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JOÃO VITOR DA SILVA

SHARP AND IMPROVED REGULARITY ESTIMATES TO FULLY NONLINEAR EQUATIONS AND FREE BOUNDARY PROBLEMS

FORTALEZA
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Thesis submitted to the Post-graduate Program of the Mathematical Department of Universidade Federal do Ceará in partial fulfillment of the necessary requirements for the degree of Ph.D in Mathematics. Area of expertise: Analysis

Advisor: Prof. Dr. Eduardo Vasconcelos Oliveira Teixeira

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I dedicate this work to my beloved parents João Virgínio da Silva and Maria Socorro Mendonça da Silva by the meaning of my life, my brothers (Ednaldo and Leond) and sisters (Eliana and Juliana), and to Jenny (Lolita), with love.

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"If I have seen further, it is by standing on the shoulders of Giants."(ISAAC NEWTON)

RESUMO

Esta tese componhe-se dos seguintes 3 manuscritos que tratam de estimativas de regularidade para equações parabólicas totalmente não-lineares e problemas elípticos de uma-fase singularmente perturbados.

Sharp regularity estimates for second order fully nonlinear parabolic equations - *Trabalho em conjunto com Eduardo V. Teixeira.*

O principal propósito do segundo capítulo é provar estimativas de regularidade precisas para soluções (no sentido da viscosidadde) de equações parabólicas totalmente não-lineares da seguinte forma:

$$\frac{\partial u}{\partial t} - F(D^2u, Du, x, t) = f(x,t) \quad \text{in} \quad Q_1 = B_1 \times (-1, 0], \tag{Eq1}$$

onde $F : \text{Sym}(n) \times \mathbb{R}^n \times Q_1 \times \to \mathbb{R}$ é um operador uniformemente elíptico e $f \in L^{p,q}(Q_1)$ (espaço de Lebesgue com normas mistas). Ressaltamos que a quantidade $\Xi(n,p,q) := \frac{n}{p} + \frac{2}{q}$ determinará (precisamente) a qual regime de regularidade uma solução deverá pertencer. Resumidamente, quando $1 < \Xi(n,p,q) < 2 - \epsilon_F$ provamos que soluções são Hölder contínuas (no sentido parabólico) para um expoente $0 < \alpha(n,p,q) < 1$. O caso $\Xi(n,p,q) = 1$ representa uma situação limítrofe crítica a qual divide a teoria de regularidade. Neste cenário obtemos uma estimativa de regularidade (universal) do tipo Log-Lipschitz precisa. Quando $0 < \Xi(n,p,q) < 1$, soluções são localmente da classe $C^{1+\sigma, \frac{1+\sigma}{2}}$. Finalmente, no "caso limite", isto é, $\Xi(n,p,q) = 0$, mostramos estimativas de regularidade $C^{1,\text{Log-Lip}}$ desde que F seja convexo na componente das matrizes Hessianas, por exemplo.

Schauder Type Estimates for "Flat" Viscosity Solutions to Non-convex Fully Nonlinear Parabolic Equations and Applications - *Trabalho em conjunto com Disson S. dos Prazeres*

Em um segundo momento (a saber, no terceiro capítulo), estabelecemos estimativas do tipo Schauder para soluções flat (ou seja, com oscilação suficientemente pequena) para equações parabólicas totalmente não-lineares (não convexas) da seguinte forma:

$$\frac{\partial u}{\partial t} - F(x, t, D^2u) = f(x,t) \quad \text{in} \quad Q_1 \tag{Eq2}$$

desde que os coeficientes de F e o termo fonte f gozem de um módulo de continuidade do tipo Dini . Além disso, provamos um resultado de regularidade parcial, bem como um teorema do tipo Evans-Krylov para essa classe de problemas. Finalmente, para problemas com dados meramente contínuos, provamos que soluções flat de (Eq2) são parabolicamnte $C^{1,\text{Log-Lip}}$ regulares.

Regularity up to the boundary for fully nonlinear singularly perturbed elliptic equations - *Trabalho em conjunto com Gleydson C. Ricarte*

Posteriormente (para ser preciso, no capítulo 4), estamos interessados em estudar a regularidade até o bordo de problemas elípticos totalmente não-linearmente de uma-fase singularmente perturbados do seguinte tipo:

$$F(x, Du^{\varepsilon}, D^2u^{\varepsilon}) = \zeta_{\epsilon}(u^{\epsilon}) \quad \text{in} \quad \Omega \subset \mathbb{R}^n \tag{Eq3}$$

onde ζ_{ε} se comporta assintoticamente como a medida δ_0 de Dirac quando ε vai para zero. Nesse contexto, estabelecemos cotas globais do gradiente (independentes do parâmetro ε) para soluções no sentido da viscosidade de (Eq3), as quais nos permitem passar o limite e obter a regularidade ótima (estimativas Lipschitz) para o problema de fronteira livre associado.

Palavras-chave: Equações elípticas totalmente não-lineares. Equações parabólicas totalmente não-lineares. Módulo de continuidade preciso. Soluções flat. Propriedades de suavidade de soluções. Problemas de uma-fase. Regularidade até o bordo. Equações singularmente perturbadas. Estimativas globais do gradiente.

ABSTRACT

The thesis consists of the following three papers on regularity estimates for fully non-linear parabolic equations and one-phase singularly perturbed elliptic problems.

