BOL. SOC. BRAS. MAT., VOL. 15 Nºs 1 e 2 (1984), 55-84

C^2 STABILITY OF CURVES WITH NON-DEGENERATE SOLUTION TO PLATEAU'S PROBLEM

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Let Γ^k, $k \geq 1$, be the set of C^k Jordan curves in $\mathbb{R}^n$ with its natural topology and let $\eta : \Gamma^1 \to \mathbb{N}^*$, $\mathbb{N}^* = \{1, 2, \ldots, \infty\}$ be the function that assigns to each $\gamma \in \Gamma^1$ the number of solutions to Plateau's problem for γ, that is, the number of minimal disks bounding γ. It is still an unanswered question whether η can reach the value ∞. Several people were able to find open and dense subsets of Γ^k for which η is finite. A result in this direction can be found in [3] where it is proved that there exists an open and dense subset of $\Gamma^\infty = \bigcap_{k=k}^{\infty} \Gamma^k$, where η is finite. Generally, the approach used for this problem assumes k large. Consider, for example, the subset $\Gamma_k \subset \Gamma^k$ of curves whose solutions to Plateau's problem are immersions. In this case A. Tromba [13] was able to show that there exists a subset Γ_k' of Γ_k open and dense in Γ_k for $k \geq 7$ where η is finite.

The aim of this paper is to present an elementary approach that also works for $k \geq 2$ and arbitrary n. In fact, we prove in §4 that there exists an open subset Γ_2' of Γ_2 where η is finite and continuous (see theorem (4.1) and corollaries (4.5-6)).

(*) Research partially supported by CNPq do Brazil.
Except for §5 this is part of my thesis [4] done during the year 1976. I thank my adviser Prof. M.P. do Carmo for suggesting me this problem and for his permanent attention.

Recebido em 19/09/84.

A similar approach is used in §5 to prove that Γ_k' (the intersection of Γ_2' with Γ_k) is open and dense in Γ_k for $k \geq 2$.

This approach also produces regularity results in a natural way. We prove in §3 (see theorem (3.1)) that for $\gamma \in \Gamma^k$, $k \geq 2$, the solutions to Plateau's problem for γ lie in the Sobolev space $H^{k+1/2}(D,\mathbb{R}^n)$ where D is the unit disk of the plane with center at the origin.

The techniques here arose from a characterization of solutions to Plateau's problem as zeroes of the function ψ defined in (1.7). This function ψ is the main tool in [4].

§1. Preliminaries

In this work we use u and v for the coordinates of the plane and we denote a complex number by $z = u+iv$, or, in polar coordinates, as $z = re^{i\theta}$, where $i^2 = -1$. The partial derivative with respect to u, for example, is $\partial/\partial u$. We also use the following operators:

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right),$$

$$\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right), \qquad (1.1)$$

$$2z\partial = r\frac{\partial}{\partial r} - i\frac{\partial}{\partial\theta}.$$

In general, we denote by df the derivative of the map f, but if the domain of f is an interval then we use f'. We use also f_θ instead of $[f(e^{i\theta})]'$ where $e^{i\theta} = \cos\theta + i\sin\theta$, $\theta \in \mathbb{R}$.

Let M be a C^∞ manifold of dimension m. We will consider the two following families of function spaces: the space $C^k(M,\mathbb{R}^n)$ of C^k maps $f: M \to \mathbb{R}^n$ with finite C^k norm, where k is a non negative real number, and the Sobolev space $H^k(M,\mathbb{R}^n)$, $k \in \mathbb{R}$, defined in [10] as $L^2_k(M\times\mathbb{R}^n)$. In our case, the manifold M will be very simple, namely the disk $D = \{z/|z| < 1\}$ or its boundary S. In the later case, the H^k norm of $f \in H^k(S,\mathbb{R}^n)$ is

(1.2) $$\|f\|_k^2 = \sum_{j=-\infty}^{\infty} (1+j^2)^k |\alpha_j|^2$$

where $\sum \alpha_j e^{ij\theta}$, $\theta \in \mathbb{R}$, is the Fourier serie of f. Actually, $C^k(M,\mathbb{R}^n)$ is a Banach space and $H^k(M,\mathbb{R}^n)$ is a Hilbert space. We will use some interesting facts about these spaces which we present here for the sake of completeness (cf. [10], [11]).

1.3. Theorem. *If $k > \ell$, then $C^k(M,\mathbb{R}^n)$ is contained in $C^\ell(M,\mathbb{R}^n)$, $H^k(M,\mathbb{R}^n)$ is contained in $H^\ell(M,\mathbb{R}^n)$, and both inclusions are completely continuous linear maps. By construction, $C^k(M,\mathbb{R}^n)$ is contained continuously in $H^k(M,\mathbb{R}^n)$ (but it is not completely continuous).*

1.4. Sobolev Immersion Theorem. *If m is the dimension of M and $k > m/2 + j + \mu$, j integer and $0 < \mu < 1$, then $H^k(M,\mathbb{R}^n)$ is contained in $C^{j+\mu}(M,\mathbb{R}^n)$ and the inclusion is completely continuous.*

1.5. Trace Theorem. *If ∂M is the boundary of M and $k > \frac{1}{2}$ then the restriction map $x \longmapsto x|\partial M$ of $C^\infty(M,\mathbb{R}^n)$ into $C^\infty(\partial M,\mathbb{R}^n)$ extends to a continuous linear map of $H^k(M,\mathbb{R}^n)$ onto $H^{k-1/2}(\partial M,\mathbb{R}^n)$.*

1.6. Theorem. *If $k > m/2$ and $|j| \le k$, then the multiplication map from $C^\infty(M,\mathbb{R}) \oplus C^\infty(M,\mathbb{R})$ into $C^\infty(M,\mathbb{R}^n)$ extends to a continuous bilinear map from $H^j(M,\mathbb{R}) \oplus H^k(M,\mathbb{R})$ to $H^j(M,\mathbb{R})$.*

Let U be an open connected and bounded subset of $\mathbb{R}^n$ and let $H^k(M,U)$, $k > m/2$, be the subset of maps $x \in H^k(M,\mathbb{R}^n)$ such that $x(M) \subset U$.

1.7. Theorem. *If $k > m/2$, then the composition map $(f,x) \to f\circ x$ of $C^{\ell+j}(U,\mathbb{R}^p) \oplus H^k(M,U)$, $\ell \le k$, into $H^\ell(M,\mathbb{R}^p)$ is of class C^j.*

As a consequence of the last theorem we obtain:

1.8. Theorem. *Let* $k > 1/2$ *be a real number, and* j, ℓ *be integers such that* $0 \leq \ell \leq \min\{j,k\}$. *Then the map*

$$\phi : C^j(S,\mathbb{R}^n) \oplus H^k(S,\mathbb{R}) \to H^\ell(S,\mathbb{R})$$

defined by

$$\phi(f,x)(z) = f(ze^{ix(z)}), \quad z \in S,$$

is of class $C^{j-\ell}$ *and*

$$d^s\phi(f,x)((f_1,x_1),\ldots,(f_s,x_s)) = \phi(d^s f,x)x_1 \cdot\ldots\cdot x_s +$$

$$+ \sum_{r=1}^{s} \phi(d^{s-1}f_r,x)x_1 \cdot\ldots\cdot \hat{x}_r \cdot\ldots\cdot x_s$$

where $\hat{x}_r$ *means the away of* x_r.

Let γ be a Jordan curve of class C^k, $k \geq 2$, embedded into $\mathbb{R}^n$. We fix an orientation for γ. The Sobolev's theorem (1.4) says that $x \in H^k(S,\mathbb{R}^n)$, $k \geq 1$, is a continuous map. We say that $x \in H^1(S,\mathbb{R}^n)$ with $x(S) = \gamma$ has degree one if is homotopic in γ to a C^k positive diffeomorphism $f : S \to \gamma$. Set, for $k \geq 1$,

$$H^k(\gamma) = \{x \in H^k(S,\mathbb{R}^n)/x \text{ has degree one and } x(S) = \gamma\}.$$

1.9. Lemma. *Let* k *and* j *be integers such that* $j > k \geq 1$ *and assume that* γ *is a Jordan curve of class* C^j. *Then* $H^k(\gamma)$ *is a* C^{j-k} *closed submanifold of* $H^k(S,\mathbb{R}^n)$.

