UNIVERSIDADE FEDERAL DO CEARÁ
CENTRO DE CIÊNCIAS
DEPARTAMENTO DE MATEMÁTICA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

LEO IVO DA SILVA SOUZA

DIFFERENTIAL OPERATORS PENALIZED BY GEOMETRIC POTENTIALS

FORTALEZA

2018

LEO IVO DA SILVA SOUZA

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Thesis submitted to the Post-graduate Program of the Mathematical Departament of Universidade Federal do Ceará in partial fulfillment of the necessary requirements for the degree of Ph.D in Mathematics. Area of expertise: Geometric Analysis

Advisor: Prof. Dr. José Fábio Bezerra Montenegro

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2018

Dados Internacionais de Catalogação na Publicação
Universidade Federal do Ceará
Biblioteca Universitária
Gerada automaticamente pelo módulo Catalog, mediante os dados fornecidos pelo(a) autor(a)

S239d Souza, Leo Ivo da Silva.
Differential operators penalized by geometric potentials / Leo Ivo da Silva Souza. – 2018.
20 f.

Tese (doutorado) – Universidade Federal do Ceará, Centro de Ciências, Programa de Pós-Graduação em Matemática , Fortaleza, 2018.
Orientação: Prof. Dr. José Fábio Bezerra Montenegro.

1. Operador. 2. Autovalor. 3. Curvatura. I. Título.

CDD 510

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Aproved in: 21/08/2018.

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I dedicate this work to all those who have contributed directly or indirectly to its completion.

ACKNOWLEDGMENTS

I would like to thank Prof. Fábio Montenegro, for the excellent guidance and support given throughout these years.

I want to thank the participating professors of the examining board Gleydson Ricarte, Luiz Antônio, Barnabé Lima, Levi Lima and Cleon Barroso for their time, for their valuable collaboration and suggestions.

I would like to thank professors Luciano Mari, Daniel Cibotaru, Diego Moreira, Marcos Melo and Jorge Herbert for the knowledge provided during their courses and the good reception they always had with me.

Futhermore, I want to thank my family, especially my father Francisco de Paulo Souza and my aunt Joana Darc Souza for the intensive support during my studies.

Also thanks to my friends Roger Oliveira, Henrique Blanco, Gilson Granja, Eduardo Garcez,Wanderley Oliveira, Diego Eloi, Diego Sousa, Nicolas Alcantara, Renivaldo Sena, JoserlanPerote, Edson Sampaio, Ederson Braga, Gleison Carneiro, Amilcar Montalban, FlavianoFrota, Fabricio Oliveira, Antônio Grangeiro, Davi Lustosa, Davi Ribeiro, Rafael Eufrásio,Jocel Faustino, Marcos Ranieri, João Nunes, Rui Paiva, Breno Rafael, Leonardo Tavares,Eurípedes Silva, Anderson Feitosa, Ivaldo Tributino, Ricardo Freire, Elisafã Braga, Air-ton Oliveira, Adriana Martins, Lisiane Castro, Letícia Holanda, Neilha Márcia, JaniellyAraújo and Elaine Sampaio for their sincere friendship and support throughout these years.

To the coordination of PGMAT-UFC, to the teachers Pacelli Bessa and AlexandreFernandes, the secretaries Andrea Costa and Jessyca Soares for all the assistance they have given over the years.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001

RESUMO

Este trabalho é apresentado em duas partes. Na primeira parte, estabelecemos a não-positividade do segundo autovalor do operador de Schrödinger $-\mathrm{div}\big(P_r\nabla\,\cdot\,\big) - W_r^2$ em uma hipersuperfície fechada Σ^n de $\mathbb{R}^{n+1}$, onde W_r é uma potência da $(r+1)$-ésima curvatura média de Σ^n que pediremos positiva. Se este eigenvalue é nulo, teremos uma caracterização da esfera. Este teorema generaliza o resultado de Harrell e Loss provado para o operador de Laplace-Beltrame penalizado pelo quadrado da curvatura média. .Na segunda parte, nós estabelecemos a não-positividade do segundo auto-valor do operador de Schrödinger $-\frac{d^2}{ds^2} - (\sqrt{F})_{\mathcal{C}}^{-2}F(\kappa)$, em uma curva fechada do plano com comprimento 2π, $F \in C^1(\mathbb{R})$ e κ é a curvatura da curva. Se este autovalor é nulo, teremos uma caracterização do círculo, que generaliza parcialmente o resultado de Harrell e Loss provado ao operador unidimensional de Laplace penalizado pelo quadrado da curvatura em curvas do plano.

