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Extended uncertainty from first principles

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ABSTRACT

A translation operator acting in a space with a diagonal metric is introduced to describe the motion of a particle in a quantum system. We show that the momentum operator and, as a consequence, the uncertainty relation now depend on the metric. It is also shown that, for any metric expanded up to second order, this formalism naturally leads to an extended uncertainty principle (EUP) with a minimum momentum dispersion. The Ehrenfest theorem is modified to include an additional term related to a tidal force arriving from the space curvature introduced by the metric. For one-dimensional systems, we show how to map a harmonic potential to an effective potential in Euclidean space using different metrics.

Matter curves the space–time in all directions leading two particles, traveling parallel to each other, to get closer or far apart as if there is a force acting between them. This is the definition of gravity in the realm of general relativity, where the space is curved in the vicinities of large densities of mass or energy. In general relativity, the metric tensor determines the geometric local structure of the curved space–time. For example, the Minkowski metric is the one used in special relativity, while the Schwarzschild metric is the most general solution to the Einstein's equation. Non-Euclidean metrics appear naturally also in very small scales where Quantum Mechanics is valid. For example, it has been used as an attempt to merge general relativity and quantum mechanics [1–5], as well as in the study of quantum systems problems with constraints [6–8]. More recently, a Schwarzschild-like metric has been used to find the quantum wave equations [9].

In a curved surface the shortest path between two points is a geodesic and the squared distance between two infinitesimally close points is given by

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu, \tag{1}$$

where $g_{\mu\nu}$ is the metric of the curved space under consideration. Here, we use a diagonal metric,

$$ds^2 = g_{xx}dx^2 + g_{yy}dy^2 + g_{zz}dz^2, \tag{2}$$

to show that an inertial force appears naturally in the quantum mechanics framework leading to a modified Ehrenfest theorem. More importantly, it is shown that the metric is responsible for a minimum momentum leading naturally to what is called extended uncertainty principle (EUP) [10].

As a first consequence of adopting Eq. (2), the space curvature leads to an internal product of the wave function given by

$$\langle \phi|\psi\rangle \equiv \int \phi^*(x,y,z)\psi(x,y,z)\sqrt{|g|}dxdydz, \tag{3}$$

where $g = \det(g_{\mu\nu})$ is the determinant of the matrix of components of the metric tensor. In this context, a particle in the vicinities of a point with coordinate x can be described by the ket $|x\rangle$ where $\hat{x}|x\rangle = x|x\rangle$. As the set $\{|x\rangle\}$ is complete, the identity operator can be written as

$$1 = \int \sqrt{|g|}dxdydz|x,y,z\rangle\langle x,y,z|, \tag{4}$$

and the scalar product in this metric for one dimension is given by $\langle x|x'\rangle = g(x)^{-1/2}\delta(x-x')$. With this metric, for a particle to go from a point x to $x + g_{xx}^{-1/2}dx$ it has to get a translation like $T_g(dx)|x\rangle = |x + g_{xx}^{-1/2}dx\rangle$. This translation is clearly non-additive and the operator can be written as

$$T_g(\vec{dr}) \equiv 1 - i\frac{\vec{\mathcal{P}}}{\hbar}\cdot\vec{dr}, \tag{5}$$

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where $\vec{\mathcal{P}}$ is a generalized momentum that generates the translation, with $[x, \mathcal{P}_x] = i\hbar g_{xx}^{-1/2}$. As a consequence, it is straightforward to show that the momentum component can be written as $\mathcal{P}_\nu = -i\hbar g_{\nu\nu}^{-1/2}\partial_\nu$ leading to an stationary equation of motion for a particle $H_g\psi = E\psi$ or,

$$-\frac{\hbar^2}{2m}\mathcal{D}^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r}), \tag{6}$$

where $\mathcal{D} = \sum_\nu g_{\nu\nu}^{-1/2}\partial_\nu$, and

$$\mathcal{D}^2 \equiv \frac{1}{\sqrt{g}}\sum_\nu \partial_\nu\sqrt{g}g^{\nu\nu}\partial_\nu, \tag{7}$$

with $g^{\nu\nu}g_{\nu\nu} = 1$, and $\nu = x, y, z$. At this point, it is important to mention that the Hamiltonian defined by

$$\mathcal{H} \equiv -\frac{\hbar^2}{2m}\mathcal{P}^2 + V(\mathbf{r}) \tag{8}$$

is Hermitian due to Eq. (4). As consequence, the probability density $\rho = \Psi(x,t)\Psi(x,t)^*$ obeys the continuity equation,

$$\frac{\partial\rho}{\partial t} + \mathcal{D}\mathcal{J} = 0, \tag{9}$$

where the probability current is now written as $\mathcal{J} = g_{xx}^{-1/2}J$. We emphasize that the translation is non-additive in this diagonal metric, the associated Schrödinger-like equation remains linear, second-order in space and first-order in time, and that the probability density is conserved in terms of a continuity equation of the same form as the standard one in Euclidean space.