Proof. Let $\pi : U \to U$ be a C^j map where U is an open subset of $\mathbb{R}^n$ containing γ such that $\pi\circ\pi = \pi$ and $\pi(U) = \gamma$. If γ is C^∞ then we may choose $\pi : U \to U$ to be a tubular neighborhood of γ. If γ is only C^j then one can use the local form of immersions together with partitions of unity to construct π. The set

$$H^k(S,U) = \{x \in H^k(S,\mathbb{R}^n) \ / \ x(S) \subset U\}$$

is an open subset of $H^k(S,\mathbb{R}^n)$. We define

$$F: H^k(S,U) \to H^k(S,U)$$

by $F(x) = \pi \circ x$. It follows from Theorem (1.7) that F is of class C^{j-k}. To conclude the proof we use the following fact: if V is an open subset of a Banach space and $F: V \to V$ is a C^k map such that $F \circ F = F$, then the image of F is a C^k submanifold. The tangent space $T_x H^k(\gamma)$ of $H^k(\gamma)$ at the point x is

$$(1.10) \qquad T_x H^k(\gamma) = \{y \in H^k(S,\mathbb{R}^n) \ / \ y(z) \in T_{x(z)}\gamma, \ z \in S\}$$

where $T_{x(z)}\gamma$ is the tangent space of γ at $x(z)$. Let $G: H^j(\gamma) \to T_x H^j(\gamma)$ be the restriction of $dF(x)$ to $H^j(\gamma)$. Then the chart at x is the restriction of G to a neighborhood of x.

Let $\{z_1,\dots,z_m\}$ be fixed points of S and $\{p_1,\dots,p_m\}$ be fixed points of γ, both in a cyclic order. Set

$$H^k(\gamma,m) = \{x \in H^k(\gamma) \ / \ x(z_r) = p_r, \ \ 1 \le r \le m\}$$

and

$$(1.11) \qquad T_x H^k(\gamma,m) = \{y \in T_x H^k(\gamma) \ / \ y(z_r) = 0, \ \ 1 \le r \le m\}$$

for some $x \in H^k(\gamma,m)$. Then $T_x H^k(\gamma,m)$ is a closed subspace of $T_x H^k(\gamma)$ of codimension m. The map G above applies a neighborhood of x in $H^k(\gamma,m)$ one-to-one and onto a neighborhood of the origin of $T_x H^k(\gamma,m)$. This proves the following:

1.12. Corollary. *$H^k(\gamma,m)$ is a closed submanifold of $H^k(\gamma)$ of class C^{j-k}.*

For each $X \in H^{k+1/2}(D,\mathbb{R}^n)$, $k \ge 2$, we can define the energy $E(X)$ of X by

$$E(X) = \frac{1}{2}\int_D \left\{\left|\frac{\partial X}{\partial u}\right|^2 + \left|\frac{\partial X}{\partial v}\right|^2\right\} du\,dv.$$

If X is harmonic, the first Green identity gives

$$2E(X) = \int_S \left\langle \frac{\partial X}{\partial r}, x\right\rangle d\theta, \quad x = X|S.$$

If $\sum \alpha_j e^{ij\theta}$ is the Fourier serie of $x = X|S$, then

$$X(re^{i\theta}) = \sum_{j=-\infty}^{\infty} r^{|j|}\alpha_j e^{ij\theta}, \quad \theta \in \mathbb{R}, \quad 0 \le r \le 1$$

and

$$\frac{\partial X}{\partial r}(re^{i\theta}) = \sum |j| r^{|j|-1}\alpha_j e^{ij\theta}$$

from where

$$E(X) = \pi \sum_{j=-\infty}^{\infty} |j|\,|\alpha_j|^2.$$

We introduce the operator $\partial_r : H^t(S,\mathbb{R}^n) \to H^{t-1}(S,\mathbb{R}^n)$, $t \in \mathbb{R}$, defined by

(1.13) $$\partial_r x = \sum |j|\alpha_j e^{ij\theta}$$

where $\sum \alpha_j e^{ij\theta}$ is the Fourier serie of $x \in H^t(S,\mathbb{R}^n)$. Observe that ∂_r is symmetric with respect to the inner product of $H^0(S,\mathbb{R}^n)$,

$$\langle \partial_r x, y\rangle_{H^0} = \langle \partial_r y, x\rangle_{H^0}, \text{ for all } x,y \in H^0(S,\mathbb{R}^n)$$

and it is a continuous linear map. If $X:\bar{D} \to \mathbb{R}^n$ is a harmonic map with finite energy then

$$\begin{aligned} E(X) &= E(x) \\ &= \frac{1}{2}\langle \partial_r x, x\rangle_{H^0}, \qquad x = X|S. \end{aligned}$$

Let ϕ be the map of Theorem 1.8 with $k = \ell = 1$. We define

(1.14) $$\varepsilon : C^j(S,\mathbb{R}^n) \oplus H^1(S,\mathbb{R}) \to \mathbb{R}, \quad j \text{ integer } \ge 2$$

by $\varepsilon(f,y) = E(\phi(f,y))$. This function ε plays an important role in this work.

1.15. Lemma. *The function* ε *is of class* C^j.

This lemma is a consequence of the following general fact. Let Y, Y_0, Y_1 and Z be Banach spaces such that Y_1 is a subspace of Y_0 and the inclusion of Y_1 into Y_0 is continuous. Let $B: Y_0 \times Y_0 \to Z$ be a continuous bilinear symmetric map and let $A: Y_1 \to Y_0$ be a continuous linear map symmetric with respect to B on the subspace Y_1 of Y_0. Now suppose we have a map $f: Y \to Y_1$ of class C^j such that, as a map from Y into Y_0, it is of class C^{j+1}. Then $F: Y \to \mathbb{R}$, $F(x) = B(Af(x), f(x))$ is of class C^{j+1}. Consider the set

$$(1.16) \qquad E^k = \{f \in C^k(S,\mathbb{R}^n) \,/\, f \text{ is embedding}\}$$

and define a C^{k-1} map $\psi: E^k \times H^1(S,\mathbb{R}) \to H^0(S,\mathbb{R})$ by

$$(1.17) \qquad \psi(f,y) = \langle \partial_r \phi(f,y), \ \phi(f',y) \rangle$$

where $(f,y) \in E^k \times H^1(S,\mathbb{R})$ and ϕ was defined in (1.8). At this point it is convenient to introduce the following notation:

$$(1.18) \qquad \begin{aligned} x &= \phi(f,y) & x_j &= \phi(f_j,y) \\ h_j &= y_j \phi(f',x), & h_{j\ell} &= y_j \phi(f'_\ell, y) \end{aligned}$$

where $y, y_j \in H^1(S,\mathbb{R})$, $f_j \in C^k(S,\mathbb{R}^n)$ and $f \in E^k$. Then, we have the following relations

$$d\varepsilon(f,y)(f_1,y_1) = \int_S y_1 \psi(f,y)\, d\theta + \langle x_1, \partial_r x \rangle_{H^0},$$

$$(1.19) \qquad d^2\varepsilon(f,y)((f_1,y_1),(f_2,y_2)) = \int_S y_1\, d\psi(f,y)(f_2,y_2)\, d\theta +$$

$$\langle \partial_r h_2, x_1 \rangle_{H^0} + \langle \partial_r x, h_{21} \rangle_{H^0} + E(x_1).$$

Let G be the set of biholomorphic maps of $\bar{D}$. The elements $w \in G$ have the representation

$$(1.20) \qquad w(z) = \rho \frac{\alpha + z}{1 - \bar{\alpha} z}, \qquad z \in \bar{D}, \qquad (\rho,\alpha) \in S \times D.$$

It is known that the energy function is invariant by conformal change of coordinates, that is, $E(X) = E(X\circ w)$, $w \in G$. If X is harmonic and $X|S = \phi(f,y)$ we obtain

$$(1.21)\qquad \begin{aligned} \varepsilon(f,\arg(y_w)) &= E(f(we^{iy\circ w})) \\ &= E(x\circ w) \\ &= E(x) \\ &= \varepsilon(f,y) \end{aligned}$$

wnere $\arg(y_w)$ is the argument of $y_w(z) = w(z)e^{iy(w(z))}$.

