $-u_\theta$ until it touches the component inside the cylinder at a first point of contact, then we get a contraction because the component inside the cylinder must be the whole tilted grim reaper by Theorem 2.13. Hence spt M_∞ is connected. □

Next we would like to apply this lemma to obtain some consequences over the behaviour of the half-hyperplane $\mathcal{H}_1$ and $\mathcal{H}_2$. More precisely, we prove that w_1 and w_2 must be parallel to u_θ. Furthermore, we prove that if the half-hyperplanes $\mathcal{H}_1$ and $\mathcal{H}_2$ are parts of the same hyperplane, then M must coincide with a hyperplane parallel to $\mathbf{e}_{n+1}$. In particular, we get a characterization of the hyperplane parallel to $\mathbf{e}_{n+1}$.

Lemma 4.2. *Let M be a hypersurface as above. Then, the normals to the boundary of the half-hyperplanes $\mathcal{H}_1$ and $\mathcal{H}_2$ must be parallel to u_θ. Moreover, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are parts of the same hyperplane Π, then M must coincide with Π.*

Proof. The proof will be by contraction. Assume that the half-hyperplane

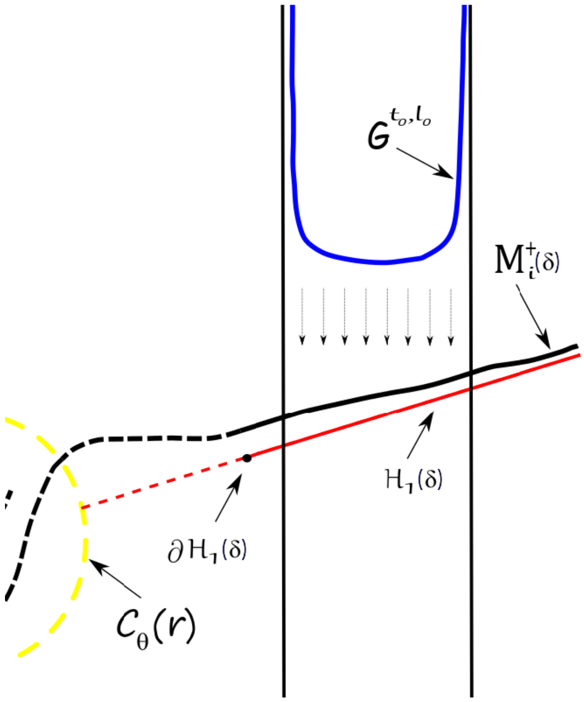
$$\mathcal{H}_1 = \{p + tw_1 \ : \ p \in \partial\mathcal{H}_1, \ t > 0\}$$

is not parallel to direction of translation u_θ. Notice that $\mathbf{e}_j$ and $E_n(\theta)$ are perpendicular to w_1 for all $j \in \{2, \ldots, n-1\}$ by our definition of C^1-asymptotic, where $E_n(\theta) := \cos(\theta)\mathbf{e}_n + \sin(\theta)\mathbf{e}_{n+1}$. In this case, w_1 form a non-vanishing angle only with $\mathbf{e}_1$, that we denote by $\alpha := \measuredangle(\mathbf{e}_1, w_1)$. Suppose that $\cos\alpha > 0$. For given real numbers t and l, we consider the tilted grim reaper cylinder:

$$\mathcal{G}^{t,l} := \left\{ F_\theta(x_1 - t, x_2, \ldots, x_n) + t\mathbf{e}_1 + lu_\theta \ : \ |x_1 - t| < \tfrac{\pi}{2\cos(\theta)}, \ (x_2, \ldots, x_n) \in \mathbb{R}^{n-1} \right\}.$$

Let w_1 be the unit inward pointing normal vector of $\partial\mathcal{H}_1$. For every $\delta > 0$ consider the closed half-hyperplanes $\mathcal{H}_1(\delta) := \{p + tw_1 \ : \ p \in \partial\mathcal{H}_1 \text{ and } t \geq \delta\}$. Consider $\mathcal{Z}^+_{1,\delta}$ denote the half-space in $\mathbb{R}^{n+1}$ which contains $\mathcal{H}_1(\delta)$ and whose boundary contains $\partial\mathcal{H}_1(\delta)$ and is perpendicular to w_1. By our assumptions about M, if δ is sufficiently large then $M_1^+(\delta) := M \cap \mathcal{Z}^+_{1,\delta}$ is sufficiently close to $\mathcal{H}_1$. From this we may conclude that there exist

Figure 10 – Transversal section of the behaviour of $M_1^+(\delta)$ and $\mathcal{G}^{t_0,l_0}$.



sufficiently large $t_0, l_0 \in \mathbb{R}$ so that $\mathcal{G}^{t_0,l_0}$ does not intersect $M_1^+(\delta)$ (see Figure 10). In fact, we can choose t_0 so that $\partial M_1^+(\delta) \cap S_{t_0} = \varnothing$, where $S_{t_0} = \left(t_0 - \frac{\pi}{2\cos(\theta)}, t_0 + \frac{\pi}{2\cos(\theta)}\right) \times \mathbb{R}^n$. Since $\mathcal{H}_1$ is not parallel to u_θ, then if we translate $\mathcal{G}^{t_0,l_0}$ into the direction of $-u_\theta$ we conclude that there exists a first l_1 such that either $\mathcal{G}^{t_0,l_0-l_1}$ and $M_1^+(\delta)$ have a point of contact or $\operatorname{dist}\left(\mathcal{G}^{t_0,l_0-l_1}, M_1^+(\delta)\right) = 0$ and $M_1(\delta) \cap \mathcal{G}^{t_0,l_0} = \varnothing$.

According to Theorem 2.1 the first case cannot be possible because of our assumptions on M. On the other hand, the second case implies that there exists a sequence $\left\{p_i = (p_i^1, \ldots, p_i^{n+1})\right\}$ in $M_1^+(\delta)$ such that:

a. The sequence $\left\{\langle p_i^{n+1}, u_\theta\rangle\right\}$ is bounded in $\mathbb{R}$;

b. $\lim_i \operatorname{dist}\left(\mathcal{G}^{t_0,l_0-l_1}, p_i\right) = 0$.

Notice that the sequence $\{p_i^1\}$ is bounded (by the asymptotic behaviour of $\mathcal{G}^{t_0,l_0-l_1}$). Thus, up to a subsequence, we can suppose $\{p_i^1\} \to p_\infty^1$ and $\{\langle p_i, u_\theta\rangle\} \to p_\infty^{u_\theta}$. Consider the sequence of hypersurfaces $\left\{M_i := M - (0, p_i^2, \cdots, p_i^{n+1}) + \langle p_i, u_\theta\rangle u_\theta\right\}$ in $\mathbb{R}^{n+1}$. By Lemma 4.1, we can suppose that $M_i \rightharpoonup M_\infty$, where M_∞ is a connected stationary integral varifold. Now Proposition 2.2 implies

$$\Theta(M_\infty, p_\infty) \geq \limsup \Theta(M_i, p_i^*) = 1,$$

where $p_i^* = p_i^1 \mathbf{e}_1 + \langle p_i, u_\theta\rangle u_\theta$. Hence it holds $p_\infty \in \operatorname{spt} M_\infty$, and it follows that $\operatorname{spt} M_\infty$ and $\mathcal{G}^{t_0,l_0-l_1}$ have a point of contact at p_∞. Therefore, by Corollary 2.1 one has

$$\operatorname{spt}\ M_\infty = \mathcal{G}^{t_0,l_0-l_1},$$

but this is impossible by our assumption about w_1 be not parallel to u_θ. Therefore $\mathcal{H}_1$ must be parallel to u_θ. Analogously, we can conclude that $\cos\alpha$ cannot be negative and that $\mathcal{H}_2$ is parallel to u_θ.

Finally, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are part of the same hyperplane Π, which we suppose to be $[\mathbf{e}_1]^\perp$ up to a rotation. We would like to conclude that the first coordinate must be constant on M. Otherwise the first coordinate x_1 takes an extreme value either at point in M or along of a sequence $\left\{p_i = (p_i^1, \dots, p_i^{n+1})\right\}$ such that $\{\langle p_i, u_\theta\rangle\} \to p_\infty^{u_\theta}$. Theorem 2.1 implies that the first case is impossible. Regarding the second case, suppose that $\{x_1(p_i)\} \to \sup_M x_1 (> 0)$ and denote $\Pi_1 := \sup\limits_M x_1\mathbf{e}_1 + \operatorname{span}[\mathbf{e}_2, \dots, \mathbf{e}_n, u_\theta]$. We consider the sequence

$$\left\{M_i := M - (0, p_i^2, \dots, p_i^{n+1}) + \langle p_i, u_\theta\rangle u_\theta\right\},$$

by Lemma 4.1, a subsequence converges to M_∞, where M_∞ is a connected stationary integral varifold, thus (reasoning as above) we have an interior point of contact between $\operatorname{spt} M_\infty$ and Π_1. So, by Corollary 2.1 we conclude that $\operatorname{spt}\ M_\infty = \Pi_1$, which is impossible. This shows that the first coordinate x_1 is constant. Therefore M must be the hyperplane Π. $\square$

We finish this subsection by getting the following version of the maximum principle.

Lemma 4.3. *Let M be a hypersurface as above and assume that the half-hyperplanes $\mathcal{H}_1$ and $\mathcal{H}_2$ are not included one inside the other. Consider a domain Σ of M (not necessarily compact) with non-empty boundary $\partial\Sigma$ such that the function $x \mapsto \langle x, u_\theta\rangle$ of Σ is bounded. Then the supremum and the infimum of the x_1-coordinate function of Σ are reached along the boundary of Σ i.e., there exists no sequence $\{p_i\}$ in the interior of Σ such that $\lim\limits_{i\to\infty} \operatorname{dist}(p_i, \partial\Sigma) > 0$ and either $\lim\limits_{i\to\infty} x_1(p_i) = \sup\limits_\Sigma x_1$ or $\lim\limits_{i\to\infty} x_1(p_i) = \inf\limits_\Sigma x_1$.*

Proof. Notice first that if there exists $q \in \operatorname{int}\Sigma$ so that $x_1(q) = \sup_\Sigma x_1$, then Theorem 2.2 gives us that $\Sigma = \Pi(x_1(q))$, where $\Pi(x_1(q)) := \{x \in \mathbb{R}^{n+1} \colon \langle x, \mathbf{e}_1\rangle = x_1(q)\}$. Thus $x_1(q) < \sup_\Sigma x_1$ for all $q \in \operatorname{int}\Sigma$. Analogously, we also see that $x_1(q) > \inf_\Sigma x_1$ for all $q \in \operatorname{int}\Sigma$.

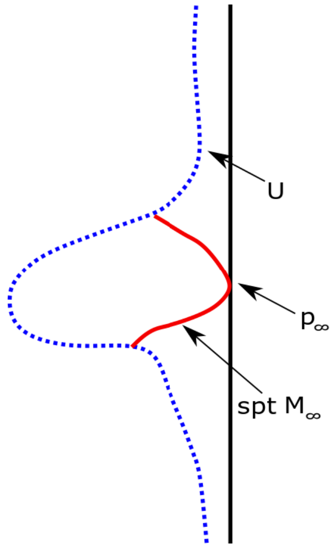
Now let us assume that there exists a sequence $\{p_n\} \subset \Sigma$ in such that way that

$$\lim_{i\to\infty} \operatorname{dist}(p_i, \partial\Sigma) > 0 \text{ and } \lim_{i\to\infty} x_1(p_i) = \sup_\Sigma x_1.$$

Consider the sequence of hypersurfaces $\left\{M_i := M - (0, p_i^2, \cdots, p_i^{n+1}) + \langle p_i, u_\theta\rangle u_\theta\right\}$ in $\mathbb{R}^{n+1}$. Naturally Lemma 4.1 says that $M_i \rightharpoonup M_\infty$, after passing to a subsequence, where M_∞ is a connected stationary integral varifold, and we may also admit that $\langle p_i, u_\theta\rangle \to p_\infty^{u_\theta}$. Now Proposition 2.2 implies

$$p_\infty = \sup_\Sigma x_1\mathbf{e}_1 + p_\infty^{u_\theta} u_\theta \in \operatorname{spt} M_\infty.$$

Figure 11 – Transversal section of the open set U (blue) and part of spt M_∞(red) inside U.



In particular, spt M_∞ touches $\Pi = \{x \in \mathbb{R}^{n+1} : \langle x, \mathbf{e}_1\rangle = \sup_\Sigma x_1\}$ at p_∞ and lies locally oneside of Π. Thus, Theorem 2.13, after a subtle choice of an open set U containing p_∞ (see Figure 11), implies $\Pi \subset$ spt M_∞, but this contrary our hypothesis over M, and in particular over the behaviour of M_∞. □

4.3 Proof of the main theorems

This part is devoted to prove the main theorems. To sake our exposition more didactic, we are going to divide our proof into three cases when $\theta \in [0, \pi/2)$, when $\theta = \pi/2$ and the minimal case.

4.3.1 Case $\theta \in [0, \pi/2)$

In this subsection we will continue by denoting $u_\theta = -\sin(\theta)\cdot\mathbf{e}_n + \cos(\theta)\cdot\mathbf{e}_{n+1}$ and $E_n(\theta) := \cos(\theta)\mathbf{e}_n + \sin(\theta)\mathbf{e}_{n+1}$, where $\theta \in [0, \pi/2)$. Our goal in this subsection is to prove the following result.

Theorem 4.1. *Let $f : M \longrightarrow \mathbb{R}^{n+1}$ be a complete, connected, properly embedded translating soliton and consider $\mathcal{C}_\theta(r) := \{x \in \mathbb{R}^{n+1} : \langle x, \mathbf{e}_1\rangle^2 + \langle u_\theta, x\rangle^2 \le r^2\}$, where $r > 0$. Assume that M is C^1-asymptotic to two half-hyperplanes outside $\mathcal{C}_\theta(r)$. Then we have one, and only one, of these two possibilities:*

(a) *Both half-hyperplanes are contained in the same hyperplane Π parallel to $\mathbf{e}_{n+1}$ and M coincides with Π;*

(b) *Both half-hyperplanes are included in different parallel hyperplanes and M coincides with a tilted grim reaper cylinder associated to θ, up to translation.*

The proof of this theorem will be done soon after many technical lemmas.

Before proceeding , we need some notations that we will use throughout the whole section. Consider the foliation of $\mathbb{R}^{n+1}$ given by

$$\Pi(t) = \left\{x \in \mathbb{R}^{n+1} \ : \ \langle x, \mathbf{e}_1 \rangle = t\right\}. \tag{57}$$

Furthermore, given $A \subset \mathbb{R}^{n+1}$ and $t \in \mathbb{R}$, we consider the sets

$$A_+(t) = \{x \in A \ : \ \langle x, \mathbf{e}_1 \rangle \geq t\}, A_-(t) = \{x \in A \ : \ \langle x, \mathbf{e}_1 \rangle \leq t\}$$

$$A^+(t) = \{x \in A \ : \ \langle x, u_\theta \rangle \geq t\}, A^-(t) = \{x \in A \ : \ \langle x, u_\theta \rangle \leq t\}.$$

Recall that we are assuming that the translating velocity is $\mathbf{e}_{n+1}$. From Lemma 4.2, we already know that the hyperplane must be different if M is not a hyperplane parallel to u_θ, so we only need to work in the case when the half-hyperplanes $\mathcal{H}_1$ and $\mathcal{H}_2$ lie in different and parallel hyperplanes to u_θ and $\mathbf{e}_{n+1}$. Thus we may assume without loss of generality that the half-hyperplanes are contained in $\Pi(-\delta)$ and $\Pi(\delta)$, for a certain $\delta > 0$. Once we have fixed these notations, our first result is to prove that both half-hyperplanes point in the same direction of u_θ.

