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SOME RIGIDITY THEOREMS IN SEMI-RIEMANNIAN WARPED PRODUCTS

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Abstract

We study the problem of uniqueness of complete hypersurfaces immersed in a semi-Riemannian warped product whose warping function has convex logarithm. By applying a maximum principle at the infinity due to S. T. Yau and supposing a natural comparison inequality between the mean curvature of the hypersurface and that of the slices of the region where the hypersurface is contained, we obtain rigidity theorems in such ambient spaces. Applications to the hyperbolic and the steady state spaces are given.

1. Introduction

The aim of this paper is to study the uniqueness of complete hypersurfaces immersed in a semi-Riemannian warped product of the type $\varepsilon I \times_f M^n$, where M^n is a connected, n-dimensional oriented Riemannian manifold, $I \subseteq \mathbf{R}$ is an open interval, $f : I \to \mathbf{R}$ a positive smooth function and $\varepsilon = \pm 1$ (for the details, see Section 3).

In the last years, many authors have approached problems in this branch. For example, we may cite the works of L. J. Alías et al [2, 3, 4, 5, 6], S. Montiel [19, 20] and A. Romero et al [9, 10, 22, 23].

More recently, the second author jointly with F. Camargo and A. Caminha obtained in [11] Bernstein-type results in two particular semi-Riemannian warped products: the hyperbolic-type and the steady state-type spaces (see Section 4).

Here, by supposing an appropriate inequality involving the mean curvature of the hypersurface and that of the slices, we are able to extend the results of [11] to the case when the ambient space is a semi-Riemannian warped product whose warping function has convex logarithm.

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In our approach, following the ideas of [11], the main analytical tool is a maximum principle at the infinity due to S. T. Yau [26], which can be seen as a sort of extension to complete (noncompact) Riemannian manifolds of the classical Hopf's maximum principle. In this setting, we explore the geometry of the vertical height function h of Riemannian immersions $\psi : \Sigma^n \to \varepsilon I \times_f M^n$ (that is, the height function with respect to the unit coordinate vector field ∂_t) to establish our rigidity results. In this setting, we will assume that the hypersurface is bounded away from the infinity of the ambient space; that is, it is contained in a slab bounded by slices $\{t_1\} \times M^n$ and $\{t_2\} \times M^n$, for some $t_1, t_2 \in I$.

We consider initially the case of the mean curvature to prove, in the Lorentzian setting, the following (cf. Theorem 4.2; see also Theorem 4.6 for the Riemannian case):

Let $\overline{M}^{n+1} = -I \times_f M^n$ be a Lorentzian warped product such that $\log f$ *is convex. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete, connected spacelike hypersurface bounded away from the infinity of $\overline{M}^{n+1}$. Suppose that the mean curvature H of Σ^n satisfies*

$$f'(h)H \geq \frac{f'^2}{f}(h) > 0. \tag{1.1}$$

If ∇h has integrable norm on Σ^n, then Σ^n is a slice.

We want to point out that the differential inequality (1.1) means that, at each point (t, x) of the spacelike hypersurface Σ^n, the absolute value of the mean curvature H can be any value greater than or equal to the absolute value of the mean curvature of the slice $\{t\} \times M^n$. Consequently, we only suppose here a natural comparison inequality between two mean curvature quantities, but we do not require H constant. In this sense, (1.1) is a mild hypothesis.

Furthermore, we use an extension of Yau's result due to F. Camargo, A. Caminha and P. Sousa [14] to treat the case of the higher order mean curvatures. More precisely, in the Lorentzian setting, we get the following (cf. Theorem 5.4; see also Theorem 5.8 for the Riemannian case):

Let $\overline{M}^{n+1} = -I \times_f M^n$ be a Lorentzian warped product with constant sectional curvature and such that $\log f$ *is convex. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete, connected spacelike hypersurface bounded away from the infinity of $\overline{M}^{n+1}$. Suppose that the mean curvature H is bounded and that, for some $1 \leq r \leq n-1$, H_r and H_{r+1} are positive and such that*

$$\frac{H_{r+1}}{H_r} \geq \frac{f'}{f}(h) > 0. \tag{1.2}$$

If h has a local minimum and ∇h has integrable norm on Σ^n, then Σ^n is a slice.

We note that, since $H_0 = 1$, the differential inequality (1.2) is a natural extension of (1.1) in the context of the higher order mean curvatures H_r defined in terms of the symmetric functions of the eigenvalues of the second fundamental form of the hypersurface (see Section 2). On the other hand, in the case $r = 1$, we observe that it is not necessary to suppose that the height function of the Σ^n has a local minimum on it (cf. Remark 5.5).

