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On Riemannian manifolds foliated by $(n-1)$-umbilical hypersurfaces

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Abstract. In this paper we define closed partially conformal vector fields and use them to give a characterization of Riemannian manifolds which admit this kind of fields as some special warped products foliated by $(n-1)$-umbilical hypersurfaces. Examples are described in space forms. In particular, closed partially conformal vector fields in Euclidean spaces are associated to the most simple foliations given by hyperspheres, hyperplanes or coaxial cylinders. Finally, for manifolds admitting such vector fields, we impose conditions for a hypersurface to be $(n-1)$-umbilical, or, in particular, a leaf of the corresponding foliation.

Keywords: vector fields, foliations, $(n-1)$-umbilical hypersurfaces

Mathematical subject classification: 53C12, 53A10.

Introduction

Riemannian and Lorentzian manifolds which admit a closed conformal vector field have been studied largely in the past few years (see [12], [13] and [1], for example). This condition is associated to manifolds which can be expressed as a warped product and to the existence of a foliation by totally umbilical hypersurfaces with constant mean curvature.

On the other hand, it is well-known that already in Riemannian space forms there are many hypersurfaces with constant mean curvature which are not totally umbilical. For example, in [15] and [5] the authors constructed rotational hypersurfaces in space forms having constant r-th curvature and being $(n-1)$-umbilical; that is, with $n-1$ equal principal curvatures. See also [10] and [11].

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These $(n-1)$-umbilical hpersurfaces were studied in [8] in the compact case. They belong to the broader class of k-umbilical submanifolds described in [4] and [6] as envelopes of spheres.

The above motivates the following question:

Given a Riemannian manifold $\overline{M}^{n+1}$, is there a vector field in $\overline{M}^{n+1}$ which determines a foliation by $(n-1)$-umbilical hypersurfaces?

Also, a natural question is the following:

In such a foliated manifold, which conditions guarantee that a hypersurface with constant mean curvature is $(n-1)$-umbilical or, in particular, a leaf of the foliation?

Here we analyze these two problems. After the Preliminaries, in Section 2 we define the following key notion: a vector field K defined on a Riemannian manifold $(\overline{M}^{n+1}, \overline{\nabla})$ is closed partially conformal if there is a unit vector field $W \in \mathfrak{X}(\overline{M})$ everywhere orthogonal to K and functions $\phi, \psi : \overline{M} \to \mathbb{R}$ such that $\overline{\nabla}_X K = \phi X$ for $\langle X, W\rangle = 0$ and $\overline{\nabla}_W K = \psi W$ or, equivalently,

$$\overline{\nabla}_X K = \phi X + (\psi - \phi)\langle X, W\rangle W$$

for each $X \in \mathfrak{X}(\overline{M})$. We then prove that this is the right tool to answer our first question as follows (see Theorem 2.4):

Let $\overline{M}^{n+1}$ be a Riemannian manifold possessing a closed partially conformal vector field K. Then the distribution $K^\perp$ defined away from the zero set of K is involutive and each leaf of the foliation $\mathcal{K}^\perp$ is a $(n-1)$-umbilical hypersurface with $n-1$ equal and constant principal curvatures.

In this result, $\mathcal{K}^\perp$ is the foliation associated to the distribution defined by taking the orthogonal complement $K^\perp$ of K.

Among some other basic facts, we prove conditions under which the leaves of $\mathcal{K}^\perp$ have constant mean curvature. (See Proposition 2.5.)

In Section 3 we show the existence of closed partially conformal vector fields in open subsets of the space forms $\mathbb{Q}_c^{n+1}$ with constant curvature c and note that these open subsets may be written as a warped product of the form $J \times (I \times_f P^{n-1})$. The relation between closed partially conformal vector fields and warped products is reinforced by proving that a manifold admitting a closed partially conformal vector field (with an additional condition) must have this product structure. More precisely, we prove the following result (see Theorem 3.3):

Let $\overline{M}^{n+1}$ be a Riemannian manifold.

1. *If* $\overline{M} = J \times (I \times_f P^{n-1})$*, then* $\overline{M}$ *admits a closed partially conformal vector field.*

2. *If* $\overline{M}$ *admits a closed partially conformal vector field* K *and the associated vector field* W *is closed conformal, then locally* $\overline{M}$ *is isometric to* $J \times (I \times_f P^{n-1})$.

As a consequence, when the ambient space is a space form, we give a description of the foliations given by partially conformal vector fields, as follows (see Corollary 3.4 and Definition 3.2):

Let K *be a closed partially conformal vector field defined in* $\mathbb{Q}_c^{n+1}$*. Suppose additionally that* W *is closed conformal. Then the associated foliation of* $\mathbb{Q}_c^{n+1}$ *is a foliation by hyperplanes, hyperspheres or tubes.*

It is worth noting that in the context of conformally flat immersions, in [8] is given a complete description of the compact $(n-1)$-umbilical hypersurfaces in space forms.

In Section 4 we return to the Euclidean space $\mathbb{R}^{n+1}$ and give a description of the foliations whose leaves have constant mean curvature and are associated to closed partially conformal vector fields, as follows (see Theorem 4.3):

Let K *be a closed partially conformal vector field in* $\mathbb{R}^{n+1}$*, whose leaves are complete and have constant mean curvature. Then the foliation* $\mathcal{K}^{\perp}$ *is a foliation by hyperplanes, by hyperspheres or by coaxial cylinders.*

In Section 5 we answer the first part of our second question: to establish conditions for an immersed hypersurface to be $(n-1)$-umbilical, as follows (see Theorem 5.3):

Let $\overline{M}^{n+1}$ *be a Riemannian manifold with non-negative Ricci curvature which admits a closed partially conformal vector field* K *and an associated vector field* W*. Let* M *be an orientable hypersurface of* $\overline{M}$ *everywhere transverse to* K*, and* N *be a unit vector field normal to* M*. Suppose that the direction determined by* $W^* = W - \langle W, N\rangle N$ *is a principal direction of* M *and that through each point of* M *passes a compact* $(n-1)$*-dimensional submanifold of* M*, everywhere orthogonal to* W^**, totally umbilical as a hypersurface in* M *and having constant mean curvature relative to* N*. Then* M *is* $(n-1)$*-umbilical.*

Finally, in Section 6 we answer the last part of our second question, namely, we give conditions for a hypersurface with constant mean curvature to be a leaf of the foliation determined by a closed partially conformal vector field, thus being $(n-1)$-umbilical.

Let $\overline{M}^{n+1}$ be a Riemannian manifold which admits a closed partially conformal vector field K and an associated closed conformal vector field W, so that by Theorem 3.3, $\overline{M}$ may be expressed locally as $J \times (I \times_f P^{n-1})$. We denote $\overline{M}_t = \{t\} \times (I \times_f P^{n-1})$. Similarly, if M is any hypersuface in $\overline{M}$, let $M_t = M \cap \overline{M}_t$. We will suppose additionally that the logarithm of the warping function f is convex. Our result reads as follows (see Theorem 6.2):

Let $\overline{M}^{n+1}$ be a Riemannian manifold which admits a closed partially conformal vector field K and an associated closed conformal vector field W, such that $\overline{M}$ is given locally by $J \times (I \times_f P^{n-1})$ with $\log f$ convex. Let M be an orientable hypersurface of $\overline{M}$, everywhere transverse to K, with constant mean curvature in $\overline{M}$.

Suppose there exists $t \in J$ such that M_t is a compact hypersurface of $\overline{M}_t$ with constant mean curvature. Suppose additionally the existence of a point $p \in M_t$ such that

1. *The unit vector $N(p)$ normal to M at p is equal to the unit vector $\widehat{K}(p)$ normal to the leaf of $\mathcal{K}^\perp$ passing through p;*

2. *Locally, M lies above the leaf of $\mathcal{K}^\perp$ passing through p with respect to K; that is, there is a neighborhood U of p in M such that each point $q \in U$ has the form $q = \phi_s(q')$, where q' is in the mentioned leaf, $s \geq 0$ and ϕ_s is the flow of K;*

3. *The derivative of $\langle N, W\rangle$ with respect to the vector $W(p)$ is positive.*

Then M coincides locally with the leaf of $\mathcal{K}^\perp$ passing through p. In particular, locally M is $(n-1)$-umbilical. Moreover, if the leaf of $\mathcal{K}^\perp$ passing through p has constant mean curvature, it coincides globally with M.