Lemma 4.4. *The two connected components of M which lie outside the cylinder $\mathcal{C}_\theta(r)$ point in the same direction of u_θ.*

Proof. First of all, notice that M cannot be asymptotic to the half-hyperplanes

$$\mathcal{H}_1 = \left\{x \in \mathbb{R}^{n+1} \ : \ \langle x, u_\theta \rangle < r_1 < 0, \ x_1 = -\delta\right\}$$

and

$$\mathcal{H}_2 = \left\{x \in \mathbb{R}^{n+1} \ : \ \langle x, u_\theta \rangle < r_2 < 0, \ x_1 = \delta\right\}.$$

This can be obtained as a consequence of Theorem 2.1, when one compares M with a suitable copy of a tilted grim reaper transverse to the hyperplane $\Pi(0)$ (as we did at the end of the proof of Lemma 4.1).

For the remaining cases, we proceed by contradiction. Suppose at first that

$$\mathcal{H}_1 = \left\{x \in \mathbb{R}^{n+1} \ : \ \langle x, u_\theta \rangle > r_1 > 0, \ x_1 = -\delta\right\}$$

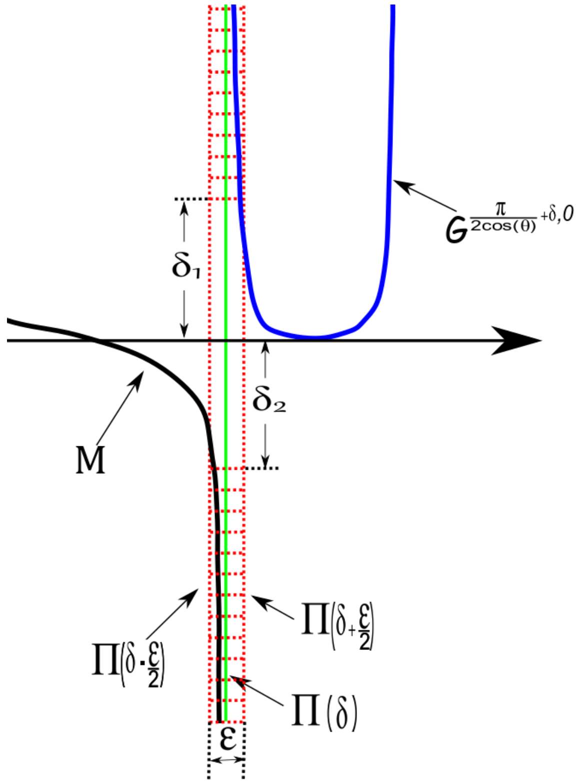
and

$$\mathcal{H}_2 = \left\{x \in \mathbb{R}^{n+1} \ : \ \langle x, u_\theta \rangle < r_2 < 0, \ x_1 = \delta\right\}$$

for some $r_1 > 0$ and $r_2 < 0$. Given t and l in $\mathbb{R}$, let $\mathcal{G}^{t,l}$ be the tilted grim reaper cylinder defined by

$$\mathcal{G}^{t,l} := \left\{F_\theta(x_1 - t, \hat{x}) + t\mathbf{e}_1 + l u_\theta \ : \ |x_1 - t| < \tfrac{\pi}{2\cos(\theta)}, \ \hat{x} \in \mathbb{R}^{n-1}\right\}, \tag{58}$$

Figure 12 – Transversal section of the M and $\mathcal{G}^{\frac{\pi}{2\cos(\theta)}+\delta,0}$.



where if we denote $\hat{x} = (x_2, \ldots, x_n)$, then $F_\theta(x_1 - t, \hat{x})$ means $F_\theta(x_1 - t, x_2, \ldots, x_n)$. Consider $\mathcal{G}^{\frac{\pi}{2\cos(\theta)}+\delta,0}$, which lie in $\left(\delta, \delta + \frac{\pi}{\cos(\theta)}\right) \times \mathbb{R}^n$ (see Figure 12). Note that it is asymptotic to the half-hyperplanes $\Pi(\delta)$ and $\Pi\left(\delta + \frac{\pi}{\cos(\theta)}\right)$. Fix $\epsilon \in (0, 2\delta)$. Using the fact that $\mathcal{G}^{\frac{\pi}{2\cos(\theta)}+\delta,0}$ is asymptotic to the half-hyperplanes outside the cylinder, then there exists $\delta_1 > r_1$, depending only on ϵ, such that[1]

$$\mathcal{G}^{\frac{\pi}{2\cos(\theta)}+\delta,0} \bigcap \mathcal{Z}^+_{\delta_1} \subset \left[\left(\delta, \delta + \tfrac{\epsilon}{2}\right) \times \mathbb{R}^n\right] \cap \{x \in \mathbb{R}^{n+1} \ : \ \langle x, u_\theta\rangle > \delta_1\}. \tag{59}$$

In turn, taking into account the asymptotic behaviour of M and our assumptions about the wings, there exists a $\delta_2 > -r_2$, depending only on ϵ, such that

$$M \bigcap \mathcal{Z}^+_{\delta_2} \subset \left[\left(\delta - \tfrac{\epsilon}{2}, \delta + \tfrac{\epsilon}{2}\right) \times \mathbb{R}^n\right] \cap \{x \in \mathbb{R}^{n+1} \ : \ \langle x, u_\theta\rangle < \delta_2\}. \tag{60}$$

From (59) and (60), there exists a $t > 0$ such that the tilted grim reaper cylinder $\mathcal{G}^{\frac{\pi}{2\cos(\theta)}+\delta+t,-\delta_1-\delta_2-1}$ satisfies

$$\mathcal{G}^{\frac{\pi}{2\cos(\theta)}+\delta+t,-\delta_1-\delta_2-1} \bigcap M = \varnothing$$

Now, since $\epsilon \in (0, 2\delta)$, there is a finite t_0 such that either M and $\mathcal{G}^{\frac{\pi}{2\cos(\theta)}+\delta+t_0,-\delta_1-\delta_2-1}$

[1] Here we are using the same notation of Lemma 4.2.

have a first point of contact or there is a sequence $\{p_i = (p_i^1, \ldots, p_i^{n+1})\}$ in M satisfying the next conditions:

i. $\{\langle p_i, u_\theta \rangle\}$ is a bounded sequence;

ii. $\{(0, p_i^2, \ldots, p_i^{n+1}) - \langle p_i, u_\theta \rangle u_\theta\}$ is an unbounded sequence;

iii.
$$\lim_i \left\{ \text{dist} \left(p_i, \mathcal{G}^{\frac{\pi}{2\cos(\theta)} + \delta + t_0, -\delta_1 - \delta_2 - 1} \right) \right\} = 0, \tag{61}$$

Notice that in this last case the sequence $\{p_i^1\}$ is bounded because to the asymptotic behaviour of M. Thus we can suppose $\{p_i^1\} \to p_\infty^1$ and $\{\langle p_i, u_\theta \rangle\} \to p_\infty^{u_\theta}$. In particular, from (61), we have

$$p_\infty^1 \mathbf{e}_1 + p_\infty^{u_\theta} u_\theta \in \mathcal{G}^{\frac{\pi}{2\cos(\theta)} + \delta + t_0, -\delta_1 - \delta_2 - 1}.$$

According to Theorem 2.1 and the asymptotic behaviour of M the first case cannot happen. Regarding the second case, let us define the sequence

$$\left\{ M_i := M - (0, p_i^2, \ldots, p_i^{n+1}) + \langle p_i, u_\theta \rangle u_\theta \right\}.$$

By Lemma 4.1, up to a subsequence, we have that $M_i \rightharpoonup M_\infty$, where M_∞ is a connected stationary integral varifold. By Proposition 2.2 one has

$$p_\infty^1 \mathbf{e}_1 + p_\infty^{u_\theta} u_\theta \in \text{spt}\, M_\infty \cap \mathcal{G}^{\frac{\pi}{2\cos(\theta)} + \delta + t, -\delta_1 - \delta_2 - 1}.$$

Thus, by Corollary 2.1 we get

$$\text{spt}\ M_\infty = \mathcal{G}^{\frac{\pi}{2\cos(\theta)} + \delta + t_0, -\delta_1 - \delta_2 - 1}.$$

But this is impossible by the asymptotic behaviour of M.

The case when

$$\mathcal{H}_1 = \left\{ x \in \mathbb{R}^{n+1} \ : \ \langle x, u_\theta \rangle < r_1 < 0, \ x_1 = -\delta \right\}$$

and

$$\mathcal{H}_2 = \left\{ x \in \mathbb{R}^{n+1} \ : \ \langle x, u_\theta \rangle > r_2 > 0, \ x_1 = \delta \right\}$$

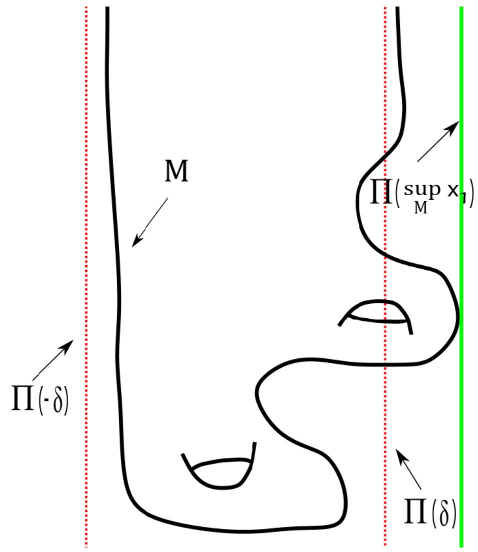
can be excluded using a symmetric argument. This concludes the proof. □

Next we would like to conclude now that M lies in the slab limited by the hyperplanes $\Pi(-\delta)$ and $\Pi(\delta)$.

Lemma 4.5. *M lies inside the slab $S := (-\delta, \delta) \times \mathbb{R}^n$.*

Proof. The proof will be done by contraction. Let us assume that $\lambda := \sup_M x_1 > \delta$. Thus, either M intersects $\Pi(\sup_M x_1)$ or $\text{dist}(M, \Pi(\sup_M x_1)) = 0$. Notice that the first case cannot be possible by Theorem 2.1. On the other hand, using the argument at the end

Figure 13 – Transversal section of the M and $\Pi(\sup_M x_1)$.



of Lemma 4.2 we see that the second case is impossible because the behaviour of M (see Figure 13). Thus, it must hold $\sup_M x_1 < \delta$. Analogously, we see that $-\delta < \inf_M x_1$. This completes the proof. □

Next we show that the distance between the two half-hyperplanes is exactly $\frac{\pi}{\cos(\theta)}$, like in the tilted grim reaper cylinder $\mathcal{G}^{0,0}$. The distance here is computed with respect to the Euclidean metric in $\mathbb{R}^{n+1}$.

Lemma 4.6. *We have* $2\delta = \frac{\pi}{\cos(\theta)}$.

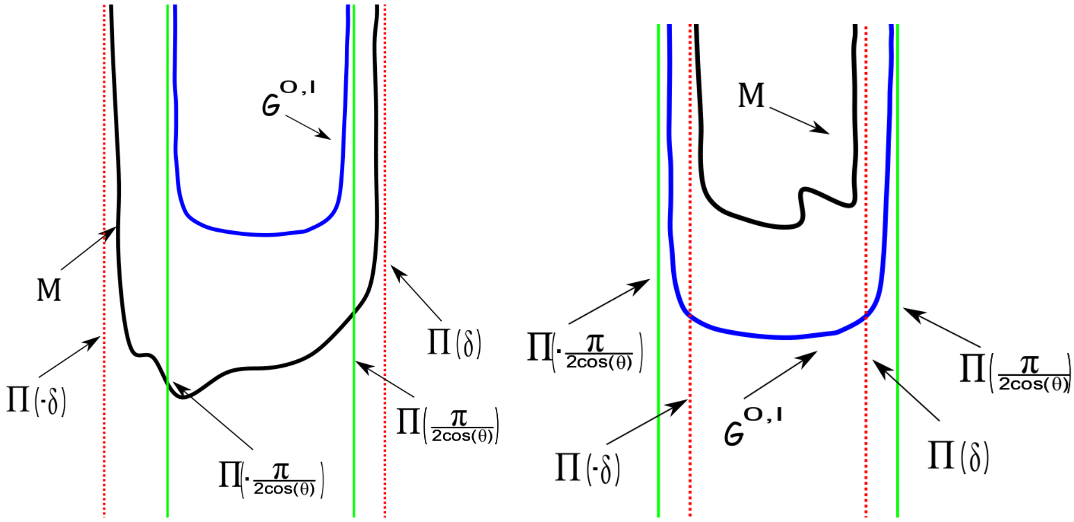
Proof. We proceed again by contradiction. Assume at first that $2\delta > \frac{\pi}{\cos(\theta)}$. By the asymptotic behaviour of M we can place a tilted grim reaper cylinder $\mathcal{G}^{0,l}$ inside S, for sufficiently large l, so that $\mathcal{G}^{0,l} \bigcap M = \varnothing$ (see Figure 14).

Next, consider $\mathcal{A} := \left\{ l \in \mathbb{R} \ : \ \mathcal{G}^{0,l} \bigcap M = \varnothing \right\}$ and let $l_0 = \inf \mathcal{A}$. Note that $l_0 > -\infty$ by the asymptotic behaviour of M. If $l_0 \notin \mathcal{A}$, then M and $\mathcal{G}^{0,l_0}$ have a point of contact. So $M = \mathcal{G}^{0,l_0}$ by Theorem 2.1, but this is impossible once $2\delta > \frac{\pi}{\cos(\theta)}$. In turn, if it holds $l_0 \in \mathcal{A}$ then $\operatorname{dist}\left(M, \mathcal{G}^{0,l_0}\right) = 0$. This means that there exists a sequence $\left\{p_i = (p_i^1, \ldots, p_i^{n+1})\right\}$ in M such that the sequences $\{p_i^1\}$ and $\{\langle p_i, u_\theta\rangle\}$ are bounded, the sequence $\left\{(0, p_i^2, \ldots, p_i^{n+1}) - \langle p_i, u_\theta\rangle u_\theta\right\}$ is unbounded and $\operatorname{dist}\left(p_i, \mathcal{G}^{0,l_0}\right) = 0$. Thus, after to a subsequence, one holds $p_i^1 \to p_\infty^1$ and $\langle p_i, u_\theta\rangle \to p_\infty^{u_\theta}$. At this time, we consider the sequence of hypersurfaces $\{M_i\}$, where

$$M_i := M - (0, p_i^2, \ldots, p_i^{n+1}) + \langle p_i, u_\theta\rangle u_\theta.$$

Using Lemma 4.1 we can suppose that $M_i \rightharpoonup M_\infty$, where M_∞ is a connected stationary integral varifold with $p_\infty^1 \mathbf{e}_1 + p_\infty^{u_\theta} u_\theta \in \operatorname{spt} M_\infty$, the last fact follows from Proposition 2.2. Hence $p_\infty^1 \mathbf{e}_1 + p_\infty^{u_\theta} u_\theta$ is a point of contact between $\operatorname{spt} M_\infty$ and $\mathcal{G}^{0,l_0}$. Thus again by Corollary 2.1 we get that $\mathcal{G}^{0,l_0} = \operatorname{spt}\ M_\infty$, which contradicts our assumptions about

Figure 14 – Transversal section of the behaviour of $\mathcal{G}^{0,l}$ with respect to M.



the behaviour of M. Consequently $2\delta \leq \frac{\pi}{\cos(\theta)}$. Comparing M with a tilted grim reaper cylinder "outside" M we conclude $2\delta = \frac{\pi}{\cos(\theta)}$. This finishes the proof. □

In the next Lemma we prove that the connected components of $M \setminus \mathcal{C}_\theta(r)$, that we will call from now on *the wings of* M, are vertical graphs. Here we come back to see M as a hypersurface in $\mathbb{R}^{n+1}$ endowed with the Euclidean metric.