It is noticeable that any similarity between the traditional formalism in Euclidean space and the equations generated in a curved space practically disappear when we bring up the Ehrenfest theorem. Using the Heisenberg equation of motion $\langle\dot{x}\rangle = \langle\{g_{xx}^{-1/2}, \mathcal{P}_x\}\rangle/2m$, where the braces represent the anticommutation relation, and considering that the metric can be Taylor expanded as

$$g_{xx}^{-1/2} = \sum_0^\infty a_n x^n, \tag{10}$$

the average force of a particle moving in one-dimension can be expressed as,

$$\frac{d\langle p_x\rangle}{dt} = -\langle\mathcal{D}_x[V(x)]\rangle + \frac{1}{2}\sum_{n=1}^\infty a_n\frac{d}{dt}\langle\{x^n, \mathcal{P}_x\}\rangle \tag{11}$$

where $\mathcal{D}_x = g_{xx}^{-1/2}\partial_x$, and we have taken $a_0 = 1$. From the classical point of view, the Lagrangian for a particle in a curved space in one-dimension is given by $\mathcal{L} = mg_{xx}\dot{x}^2/2 - V(x)$ leading to the following equation of motion

$$\ddot{x} + \Gamma^x_{xx}\dot{x}^2 + g^{xx}\partial_x V(x) = 0, \tag{12}$$

where Γ^x_{xx} is the Christoffel symbol of second kind given by $\Gamma^x_{xx} = g^{xx}\partial_x g_{xx}/2$. Equation (12) is the classical correspondent of Eq. (11). Up to $n = 1$ in Eq. (10), we obtain $g_{xx}^{-1/2} = 1 + \gamma x$ when considering $a_1 = \gamma$, so that Eq. (11) becomes

$$\frac{d\langle p_x\rangle}{dt} = -\langle(1 + 2\gamma x)\partial_x V(x)\rangle + \frac{\gamma}{m}\langle\mathcal{P}_x^2\rangle, \tag{13}$$

and for $\gamma = 0$ the Newton's law for the particle is recovered.

In the absence of any potential ($V(x) = 0$), a free particle feels a force arising purely from the space geometry given by

$$\vec{F} = \frac{\gamma m}{1 + \gamma x}\dot{x}^2\hat{x}, \tag{14}$$

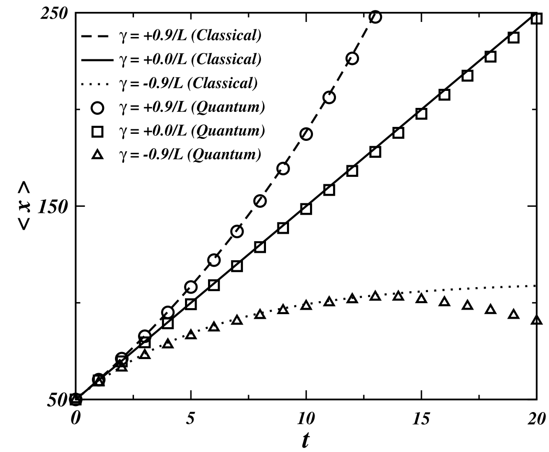

Fig. 1. The average value for a free particle taking the metric up to its first derivative in a Taylor expansion.

where v_0 is the particle velocity. To illustrate the result of Eq. (14), we plot in Fig. 1 the average position of a free particle starting from a point x_0 and initial velocity v_0, where its wavefunction is given by

$$\Psi(x, 0) = A\exp\left[-\left(\frac{\eta - \eta_0}{\sigma}\right)^2\right]\exp(ik\eta), \tag{15}$$

the coordinate $\eta = \ln(1 + \gamma x)/\gamma$ comes naturally from the change of coordinates

$$\eta(x) \equiv \int\sqrt{g(x)}dx, \tag{16}$$

and $\hbar k = mv_0/(1 + \gamma x_0)$. From Fig. 1 one can see that, for positive (negative) values of γ the particle accelerates (decelerates) with time. The classical particle position expression shown in Fig. 1 comes from the solution of Eq. (12) for $V(x) = 0$;

$$x = \langle x\rangle = \frac{1 + \gamma x_0}{\gamma}\exp\frac{\gamma v_0}{1 + \gamma x_0}t - \frac{1}{\gamma}. \tag{17}$$

As depicted, the classical and the quantum solutions fit almost perfectly. In order to find the average position of the quantum particle, we need to impose finite boundaries in the evolution of the wavefunction, which corresponds exactly to the origin of the mismatch in Fig. 1 for negative γ.