Sharp regularity estimates for second order fully nonlinear parabolic equations - *Joint work with Eduardo V. Teixeira.*

The purpose of the first chapter is prove sharp regularity estimates for viscosity solutions to fully non-linear parabolic equations of the form

$$\frac{\partial u}{\partial t} - F(D^2u, Du, x, t) = f(x,t) \quad \text{in} \quad Q_1 = B_1 \times (-1, 0], \tag{Eq1}$$

where F is a uniformly elliptic operator and $f \in L^{p,q}(Q_1)$. The quantity $\Xi(n,p,q) := \frac{n}{p} + \frac{2}{q}$ determines which regularity regime a solution to (Eq1) belongs to. We prove that when $1 < \Xi(n,p,q) < 2 - \epsilon_F$, solutions are parabolic-Hölder continuous for a sharp, quantitative exponent $0 < \alpha(n,p,q) < 1$. The case $\Xi(n,p,q) = 1$ is a critical borderline situation as it divides the regularity theory. In this scenario, we obtain a sharp universal Log-Lipschitz regularity estimate. When $0 < \Xi(n,p,q) < 1$, solutions are locally of class $C^{1+\sigma, \frac{1+\sigma}{2}}$ and in the limiting case $\Xi(n,p,q) = 0$, we show $C^{1,\text{Log-Lip}}$ regularity estimates provided F is convex in the Hessian argument for example.

Schauder Type Estimates for "Flat" Viscosity Solutions to Non-convex Fully Nonlinear Parabolic Equations and Applications - *Joint work with Disson S. dos Prazeres*

In a second moment we establish Schauder type estimates for flat solutions to non-convex fully non-linear parabolic equations of the following form

$$\frac{\partial u}{\partial t} - F(x, t, D^2u) = f(x,t) \quad \text{in} \quad Q_1 \tag{Eq2}$$

provided the coefficients of F and the source f are Dini continuous. Furthermore, we prove a partial regularity result, as well as a theorem of Evans-Krylov type. Finally, for problems with merely continuous data we prove that flat solutions to (Eq2) are parabolic $C^{1,\text{Log-Lip}}$ smooth.

Regularity up to the boundary for fully nonlinear singularly perturbed elliptic equations - *Joint work with Gleydson C. Ricarte*

Posteriorly, we are interested in studying regularity up to the boundary for one-phase singularly perturbed fully non-linear elliptic problems

$$F(x, Du^{\varepsilon}, D^2u^{\varepsilon}) = \zeta_{\epsilon}(u^{\epsilon}) \quad \text{in} \quad \Omega \subset \mathbb{R}^n \tag{Eq3}$$

where ζ_{ε} behaves asymptotically as the Dirac measure δ_0 as ε goes to zero. We shall establish global gradient bounds independent of the parameter ε to viscosity solutions to (Eq3), which allow us to pass the limit and obtain optimal regularity for free boundary problem.

Keywords: Fully nonlinear elliptic equations. Fully nonlinear parabolic equations. Sharp moduli of continuity. Flat solutions. Smoothness properties of solutions. One-phase problems. Regularity up to the boundary. Singularly perturbed equations. Global gradient estimates.

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SUMÁRIO

1 INTRODUCTION

The study of second order parabolic equations plays a fundamental role in the development of several fields in pure and applied mathematics, such as differential geometry, functional and harmonic analysis, infinite dimensional dynamical systems, probability, as well as in mechanics, thermodynamics, electromagnetism, among others. The non-homogeneous heat equation,

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in} \quad Q_1 = B_1 \times (-1, 0], \tag{1.1}$$

where $f \in L^p(Q_1)$, $p > \frac{n+2}{2}$, represents the simplest linear prototype. Its mathematical analysis goes back to 19th century and the regularity theory for such an equation is nowadays fairly complete.

The fully nonlinear parabolic theory is quite more recent. The Krylov-Safonov's fundamental works in KRYLOV and SAFONOV (1979) and KRYLOV and SAFONOV (1980) on linear, non-divergence form elliptic/parabolic equations set the beginning of the development of the regularity theory for viscosity solutions to fully nonlinear parabolic equations. Since then this has been a central subject of research. In effect, L. Wang in WANG (1992a,b) proves Harnack inequality and $C^{1+\alpha, \frac{1+\alpha}{2}}$ estimates for fully nonlinear parabolic equations as follows $\frac{\partial u}{\partial t} - F(D^2u, Du, x, t) = f \quad \text{in} \quad Q_1$, and M. Crandall *et al* in CRANDALL, KOCAN, and ŚWIECH (2000) develop an L^p-viscosity theory, see also Imbert-Silvestre's survey in IMBERT and SILVESTRE (2013) as regards to existence, comparison and Hölder regularity of viscosity solutions. With regards to higher regularity estimates, N. Krylov in KRYLOV (1982) and KRYLOV (1983) obtains $C^{2+\alpha, \frac{2+\alpha}{2}}$ estimates for solutions to $\frac{\partial u}{\partial t} - F(D^2u) = 0$, under convexity assumptions (see also (WANG, 1992b, Section 4.3) for similar results). Finally, we must comment that Caffarelli-Stefanelli in CAFFARELLI and STEFANELLI (2008) exhibit solutions to uniform parabolic equations that are not $C^{2,1}$, thereby showing the impossibility (in general) of an existence theory for classical solutions to such parabolic equations.

Let us remember that parabolic equations in non-divergence form involving sources with mixed integrability conditions, namely

$$\frac{\partial v}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t) D_{ij} v = f \in L^{p,q}(Q_1),$$

have also been fairly well studied in the literature in the last decades. Existence in suitable parabolic Sobolev spaces has been proven by N. Krylov, see KRYLOV (2008, 2007), see also the sequence of works by D. Kim KIM (2008, 2010). Insofar as regularity estimates are concerned, only qualitative results are available when p and q are sufficient large. Nonetheless, as in a number of physical, geometric and free boundary problems, obtaining a quantitative sharp regularity estimate for solutions is decisive for a finer analysis. Therefore, the purpose of the second chapter in this thesis is to obtain sharp moduli of continuity of solutions for second order parabolic equation as follows

$$\frac{\partial u}{\partial t} - F(D^2u, Du, x, t) = f(x,t) \quad \text{in} \quad Q_1,$$

involving sources with mixed norms, for which the corresponding estimates depend only on dimension, p, q and universal parameters.