Lemma 4.7. *If $t > 0$ is sufficiently large, then the two connected components of $M^+(t)$ are vertical graphs over an open subset of the hyperplane $[\mathbf{e}_{n+1}]^\perp$.*

Proof. Observe first that the C^1-asymptotic implies that if we take a sufficiently large t, then

$$M^+(t) \subset M_+\left(\frac{\pi}{2\cos(\theta)} - \tau\right) \cup M_-\left(-\frac{\pi}{2\cos(\theta)} + \tau\right),$$

for a small enough $\tau > 0$. Therefore, we only need to prove that if δ is small enough, then $M_+\left(\frac{\pi}{2\cos(\theta)} - \tau\right)$ is a graph over a subset of $[\mathbf{e}_{n+1}]^\perp$. The case of $M_-\left(-\frac{\pi}{2\cos(\theta)} + \tau\right)$ is treated in a similar way.

Fix a sufficiently small $\epsilon > 0$, with $\epsilon < \frac{1}{8}$. Since $\mathcal{G}\,(= \mathcal{G}^{0,0})$ and $M \setminus \mathcal{C}_\theta(r)$ are C^1-asymptotic to the same half-hyperplane contained in $\Pi\left(\frac{\pi}{2\cos(\theta)}\right)$ by hypothesis and Example 4.2 we can represent $M_+\left(\frac{\pi}{2\cos(\theta)} - \tau\right)$ as a graph over $\mathcal{G}$. Hence, we can find a smooth map

$$\varphi : T_\tau := \left(\frac{\pi}{2\cos(\theta)} - \tau, \frac{\pi}{2\cos(\theta)}\right) \times \mathbb{R}^{n-1} \to \mathbb{R}$$

such that

$$\sup_{T_\tau} |\varphi| < \epsilon \text{ and } \sup_{T_\tau} |D\varphi| < \epsilon, \tag{62}$$

and the map $\widetilde{F} : T_\tau \times \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n+1}$ given by

$$\widetilde{F} = F_\theta + \varphi\nu_\theta, \tag{63}$$

is a parametrization of $M_+\left(\frac{\pi}{2\cos(\theta)}-\tau\right)$, where F_θ is the parametrization given by (53) and

$$\nu_\theta(x_1,\ldots,x_n)=\sin(x_1\cos(\theta))\mathbf{e}_1-\cos(x_1\cos(\theta))u_\theta.$$

Next, we consider the projection $\Pi:\mathbb{R}^{n+1}\longrightarrow\mathbb{R}^n$ given by $\Pi(x_1\ldots,x_{n+1})=(x_1\ldots,x_n)$ and its restriction

$$\widetilde{\Pi}:=\Pi\Big|_{\mathrm{int}\left(M_+\left(\frac{\pi}{2\cos(\theta)}-\tau\right)\right)}:\mathrm{int}\left(M_+\left(\frac{\pi}{2\cos(\theta)}-\tau\right)\right)\longrightarrow T_\tau. \tag{64}$$

Note that the image of $\widetilde{\Pi}$ lies on T_τ, because for all $x\in\mathrm{int}\left(M_+\left(\frac{\pi}{2\cos(\theta)}-\tau\right)\right)$ we have

$$\frac{\pi}{2\cos(\theta)}-\tau<\langle x,\mathbf{e}_1\rangle<\frac{\pi}{2\cos(\theta)},$$

by the definition of $M_+\left(\frac{\pi}{2\cos(\theta)}-\tau\right)$. The idea here is to show that $\widetilde{\Pi}$ is a diffeomorphism. To deduce this, by a standard topological argument, we only must check that:

1. $\widetilde{\Pi}$ is a proper covering map;
2. $\mathrm{int}\left(M_+\left(\frac{\pi}{2\cos(\theta)}-\tau\right)\right)$ is path connected.

First, let us show that $\widetilde{\Pi}$ is a local diffeomorphism. Equivalently, let us show that $H>0$ on $M_+\left(\frac{\pi}{2\cos(\theta)}-\tau\right)$. The proof of this fact will follow from the next claim.

Claim 4.2. *The unit normal $N_{\widetilde{F}}$ along of $\widetilde{F}$ is given by the formulae*

$$D\cdot N_{\widetilde{F}}=AE_1(\theta)+B\textstyle\sum_{j=2}^{n-1}(-1)^{j\left[n-\frac{j+1}{2}\right]}\partial_{x_j}\varphi\mathbf{e}_j+C\partial_{x_n}\varphi E_n(\theta)+B\nu_\theta, \tag{65}$$

where

$$E_1(\theta):=\cos(x_1\cos(\theta))\mathbf{e}_1+\sin(x_1\cos(\theta))u_\theta, \tag{66}$$

$$A:=(-1)^{n-2}\left(\sin(\theta)\sin(x_1\cos(\theta))\partial_{x_n}\varphi-\cos(x_1\cos(\theta))\partial_{x_1}\varphi\right), \tag{67}$$

$$B:=1+\varphi\cos(\theta)\cos(x_1\cos(\theta)), \tag{68}$$

$$C:=(-1)^{n-1}\cos(\theta)\left(1+\varphi\cos(\theta)\cos(x_1\cos(\theta))\right) \tag{69}$$

and

$$D^2:=\left\{\begin{array}{rc} & [\sin(\theta)\sin(x_1\cos(\theta))\partial_{x_n}\varphi-\cos(x_1\cos(\theta))\partial_{x_1}\varphi]^2\\ + & [1+\varphi\cos(\theta)\cos(x_1\cos(\theta))]^2\left[1+\sum_{j=2}^n\left(\partial_{x_j}\varphi\right)^2\right]\\ + & \cos^2(\theta)\,[1+\varphi\cos(\theta)\cos(x_1\cos(\theta))]^2\left(\partial_{x_n}\varphi\right)^2\end{array}\right. \tag{70}$$

Proof of the Claim 4.2. Here we will use the following positive orthonormal basis for $\mathbb{R}^{n+1}$

$$\left\{\begin{array}{cc} E_1(\theta)=\cos(x_1\cos(\theta))\mathbf{e}_1+\sin(x_1\cos(\theta))u_\theta, & E_j:=\mathbf{e}_j,\quad j\in\{2,\ldots,n-1\}\\ E_n(\theta):=\cos(\theta)\mathbf{e}_n+\sin(\theta)\mathbf{e}_{n+1}, & \nu_\theta\end{array}\right\} \tag{71}$$

Before we figure out $N_{\widetilde{F}}$, let us observe the following equalities.

$$\partial_{x_1} F_\theta = \frac{E_1(\theta)}{\cos(x_1\cos(\theta))} + \tan(\theta)\tan(x_1\cos(\theta))E_n(\theta),\ \partial_{x_j} F_\theta = E_j \text{ for } j \in \{2,\dots,n-1\},$$

$$\partial_{x_n} F_\theta = \frac{1}{\cos(\theta)} E_n(\theta),\ \partial_{x_1}\nu_\theta = \cos(\theta)E_1(\theta),\ \text{and } \partial_{x_j}\nu_\theta = 0 \text{ for all } j \in \{2,\dots,n\}.$$

From these equalities follow

$$\begin{aligned}
\partial_{x_1}\widetilde{F} &= \partial_{x_1}F_\theta + \partial_{x_1}\varphi\nu_\theta + \varphi\partial_{x_1}\nu_\theta \\
&= \frac{E_1(\theta)}{\cos(x_1\cos(\theta))} + \tan(\theta)\tan(x_1\cos(\theta))E_n(\theta) + \partial_{x_1}\varphi\nu_\theta + \varphi\cos(\theta)E_1(\theta) \\
&= [1+\varphi\cos(\theta)\cos(x_1\cos(\theta))]\,\frac{E_1(\theta)}{\cos(x_1\cos(\theta))} + \tan(\theta)\tan(x_1\cos(\theta))E_n(\theta) + \partial_{x_1}\varphi\nu_\theta \\
&= \alpha E_1(\theta) + \beta E_n(\theta) + \partial_{x_1}\varphi\nu_\theta,
\end{aligned}$$

where

$$\alpha := \frac{1+\varphi\cos(\theta)\cos(x_1\cos(\theta))}{\cos(x_1\cos(\theta))}$$

and

$$\beta := \tan(\theta)\tan(x_1\cos(\theta))$$

$$\partial_{x_j}\widetilde{F} = \partial_{x_j}F_\theta + \partial_{x_j}\varphi\nu_\theta = E_j + \partial_{x_j}\varphi\nu_\theta \text{ for } j \in \{2,\dots,n-1\}$$

and

$$\partial_{x_n}\widetilde{F} = \partial_{x_n}F_\theta + \partial_{x_n}\varphi\nu_\theta = \frac{1}{\cos(\theta)}E_n(\theta) + \partial_{x_n}\varphi\nu_\theta.$$

These equalities together imply that $X := \partial_{x_1}\widetilde{F} \wedge \ldots \wedge \partial_{x_n}\widetilde{F}$ has the following expression

$$X = \begin{cases} \frac{(-1)^{n-2}}{\cos(\theta)}\left\{\beta\partial_{x_n}\varphi\cos(\theta) - \partial_{x_1}\varphi\right\}E_1(\theta) \\ +\frac{\alpha}{\cos(\theta)}\sum_{j=2}^{n-1}(-1)^{j\left[n-\frac{j+1}{2}\right]}\partial_{x_j}\varphi E_j \\ +\frac{\alpha}{\cos(\theta)}\nu_\theta + (-1)^{n-1}\alpha\partial_{x_n}\varphi E_n(\theta) \end{cases} \tag{72}$$

Indeed, straightforward calculation gives

$$
\begin{aligned}
X &= (\alpha E_1(\theta) + \beta E_n(\theta) + \partial_{x_1}\varphi\nu_\theta) \wedge \ldots \wedge (E_{n-1} + \partial_{x_{n-1}}\varphi\nu_\theta) \wedge \left(\frac{1}{\cos(\theta)}E_n(\theta) + \partial_{x_n}\varphi\nu_\theta\right) \\
&= \left\{ \begin{array}{l} \alpha E_1(\theta) \wedge (E_2 + \partial_{x_2}\varphi\nu_\theta) \wedge \ldots \wedge (E_{n-1} + \partial_{x_{n-1}}\varphi\nu_\theta) \wedge \left(\frac{1}{\cos(\theta)}E_n(\theta) + \partial_{x_n}\varphi\nu_\theta\right) \\ +\beta\partial_{x_n}\varphi E_n(\theta) \wedge E_2 \wedge \ldots \wedge E_{n-1} \wedge \nu_\theta + \frac{\partial_{x_1}\varphi}{\cos(\theta)}\nu_\theta \wedge E_2 \wedge \ldots \wedge E_{n-1} \wedge E_n(\theta) \end{array} \right. \\
&= \left\{ \begin{array}{l} \alpha E_1(\theta) \wedge (E_2 + \partial_{x_2}\varphi\nu_\theta) \wedge \ldots \wedge (E_{n-1} + \partial_{x_{n-1}}\varphi\nu_\theta) \wedge \left(\frac{1}{\cos(\theta)}E_n(\theta) + \partial_{x_n}\varphi\nu_\theta\right) \\ +\beta\partial_{x_n}\varphi(-1)^{n-2}E_2 \wedge \ldots \wedge E_n(\theta) \wedge \nu_\theta + \frac{\partial_{x_1}\varphi}{\cos(\theta)}(-1)^{n-1}E_2 \wedge \ldots \wedge E_n(\theta) \wedge \nu_\theta \end{array} \right. \\
&= \left\{ \begin{array}{l} \alpha E_1(\theta) \wedge (E_2 + \partial_{x_2}\varphi\nu_\theta) \wedge \ldots \wedge (E_{n-1} + \partial_{x_{n-1}}\varphi\nu_\theta) \wedge \left(\frac{1}{\cos(\theta)}E_n(\theta) + \partial_{x_n}\varphi\nu_\theta\right) \\ +(-1)^{n-2}\left\{\beta\partial_{x_n}\varphi - \frac{\partial_{x_1}\varphi}{\cos(\theta)}\right\} E_1(\theta) \end{array} \right. \\
&= \left\{ \begin{array}{l} (-1)^{n-2}\left\{\beta\partial_{x_n}\varphi - \frac{\partial_{x_1}\varphi}{\cos(\theta)}\right\} E_1(\theta) + \alpha E_1(\theta) \wedge \ldots \wedge E_{n-1} \wedge \left(\frac{E_n(\theta)}{\cos(\theta)} + \partial_{x_n}\varphi\nu_\theta\right) \\ +\frac{\alpha}{\cos(\theta)}\sum_{j=2}^{n-1}\partial_{x_j}\varphi E_1 \wedge \ldots \wedge E_{j-1} \wedge \nu_\theta \wedge E_{j+1} \wedge \ldots \wedge E_n \end{array} \right. \\
&= \left\{ \begin{array}{l} (-1)^{n-2}\left\{\beta\partial_{x_n}\varphi - \frac{\partial_{x_1}\varphi}{\cos(\theta)}\right\} E_1(\theta) + \frac{\alpha}{\cos(\theta)}E_1(\theta) \wedge E_2 \wedge \ldots \wedge E_{n-1} \wedge E_n(\theta) \\ +\alpha\partial_{x_n}\varphi E_1(\theta) \wedge E_2 \wedge \ldots \wedge E_{n-1} \wedge \nu_\theta \\ +\frac{\alpha}{\cos(\theta)}\sum_{j=2}^{n-1}(-1)^{j\left[n-\frac{j+1}{2}\right]}\partial_{x_j}\varphi E_{j+1} \wedge \ldots \wedge E_n(\theta) \wedge \nu_\theta \wedge E_1(\theta) \wedge \ldots \wedge E_{j-1} \end{array} \right. \\
&= \left\{ \begin{array}{l} \frac{(-1)^{n-2}}{\cos(\theta)}\left\{\beta\partial_{x_n}\varphi\cos(\theta) - \partial_{x_1}\varphi\right\} E_1(\theta) + \frac{\alpha}{\cos(\theta)}\sum_{j=2}^{n-1}(-1)^{j\left[n-\frac{j+1}{2}\right]}\partial_{x_j}\varphi E_j \\ +\frac{\alpha}{\cos(\theta)}\nu_\theta + (-1)^{n-1}\alpha\partial_{x_n}\varphi E_n(\theta) \end{array} \right. \qquad (73)
\end{aligned}
$$

Consequently, it holds

$$
\begin{aligned}
\langle X, X\rangle &= \frac{1}{\cos^2(\theta)}\left\{\beta\partial_{x_n}\varphi\cos(\theta) - \partial_{x_1}\varphi\right\}^2 + \frac{\alpha^2}{\cos^2(\theta)}\sum_{j=2}^{n-1}(\partial_{x_j}\varphi)^2 + \frac{\alpha^2}{\cos^2(\theta)} + \alpha^2(\partial_{x_n}\varphi)^2 \\
&= \frac{1}{\cos^2(\theta)}\left\{\left\{\beta\partial_{x_n}\varphi\cos(\theta) - \partial_{x_1}\varphi\right\}^2 + \alpha^2\left(\sum_{j=2}^{n-1}(\partial_{x_j}\varphi)^2 + 1\right) + \alpha^2(\partial_{x_n}\varphi)^2\cos^2(\theta)\right\} \\
&= \frac{D^2}{\cos^2(\theta)\cos^2(x_1\cos(\theta))}, \qquad (74)
\end{aligned}
$$

in the last equality we have used the definition of α and β. Using again the definition of β together with (72) and (74) we get the expression to $N_{\tilde{F}}$, and this finishes the proof of the claim. □

Now the equality (65) and (54) with $v = \mathbf{e}_{n+1}$ imply

$$\frac{D\cdot H}{\xi} = \left\{ \begin{array}{c} 1+\varphi\cos(\theta)\cos(x_1\cos(\theta)) + (-1)^n \sin(x_1\cos(\theta))\partial_{x_1}\varphi \\ +(-1)^n\sin(\theta)\cos(x_1\cos(\theta))\partial_{x_n}\varphi + (-1)^n\cos(\theta)\sin(\theta)\varphi\partial_{x_n}\varphi \end{array} \right. , \tag{75}$$

where $\xi := \cos\theta\cos(x_1\cos(\theta))$. Thus, by our assumptions about ϵ, φ and $D\varphi$ we see that $H(p) > 0$ at all $p \in M_+\left(\frac{\pi}{2\cos(\theta)} - \tau\right)$. Hence, $\widetilde{\Pi}$ is a local diffeomorphism.