Next, as an example, we consider a parabolic potential $V(x) \propto x^2$ and study how the particle's energy is modified by this confinement in a space with a non-Euclidean metric. In order to do this we write the stationary Schrödinger equation in terms of η

$$-\frac{\hbar^2}{2m}\frac{d^2}{d\eta^2}\psi(\eta) + V_{eff}(\eta)\psi(\eta) = E\psi(\eta), \tag{18}$$

with the parabolic potential written in the new coordinates η, i.e., the effective potential expressed in terms of Eq. (18). For the metric $g_{xx}^{-1/2} = 1 + \gamma x$ the parabolic potential becomes a Morse-like potential [12]. It is interesting to note that, depending on the metric, it is possible to map the parabolic potential to other well known potentials frequently used in physics and chemistry. For example, if one considers $g(x) = 4x^2$, then $\eta(x) = x^2$ and $V_{eff}(\eta) \propto \eta$. Therefore, a particle under a parabolic potential in a space with metric given by $ds^2 = 4x^2dx^2$ behaves like a free particle in the presence of an electric field. It is also possible to map a parabolic potential to a Coulombian one using $g(x) = 4/x^6$. Table 1 resumes the mapping between some special metrics and the effective potentials.

Table 1
Effective potentials $V_{eff}(\eta)$ generated from the parabolic potential $V(x) \propto x^2$ for different spatial metrics.

$V(x)$	$V_{eff}(\eta)$	$g(x)$
x^2	η^2/a^2	a^2
x^2	η	$4x^2$
x^2	$-1/\eta$	$4/x^6$
x^2	$(e^{\gamma\eta}-1)^2/\gamma^2$	$1/(1+\gamma x)^2$

The metric effect goes beyond the forces acting on a particle and reaches the foundations of the quantum theory. In quantum mechanics with Euclidean metric, the uncertainty in the position of the particle x can become very small while the momentum p increases to very large values and vice-versa. This is the celebrated Heisenberg uncertainty principle that prevents the existence of a minimum length scale which is essential in different areas of physics, such as relativity [13], string theory [14], and quantum gravity [15]. In this sense, a modification of basic quantum principles is necessary to properly approach the interaction of matter and fields. Snyder [16] was the first to propose the continuity of space–time at a high energy limit for which the effects of gravity become so important that would result in the discreetness of the space–time. As a consequence, the Heisenberg uncertainty principle should be modified to the so-called Generalized Uncertainty Principle (GUP), where there is a nonzero minimal uncertainty in position measurements.

Under this framework, the commutation relation $[x, p] = i\hbar(1 + \alpha^2 p^2)$ leads to the GUP,

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}\left[1 + \alpha^2\left(\langle p\rangle^2 + \Delta p^2\right)\right], \tag{19}$$

with the smallest uncertainty in position being $\Delta x_0 = \hbar\alpha$, where α is a characteristic length. For small values of α, the traditional Heisenberg commutation relation is recovered. This is a well stablished theory [17–19] that has been used to study deformed quantum mechanics [20,21], string theory [22,23], quantum gravity [24–26], and black-hole thermodynamics [27–30]. Recently there has been attempt to test the GUP through experiments in quantum optics [31,32]. From a different perspective, when considering large distances, the curvature of spacetime becomes important, no Euclidean reference system exists, and there is a limit to the precision for which momenta can be defined. As already mentioned, the so called extended uncertainty principle [33] can be expressed by

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}\left[1 + \alpha'^2\left(\langle x\rangle^2 + \Delta x^2\right)\right]. \tag{20}$$

Equation (20) is essential to study the deSitter black hole thermodynamics since it is used to give a more symmetric description of GUP [10,27,34,35].

We now show that the EUP can be derived naturally from the commutation relation developed here. Using the general expression for the uncertainty between two observables

$$\Delta x \Delta \mathcal{P}_x \geq \frac{1}{2}|\langle[x, \mathcal{P}_x]\rangle|, \tag{21}$$

$$\Delta x \Delta \mathcal{P}_x \geq \frac{\hbar}{2}|\langle g_{xx}^{-1/2}\rangle|, \tag{22}$$

and taking into account the metric expansion up to $n = 2$, $g_{xx}^{-1/2} = 1 + \gamma x + \beta^2 x^2$, the uncertainty becomes

$$\Delta x \Delta \mathcal{P}_x \geq \frac{\hbar}{2}\left|1 + \gamma\langle x\rangle + \beta^2\langle x^2\rangle\right|, \tag{23}$$