In this first moment, we define the quantity

$$\Xi(n,p,q) := \frac{n}{p} + \frac{2}{q},$$

which to determinate the sharp regularity regime in which viscosity solutions belong to. The first quantitative regularity result we show states that if $1 < \Xi(n,p,q) < \frac{n+2}{\mathfrak{p}_0}$, where $\frac{n+2}{2} \leq \mathfrak{p}_0 < n+1$ is a universal constant[1], then solutions are α-Hölder continuous in the parabolic sense for the sharp exponent $\alpha := 2 - \Xi(n,p,q)$ (see Section 2.2 for the treatment of this case).

Intuitively, as $\Xi(n,p,q)$ decreases, one should expect that regularity estimates of solutions improve . The borderline is $\Xi(n,p,q) = 1$, where we prove that solutions are Log-Lipschitz continuous in the parabolic sense (see Section 2.3 for this analysis). This result is a further quantitative improvement to the fact that $u \in C^{\alpha, \frac{\alpha}{2}}_{loc}(Q_1)$ for any $0 < \alpha < 1$.

When $0 < \Xi(n,p,q) < 1$, we show that solutions are $C^{1+\beta, \frac{1+\beta}{2}}$, for $\beta \leq 1 - \Xi(n,p,q)$ (see Section 2.4 for this case). Qualitative results, when $p = q > n+1$, were previously obtained by M. Crandall *et al* (CRANDALL, KOCAN, and ŚWIECH, 2000, Section 7) and L. Wang (WANG, 1992b, Section 1.2).

Finally, we deal with the upper borderline case, $f \in \text{BMO}(Q_1)$. Under appropriate higher *a priori* estimates on F, we show that solutions are $C^{1,\text{Log-Lip}}_{loc}(Q_1)$ in the parabolic sense (see Section 2.5 for this approach). Particularly, $u \in C^{1+\alpha, \frac{1+\alpha}{2}}_{loc}(Q_1)$ for any $0 < \alpha < 1$.

The table below provides a global picture of the parabolic regularity theory for equations with anisotropic sources, in comparison with the sharp elliptic estimate from TEIXEIRA (2006):

Here, $\varsigma := 2 - \Xi(n,p,q)$ and $\mu := \min\{\alpha^-, 1 - \Xi(n,p,q)\}$. Moreover, α^- means $\alpha - \zeta$ for every $0 < \zeta < \alpha$ and $\varepsilon^2 \in \left(0, \frac{n}{2}\right)$ is a universal constant.

[1]The universal constant $\mathfrak{p}_0$ is one which gives the minimal range for which the Aleksandrov-Bakelman-Pucci-Krylov-Tso maximum principle holds for L^p-viscosity solutions provided $p > \mathfrak{p}_0$ (cf. (CRANDALL, KOCAN, and ŚWIECH, 2000, Section 2) for more details).

[2]Here ε is the Escauriaza's universal constant which provides the minimal range which the Caffarelli's $W^{2,p}$ theory (resp. Harnack inequality and Hölder regularity) holds for viscosity solutions to fully nonlinear elliptic equations, since $p \geq n - \varepsilon$ (see (ESCAURIAZA, 1993, Theorem 1 and Lemmas 1 and 2) for more details).

$f \in L^p(B_1)$	Regularity of u	$f \in L^{p,q}(Q_1)$	Regularity of u
$n-\varepsilon \leq p < n$	$C^{0,2-\frac{n}{p}}_{loc}(B_1)$	$1 < \Xi(n,p,q) < \frac{n+2}{\mathfrak{p}_0}$	$C^{\varsigma,\frac{\varsigma}{2}}_{loc}(Q_1)$
$p = n$	$C^{0,\text{Log-Lip}}_{loc}(B_1)$	$\Xi(n,p,q) = 1$	$\text{par} - C^{0,\text{Log-Lip}}_{loc}(Q_1)$
$p > n$	$C^{1,\min\left\{\alpha^-,1-\frac{n}{p}\right\}}_{loc}(B_1)$	$0 < \Xi(n,p,q) < 1$	$C^{1+\mu,\frac{1+\mu}{2}}_{loc}(Q_1)$
$\text{BMO} \supsetneq L^\infty$	$C^{1,\text{Log-Lip}}_{loc}(B_1)$	$\text{BMO} \supsetneq L^\infty$	$\text{par} - C^{1,\text{Log-Lip}}_{loc}(Q_1)$
Elliptic Theory	**X**	**Parabolic Theory**	

Tabela 1: Sharp regularity estimates in the elliptic and parabolic scenarios

It is interesting to note that the parabolic regularity estimates agree with its elliptic counterpart provided $f \in L^{p,\infty}(Q_1)$.

Next picture shows the critical surfaces and the regions they define for the optimal regularity estimates available for solutions to (Eq).