The previous argument also implies that $\widetilde{\Pi}$ is onto as follows: Otherwise, it there would be a vertical cylinder which intersects T_τ but it would not intersect the set $M_+\left(\frac{\pi}{2\cos(\theta)} - \tau\right)$. Taking into account the asymptotic behaviour of M, we could translate horizontally this cylinder until having a first contact with

$$\operatorname{int}\left(M_+\left(\frac{\pi}{2\cos(\theta)} - \tau\right)\right).$$

At this first contact the normal vector field to M would be horizontal, which is absurd because we have proved that $H > 0$ on $M_+\left(\frac{\pi}{2\cos(\theta)} - \tau\right)$.

Finally, let us check that $\widetilde{\Pi}$ is proper. Let $K \subset T_\tau$ a compact set and $\{p_i\}_{i\in\mathbb{N}}$ be a sequence on $\widetilde{\Pi}^{-1}(K)$. Note that the sequence $\{p_i\}_{i\in\mathbb{N}}$ is bounded, because of the asymptotic behaviour of M and the fact that $\operatorname{dist}(K, \partial T_\tau) > 0$. So, up to a subsequence, we can assume that $p_i \to p_\infty$. Since the set $\widetilde{\Pi}^{-1}(K)$ is closed, it follows that $p_\infty \in \widetilde{\Pi}^{-1}(K)$. This proves that $\widetilde{\Pi}^{-1}(K)$ is compact.

At this point, we have that any connected component of $\operatorname{int}\left(M_+\left(\frac{\pi}{2\cos(\theta)} - \tau\right)\right)$ is a graph over T_τ. But only one of them contains the wing. This means that if there were another connected component, Σ, then the function $x \mapsto \langle x, u_\theta\rangle$ would be bounded on Σ and $\partial\Sigma \subset \Pi\left(\frac{\pi}{2\cos(\theta)} - \tau\right)$, which is impossible by Lemma 4.3. Repeating the same argument we should obtain that $M_-\left(-\frac{\pi}{2\cos(\theta)} + \tau\right)$ is smooth *vertical graph* over a subset of the hyperplane $[\mathbf{e}_{n+1}]^\perp$. □

Now we are going to show that is possible to place a tilted grim reaper cylinder below M. This means that M lies in the convex region limited by the tilted grim reaper cylinder. Henceforth, up to a translation, we will assume that $\inf_M\langle x, u_\theta\rangle = 0$.

Lemma 4.8. *There is a tilted grim reaper cylinder that contains M "inside" it, i.e., M lies in the convex region of the complement of a tilted grim reaper cylinder.*

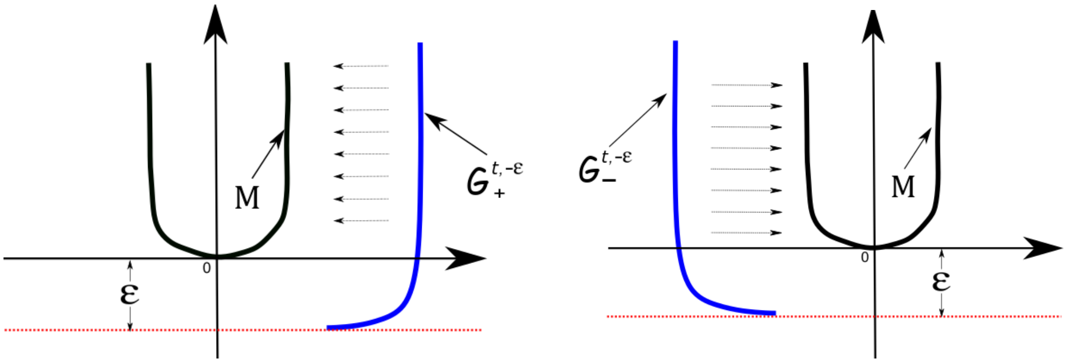
Proof. Consider the family of "half"-tilted grim reaper cylinders

$$\mathcal{G}_\pm^{t,-\epsilon} := \left\{x \in \mathcal{G}^{0,-\epsilon} \ : \ \pm\langle x, \mathbf{e}_1\rangle \geq 0\right\} \pm t\mathbf{e}_1 \tag{76}$$

where $\epsilon > 0$ is fixed and $t \in [0, \infty)$.

Let us work with the "half"-tilted grim reaper cylinder $\mathcal{G}_+^{t,-\epsilon}$. By taking a

Figure 15 – Transversal section of the behaviour of $\mathcal{G}_+^{t,-\epsilon}$ and $\mathcal{G}_-^{t,-\epsilon}$ with respect to M.



sufficiently large t_0, we obtain $\mathcal{G}_+^{t_0,-\epsilon} \cap M = \varnothing$. Hence the set $\mathcal{A}$ defined by

$$\mathcal{A} := \{t \in [0,\infty) \ : \ \mathcal{G}_+^{t,-\epsilon} \cap M = \varnothing\}$$

is not empty. Take $s_0 = \inf \mathcal{A}$. We claim that $s_0 = 0$. Otherwise, we have two possibilities for $s_0 > 0$: either $s_0 \in \mathcal{A}$ or $s_0 \notin \mathcal{A}$. If $s_0 \notin \mathcal{A}$ then $\mathcal{G}_+^{t,-\epsilon} \cap M \neq \varnothing$ and since $\partial\mathcal{G}_+^{t,-\epsilon} \cap M = \varnothing$, we conclude that $\mathcal{G}_+^{t,-\epsilon} \subset M$, by Theorem 2.1, but this is absurd because

$$0 = \inf_M \langle x, u_\theta\rangle > \inf_{\mathcal{G}^{t,-\epsilon}} \langle x, u_\theta\rangle = -\epsilon.$$

In turn, if $s_0 \in \mathcal{A}$ then $\operatorname{dist}\left(\mathcal{G}_+^{t,-\epsilon}, M\right) = 0$. This tells us that there exists a sequence $\left\{p_i = (p_i^1, \ldots, p_i^{n+1})\right\}$ in M such that:

1. $\lim_i \operatorname{dist}\left(p_i, \mathcal{G}_+^{t,-\epsilon}\right) = 0$;
2. $\{p_i^1\} \to p_\infty^1$ and $a < p_i^1 - t < b$, where $0 < a < b < \frac{\pi}{2\cos(\theta)}$ are constants;
3. $\{\langle p_i, u_\theta\rangle\} \to p_\infty^{u_\theta}$;
4. The sequence $\left\{(0, p_i^2, \ldots, p_i^{n+1}) - \langle p_i, u_\theta\rangle u_\theta\right\}$ is unbounded.

In this case, we consider again the sequence of hypersurfaces

$$\left\{M_i = M - (0, p_i^2, \ldots, p_i^{n+1}) + \langle p_i, u_\theta\rangle u_\theta\right\}.$$

Naturally, by Lemma 4.1 we may admit $M_i \rightharpoonup M_\infty$, where M_∞ is a connected stationary integral varifold. Notice that we would have $p_\infty^1 \mathbf{e}_1 + p_\infty^\theta u_\theta \in \mathcal{G}_+^{t,-\epsilon} \cap \operatorname{spt} M_\infty$, so Corollary 2.1 implies $\mathcal{G}_+^{t,-\epsilon} = M_\infty$. But this again contradicts the asymptotic behaviour of M. Therefore $\inf \mathcal{A} = 0$, and

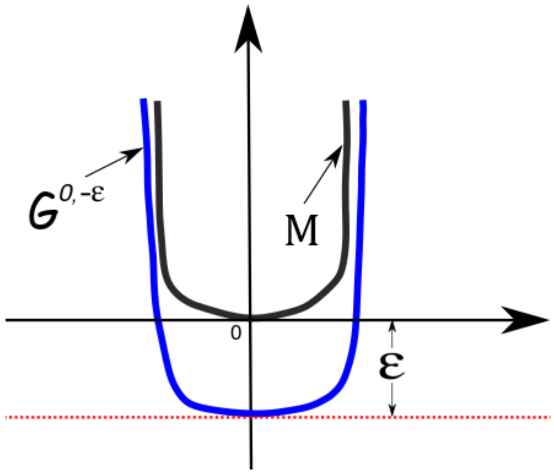
$$\mathcal{G}_+^{0,-\epsilon} \cap M = \varnothing,$$

for all $\epsilon > 0$. A similar argument allows us to conclude that $\mathcal{G}_-^{0,-\epsilon} \cap M = \varnothing$. Thus $\mathcal{G}^{0,-\epsilon} \cap M = \varnothing$.

This completes the proof. □

As an application of the previous two lemmas, we shall conclude now that the hypersurface M is itself in vertical a graph over a slap on the hyperplane $[\mathbf{e}_{n+1}]^\perp$.

Figure 16 – Transversal section of the behaviour of $\mathcal{G}^0$ with respect to M.



Lemma 4.9. *M is a vertical graph over* $\left(-\frac{\pi}{2\cos\theta}, \frac{\pi}{2\cos\theta}\right) \times \mathbb{R}^{n-1}$.

Proof. For each $i \in \mathbb{N}$ consider the sets

$$T_i := \left\{v \in \mathbb{R}^{n+1} \ : \langle v, E_n(\theta)\rangle \geq i\right\},$$

where $E_n(\theta) := \cos(\theta)\mathbf{e}_n + \sin(\theta)\mathbf{e}_{n+1}$, and call $\alpha := \lim_i \inf_{T_i \cap M} \langle x, u_\theta\rangle$. Consider a sequence $\left\{p_i = \left(p_i^1, \ldots, p_i^{n+1}\right)\right\}$ in M such that:

i. $p_i \in T_i \cap M$ and $\langle p_i, u_\theta\rangle - \inf_{T_i \cap M}\langle x, u_\theta\rangle < \frac{1}{i}$

ii. $\{p_i^1\} \to p_\infty^1$ and $-\frac{\pi}{2\cos(\theta)} < p_\infty^1 < \frac{\pi}{2\cos(\theta)}$;

iii. $\{\langle p_i, u_\theta\rangle\} \to \alpha$.

Consider the sequence of hypersurfaces

$$\left\{M_i = M - (0, p_i^2, \ldots, p_i^{n+1}) + \langle p_i, u_\theta\rangle u_\theta\right\}.$$

We know from Lemma 4.1 that, up to a subsequence, $M_i \rightharpoonup M_\infty$, where M_∞ is a connected stationary integral varifold. Since $p_\infty^1 \mathbf{e}_1 + \alpha u_\theta \in \operatorname{spt} M_\infty$, it follows that $\inf_{\operatorname{spt} M_\infty} \langle x, u_\theta\rangle \leq \alpha$.

We would like to conclude that $\alpha = \inf_{\operatorname{spt} M_\infty} \langle x, u_\theta\rangle$. Indeed, take any $p \in \mathbb{R}^{n+1}$ such that $\langle p, u_\theta\rangle < \alpha$. Let $B_r(p)$ be the open ball in $\mathbb{R}^{n+1}$, where $r \in \left(0, \frac{\alpha - \langle p, u_\theta\rangle}{4}\right)$. Note that $B_r(p) \cap \Pi_\alpha = \varnothing$, where $\Pi_\alpha = [u_\theta]^\perp + \alpha u_\theta$. Take any $\epsilon \in \left(0, \frac{\alpha - \langle p, u_\theta\rangle)}{4}\right)$. By the definition of α, there is an i_0 such that if $i > i_0$ then

$$\inf_{T_i \cap M} \langle x, u_\theta\rangle > \alpha - \epsilon > 0. \tag{77}$$

Notice that (77) implies that if $i > i_0$ one has

$$\mu_{M_i}(B_r(p)) = \int_{B_r(p)} d\mu_{M_i} = 0,$$

where μ_{M_i} denotes the weight measure associated to the varifold M_i in $\mathbb{R}^{n+1}$ with the Ilmanen's metric. This implies that $\mu_{M_\infty}(B_r(p)) = 0$, and so $p \notin \operatorname{spt} M_\infty$. Consequently $\alpha = \inf_{\operatorname{spt} M_\infty} \langle x, u_\theta \rangle$. As we are going to see, this equality implies that spt M_∞ is the element of the family of the grim reaper cylinder associated to θ with u_θ height α.

Claim 4.3. *M_∞ coincides with the tilted grim reaper cylinder associated to θ with u_θ height α into the direction of u_θ.*

Proof of the Claim 4.3. The proof follows by using a similar idea as in Lemma 4.8, (see Figure 15). Consider the "half"-tilted grim reaper cylinder

$$\mathcal{G}_+^{t,\alpha-\epsilon} := \left\{ x \in \mathcal{G}^{0,\alpha-\epsilon} \ : \ \langle x, \mathbf{e}_1 \rangle \geq 0 \right\} + t\mathbf{e}_1 \tag{78}$$

where $\epsilon > 0$ and $t \in [0, \infty)$. Naturally, we can take a sufficiently large t_0 in such that way that

$$\mathcal{G}_+^{t_0,\alpha-\epsilon} \cap \operatorname{spt} M_\infty = \varnothing,$$

by Lemma 4.8. Next, we consider the set

$$\mathcal{A} = \{ t \in [0, +\infty) \ : \ \mathcal{G}_+^{t,\alpha-\epsilon} \cap \operatorname{spt} M_\infty = \varnothing \},$$

which is non-empty. We would like to show that $\inf \mathcal{A} = 0$. Indeed, otherwise, $s_0 = \inf \mathcal{A} > 0$ satisfy one of the following conditions:

a. $\mathcal{G}_+^{s_0,\alpha-\epsilon}$ and spt M_∞ have a point of contact;

b. $\operatorname{dist}\left(\mathcal{G}_+^{s_0,\alpha-\epsilon}, \operatorname{spt} M_\infty\right) = 0$.

According to Corollary 2.1 and Lemma 4.8, the first case is not possible. Regarding the second case, by Lemma 4.8 there exists a sequence $\left\{ z_i = (z_i^1, \ldots, z_i^{n+1}) \right\}$ in spt M_∞ such that:

i. $\lim_i \operatorname{dist}\left(z_i, \mathcal{G}_+^{s_0,\alpha-\epsilon}\right) = 0$;

ii. $\{z_i^1\} \to z_\infty^1$ and $a < z_i^1 - t < b$ where $0 < a < b < \frac{\pi}{2\cos(\theta)}$ are constants;

iii. $\{\langle z_i, u_\theta \rangle\} \to z_\infty^{u_\theta}$;

iv. The sequence $\left\{ (0, z_i^2, \ldots, z_i^{n+1}) - \langle z_i, u_\theta \rangle u_\theta \right\}$ is unbounded;

v. $\Theta(\operatorname{spt} M_\infty, z_i) \geq 1$.

