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On the existence of complete bounded minimal surfaces in $\mathbb{R}^n$.

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In this note we obtain partial results on the following conjecture, attributed to Calabi [2]: a complete minimal surface in R^3 is not contained in a ball or a half-space, unless it is a plane. In what follows let $n \geq 3$ be an integer and R^n_+ be the set $\{(x_1, \ldots, x_n) \mid x_i > 0,\ i = 1, \ldots, n\}$. We prove the following

Theorem. *There is no complete minimal surface with bounded curvature immersed in R^n_+. In particular, a complete minimal surface in R^n with bounded curvature is an unbounded subset of R^n.*

The proof involves the use of Herglotz's theorem on the boundary behaviour of positive harmonic functions on the unit disc and the analysis of the gradient flow of a function closely related to the Euclidean distance.

§1. Some Lemmas.

Suppose there is a complete minimal surface M immersed in R^n_+. By the Riemann-Koebe theorem the universal covering $\tilde{M}$ of M is either the complex plane or the unit disc. Since the coordinate functions of the immersion are harmonic and positive the first alternative is ruled out. Hence, in order to prove the theorem it suffices to show that a conformally flat metric g on the unit disc D with bounded curvature cannot be realized by a minimal immersion into R^n_+. Let us suppose, by way of contradiction, that $I : (D, g) \to R^n_+$ is such an immersion.

Lemma 1. (Herglotz's theorem, [1] page 38). *A positive harmonic function on D has finite radial limits at almost every point of $\partial D = \{z \mid |z| = 1\}$.*

Let X be a vector field on a manifold and suppose that the trajectory of X throught p, $x_p(t) = x(t, p)$ is defined for all $t \geq 0$. The ω-limit of p

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After This paper was submitted, our attention was called to the paper "The Exterior diameter of an immersed Remannian manifold", by Ju. A. Aminov, Math. USSR Sbornik, vol. 2.1 (1973) 449-454, which contains, with a different proof, a weaker form of our theorem.

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is the set $\omega(p) = \bigcap_{t \geq 0} \overline{x_p([t, \infty))}$. A singularity of X is a point where it vanishes. The next lemma follows essentially from the observation that a trajectory of a gradient field cannot intersect the same level line of the function twice.

Lemma 2. *Let X be a gradient field on a Riemannian manifold and suppose that X generates a flow $x_p(t) = x(t, p)$, defined for all $t \geq 0$. Then $\omega(p)$ is either empty, or consists entirely of singularities of X.*

By lemma 1 it is possible to choose a point $z_0 \in \partial D$ for which the radial limits at z_0 exist for every coordinate function of the immersion $I:(D,g) \to R^n_+$. Let $p_0 \in R^n$ be the radial limit of I at z_0. Let $f(p) = \|I(p) - p_0\|^2$, K = curvature of g, $K_0 = \sup|K|$, and $D_\varepsilon = f^{-1}([0,\varepsilon])$. Also, let α be the (vector-valued) second fundamental form of I.

Lemma 3. *The estimate Hess $f(p)(X,X) \geq 2\|X\|^2(1 - \sqrt{2\,\varepsilon K_0})$ holds for every $X \in T_pD$, $p \in D_\varepsilon$ and $0 < \varepsilon < \dfrac{1}{2K_0}$.*

Proof. A straightforward computation shows that

$$\frac{1}{2} \text{ Hess } f(p)(X,X) = \|X\|^2 + \langle \alpha(X,X), I(p) - p_0 \rangle.$$

The result now follows from Schwarz's inequality by noting that $\|\alpha\|^2 = 2|K|$.

Let $x_p(t) = x(t,p)$ be the flow of $-\operatorname{grad} f$ (in the metric g). The set D_ε is invariant under the flow, that is, $x(t \times D_\varepsilon) \subset D_\varepsilon$ for all $t \geq 0$ such that $x(t,p)$ is defined. It follows from this, the completeness of (D,g), and the fact that $-\operatorname{grad} f$ is bounded on D_ε, that $x(t,p)$ is actually defined for all positive t.