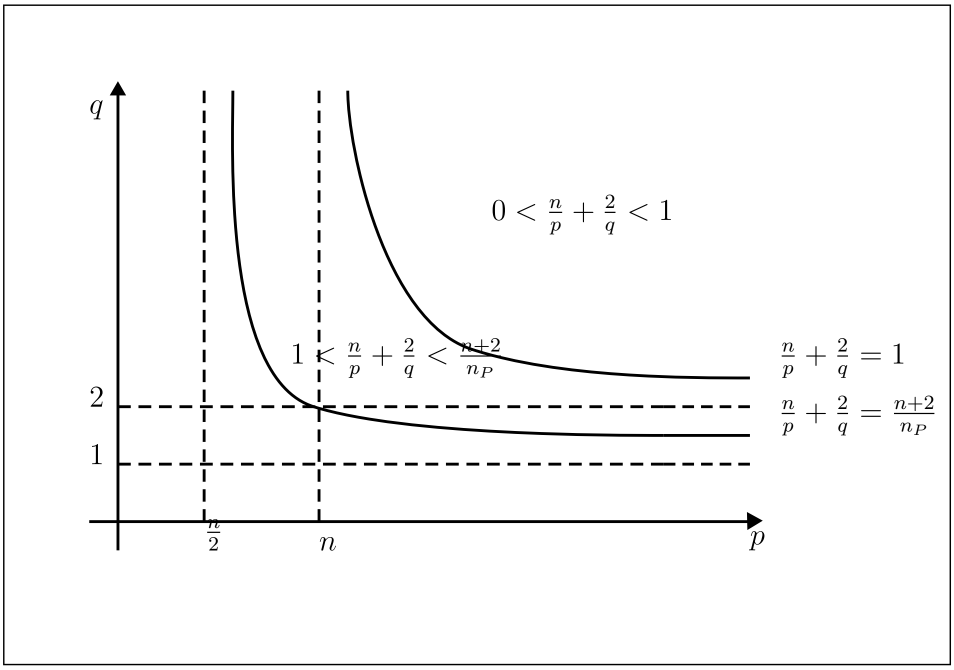

Figura 1: Critical surfaces for optimal regularity estimates.

This summarizes the content of the first chapter of this thesis.

In the third chapter of the thesis we study interior regularity estimates for solutions of the following class of fully non-linear parabolic equations

$$\frac{\partial u}{\partial t} - F(x,t,D^2u) = \mathscr{F}(x,t,u,Du). \tag{1.2}$$

Under continuous differentiability with respect to the matrix variable and appropriate continuity assumptions on the coefficients and source function, we prove Schauder type estimates for parabolic δ-flat solutions, namely solutions whose oscillation is small enough, $\|u\| \leq \delta \ll 1$.

The non-linear operator $F\colon B_1 \times (-1,0] \times \text{Sym}(n) \to \mathbb{R}$ is assumed to be uniformly elliptic and Lipschitz, i.e., there exist constants $\Lambda \geq \lambda > 0$ such that for any $M, P \in \mathrm{Sym}(n)$, with $P \geq 0$ and all $(x,t) \in Q_1$ there holds

$$\lambda \|P\| \leq F(x,t,M+P) - F(x,t,M) \leq \Lambda \|P\|. \tag{1.3}$$

As a prologue to our researches, let us remember (briefly) the historical breakthroughs in relation to the regularity theory for non-divergence form parabolic equations: Fully non-linear parabolic equations have received great attention since the early 80s. The *Magnum Opus* of such a theory is the following fundamental Theorem due to Krylov and Safonov:

Theorem 1.1 (**Krylov-Safonov's Harnack inequality** KRYLOV and SAFONOV (1980), M. Crandall *et al*CRANDALL, KOCAN, and ŚWIECH (2000) and L. Wang WANG (1992a))**.** *Let v be a bounded non-negative viscosity solution to*

$$\frac{\partial v}{\partial t} - a^{ij}(x,t)D_{ij}v = 0 \quad in \quad Q_1 \tag{1.4}$$

with $\lambda I \leq a_{ij} \leq \Lambda I$. Then v satisfies the parabolic Harnack inequality. In particular, it is Hölder continuous and

$$\|v\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{1/2})} \leq C(n,\lambda.\Lambda)\|v\|_{L^\infty(Q_1)}.$$

It also follows by Krylov-Safonov's Harnack inequality, that solutions to constant coefficients equations are locally differentiable. That is:

Theorem 1.2 ($C^{1+\alpha,\frac{1+\alpha}{2}}$ **estimates**, M. Crandall *et al* CRANDALL, KOCAN, and ŚWIECH (2000) and L. Wang WANG (1992b))**.** *Let v be a bounded viscosity solution to*

$$\frac{\partial v}{\partial t} - F(D^2 v) = 0 \quad in \quad Q_1 \tag{1.5}$$

for an F uniformly elliptic operator. Then, there exist constants $C > 0$ and $0 < \alpha < 1$ depending only on universal parameters such that

$$\|v\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(Q_{1/2})} \leq C\|v\|_{L^\infty(Q_1)}.$$

It is also well established that, under suitable assumptions on $\mathscr{F}: B_1 \times (-1,0] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, viscosity solutions to (1.2) are locally of class $C^{1+\alpha,\frac{1+\alpha}{2}}$, see CRANDALL, KOCAN, and ŚWIECH (2000) and WANG (1992b). Hence, one can regard the source $\mathscr{F}(x,t,u,Du)$ in (1.2) simply as a continuous function $f(x,t)$. Therefore, hereafter, the equation (1.2) will be rewritten as

$$\frac{\partial u}{\partial t} - F(x,t,D^2u) = f(x,t) \quad \text{in} \quad Q_1. \tag{1.6}$$

It is important to highlight that $C^{1+\alpha,\frac{1+\alpha}{2}}$ is optimal. The fact that such a regularity is optimal (without convexity assumption of F) is due to remarkable (elliptic) examples due to Nadirashvili and Vlăduţ in NADIRASHIVILI and VLĂDUŢ (2007), NADIRASHIVILI and VLĂDUŢ (2008), NADIRASHIVILI and VLĂDUŢ (2011) and NADIRASHIVILI and VLĂDUŢ (2013). See also the counterexample due to Caffarelli and Stefanelli in CAFFARELLI and STEFANELLI (2008), which shows that $C^{2,1}$ regularity is generally not to be expected. Classical solutions are granted upon convexity (concavity) assumption on F.