Moreover, by observing that the family of spaces $\varepsilon I \times_{e^t} M^n$ include the $(n+1)$-dimensional steady state space $\mathscr{H}^{n+1}$ as well as the $(n+1)$-dimensional hyperbolic space $\mathbf{H}^{n+1}$, we give applications in each of these ambient spaces (cf. Sections 4 and 5).

2. Preliminaries

Let $\overline{M}^{n+1}$ be a connected semi-Riemannian manifold with metric $\bar{g} = \langle\, ,\rangle$ of index $\nu \leq 1$, and semi-Riemannian connection $\overline{\nabla}$. For a vector field $X \in \mathfrak{X}(\overline{M})$, let $\varepsilon_X = \langle X, X\rangle$; X is a *unit* vector field if $\varepsilon_X = \pm 1$, and *timelike* if $\varepsilon_X = -1$.

In all that follows, we consider *Riemannian immersions* $\psi : \Sigma^n \to \overline{M}^{n+1}$, namely, immersions from a connected, n-dimensional orientable differentiable manifold Σ^n into $\overline{M}$, such that the induced metric $g = \psi^*(\bar{g})$ turns Σ into a Riemannian manifold (in the Lorentz case $\nu = 1$ and we refer to (Σ^n, g) as a *spacelike* hypersurface of $\overline{M}$), with Levi-Civita connection ∇. We orient Σ^n by the choice of a unit normal vector field N on it.

In this setting, if we let A denote the corresponding shape operator, then, at each $p \in \Sigma^n$, A restricts to a self-adjoint linear map $A_p : T_p\Sigma \to T_p\Sigma$.

For $0 \leq r \leq n$, let $S_r(p)$ denote the r-th elementary symmetric function of the eigenvalues of A_p; in this way one gets n smooth functions $S_r : \Sigma^n \to \mathbf{R}$, such that

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by definition. If $p \in \Sigma^n$ and $\{e_k\}$ is a basis of $T_p\Sigma$ formed by eigenvectors of A_p, with corresponding eigenvalues $\{\lambda_k\}$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \dots, \lambda_n),$$

where $\sigma_r \in \mathbf{R}[X_1, \dots, X_n]$ is the r-th elementary symmetric polynomial on the indeterminates $X_1, \dots, X_n$.

Also, we define the r-th mean curvature H_r of ψ, $0 \leq r \leq n$, by

$$\binom{n}{r} H_r = \varepsilon_N^r S_r = \sigma_r(\varepsilon_N \lambda_1, \dots, \varepsilon_N \lambda_n).$$

We observe that $H_0 = 1$ and H_1 is the usual mean curvature H of Σ^n. Moreover, when the ambient space has constant sectional curvature $\bar{\kappa}$, it follows from the Gauss equation that

$$R = \varepsilon_N(\bar{\kappa} - H_2), \tag{2.1}$$

where R is the normalized scalar curvature of Σ^n.

We also observe that the Hilbert-Schmidt norm of the shape operator A of Σ^n is given by

$$|A|^2 = n^2H^2 - n(n-1)H_2. \tag{2.2}$$

For $0 \le r \le n$, one defines the r-th Newton transformation P_r on Σ^n by setting $P_0 = I$ (the identity operator) and, for $1 \le r \le n$, via the recurrence relation

$$P_r = \varepsilon_N^r S_r I - \varepsilon_N A P_{r-1}. \tag{2.3}$$

A trivial induction shows that

$$P_r = \varepsilon_N^r (S_r I - S_{r-1}A + S_{r-2}A^2 - \cdots + (-1)^r A^r),$$

so that Cayley-Hamilton theorem gives $P_n = 0$. Moreover, since P_r is a polynomial in A for every r, it is also self-adjoint and commutes with A. Therefore, a basis of $T_p\Sigma$ diagonalizing A at $p \in \Sigma^n$ also diagonalizes all of the P_r at p. Let $\{e_k\}$ be such a basis. Denoting by A_i the restriction of A to $\langle e_i \rangle^\perp \subset T_p\Sigma$, it is easy to see that

$$\det(tI - A_i) = \sum_{k=0}^{n-1} (-1)^k S_k(A_i) t^{n-1-k},$$

where

$$S_k(A_i) = \sum_{\substack{1 \le j_1 < \cdots < j_k \le n \\ j_1, \ldots, j_k \ne i}} \lambda_{j_1} \cdots \lambda_{j_k}.$$

It is also immediate to check that $P_r e_i = \varepsilon_N^r S_r(A_i) e_i$, so that an easy computation (cf. Lemma 2.1 of [7]) gives the following

LEMMA 2.1. *With the above notations, the following formulas hold*:
(a) $S_r(A_i) = S_r - \lambda_i S_{r-1}(A_i)$;
(b) $\mathrm{tr}(P_r) = \varepsilon_N^r \sum_{i=1}^n S_r(A_i) = \varepsilon_N^r (n-r) S_r = b_r H_r$;
(c) $\mathrm{tr}(AP_r) = \varepsilon_N^r \sum_{i=1}^n \lambda_i S_r(A_i) = \varepsilon_N^r (r+1) S_{r+1} = \varepsilon_N b_r H_{r+1}$,
where $b_r = (n-r)\binom{n}{r}$.