Unfortunately $w \longmapsto y_w$, $w \in G$, $y \in H^1(S,\mathbb{R})$ is not smooth. However, the G-action has some consequences on ψ as we can see in the following result:

1.22. Proposition. *The subspace of* $H^0(S,\mathbb{R})$ *spanned by* $\{1 + y_0,\ (1+y_\theta)\cos\theta,\ (1+y_\theta)\sin\theta\}$, *is orthogonal to the image of* $d\psi(f,y)$, *for each* $(f,y) \in E^k \times H^1(S,\mathbb{R})$.

Proof. We consider, in the group G, the differential structure induced from $S\times D$ by representation (1.20). Let W_s be a differentiable curve on G with $W_0(z) = z$, that is,

$$W_s(z) = \rho_s \frac{\alpha_s + z}{1+\bar{\alpha}_s z}, \qquad z \in \bar{D},$$

where (ρ_s,α_s) is a differentiable curve in $S\times D$ with $(\rho_0,\alpha_0) = (1,0)$. Then

$$\begin{aligned} \frac{d}{ds} w_s\Big|_{s=0} &= -i(\rho_0' + \alpha_0'\bar{z} - \bar{\alpha}_0' z)iz \\ &= (c+b\cos\theta - a\sin\theta)(-\sin\theta,\ \cos\theta) \end{aligned}$$

where $\rho_0' = ic$ and $\alpha_0' = \frac{1}{2}(a+ib)$. Then the tangent space $T_{w_0}G$ is generated by $\{iz,\ \cos\theta\, iz,\ \sin\theta\, iz\}$.

If y is of class C^∞ then $s \longmapsto \arg(y_{w_s})$ is a diferentiable curve in $H^1(S,R)$ with velocity

$$\frac{d}{ds}\arg(y_{w_s})\Big|_{s=0} = (1+y_\theta)t$$

where t is a linear combination of 1, $\sin\theta$, and $\cos\theta$. Taking derivatives in (1.21) we get

$$0 = d^2\varepsilon(f,y)((f_1,y_1),\ (0,(1+y_\theta)t))$$

$$= \int_S (1+y_\theta)t\cdot d\psi(f,y)(f_1,y_1)d\theta, \quad \text{by (1.19).}$$

This last equality extends, by limits, for each $y \in H^1(S,\mathbb{R})$.

§2. The Second Variation of Energy

Let $\mathcal{D}$ be the disk D with the natural Riemann surface structure. A *generalized minimal surface* is a harmonic map $X:\mathcal{D} \to \mathbb{R}^n$ such that

$$\langle\partial X,\partial X\rangle = \frac{1}{4}\left(\left|\frac{\partial X}{\partial u}\right|^2 - \left|\frac{\partial X}{\partial v}\right|^2 - 2i\left\langle\frac{\partial X}{\partial u},\frac{\partial X}{\partial v}\right\rangle\right) = 0,$$

that is, X is harmonic and conformal.

Let $\gamma \subset \mathbb{R}^n$ be a Jordan curve. A *solution to Plateau's problem* for γ is a generalized minimal surface $X:\mathcal{D} \to \mathbb{R}^n$ such that

(I) X extends to a continuous map from the closure $\bar{D}$ of D into $\mathbb{R}^n$ and

(II) X restricted to the boundary S of $\mathcal{D}$ is a homeomorphism between S and γ.

There are several results about the class of differentiability of a solution to Plateau's problem for γ (see [6] for reference). We report here a result of Nitsche [8] for $\gamma \subset \mathbb{R}^3$ which can also be proved for $\gamma \subset \mathbb{R}^n$ with some slight modifications.

2.1. Theorem. ([8] th. 1). Let $\gamma \subset \mathbb{R}^n$ be a Jordan curve of class $C^{k+\mu}$, k integer ≥ 1 and $0 < \mu < 1$. Then there is a constant τ, depending only on the geometry of γ such that

$$\|X\|_{C^{k+\delta}} \leq \tau, \quad 0 \leq \delta < \mu,$$

for all solutions X to Plateau's problem for γ satisfying a three point condition.

Let $f:S \to \mathbb{R}^n$ be a C^2 embedding with image γ and let $H^1(\gamma)$ be the manifold of Lemma 1.9. Then the map $\phi(f,y)$ defined in Theorem 1.8 for $y \in H^1(S,\mathbb{R})$, is a global parametrization of $H^1(\gamma)$.

2.2. Lemma. *Let $\gamma \subset \mathbb{R}^n$ be a C^2 Jordan curve. Let $X \in H^{\frac{3}{2}}(D,\mathbb{R}^n)$ be a harmonic map and x be its restriction to S. If $x=\phi(f,y)$ where $y \in H^1(S,\mathbb{R})$ and f is a C^2 diffeomorphism between S and γ then the following assertions are equivalent:*

(a) *$X:D \to \mathbb{R}^n$ is a generalized minimal surface,*

(b) *$\langle\partial_r x, x_\theta\rangle = 0$, in the complement of a subset of S with Lebesgue measure zero,*

(c) *$\frac{\partial\varepsilon}{\partial y}(f,y) = 0$, ε as defined in (1.14).*

Proof. Set $\omega(z) = \langle\partial X(z),\partial X(z)\rangle$, for $z \in D$. Then ω is holomorphic and, in polar coordinates, it satisfies

$$4z^2\omega = \left|r\frac{\partial X}{\partial r}\right|^2 - \left|\frac{\partial X}{\partial\theta}\right|^2 - 2i\langle r\frac{\partial X}{\partial r}, \frac{\partial X}{\partial\theta}\rangle.$$

By Theorem 2.1 the restriction of $\langle r\frac{\partial X}{\partial r}, \frac{\partial X}{\partial\theta}\rangle$ to S is precisely $\langle\partial_r x, x_\theta\rangle$. Then (b) holds only if $4z^2\omega$ is constant. Taking $z = 0$ we conclude that (a) and (b) are equivalent.

Now, by taking the y derivative of ε, we get

$$\frac{\partial\varepsilon}{\partial y}(f,y)t = \int_S \langle\partial_r x, d\phi(f,y)(0,t)\rangle d\theta, \quad t \in H^1(S,\mathbb{R}).$$

Let ν be the unit vector field oriented in the positive sense and let $\nu(x) = \nu \circ x$ be the composition of ν with x. Then

$$d\phi(f,y)(0,t) = t\alpha\nu(x)$$

where $t \in H^1(S,\mathbb{R})$ and $\alpha(z) = |f'(ze^{iy(z)})|$, $z \in S$. Thus

$$\frac{\partial\varepsilon}{\partial y}(f,y)t = \int_S \langle\partial_r x, \nu(x)\rangle t\alpha d\theta, \quad t \in H^1(S,\mathbb{R}). \tag{2.3}$$

Since $H^1(S,\mathbb{R})$ is a dense subspace of $H^0(S,\mathbb{R})$ and $\alpha(z) \neq 0$ for all $z \in S$, it follows that (c) is equivalent to

$$\langle\partial_r x, \nu(x)\rangle = 0, \quad \text{almost everywhere.} \tag{2.4}$$

By Theorem 2.1 the coordinates of the holomorphic curve $2z\partial X$ lie in some Hardy space H^μ with $\mu = 2$. If $|\frac{\partial X}{\partial\theta}| = 0$ in a subset of S with positive Lebesgue measure we get that $\partial X/\partial\theta$ is constant, which is impossible (see [14] p. 137). The equivalence between (b) and (c) now follows from $\langle\partial_r x, x_\theta\rangle = |x_\theta|\langle\partial_r x, \nu(x)\rangle$.