Theorem 1.3 (**Evans-Krylov Theorem, $C^{2+\alpha,\frac{2+\alpha}{2}}$ estimates**, EVANS (1982) and KRYLOV (1983) and WANG (1992b))**.** *Let v be a bounded viscosity solution to* (1.5)*, where F is a convex (concave) operator. Then, there exist constants $C>0$ and $0<\alpha<1$ depending only on universal parameters such that*

$$\|v\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_{1/2})} \leq C\|v\|_{L^\infty(Q_1)}.$$

Taking into account the impossibility of a general existence theory for classical solutions to fully nonlinear parabolic equations, to obtain additional conditions on F, f and u in order to establish local $C^{2+\alpha,\frac{2+\alpha}{2}}$ estimates have become a central theme of research in the last years. In this direction, the monumental article SAVIN (2007) is the pioneering work as regards to flat viscosity solutions in the elliptic case (cf. WANG (2013) for its parabolic counterpart). Afterwards, DOS PRAZERES and TEIXEIRA (2016) arisen with a remodelled systematic approach based on geometric tangential analysis (in short GTA) for such subject also in elliptic scenario. We also recommend to the reader TEIXEIRA (2014), TEIXEIRA (2015), TEIXEIRA (2016) and TEIXEIRA and URBANO (2014) as another insight and motivation for this GTA's topic in the elliptic and parabolic settings. Regarding that research topic, our work treats with non-rigid assumptions on data, nonlinearity and solutions in order to recover classical regularity estimates to viscosity solutions to (1.6) (compare with ZOU and CHEN (2002), KRYLOV (1983), TIAN and WANG (2013), (WANG, 1992b, Section 1.1) and (WANG, 1992b, Section 4.3), which work under convexity/concavity assumptions on F). Furthermore, we point out that different from its elliptic counterpart (cf. DOS PRAZERES and TEIXEIRA (2016)), this parabolic version involves further extensions and difficulties that are treated and resolved throughout this thesis.

Therefore, we establish sharp and improved regularity estimates for solutions with small oscillation (flat) to (1.6), provided the data f and F are, in some sense, *parabolically Dini continuous* functions, see ZOU and CHEN (2002), KOVATS (1999) and TIAN and WANG (2012) for some surveys on this topic. Such regularity issues appear in many contexts such as Mathematical Physics, stochastic process, Geometric Analysis, Free boundary problems, because they allow us to access the high order estimates, as well as point out the precise modulus of continuity of solutions in terms of regularity of the medium and the source for problems governed by equations of the form (1.6).

It is worth to remembering that, Schauder type estimates for solutions of (1.6) with Dini continuous sources f have been considerably well studied in the literature. For linear equations,

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \mathfrak{a}_{ij}(x,t)\partial_{ij} u = f(x,t) \quad \text{in} \quad Q_1$$

the modulus of continuity of $\frac{\partial u}{\partial t}$ and D^2u are well-known, see for example ZOU and CHEN (2002), TIAN and WANG (2012) and WANG (2006). Concerning fully nonlinear equations, Kovats in KOVATS (1999), studied classical solutions of the constant coefficient model

$$F(x, D^2u) = f(x) \quad \text{in} \quad B_1, \tag{1.7}$$

by means of polynomial approximation methods, maximum principle tools and Evans-Krylov Theorem, where F is assumed to be convex and f is Dini continuous (in the L^n-average sense). Moreover, Kovats developed the *Dini-Campanato spaces theory*, which naturally generalize the Hölder spaces, and applied this result to establish Schauder type estimates for such solutions of (1.7).

As for parabolic equations, recently Zou and Chen in ZOU and CHEN (2002) have reproduced similar results under suitable Dini continuity assumptions and $C^{2+\alpha, \frac{2+\alpha}{2}}$ *a priori* estimates for $\frac{\partial \mathfrak{h}}{\partial t} - F(x_0, t_0, D^2\mathfrak{h}) = 0$. Finally, more recently an original approach was developed by Tian and Wang in TIAN and WANG (2012), see also Wang in WANG (2006), using maximum principle, derivative estimates and polynomial approximation in the linear and fully nonlinear setting for elliptic and parabolic operators which are $C^{1,1}$ and have $C^{2+\alpha}$ *a priori* estimates. Wang in WANG (2006) also studied the borderline case, when f is merely continuous. In this scenery, under the previous hypotheses, gradients of solutions to (1.7) have Log-Lipschitz modulus of continuity, see also (DA SILVA and TEIXEIRA, 2017, Section 6) and (TEIXEIRA, 2014, Section 5) for similar results.

Therefore, given a flat viscosity solution to (1.6) with source f and medium (coefficients of F) been Dini continuous functions, namely $C^{0,\tau}(Q_1)$ with $\displaystyle\int_0^1 \frac{\omega(r)}{r} dr$ finite, then $u \in C^{2,1,\psi}_{loc}(Q_1)$, where

$$\psi(s) := s\int_s^1 \frac{\tau(r)}{r^2} dr + \int_0^1 \frac{\tau(r)}{r} dr.$$

Particularly, we recover classical Schauder estimates, when $\tau(s) = s^\alpha$ for $0 < \alpha < 1$.

As consequences of the our regularity estimates we are able to prove a partial regularity result when the source is a Lipschitz function, as well as a theorem of Evans-Krylov type. Similar results can be obtained for regularity to some geometric flows for certain kind of manifolds with geometric restrictions (cf. HUISKEN (1987) and TIAN and WANG (2013))). Moreover, Lipschitz logarithmic estimates are obtained, when the Dini condition fails, compare with (DA SILVA and TEIXEIRA, 2017, Section 6).

This summarizes the content of the third chapter of this thesis.

In the last part of our thesis we establish global gradient estimates for singularly perturbed fully nonlinear elliptic equations, which arise of certain models from combustion theory. In the following a brief resume and overview on the problem: Throughout the last three decades or so, variational problems involving singular PDEs have received a warm attention as they often come from the theory of critical points of non-differentiable functionals. The pioneering work of Alt-Caffarelli ALT and CAFFARELLI (1981) marks the beginning of such a theory by carrying out the variational analysis of the minimization problem

$$\min \int_{\Omega} \left(\frac{1}{2} |\nabla v|^2 + \chi_{\{v>0\}} \right) dx,$$

among competing functions with the same non-negative Dirichlet boundary condition.