Palavras-chave: Operador. Autovalor. Curvatura.

ABSTRACT

This paper is presented in two parts. In the first part, we establish the non-positivity of the second eigenvalue of the Schrödinger operator $-\mathrm{div}\big(P_r\nabla\,\cdot\,\big) - W_r^2$ on a closed hypersurface Σ^n of $\mathbb{R}^{n+1}$, where W_r is a power of the $(r+1)$-th mean curvature of Σ^n which we will ask to be positive. If this eigenvalue is null, we will have a characterization of the sphere. This theorem generalizes the result of Harrell and Loss proved to the Laplace-Beltrame operator penalized by the square of the mean curvature. In the second part, we established the non-positivity of the second auto-value of the Schödinger operator $-\frac{d^2}{ds^2} - (\sqrt{F})_{\mathcal{C}}^{-2}F(\kappa)$, in a closed curve of the plane with length 2π, $F \in C^1(\mathbb{R})$ and κ is the curvature of the curve. If this eigenvalue is null, we will have a characterization of the circle, which generalizes partially the result of Harrell and Loss proved to the one-dimensional Laplace operator penalized by the square of the curvature in curves of the plane.

Keywords: Operator. Eigenvalue. Curvature.

SUMÁRIO

1 INTRODUCTION

This result is based on the ideas presented by Harrell and Loss in (1998), where we obtain an elegant and more simplified proof that allowed us to generalize their results to a more general class of operators, L_r penalized by a power of $(r+1)-th$ mean curvature. In 1997, Harrell and Loss, obtained the following rigidity result.

Theorem 1.1. *Let Ω a smooth compact oriented hypersurface of dimension d immersed in $\mathbb{R}^{d+1}$; in particular self-intersections are allowed. The metric on that surface is the standard Euclidean metric inherited from $\mathbb{R}^{d+1}$. Then the second eigenvalue λ_2 of the operator*

$$H = -\Delta - \frac{1}{d}h^2$$

is strictly negative unless Ω is a sphere, in which case λ_2 equals zero.

The goal of this paper is to extend this result for a more general class of elliptic geometric operators. To present our main result, we need to introduce some definitions and notations.

Let ϕ: $M^n \to \overline{M}^{n+1}$ be an isometric immersion, and denote by $\mathbf{A}$ the second fundamental form associated to ϕ. It is known that $\mathbf{A}$ has n-geometric invariants. They are given by the elementary symmetric functions S_r of the principal curvatures $\kappa_1, \ldots, \kappa_n$ as follows:

$$S_r := \sum_{i_1 < \cdots < i_r} \kappa_{i_1} \ldots \kappa_{i_r} \quad (1 \leq r \leq n).$$

The r-curvature H_r of ϕ is then defined by

$$H_r := \frac{S_r}{\binom{n}{r}}.$$

Notice that H_1 corresponds to the mean curvature and H_n the Gauss-Kronecker curvature of ϕ. The Newton's transformations of ϕ are the operators P_r defined inductively by

$$\begin{cases} P_r = S_r I - \mathbf{A} P_{r-1}, \\ P_0 = I. \end{cases}$$

2 ON THE L_r-OPERATORS PENALIZED BY $(r+1)$-MEAN CURVATURE

The so-called L_r-operators are defined by $L_r := \operatorname{div}\big(P_r \nabla \cdot\big)$. It is known that if every H_r is positive, then L_r is elliptic by Proposition 3.2 in (BARBOSA e COLARES, 1997).

Let Σ be a compact hypersurface of $\mathbb{R}^{n+1}$ with the operator L_r being elliptic, we have that $-L_r$ is a positive, unbounded, self-adjoint operator with the spectrum formed only by eigenvalues

$$\sigma(-L_r) = \{0 = \lambda_1(-L_r) < \lambda_2(-L_r) \leq ...\}.$$

We consider the following class of Schrödinger operators

$$\mathcal{L}_r := -L_r - W_r^2.$$

where the potential $W_r = \big(c_r\, H_{r+1}^{\frac{r+2}{r+1}}\big)^{1/2}$ and $c_r = (n-r)\binom{n}{r}$, with $0 \leq r \leq n-1$. Now we can present the main result of this thesis.