Here we are using that on spt M_∞ we have $\Theta(M_\infty, p) \geq 1$ at all $p \in \operatorname{spt} M_\infty$. At this point, let us consider the sequence

$$\left\{ \mathfrak{M}_i := M_\infty - (0, z_i^2, \ldots, z_i^{n+1}) + \langle z_i, u_\theta \rangle u_\theta \right\}.$$

Claim 4.4. *$\{\mathfrak{M}_i\}$ has locally bounded area.*

Proof of the Claim 4.4. Firstly observe that each $\mathfrak{M}_i$ is the limit weakly* of the sequence $\{M_j - (0, z_i^2, \ldots, z_i^{n+1}) + \langle z_i, u_\theta \rangle u_\theta\}$ as $j \to +\infty$. Secondly, we know that outside $\mathcal{C}_\theta(r)$

the following estimative holds

$$\mathcal{A}_g[(M_j - (0, z_i^2, \ldots, z_i^{n+1}) + \langle z_i, u_\theta \rangle u_\theta) \cap B_r(p)] \le \frac{1}{2}\mathcal{A}_g[\partial B_\epsilon(p)] \tag{79}$$

for all j and i by Proposition 4.2, where $p \in \mathbb{R}^{n+1} \setminus \mathcal{C}_\theta(r)$ and ϵ is taken small enough so that $B_\epsilon(p)$ does not intersect $\mathcal{C}_\theta(r)$. Thus, taking the limit at (79) as $j \to +\infty$ one obtains

$$\mu_{\mathfrak{M}_i}[B_r(p)] \le \frac{1}{2}\mathcal{A}_g[\partial B_\epsilon(p)] \text{ for all } j.$$

Consequently, the are blow-up set $\mathcal{B}$ of $\{\mathfrak{M}_i\}$ lies inside $\mathcal{C}_\theta(r)$. Proceeding as in Lemma 4.1 we shall conclude that $\mathcal{B} = \varnothing$. □

By the previous claim, observing that each $\mathfrak{M}_i$ satisfies the conditions of Theorem 2.9 outside $\mathcal{C}_\theta(r)$ and inside $\mathcal{C}_\theta(r)$ it is just an n-dimensional stationary integral varifold. So we can apply Theorem 2.9 and Theorem 2.8 together to conclude that, after to passing to a subsequence, $\mathfrak{M}_i \rightharpoonup \mathfrak{M}_\infty$, where $\mathfrak{M}_\infty$ is a connected stationary integral varifold. This last fact can be obtained by arguing as in the proof of Lemma 4.1. Notice that Proposition 2.2 implies $p_\infty \in \operatorname{spt} M_\infty$, consequently it holds

$$z_\infty^1 \mathbf{e}_1 + z_\infty^{u_\theta} u_\theta \in \operatorname{spt} \mathfrak{M}_\infty \cap \ \mathcal{G}_+^{s_0, \alpha-\epsilon}.$$

Moreover, note that the item ii implies that $z_\infty^1 \mathbf{e}_1 + z_\infty^{u_\theta} u_\theta$ is an interior point of $\mathcal{G}_+^{s_0,\alpha-\epsilon}$. Therefore, by Corollary 2.1 and Lemma 4.8 we arrive at a contraction since $\inf_{\operatorname{spt} M_\infty} \langle x, u_\theta \rangle = \alpha$. Thus, $\inf \mathcal{A} = 0$ and

$$\mathcal{G}_+^{0,\alpha-\epsilon} \cap \operatorname{spt} M_\infty = \varnothing,$$

because $\epsilon > 0$ and $\inf_{\operatorname{spt} M_\infty} \langle x, u_\theta \rangle = \alpha$. Similarly, we deduce that

$$\mathcal{G}_-^{0,\alpha-\epsilon} \cap \operatorname{spt} M_\infty = \varnothing.$$

Hence $\mathcal{G}^{0,\alpha-\epsilon} \cap \operatorname{spt} M_\infty = \varnothing$.

Now, letting $\epsilon \to 0^+$ and using the fact that

$$\inf_{\operatorname{spt} M_\infty} \langle x, u_\theta \rangle = \min_{\operatorname{spt} M_\infty} \langle x, u_\theta \rangle = \alpha$$

we conclude that $\operatorname{spt} M_\infty$ touches the tilted grim reaper cylinder $\mathcal{G}^{0,\alpha}$ at $p_\infty^1 \mathbf{e}_1 + \alpha u_\theta$. In particular, by Corollary 2.1 we conclude that $\operatorname{spt}\ M_\infty = \mathcal{G}^{0,\alpha}$. This concludes the proof of our claim. Notice that since M_i converges weakly* to $\operatorname{spt}\ M_\infty = \mathcal{G}^{0,\alpha}$ then M_i converges as set to $\mathcal{G}^{0,\alpha}$ and the multiplicity is one outside $\mathcal{C}_\theta(r)$ because $\operatorname{sing} M_\infty = \varnothing$ and each M_i is a horizontal graph outside $\mathcal{C}_\theta(r)$. Thus, all conditions of Theorem 2.12 are satisfied, consequently one has $M_i \to \mathcal{G}^{0,\alpha}$, with multiplicity one everywhere. □

In turn, consider the sets

$$S_i := \{v \in \mathbb{R}^{n+1} \ : \langle v, E_n(\theta)\rangle \leq -i\},$$

where $i \in \mathbb{N}$, and take $\beta = \lim_i \inf_{S_i \cap M} \langle x, u_\theta \rangle$. Let $\{q_i = (q_i^1, \ldots, q_i^{n+1})\}$ be a sequence in M such that:

i. $q_i \in S_i \cap M$ and $\langle q_i, u_\theta\rangle - \inf_{S_i\cap M}\langle x, u_\theta\rangle < \frac{1}{i}$
ii. $\{q_i^1\} \to q_\infty^1$ and $-\frac{\pi}{2\cos(\theta)} < q_\infty^1 < \frac{\pi}{2\cos(\theta)}$;
iii. $\{\langle q_i, u_\theta\rangle\} \to \beta$.

Then, reasoning as before, we obtain

$$N_i := M - (0, q_i^2, \ldots, q_i^{n+1}) + \langle q_i, u_\theta\rangle u_\theta \longrightarrow \mathcal{G}^{0,\beta},$$

with multiplicity one everywhere.

By Lemma 4.7, we know that there exists a sufficiently large t_0, so that $M^+(t_0)$ is a graph over an open set in the hyperplane $[\mathbf{e}_{n+1}]^\perp$. Moreover, we can choose at the same time a small enough $\tau > 0$ so that

$$M_+\left(\frac{\pi}{2\cos(\theta)} - 2\tau\right) \cup M_+\left(-\frac{\pi}{2\cos(\theta)} + 2\tau\right) \subset M^+(t_0).$$

Hence, these together with $M_i \to \mathcal{G}^{0,\alpha}$ and $N_i \to \mathcal{G}^{0,\beta}$ imply that there is $i_0 \in \mathbb{N}$ such that:

a. There exist strictly increasing sequences of positive numbers $\{m_i^1\}$, $\{m_i^2\}$, $\{n_i^1\}$ and $\{n_i^2\}$ so that
$$m_i^1 < m_i^2 \ \text{ and } \ -n_i^1 < -n_i^2, \quad \text{for all } i > i_0;;$$

b. There exist smooth functions:

$$\varphi_i : \left(-\frac{\pi}{2\cos(\theta)} + \tau, \frac{\pi}{2\cos(\theta)} - \tau\right) \times \left(m_i^1, m_i^2\right)^{n-1} \longrightarrow \mathbb{R} \tag{80}$$

and

$$\phi_i : \left(-\frac{\pi}{2\cos(\theta)} + \tau, \frac{\pi}{2\cos(\theta)} - \tau\right) \times \left(-n_i^1, -n_i^2\right)^{n-1} \longrightarrow \mathbb{R} \tag{81}$$

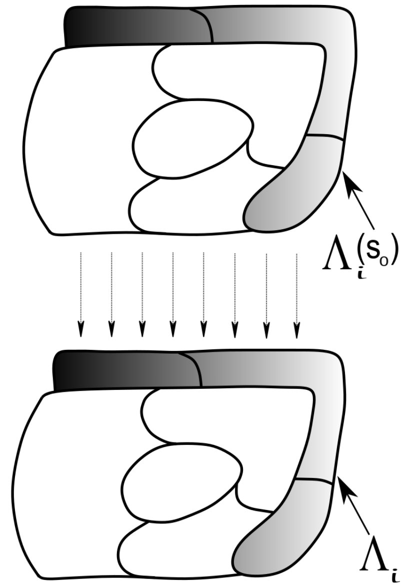
satisfying

$$|\varphi_i| < \tfrac{1}{i} \ , \ |D\varphi_i| < \tfrac{1}{i} \ , \ |\phi_i| < \tfrac{1}{i} \text{ and } |D\phi_i| < \tfrac{1}{i} \text{ for all } i > i_0 \tag{82}$$

and such that the hypersurfaces

$$R_i := \left\{ \begin{array}{ccc} x = (x_1, \ldots, x_{n+1}) \in M & : & -\frac{\pi}{2\cos(\theta)} + \tau < x_1 < \frac{\pi}{2\cos(\theta)} - \tau \\ (x_2, \ldots, x_{n-1}) \in (m_i^1, m_i^2)^{n-2} & , & \langle x, E_n(\theta)\rangle \in (m_i^1, m_i^2) \end{array} \right\}$$

Figure 17 – Picture of Λ_i and $\Lambda_i(s_0)$.



and

$$L_i := \left\{ \begin{array}{ccc} x = (x_1, \ldots, x_{n+1}) \in M & : & -\frac{\pi}{2\cos(\theta)} + \tau < x_1 < \frac{\pi}{2\cos(\theta)} - \tau \\ (x_2, \ldots, x_{n-1}) \in (-n_i^1, -n_i^2)^{n-2} & , & \langle x, E_n(\theta) \rangle \in (-n_i^1, -n_i^2) \end{array} \right\}$$

can be written as graphs of functions φ_i and ϕ_i, respectively, over the corresponding pieces of the tilted grim reaper cylinder as in the proof of Lemma 4.7, where $E_n(\theta) := \cos(\theta)\mathbf{e}_n + \sin(\theta)\mathbf{e}_{n+1}$.

Now following the same idea as in Lemma 4.7, we see that R_i and L_i are smooth vertical graphs over domains in the hyperplane $[\mathbf{e}_{n+1}]^\perp$ (for i_0 large enough). Note that R_i and L_i are connected because they are graphs over the connected sets and the convergences $M_i \to \mathcal{G}^{0,\alpha}$ and $N_i \to \mathcal{G}^{0,\beta}$ have multiplicity one. Finally, let us consider the exhaustion $\{\Lambda_i\}$ of M by compact sets given by

$$\Lambda_i := \left\{ \begin{array}{ccc} x = (x_1, \ldots, x_{n+1}) \in M & : & (x_2, \ldots, x_{n-1}) \in [-a_i, b_i]^{n-2} \\ \langle x, E_n(\theta) \rangle \in [-a_i, b_i] & , & \langle x, u_\theta \rangle \leq i\} \end{array} \right\} \tag{83}$$

where $a_i = \frac{n_i^1 + n_i^2}{2}$ and $b_i = \frac{m_i^1 + m_i^2}{2}$.

Since $M^+(t_0)$, R_i and L_i are vertical graphs, then a small strip B_i around the boundary of Λ_i is a graph over the hyperplane $[\mathbf{e}_{n+1}]^\perp$. Now we would like to use a Rado's argument to conclude that in fact each Λ_i must be a vertical graph over a subset of the hyperplane $[\mathbf{e}_{n+1}]^\perp$ if $i > i_0$. Indeed, assume to the contrary that this is not true. Consider the family

$$\{\Lambda_i(s) := \Lambda_i + s\mathbf{e}_{n+1}\}_{s \in \mathbb{R}}$$

of translations of Λ_i into the direction of $\mathbf{e}_{n+1}$. Since Λ_i is compact there exists a sufficiently large s_0 so that

$$\Lambda_i(s_0) \cap \Lambda_i = \varnothing.$$

Now move $\Lambda_i(s_0)$ back into the direction of $-\mathbf{e}_{n+1}$(see Figure 17). Since Λ_i is not a graph and $B_i \cap \{B_i + s\mathbf{e}_{n+1}\} = \varnothing$, because B_i is a graph over a subset of $[\mathbf{e}_{n+1}]^\perp$. Then there exists a $s_1 \in (0, s_0)$ such that $\Lambda_i(s_1)$ has a point of contact at interior with Λ_i. Therefore $\Lambda_i(s_1) = \Lambda_i$, but this gives us to a contraction. Hence, each Λ_i must be a graph only continuous. However, since $\bigcup_i \Lambda_i = M$, then M is also a vertical graph. Notice that this argument only allows to conclude that M is a continuous *vertical graph* and it is a smooth *vertical graph* at its wings. In particular, taking a subtle orientation on M we see that the mean curvature H is positive along of the wings of M. Now, as M is a continuous *vertical graph* we can orient M in such that way that $H = \langle N, \mathbf{e}_{n+1}\rangle \geq 0$ on M, but by Proposition 2.3 H satisfies

$$\Delta H + \langle \nabla H, \nabla x_{n+1}\rangle = -|A|^2 H \leq 0.$$

Consequently, the maximum principle implies $H > 0$,i. e. M is a smooth *vertical graph.* To finalize the proof, notice that the argument of the Lemma 4.7 allows us to conclude that the restriction of projection over $[e_{n+1}]\perp$ on M is onto over $\left(-\frac{\pi}{2\cos\theta}, \frac{\pi}{2\cos\theta}\right) \times \mathbb{R}^{n-1}$. □

Since the mean curvature $H > 0$ on M, then given any $v \in \mathbb{R}^{n+1}$ if $N_v = \langle N, v\rangle$, then $h_v = \frac{N_v}{H}$ are well defined on whole M, where N unit normal along of M and H is the mean curvature of M. At that moment, we will consider the stand Euclidean metric in $\mathbb{R}^{n+1}$.