Since the very beginning it has been well established that such discontinuous minimization problems could be treated by penalization methods. Lewy-Stampacchia, Kinderlehrer-Nirenberg, Caffarelli among others were the precursors of such an approach to the study of problem $\Delta u^{\epsilon} = \zeta_{\varepsilon}(u^{\varepsilon})$ over of 70s and 80s. Linear problems in non-divergence form was firstly considered by Berestycki *et al* in BERESTYCKI, CAFFARELLI, and NIRENBERG (1990). Teixeira in TEIXEIRA (2006) started the journey of investigation into fully non-linear elliptic equations via singular perturbation methods:

$$F(x, D^2 u^{\varepsilon}) = \zeta_{\varepsilon}(u^{\varepsilon}) \quad \text{in} \quad \Omega,$$

where $\zeta_{\varepsilon} \sim \varepsilon^{-1}\chi_{(0,\varepsilon)}$. The problem appears in nonlinear formulations of high energy activation models, see RICARTE and TEIXEIRA (2011) and TEIXEIRA (2006). It can also be employed in the analysis of over-determined problems as follows. Given $\Omega \subset \mathbb{R}^n$ a domain and a non-negative function $g\colon \Omega \to \mathbb{R}$, the question of finding a compact hyper-surface $\partial\Omega' \subset \Omega$ such that the following elliptic boundary value problem

$$\begin{cases} F(x, Du, D^2u) &= 0 \quad \text{in} \quad \Omega \setminus \Omega' \\ u(x) &= g(x) \quad \text{on} \quad \partial\Omega \\ u(x) &= 0 \quad \text{on} \quad \partial\Omega', \end{cases} \tag{1.8}$$

plays a crucial role in geometry and mathematical physics.

In RICARTE and TEIXEIRA (2011), several analytical and geometrical properties of such a problem were established. Notwithstanding, regularity up to the boundary for approximating solutions has not been proven in the literature yet. This is the key goal of the fourth chapter of the thesis. More precisely, we prove a uniform gradient estimate up to the boundary for viscosity solutions of the singular perturbation problem

$$\begin{cases} F(x, Du^{\varepsilon}, D^2u^{\epsilon}(x)) & = & \zeta_{\epsilon}(u^{\epsilon}) & \text{in} & \Omega \\ u^{\epsilon}(x) & = & g(x) & \text{on} & \partial\Omega, \end{cases} \tag{E_ϵ}$$

where the singular reaction term $\zeta_{\varepsilon}(s) = \frac{1}{\varepsilon}\zeta\left(\frac{s}{\varepsilon}\right)$ for some non-negative $\zeta \in C_0^{\infty}([0,1])$, a parameter $\varepsilon > 0$, a non-negative $g \in C^{1,\gamma}(\overline{\Omega})$, with $0 < \gamma < 1$, and, a bounded $C^{1,1}$ domain Ω (or $\partial\Omega$ for short). In other words,

$$\|Du^{\varepsilon}\|_{L^{\infty}(\overline{\Omega})} \leq C(n, \lambda, \Lambda, b, \|\zeta\|_{L^{\infty}([0,1])}, \|g\|_{C^{1,\gamma}(\overline{\Omega})}, \Omega).$$

It is worth to highlight that our estimates generalize the correspondent local ones, see TEIXEIRA (2006) and RICARTE and TEIXEIRA (2011). As a consequence we are able to obtain existence for corresponding free boundary problem.

Our approach follows the pioneering work of Gurevich GUREVICH (1999), where it was introduced a new strategy to investigate uniform estimate up to boundary of two-phase singular perturbation problems involving linear elliptic operators of type $\mathcal{L}u = \partial_i(a_{ij}\partial_j u)$. This method has been successfully applied by Karakhanyan in KARAKHANYAN (2006) for the one-phase problem in the case involving non-linear singular/degenerate elliptic operators of the p-Laplace type, $\Delta_p u^{\epsilon} = \zeta_{\epsilon}(u^{\epsilon})$.

2 SHARP MODULI OF CONTINUITY FOR FULLY NONLINEAR PARABOLIC EQUATIONS

In the present chapter we shall prove sharp regularity estimates for viscosity solutions of the following fully non-linear parabolic form

$$\frac{\partial u}{\partial t} - F(D^2u, Du, x, t) = f(x,t) \quad \text{in} \quad Q_1, \tag{Eq}$$

where F is a uniformly elliptic operator and f belongs to Anisotropic Lebesgue space $L^{p,q}(Q_1)$. The quantity $\Xi(n,p,q) := \frac{n}{p} + \frac{2}{q}$ is pivotal in our results, because it classifies the regularity regime of solution to (Eq).

The chapter is organized according the following way: In Section 2.1 we introduce the language of this chapter, as well as we prove a fundamental result for our purpose, Lemma 2.6. In Section 2.2 we establish the optimal Hölder regularity of solutions. In Section 2.3 we treat a borderline case for the regularity theory, in this case we obtain optimal Lipschitz logarithmic modulus of continuity for solutions. In sequel, Section 2.4, we comment how to obtain the optimal results for the $C^{1+\sigma}$ regularity estimates. To finish, in Section 2.5 we study the upper borderline case for the regularity theory, as result we we obtain a C^1-Lipschitz logarithmic modulus of continuity.