1 Preliminaries

We will denote by $\overline{M}^{n+1}$ a Riemannian manifold with metric $\langle\ ,\ \rangle$, connection $\overline{\nabla}$ and curvature tensor $\overline{R}$. Also, $\mathfrak{X}(\overline{M})$ will stand for the module of vector fields defined in $\overline{M}$. For any submanifold of $\overline{M}$, its induced Riemannian connection will be symbolized by ∇. All manifolds are supposed to be connected, including the leaves associated to some foliations to be defined later.

For each $K \in \mathfrak{X}(\overline{M})$ we denote by $K^\perp$ the n-dimensional distribution defined at each point by taking the orthogonal complement to K. In this context, if a distribution is involutive, we will denote the associated foliation by a similar notation with calligraphic style; for example, if the distribution $K^\perp$ is involutive, the corresponding foliation is denoted by $\mathcal{K}^\perp$.

Also, we introduce the notion of a k-umbilical hypersurface as follows.

Definition 1.1. Let $\overline{M}^{n+1}$ be a Riemannian manifold. A hypersurface M is k-umbilical if there is a unit vector field $\widehat{K}$ normal to M, a k-dimensional distribution $\mathcal{D} \subset TM$, as well as a C^∞ function ϕ such that

$$\mathcal{D} = \left\{X \in TM \mid \overline{\nabla}_X \widehat{K} = \phi X\right\}. \tag{1}$$

This concept was analyzed in detail in [4] and [6] (see also [16]). For example, it is known that every k-umbilical hypersurface must be an envelope of a family of k-spheres. Also, it is worth noting that in the context of conformally flat immersions, in [8] is given a complete description of the compact $(n-1)$-umbilical hypersurfaces in space forms.

We will denote by $\mathbb{Q}_c^{n+1}$ the complete, simply connected Riemannian manifold of constant curvature c. That is, for $c = 0$ we have the Euclidean space $\mathbb{R}^{n+1}$, for $c > 0$ we use the sphere $\mathbb{S}_c^{n+1}$, and for $c < 0$ we get the hyperbolic space $\mathbb{H}_c^{n+1}$.

2 Basic facts

We define here the vector fields which will prove to be adequate for our purposes.

Definition 2.1. Let $\overline{M}^{n+1}$ be a Riemannian manifold and $K \in \mathfrak{X}(\overline{M})$. We say that K is *closed, partially conformal* in $\overline{M}$ if there is a unit vector field $W \in \mathfrak{X}(\overline{M})$ everywhere orthogonal to K and functions $\phi, \psi : \overline{M} \to \mathbb{R}$ such that $\overline{\nabla}_X K = \phi X$ for $\langle X, W\rangle = 0$ and $\overline{\nabla}_W K = \psi W$ or, equivalently,

$$\overline{\nabla}_X K = \phi X + (\psi - \phi)\langle X, W\rangle W \tag{2}$$

for each $X \in \mathfrak{X}(\overline{M})$. W is called the vector field *associated* to K.

This notion is intimately related with that of closed conformal vector fields (those satisfying $\phi = \psi$) analyzed in detail in [12]. In that paper, Montiel proved that the set of zeroes of a non-null closed conformal vector field is discrete, so in the compact setting this set is finite. As it turned out by analyzing some examples (see Section 3), the set $\mathcal{Z}(K)$ of zeroes of closed partially conformal vector fields could be extremely large, so we will assume hereafter that $\mathcal{Z}(K)$ can be at most a discrete subset or a union of curve segments.

Away from $\mathcal{Z}(K)$ we may define the unit vector field $\widehat{K} = K/|K|$. It is easy to see from the definition that

$$\begin{aligned} \overline{\nabla}_W \widehat{K} &= \frac{\psi}{|K|} W, \\ \overline{\nabla}_X \widehat{K} &= \frac{\phi}{|K|} X, \quad \text{if } \langle X, K \rangle = \langle X, W \rangle = 0, \\ \overline{\nabla}_{\widehat{K}} \widehat{K} &= 0, \end{aligned} \tag{3}$$

so that $\widehat{K}$ defines a unit speed geodesic flow. We can summarize (3) with the following expression:

$$\overline{\nabla}_X \widehat{K} = \frac{1}{|K|}\big(\phi X + (\psi - \phi)\langle X, W \rangle W\big) - \phi \frac{\langle K, X \rangle}{|K|^3} K.$$

Note that ψ is related to the normal curvature κ of the integral curves of W, since

$$\kappa = \langle \overline{\nabla}_W W, \widehat{K} \rangle = -\langle W, \overline{\nabla}_W \widehat{K} \rangle = -\frac{\psi}{|K|}.$$

A closed partially conformal vector field K and its associated vector field W give rise to three important distributions, namely $K^\perp$, $W^\perp$ and $K^\perp \cap W^\perp$. Along this section we collect some basic facts about these distributions, which may be well-known, but we included them here for completeness.

Proposition 2.2. *$K^\perp$ is an involutive distribution. Moreover, each leaf of the foliation determined by $K^\perp$ in $\overline{M} \setminus \mathcal{Z}(K)$ is $(n-1)$-umbilical (see Definition* 1.1*).*

Proof. Let X, Y be vector fields in $K^\perp$. Using (2), we have

$$\begin{aligned} \langle [X, Y], K \rangle &= \langle \overline{\nabla}_X Y - \overline{\nabla}_Y X, K \rangle = -\langle Y, \overline{\nabla}_X K \rangle + \langle X, \overline{\nabla}_Y K \rangle \\ &= -\langle Y, \phi X + (\psi - \phi)\langle X, W \rangle W \rangle + \langle X, \phi Y + (\psi - \phi)\langle Y, W \rangle W \rangle = 0, \end{aligned}$$

which means that $[X, Y] \in K^\perp$. Therefore, $K^\perp$ is involutive. Denote by $\mathcal{K}^\perp$ the corresponding foliation defined in $\overline{M} \setminus \mathcal{Z}(K)$.

Now, fix a leaf of $\mathcal{K}^\perp$ in $\overline{M} \setminus \mathcal{Z}(K)$, so that the vector field $\widehat{K}$ restricted to the leaf is its unit normal. By (3), we have

$$\overline{\nabla}_X \widehat{K} = \frac{\phi}{|K|} X \quad \text{for } X \in K^\perp \cap W^\perp,$$

then the $(n-1)$-dimensional distribution $K^\perp \cap W^\perp$ satisfies (1) and the leaf is $(n-1)$-umbilical, as desired. □

Thus, each leaf of $\mathcal{K}^\perp$ is $(n-1)$-umbilical and at each point has $n-1$ equal principal curvatures given in (3). We will prove in Theorem 2.4 that in fact these curvatures are constant along each leaf, for which we present the crucial step of the proof as the following auxiliary result.

Lemma 2.3. *The functions $|K|^2$, ϕ and $K\phi$ are constant along each leaf of $\mathcal{K}^\perp$.*

Proof. Note first that for each $X \in \mathfrak{X}(\overline{M})$ we have

$$\langle \operatorname{grad} |K|^2, X\rangle = X(|K|^2) = 2\langle K, \phi X + (\psi - \phi)\langle X, W\rangle W\rangle = \langle 2\phi K, X\rangle,$$

which implies

$$\operatorname{grad} |K|^2 = 2\phi K.$$

In particular, $X(|K|^2) = 0$ for each vector field $X \in K^\perp$, so that $|K|^2$ is constant along each leaf of $\mathcal{K}^\perp$.