Associated to each Newton transformation P_r one has the second order linear differential operator $L_r : \mathscr{D}(\Sigma) \to \mathscr{D}(\Sigma)$, given by

$$L_r(f) = \mathrm{tr}(P_r \,\mathrm{Hess}\, f).$$

For a smooth $\varphi : \mathbf{R} \to \mathbf{R}$ and $h \in \mathscr{D}(\Sigma)$, it follows from the properties of the Hessian of functions that

$$L_r(\varphi \circ h) = \varphi'(h) L_r(h) + \varphi''(h) \langle P_r \nabla h, \nabla h \rangle. \tag{2.4}$$

Furthermore, we observe that

$$L_r(f) = \mathrm{tr}(P_r \text{ Hess } f) = \sum_{i=1}^{n} \langle P_r(\nabla_{e_i}\nabla f), e_i\rangle$$

$$= \sum_{i=1}^{n} \langle \nabla_{e_i}\nabla f, P_r(e_i)\rangle = \sum_{i=1}^{n} \langle \nabla_{P_r(e_i)}\nabla f), e_i\rangle = \mathrm{tr}(\text{Hess } f \circ P_r),$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame on Σ^n. Moreover, according [24], when the space ambient has constant sectional curvature we also have

$$\mathrm{div}(P_r(\nabla f)) = \sum_{i=1}^{n} \langle (\nabla_{e_i} P_r)(\nabla f), e_i\rangle + \sum_{i=1}^{n} \langle P_r(\nabla_{e_i}\nabla f), e_i\rangle$$

$$= \langle \mathrm{div}\, P_r, \nabla f\rangle + L_r(f) = L_r(f).$$

Consequently, we conclude that the operator L_r is elliptic if and only if P_r is positive definite. We observe that $L_0 = \Delta$ is always elliptic. The next lemma gives a geometric condition which guarantees the ellipticity of L_1.

Lemma 2.2 (Lemma 3.2 of [3]). *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a Riemannian immersion in a semi-Riemannian manifold $\overline{M}^{n+1}$. If $H_2 > 0$ on Σ^n, then L_1 is elliptic or, equivalently, P_1 is positive definite (for a appropriate choice of the Gauss map N).*

When $r \geq 2$, the following lemma establishes sufficient conditions to guarantee the ellipticity of L_r (for the proof see [7], Proposition 3.2).

Lemma 2.3 (Lemma 3.3 of [3]). *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a Riemannian immersion in a semi-Riemannian manifold $\overline{M}^{n+1}$. If there exists an elliptic point of Σ^n, with respect to an appropriate choice of the Gauss map N, and $H_{r+1} > 0$ on Σ^n, for $2 \leq r \leq n-1$, then for all $1 \leq k \leq r$ the operator L_k is elliptic or, equivalently, P_k is positive definite (for a appropriate choice of the Gauss map N, if k is odd).*

Here, by an elliptic point in a Riemannian immersion $\psi : \Sigma^n \to \overline{M}^{n+1}$ into a semi-Riemannian manifold $\overline{M}^{n+1}$, we mean a point $p_0 \in \Sigma^n$ where all principal curvatures $\kappa_i(p_0)$ have the same sign.

3. Semi-Riemannian warped products

In order to study semi-Riemannian warped products, we define conformal vector fields. A vector field V on $\overline{M}^{n+1}$ is said to be *conformal* if

$$\mathscr{L}_V\langle\, ,\rangle = 2\phi\langle\, ,\rangle \tag{3.1}$$

for some function $\phi \in C^\infty(\overline{M})$, where $\mathscr{L}$ stands for the Lie derivative of the metric of $\overline{M}$. The function ϕ is called the *conformal factor* of V.

Since $\mathscr{L}_V(X) = [V, X]$ for all $X \in \mathfrak{X}(\overline{M})$, it follows from the tensorial character of $\mathscr{L}_V$ that $V \in \mathfrak{X}(\overline{M})$ is conformal if and only if

$$\langle \overline{\nabla}_X V, Y \rangle + \langle X, \overline{\nabla}_Y V \rangle = 2\phi \langle X, Y \rangle, \tag{3.2}$$

for all $X, Y \in \mathfrak{X}(\overline{M})$. In particular, V is a Killing vector field relatively to $\bar{g}$ if $\phi \equiv 0$.