Lemma 4.10. *The function* $h_v = \frac{N_v}{H}$ *satisfies the following linear* PDE *in* M

$$\Delta h_v + \langle \nabla h_v, \nabla(x_{n+1} + 2\log H)\rangle = 0. \tag{84}$$

Proof. To deduce this, notice firstly that since v and $\mathbf{e}_{n+1}$ are Killing fields in $\mathbb{R}^{n+1}$, then by Proposition 2.3 we have

$$\Delta N_v + \langle \nabla N_v, \nabla x_{n+1}\rangle + |A|^2 N_v = 0 \text{ and } \Delta H + \langle \nabla H, \nabla x_{n+1}\rangle + |A|^2 H = 0.$$

These equality together with $\nabla h_v = \dfrac{1}{H}\nabla N_v - \dfrac{N_v}{H^2}\nabla H$ imply

$$\begin{aligned}
\Delta h_v &= \Delta\left(\frac{N_v}{H}\right) = \operatorname{div}(\nabla h_v) = \operatorname{div}\left(\frac{\nabla N_v}{H} - \frac{N_v}{H^2}\nabla H\right)\\
&= \operatorname{div}\left(\frac{\nabla N_v}{H}\right) - \operatorname{div}\left(\frac{N_v}{H^2}\nabla H\right)\\
&= \frac{\Delta N_v}{H} - \left\langle \nabla N_v, \frac{\nabla H}{H^2}\right\rangle - \frac{N_v}{H^2}\Delta H - \left\langle \nabla\left(\frac{N_v}{H^2}\right), \nabla H\right\rangle\\
&= \frac{\Delta N_v}{H} - 2\left\langle \nabla N_v, \frac{\nabla H}{H^2}\right\rangle - \frac{N_v}{H^2}\Delta H + 2\frac{N_v}{H^3}\left\langle \nabla H, \nabla H\right\rangle\\
&= -\frac{\langle \nabla N_v, \nabla x_{n+1}\rangle}{H} - 2\left\langle \nabla N_v, \frac{\nabla H}{H^2}\right\rangle + \frac{N_v}{H^2}\langle \nabla H, \nabla x_{n+1}\rangle + 2\frac{N_v}{H^3}\left\langle \nabla H, \nabla H\right\rangle\\
&= -\left\langle \frac{\nabla N_v}{H} - \frac{N_v}{H^2}\nabla H, \nabla x_{n+1}\right\rangle - 2\left\langle \frac{\nabla N_v}{H} - \frac{N_v}{H^2}\nabla H, \frac{\nabla H}{H}\right\rangle\\
&= -\langle \nabla h_v, \nabla x_{n+1}\rangle - 2\langle \nabla h_v, \nabla \log H\rangle = -\langle \nabla h_v, \nabla(x_{n+1} + 2\log H)\rangle .
\end{aligned}$$

This completes the proof. □

Before we prove that for some choose of the vector v the function h_v go to zero at the end of M, we need to prove M is in fact C^2-asymptotic to two half-hyperplanes with respect to the Euclidean metric. This is the statement of the next result.

Lemma 4.11. *The hypersurface M is C^2-asymptotic outside the cylinder to two half-hyperplanes with respect to the Euclidean metric.*

Proof. To prove this lemma, we will need of the following fact.

Claim 4.5. *There exist a tilted grim reaper cylinder inside the region that lie "above" M. This means that M lies in the region concave in $\mathbb{R}^{n+1}$ limited by that tilted grim reaper.*

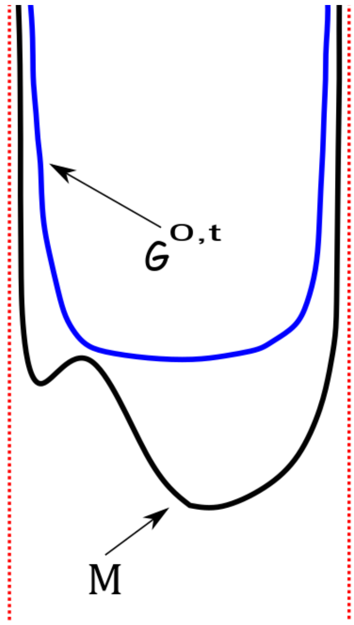
Proof of the Claim 4.5. Indeed, by our hypothesis over M we know that if t is sufficiently large then $M^+(t)$ is graph over the hyperplane $\Pi(0)$, so we fix such t. Next we consider the tilted grim reaper $\mathcal{G}^{0,t}$. We will show that it lies in the region above M (see Figure 18). Following the idea from of the Lemma 4.8, let us consider the family of "half"-M

$$\{M_*(s) := M_+(0) + s\mathbf{e}_1\}_{s\in[0,+\infty)} .$$

Taking into account the asymptotic behaviour of M, there exists a sufficiently large $s_0 > 0$ so that $M_*(s_0) \cap \mathcal{G}^{0,t} = \varnothing$. Arguing as in Lemma 4.8 and using the fact that $M^+(t)$ is graph over $\Pi(0)$, we shall conclude that $\inf \mathcal{A} = 0$, where

$$\mathcal{A} := \left\{ s \in [0, +\infty) \ : \ M_*(s) \cap \mathcal{G}^{0,t} = \varnothing \right\} .$$

Figure 18 – Transversal section of the behaviour of M and $\mathcal{G}^{0,t}$.



Hence, one holds $M_+(0) \cap \mathcal{G}^{0,t} = \varnothing$. In turn, the same argument applying to the family

$$\{M^*(s) := M_-(0) - s\mathbf{e}_1\}_{s\in[0,+\infty)},$$

proves $M_-(0) \cap \mathcal{G}^{0,t} = \varnothing$. Therefore $M \cap \mathcal{G}^{0,t} = \varnothing$ and this proves the claim. □

In order of proof that M is in fact C^2-asymptotic to the half-hyperplanes $\Pi\left(-\frac{\pi}{2\cos(\theta)}\right)$ and $\Pi\left(\frac{\pi}{2\cos(\theta)}\right)$ in the sense of Definition 4.1, let us work with the wing of M which is C^1-close to the half-hyperplane $\mathcal{H}_1$ of $\Pi\left(\frac{\pi}{2\cos(\theta)}\right)$. As we know, given $\epsilon > 0$, there exists $\delta > 0$ so that M can be represent a graph of φ defined over $\mathcal{H}_1$, with $\sup_{\mathcal{H}_1(\delta)} |\varphi| < \epsilon$ and $\sup_{\mathcal{H}_1(\delta)} |D\varphi| < \epsilon$, where D indicates the Euclidean derivative. Arguing by contraction, that is, let us suppose that there exist $\epsilon > 0$ and a sequence $\{p_i\}$ in M such that:

$$|D^2\varphi(p_i)| \geq \epsilon \text{ and } \langle p_i, u_\theta\rangle \to \infty. \tag{85}$$

Consider the sequence $\{M_i := M - p_i\}$. Fix some $s > 0$ small enough so that the intersection of the geodesic $B_s(0)$ with M_i has only one connected component, and we denote it by $S_i = B_s(0) \cap M_i$. Thus, $\{S_i\}$ is a sequence of stable hypersurfaces in $B_s(0)$ with locally bounded area, by Proposition 4.1 and Proposition 4.2, so by Theorem 2.9 we may suppose $S_i \rightharpoonup S_\infty$, where S_∞ is an n-dimensional stationary integral varifold in $B_s(0)$ so that $0 \in \operatorname{spt} S_\infty$, by Proposition 2.2. Now using the fact that M lies in the concave region limited by $\mathcal{G}^{0,t}$ we conclude that $\operatorname{spt} S_\infty \subset \Pi(0) \cap B_s(0)$, but as $S_i \to \operatorname{spt} S_\infty$ with multiplicity one, since each S_i is a *horizontal graph*, then by Theorem 2.12 we conclude $S_i \to \Pi(0) \cap B_s(0) = \operatorname{spt} S_\infty$ with multiplicity one everywhere, with respect to the Ilmanen metric. Notice that we can represent S_i are the graph of a function $\varphi_i(\cdot) = \varphi(\cdot + p_i) - \varphi(p_i)$

which is defined on an open subset of $\Pi(0)$ that contain the origin.

Next, consider a small geodesic cylinder $W_{r,\epsilon}(p)$ over $D_r(p) \subset \mathrm{T}\Pi(0)$, with respect to Ilmanen's metric. By definition of convergence in the $C^\infty-$topology, there exist sufficiently large $i_0 \in \mathbb{N}$ so that for all $i > i_0$ the set $W_{r,\epsilon}(p) \cap M_i$ is a graph of a function η_i defined over $D_r(p) \subset \Pi(0)$ such that $\sup_{D_r(p)\cap\Pi(0)} |D^l \eta_i| < \epsilon/8$, for all $l \in \mathbb{N}$. Notice that the hyperplanes parallel to $\mathbf{e}_{n+1}$ are totally geodesic and $\mathbf{e}_1$ is the unit normal vector to $\mathrm{T}_0\Pi(0)$ and we have the following relation between φ_i and η_i:

$$\varphi_i(\exp_0(q + \eta_i(q)\mathbf{e}_1) - \langle \exp_0(q + \eta_i(q)\mathbf{e}_1), \mathbf{e}_1\rangle \mathbf{e}_1) = \langle \exp_0(q + \eta_i(q)\mathbf{e}_1), \mathbf{e}_1\rangle,$$

where $\exp_0$ denotes the exponential map of $\mathbb{R}^{n+1}$ at 0 with respect to the metric g. Thus differentiating twice this expression with respect to a geodesic frame at 0 and evaluating at $q = 0$, we deduce that

$$\langle D^2 \exp_0(\overline{u}_i, \overline{w}_i), \mathbf{e}_1\rangle + D^2\eta_i(u, w) = \left\{ \begin{array}{c} d\varphi_i[D^2 \exp_0(\overline{u}_i, \overline{w}_i) - \langle D^2 \exp_0(\overline{u}_i, \overline{w}_i), \mathbf{e}_1\rangle \mathbf{e}_1] \\ +D^2\varphi_i(u, w) \end{array} \right\}$$

where $\overline{u}_i := u + d\eta_i(u)\mathbf{e}_1$ and $u \in \mathrm{T}_0\Pi(0)$. From this expression, the control on the C^∞ norm of η_i, the C^1 norm of φ and using that $\Pi(0)$ is totally geodesic we get a contraction with $|D^2\varphi_i(0)| = |D^2\varphi(p_i)| \geq \epsilon$, if i is sufficiently large. This proves the lemma. □

Next, let us set $h_j := \frac{\langle N, \mathbf{e}_j\rangle}{H}$, where $j \in \{2, \dots, n-1\}$ and $h_n = \frac{\langle N, E_n(\theta)\rangle}{H}$. Using the previous lemma, we can obtain some information about the behaviour of the functions h_j at the ends of M.

Lemma 4.12. *The functions h_j, $j \in \{2, \dots, n\}$, tend to zero as we approach the end of M.*

Proof. The proof we will be done as follows: consider the exhaustion $\{\Lambda_i\}$ given by (83). Notice that the boundary of each Λ_i consists of the following $2n - 1$ regions

$$\Lambda_i^1 := \left\{ \begin{array}{ccc} x = (x_1, \dots, x_{n+1}) \in M & : & (x_2, \dots, x_{n-1}) \in [-a_i, b_i]^{n-2} \\ \langle x, E_n(\theta)\rangle \in [-a_i, b_i] & , & \langle x, u_\theta\rangle = i \end{array} \right\}$$

$$\Lambda_i^{-2} := \left\{ \begin{array}{ccc} x = (x_1, \dots, x_{n+1}) \in M & : & \langle x, u_\theta\rangle \leq i \ , \ x_2 = -a_i \\ (x_3, \dots, x_{n-1}) \in [-a_i, b_i]^{n-3} & , & \langle x, E_n(\theta)\rangle \in [-a_i, b_i] \end{array} \right\}$$

$$\Lambda_i^{2} := \left\{ \begin{array}{ccc} x = (x_1, \dots, x_{n+1}) \in M & : & \langle x, u_\theta\rangle \leq i \ , \ x_2 = b_i \\ (x_3, \dots, x_{n-1}) \in [-a_i, b_i]^{n-3} & , & \langle x, E_n(\theta)\rangle \in [-a_i, b_i] \end{array} \right\}$$

$$\vdots$$

$$\Lambda_i^{-n} := \left\{ \begin{array}{ccc} x = (x_1, \dots, x_{n+1}) \in M & : & \langle x, u_\theta\rangle \leq i \\ (x_2, \dots, x_{n-1}) \in [-a_i, b_i]^{n-2} & , & \langle x, E_n(\theta)\rangle = -a_i \end{array} \right\}$$

and

$$\Lambda_i^n := \left\{ \begin{array}{ccc} x = (x_1, \dots, x_{n+1}) \in M & : & \langle x, u_\theta \rangle \leq i \\ (x_2, \dots, x_{n-1}) \in [-a_i, b_i]^{n-2} & , & \langle x, E_n(\theta) \rangle = b_i \end{array} \right\}$$

Next we would like to study the behaviour of h_j at a small strip around the boundary of Λ_i. Let us begin our study at the connected component Λ_i^1. Consider any sufficiently small $\epsilon > 0$. Taking into account Lemma 4.11 and the definition of $M^+(t)$, we can use a similar argument as at the proof of Lemma 4.7 to guarantee the existence of a sufficiently large $i_1(> i_0)$, a sufficiently small $\tau > 0$ and a smooth function φ defined on the strip

$$\mathcal{S}_\tau := \left[\left(-\tfrac{\pi}{2\cos(\theta)}, -\tfrac{\pi}{2\cos(\theta)} + \tau \right) \cup \left(\tfrac{\pi}{2\cos(\theta)} - \tau, \tfrac{\pi}{2\cos(\theta)} \right) \right] \times \mathbb{R}^{n-1}$$

satisfying

$$\sup_{\mathcal{S}_\tau} |\varphi| < \epsilon, \quad \sup_{\mathcal{S}_\tau} |D\varphi| < \epsilon \;\text{ and }\; \sup_{\mathcal{S}_\tau} |D^2\varphi| < \epsilon \tag{86}$$

and such that $M^+(i_1)$ is a graph of this function over the corresponding strip in the tilted grim reaper cylinder. From (65) and (75) we obtain

$$h_j = \frac{\alpha(j)}{\cos(\theta)} \frac{\partial_{x_j}\varphi}{\cos(x_1\cos(\theta))} \frac{1 + \varphi\cos(\theta)\cos(x_1\cos(\theta))}{1 + \sigma(\varphi, D\varphi)}, \tag{87}$$

where $\alpha(j) = (-1)^{j\left[n - \frac{j+1}{2}\right]}$, if $j \in \{2, \dots, n-1\}$ and $\alpha(n) = (-1)^{n-1}\cos(\theta)$. Here

$$\sigma(\varphi, D\varphi) := \left\{ \begin{array}{cc} (-1)^n & [\sin(\theta)(1 + \varphi\cos(\theta))\partial_{x_n}\varphi + \sin(x_1\cos(\theta))\partial_{x_1}\varphi] \\ + & \varphi\cos(\theta)\cos(x_1\cos(\theta)) \end{array} \right\}.$$

In turn, using the fact that $M^+(i_1)$ is a graph over the tilted grim reaper cylinder and it is C^2−asymptotic to the half-hyperplane, we conclude that for all fixed $(x_2, \dots, x_n)$ we have

$$\lim_{x_1 \to \frac{\pi}{2\cos(\theta)}^-} |\varphi| = \lim_{x_1 \to \frac{\pi}{2\cos(\theta)}^-} |D\varphi| = 0.$$

Therefore

$$|\partial_{x_j}\varphi(x_1, x_2, \dots, x_n)| = \left| -\int_{x_1}^{\frac{\pi}{2\cos(\theta)}} \partial_{x_j x_1}\varphi(x, x_2, \dots, x_n)dx \right| \leq \left(\tfrac{\pi}{2\cos(\theta)} - x_1 \right)\epsilon. \tag{88}$$

So, by (86), (87) and (88) we obtain that $|h_j(x)| < o(\epsilon)$, for all x near Λ_i^1, here $o(\epsilon)$ denotes a term that goes to zero as $\epsilon \to 0$. Thus

$$\sup_{N(\Lambda_i^1)} |h_j| < o(\epsilon) \tag{89}$$

where $N(\Lambda_i^1)$ is a small neighbourhood the Λ_i^1 in Λ_i, if $i > i_1$.