The Hessian of $|K|^2$ is given by (see [14], p. 86, for example):

$$\begin{aligned} \operatorname{Hess} |K|^2(X, Y) &= \langle \overline{\nabla}_X(\operatorname{grad} |K|^2), Y\rangle \\ &= \langle \overline{\nabla}_X(2\phi K), Y\rangle = 2\langle (X\phi)K + \phi\overline{\nabla}_X K, Y\rangle. \end{aligned}$$

Using (2), we have

$$\operatorname{Hess} |K|^2(X, Y) = 2\langle (X\phi)K + \phi(\phi X + (\psi - \phi)\langle X, W\rangle W), Y\rangle.$$

Now we substitute $Y = K$, $X \in K^\perp$ and the above expression in

$$\operatorname{Hess} |K|^2(X, K) = \operatorname{Hess} |K|^2(K, X)$$

to obtain easily that $(X\phi)\langle K, K\rangle = 0$. Since we are working away from $\mathcal{Z}(K)$, we have $X\phi = 0$ and then $\langle \operatorname{grad} \phi, X\rangle = 0$ for each vector field $X \in K^\perp$. On the other hand, we have

$$\langle \operatorname{grad} \phi, \widehat{K}\rangle = \widehat{K}\phi = \frac{K\phi}{|K|},$$

which implies that $\operatorname{grad} \phi$ is given by

$$\operatorname{grad} \phi = (\widehat{K}\phi)\widehat{K} = \frac{K\phi}{|K|^2}K.$$

Again, let $X \in K^\perp$. Then

$$0 = K\langle \operatorname{grad} \phi, X\rangle = (\operatorname{Hess} \phi)(K, X) + \langle \operatorname{grad} \phi, \overline{\nabla}_K X\rangle.$$

The second term in the right hand side vanishes, since

$$\langle K, \overline{\nabla}_K X\rangle = -\langle \overline{\nabla}_K K, X\rangle = -\phi\langle K, X\rangle = 0.$$

Hence $(\operatorname{Hess}\phi)(K, X) = 0$. This fact implies

$$\begin{aligned} X(K\phi) &= X\langle \operatorname{grad}\phi, K\rangle \\ &= (\operatorname{Hess}\phi)(K, X) + \langle \operatorname{grad}\phi, \overline{\nabla}_X K\rangle = \langle \operatorname{grad}\phi, \overline{\nabla}_X K\rangle, \end{aligned}$$

but again this last term vanishes by the fact that

$$\langle K, \overline{\nabla}_X K\rangle = \frac{1}{2}X(|K|^2) = 0.$$

In consequence, $X(K\phi) = 0$ for each $X \in K^\perp$ and $K\phi$ is constant along each leaf of $\mathcal{K}^\perp$. □

Theorem 2.4. *Let $\overline{M}^{n+1}$ be a Riemannian manifold possessing a closed partially conformal vector field K. Then the distribution $K^\perp$ defined in the set $\overline{M} \setminus \mathcal{Z}(K)$ is involutive and each leaf of the foliation $\mathcal{K}^\perp$ is a $(n-1)$-umbilical hypersurface with $n-1$ equal and constant principal curvatures.*

Proof. By Proposition 2.2, $K^\perp$ is involutive and each leaf of the foliation $\mathcal{K}^\perp$ defined in $\overline{M} \setminus \mathcal{Z}(K)$ is a $(n-1)$-umbilical hypersurface with $n-1$ equal principal curvatures.

Let W be the vector field associated to K and let us use the frame $E_1, \ldots, E_{n-1}, E_n = W$ and $E_{n+1} = \widehat{K}$, where $E_1, \ldots, E_n$ correspond to the principal directions on the leaf of $\mathcal{K}^\perp$. The principal curvatures of this leaf are given by

$$\kappa_i = -\langle \overline{\nabla}_{E_i}\widehat{K}, E_i\rangle = -\frac{\phi}{|K|}, \quad \kappa_n = -\langle \overline{\nabla}_{E_n}\widehat{K}, E_n\rangle = -\frac{\psi}{|K|}, \tag{4}$$

where $i = 1, \ldots, n-1$. By Lemma 2.3, ϕ and $|K|$ are constant along each leaf of $\mathcal{K}^\perp$. Hence the first $n-1$ principal curvatures are constant along each leaf. □

One may ask whether the function ψ in Definition 2.1 is constant along each leaf of $\mathcal{K}^\perp$. We answer this question in the following proposition.

Proposition 2.5. *Let $\overline{M}^{n+1}$ be a Riemannian manifold possessing a closed partially conformal vector field K with associated vector field W. Given a leaf of $\mathcal{K}^\perp$, if any of the following functions is constant along this leaf, then the same happens with the other three:*

1. *The function* ψ *given in the Definition* 2.1*;*
2. *The principal curvature* $-\psi/|K|$*;*
3. *The divergence of* K*;*
4. *The mean curvature* H *of the leaf.*

Proof. We use the frame $E_1, \ldots, E_{n-1}, E_n = W$ and $E_{n+1} = \widehat{K}$ given in the proof of Theorem 2.4 and the properties of K to obtain the expression of the divergence of K:

$$\operatorname{div} K = \sum_{i=1}^{n+1} \langle \overline{\nabla}_{E_i} K, E_i \rangle = n\phi + \psi.$$

On the other hand, using the expressions of the principal curvatures given in (4) we have that the mean curvature H of M is

$$nH = -(n-1)\frac{\phi}{|K|} - \frac{\psi}{|K|}. \tag{5}$$

By Lemma 2.3, ϕ and $|K|$ are constant along M, so these formulas prove the proposition. □

Now we turn to the analysis of the distributions $W^\perp$ and $K^\perp \cap W^\perp$. The following proposition imposes a condition on W similar to that of K for $W^\perp$ to be involutive.

Proposition 2.6. *Let* $\overline{M}^{n+1}$ *be a Riemannian manifold possessing a closed partially conformal vector field* K *with associated vector field* W*. The distribution* $W^\perp$ *is involutive if there exists a function* $\sigma : \overline{M} \to \mathbb{R}$ *such that*

$$\overline{\nabla}_X W = \sigma X \qquad \text{for } X \in W^\perp. \tag{6}$$

Proof. Suppose that (6) holds. If $X, Y \in W^\perp$, then

$$\langle [X, Y], W \rangle = \langle \overline{\nabla}_X Y - \overline{\nabla}_Y X, W \rangle = -\langle X, \overline{\nabla}_Y W \rangle + \langle Y, \overline{\nabla}_X W \rangle = 0,$$

and so $W^\perp$ is involutive. □

Corollary 2.7. *Let* $\overline{M}^{n+1}$ *be a Riemannian manifold possessing a closed partially conformal vector field* K *with associated vector field* W*. If* W *satisfies* (6)*, then the distribution defined by* $K^\perp \cap W^\perp$ *is involutive.*

Remark 2.8. We will suppose later that condition (6) will hold for every vector field $X \in \mathfrak{X}(\overline{M})$; in fact, we will suppose that W is closed conformal. It can be easily seen that W is closed conformal and has constant norm if and only if W is parallel (thus, $\sigma = 0$ in (6)). This in turn implies that W is a Killing vector field.

Suppose K is a partially conformal vector field and that the associated vector field W satisfies (6). Theorem 2.4, Proposition 2.6 and Corollary 2.7 imply respectively that $K^\perp$, $W^\perp$ and $K^\perp \cap W^\perp$ define corresponding foliations on $\overline{M}^{n+1}$, which will be denoted by calligraphic letters $\mathcal{K}^\perp$, $\mathcal{W}^\perp$ and $\mathcal{K}^\perp \cap \mathcal{W}^\perp$, respectively.

To close this section, we analyze the distribution $K^\perp \cap W^\perp$ in an ambient space with constant curvature. It is already known that in this case this distribution is involutive (see [16]), but we will include the proof for completeness. Also, we give a geometric condition for the constancy of the function ψ given in Definition 2.1 along a leaf of the foliation $\mathcal{K}^\perp \cap \mathcal{W}^\perp$.

Proposition 2.9. *Let K be a closed partially conformal vector field with associated vector field W, defined in an open set of a Riemannian manifold $\overline{M}^{n+1}$ of constant curvature. Then the distribution $K^\perp \cap W^\perp$ is involutive.*

Moreover, the function ψ given in Definition 2.1 is constant along a leaf of the foliation $\mathcal{K}^\perp \cap \mathcal{W}^\perp$ if and only if for each point p of this leaf, either p is an umbilical point of the leaf of $\mathcal{K}^\perp$ passing through p or the integral curve of W passing through p is a geodesic of the leaf of $\mathcal{K}^\perp$ at p.