2.1 Definitions and preliminary results

Throughout this chapter $F\colon Sym(n) \times \mathbb{R}^n \times B_1(0) \times (-1,0] \longrightarrow \mathbb{R}$ is a fully nonlinear uniformly elliptic operator with respect to the Hessian argument and Lipschitz with respect to gradient dependence. That is, there are constants $\Lambda \geq \lambda > 0$ and $\Gamma \geq 0$ such that for all $Z, W \in R^n$ and $M, N \in \mathrm{Sym}(n)$, space of $n \times n$ symmetric matrices, with $M \geq N$, there holds

$$\mathcal{P}^-_{\lambda,\Lambda}(M-N) - \Gamma|Z-W| \leq F(M,Z,x,t) - F(N,W,x,t) \leq \mathcal{P}^+_{\lambda,\Lambda}(M-N) + \Gamma|Z-W|. \tag{2.1}$$

Hereafter, $\mathcal{P}^{\pm}_{\lambda,\Lambda}$ denote the Pucci's extremal operators:

$$\mathcal{P}^+_{\lambda,\Lambda}(M) := \lambda \cdot \sum_{e_i<0} e_i + \Lambda \cdot \sum_{e_i>0} e_i \quad \text{and} \quad \mathcal{P}^-_{\lambda,\Lambda}(M) := \lambda \cdot \sum_{e_i>0} e_i + \Lambda \cdot \sum_{e_i<0} e_i$$

where $\{e_i : 1 \leq i \leq n\}$ are the eigenvalues of M. Any operator F which satisfies the condition (2.1) will be referred in this chapter as a $(\lambda, \Lambda, \Gamma)$-*parabolic operator*. Following classical terminology, any constant or mathematical term which depends only on dimension and of the parameters λ, Λ and Γ will be called *universal*.

We can (and will) always assume that F is normalized in the sense

$$F(0,0,x,t)=0, \tag{2.2}$$

and, unless otherwise stated, conditions (2.1) and (2.2) are always assumed throughout the text; sometimes we will refer F as a normalized (λ,Λ,Γ) operator.

Equations and problems studied here are designed in the $(n+1)$-dimensional Euclidean space, $\mathbb{R}^{n+1}$. The semi-open cylinder is denoted by $Q_r(x_0,\tau)=B_r(x_0)\times(\tau-r^2,\tau]$. For simplicity we refer $Q_1(0,0)=Q_1$. The *parabolic distance* between the points $P_1=(x_1,t_1)$ and $P_2=(x_2,t_2)\in Q_1$ is defined by

$$d_{\text{par}}(P_1,P_2):=\sqrt{|x_1-x_2|^2+|t_1-t_2|}.$$

For a function $u\colon Q_1\to\mathbb{R}$ the semi-norm and norm for the *parabolic Hölder space* are defined respectively by

$$[u]_{C^{\alpha,\frac{\alpha}{2}}(Q_1)}:=\sup_{\substack{(x,t),(y,s)\in Q_1\\(x,t)\neq(y,s)}}\frac{|u(x,t)-u(y,s)|}{d_{\text{par}}((x,t),(y,s))^{\alpha}}\quad\text{and}\quad\|u\|_{C^{\alpha,\frac{\alpha}{2}}(Q_1)}:=\|u\|_{C^0(Q_1)}+[u]_{C^{\alpha,\frac{\alpha}{2}}(Q_1)}.$$

Under finiteness of such a norm one concludes that u is α-Hölder continuous with respect to the spatial variables and $\frac{\alpha}{2}-$Hölder with respect to the temporal variable.

We say that u is locally Log-Lipschitz continuous (in the parabolic sense) if the following quantity

$$[u]_{\text{par}-C^{0,\text{Log-Lip}}(Q_r(x_0,t_0))}:=\sup_{\substack{(x,t),(y,s)\in Q_r(x_0,t_0)\\(x,t)\neq(y,s)}}\frac{|u(x,t)-u(y,s)|}{r\log r^{-1}}\quad\forall\ r\ll 1.$$

is finite for $(x_0,t_0)\in Q_1$. Moreover, the corresponding parabolic Log-Lipschitz norm is given by

$$\|u\|_{\text{par}-C^{0,\text{Log-Lip}}(Q_r(x_0,t_0))}:=\|u\|_{C^0(Q_r(x_0,t_0))}+[u]_{\text{par}-C^{0,\text{Log-Lip}}(Q_r(x_0,t_0))}.$$

In what follows, $C^{1+\alpha,\frac{1+\alpha}{2}}(Q_1)$ denotes the space of u whose spacial gradient $Du(x,t)$ there exists in the classical sense for every $(x,t)\in Q_1$ and such that

$$\begin{aligned}\|u\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(Q_1)} &:= \|u\|_{L^\infty(Q_1)}+\|Du\|_{L^\infty(Q_1)}\\ &+\sup_{\substack{(x,t),(y,s)\in Q_1\\(x,t)\neq(y,s)}}\frac{|u(x,t)-[u(y,\tau)-Du(y,s)\cdot(x-y)]|}{d_{\text{par}}^{1+\alpha}((x,t),(y,s))}\end{aligned}$$

is finite. It is easy to verify that $u \in C^{1+\alpha,\frac{1+\alpha}{2}}(Q_1)$ implies every component of Du is $C^{0,\alpha}(Q_1)$, and u is $\frac{1+\alpha}{2}-$Hölder continuous in the variable t, see for instance (CRANDALL, KOCAN, and ŚWIECH, 2000, Section 1).

Now, we say that u is locally $C^{1,\text{Log-Lip}}$ continuous (in the parabolic sense) if the quantity

$$[u]_{\text{par}-C^{1,\text{Log-Lip}}(Q_r(x_0,t_0))} := \sup_{\substack{(x,t),(y,s)\in Q_r(x_0,t_0)\\ (x,t)\neq(y,s)}} \frac{|u(x,t)-[u(y,s)+Du(y,s).(x-y)]|}{r^2\log r^{-1}}$$

is finite. Moreover, its parabolic $C^{1,\text{Log-Lip}}-$norm is given by

$$\|u\|_{\text{par}-C^{1,\text{Log-Lip}}(Q_r(x_0,t_0))} := \|u\|_{C^0(Q_r(x_0,t_0))} + \|Du\|_{L^\infty(Q_r(x_0,t_0))} + [u]_{\text{par}-C^{1,\text{Log-Lip}}(Q_r(x_0,t_0))}.$$

A function u belongs to the Sobolev space $W^{2,1,p}(Q_1)$ if it satisfies $u, Du, D^2u, u_t \in L^p(Q_1)$. The corresponding norm is given by

$$\|u\|_{W^{2,1,p}(Q_1)} = \left[\|u\|^p_{L^p(Q_1)} + \|u_t\|^p_{L^p(Q_1)} + \|Du\|^p_{L^p(Q_1)} + \|D^2u\|^p_{L^p(Q_1)}\right]^{\frac{1}{p}}$$

It follows by Sobolev embedding that if $p > \frac{n+2}{2}$ then $W^{2,1,p}(Q_1)$ is continuously embedded in $C^0(Q_1)$. Also, $u \in W^{2,1,p}_{loc}(Q_1)$ implies that u is twice parabolically differentiable a.e., see for more details CRANDALL *et al.* (1998).