2.1 Principal Theorem

Theorem 2.1. *Let Σ be a n-dimensional closed hypersurface embedded in $\mathbb{R}^{n+1}$. Assume that $H_{r+1} > 0$. Then the second eigenvalue of $\mathcal{L}_r$, $\lambda_2(\mathcal{L}_r)$ is strictly negative unless Σ is a sphere, in which case $\lambda_2(\mathcal{L}_r)$ equals zero.*

Note that the potential W_r^2 has the dimension $(\mathrm{vol}(\Sigma))^{-(r+2)}$, the same as the differential operator. As a consequence, the number of negative eigenvalues is independent of the volume of the hypersurface.

For the above proof, the following lemma will be used

Lemma 2.1. *Let Σ be a n-dimensional closed hypersurface embedded in $\mathbb{R}^{n+1}$ with $H_{r+1} > 0$ and consider the operator $\mathcal{L}_r = -L_r - W_r^2$. Suppose there $f \in L^2(\Sigma)$ satisfying:*

(1) $\langle f, W_r \rangle = 0$;

(2) $\langle R_0(W_r f), W_r f \rangle > \|f\|_2^2$,

where $\langle\ ,\ \rangle$ is the inner product in $L^2(\Sigma)$, $R_0 = \big(-L_r|_{[1]^\perp}\big)^{-1}$ and

$$[1]^\perp = \{u \in L^2(\Sigma); \langle u, 1 \rangle = 0\}.$$

Then the operator $\mathcal{L}_r$ has two negative eigenvalues.

Proof. Note that $\langle \mathcal{L}_r 1, 1 \rangle < 0$. To prove the lemma, we must find another function such that $\mathcal{L}_r$ is negative and then, apply the Min-Max Principle.

Let $\alpha = \sup\{\langle R_0(W_r g), W_r g\rangle; \|g\|_2 = 1, \langle g, W_r\rangle = 0\}$ be, note that $\alpha > 1$, then $\alpha = 1+\epsilon$ for $\epsilon > 0$. Define the functional in $L^2(\Sigma)$ as being

$$F(g) = \langle R_0(W_r g), W_r g\rangle, \tag{1}$$

$$G(g) = \|g\|_2^2 \tag{2}$$

and

$$J(g) = \langle g, W_r\rangle. \tag{3}$$

By Lagrange Multipliers method, there exists $u \in L^2(\Sigma)$ with $\|u\| = 1$ and $\langle u, W_r\rangle = 0$, such that

$$F'(u) = \alpha G'(u) + \beta J'(u). \tag{4}$$

The above functional equation gives us the following Euler-Lagrange equation

$$W_r R_0(W_r u) = \alpha u + \beta W_r. \tag{5}$$

Rewriting the above equation in the differential form, we have the following partial differential equation

$$W_r u = -\alpha L_r\left(\frac{u}{W_r}\right). \tag{6}$$

Using the fact that $\alpha = 1+\epsilon$, we have to

$$\mathcal{L}_r\left(\frac{u}{W_r}\right) = \epsilon L_r\left(\frac{u}{W_r}\right). \tag{7}$$

Thus we conclude that

$$\begin{aligned}\left\langle \mathcal{L}_r\left(\frac{u}{W_r}\right), \left(\frac{u}{W_r}\right)\right\rangle &= \epsilon\left\langle L_r\left(\frac{u}{W_r}\right), \left(\frac{u}{W_r}\right)\right\rangle \\ &= -\epsilon\left\langle P_r\nabla\left(\frac{u}{W_r}\right), \nabla\left(\frac{u}{W_r}\right)\right\rangle < 0.\end{aligned} \tag{8}$$

Using the Min-Max characterization we have to

$$\lambda_2(\mathcal{L}_r) = \min_{\substack{V\subset H^2(\Sigma)\\ dim V=2}} \max_{\substack{v\in V\\ v\neq 0}} \left\{\frac{\langle \mathcal{L}_r v, v\rangle}{\|v\|^2}\right\}. \tag{9}$$

Let $V_0 = [1, u/W_r]$ be, we have $\dim V_0 = 2$ and $\langle \mathcal{L}_r v, v\rangle < 0$ for all $v \in V_0$. In fact $\dim V_0 = 2$, otherwise we would have $W_r = 0$, which contradicts the fact that $H_{r+1} > 0$.