Proof. Consider the leaf of $\mathcal{K}^\perp$ passing through p. Thus, the $(n-1)$-dimensional distribution $K^\perp \cap W^\perp$ is defined by those vector fields X tangent to the leaf such that

$$\overline{\nabla}_X K = \phi X. \tag{7}$$

To prove that $K^\perp \cap W^\perp$ is involutive, take X, Y tangent to the leaf satisfying (7). Thus we have

$$\begin{aligned} \overline{\nabla}_{[X,Y]} K &= \overline{\nabla}_X \overline{\nabla}_Y K - \overline{\nabla}_Y \overline{\nabla}_X K + \overline{R}(X,Y)K \\ &= \overline{\nabla}_X(\phi Y) - \overline{\nabla}_Y(\phi X) + c(\langle K, X\rangle Y - \langle K, Y\rangle X) \\ &= \phi[X,Y] + (X\phi)Y - (Y\phi)X = \phi[X,Y]. \end{aligned}$$

Here, $\overline{R}$ denotes the curvature tensor and c is the (constant) curvature of $\overline{M}^{n+1}$. In the last equality we used the fact that ϕ is constant along the leaf of $\mathcal{K}^\perp$. Then $[X, Y]$ satisfies (7) and $K^\perp \cap W^\perp$ is involutive.

Take $u_1, \ldots, u_{n-1}, t$ coordinates on the leaf of $\mathcal{K}^\perp$ passing through p such that the vector fields $\partial/\partial u_i = E_i \in K^\perp \cap W^\perp$ and $\partial/\partial t$ is a multiple of W say, $\partial/\partial t = \lambda W$, $\lambda \neq 0$. Note that $[\partial/\partial t, E_i] = 0$ for $i = 1, \ldots, n-1$; from this it is easy to check that $[W, E_i]$ is a multiple of W, namely,

$$[W, E_i] = \frac{E_i(\lambda)}{\lambda} W.$$

Substituting this expression in (2), we obtain

$$\overline{\nabla}_{[W,E_i]} K = \psi [W, E_i].$$

On the other hand, we have

$$\overline{\nabla}_W \overline{\nabla}_{E_i} K = (W\phi) E_i + \phi \overline{\nabla}_W E_i = \phi \overline{\nabla}_W E_i$$

and

$$\overline{\nabla}_{E_i} \overline{\nabla}_W K = (E_i \psi) W + \psi \overline{\nabla}_{E_i} W.$$

Thus, we obtain the following expression for the curvature tensor:

$$\begin{aligned} \overline{R}(W, E_i)K &= -\overline{\nabla}_W \overline{\nabla}_{E_i} K + \overline{\nabla}_{E_i} \overline{\nabla}_W K + \overline{\nabla}_{[W,E_i]} K \\ &= -\phi \overline{\nabla}_W E_i + (E_i \psi) W + \psi \overline{\nabla}_{E_i} W + \psi [W, E_i] \\ &= (\psi - \phi) \overline{\nabla}_W E_i + (E_i \psi) W. \end{aligned}$$

Using that the ambient space has constant curvature c we have that

$$\overline{R}(W, E_i)K = c\,(\langle K, W\rangle E_i - \langle K, E_i\rangle W) = 0,$$

so that

$$(E_i \psi) W = (\phi - \psi) \overline{\nabla}_W E_i.$$

Taking the scalar product with the unit vector field W and using a standard calculation, we have

$$E_i \psi = (\phi - \psi) \langle \overline{\nabla}_W E_i, W\rangle = (\psi - \phi) \langle \overline{\nabla}_W W, E_i\rangle. \tag{8}$$

Suppose p is an umbilical point of a leaf of $\mathcal{K}^\perp$; then $\psi(p) = \phi(p)$ and $(E_i \psi)(p) = 0$ for each $i = 1, \ldots, n-1$. On the other hand, if the integral curve of W passing through p is a geodesic of the leaf of $\mathcal{K}^\perp$ at p, then $\overline{\nabla}_W W(p)$ is orthogonal to this leaf and again $(E_i \psi)(p) = 0$ for $i = 1, \ldots, n-1$. Since by hypothesis one of these conditions holds, then the right hand side of (8) is always zero and we have that $E_i\psi \equiv 0$ for $i = 1, \ldots, n-1$. Then ψ is constant along each leaf of $\mathcal{K}^\perp \cap \mathcal{W}^\perp$.

Conversely, suppose that $E_i(\psi)(p) = 0$ for each $i = 1, \ldots, n-1$ and $\psi(p) \neq \phi(p)$. Then

$$\langle \overline{\nabla}_W W, E_i \rangle(p) = 0, \quad \text{for } i = 1, \ldots, n-1,$$

and $\overline{\nabla}_W W(p)$ is orthogonal to each $E_i(p)$, but also it is orthogonal to $W(p)$, since $\langle \overline{\nabla}_W W, W \rangle = 0$ everywhere; thus $\overline{\nabla}_W W(p)$ is orthogonal to the leaf of $\mathcal{K}^\perp$ and the integral curve of W passing through p is a geodesic of this leaf at p, as desired. □

3 Examples

The manifolds we are interested in are those which admit a closed partially conformal vector field. We will give some examples of these vector fields defined in open subsets of the space forms $\mathbb{Q}_c^{n+1}$.

The motivation here was to generalize the basic example consisting on a foliation of $\mathbb{R}^{n+1}$ by cylinders, each one equidistant to a line. The generalization may be described as follows.

Let γ be a curve in $\mathbb{Q}_c^{n+1}$ and take $r(\cdot) = d(\cdot, \gamma)$; that is, the distance to γ. Also, define the function S_c by

$$S_c(r) = \begin{cases} r, & c = 0, \\ \sin(r\sqrt{c})/\sqrt{c}, & c > 0, \\ \sinh(r\sqrt{-c})/\sqrt{-c}, & c < 0. \end{cases}$$

Proposition 3.1. *The vector field defined by*

$$K = S_c(r)\,\mathrm{grad}\, r$$

is closed partially conformal, defined in an open subset of $\mathbb{Q}_c^{n+1} \setminus \gamma$.

Proof. Given $p \in \mathbb{Q}_c^{n+1} \setminus \gamma$, let P be the totally geodesic hypersurface passing through p and orthogonal to γ.

We take an orthogonal frame $E_1, \ldots, E_{n-1}, E_{n+1}$ in $P \setminus \gamma$ with $E_{n+1} = \mathrm{grad}\, r$. Note that

$$\overline{\nabla}_{E_{n+1}} \mathrm{grad}\, r = 0 \quad \text{and so} \quad \overline{\nabla}_{E_{n+1}} K = S_c' E_{n+1}, \tag{9}$$

where we denote by $\overline{\nabla}$ the connection of $\mathbb{Q}_c^{n+1}$. For each $i = 1, \ldots, n-1$ write

$$E_i = \langle \mathrm{grad}\, r, E_i \rangle\, \mathrm{grad}\, r + v_i,$$

where v_i belongs to the plane spanned by E_i and $\operatorname{grad} r$. Then

$$\overline{\nabla}_{E_i} \operatorname{grad} r = \langle \operatorname{grad} r, E_i \rangle \overline{\nabla}_{\operatorname{grad} r} \operatorname{grad} r + \overline{\nabla}_{v_i} \operatorname{grad} r = \frac{S_c'}{S_c} v_i .$$

We finally obtain, for $i = 1, \dots, n-1$,

$$\overline{\nabla}_{E_i} K = S_c' \big(\langle \operatorname{grad} r, E_i \rangle \operatorname{grad} r + E_i - \langle \operatorname{grad} r, E_i \rangle \operatorname{grad} r \big) = S_c' E_i . \tag{10}$$

As P is totally geodesic, equations (9) and (10) remain valid when we replace the connection $\overline{\nabla}$ of $\mathbb{Q}_c^{n+1}$ by the induced connection ∇ in P. These equations show that K is a closed conformal vector field while restricted to P. Hence, by Proposition 1 in [12], K restricted to P determines a foliation by totally umbilical $(n-1)$-dimensional submanifolds.