Let M^n be a connected, n-dimensional oriented Riemannian manifold, $I \subseteq \mathbf{R}$ an interval and $f : I \to \mathbf{R}$ a positive smooth function. In the product differentiable manifold $\overline{M}^{n+1} = I \times M^n$, let π_I and π_M denote the projections onto the I and M factors, respectively. A particular class of semi-Riemannian manifolds having conformal fields is the one obtained by furnishing $\overline{M}$ with the metric

$$\langle v, w \rangle_p = \varepsilon \langle (\pi_I)_* v, (\pi_I)_* w \rangle + f(p)^2 \langle (\pi_M)_* v, (\pi_M)_* w \rangle,$$

for all $p \in \overline{M}$ and all $v, w \in T_p \overline{M}$, where $\varepsilon = \varepsilon_{\partial_t}$ and ∂_t is the standard unit vector field tangent to I. Moreover (cf. [19] and [20]), the vector field

$$V = (f \circ \pi_I) \partial_t$$

is conformal and closed (in the sense that its dual 1-form is closed), with conformal factor $\phi = f' \circ \pi_I$, where the prime denotes differentiation with respect to $t \in I$. Such a space is a particular instance of a semi-Riemannian *warped product*, and, from now on, we shall write $\overline{M}^{n+1} = \varepsilon I \times_f M^n$ to denote it.

If $\psi : \Sigma^n \to \varepsilon I \times_f M^n$ is a Riemannian immersion, with Σ^n oriented by the unit vector field N, one obviously has $\varepsilon = \varepsilon_{\partial_t} = \varepsilon_N$.

Remark 3.1. For $t_0 \in \mathbf{R}$, we orient the *slice* $\Sigma^n_{t_0} = \{t_0\} \times M^n$ by using the unit normal vector field ∂_t. According to [6], Σ_{t_0} has constant r-th mean curvature $H_r = -\varepsilon \left(\dfrac{f'(t_0)}{f(t_0)} \right)^r$ with respect to ∂_t (see also [19] and [20]).

Now, let h denote the (vertical) height function naturally attached to Σ^n, namely, $h = (\pi_I)|_\Sigma$. Let $\overline{\nabla}$ and ∇ denote gradients with respect to the metrics of $\varepsilon I \times_f M^n$ and Σ^n, respectively. A simple computation shows that the gradient of π_I on $\varepsilon I \times_f M^n$ is given by

$$\overline{\nabla} \pi_I = \varepsilon \langle \overline{\nabla} \pi_I, \partial_t \rangle = \varepsilon \partial_t, \tag{3.3}$$

so that the gradient of h on Σ^n is

$$\nabla h = (\overline{\nabla} \pi_I)^\top = \varepsilon \partial_t^\top = \varepsilon \partial_t - \langle N, \partial_t \rangle N. \tag{3.4}$$

In particular, we get

$$|\nabla h|^2 = \varepsilon (1 - \langle N, \partial_t \rangle^2), \tag{3.5}$$

where $|\ |$ denotes the norm of a vector field on Σ^n.

In the Lorentzian setting, the following result is a particular case of one obtained by the first author jointly with L. J. Alías (cf. [3], Lemma 4.1).

Lemma 3.2. *Let* $\psi : \Sigma^n \to \varepsilon I \times_f M^n$ *be a Riemannian immersion. If* $h = (\pi_I)|_\Sigma : \Sigma^n \to I$ *is the height function of* Σ^n, *then*

$$L_r(h) = (\log f)'(\varepsilon \operatorname{tr} P_r - \langle P_r \nabla h, \nabla h \rangle) + \langle N, \partial_t \rangle \operatorname{tr}(AP_r). \tag{3.6}$$

Remark 3.3. In [11], the second author jointly with F. Camargo and A. Caminha have presented an alternative proof of the previous lemma.

4. Rigidity theorems in semi-Riemannian warped products

In this section, we will apply the results that we have discussed in the previous sections to study the rigidity of Riemannian immersions into semi-Riemannian warped products $\varepsilon I \times_f M^n$, where M^n is a complete Riemannian manifold. Initially, we will consider the Lorentzian setting.

In order to prove our rigidity results, we will use the following result due to S. T. Yau. In what follows, $\mathscr{L}^1(\Sigma)$ denotes the space of Lebesgue integrable functions on Σ^n.