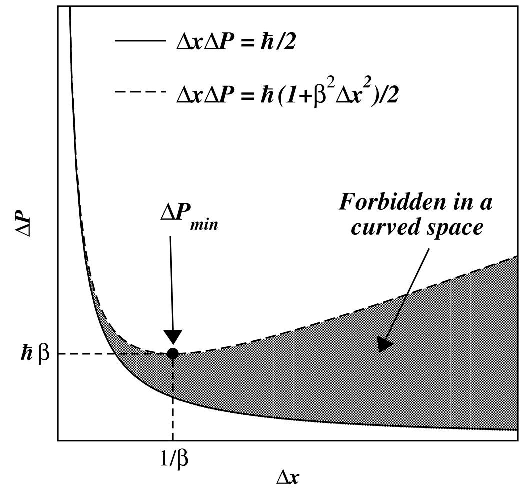

Fig. 2. The Heisenberg uncertainty and the extended uncertainty principle are plotted with solid and dashed lines respectively. The shaded area in the figure represents states not allowed for any metric expanded after first order.

and using $\Delta x^2 = \langle x^2\rangle - \langle x\rangle^2$, we get

$$\Delta x \Delta \mathcal{P}_x \geq \frac{\hbar}{2}\left|1 + \gamma\langle x\rangle + \beta^2\left(\langle x\rangle^2 + \Delta x^2\right)\right|. \tag{24}$$

Solving the above equation for Δx we obtain

$$\Delta x = \frac{\Delta \mathcal{P}_x}{\hbar\beta^2} \pm \frac{1}{\beta^2}\sqrt{\frac{\Delta \mathcal{P}_x^2}{\hbar^2\beta^2} - 1 - \gamma\langle x\rangle - \beta^2\langle x\rangle^2}, \tag{25}$$

leading to a minimum momentum $\Delta\mathcal{P}_x$ that depends on γ and β

$$\Delta \mathcal{P}_{xmin} = \hbar\beta\sqrt{1 + \gamma\langle x\rangle + \beta^2\langle x\rangle^2}, \tag{26}$$

where the parameter β has a unit of inverse of length. For the particular case in which the position average value is zero, $\langle x\rangle = 0$, Eq. (20) is recovered leading to the minimum momentum $\Delta\mathcal{P}_{xmin} = \hbar\beta$. This is shown in Fig. 2, where the Heisenberg uncertainty relation is plotted along with the modified relation found in Eq. (26). The dark region in Fig. 2 is forbidden for measurements with any metric expanded after first order.

In summary, we have developed a quantum mechanics formalism for a non-Euclidean space with a diagonal metric. The main tool for this formalism is a position dependent translation operator that is responsible for the modified commutation relation between position and momentum, $[x, \mathcal{P}_x] = i\hbar g_{xx}^{-1/2}$. This modified commutation relation leads to an external force acting on a particle due to the space metric. This extra force changes the potential submitted to the particle resulting in an effective potential. For example, depending on the metric used, the harmonic potential can be turned into another potential. The most surprising one is the Morse potential that arises when we take the first two terms in the metric expansion $g_{xx}^{-1/2} = 1 + \gamma x + \beta^2 x^2 + \cdots$.

Another important result, when taking the expansion up to the second order, is that the uncertainty relation is exactly the one suggested previously as a natural term to symmetrize the generalized uncertainty principle. It is important to mention that both GUP and EUP are derived in the literature using modified commutation relations for position and momentum introduced *ad hoc*, while here the EUP clearly arises naturally from the first terms in the expansion of any metric. This is important because, in general, the corrections to the Schwarzschild temperature black-holes can be calculated by introducing the gravitational interaction as an external force on a flat background, and neglecting the curvature of spacetime [10,30]. In our case, the metric imposes a curvature, and the correction to the usual Hawking temperature $T \approx \frac{1}{4\pi r_+}$ is given by a term proportional to the minimum momentum determined by the EUP. Here r_+ corresponds to the radius of the horizon of the black-hole. Therefore, our formalism discloses a connection between the cosmological constant and the minimum momentum.

One last aspect to mention is that the displacement operator described by our formalism is a q-exponential function, namely, an important ingredient of Tsallis thermostatistics [36–38]. This therefore might provide an interesting connection between our formalism and the thermodynamics of nonextensive systems like, for example, black-holes. Finally, our formalism leads to a modified Schrödinger equation with a position-dependent mass and a particular kinetic operator that emerges naturally [11]. As a consequence, our approach provides a first-principles interpretation for the concept of effective mass widely and successfully used to model electronic transport in semiconductor heterostructures, showing very good agreement with experimental results [39,40].

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