Let $v \in V_0$ be then we have $v = a \cdot 1 + b \cdot \dfrac{u}{W_r}$, and

$$\langle \mathcal{L}_r v, v\rangle = a^2\langle \mathcal{L}_r 1, 1\rangle + 2ab\left\langle \mathcal{L}_r 1, \left(\frac{u}{W_r}\right)\right\rangle + b^2\left\langle \mathcal{L}_r\left(\frac{u}{W_r}\right), \left(\frac{u}{W_r}\right)\right\rangle,$$

and

$$2ab\left\langle \mathcal{L}_r 1, \left(\frac{u}{W_r}\right)\right\rangle = 2ab\left\langle -W_r^2, \left(\frac{u}{W_r}\right)\right\rangle = -2ab\langle W_r, u\rangle = 0.$$

Then we have to $\langle \mathcal{L}_r v, v\rangle < 0$ for all $v \in V_0$. Choosing $V = V_0$, we have to $\lambda_2(\mathcal{L}_r) < 0$.

Hence, the operator $\mathcal{L}_r$ has more than one negative eigenvalue, if there is $f \in L^2(\Sigma)$ satisfying (1) and (2). □

2.1.1 Proof of the principal theorem

Now we give a proof for the Theorem 1.2. Let $\phi : \Sigma^n \to \mathbb{R}^{n+1}$ be an isometric immersion. By (ALENCAR, DO CARMO, e ROSENBERG, 1993), we have the following equation satisfied:

$$-L_r\phi = c_r H_{r+1} N, \tag{10}$$

where N is the normal vector of the surface. Thus, each coordinate satisfies

$$-L_r\phi_i = c_r H_{r+1} N_i,$$

with $i \in \{1, ..., n+1\}$. Denote by

$$(\phi_i)_\Sigma := \frac{1}{\mathrm{vol}(\Sigma)}\int_\Sigma \phi_i d\Sigma,$$

and $(\phi)_\Sigma := ((\phi_1)_\Sigma, ..., (\phi_{n+1})_\Sigma)$. Choosing f_i so that

$$f_i W_r = c_r H_{r+1} N_i,$$

we have

$$f_i = (c_r H_{r+1}^{\frac{r}{r+1}})^{\frac{1}{2}} N_i,$$

and $\langle f_i, W_r\rangle = 0$, by (2.10).

Observe that

$$R_0(W_r f_i) = R_0(c_r H_{r+1} N_i) = R_0(-L_r(\phi_i - (\phi_i)_\Sigma)) = \phi_i - (\phi_i)_\Sigma.$$

By multiplying both sides by $W_r f_i$ and using Divergence Theorem, we conclude that

$$\langle R_0(W_r f_i), W_r f_i\rangle_2 = \langle P_r \nabla\phi_i, \nabla\phi_i\rangle_2 = \int_\Sigma c_r H_{r+1}(\phi_i - (\phi_i)_\Sigma) N_i d\Sigma.$$

Summing up both sides with i varying from 1 to $n+1$, we have

$$\sum_{i=1}^{n+1}\langle R_0(W_rf_i),W_rf_i\rangle_2=\sum_{i=1}^{n+1}\langle P_r\nabla\phi_i,\nabla\phi_i\rangle_2=\int_\Sigma c_rH_{r+1}\langle\phi-(\phi)_\Sigma,N\rangle d\Sigma.$$

In (ALENCAR, DO CARMO, e ROSENBERG, 1993), we find the Minkowski's integral formula

$$\int_\Sigma H_r d\Sigma-\int_\Sigma H_{r+1}\langle\phi-(\phi)_\Sigma,N\rangle d\Sigma=0.$$

Thus, replacing the previous expression, we have

$$\sum_{i=1}^{n+1}\langle R_0(W_rf_i),W_rf_i\rangle_2=\sum_{i=1}^{n+1}\langle P_r\nabla\phi_i,\nabla\phi_i\rangle_2=\int_\Sigma c_rH_r d\Sigma.$$