Now, in $\mathbb{Q}_c^{n+1} \setminus \gamma$ define $E_n = W$ as a unit vector field satisfying

$$\langle W, K \rangle = 0 \quad \text{and} \quad \langle W, E_i \rangle = 0, \quad i = 1, \dots, n-1.$$

Fix a point $p \in \mathbb{Q}_c^{n+1} \setminus \gamma$ and let M be the hypersurface generated by taking the totally umbilical submanifold of P passing through p and moving it following the flow of W. Equations (9) and (10) mean that the vector fields $E_1, \dots, E_{n-1}$ determine principal directions of M. The vector field W, being orthogonal to them, defines also a principal direction of M and we obtain

$$\overline{\nabla}_W K = \psi W \tag{11}$$

for some function ψ. From (9), (10) and (11) we have that K is a closed partially conformal vector field defined in an open set of $\mathbb{Q}_c^{n+1} \setminus \gamma$. □

Definition 3.2. We will say that each hypersurface obtained by the procedure described above is a *tube* around the curve γ. If γ is a geodesic in $\mathbb{Q}_c^{n+1}$, we will say that the hypersurface is a *cylinder*.

As suggested by our examples, there is a close relation between closed partially conformal vector fields and a certain product structure on the ambient space. In fact, we have the following result.

Theorem 3.3. *Let $\overline{M}^{n+1}$ be a Riemannian manifold.*

1. *If $\overline{M} = J \times (I \times_f P^{n-1})$, then $\overline{M}$ admits a closed partially conformal vector field.*

2. *If $\overline{M}$ admits a closed partially conformal vector field K and the associated vector field W is closed conformal, then locally $\overline{M}$ is isometric to $J \times (I \times_f P^{n-1})$.*

Proof. Suppose first that $\overline{M} = J \times (I \times_f P^{n-1})$. Take coordinates $t \in J$, $s \in I$ and define

$$K = f(s)\frac{\partial}{\partial s} \qquad \text{and} \qquad W = \frac{\partial}{\partial t}.$$

By the product structure, every vector field $V \in \mathfrak{X}(\overline{M})$ can be expressed as

$$V = a\frac{\partial}{\partial t} + b\frac{\partial}{\partial s} + X,$$

where X is a lifting to $\overline{M}$ of a field tangent to P. Taking $a = 0$, we have

$$\begin{aligned}
\overline{\nabla}_{b\frac{\partial}{\partial s}+X} K &= \overline{\nabla}_{b\frac{\partial}{\partial s}+X}\left(f\frac{\partial}{\partial s}\right) \\
&= b\overline{\nabla}_{\frac{\partial}{\partial s}}\left(f\frac{\partial}{\partial s}\right) + \overline{\nabla}_X\left(f\frac{\partial}{\partial s}\right) \\
&= b\left(f\overline{\nabla}_{\frac{\partial}{\partial s}}\left(\frac{\partial}{\partial s}\right) + f'(s)\frac{\partial}{\partial s}\right) + \frac{f}{f}f'(s)X \\
&= f'(s)\left(b\frac{\partial}{\partial s} + X\right).
\end{aligned}$$

where we have used the formula $\overline{\nabla}_X Y = \frac{Yf}{f}X$ (see [14], p. 206, Prop. 35) and the fact that $\overline{\nabla}_{\frac{\partial}{\partial s}}\left(\frac{\partial}{\partial s}\right) = 0$. Then the first condition in (2) holds with $\phi = f'(s)$. On the other hand, it is clear that

$$\overline{\nabla}_{\frac{\partial}{\partial t}} K = 0,$$

and so the second condition in (2) holds with $\psi = 0$. Then, K is closed, partially conformal, proving the first part of this Theorem.

For the second part of the Theorem, suppose that W is closed conformal. By Remark 2.8, condition (6) holds with $\sigma = 0$ and thus the distribution $W^\perp$ is involutive. Then K is a closed conformal vector field when restricted to a leaf of $\mathcal{W}^\perp$. By a result proved by Montiel in [12] (p. 721), this leaf is isometric to a warped product $I \times_f P^{n-1}$. Since W is also Killing, each leaf of $\mathcal{W}^\perp$ is isometric to each other; thus following the flow defined by the unit vector field W, we obtain a (local) isometry between $\overline{M}$ and $J \times \left(I \times_f P^{n-1}\right)$. □

As a consequence of the theorem above, we may give a first characterization of the foliations given by closed partially conformal vector fields in space forms.

Corollary 3.4. *Let K be a closed partially conformal vector field defined in $\mathbb{Q}_c^{n+1}$. Suppose additionally that W is closed conformal. Then the associated foliation of $\mathbb{Q}_c^{n+1}$ is a foliation by hyperplanes, hyperspheres or tubes.*

Proof. Since $W^\perp$ is involutive, the second part of Theorem 3.3 implies that locally $\mathbb{Q}_c^{n+1}$ must have the form $J \times (I \times_f P^{n-1})$. Thus $I \times_f P^{n-1}$ has constant curvature. By a remark by Sánchez in [17], P^{n-1} has also constant curvature and so it must be an open set of a $(n-1)$-dimensional sphere or plane, thus giving the result. □

Remark 3.5. For later use, we point out that the proof of Theorem 3.3 gives

$$|K| = f(s), \quad \phi = f'(s) \quad \text{and} \quad \psi = 0;$$

hence the mean curvature of a leaf of $\mathcal{K}^\perp$ is given by

$$nH = -(n-1)\frac{\phi}{|K|} = -(n-1)\frac{f'}{f} = -(n-1)(\log f)'. \tag{12}$$

4 Partially conformal vector fields in $\mathbb{R}^{n+1}$

Here we classify the constant mean curvature complete leaves of the foliation $\mathcal{K}^\perp$ associated to a closed partially conformal vector field K in the Euclidean space $\mathbb{R}^{n+1}$ into four types: hyperplanes, hyperspheres and products of the form $\mathbb{R}^{n-1} \times \mathbb{S}^1$ or $\mathbb{R} \times \mathbb{S}^{n-1}$. Recall that if we consider a leaf of $\mathcal{K}^\perp$, Lemma 2.3 implies that the function ϕ is constant along the leaf, while if the leaf has constant mean curvature, Proposition 2.5 implies that the function ψ is constant along this leaf.

Lemma 4.1. *Let K be a closed partially conformal vector field in $\mathbb{R}^{n+1}$. Suppose that each leaf of $\mathcal{K}^\perp$ is complete and has constant mean curvature. Take $p \in \mathbb{R}^{n+1} \setminus \mathcal{Z}(K)$ and let $\phi = \phi(p)$, $\psi = \psi(p)$ be the functions given in (2), constant along the leaf of $\mathcal{K}^\perp$ passing through p.*

1. *If $\phi = 0 = \psi$, then the leaf is a hyperplane.*
2. *If $\phi = 0$ and $\phi \neq \psi$, then the leaf is a cylinder $\mathbb{R}^{n-1} \times \mathbb{S}^1$.*
3. *If $\phi \neq 0$ and $\phi = \psi$, then the leaf is a hypersphere.*
4. *If $\phi \neq 0$ and $\phi \neq \psi$, then the leaf is a cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$.*

Proof. The interesting case is the last one. Here we use a result by do Carmo and Dajczer [7] stating that the leaf of $\mathcal{K}^\perp$ through p must be a rotation hypersurface. As ψ is the (constant) curvature of the profile curve generating the leaf in the orbit space, this profile curve is (part of) a circle or a line. Since $\phi \neq 0$ and $\phi \neq \psi$, the only case giving a complete hypersurface with constant mean curvature occurs when the profile curve is a line parallel to the rotation axis, so that the leaf is a cylinder. □

Before stating our next lemma, we will call a leaf of $\mathcal{K}^\perp$ *non-singular* if there is a point $p \notin \mathcal{Z}(K)$ on the leaf. By Lemma 4.1, there are only four types of non-singular leaves in our setting.