Lemma 4.1 (Corollary on page 660 of [26]). *Let* Σ^n *be an n-dimensional complete Riemannian manifold. If* $g : \Sigma^n \to \mathbf{R}$ *is a smooth subharmonic function such that* $|\nabla g| \in \mathscr{L}^1(\Sigma)$, *then* g *must be actually harmonic.*

According [1], we say that a spacelike hypersurface $\psi : \Sigma^n \to -I \times_f M^n$ is *bounded away from the future infinity* of $-I \times_f M^n$ if there exists $\bar{t} \in I$ such that

$$\psi(\Sigma) \subset \{(t, x) \in -I \times_f M^n; t \le \bar{t}\}.$$

Analogously, we say that Σ^n is *bounded away from the past infinity* of $-I \times_f M^n$ if there exists $\underline{t} \in I$ such that

$$\psi(\Sigma) \subset \{(t, x) \in -I \times_f M^n; t \ge \underline{t}\}.$$

Finally, Σ^n is said to be *bounded away from the infinity* of $-I \times_f M^n$ if it is both bounded away from the past and future infinity of $-I \times_f M^n$. In other words, Σ^n is bounded away from the infinity if there exist $\underline{t} < \bar{t}$ such that $\psi(\Sigma)$ is contained in the slab bounded by the slices $\{\underline{t}\} \times M^n$ and $\{\bar{t}\} \times M^n$.

Now, we can state and prove our results. As before, h is the height function of Σ^n.

Theorem 4.2. *Let* $\overline{M}^{n+1} = -I \times_f M^n$ *be a Lorentzian warped product such that* $\log f$ *is convex. Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *be a complete, connected spacelike hypersurface bounded away from the infinity of* $\overline{M}^{n+1}$. *Suppose that the mean curvature* H *of* Σ^n *satisfies*

$$f'(h)H \ge \frac{f'^2}{f}(h) > 0. \tag{4.1}$$

If $|\nabla h| \in \mathscr{L}^1(\Sigma)$, *then* Σ^n *is a slice.*

Proof. If N is the Gauss map such that $\langle N, \partial_t \rangle < 0$, then, by applying the reverse Cauchy's inequality, we have

$$\langle N, \partial_t \rangle \leq -1. \tag{4.2}$$

From (2.4), $\Delta f(h) = f''(h)|\nabla h|^2 + f'(h)\Delta h$. Thus, with the aid of Lemmas 2.1 and 3.2, we obtain

$$\Delta f(h) = \left(\frac{f''f - f'^2}{f}(h)\right)|\nabla h|^2 - n\left(\frac{f'^2}{f}(h) + f'(h)H\langle N, \partial_t \rangle\right). \tag{4.3}$$

Thus, since $\log f$ is convex, from the hypothesis (4.1) and inequality (4.2) we get

$$\Delta f(h) \geq 0.$$

On the other hand, since $|\nabla h|$ is integrable and Σ^n is bounded away from the infinity of $-I \times_f M^n$, we get $|\nabla f(h)| = |f'(h)|\,|\nabla h|$ also integrable on Σ^n.

Consequently, $f(h)$ is a subharmonic function on Σ^n whose gradient has integrable norm. Since Σ^n is complete, it follows from Lemma 4.1 that $f(h)$ is actually harmonic.

Now, suppose by contradiction, that exists $p \in \Sigma^n$ such that $|\nabla h|(p) > 0$. Back to formula (4.3), we get

$$f''f - f'^2 = 0$$

on an open subset of I and, hence, $f(t) = \alpha e^{\beta t}$ for some nonzero constants α and β. Thus, back again to formula (4.3) and taking into account once more the hypothesis (4.1), we have

$$-\beta = H\langle N, \partial_t \rangle \leq -H \leq -\beta,$$

so that $\langle N, \partial_t \rangle \equiv -1$ and $H \equiv \beta$. So, it follows from (3.5) and the connectedness of Σ^n that it is a slice of $-I \times_{\alpha e^{\beta t}} M^n$, and we arrive at a contradiction. Therefore, $|\nabla h| \equiv 0$, i.e., Σ^n is a slice of $-I \times_f M^n$. □

Remark 4.3. An interesting special case of Theorem 4.2 is that of the $(n+1)$-dimensional *steady state space*, i.e., the warped product

$$\mathscr{H}^{n+1} = -\mathbf{R} \times_{e^t} \mathbf{R}^n,$$

which is isometric to an open subset of the de Sitter space $\mathbf{S}_1^{n+1}$. In this case, the slice Σ_{t_0} is isometric to $\mathbf{R}^n$ and is called a *hyperplane* of $\mathscr{H}^{n+1}$ (cf. [1]).

The importance of considering $\mathscr{H}^{n+1}$ comes from the fact that, in Cosmology, $\mathscr{H}^4$ is the steady state model of the universe proposed by H. Bondi and T. Gold [8], and F. Hoyle [18], when looking for a model of the universe which looks the same not only at all points and in all directions (that is, spatially isotropic and homogeneous), but also at all times (cf. [25], Section 14.8, and [17], Section 5.2).

From Theorem 4.2, we get the following

Corollary 4.4 (Theorem 1.1 of [11]). *Let* $\psi : \Sigma^n \to \mathscr{H}^{n+1}$ *be a complete, connected spacelike hypersurface bounded away from the infinity of* $\mathscr{H}^{n+1}$*, with mean curvature* $H \geq 1$*. If* $|\nabla h| \in \mathscr{L}^1(\Sigma)$*, then* Σ^n *is a hyperplane of* $\mathscr{H}^{n+1}$.