By (ALENCAR, DO CARMO, e ROSENBERG, 1993) using the classical inequality $H_r^{\frac{1}{r}}\geq H_{r+1}^{\frac{1}{r+1}}$, for $r\geq 1$, we have

$$\sum_{i=1}^{n+1}\langle R_0(W_rf_i),W_rf_i\rangle_2=\int_\Sigma c_rH_r d\Sigma\geq\int_\Sigma c_rH_{r+1}^{\frac{r}{r+1}}d\Sigma=\sum_{i=1}^{n+1}\int_\Sigma c_rH_{r+1}^{\frac{r}{r+1}}N_i^2 d\Sigma$$

$$=\sum_{i=1}^{n+1}\|f_i\|_2^2.$$

Remark. *If $r=0$, we have written the sums above being identical and the only step that does not appear is the gap between the curvatures, however it is easy to see that the rest of the argument is following analogous to other cases.*

Define $d_i=\langle R_0(W_rf_i),W_rf_i\rangle_2-\|f_i\|_2^2$, thus $\sum\limits_{i=1}^{n+1}d_i\geq 0$ and then two possibilities may occur:

(i) There is $i\in\{1,...,n+1\}$ such that $d_i>0$;

(ii) $d_i=0$, for all $i\in\{1,...,n+1\}$.

If (i) occurs, we have f_i satisfies the hypotheses (1) and (2) of the Lemma 2.1 and therefore

$$\lambda_2(\mathcal{L}_r)<0.$$

If (ii) occurs, we have all the d_i void. In this case we use Lagrange multipliers. Now consider the functionals $\Psi,\Phi:L^2(\Sigma)\to\mathbb{R}$ given by

$$\Psi(f)=\langle R_0(W_rf),W_rf\rangle-\|f\|_2^2,\qquad\Phi(f)=\langle W_r,f\rangle_2$$

and the set of constraints

$$S = \{f \in L^2(\Sigma);\ \Phi(f) = \langle W_r, f\rangle_2 = 0\}.$$

We have to study two possibilities:
(a) $\inf\{\Psi(f);\ f \in S\} < 0$ or
(b) $\inf\{\Psi(f);\ f \in S\} = 0$.

In the first case, there is function $f \in S$ such that $\Psi(f) < 0$, and f is a critical function for Ψ on S. Then the method of Lagrange multipliers have to exist $\Gamma \in \mathbb{R}$, such that

$$\Psi'(f) = \Gamma\Phi'(f)$$

which resulted in the following Euler-Lagrange equation

$$W_r R_0(W_r f) - f = \Gamma W_r.$$

Multiplying both sides of the above equation for $f \in S$ and integrating, we have

$$0 = \Gamma\langle W_r, f\rangle = \langle R_0(W_r f), W_r f\rangle - \|f\|_2^2 < 0.$$

This is a contradiction, and the case (a) not occuring. In the second case, we have seen that each $f_i \in S$ and $\Psi(f_i) = \inf\{\Psi(f);\ f \in S\} = 0$. By the Method of Lagrange Multipliers, there exists $\Gamma \in \mathbb{R}$ such that $\Psi'(f_i) = \Gamma\Phi'(f_i)$. Hence, we obtain that each f_i satisfies the following Euler-Lagrange equation,

$$W_r R_0(W_r f_i) = f_i + \Gamma W_r,$$

therefore we conclude that

$$W_r(R_0(W_r f_i) - \Gamma) = f_i,$$

$$W_r(\phi_i - (\phi_i)_\Sigma - \Gamma) = f_i,$$

then

$$\phi_i - (\phi_i)_\Sigma - \Gamma = \frac{f_i}{W_r} = H_{r+1}^{-\frac{1}{r+1}} N_i.$$

Thus, have its version vector

$$\phi - (\phi)_\Sigma - \Gamma = H_{r+1}^{-\frac{1}{r+1}} N.$$

Differentiating the above expression along any curve Σ, we conclude that the derivative of $H_{r+1}^{-\frac{1}{r+1}}$ is zero, so H_{r+1} is constant, then Σ is a sphere by Alexandrov's Theorem in (ROS, 1987).