Let X be a solution to Plateau's problem to γ and $x = X|S$. A *variation of X by harmonic maps with variational fields* $Y_1,\ldots,Y_r$ is a differentiable map $F: I^r \to H^{\frac{3}{2}}(D,\mathbb{R}^n)$ where I is the interval $(-\delta,\delta)$, $\delta > 0$, such that

(2.5a) $F(t)$ apply S over γ for all $t \in I^r$

(2.5b) $F(0) = X$ and $\frac{\partial F}{\partial t_j}(0) = Y_j$, $1 \leq j \leq r$.

The trace map from $H^{\frac{3}{2}}(D,\mathbb{R}^n)$ into $H^1(S,\mathbb{R}^n)$ gives the following equivalence: F is a variation of X by harmonic maps with variational fields Y_j if and only if the trace of F is $\phi\circ F_0$, where $F_0: I^r \to C^k(S,\mathbb{R}^n) \times H^1(S,\mathbb{R})$ satisfies

(2.5a)' $\phi(F_0(0)) = X|S$,

(2.5b)' $Y_j|S = d\phi(F_0(0))(0,y_j)$, $y_j \in H^1(S,\mathbb{R})$.

The second variation of energy is, by definition,

$$(2.6) \qquad E''_{\gamma,X}(Y_1,Y_2) = \frac{\partial^2}{\partial t_1 \partial t_2} E(F(t_1,t_2))\Big|_{t_1=t_2=0},$$

where F is a variation of X by harmonic maps with variational fields Y_1 and Y_2.

Let γ be a C^k Jordan curve. We define a linear map Ω in $H^0(S,\mathbb{R}^n)$ by

$$(2.7) \qquad \Omega(y) = \langle y,\nu(x)\rangle\nu(x), \quad y \in H^0(S,\mathbb{R}^n)$$

where $\nu(x) = \nu_0 x$ is the unit tangent field of γ composed with x. By Theorem 1.6 Ω is continuous. Let T_x be the image of $H^0(S,\mathbb{R}^n)$ by Ω. Let k be the curvature vector of γ and $k(x) = k\circ x$. We define the operator $\Lambda_{\gamma,x}: T_x H^1(\gamma) \to T_x$ by

$$\Lambda_{\gamma,x} y = \Omega(\partial_r y) + \langle\partial_r x, k(x)\rangle y, \qquad y \in T_x H^1(\gamma).$$

2.8. Proposition. *Let $X \in H^{\frac{3}{2}}(D,\mathbb{R}^n)$ be a harmonic map spanning γ. If X is a critical point of the energy function for variation by harmonic maps then*

$$E''_{\gamma,x}(Y_1,Y_2) = \langle\Lambda_{\gamma,x} y_1, y_2\rangle_{H^0}$$

$$= \int_S \langle\partial_r y_1 + \langle\partial_r x, k(x)\rangle y_1, y_2\rangle d\theta$$

where $x = X|S$ and $y_j = Y_j|S$, $j = 1,2$.

Proof. Let $(f,x_0) \in C^k(S,\mathbb{R}^n) \times H^1(S,\mathbb{R})$ such that $\phi(f,x_0)=x$. Then

$$E''_{\gamma,X}(Y_1,Y_2) = d^2\varepsilon(f,x_0)((0,y_1),(0,y_2))$$

$$= \langle\partial_r y_1, y_2\rangle_{H^0} + \langle\partial_r x, \phi(f'',x_0) y_1 y_2\rangle_{H^0}.$$

By (2.4), $\langle\partial_r x, \nu(x)\rangle = 0$, from where

$$\langle\partial_r x, \phi(f'',x_0)\rangle = \langle\partial_r x, k(x)\rangle|\phi(f',x_0)|^2.$$

Substituting this expression we get

$$E''_{\gamma,X}(Y_1,Y_2) = \langle \Lambda_{\gamma,x} y_1, y_2 \rangle_{H^0}$$

as we wanted.

Let $x = X|S$, where X is a generalized minimal surface bounding a C^2 curve γ. Then the Theorem 2.1 says that x_θ lies in the Lebesgue space L_∞. It follows from the proof of Lemma 2.2 that $|\partial_r x| = |x_\theta|$, that is, $\partial_r x$ also lies in L_∞. Hence the operator $\Lambda_{\gamma,x}$ satisfies the Gärding inequality

$$(2.9) \qquad \langle \Lambda_{\gamma,x} y, y \rangle_{H^0} \geq \|y\|^2_{H^{\frac{1}{2}}} - C\|y\|^2_{H^0},$$

where $y \in T_x H^1(\gamma)$ and C is a constant.

2.10. Proposition. *Let γ be a Jordan curve of class C^k, $k \geq 2$. Let $x = X|S$ and X be a solution to Plateau's problem for γ. Then*

(a) $\Lambda_{\gamma,x}: T_x H^1(\gamma) \subset T_x \to T_x$ *is self adjoint,*

(b) *The spectrum of $\Lambda_{\gamma,x}$ is an increasing sequence of real numbers without accumulation points, that is, $\lambda_1 < \lambda_2 < \ldots$, $\lim \lambda_n = \infty$, and the λ_n-space has finite dimension,*

(c) $\Lambda_{\gamma,x}$ *is a Fredholm operator of index zero*

(d) *The eigenvalues of $\Lambda_{\gamma,x}$ lie in $H^{k-1}(S,\mathbb{R}^n)$.*

The proof of this proposition is an easy variation of standard methods in the theory of elliptic operators and it is included in Appendix A for the sake of completeness.

2.11. Example. Let $\gamma = S$. We know that $X(z) = z$, $z \in \bar{D}$, is a solution to Plateau's problem to S. Set $x = X|S$. Define

$$\alpha_n = \begin{cases} \frac{i}{\sqrt{2\pi}} e^{i\theta} & n = 0 \\ \frac{i}{\sqrt{\pi}} \cos n\theta e^{i\theta}, & n = 1,2,\dots, \end{cases}$$

$$\beta_n = \frac{i}{\sqrt{\pi}} \sin n\theta e^{i\theta}, \quad n = 1,2,\dots.$$

Then

$$\Lambda_{S,X}\alpha_n = \begin{cases} 0 & n = 0 \\ (n-1)\beta_n, & n \geq 1 \end{cases}$$

$$\Lambda_{S,X}\beta_n = (n-1)\beta_n, \quad n \geq 1,$$

that is, the spectrum of $\Lambda_{S,X}$ is $\{0,1,2,\dots\}$, where the 0-space has dimension three and the n-space, $n \geq 2$, has dimension two.

Proof. We have

$$\Lambda_{S,X}h = \langle\partial_r h, x_\theta\rangle x_\theta - h, \qquad h \in T_x H^1(S).$$

We are interested in $h = \mathrm{Re}(z^n)x_\theta$ or $h = Im(z^n)x_\theta$. Set

$$h_n = (z^n + \bar{z}^n)x_\theta, \quad z \in S, \ n \geq 0.$$

Let $\zeta = \partial X = (\frac{1}{2}, \frac{-i}{2})$ and $x_\theta = iz\zeta - i\bar{z}\bar{\zeta}$. Then

$$h_n = \frac{1}{2}\left(i(z^n+\bar{z}^n)(z-\bar{z}), (z^n+\bar{z}^n)(z+\bar{z})\right).$$

It follows from $z\bar{z} = 1$ that the harmonic extension X_n of h_n to D is

$$X = \begin{cases} (i(z-\bar{z}), z+\bar{z}), & n = 0 \\ \frac{1}{2}(i(z^{n+1}+\bar{z}^{n-1}-z^{n-1}-\bar{z}^{n+1}), \ z^{n+1}+\bar{z}^{n-1}+z^{n-1}+\bar{z}^{n+1}), & n \geq 1 \end{cases}$$

Now $r\frac{\partial}{\partial r} = z\partial + \bar{z}\bar{\partial}$ implies

$$r\,\frac{\partial X_n}{\partial r} = \begin{cases} X_0, & n = 0 \\ nX_n - (\mathrm{Im}(z^{n+1}+z^{n-1}),\ \mathrm{Re}(z^{n-1}-z^{n+1})), & n \geq 1. \end{cases}$$

Using that $x_\theta = \frac{1}{2}\,(\mathrm{i}(z-\bar{z}), z+\bar{z})$, we get

$$\langle x_\theta, (\mathrm{Im}(z^{n+1}+z^{n-1}),\ \mathrm{Re}(z^{n-1}-z^{n+1}))\rangle = 0, \quad z \in S.$$

Therefore

$$\Omega(\partial_r h_n) = \begin{cases} h_0, & n = 0 \\ nh_n, & n \geq 1 \end{cases}$$

from where $\Lambda_{S,X}h_0 = 0$ and $\Lambda_{S,X}h_n = (n-1)h_n$, $n \geq 1$. Analogously we obtain $\Lambda_{S,X}h_n^* = (n-1)h_n^*$ for $h_n^* = \mathrm{Im}(z^n)x_\theta$. Since $\{h_n, h_n^*\}$ is a complete orthonormal system of T_x, we see that the spectrum of $\Lambda_{S,X}$ is exactly $\{0,1,2,\ldots\}$.