Remark 4.5. Following the ideas of A. L. Albujer and L. J. Alías [1], we can consider a natural extension of the steady state space $\mathscr{H}^{n+1}$, the so-called *steady state-type* spacetimes $-\mathbf{R} \times_{e^t} M^n$, where M^n is a connected n-dimensional Riemannian manifold. For instance, when M^n is the flat n-torus we get the de Sitter cusp as defined in [16]. We observe that, from Lemma 7 of [1], if a steady state type spacetime admits a complete spacelike hypersurface which is bounded away from the future infinity, then its Riemannian fiber M^n is necessarily complete. In this setting, we observe that Corollary 4.4 admits an extension for such ambient spaces (cf. [11], Theorem 3.1).

Now, we turn our attention to Riemannian warped products $I \times_f M^n$, where M^n is a complete, connected Riemannian manifold. In this setting, similarly to the Lorentzian case, we say that a complete hypersurface $\psi : \Sigma^n \to I \times_f M^n$ is *bounded away from the infinity* of $I \times_f M^n$ if there exist $\underline{t} < \bar{t}$ such that $\psi(\Sigma)$ is contained in the slab bounded by the slices $\Sigma_{\underline{t}}$ and $\Sigma_{\bar{t}}$.

In the Riemannian setting, the analogue of Theorem 4.2 is given by

Theorem 4.6. *Let* $\overline{M}^{n+1} = I \times_f M^n$ *de a Riemannian warped product such that* $\log f$ *is convex. Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *be a complete, connected hypersurface bounded away from the infinity of* $\overline{M}^{n+1}$*. Suppose that the mean curvature* H *of* Σ^n *satisfies*

$$\frac{f'^2}{f}(h) \geq f'(h)H > 0. \tag{4.4}$$

If $|\nabla h| \in \mathscr{L}^1(\Sigma)$*, then* Σ^n *is a slice.*

Proof. If we choose the Gauss map N of Σ in such a way that $\langle N, \partial_t \rangle < 0$, then Cauchy-Schwarz inequality gives

$$\langle N, \partial_t \rangle \geq -1. \tag{4.5}$$

Since we are supposing that the mean curvature H of Σ^n satisfies (4.4), inequality (5.5) gives $\frac{f'^2}{f}(h) + f'(h)H\langle N, \partial_t \rangle \geq 0$. Consequently, since $\log f$ is convex, from Lemma 3.2 and equation (2.4) we obtain that

$$\Delta f(h) = \left(\frac{f''f - f'^2}{f}(h) \right) |\nabla h|^2 + n\left(\frac{f'^2}{f}(h) + f'(h)H\langle N, \partial_t \rangle \right) \geq 0.$$

At this point, if we follow essentially the same arguments employed in the last part of the proof of Theorem 4.2, we conclude that Σ^n is a slice of $I \times_f M^n$. □

We observe that the $(n+1)$-dimensional hyperbolic space $\mathbf{H}^{n+1}$ is isometric to $\mathbf{R} \times_{e^t} \mathbf{R}^n$, an explicit isometry being found in [4]. It can be easily seen from such isometry that the slices $\Sigma_{t_0} = \{t_0\} \times \mathbf{R}^n$ of the warped product model of the hyperbolic space are precisely the horospheres.

As a consequence of Theorem 4.6, we obtain the following

Corollary 4.7 (Theorem 1.2 of [11]). *Let $\psi : \Sigma^n \to \mathbf{H}^{n+1}$ be a complete, connected hypersurface bounded away from the infinity of $\mathbf{H}^{n+1}$, with mean curvature $0 < H \leq 1$. If $|\nabla h| \in \mathscr{L}^1(\Sigma)$, then Σ^n is a horosphere of $\mathbf{H}^{n+1}$.*

Remark 4.8. As in Remark 4.5, we observe that Corollary 4.7 can be extended to the context of the so-called *hyperbolic-type* spaces $\mathbf{R} \times_{e^t} M^n$, where M^n is a complete, connected n-dimensional Riemannian manifold (cf. [11], Theorem 3.4).

Remark 4.9. Related with Theorems 4.2 and 4.6 we note that, when the Riemannian immersion $\psi : \Sigma^n \to \overline{M}^{n+1}$ is closed, $\psi(\Sigma)$ is naturally bounded away from the infinity of $\overline{M}^{n+1}$ and the hypothesis that $|\nabla h| \in \mathscr{L}^1(\Sigma)$ is immediately satisfied. In this case the proofs of such theorems follow, in fact, from the Hopf's maximum principle.