Lemma 4.2. *Let K be a closed partially conformal vector field in $\mathbb{R}^{n+1}$ whose leaves are complete and have constant mean curvature. Then all non-singular leaves have the same type. Moreover, when all non-singular leaves are cylinders, they are coaxial.*

Proof. Suppose first that there is a non-singular leaf of the type $\mathbb{R} \times \mathbb{S}^{n-1}$ and take a point p in that leaf. Along the $(n-1)$-sphere contained in the leaf and passing through p the vector field $\widehat{K}$ normal to the leaf lies in the same n-plane containing the $(n-1)$-sphere. By equation (3), the flow of $\widehat{K}$ is a geodesic flow in $\mathbb{R}^{n+1}$, which implies that the restriction of $\widehat{K}$ to this n-plane stays everywhere tangent to the plane and thus it is a closed conformal field. Proposition 2 of [12] implies that the leaves of the foliation obtained by restricting K to the n-plane are concentric $(n-1)$-spheres. This fact shows that no sequence of these cylinders may converge to a cylinder $\mathbb{R}^{n-1} \times \mathbb{S}^1$ nor to a hyperplane. Because of non-compactness, this sequence can not converge either to a hypersphere. Also, we have that all leaves of $\mathcal{K}^\perp$ are cylinders $\mathbb{R} \times \mathbb{S}^{n-1}$ with the same rotation axes.

Similarly, if the leaf of $\mathcal{K}^\perp$ passing through p is a hyperplane, take P as the n-plane containing p and orthogonal to W. Applying the argument above, the leaves defined by K in P are $(n-1)$-planes. As this happens for every p on the leaf, $\mathcal{K}^\perp$ is a foliation by hyperplanes.

Next, if a leaf is a hypersphere, take P as the hyperplane defined before, so that the foliation on P is by $(n-1)$-spheres. As we have seen, by compactness the leaves of $\mathcal{K}^\perp$ can not be cylinders of the type $\mathbb{R} \times \mathbb{S}^{n-1}$, nor a hypersurface of the other types. □

Combining the Lemmas 4.1 and 4.2, we have

Theorem 4.3. *Let K be a closed partially conformal vector field in $\mathbb{R}^{n+1}$, whose leaves are complete and have constant mean curvature. Then the foliation $\mathcal{K}^{\perp}$ is a foliation by hyperplanes, by hyperspheres or by coaxial cylinders.*

5 Conditions for a hypersurface to be $(n-1)$-umbilical

In this section we will work with a manifold admitting a closed partially conformal vector field to answer the second question posed in the Introduction, imposing conditions to guarantee that a hypersurface is $(n-1)$-umbilical. First we will give two technical lemmas.

Let $\overline{M}^{n+1}$ be a Riemannian manifold which admits a closed partially conformal vector field K and an associated vector field W. Let M be an orientable hypersurface of $\overline{M}$, everywhere transverse to K, and N be a unit vector field normal to M. Note that the transversality implies that the vector field $W^* = W - \langle W, N\rangle N$ is everywhere different from zero.

Lemma 5.1. *Let $\overline{M}^{n+1}$ be a Riemannian manifold satisfying the conditions given in the last paragraph. Let $A(X) = -\overline{\nabla}_X N$ be the shape operator corresponding to N and take an orthonormal frame $E_1, \ldots, E_n$ of eigenvectors of A with eigenvalues $\lambda_1, \ldots, \lambda_n$. Suppose that $E_n = W^*/|W^*|$ and define*

$$\begin{aligned} K^T &= K - \langle K, N\rangle N - \langle K, E_n\rangle E_n, \\ (A(K^T))^T &= A(K^T) - \langle A(K^T), E_n\rangle E_n. \end{aligned}$$

Then

$$\sum_{i=1}^{n-1} \langle \overline{\nabla}_{E_i}((A(K^T))^T), E_i\rangle$$

is equal to

$$K^T\left(\sum_{i=1}^{n-1} \lambda_i\right) - \mathrm{Ric}\,(K^T, N) + \sum_{i=1}^{n-1} \langle A(E_i), \overline{\nabla}_{E_i}(K^T)\rangle. \tag{13}$$

Proof. Fix $i \in \{1, \ldots, n-1\}$. By definition of $(A(K^T))^T$, we have that

$$\overline{\nabla}_{E_i}((A(K^T))^T)$$

is equal to

$$\overline{\nabla}_{E_i}(A(K^T)) - \langle A(K^T), E_n\rangle \overline{\nabla}_{E_i}(E_n) - E_i(\langle A(K^T), E_n\rangle) E_n.$$

Note that the hypothesis $A(E_n) = \lambda_n E_n$ implies

$$\langle A(K^T), E_n\rangle = \langle K^T, A(E_n)\rangle = \lambda_n \langle K^T, E_n\rangle = 0,$$

and hence we have

$$\begin{aligned}
\langle \overline{\nabla}_{E_i}((A(K^T))^T), E_i\rangle &= \langle \overline{\nabla}_{E_i}(A(K^T)), E_i\rangle \\
&= \langle \overline{\nabla}_{K^T}(A(E_i)), E_i\rangle + \langle \overline{R}(K^T, E_i)E_i, N\rangle \\
&\quad - \langle A(E_i), [K^T, E_i]\rangle \\
&= K^T(\langle A(E_i), E_i\rangle) - \langle \overline{R}(K^T, E_i)N, E_i\rangle \\
&\quad + \langle A(E_i), \overline{\nabla}_{E_i}(K^T)\rangle - 2\langle A(E_i), \overline{\nabla}_{K^T}(E_i)\rangle, \\
&= K^T(\langle A(E_i), E_i\rangle) - \langle \overline{R}(K^T, E_i)N, E_i\rangle \\
&\quad + \langle A(E_i), \overline{\nabla}_{E_i}(K^T)\rangle,
\end{aligned}$$

where we have used the Codazzi equation and the fact that

$$\langle A(E_i), \overline{\nabla}_{K^T}(E_i)\rangle = \lambda_i \langle E_i, \overline{\nabla}_{K^T}(E_i)\rangle = 0.$$

We just add in the expression above from $i = 1$ to $n - 1$ to obtain our result. □

Following [3], we will use the expression given in equation (13) to characterize a $(n-1)$-umbilical hypersurface.

Lemma 5.2. *Under the same hypothesis of Lemma* 5.1*, the expression*

$$\sum_{i=1}^{n-1} \langle H_0 \overline{\nabla}_{E_i}(K^T) - \overline{\nabla}_{E_i}((A(K^T))^T), E_i\rangle,$$

where

$$(n-1)H_0 = \sum_{i=1}^{n-1} \lambda_i,$$

is equal to

$$-K^T((n-1)H_0) + \operatorname{Ric}(K^T, N) - \sum_{i=1}^{n-1} \Big(\langle (H_0 I - A)(E_i), \overline{\nabla}_{E_i}(K^T - K)\rangle\Big). \quad (14)$$

Proof. By (13) we just have to analyze

$$\sum_{i=1}^{n-1}\big(\langle H_0\,\overline{\nabla}_{E_i}(K^T),\,E_i\rangle-\langle A(E_i),\overline{\nabla}_{E_i}(K^T)\rangle\big)\,.$$

First, we have

$$\langle H_0\,\overline{\nabla}_{E_i}(K^T),\,E_i\rangle-\langle A(E_i),\overline{\nabla}_{E_i}(K^T)\rangle=\langle\overline{\nabla}_{E_i}(K^T),(H_0I-A)(E_i)\rangle.$$

Then, using the facts that K is a closed partially conformal vector field and that $\langle E_i,W\rangle=0$ for all $i=1,\ldots,n-1$, we obtain

$$\begin{aligned}\langle\overline{\nabla}_{E_i}(K^T),(H_0I-A)(E_i)\rangle &= \langle\overline{\nabla}_{E_i}(K+K^T-K),(H_0I-A)(E_i)\rangle\\ &= \langle\overline{\nabla}_{E_i}K+\overline{\nabla}_{E_i}(K^T-K),(H_0I-A)(E_i)\rangle\\ &= \phi\langle E_i,(H_0I-A)(E_i)\rangle\\ &\quad+\langle\overline{\nabla}_{E_i}(K^T-K),(H_0I-A)(E_i)\rangle.\end{aligned}$$

Now we add from $i=1$ to $n-1$ and note that

$$\sum_{i=1}^{n-1}\langle E_i,(H_0I-A)(E_i)\rangle=(n-1)H_0-(n-1)H_0=0$$

to finish the proof. □

Before stating our theorem on conditions for a hypersurface to be $(n-1)$-umbilical, we define a convenient notion of constant mean curvature. Let $M_0^{n-1}\subset\overline{M}^{n+1}$ be a submanifold of $\overline{M}$, N a unit vector field everywhere normal to M_0 and E_i, $i=1,\ldots,n-1$ an orthonormal frame defined along M_0. The *mean curvature* H_0 of M_0 relative to N is, by definition,

$$(n-1)H_0=-\sum_{i=1}^{n-1}\langle\overline{\nabla}_{E_i}N,E_i\rangle=\sum_{i=1}^{n-1}\lambda_i.$$

where λ_i are the eigenvalues of the shape operator A_0 of M_0 relative to N. We say that M_0 has *constant mean curvature relative to* N if the above sum is constant along M_0.