Now we are going to work with the components of $\partial\Lambda_i$ that intersect $M^-(i_1)$. Since R_i and L_i are C^1-close to a strip in the tilted grim reaper cylinder, there is a sufficiently large i_2 such that $R_i \cap \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}; \langle x, u_\theta\rangle \leq i_1\}$ is a graph over the strip in the tilted grim reaper cylinder of a function φ_i defined in the strip

$$G_\tau := \left(-\frac{\pi}{\cos(\theta)2} + \frac{\tau}{2}, \frac{\pi}{2\cos(\theta)} - \frac{\tau}{2}\right) \times (m_i^1, m_i^2)^{n-1}$$

satisfying the following properties

$$\sup_{G_\tau} |\varphi_i| < \epsilon \ \text{ and} \sup_{G_\tau} |D\varphi_i| < \epsilon. \tag{90}$$

The same estimate is true for L_i. Furthermore, since $\cos(x_1\cos(\theta)) > \kappa > 0$ in G_τ, for a suitable constant κ, then (90) and (87) gives us that $\sup_{G_\tau} |h_j| < o(\epsilon)$. Hence

$$\sup_{N\left(\Lambda_i^{\pm k}\right)} |h_j| < o(\epsilon), \tag{91}$$

where $k \in \{2, \ldots, n\}$ and $N\left(\Lambda_i^{\pm k}\right)$ is a small neighbourhood of the $\Lambda_i^{\pm k}$ in Λ_i. Hence for (89) and (91) we have $\sup_{N(\partial\Lambda_i)} |h_j| < o(\epsilon)$, for any $i \in \mathbb{N}$, $i > \max\{i_1, i_2\}$. □

This lemma is the last ingredient that we need to prove Theorem 4.1. Here we come back to adopt the Euclidean metric in $\mathbb{R}^{n+1}$.

Proof of Theorem 4.1. Recall that we are assuming that M is asymptotic to two half-hyperplanes that are contained in different hyperplanes and that $\inf_M(\langle x, u_\theta\rangle) = 0$. According to Lemma 4.12 there is an interior point where h_j has an extremum. Then, because h_j is a solution of (84), we can apply Hopf's maximum principle to conclude that $h_j = 0$, that is, $\xi_j = 0$ on M for all $j \in \{2, \ldots, n\}$. In particular, each $\mathbf{e}_j$ and $E_n(\theta)$ are tangent vectors of M for $j \in \{2, \ldots, n-1\}$ at all point of M. Thus, we can consider a global orthonormal basis on M, $\{E_1,\ E_j = \mathbf{e}_j;\ j \in \{2, \ldots, n-1, \}; E_n(\theta)\}$, where $E_1 = E_2 \wedge \ldots \wedge E_n \wedge N$. Differentiating each N_j, $j \in \{2, \ldots, n\}$, with respect to E_k, $k \in \{1, \ldots, n\}$ one deduces

$$0 = E_k(N_j) = E_k\langle N, E_j\rangle = \langle \nabla_{E_k} N, E_j\rangle = -A(E_k, E_j).$$

Hence,

$$|A|^2 = \sum_{i,j} A(E_j, E_k)^2 = A(E_1, E_1)^2 = H^2.$$

Therefore, by Theorem B in (MARTÍN, SAVAS-HALILAJ, and SMOCZYK, 2015), we conclude that $M = \mathcal{G}^{0,0}$, because we are assuming that $0 = \inf_M \langle x, u_\theta\rangle$. □

4.3.1.1 *Topological consequences*

In this little part we prove that the number of half-hyperplanes at Theorem 4.1 cannot be odd. So before proceeding, we need to do a little modification at definition 4.1.

Definition 4.2. *We say that a smooth hypersurface $M \subset \mathbb{R}^{n+1}$ is C^k-asymptotic outside a cylinder to k half-hyperplanes $\mathcal{H}_1,\ \mathcal{H}_2, \ldots,\ \mathcal{H}_k$ if there exists a solid cylinder $\mathcal{C}$ such that:*

i. The boundary of the solid cylinder $\mathcal{C}$ contains the boundaries of the half-hyperplanes $\mathcal{H}_i$ for all i,

ii. $M \setminus \mathcal{C}$ consists of k connected components $M_1, \ldots, M_k$ which are C^k-asymptotic to $\mathcal{H}_1, \ldots, \mathcal{H}_k$, respectively.

Define this we will prove the non-existence theorem. Here no conditions are required over the cylinder.

Proposition 4.3. *There not exist a complete, connected, properly embedded translating soliton in $\mathbb{R}^{n+1}$ which is C^1-asymptotic to k half-hyperplanes outside a solid cylinder $\mathcal{C}$, if k is odd.*

Proof. The proof follows from the topological result that says that a properly embedded hypersurface in $\mathbb{R}^{n+1}$ must intersect any transversal loop at an even number counting with multiplicity. So if such M existed, then we should be able to construct a curve α around the cylinder $\mathcal{C}$ which is transversal to M, and it intersects M at an odd number of points. □

4.3.2 Case $\theta = \pi/2$

Now we are going to work in the case when the cylinder is vertical, i.e. the axis of the cylinder is parallel to the translating velocity $\mathbf{e}_{n+1}$. So first of all, let us point out the following version of Lemma 4.1 in this setting.

Lemma 4.13. *Let $M^n \subset \mathbb{R}^{n+1}$ be a complete, connected, properly embedded translating soliton and $\mathcal{C}_{\pi/2}(r) := \{x \in \mathbb{R}^{n+1} \ :\ \langle x, \mathbf{e}_1\rangle^2 + \langle x, \mathbf{e}_n\rangle^2 \leq r^2\}$, for $r > 0$. Assume that M is C^1-asymptotic to two half-hyperplanes outside $\mathcal{C}_{\pi/2}(r)$. Suppose that $\{b_i\}_{i\in\mathbb{N}}$ is a sequence in $[\mathbf{e}_1, \mathbf{e}_n]^{\perp}$ and let $\{M_i\}_{i\in\mathbb{N}}$ be a sequence of hypersurfaces given by $M_i := M + b_i$. Then there exist a connected stationary integral varifold M_∞ and a subsequence $\{M_{i_k}\} \subset \{M_i\}$ so that*

(i) $M_{i_k} \overset{*}{\rightharpoonup} M_\infty$ *in* $\mathbb{R}^{n+1}$;

(ii) $\text{sing } M_\infty$ *satisfies* $\mathcal{H}^{n-7+\beta}(\text{sing } M_\infty \cap (\mathbb{R}^{n+1} \setminus \mathcal{C}(r))) = 0$ *for all* $\beta > 0$ *if* $n \geq 7$, $\text{sing } M_\infty \cap (\mathbb{R}^{n+1} \setminus \mathcal{C}(r))$ *is discrete if* $n = 7$ *and* $\text{sing } M_\infty \cap (\mathbb{R}^{n+1} \setminus \mathcal{C}(r)) = \varnothing$ *if* $1 \leq n \leq 6$;

(iii) $M_{i_k} \to \text{spt}\, M_\infty$ *in* $\mathbb{R}^{n+1} \setminus (\mathcal{C}(r) \cup \text{sing } M_\infty)$.

Proof. The proof works exactly as in the case $\theta < \pi/2$ at Lemma 4.1, except the proof that the sequence $\{M_i\}$ has locally bounded area. More precisely, when we would like to conclude that the area blow-up set associated to the sequence $\{M_i\}$ is empty. In order to prove that this fact, we use as barriers the family $P_\lambda = W^2_\lambda \times \mathbb{R}^{n-2}$ (cylinders over the translating catenoid of dimension 2), for a sufficiently large $\lambda > 0$ so that the cylinder lies inside the neck of $P_\lambda = W^2_\lambda \times \mathbb{R}^{n-2}$. Hence, if the set of area blow-up is not empty, then we could move $P_\lambda = W^2_\lambda \times \mathbb{R}^{n-2}$ until we get a first finite contact point with the area blow-up set, which is impossible by Theorem 2.11. The remaining conditions may be obtained by arguing as in the proof of Lemma 4.1. □

Remark 4.3. *Let Σ be a translating soliton in $\mathbb{R}^3$. The cylinder over Σ denoted by $\Sigma \times \mathbb{R}^{n-2}$ is defined as follows: if $X : \Sigma \times \mathbb{R}^3$ is an immersion of Σ in $\mathbb{R}^3$, where $X = (X_1, X_2, X_3)$, then $\tilde{X}(p, x_1, \ldots, x_{n-2}) = (X_1(p), X_2(p), x_1, \ldots, x_{n-2}, X_3(p))$ is an immersion of $\Sigma \times \mathbb{R}^{n-2}$. Moreover, a simple computation proves that $\tilde{X}(\Sigma \times \mathbb{R}^{n-2})$ is a translating soliton in $\mathbb{R}^{n+1}$ with respect to $\mathbf{e}_{n+1}$. Such translating soliton is called the cylinder over Σ.*

Once we have proven this version of compactness lemma, we can prove the main result of this part.

Theorem 4.2. *Let $M^n \subset \mathbb{R}^{n+1}$ be a complete, connected, properly embedded translating soliton and $\mathcal{C}_{\pi/2}(r) = \{x \in \mathbb{R}^{n+1} : \langle x, \mathbf{e}_1\rangle^2 + \langle x, \mathbf{e}_n\rangle^2 \leq r^2\}$, for $r > 0$. Assume that M is C^1-asymptotic to two half-hyperplanes outside $\mathcal{C}_{\pi/2}(r)$. Then M must coincide with a hyperplane parallel to $\mathbf{e}_{n+1}$.*

Proof. We start by proving that $\mathcal{H}_1$ and $\mathcal{H}_2$ are parallel.

Claim 4.6. *The half-hyperplanes $\mathcal{H}_1$ and $\mathcal{H}_2$ are parallel.*

Proof of the Claim 4.6. Otherwise, we could take a hyperplane parallel to $\mathbf{e}_{n+1}$, Γ, such that it does not intersect M and such that the normal vector v to Γ is not perpendicular to w_1 and w_2. Translating Γ by $t_0 \in \mathbb{R}$ in the direction of v until we get a hyperplane $\Gamma_{t_0} := \Gamma + t_0 v$ in such that way either Γ_{t_0} and M have a first point of contact or $\operatorname{dist}(\Gamma_{t_0}, M) = 0$ and $\Gamma_{t_0} \cap M = \varnothing$. The first case is not possible by Theorem 2.1. Regarding the second case, if we argue as in Lemma 4.2, we shall see that this case is also impossible.

Notice that we cannot have either $\mathcal{H}_1 \subset \mathcal{H}_2$ or $\mathcal{H}_2 \subset \mathcal{H}_1$, because in these cases we could take a hyperplane parallel to $\mathbf{e}_{n+1}$, Υ, whose normal is exactly w_1 and do not intersect M. Now we could move Υ into direction of w_1 until there exists $t_0 > 0$ such that either $\Upsilon + t_0 w_1$ and M have a first point of contact or $\{\Upsilon + t_0 w_1\} \cap M = \varnothing$ and $\operatorname{dist}(\Upsilon + t_0 w_1, M) = 0$. Reasoning as in the above paragraph, we can conclude that both situations are impossible. □

Denote by Π_1 and Π_2 the hyperplanes that contain the half-hyperplanes $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, notice that the previous claim implies that Π_1 and Π_2 are parallel.

Moreover, if they are different, then the proof of Lemma 4.5 implies that M lies in the slab between Π_1 and Π_2, moreover M does touch Π_j, unless $M = \Pi_j$. Notice that if $\Pi_1 = \Pi_2$, then using the strategy at the end of Lemma 4.2 we shall conclude that M coincides with $\Pi_1 = \Pi_2$. So we only need to prove that $\Pi_1 = \Pi_2$. Suppose that this is not the case.

Claim 4.7. *For all $s \geq r$ fixed we have* $\mathrm{dist}(M \cap \mathcal{C}(s)\Pi_i) > 0$.

Proof of the Claim 4.7. Otherwise, we could find a sequence $\{p_i = (p_1^i, \ldots, p_{n+1}^i)\}$ in $M \cap \mathcal{C}(s)$ so that $\mathrm{dist}(p_i, \Pi_i) = 0$, so considering the sequence of hypersurfaces $\{M_i := M - (0, p_2^i, \ldots, p_{n-2}^i, 0, p_{n+1}^i)\}$ by Lemma 4.13 we would have that $M_i \stackrel{*}{\rightharpoonup} M_\infty$, after passing to a subsequence, where M_∞ is a connected n-dimensional stationary integral varifold. Using that $\{p_i\}$ lies in $\mathcal{C}(s)$ we may also suppose $\langle p_i, \mathbf{e}_1 \rangle \to a_1$ and $\langle p_i, \mathbf{e}_n \rangle \to a_n$. Now $(a_1, 0, \ldots, 0, a_n, 0) \in \mathrm{spt}\ M_\infty \cap \Pi_1$ by Proposition 2.2. So by Corollary 2.1 we would have $\mathrm{spt}\, M_\infty = \Pi_i$, which is impossible because that $\Pi_1 \neq \Pi_2$ and part of $\mathrm{spt}\, M_\infty$ is close to Π_1 and Π_2. □

We know that $M \setminus \mathcal{C}(r) = \mathrm{Graph}[u_1] \cup \mathrm{Graph}[u_2]$, where $u_i : \mathcal{H}_i \to \mathbb{R}$ and it holds

$$\sup_{\mathcal{H}_i(\delta)} |u_i| < \epsilon \text{ and } \sup_{\mathcal{H}_i(\delta)} |Du_i| < \epsilon,$$

where δ depends on ϵ and $\delta \to +\infty$ as $\epsilon \to 0$. Fix some $s > r$ and define

$$\epsilon = \frac{1}{10} \min_i \{\mathrm{dist}\{M \cap \mathcal{C}(s), \Pi_i\}\} > 0.$$

New take $\delta > 0$ so that

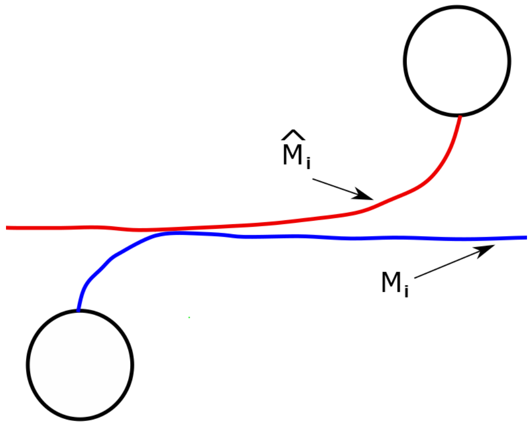
$$\sup_{\mathcal{H}_i(\delta)} |u_i| < \epsilon \text{ and } \sup_{\mathcal{H}_i(\delta)} |Du_i| < \epsilon.$$

With these choices, we will attain at a contradiction with $\Pi_1 \neq \Pi_2$ as follows: let ν be the unit normal vector to Π_1 which point outside to the slab limited by Π_1 and Π_2. Next call $s_0 = \mathrm{dist}(\Pi_1, \Pi_2) > 0$, and notice that for this choice of s_0 we have that $M + s_0\nu$ does not intersect the slab limited by Π_1 and Π_2, but the wing of $M + s_0\nu$ correspondents to $\mathcal{H}_2 + s_0\nu$ asymptotic a half-hyperplane in Π_1 with unit inward normal to its boundary is $-w_1$, where w_1 denotes the upward unit normal to $\partial\mathcal{H}_1$. Define $M_\epsilon := \{x \in M \colon \min\{\mathrm{dist}(x, \Pi_1), \mathrm{dist}(x, \Pi_2)\} \geq \epsilon\}$. By what we have seen above it holds $M \cap \mathcal{C}(s) \subset M_\epsilon$. Now consider a sufficiently large $t_0 > 0$ so that $M_\epsilon + s_0\nu + t_0 w_1$ lies in $\mathcal{Z}^+_{1,2\delta}$ (see Lemma 4.2).

Define the set

$$\mathcal{A} := \{s \in [0, s_0] \colon (M + s\nu + t_0 w_1) \cap M = \varnothing\}.$$

Figure 19 – Transversal section of the behaviour of M_i(blue) and $\widehat{M_i}$(red).