3. Branch points and Jacobi fields of energy

Let γ be of class C^2 and $X:D \to \mathbb{R}^n$ be a solution to Plateau's problem for γ. By Nitsche's theorem 2.1 we have that the holomorphic curve $\partial X(z)$, $z \in D$, is bounded. Thus $\partial X = B\omega$, where B is a Blaschke product and $\omega: D \to \mathbb{C}^n$ is a holomorphic curve without zeros. The branch points of X are, by definition, the zeros of ∂X (or B) and, if $z_0 \in \bar{D}$ is a branch point of X, its order is the lowest integer m_0 such that

$$\lim_{z\to z_0} \frac{|\zeta(z)|}{|z-z_0|^t} = \begin{cases} 0, & 0 \leq t < m_0, \\ \infty, & t > m_0, \end{cases} \qquad z \in D.$$

Of course, if the branch point z_0 lies in D, its order is the multiplicity of z_0 as a zero of ∂X. In this definition the order of a branch point can be infinite if B is an arbitrary Blaschke function. For example,

$$B(z) = \prod_{n=1}^{\infty} \left(\frac{z_n - z}{1-z_n z}\right)^n, \qquad z \in D,$$

where $z_n = 1-e^{-2n}$, and $z_0 = 1$. Nevertheless, this is impossible if B is a Blaschke product of solution to Plateau's problem.

We will give here some relations between the kernel of $\Lambda_{\gamma,x}$ and the branches of X. To do that we need a regularity result which can be seem as a complement to Nitsche's theorem.

3.1. Theorem. *Let* $\gamma \subset \mathbb{R}^n$ *be of class* C^k, $k \geq 2$, *and* X *be a solution to Plateau's problem for* γ. *If* $x = X|S$ *then* x_θ, $\sin\theta x_\theta$ *and* $\cos\theta x_\theta$ *lie in the kernel of* $\Lambda_{\gamma,x}$. *In particular,* $x \in H^k(S,\mathbb{R}^n)$ *or, equivalently,* $X \in H^{k+1/2}(D,\mathbb{R}^n)$.

Proof. Let $x = \phi(f,y)$, $(f,y) \in E^k \times H^1(S,\mathbb{R})$. From (1.19) and Lemma 2.2 we obtain

$$\psi(f,y) = 0,$$

and

$$\frac{\partial\psi}{\partial y}(f,y)(0,y_1) = \langle\Lambda_{\gamma,x}h_1, \phi(f',y)\rangle,$$

where $h_1 = y_1\phi(f',y)$. By Proposition 1.22 we have

$$0 = \int_S a(1+y_\theta)\frac{\partial\psi}{\partial y}(f,y)(0,y_1), \quad \forall\ y_1 \in H^1(S,\mathbb{R})$$

$$= \langle\Lambda_{\gamma,x}h_1, ax_\theta\rangle_{H^0}, \quad \forall\ h_1 \in T_x H^1(\gamma)$$

where $a \in \{1, \sin\theta, \cos\theta\}$. We conclude from Proposition 2.10 that $ax_\theta \in \operatorname{Ker}\Lambda_{\gamma,x}$. In particular, $x_\theta \in H^{k-1}(S,\mathbb{R}^n)$.

There is a description of the kernel of $\Lambda_{\gamma,x}$ found by R. Böhme ([1] SATZ 6) for smooth solutions to Plateau's problem. After Theorem 3.1 we can extend this description to solutions for curves of class C^2.

3.2. Lemma ([1]). *Let* $\gamma \subset \mathbb{R}^n$ *be a curve of class* C^2. *Let* X *be a solution to Plateau's problem to* γ *and set* $x = X|S$. *If* $y \in T_x H^1(\gamma)$ *and* $Y:\bar{D} \to \mathbb{R}^n$ *is its harmonic extension to* $\bar{D}$, *then the following assertions are equivalent:*

(a) $y \in \operatorname{Ker} \Lambda_{\gamma,x}$,

(b) $\langle \partial_r y, x_\theta \rangle + \langle \partial_r x, y_\theta \rangle = 0$,

(c) $\langle \partial Y, \partial X \rangle = 0$.

The key point to extend Böhme's proof to this case is the existence of the trace of $4z^2 \langle \partial Y, \partial X \rangle$ which lies in some Hardy space H^2. The item (b) is exactly the imaginary part of the trace of this holomorphic curve.

3.3. Proposition. *Let* $\gamma \subset \mathbb{R}^n$ *be a Jordan curve of class* C^2 *and* X *be a solution to Plateau's problem for* γ. *Then* X *has only a finite number of branch points* $z_1,\dots,z_p$ *in* D *and* $z_{p+1},\dots,z_{p+q}$ *in* S. *Moreover if* m_j *is the order of* z_j, *then*

$$\dim(\operatorname{Ker} \Lambda_{\gamma,x}) \geq 3 + 2 \sum_{j=1}^{p} m_j + q.$$

Proof. Let $\{z_1,\dots,z_p\} \subset D$ and $\{t_1,\dots,t_q\} \subset S$ be branch points of X, with orders $m_1,\dots,m_p$ and $m_{p+1},\dots,m_{p+q}$, respectively. Define $y:S \to \mathbb{C}$ by

$$y(z) = \prod_{j=1}^{p} \left(\frac{z - z_j}{1-\bar{z}_j z}\right)^{s_j} \cdot \prod_{j=1}^{q} (t_j - z)^{-r_j}$$

where $0 \leq s_j \leq m_j$, $j = 1,2,\dots,p$, and $0 \leq r_j \leq m_{p+j}$, $j = 1,2,\dots,q$. We will show that the real and the imaginary parts of yx_θ both lie in $\operatorname{Ker} \Lambda_{\gamma,x}$. We have

$$x_\theta = iz\partial X - i\bar{z}\bar{\partial} X$$

for almost all $z \in S$ and

$$\partial X = \prod_{j=1}^{p} (z-z_j)^{m_j} \cdot \prod_{j=1}^{q} (z-t_j)^{m_{p+j}} \cdot Q(z)$$

where $Q:D \to \mathbb{C}^n$ is a holomorphic curve. Then

$$y(z)x_\theta(z) = izy(z)\partial X(z) - i\bar{z}y(z)\bar{\partial}X(z)$$

for almost all $z \in S$. The harmonic extension of $iy\partial X$ to $\bar{D}$ is trivial. We obtain from $z\bar{z} = 1$ that

$$\bar{z}\bar{y}\bar{\partial}X = \bar{z} \prod_{j=1}^{p} (1-z_j\bar{z})^{s_j}(\bar{z}-\bar{z}_j)^{m_j-s_j} \cdot \prod_{j=1}^{q} (\bar{t}_j\bar{z})^{r_j}(\bar{z}-\bar{t}_j)^{m_{p+j}-r_j} \cdot \bar{Q}.$$

Then the harmonic extension of $\bar{z}\bar{y}\bar{\partial}X$ to D is the right side of the last equality. Let Y be the harmonic extension of yx_θ. Then $\partial Y = \partial(izy\partial X)$ and

$$<\partial Y, \partial X> = \partial(izy)<\partial X, \partial X> + izy<\partial X, \partial^2 X>.$$

Now, $<\partial X, \partial X> = <\partial^2 X, \partial X> = 0$, and from Lemma 3.2 we get that the real and the imaginary parts of yx_θ belong to Ker $\Lambda_{\gamma,x}$. Now, the proof of the proposition follows from simple results on complex functions.