5. Extensions to the r-th mean curvatures

In order to obtain extensions of the theorems of the previous section to the case of the r-th mean curvatures, we will use the following extension of Lemma 4.1 to the context of the L_r operators due to F. Camargo, A. Caminha and P. Sousa.

Lemma 5.1 (Corollary 1 of [14]). *Let $\overline{M}^{n+1}$ have constant sectional curvauture, and $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete Riemannian immersion with bounded second fundamental form. Let also $g : \Sigma^n \to \mathbf{R}$ be a smooth function such that $|\nabla g| \in \mathscr{L}^1(\Sigma)$. If $L_r g$ does not change sign on Σ^n, then $L_r g = 0$ on Σ^n.*

Remark 5.2. From Proposition 7.42 of [21], the condition of that a semi-Riemannian warped product $\varepsilon I \times_f M^n$ to have constant sectional curvature implies that the sectional curvatures of its fibre M^n must be nonnegative.

We will also need a sufficient condition to guarantee the existence of an elliptic point in our Riemannian immersions. In what follows, we quote the semi-Riemannian version of Lemma 5.4 of [2] due to the first author jointly with L. J. Alías and A. Brasil Jr.

Lemma 5.3. *Let $\overline{M}^{n+1} = \varepsilon I \times_f M^n$ be a semi-Riemannian warped product, and $\psi : \Sigma^n \to \overline{M}^{n+1}$ a Riemannian immersion. If $-\varepsilon f(h)$ attains a local minimum at some $p \in \Sigma^n$, such that $f'(h(p)) \neq 0$, then p is an elliptic point for Σ^n.*

Now, we are in condition to state and proof our next results.

Theorem 5.4. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a Lorentzian warped product with constant sectional curvature and such that* $\log f$ *is convex. Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *be a complete, connected spacelike hypersurface bounded away from the infinity of* $\overline{M}^{n+1}$. *Suppose that the mean curvature* H *is bounded and that, for some* $1 \le r \le n-1$, H_r *and* H_{r+1} *are positive and such that*

$$\frac{H_{r+1}}{H_r} \ge \frac{f'}{f}(h) > 0. \tag{5.1}$$

If h *has a local minimum on* Σ^n *and* $|\nabla h| \in \mathscr{L}^1(\Sigma)$, *then* Σ^n *is a slice.*

Proof. As in the proof of Theorem 4.2, if N is the Gauss map such that $\langle N, \partial_t \rangle < 0$, then, by applying the reverse Cauchy's inequality, we have

$$\langle N, \partial_t \rangle \le -1. \tag{5.2}$$

From (2.4), $L_r f(h) = f''(h)|\nabla h|^2 + f'(h)L_r h$. Thus, with the aid of Lemmas 2.1 and 3.2, we obtain

$$L_r f(h) = \left(\frac{f''f - f'^2}{f}(h)\right)\langle P_r \nabla h, \nabla h\rangle - b_r f'(h) H_r \left(\frac{f'}{f}(h) + \frac{H_{r+1}}{H_r}\langle N, \partial_t\rangle\right), \tag{5.3}$$

where $b_r = (n-r)\binom{n}{r}$.

On the other hand, by using Lemmas 5.3 and 2.3, we guarantee that P_r is positive definite. Thus, since $\log f$ is convex, from hypothesis (5.1) and inequality (5.2), we get

$$L_r f(h) \ge 0.$$

On the other hand, since $|\nabla h|$ is integrable and Σ^n is bounded away from the infinity of $-I \times_f M^n$, we get $|\nabla f(h)| = |f'(h)|\,|\nabla h|$ also integrable on Σ^n. Moreover, since Lemma 2.3 also guarantees that $H_2 > 0$, from equation (2.2) we see that the boundedness of H implies that Σ^n has bounded second fundamental form. Consequently, from Lemma 5.1, we have that $L_r f(h) = 0$ on Σ^n.

Now, if we follow the same steps of last part of the proof of Theorem 4.2, we conclude that Σ^n is a slice of $-I \times_f M^n$. □

Remark 5.5. Taking into account Lemma 2.2, from the proof of Theorem 5.4 we easily see that in the case $r = 1$ it is not necessary to suppose that the height function of the Riemannian immersion has a local minimum on it.

From the proof of Theorem 5.4, we obtain the following results:

Corollary 5.6 (Theorem 3.6 of [11]). *Let* M^n *be a Riemmanian space form of zero sectional curvature and* $\psi : \Sigma^n \to -\mathbf{R} \times_{e^t} M^n$ *be a complete, connected* *spacelike hypersurface with bounded second fundamental form and bounded away from the infinity of* $-\mathbf{R} \times_{e^t} M^n$. *If* $|\nabla h| \in \mathscr{L}^1(\Sigma)$ *and* $0 < H_r \leq H_{r+1}$ *on* Σ^n, *then* Σ^n *is a slice of* $-\mathbf{R} \times_{e^t} M^n$.