In fact, in this case we have $\lambda_2(\mathcal{L}_r) = 0$, as we have

$$W_r(\phi_i - (\phi_i)_\Sigma - \Gamma) = f_i,$$

and multiplying both sides by the expression W_r, we obtain

$$W_r^2(\phi_i - (\phi_i)_\Sigma - \Gamma) = W_r f_i = -L_r(\phi_i - (\phi_i)_\Sigma - \Gamma),$$

thus $\psi = \phi_i - (\phi_i)_\Sigma - \Gamma$ is the second eigenfunction of $\mathcal{L}_r = -L_r - W_r^2$, and $\mathcal{L}_r\psi = 0$.

Define the operator $T_r = -L_r - c_r\|\mathbf{A}\|^{r+2}$.

Corollary 2.1. *Under the same conditions of Theorem 1.2, $\lambda_2(T_r) \leq 0$, with equality if and only if Σ is a round sphere.*

The proof of the corollary follows immediately from the Jensen's inequality and the min-max principle. This finishes the proof.

3 ON THE UNIDIMENSIONAL LAPLACE OPERATOR PENALIZED BY A FUNCTION OF THE CURVATURE

This result is based on the ideas presented by Harrell and Loss in (1998). In this section we will consider a family of operators in a smooth and closed curve of the plane with length 2π and study the non-positivity of the second eigenvalue of these operators, characterizing the circle when the second auto value is zero.

Theorem 3.1. *Let $\mathcal{C}$ be a smooth, closed, simple curve in the plane with length 2π and with curvature κ. We consider $F \in C^1(\mathbb{R})$ a function such that $F \geq 0$, where F, F' vanish only in 0, and the operator*

$$L_F := -\frac{d^2}{ds^2} - (\sqrt{F})_{\mathcal{C}}^{-2} F(\kappa),$$

with $(\sqrt{F})_{\mathcal{C}} = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{F(\kappa)}dt$. Then, the second eigenvalue of L_F is less than or equal to 0, with equality if and only if $\mathcal{C}$ is a circle.

For the above proof, the following lemma will be used.

Lemma 3.1. *Let $\mathcal{C}$ be a smooth, closed, simple curve in the plane and consider the operator $L_F := -\frac{d^2}{ds^2} - (\sqrt{F})_{\mathcal{C}}^{-2} F(\kappa)$. Suppose there $f \in L^2(\mathcal{C})$ satisfying:*

(1) $\langle f, (\sqrt{F})_{\mathcal{C}}^{-1} \sqrt{F(\kappa)} \rangle_2 = 0$;

(2) $\langle R_0((\sqrt{F})_{\mathcal{C}}^{-1} \sqrt{F(\kappa)} f), (\sqrt{F})_{\mathcal{C}}^{-1} \sqrt{F(\kappa)} f \rangle_2 > \|f\|_2^2$,

where $\langle\ ,\ \rangle_2$ is the inner product in $L^2(\mathcal{C})$, $R_0 = \left(-\frac{d^2}{ds^2}|_{[1]^\perp}\right)^{-1}$ and

$$[1]^\perp = \{u \in L^2(\mathcal{C}); \langle u, 1 \rangle_2 = 0\}.$$

Then the operator L_F has two negative eigenvalues.

The proof of the lemma is analogous to the case where $F(x) = x^2$.

Now we show the Theorem 1.

Proof. Let $\phi_1, \phi_2, \Phi_1, \Phi_2 : [0, 2\pi] \to \mathbb{R}$ the functions defined by

$$\phi_1(s) := \cos\left((\sqrt{F})_{C}^{-1} \int_0^s \sqrt{F(\kappa)}dt\right),$$

$$\phi_2(s) := \sin\left((\sqrt{F})_{C}^{-1} \int_0^s \sqrt{F(\kappa)}dt\right),$$

$$\Phi_1(s) := \int_0^s \phi_1(t)dt \quad \text{and} \quad \Phi_2(s) := \int_0^s \phi_2(t)dt.$$

Observe that

$$\Phi_1' = \phi_1, \Phi_2' = \phi_2, \Phi_1'' = \phi_1' = -(\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}\phi_2 \text{ and } \Phi_2'' = \phi_2' = (\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}\phi_1.$$