At this point we are in position to define the index and a degenerated solution to Plateau's problem.

We say that X is a *non-degenerate solution to Plateau's problem for* γ if the kernel of $\Lambda_{\gamma,X|S}$ has dimension 3. The *index of* X is the dimension of the subspace of $T_{X|S}H^1(\gamma)$ generated by the eigenvectors whose eigenvalues are negative.

The harmonic maps $Y:\bar{D} \to \mathbb{R}^n$ such that $Y|S \in$ Ker $\Lambda_{\gamma,X|S}$ are called the *Jacobi fields* of the energy.

3.4. Remark. If X is a non-degenerate solution to Plateau's problem then X is an immersion (see Prop. 3.3). In this case, there is a nice relation between Jacobi fields for the energy and for the area. We prove in [5] that, if $Y:\bar{D} \to \mathbb{R}^n$ is a Jacobi field for energy and $A(z)$, $z \in \bar{D}$, is the orthogonal projection of $Y(z)$ in the subspace of $\mathbb{R}^n$ orthogonal to $T_{X(z)}X(\bar{D})$, then A is a Jacobi field for the area. Moreover each Jacobi field for the area can be obtained in this way. If we consider only solutions in $\mathbb{R}^3$, then there is a complete description of relations between second variations of the area and the energy due to K. Schüffler [9].

3.5. Remark. Let $z = 1$ be a branch point of X with order k. Then

$$\left(\frac{\sin\theta}{1-\cos\theta}\right)^j x_\theta, \qquad 1 \le j \le k,$$

are Jacobi fields for the energy, that is, each boundary branch point of order k produces k linearly independent Jacobi fields. In contrast, an interior branch point of the same order produces $2k+1$ Jacobi fields.

4. Stability of non-degenerate solutions

Let E^k be the set of maps $f \in C^k(S,\mathbb{R}^n)$ which are embeddings and consider $x \in H^1(S,\mathbb{R}^n)$ such that its harmonic extension $X:\bar{D} \to \mathbb{R}^n$ is a solution to Plateau's problem for $f(S)$, $f \in E^k$. Let $U \ni x$ be an open set of $H^1(S,\mathbb{R}^n)$. We see from (1.21) that the conformal action of $S\times D$ into $H^1(S,\mathbb{R}^n)$ produces an orbit $O(x)$ (intersecting U) whose elements are trace of reparametrizations of X. We say that x *is the unique solution to Plateau's problem for* $f(S)$ *that lies in* U if no other orbit of solutions for $f(S)$ intersects U.

4.1. Theorem. *Let* $f \in E^k$, $k \ge 2$, *and* x_0 *be the trace of a non-degenerate solution* X_0 *to Plateau's problem for* $f_0(S)$. *Then there are open sets* $W_0 \ni f_0$ *in* E^k, $U_0 \ni x_0$ *in* $H^1(S,\mathbb{R}^n)$ *and a* C^{k-1} map $\Phi: W_0 \to U_0$ *such that:*

(a) $\Phi(f)$, $f \in W_0$, *is the trace of a non-degenerate solution to* $f(S)$ *and its index is equal to the index of* X_0,

(b) $\Phi(f)$, $f \in W_0$, *is the unique solution to Plateau's problem for* $f(S)$ *which lies in* U_0.

Proof. Let ε and ψ be the maps defined in (1.14) and (1.17). We saw in the proof of Theorem 3.1 that $x = \phi(f,y)$ is the trace of a generalized minimal surface bounding $f(S)$ if and only if $\psi(f,y) = 0$. In this case we have

$$d^2\varepsilon(f,y)((0,y_1),(0,y_2)) = \langle \Lambda_{f(S),x} h_1, h_2 \rangle_{H^0},$$

where $h_j = y_j \phi(f',y)$, $j = 1,2$. Hence

$$(4.2) \qquad y_2 \frac{\partial\psi}{\partial y}(f,y)(y_1) = \langle \Lambda_{f(S),x} h_1, h_2 \rangle,$$

that is, $\partial\psi/\partial y$ is a Fredholm operator (cf. Proposition 2.10). Therefore $\partial\psi/\partial y$ is Fredholm in a neighborhood of (f_0,y_0) where $x_0 = \phi(f_0,y_0)$. By Proposition 1.22 and 2.10

$$\dim(\mathrm{Ker}\ \frac{\partial\psi}{\partial y}(f,y)) \geq 3$$

for (f,y) in $E^k \times H^1(S,\mathbb{R}^n)$. We also have, for (f,y) near to (t_0,y_0), that

$$\dim(\mathrm{Ker}\ \frac{\partial\psi}{\partial y}(f,y)) \leq \dim(\ker \frac{\partial\psi}{\partial y}(f_0,y_0)) = 3,$$

because of Fredholm properties. Then the kernel of $\partial\psi/\partial y$ has constant dimension 3 in a neighborhood of (f_0,y_0). Applying the post theorem we get three neighborhoods $W_0 \ni f_0$ in E^k, $V_1 \ni (f_0,y_0)$ in $E^k \times H^1(S,\mathbb{R}^n)$, V_0 in a three dimension subspace of $H^1(S,\mathbb{R})$ and a C^{k-1} map $F: W_0 \times V_0 \to H$, H a complement of the subspace of $H^1(S,\mathbb{R}^n)$ containings V_0, such that the solutions of

$$(4.3) \qquad \psi(p) = 0, \quad p \in V_1,$$

are $P = (f,v,F(f,v))$, $(f,v) \in W_0 \times V_0$. The maps searched in the theorem is $\Phi(f) = \phi(f,v\ ,F(f,v\ ))$, where $f \in W_0$ and v_0 is a fixed point of V_0.

For each $f \in E^k$ the map $y \longmapsto \phi(f,y)$ is a diffeomorphism between $H^1(S,\mathbb{R})$ and $H^1(f(S))$. Since $H^1(f(S))$ is a submanifold of $H^1(S,\mathbb{R}^n)$ and ϕ is of class C^1, it is possible to prove the existence of an open ball $U_0 \ni x_0$ in $H^1(S,\mathbb{R}^n)$ such that for f near f_0 and $\phi(f,y) \in U_0$ we obtain that y is near y_0. Then the trace of the solutions to Plateau's problem for $f(S)$, f near f_0, has the expression found in (4.3).

The assertion about the index follows from the continuity of $\Lambda_{f(S),x}$ with respect to the parameters (f,y), where $x = \phi(f,y)$.

Let Γ^k, $k \geq 2$, be the set of C^k Jordan curves in $\mathbb{R}^n$. We identify Γ^k with the quotient of E^k by the relation: $f \sim g$ if $f(S) = g(S)$ and we bring the topology of E^k to Γ^k.

4.4. Corollary. *Let* $\gamma_0 \in \Gamma^k$, $k \geq 2$, *and* X_0 *be a non-degenerate solution to Plateau's problem for* γ_0. *Set* $x_0 = X|S$. *Then there are open sets* $W_0 \ni \gamma_0$ *in* Γ^k *and* $U_0 \ni x_0$ *in* $H^1(S,\mathbb{R}^n)$ *and a continuous* map $\Phi: W_0 \to U_0$ *such that:*

(a) $\Phi(\gamma)$, $\gamma \in W_0$, *is the unique trace of the solution to Plateau's problem for* γ *that lies in* U_0,

(b) *the solution for* $\gamma \in W_0$ *in* (a) *is non-degenerate and has the same index as* X_0.