From Theorem 5.4 and equation (2.1), we obtain a sort of extension of the Theorem 1.1 of [12] and of Theorem 3.1 of [13]. In what follows, the height function of the hypersurface is defined with respect to the unit vector field which is normal to the foliation of $\mathscr{H}^{n+1}$ by hyperplanes.

Corollary 5.7. *Let* Σ^n *be a complete, connected spacelike hypersurface, which lies lies between two hyperplanes of* $\mathscr{H}^{n+1}$. *Suppose that the mean curvature* H *is positive and bounded and that the scalar curvature* R *satisfies* $R + H \leq 1$. *If the gradient of the height function has integrable norm on* Σ^n, *then* Σ^n *is a hyperplane of* $\mathscr{H}^{n+1}$.

The Riemannian version of Theorem 5.4 is given below.

Theorem 5.8. *Let* $\overline{M}^{n+1} = I \times_f M^n$ *de a Riemannian warped product with constant sectional curvature and such that* $\log f$ *is convex. Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *be a complete, connected hypersurface bounded away from the infinity of* $\overline{M}^{n+1}$. *Suppose that the mean curvature* H *is bounded and that, for some* $1 \leq r \leq n-1$, H_r *and* H_{r+1} *are positive and such that*

$$\frac{f'}{f}(h) \geq \frac{H_{r+1}}{H_r} > 0. \tag{5.4}$$

If h *has a local maximum on* Σ^n *and* $|\nabla h| \in \mathscr{L}^1(\Sigma)$, *then* Σ^n *is a slice.*

Proof. As in the proof of Theorem 4.6, we choose the Gauss map N of Σ in such a way that $\langle N, \partial_t \rangle < 0$, then Cauchy-Schwarz inequality gives

$$\langle N, \partial_t \rangle \geq -1. \tag{5.5}$$

From hypothesis (5.4) and inequality (5.5), we have that

$$\frac{f'}{f}(h) + \frac{H_{r+1}}{H_r}\langle N, \partial_t \rangle \geq 0.$$

On the other hand, by using Lemmas 5.3 and 2.3, we guarantee that P_r is positive definite. Consequently, since $\log f$ is convex, from Lemma 3.2 and equation (2.4) we obtain that

$$L_r f(h) = \left(\frac{f''f - f'^2}{f}(h)\right)\langle P_r \nabla h, \nabla h \rangle + b_r f'(h) H_r \left(\frac{f'}{f}(h) + \frac{H_{r+1}}{H_r}\langle N, \partial_t \rangle\right) \geq 0,$$

where $b_r = (n-r)\binom{n}{r}$.

Therefore, by following the same arguments employed in the last part of the proof of Theorem 5.4, we conclude that Σ^n is a slice of $I \times_f M^n$. □

Remark 5.9. As we have observed in Remark 5.5, taking into account once more Lemma 2.2, from the proof of Theorem 5.8 we see that in the case $r = 1$ it is not necessary to suppose that the height function of the Riemannian immersion has a local maximum on it.

As in the Lorentzian setting, from the proof of Theorem 5.8 we obtain the following

Corollary 5.10 (Theorem 3.7 of [11]). *Let M^n be a Riemmanian space form of zero sectional curvature and $\psi : \Sigma^n \to \mathbf{R} \times_{e^t} M^n$ be a complete, connected hypersurface with bounded second fundamental form and bounded away from the infinity of $\mathbf{R} \times_{e^t} M^n$. If $|\nabla h| \in \mathscr{L}^1(\Sigma)$ and $H_r \geq H_{r+1} > 0$ on Σ^n, then Σ^n is a slice of $\mathbf{R} \times_{e^t} M^n$.*

From Theorem 5.8 and taking into account once more equation (2.1), we also get the following extension of the classical theorem due to S. Y. Cheng and S. T. Yau in [15]. Here, analogously to Corollary 5.7, the height function of the hypersurface is defined with respect to the unit vector field which is normal to the foliation of $\mathbf{H}^{n+1}$ by horospheres.

Corollary 5.11. *Let Σ^n be a complete, connected hypersurface, which lies lies between two horospheres of $\mathbf{H}^{n+1}$. Suppose that the mean curvature H is positive and bounded and that the scalar curvature R satisfies $0 < R + 1 \leq H$. If the gradient of the height function has integrable norm on Σ^n, then Σ^n is a horosphere of $\mathbf{H}^{n+1}$.*

Remark 5.12. In the setting of Theorems 5.4 and 5.8, if the Riemannian immersion $\psi : \Sigma^n \to \overline{M}^{n+1}$ is closed, since from Lemma 2.3 L_r is an elliptic operator, the result follows in fact from the Hopf's maximum principle.

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