Then

$$R_0((\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}\phi_2) = R_0(-\Phi_1'') = \Phi_1 - (\Phi_1)_{\mathcal{C}}$$

and

$$R_0((\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}\phi_1) = R_0(\Phi_2'') = (\Phi_2)_{\mathcal{C}} - \Phi_2.$$

Therefore

$$\langle R_0((\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}\phi_1), (\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}\phi_1)\rangle_2 = \|\phi_2\|_2^2$$

and

$$\langle R_0((\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}\phi_2), (\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}\phi_2)\rangle_2 = \|\phi_1\|_2^2.$$

Define $d_i = \langle R_0((\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}\phi_i), (\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}\phi_i)\rangle_2 - \|\phi_i\|_2^2$, thus $d_1 + d_2 = 0$ and then two possibilities may occur:

(i) There is $i \in \{1, 2\}$ such that $d_i > 0$;

(ii) $d_i = 0$, for $i \in \{1, 2\}$.

If (i) occurs, we have ϕ_i satisfies the hypotheses (1) and (2) of the Lemma 2 and therefore $\lambda_2(L_F) < 0$.

If (ii) occurs, we have all the d_i void. In this case we use Lagrange multipliers.

Now consider the functionals $\Psi, \Theta : L^2(\mathcal{C}) \to \mathbb{R}$ given by

$$\Psi(f) = \langle R_0((\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}f), (\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}f\rangle_2 - \|f\|_2^2, \qquad \Theta(f) = \langle (\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}, f\rangle_2$$

and the set of constraints

$$S = \{f \in L^2(\mathcal{C});\ \Theta(f) = \langle (\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}, f\rangle_2 = 0\}.$$

We have to study two possibilities:

(a) $\inf\{\Psi(f);\ f \in S\} < 0$ or

(b) $\inf\{\Psi(f);\ f \in S\} = 0$.

In the first case, there is function $f \in S$ such that $\Psi(f) < 0$, and f is a critical function for Ψ on S. Then the method of Lagrange multipliers have to exist $\Gamma \in \mathbb{R}$, such that

$$\Psi'(f) = \Gamma\Theta'(f)$$

which resulted in the following Euler-Lagrange equation

$$(\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}R_0((\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}f) - f = \Gamma(\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}.$$

Multiplying both sides of the above equation for $f \in S$ and integrating, we have

$$0 = \Gamma\langle(\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}, f\rangle_2 = \langle R_0((\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}f), (\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}f\rangle_2 - \|f\|_2^2 < 0.$$

This is a contradiction, and the case (a) not occurs.

In the second case, we have seen that each $\phi_i \in S$ and $\Psi(\phi_i) = \inf\{\Psi(f);\ f \in S\} = 0$. By the Method of Lagrange Multipliers, there exists $\Gamma \in \mathbb{R}$ such that $\Psi'(\phi_i) = \Gamma\Theta'(\phi_i)$. Hence, we obtain that each ϕ_i satisfies the following Euler-Lagrange equation,

$$(\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}R_0((\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}\phi_i) = \phi_i + \Gamma(\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)},$$

therefore we conclude that

$$(\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}(R_0((\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}\phi_i) - \Gamma) = \phi_i,$$

$$(\sqrt{F})_{\mathcal{C}}^{-1}\sqrt{F(\kappa)}(\Phi_j - (\Phi_j)_{\mathcal{C}} - \Gamma) = \phi_i$$

Differentiating the above expression we have that

$$\phi_j = [(\sqrt{F})_{\mathcal{C}}(\sqrt{F(\kappa)})^{-1}]'\phi_i + \phi_j.$$

Therefore,

$$[(\sqrt{F})_{\mathcal{C}}(\sqrt{F(\kappa)})^{-1}]' = 0 \Rightarrow \kappa'(s)F'(\kappa) = 0$$

such as $\kappa > 0, F'(\kappa) \neq 0$ we have $\kappa' = 0$, that is κ is constant and therefore $\mathcal{C}$ is a circle. □

4 CONCLUSION

In this thesis, we present some results based on the work of Harrell and Loss.

In the first part we generalize the result in hypersurfaces submerged with $(r+1)$-positive mean curvature, for the operator L_r penalized by a power of this curvature.

In the second part we generalize the result into curves of the plane with length 2π, for the one-dimensional laplace operator penalized by a function of the curvature of the curve.

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