Theorem 5.3. *Let $\overline{M}^{n+1}$ be a Riemannian manifold with non-negative Ricci curvature which admits a closed partially conformal vector field K and an associated vector field W. Let M be an orientable hypersurface of $\overline{M}$ everywhere* *transverse to K and N be a unit vector field normal to M. Suppose that the direction determined by $W^* = W - \langle W, N\rangle N$ is a principal direction of M and that through each point of M passes a compact $(n-1)$-dimensional submanifold of M, everywhere orthogonal to W^*, totally umbilical as a hypersurface in M and having constant mean curvature relative to N. Then M is $(n-1)$-umbilical.*

Before starting the proof, we remark that each of the $(n-1)$-dimensional submanifolds given in the statement of the theorem has codimension 2; we will prove that if such a submanifold is umbilical in M (i.e., using the normal vector field $W^*/|W^*|$), then it is $(n-1)$-umbilical in $\overline{M}^{n+1}$ (i.e., relative to the normal vector field N).

Proof. We will use the results and notations of the last two lemmas. Fix $p \in M$ and let M_0 the compact $(n-1)$-dimensional submanifold of M passing through p, satisfying the hypothesis of the theorem.

If $E_1, \ldots, E_{n-1}, E_n = W^*/|W^*|$ is an orthonormal frame of eigenvectors of A in a neighborhood of p in M and $\lambda_1, \ldots, \lambda_n$ their corresponding eigenvalues, then

$$\begin{aligned} &H_0 \operatorname{div}_{M_0}(K^T) - \operatorname{div}_{M_0}((A(K^T))^T) \\ &= \sum_{i=1}^{n-1} \langle H_0 \overline{\nabla}_{E_i}(K^T) - \overline{\nabla}_{E_i}((A(K^T))^T), E_i\rangle \\ &= -K^T((n-1)H_0) + \operatorname{Ric}(K^T, N) \\ &\quad - \sum_{i=1}^{n-1} \langle (H_0 I - A)(E_i), \overline{\nabla}_{E_i}(K^T - K)\rangle. \end{aligned}$$

Since M_0 has constant mean curvature relative to N, H_0 is constant along M_0 and the first term vanishes. Now let us see what happens with

$$\begin{aligned} &-\sum_{i=1}^{n-1} \langle (H_0 I - A)(E_i), \overline{\nabla}_{E_i}(K^T - K)\rangle \\ &= \sum_{i=1}^{n-1} \langle (H_0 I - A)(E_i), \overline{\nabla}_{E_i}(\langle K, N\rangle N + \langle K, E_n\rangle E_n)\rangle \\ &= \langle K, N\rangle \sum_{i=1}^{n-1} \langle (H_0 I - A)(E_i), \overline{\nabla}_{E_i} N\rangle \\ &\quad + \langle K, E_n\rangle \sum_{i=1}^{n-1} \langle (H_0 I - A)(E_i), \overline{\nabla}_{E_i} E_n\rangle. \end{aligned}$$

For the first term of this last summation, we have

$$\sum_{i=1}^{n-1}\langle (H_0I - A)(E_i), \overline{\nabla}_{E_i}N\rangle = \sum_{i=1}^{n-1}\lambda_i^2 - (n-1)H_0^2 = \mathrm{Tr}(A_0^2) - (n-1)H_0^2,$$

where A_0 is the shape operator of M_0 relative to N. This last expression is easily seen to be globally defined and non-negative. Moreover, it is equal to zero if and only if M is $(n-1)$-umbilical.

As for the second term, we use the umbilicity of M_0 in M, which means that $\overline{\nabla}_X E_n = \kappa X$ for each vector field X tangent to M_0. We obtain

$$\sum_{i=1}^{n-1}\langle (H_0I - A)(E_i), \overline{\nabla}_{E_i}E_n\rangle = \kappa\sum_{i=1}^{n-1}\langle (H_0I - A)(E_i), E_i\rangle = 0.$$

Gathering all the information, we obtain

$$\begin{aligned} &H_0\,\mathrm{div}_{M_0}(K^T) - \mathrm{div}_{M_0}((A(K^T))^T)\\ &= \mathrm{Ric}(K^T, N) + \langle K, N\rangle(\mathrm{Tr}(A_0^2) - (n-1)H_0^2). \end{aligned}$$

Integrating over M_0, we have

$$0 = \int_{M_0}\left(\mathrm{Ric}(K^T, N) + \langle K, N\rangle\left(\mathrm{Tr}(A_0^2) - (n-1)H_0^2\right)\right).$$

Since M is everywhere transversal to K, $\langle K, N\rangle$ does not change sign along M, and we may suppose that it is positive everywhere. Since Ric is non-negative, we deduce that $\mathrm{Tr}(A_0^2) - (n-1)H_0^2$ vanishes identically and then M is $(n-1)$-umbilical. □

To close this section, we remark that the hypothesis over the Ricci curvature can be changed by that of K^T being orthogonal to $\overline{\mathrm{Ric}}(N)$, where $\overline{\mathrm{Ric}}$ is the Ricci operator of the ambient space $\overline{M}^{n+1}$. This condition was used before in other contexts; see, for example, [1], p. 475.

6 Conditions for a constant mean curvature hypersurface to be a leaf

Hereafter we suppose that $\overline{M}^{n+1}$ is a Riemannian manifold which admits a closed partially conformal vector field K with associated vector field W such that W is closed conformal. By the second part of Theorem 3.3, $\overline{M}$ is locally isometric to $J \times (I \times_f P^{n-1})$. For each $t \in J$, let $\overline{M}_t = \{t\} \times (I \times_f P^{n-1})$. Also, if M is any hypersuface in $\overline{M}$ which intersects transversally with $\overline{M}_t$, we denote $M_t = M \cap \overline{M}_t$. In the following proposition we obtain an expression for the mean curvature of M.

Proposition 6.1. *Let $\overline{M}^{n+1}$ be a manifold which admits a closed partially conformal vector field K with associated vector field W such that W is closed conformal. Let M be an orientable hypersuface of $\overline{M}$, everywhere transverse to K. Using the notation given before this proposition, suppose also that for each $t \in J$ the $(n-1)$-dimensional submanifold M_t is contained in one leaf of $\mathcal{K}^\perp \cap \mathcal{W}^\perp$. Then the mean curvature H_M of M is given by*

$$nH_M = -(n-1)\langle N, \widehat{K}\rangle \frac{\phi}{|K|} - \frac{\tilde{E}_n\langle N, W\rangle}{\sqrt{1-\langle N, W\rangle^2}}, \tag{15}$$

where $N = \langle N, \widehat{K}\rangle\, \widehat{K} + \langle N, W\rangle\, W$ and $\widetilde{E}_n = -\langle N, W\rangle\, \widehat{K} + \langle N, \widehat{K}\rangle\, W$.

Proof. We use a frame field such that $E_1, \ldots, E_{n-1}$ span $K^\perp \cap W^\perp$, $E_n = W$ and $E_{n+1} = \widehat{K}$. Note that the fact that M_t is contained in one leaf of $\mathcal{K}^\perp \cap \mathcal{W}^\perp$ implies that the vector field N is in fact a unit vector field everywhere normal to M.

For $i = 1, \ldots, n-1$ we have

$$\begin{aligned} \overline{\nabla}_{E_i} N &= \langle N, \widehat{K}\rangle\, \overline{\nabla}_{E_i} \widehat{K} + \langle N, W\rangle\, \overline{\nabla}_{E_i} W + (E_i\langle N, \widehat{K}\rangle)\, \widehat{K} + (E_i\langle N, W\rangle)\, W \\ &= \langle N, \widehat{K}\rangle \frac{\phi}{|K|} E_i + (E_i\langle N, \widehat{K}\rangle)\, \widehat{K} + (E_i\langle N, W\rangle)\, W, \end{aligned}$$

where we used the fact that W is parallel (see Remark 2.8).