4.5. Corollary. *If* $\gamma_0 \in \Gamma^k$, $k \geq 2$, *has only non-degenerate solutions to Plateau's problem, then* γ_0 *has a finite number* n_0 *of solutions and there is a neighborhood* $W_0 \ni \gamma_0$ *in* Γ^k *such that*

(a) *Each curve* $\gamma \in W_0$ *has exactly* n_0 *solutions and all of them are non-degenerate,*

(b) *Solutions of* $\gamma \in W_0$ *close to a solution to* γ_0 *have the same index.*

Proof. We can impose a global condition of three points to each solution to γ_0. By Nitsche's theorem 2.1 the set of solutions to Plateau's problem to γ_0 is compact in $C^{1,\mu}(S,\mathbb{R}^n)$. Theorem 4.1 says that each solution is isolated, and then there is only a finite number. Applying Corollary 4.6 we find an open set U of $H^1(S,\mathbb{R}^n)$ containing all solutions for γ_0 and an open set $W_0 \ni \gamma_0$

such that each curve $\gamma \in W_0$ satisfies (a) and (b) in U. It is a classical result that if $\gamma_n \in \Gamma^k$ converge to γ_0 in the C^2-topology (for exemple) then the solutions to Plateau's problem for γ_n converge to solutions for γ_0 in a $C^{1,\mu}$-topology (this also follows from Nitsche's theorem). Then if we lessen W_0 we find that each solution to Plateau's problem for $\gamma \in W_0$ has trace in U.

4.6. Corollary. *The set $\Gamma_2' \subset \Gamma^2$ of curves such that all solutions are non-degenerate is an open set of Γ^2 and the number of solutions is a continuous function on Γ_2'.*

5. Density

Let $\Gamma_k \subset \Gamma^k$, $k \geq 2$, be the subset of those Jordan curves whose solutions to Plateau's problem are immersions. Tromba called this set the fine embeddings (see [13] p. 95). Let $\Gamma_k' \subset \Gamma_k$ be the subset of curves whose solutions are non-degenerate. Set $\Gamma_\infty = \bigcap_{k \geq 2} \Gamma_k$, and $\Gamma_\infty' = \bigcap_{k \geq 2} \Gamma_k'$, both with the C^∞ topology.

In an analogous way we can define sets H_k', H_k and H^k substituting the C^k class of Jordan curves by the set of images of embeddings $f \in H^k(S, \mathbb{R}^n)$. In [13] the following result was proved.

5.1. Theorem. (A. Tromba). *H_k' is open and dense in H_k for all $k \geq 7$.*

5.2. Remark. Corollary 4.7 says that each curve of Γ_k' bounds a finite number of solutions to Plateau's problem. We also have, from this corollary, that Γ_2' is open in Γ_2. It follows from the continuous inclusion of Γ^k in $\Gamma^{k'}$, for $k \geq k'$, that Γ_k' is open in Γ_k for all $k \geq 2$. In this sense, Theorem 4.1 improves Theorem 5.1.

The next result is a Corollary to Theorem 5.1. Here, we will give a simple proof by using the techniques of the proceding section.

5.3. Theorem. *Γ'_∞ is open and dense in Γ_∞. In fact, Γ'_k is open and dense in Γ_k for any $k \geq 2$.*

Let M be the subset of $(f,y) \in E^k \times H^1(S,\mathbb{R})$ such that $\phi(f,y)$ is the trace of a generalized minimal surface without branch point. The idea of the proof of the theorem consists in showing that M is a submanifold of class C^{k-1} and that the projection $\pi: M \to E^k$, $\pi(f,y) = f$, is Fredholm of index 3. The conclusion of the proof follows from Sard's theorem, for $k \geq 5$.

Let $\psi: E^k \times H^1(S,\mathbb{R}) \to H^0(S,\mathbb{R})$ be the map defined in (1.14). The set M is a subset of $\psi^{-1}(0)$. Therefore, the image of $\frac{\partial\psi}{\partial y}(f,y)$, $(f,y) \in M$, is contained in the image of $d\psi(f,y)$, it is closed and has finite codimension (see Proposition 2.10 and 4.2). Then the image of $d\psi(f,y)$ is also closed and its orthogonal complement is contained in the kernel of $\frac{\partial\psi}{\partial y}(f,y)$. For the next computation it is convenient to go back to the notation (1.18). Now taking the derivative of ψ we get

$$\begin{aligned} d\psi(f,y)(f_1,y_1) &= \langle \partial_r \phi(f,y)(f_1,y_1),\ \phi(f',y)\rangle + \\ &\quad \langle \partial_r x,\ \phi(f'',y)y_1 + \phi(f',y)\rangle \\ &= \langle \Lambda_{f(s),x} h_1 + \partial_r x_1,\ \phi(f',y)\rangle + \qquad (5.4) \\ &\quad \langle \partial_r x,\ \phi(f_1',y)\rangle, \end{aligned}$$

from where

$$\int_S y_2 d\psi(f,y)(f_1,y_1)d\theta = \langle \Lambda_{f(S),x} h_1, h_2\rangle_{H^0} + \langle \partial_r x_1, h_2\rangle_{H^0} + \langle \partial_r x, h_{21}\rangle_{H^0}.$$

If y_2 is orthogonal to the image of $d\psi(f,y)$, then $\Lambda_{f(S),x} h_2 = 0$ and the last equation becomes

$$\langle \partial_r h_2, x_1\rangle_{H^0} + \langle y_2 \partial_r x, \phi(f_1',y)\rangle_{H^0} = 0, \qquad \forall f_1.$$

We obtain $x_{1\theta} = (1+y_\theta)\phi(f_1', y)$. Now integrating by parts gives us

$$<\partial_r h_2 - \frac{\partial}{\partial\theta}\left[\frac{y_2}{1+y_\theta}\,\partial_r x\right],\ x_1>_{H^0} = 0$$

The set of $x_1 = \phi(f_1, y)$ with $f_1 \in C^2(S,\mathbb{R}^n)$ is dense in $H^0(S,\mathbb{R}^n)$ because $x_1(z) = f_1(ze^{iy(z)})$, $z \in S$, and $ze^{iy(z)}$ is a homeomorphism of S with vanishing derivatives in a set of Lebesgue measure zero. It contains, for example, each H^2 map whose support doesn't intersect the zeros of derivatives of $ze^{iy(z)}$. Hence the last equality is equivalent to

$$\partial_r\left[\frac{y_2}{1+y_\theta}\,x_\theta\right] - \frac{\partial}{\partial\theta}\left[\frac{y_2}{1+y_\theta}\,\partial_r x\right] = 0.$$

If x is the trace of the generalized minimal surface then $y \in H^2(S,\mathbb{R})$ by Theorem 3.1. If, in addition, this surface has no branch points at the boundary, then $1 + y_\theta$ has no zeros. Thus multiplication by $1 + y_\theta$ is an isomorphism of $H^1(S,\mathbb{R})$ and, in particular, there is $w \in H^1(S,\mathbb{R})$ such that $y_2 = (1+y_\theta)w$. Therefore the last equality becomes the Tromba's fundamental transversality equation:

$$\partial_r(wx_\theta) - \frac{\partial}{\partial\theta}(w\,\partial_r x) = 0 \tag{5.5}$$

whose solution for w is the space generated by 1, $\sin\theta$, and $\cos\theta$ (see [13] pages 94-96). Then the codimension of $d\psi(f,y)$, $(f,y) \in M$, is three and by Proposition 1.22 the codimension of the image of $d\psi$ is at least three. We conclude that there exists a neighborhood U of M where $d\psi(f,y)$, $(f,y) \in U$ has a closed image with codimension three.

Let $(f,y) \in M$. We define V_0 as the subspace of $H^1(S,\mathbb{R})$ generated by $\{1+y_\theta, (1+y_\theta)\sin\theta, (1+y_\theta)\cos\theta\}$ and let V_1 be the complement of the kernel of $\frac{\partial\psi}{\partial y}(f,y)$. Let $F_0 \subset C^k(S,\mathbb{R}^n)$ be a finite dimensional subspace such that $d\psi(f,y)$ is an isomorphism of $F_0 \times V_1$ over its image. Now we observe that F_0 is finite dimensional and therefore it has a complement F_1 in $C^k(S,\mathbb{R}^n)$. By the post theorem we obtain that M is locally a graphic of a C^{k-1} map $g:W \subset F_1 \times V_0 \to F_0 \times V_1$. Therefore M is a C^{k-1} submanifold. We also get the following characterization of non-degenerate solutions:

(5.6) $\phi(f,y)$ is the trace of a non-degenerate solution to Plateau's problem for $f(S)$ if and only if the dimension of F_0 is zero.