Lemma 4. *There exist $\varepsilon_0 \in \left(0, \dfrac{1}{2\varepsilon_0 K_0}\right)$ and $p \in D_{\varepsilon_0}$ for which the orbit $x_p(t)$ of the field $-$ grad f has infinite length.*

Proof. For any $\varepsilon > 0$ let $\tilde{D}_\varepsilon$ be the connected component of D_ε that contains a segment of the ray joining 0 to z_0. The existence of $\tilde{D}_{\varepsilon_0}$ is guaranteed by lemma 1. We divide the proof in two cases:

1) There exist an $\varepsilon_0 \in \left(0, \dfrac{1}{2K_0}\right)$ for which $\tilde{D}_{\varepsilon_0}$ contains no critical points of f.

In this case, we assert, the orbit $x(t,p)$ through any point of $\tilde{D}_{\varepsilon_0}$ has infinite length. If not, the completeness of (D,g) would show that $\omega(p) \neq \varnothing$.

Since $\omega(p) \subset \tilde{D}_{\varepsilon_0}$, lemma 2 would show that $\tilde{D}_{\varepsilon_0}$ contains singularities of grad f, a contradiction.

2) Suppose now that for each $\varepsilon \in \left(0, \frac{1}{2K_0}\right)$ the set $\tilde{D}_{\varepsilon_0}$ contains critical points of f. Since $\tilde{D}_{\varepsilon_1} \supset \tilde{D}_{\varepsilon_2}$ whenever $\varepsilon_1 > \varepsilon_2$ we see that any $\tilde{D}_\varepsilon$ contains infinitely many critical points of f. Choose $\varepsilon_0 \in \left(0, \frac{1}{2K_0}\right)$ and let C be the set of critical points of f which are in $\tilde{D}_{\varepsilon_0}$. For any $p \in C$ let $A(p) = \{q \in \tilde{D}_{\varepsilon_0} \mid \omega(q) = p\}$. By lemma 3 any point $p \in C$ is a (local) strict minimum. It follows from this and the continuity of flow that $A(p)$ is an open (non-empty) set. Besides, $A(p) \cap A(q) = \emptyset$ if $p \neq q$. Since $\tilde{D}_{\varepsilon_0}$ is connected there is some $p_0 \in \tilde{D}_{\varepsilon_0} / \bigcap_{p \in C} A(p)$. We assert that $x(t, p_0)$ has infinite length. Indeed, if this were not the case we would have $\omega(p_0) \neq \emptyset$. Again, since all critical points of $\tilde{D}_{\varepsilon_0}$ are strict minima, the set $\omega(p_0)$ cannot contain more than one point. We would therefore conclude that $\omega(p_0) \in C$, a contradiction.

§2. Proof of the theorem.

The proof of the theorem can now be easily finished. Let $x_p(t)$ be the orbit of $-$ grad f given by lemma 4. Let $y(t)$ be a reparametrization of $x_p(t)$ by arc-length. Hence y is defined in $[0, \infty)$. Let $h : [0, \infty) \to R$ be given by $h(t) = f(y(t))$. Simple computations show that $h' = -\|\text{grad } f\|^2$ and $h'' = \text{Hess } f(y', y')$. On the other hand, by lemma 3 we have $h'' \geq$ $\geq 2(1 - \sqrt{2\varepsilon_0 K_0} > 0$. The last inequality implies that $h(t)$ will ultimately grow like t^2. But this contradicts the fact that h is decreasing.

References

[1] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Inc. Englewood Cliffs, N. J. (1962).

[2] W. Meeks III, *Lectures on Plateau's Problem*, II Escola de Geometria Diferencial, held in the Universidade Federal do Ceará, Brasil (1978).

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