Let $s_1 := \inf \mathcal{A} > 0$, since before we arrive at 0 we must have

$$(M + s\nu + t_0 w_1) \cap M \neq \varnothing,$$

because our supposition about s_0 and ϵ. We have two possibilities for s_1: either $s_1 \notin \mathcal{A}$ or $s_1 \in \mathcal{A}$. The first case implies that $M + s_1\nu + t_0 w_1$ and M have points of contact, which is impossible by the maximum principle and our hypothesis over M. Consequently, it holds $s_1 \in \mathcal{A}$. Turn out that this implies

$$\operatorname{dist}(M + s_1\nu + t_0 w_1, M) = 0$$

and $\{M + s_1\nu + t_0 w_1\} \cap M = \varnothing$. This fact together our choice of ϵ imply that there exist sequences $\{p_i\}$ in $M \backslash \mathcal{C}(s)$ and $\{q_i\}$ in $(M \backslash \mathcal{C}(s)) + s_1\nu + t_0 w_1$ such that $\operatorname{dist}(p_i, \mathcal{C}(s) \cap M) > 2\epsilon$, $\operatorname{dist}(q_i, \mathcal{C}(s) \cap M) > 2\epsilon$, $\operatorname{dist}(p_i, \mathcal{C}(s) \cap M + s_1\nu + t_0 w_1) > 2\epsilon$, $\operatorname{dist}(q_i, \mathcal{C}(s) \cap M + s_1\nu + t_0 w_1) > 2\epsilon$ and $\operatorname{dist}(p_i, q_i) = 0$. Observe that we can assume that $\{\langle q_i, \mathbf{e}_1 \rangle\}, \{\langle p_i, \mathbf{e}_1 \rangle\} \to a$ and $\{\langle q_i, \mathbf{e}_n \rangle\}, \{\langle p_i, \mathbf{e}_n \rangle\} \to b$.

In $\mathbb{R}^{n+1} \setminus (\mathcal{C}(s) \cup \mathcal{C}(s) + s_1\nu + t_0 w_1)$ consider the following sequences

$$\{M_i := (M^1 \setminus \mathcal{C}(s)) - (0, p_2, \ldots, p_{n-1}, 0, p_{n+1})\}$$

and

$$\{\widehat{M_i} := (M^2 \setminus \mathcal{C}(s)) + s_1\nu + t_0 w_1 - (0, q_2, \ldots, q_{n-1}, 0, q_{n+1})\},$$

where M_i indicates the wing of M which is asymptotic to $\mathcal{H}_i$ (see Figure 19). In particular M_i and $\widehat{M_i}$ are stable hypersurface and $\{M_i\}$ and $\{\widehat{M_i}\}$ have locally bounded area, by Proposition 4.1 and Proposition 4.2

Turn out that Theorem 2.9 and Proposition 2.2 imply, up to a subsequence, that $M_i \rightharpoonup M_\infty$ and $\widehat{M_i} \rightharpoonup \widehat{M}_\infty$, where M_∞ and $\widehat{M}_\infty$ are connected stable integral varifolds, and $(a, 0, \ldots, 0, b, 0) \in \operatorname{spt}\ M_\infty \cap \operatorname{spt} \widehat{M}_\infty$. Here the connectedness can be deduced by arguing as in Lemma 4.1. On the other hand, Theorem 2.14 implies that reg M_∞ and reg $\widehat{M}_\infty$ are connected subset of $\mathbb{R}^{n+1} \setminus (\mathcal{C}(s) \cup \mathcal{C}(s) + s_1\nu + t_0 w_1)$. Consequently, the asymptotic behaviour of spt M_∞ and spt $\widehat{M}_\infty$ imply that reg M_∞ does not intersect reg $\widehat{M}_\infty$. Thus it holds $\operatorname{spt} M_\infty \cap \operatorname{spt} \widehat{M}_\infty \subset \operatorname{sing} M_\infty \cup \operatorname{sing} \widehat{M}_\infty$. In particular, $\mathcal{H}^{n-1}(\operatorname{spt} M_\infty \cap \operatorname{spt} \widehat{M}_\infty) = 0$, so by Theorem 2.15 we have $\operatorname{spt} M_\infty \cap \operatorname{spt} \widehat{M}_\infty = \varnothing$, which is impossible since $(a, 0, \ldots, 0, b, 0) \in \operatorname{spt} M_\infty \cap \operatorname{spt} \widehat{M}_\infty$. Therefore, we must have $\Pi_1 = \Pi_2$, and consequently $M = \Pi_1$. □

4.3.3 The minimal case

In this little subsection we are going to adapt the argument of the subsection 4.3.2 to the minimal case. Here we are considering the Euclidean metric in $\mathbb{R}^{n+1}$.

Lemma 4.14. *Let $M^n \subset \mathbb{R}^{n+1}$ be a complete, connected, properly embedded minimal hypersurface and $\mathcal{C}(r) := \{x \in \mathbb{R}^{n+1} : \langle x, \mathbf{e}_1\rangle^2 + \langle x, \mathbf{e}_n\rangle^2 \le r^2\}$, for $r > 0$. Assume that M is C^1-asymptotic to two half-hyperplanes outside $\mathcal{C}(r)$. Suppose that $\{b_i\}_{i\in\mathbb{N}}$ is a sequence in $[\mathbf{e}_1, \mathbf{e}_n]^\perp$ and let $\{M_i\}_{i\in\mathbb{N}}$ be a sequence of minimal hypersurfaces given by $M_i := M + b_i$. Then there exist a connected stationary integral varifold M_∞ and a subsequence $\{M_{i_k}\} \subset \{M_i\}$ so that*

- *(i)* $M_{i_k} \overset{*}{\rightharpoonup} M_\infty$ *in* $\mathbb{R}^{n+1}$;
- *(ii)* sing M_∞ *satisfies* $\mathcal{H}^{n-7+\beta}(\operatorname{sing} M_\infty \cap (\mathbb{R}^{n+1} \setminus \mathcal{C}(r))) = 0$ *for all* $\beta > 0$ *if* $n \ge 7$, $\operatorname{sing} M_\infty \cap (\mathbb{R}^{n+1} \setminus \mathcal{C}(r))$ *is discrete if* $n = 7$ *and* $\operatorname{sing} M_\infty \cap (\mathbb{R}^{n+1} \setminus \mathcal{C}(r)) = \varnothing$ *if* $1 \le n \le 6$;
- *(iii)* $M_{i_k} \to \operatorname{spt} M_\infty$ *in* $\mathbb{R}^{n+1} \setminus (\mathcal{C}(r) \cup \operatorname{sing} M_\infty)$.

Proof. The proof of this fact follows the same strategy of the proof of Lemma 4.13. The unique differ is when we want to conclude that the are blow-up set is empty. In this case, we shall use the barriers $C(\lambda) \times \mathbb{R}^{n-2}$ to conclude this, here $C(\lambda)$ indicates the catenoid in $\mathbb{R}^3$ with neck λ. □

The proof of the next result is exactly the same proof given for Theorem 4.2.

Theorem 4.3. *Let $M^n \subset \mathbb{R}^{n+1}$ be a complete, connected, properly embedded minimal hypersurface and $\mathcal{C} := \{x \in \mathbb{R}^{n+1} : \langle x, \mathbf{e}_1\rangle^2 + \langle x, \mathbf{e}_n\rangle^2 \le r^2\}$, for $r > 0$. Assume that M is C^1-asymptotic to two half-hyperplanes outside $\mathcal{C}$. Then M must coincide with a hyperplane parallel to $\mathbf{e}_{n+1}$.*

Remark 4.4. *It was proved by HOFFMAN and MEEKS III (1990) that if M^2 is a minimal surface in $\mathbb{R}^3$ which lies in half-space, then M^2 is a plane. Their method was to use in a clever way part of the catenoid to get a contraction, if M^2 is not a plane. In fact, their proof also implies Theorem 4.3, if we use part of the catenoid product with $\mathbb{R}^{n-2}$ exactly as they did.*

5 CONCLUSION

In this thesis, we have obtained several results about translating solitons for the mean curvature flow. We have divided our study into two central parts: Jenkins-Serrin problem in $M \times \mathbb{R}$ and the characterization of the family of the tilted grim reaper cylinders in $\mathbb{R}^{n+1}$.

With respect to the Jenkins-Serrin problem we have divided our study into two different cases, the horizontal one and the vertical one. About the horizontal one we have obtained the following general result.

Theorem 5.1. *Let $\Omega \subset \mathbb{P}$ be an admissible domain such that for any admissible polygon $\mathcal{P} \subset \overline{\Omega}$ we have*

$$2\alpha_f(\mathcal{P}) < \mathfrak{L}_f[\partial\mathcal{P}] \quad \text{and} \quad 2\beta_f(\mathcal{P}) < \mathfrak{L}_f[\partial\mathcal{P}]. \tag{92}$$

Then

(a) If $\{C_k\} \neq \varnothing$ and $c_k\colon C_k \to \mathbb{R}$ are given continuous functions, then there exists an unique Jenkins-Serrin solution of (16) *with continuous boundary data c_k.*

(b) If $\{C_k\} = \varnothing$ and $\alpha_f(\Omega) = \beta_f(\Omega)$, then there exists an unique Jenkins-Serrin solution of (16) *up to translation.*

Furthermore, if u is the unique Jenkins-Serrin solution of (16) *with continuous boundary data*

$$c_k\colon C_k \to \mathbb{R}$$

and if $\{C_k\} \neq \varnothing$, then inequalities (92) *hold for all admissible polygon $\mathcal{P}$ in Ω, and if $\{C_k\} = \varnothing$ then we also have $\alpha_f(\Omega) = \beta_f(\Omega)$.*

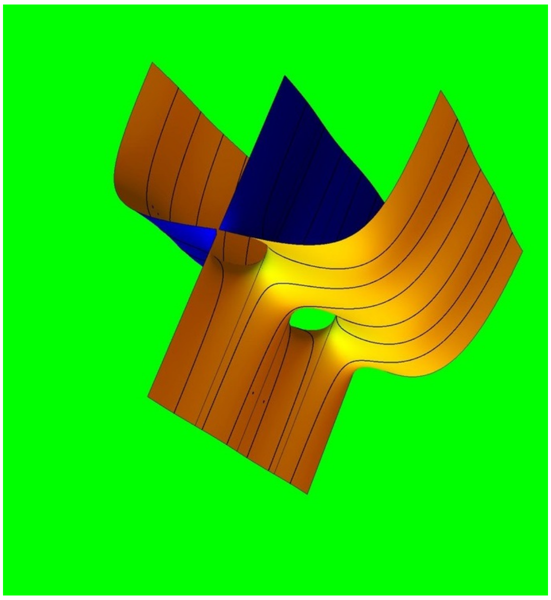
Unfortunately, in the vertical case we only could give the existence Jenkins-Serrin solution type I when M has non-positive sectional curvatures and is rotationally symmetric.

Theorem 5.2 (Existence of Jenkins-Serrin graph type I)**.** *Let $\Omega \subset M$ be an admissible domain with $\{B_i\} = \varnothing$. Given any continuous data $c_k\colon C_k \to \mathbb{R}$, there exists a Jenkins-Serrin solution $u\colon \Omega \to \mathbb{R}$ for the translating soliton equation with continuous data $u|_{C_k} = c_k$, if for any admissible polygon $\mathcal{P}$ we have*

$$2\alpha(\mathcal{P}) < \mathfrak{L}_\sigma(\mathcal{P}). \tag{93}$$

As we have mentioned earlier, the problem in this setting is because the vector field ∂_t is only conformal in $M \times \mathbb{R}$ with the Riemannian metric g_c. So when we try to use the flux formula we always get a quantity that depends on the function, in fact, its gradient. Maybe we could approach this problem by using the tools from the work of MASSARI (1977), but it is not clear that is possible to do that. However, as we have mentioned earlier too, HOFFMAN, MARTÍN, and WHITE (2019) gave an example of

Figure 20 – Nguyen's trident translator.



Jenkins-Serrin solution over a rhombus without continuous data in $\mathbb{R}^3$. Nevertheless, the construction of their depends on the result of classification obtained by HOFFMAN, ILMANEN, MARTÍN, and WHITE (2019), so it is not clear if their approaching can be done into other spaces.

About the result of characterization in $\mathbb{R}^{n+1}$, we have proved the following general result.

Theorem 5.3. *Let $M \hookrightarrow \mathbb{R}^{n+1}$ be a complete, connected, properly embedded translating soliton and consider the cylinder $\mathcal{C}_\theta(r) := \{x \in \mathbb{R}^{n+1} \ : \ \langle x, \mathbf{e}_1\rangle^2 + \langle u_\theta, x\rangle^2 \leq r^2\}$, where $r > 0$. Assume that M is C^1-asymptotic to two half-hyperplanes outside $\mathcal{C}_\theta(r)$.*

i. If $\theta \in [0, \pi/2)$, then we have one, and only one, of these two possibilities:

a. Both half-hyperplanes are contained in the same hyperplane Π parallel to $\mathbf{e}_{n+1}$ and M coincides with Π;

b. The half-hyperplanes are included in different parallel hyperplanes and M coincides with a vertical translation of the tilted grim reaper cylinder associated to θ.

ii. If $\theta = \pi/2$, then M coincides with a hyperplane parallel to $\mathbf{e}_{n+1}$.

Indeed, Theorem 5.3 is sharp in several senses. If we increase the number of half-hyperplanes then there are a lot of counterexamples, this number cannot be odd by Proposition 4.3. The cylinder over the pitchfork translator obtained recently by HOFFMAN, MARTÍN, and WHITE (2019) is an example of a complete, connected, properly embedded translating soliton which is C^1-asymptotic to four half-hyperplanes outside a cylinder in $\mathbb{R}^{n+1}$ (See Figure 6). In general, the cylinder over the examples obtained by NGUYEN (2009), NGUYEN (2013) and NGUYEN (2015) give similar examples which are C^1-asymptotic to $2k$ half-hyperplanes outside a cylinder, for any $k \geq 2$ (See Figure 20 for a picture of Nguyen's trident translator). The examples given by Nguyen have infinity topology, however the pitchfork translator is simply connected. Hence, we cannot increase the number of half-hyperplanes at Theorem 4.1. On the other hand, the hypothesis about the asymptotic behaviour outside a cylinder is also necessary as it is shown by the examples obtained by HOFFMAN, ILMANEN, MARTÍN, and WHITE (2019).

Moreover, as a consequence of our approaching we also have getter the following consequence in the minimal case.

Theorem 5.4. *Let $M^n \subset \mathbb{R}^{n+1}$ be a complete, connected, properly embedded minimal hypersurface and $\mathcal{C} := \{x \in \mathbb{R}^{n+1} \;:\; \langle x, \mathbf{e}_1\rangle^2 + \langle x, \mathbf{e}_n\rangle^2 \leq r^2\}$, for $r > 0$. Assume that M is C^1-asymptotic to two half-hyperplanes outside $\mathcal{C}$. Then M must coincide with a hyperplane parallel to $\mathbf{e}_{n+1}$.*

Thus, we cannot improve Theorem 5.3 and Theorem 5.4. However, in $\mathbb{H}^{n+1} \times \mathbb{R}$ we have proved the existence of an example that looks like the grim reaper cylinder in $\mathbb{R}^{n+2}$. Actually, this example is the authentic grim reaper cylinder with respect to a specific vector in $\mathbb{H}^{n+1} \times \mathbb{R}$ seen as $\mathbb{R}^{n+2}$ with the metric

$$g := e^{2x_{n+1}}(dx_1^2 + \cdots + dx_n^2) + dx_{n+1}^2 + dx_{n+2}^2.$$

So we can ask if the analogy of Theorem 5.3 and Theorem 5.4 are true is this space.

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