Obviously the projection $\pi:M \to E^k$ is a C^{k-1} Fredholm map of index 3. We also get that π is regular at $(f,y) \in M$ if and only if $\phi(f,y)$ is the trace of a non-degenerate solution to Plateau's problem for $f(S)$, that is, $\dim F_0 = 0$. To complete the proof we take $k \geq 5$ and apply Sarde's theorem. The assertion about the density and openness of Γ'_k for $2 \leq k \leq 4$ now follows from Corollary 4.7 and the fact that the inclusion of E^k into $E^{k'}$ is dense if $k \geq k'$.

It is interesting to summarize here what we have done in the proof of Theorem 5.3.

5.7. Proposition. *Let M be the set of (f,y) in $E^k \times H^1(S^1,\mathbb{R})$ such that $\phi(f,y)$ is the trace of a generalized minimal surface free of branch points up to the boundary. Then, M is a submanifold of class C^{k-1} and the projection* map $\pi:M \to E^k$, *$\pi(f,y) = f$, for $(f,y) \in M$, is Fredholm of index 3 and class C^{k-1}.* A *point $(f,y) \in M$ is a regular point for π if and only if $\phi(f,y)$ is the trace of a non-degenerate solution to Plateau's problem for $f(S^1)$.*

Remark. It is possible to impose a three point condition on M and get π with index zero.

Because ψ applies $E^{k+j} \times H^j(S,\mathbb{R})$ into $H^j(S,\mathbb{R}^n)$ and is of class C^k for $j \geq 1$, it is easy to conclude that:

5.8 Corollary. *M is a C^k submanifold of $E^{k+j} \times H^j(S,\mathbb{R})$ and the same conclusion of Proposition* 5.7 *holds.*

Appendix A: Proof of Proposition 2.10.

Let H_t, $t \in \mathbb{R}$, be a chain of Hilbert spaces and $\Lambda : H_t \to H_{t-k}$ be an operator (of order k) such that:

(A.1) If $t > t'$ then H_t is dense subset of $H_{t'}$, and the inclusion of H_t into $H_{t'}$ is a compact map.

(A.2) H_{-t}, for $t > 0$, is the dual of H_t with respect to the inner product of H_0.

(A.3) The image $(\Lambda+\lambda)H_{t+k}$ of H_{t+k} by $\Lambda+\lambda$, $\lambda \in \mathbb{R}$, is a closed subspace of H_t, for $t \geq 0$.

(A.4) Λ is a symmetric operator satisfying the Garding inequality

$$<\Lambda h, h>_{H^0} \geq c_0 |h|^2_{H_{k/2}} - c_1 |h|^2_{H_0}$$

where c_0 and c_1 are constants.

Under these conditions, the operator Λ satisfies the properties of Proposition 2.10. The proof of this fact is standard and can be found in textbooks about elliptic operators like [7]. In fact, a more general result can be proved. The argument can be summarized as follows:

First step: We start setting $\Sigma = \Lambda + \lambda$ where λ is a real number so large that the following inequality holds

(A.5) $$<\Sigma h, h> \geq C_2 |h|^2_{H_{k/2}}, \quad h \in H_k,$$

for some constant C_2. The Lax-Milgran lemma implies that for each $y \in H_0$, there is $h \in H_{k/2}$ such that $\Sigma h = y$ (in H_{-k}). Then $\Sigma : H_{k/2} \to H_{-k/2}$ is an isomorphism. In particular the image ΣH_k is dense in H_0. The property (A.3) saies that $\Sigma : H_k \to H_0$ is an isomorphism. Therefore $\Sigma : H_k \subset H_0 \to H_0$ is self adjoint. We also have that $\Sigma^j : H_{jk} \to H_0$ is an isomorphism over the image for all $j \geq 1$. If the image $\Sigma^j H_{jk}$ is not dense in H_0 then there exists $h_0 \in H_{jk/2}$ such that $\langle \Sigma^j h, h_0 \rangle_0 = 0$, for all $h \in H_{jk}$. Taking a sequence $h_n \in H_{jk}$ converging to h_0 in $H_{jk/2}$ we find that $\langle h_0, \Sigma^j h_0 \rangle_0 = 0$. If j is even it is easy to conclude that $h_0 = 0$. For odd j we get the same conclusion applying (A.5).

Second step: It follows from (A.5) that the inverse Σ^{-1} of Σ is a continuous linear map from H_0 into $H_{k/2}$. Let $\Sigma_0 : H_0 \to H_0$ be the composition of Σ^{-1} with the inclusion of $H_{k/2}$ into H_0. Then Σ_0 is a continuous compact positive defined self adjoint operator. Applying the spectral theory to Σ_0 we get the properties (b) and (c) of Proposition 2.10, regardless of the fact: $\Sigma_0 h = \delta h$ if and only if $\Lambda h = (1/\delta - \lambda)h$.

Third step. By the first step we have that the solutions of $\Lambda h = \lambda h$ (or equivalently, $\Sigma h = \lambda' h$) lie in the intersection $\bigcap H_{jk}$ for all $j \geq 1$.

Now we will prove Proposition 2.10. Let Ω be defined as in (2.7) and let H_t be the image by Ω of the Sobolev space $H^t(S, \mathbb{R}^n)$. Then H_t has the properties (A.1) and (A.2) and $\Lambda_{\gamma,x}$ satisfies (A.4). Therefore it is enough to prove (A.3) for $\Lambda_{\gamma,x}$.

Let $x \in H^t(S, \mathbb{R}^n)$, $t \geq \frac{1}{2}$, and $X \in H^{(t+1)/2}(D, \mathbb{R}^n)$ be the harmonic extension of x to $\bar{D}$. If $x = \Sigma \alpha_j e^{ij\theta}$ then

$$X = \Sigma r^{|z|} \alpha_j e^{ij\theta}, \quad \theta \in \mathbb{R}, \quad 0 \leq r \leq 1.$$

Let X_r, $0 < r < 1$, be the restriction of X to the disk

$$D_r = \{z \in \mathbb{C} / |z| \leq r < 1\}.$$

Then

$$(A.6) \qquad |x|_t \leq \sqrt{1-r}\,|x|_t + |X_r|_{H^{t+1}(D_r)}, \qquad r < 1.$$

To prove this, observe that the trace map is an isomorphism between $H^{(t+1)/2}(\partial D_r, \mathbb{R}^n)$ and the subspace of harmonic maps of $H^{t+1}(D_r, \mathbb{R}^n)$. Then

$$\begin{aligned}|x|_t^2 &= \sum_j (1+j^2)^t |\alpha_j|^2 \\ &= \sum_j (1+j^2)^t (1-r^{2|j|}) |\alpha_j|^2 + \sum_j (1+j^2)^t |r^{|j|}\alpha_j|^2 \\ &\leq (1-r)|x|_t^2 + |\text{trace } X_r|^2_{(t+1)/2} \\ &= (1-r)|x|_t^2 + |X_r|^2_{H^{t+1}(D_r)}\end{aligned}$$

as we wished.

Let $\Sigma = \Lambda_{\gamma,x} + \lambda$ as in the first step. We will prove that the image of H_{t+1} by Σ is a closed subspace of H_t, $t \geq 0$. If this is not the case, there are $h_n \in H_{t+1}$ such that $|h_n|_{t+1} = 1$ and Σh_n converge to zero in H_t. Let X_n be the harmonic extensions of h_n to D. By (A.5) we have that X_n converges to zero in $H^1(D,\mathbb{R}^n)$. Then, for $r < 1$, the restriction $X_n|D_r$ is a sequence in $H^k(D_r,\mathbb{R}^n)$, $k \geq 0$, convergint to zero (this follows, from example from a direct computation of the Poission integral and the fact that the trace of X converges to zero in $H^0(S,\mathbb{R}^n)$). Then $X_n|D_r$, $r < 1$, converges to zero in $H^{t+2}(D_r, \mathbb{R}^n)$ and we get contradiction on (A.6). Therefore $\Sigma: H_{t+1} \to H_t$ is an isomorphism over its image.

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