We take the scalar product with E_i to obtain

$$\langle \overline{\nabla}_{E_i} N, E_i\rangle = \langle N, \widehat{K}\rangle \frac{\phi}{|K|}. \tag{16}$$

Now we use the vector field $\widetilde{E}_n = -\langle N, W\rangle\, \widehat{K} + \langle N, \widehat{K}\rangle\, W$ to obtain

$$\begin{aligned} \overline{\nabla}_{\widetilde{E}_n} N &= \langle N, \widehat{K}\rangle \overline{\nabla}_{\widetilde{E}_n} \widehat{K} + \langle N, W\rangle \overline{\nabla}_{\widetilde{E}_n} W + (\widetilde{E}_n\langle N, \widehat{K}\rangle)\widehat{K} + (\widetilde{E}_n\langle N, W\rangle)W \\ &= (\widetilde{E}_n\langle N, \widehat{K}\rangle)\, \widehat{K} + (\widetilde{E}_n\langle N, W\rangle)\, W, \end{aligned}$$

where we have used that W is parallel, $\psi = 0$ and $\overline{\nabla}_{\widehat{K}} \widehat{K} = 0$; taking the scalar product with $\widetilde{E}_n$ we have

$$\langle \overline{\nabla}_{\widetilde{E}_n} N, \widetilde{E}_n\rangle = (\tilde{E}_n\langle N, W\rangle)\langle N, \widehat{K}\rangle - (\tilde{E}_n\langle N, \widehat{K}\rangle)\langle N, W\rangle. \tag{17}$$

We may simplify the last two terms as follows. Since

$$\langle N, \widehat{K}\rangle^2 + \langle N, W\rangle^2 = 1,$$

we take the derivative of this expression with respect to $\widetilde{E}_n$ and substitute the result in (17) to obtain

$$(\tilde{E}_n\langle N, W\rangle)\langle N, \widehat{K}\rangle - (\tilde{E}_n\langle N, \widehat{K}\rangle)\langle N, W\rangle = \frac{\tilde{E}_n\langle N, W\rangle}{\sqrt{1-\langle N, W\rangle^2}}.$$

Using the expression above and (16), we conclude that the mean curvature of M,

$$nH_M = -\sum_{i=1}^{n}\langle \overline{\nabla}_{E_i} N, E_i\rangle$$

is given by (15). □

In our last theorem we will impose a condition on the warping function f of the local expression $J \times (I \times_f P^{n-1})$ of the ambient space $\overline{M}$, namely, that $\log f$ is convex and give geometric conditions under which a hypersurface is actually a leaf of the foliation $\mathcal{K}^\perp$ determined by a closed partially conformal vector field K defined on $\overline{M}$.

Theorem 6.2. *Let $\overline{M}^{n+1}$ be a Riemannian manifold which admits a closed partially conformal vector field K and an associated closed conformal vector field W, such that $\overline{M}$ is given locally by $J \times (I \times_f P^{n-1})$ with $\log f$ convex. Let M be an orientable hypersurface of $\overline{M}$, everywhere transverse to K, with constant mean curvature in $\overline{M}$.*

Suppose there exists $t \in J$ such that M_t is a compact hypersurface of $\overline{M}_t$ with constant mean curvature. Suppose additionally the existence of a point $p \in M_t$ such that

1. *The unit vector $N(p)$ normal to M at p is equal to the unit vector $\widehat{K}(p)$ normal to the leaf of $\mathcal{K}^\perp$ passing through p,*
2. *Locally, M lies above the leaf of $\mathcal{K}^\perp$ passing through p with respect to K; that is, there is a neighborhood U of p in M such that each point $q \in U$ has the form $q = \phi_s(q')$, where q' is in the mentioned leaf, $s \geq 0$ and ϕ_s is the flow of K,*
3. *The derivative of $\langle N, W\rangle$ with respect to the vector $W(p)$ is positive.*

Then M coincides locally with the leaf of $\mathcal{K}^\perp$ passing through p. In particular, locally M is $(n-1)$-umbilical. Moreover, if the leaf of $\mathcal{K}^\perp$ passing through p has constant mean curvature, it coincides globally with M.

Proof. We observe first that the hypothesis $\log f$ convex is equivalent to $\mathcal{H}' \geq 0$, where $\mathcal{H} = f'/f = (\log f)'$. Also, the fact that M is everywhere transversal to K implies that the angle θ between the normal to M_t in $\overline{M}_t$ and K (which is tangent to $\overline{M}_t$) does not change sign. By Theorem 4 in Alías-Dacjzer [2], M_t is a leaf of $\mathcal{K}^\perp$ in $\overline{M}_t$.

Since M_t is clearly contained in a leaf of $\mathcal{W}^\perp$, we have that M_t is contained in a leaf of $\mathcal{K}^\perp \cap \mathcal{W}^\perp$. From Proposition 6.1, the (constant) mean curvature of M is given by

$$nH_M = -(n-1)\langle N, \widehat{K}\rangle \frac{\phi}{|K|} - \frac{\tilde{E}_n\langle N, W\rangle}{\sqrt{1-\langle N, W\rangle^2}},$$

where $N = \langle N, \widehat{K}\rangle\, \widehat{K} + \langle N, W\rangle\, W$ is normal to M and $\widetilde{E}_n = -\langle N, W\rangle \widehat{K} + \langle N, \widehat{K}\rangle W$. Since $\widehat{K}(p) = N(p)$, we have $\widetilde{E}_n(p) = W(p)$. Thus, hypothesis 3 and continuity imply that $\langle N, \widehat{K}\rangle$ and $\tilde{E}_n\langle N, W\rangle$ are positive in a neighborhood of p in M. Note also that we may suppose that $\phi \leq 0$ or equivalently that $H_M \geq 0$, since this sign depends on choosing one of the closed partially conformal fields K or $-K$. Taking these facts into account, at each point q in this neighborhood we have that

$$nH_M \leq -(n-1)\frac{\phi}{|K|}(q).$$

By Remark 3.5, the right hand side of this inequality is the mean curvature of a leaf of $\mathcal{K}^\perp$ through q; that is, the mean curvature of M at q is less than or equal to the mean curvature of the leaf of $\mathcal{K}^\perp$ through q.

We want to compare the mean curvatures of the leaf of $\mathcal{K}^\perp$ through p and that of the leaf of $\mathcal{K}^\perp$ through q. Note first that our hypothesis 2 implies that the leaf through q lies above the leaf through p. Using again equation (12) in Remark 3.5, the mean curvature of the leaves of $\mathcal{K}^\perp$ is given by

$$nH = -(n-1)\frac{f'}{f} = -(n-1)(\log f)'.$$

Now $\log f$ convex implies that nH is a non-increasing function on I, which implies in turn that the mean curvature of the leaf passing through q is less than or equal to the mean curvature of the leaf passing through p.

In short, in a neighborhood of p it happens that M lies above the leaf through p with respect to K and the mean curvature of M is less than or equal to the mean curvature of this leaf. By the tangency principle (see for example [9]), these hypersurfaces must coincide locally. In particular, locally M is $(n-1)$-umbilical. Finally, if both hypersurfaces have constant mean curvature, it is known that they must coincide globally. □

Corollary 6.3. *Let K be a closed partially conformal vector field defined in an open set of $\mathbb{R}^{n+1}$ with associated closed conformal vector field W. Let $M^n \subset \mathbb{R}^{n+1}$ be an orientable hypersurface satisfying the hypotheses of Proposition* 6.1*, as well as the conditions in Theorem* 6.2*. Then M is locally a hyperplane, a hypersphere or a cylinder.*

Proof. From Theorem 6.2, we know that M coincides locally with the leaf of $\mathcal{K}^{\perp}$ passing through p. Since M has constant mean curvature, by Proposition 2.5 we have that the function ψ associated to K is locally constant. By the local version of Lemma 4.1, M is locally a hyperplane, a hypersphere or a cylinder. □

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