Definition 2.1 ($L^{\mathrm{P}}-$**viscosity solutions).** Let $\mathcal{G}\colon Sym(n)\times\mathbb{R}^n\times B_1(0)\times(-1,0]\to\mathbb{R}$ be a uniformly elliptic operator, $\mathrm{P} > \frac{n+2}{2}$ and $f \in L^{\mathrm{P}}_{loc}(Q_1)$. We say that a function $u \in C^0(Q_1)$ is an $L^{\mathrm{P}}-$viscosity sub-solution (respectively super-solution) to

$$\frac{\partial u}{\partial t}(x,t) - \mathcal{G}\left(D^2u(x,t), Du(x,t), x, t\right) = f(x,t) \quad in \quad Q_1 \tag{2.3}$$

if for all $\varphi \in W^{2,1,\mathrm{P}}_{loc}(Q_1)$ whenever $\varepsilon > 0$ and $\mathcal{O} \subset Q_1$ is an open set and

$$\frac{\partial \varphi}{\partial t}(x,t) - \mathcal{G}\left(D^2\varphi(x,t), D\varphi(x,t), x, t\right) - f(x,t) \geq \varepsilon \quad (resp. \leq -\varepsilon) \quad a.e. \quad in \quad \mathcal{O}$$

then $u-\varphi$ cannot attains a local maximum (resp. minimum) in $\mathcal{O}$. In an equivalent manner, u is an $L^{\mathrm{P}}-$viscosity sub-solution (resp. super-solution) if for all test function $\varphi \in W^{1,2,\mathrm{P}}_{loc}(Q_1)$ and $(x_0,t_0) \in Q_1$ at which $u-\varphi$ attain a local maximum (resp. minimum) one has

$$\begin{cases} [\\ (x,t) \to (x_0,t_0)]\text{essliminf} \left[\dfrac{\partial\varphi}{\partial t}(x,t) - \mathcal{G}\left(D^2\varphi(x,t), D\varphi(x,t), x, t\right) - f(x,t)\right] \leq 0 \\ [\\ (x,t) \to (x_0,t_0)]\text{esslimsup} \left[\dfrac{\partial\varphi}{\partial t}(x,t) - \mathcal{G}\left(D^2\varphi(x,t), D\varphi(x,t), x, t\right) - f(x,t)\right] \geq 0 \end{cases} \tag{2.4}$$

Finally we say that u is an $L^{\mathrm{P}}-$viscosity solution to (2.3) if it is both an $L^{\mathrm{P}}-$viscosity super-solution and an $L^{\mathrm{P}}-$viscosity sub-solution.

Remark 2.2. We say that a function $u \in C^0(Q_1)$ is a C^0-viscosity solution to (2.3) when the sentences in (2.4) are evaluated point-wisely for all "test function" $\varphi \in C^{2,1}_{loc}(Q_1)$. This is the notion used by Imbert-Silvestre in IMBERT and SILVESTRE (2013) and Wang in WANG (1992a,b).

According to (CRANDALL, KOCAN, and ŚWIECH, 2000, Section 6) (see also (WANG, 1992a, Section 5)) for a fixed $(X_0, t_0) \in Q_1$, we measure the oscillation of the coefficients of F around (X_0, t_0) by the quantity

$$\Theta_F(x_0,t_0,x,t) := \sup_{M \in Sym(n)} \frac{|F(M,0,x,t) - F(M,0,x_0,t_0)|}{\|M\| + 1}. \tag{2.5}$$

Moreover, for notation purposes, we shall often write $\Theta_F(0,0,x,t) = \Theta_F(x,t)$.

We recall that a function f is said to belong to the *Anisotropic Lebesgue* space, $L^{p,q}(Q_1)$ if

$$\|f\|_{L^{p,q}(Q_1)} := \left(\int_{-1}^{0}\left(\int_{B_1}|f(x,t)|^p dx\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}} < +\infty.$$

This is a Banach space when endowed with the norm above. When $p = q$, this is the standard definition of L^p spaces. The definition are naturally extended when either p or q are infinity. It is plain to verify that $L^{p,q}(Q_1) \subset L^s(Q_1)$ for $s := \min\{p,q\}$.

We recall again the existence of the constant $\mathfrak{p}_0$, satisfying $\frac{n+2}{2} \leq \mathfrak{p}_0 < n+1$, for which Harnack inequality (resp, Hölder regularity) holds for $L^{\mathrm{P}}-$viscosity solutions, provided $\mathrm{P} > \mathfrak{p}_0$, see for instance (CRANDALL, KOCAN, and ŚWIECH, 2000, Section 5). The following compactness result becomes then available:

Proposition 2.3 (**Compactness of solutions).** *Let u be an $L^{\mathrm{P}}-$viscosity solution to* (Eq) *in Q_r under the assumption* $\mathrm{P} \geq \min\{p,q\} > \mathfrak{p}_0$. *Then u is locally of class $C^{\beta,\frac{\beta}{2}}$ for some* $0 < \beta < 1$ *and*

$$\|u\|_{C^{\beta,\frac{\beta}{2}}(Q_r)} \leq C(n,\lambda,\Lambda,\Gamma) r^{-\beta}\left(\|u\|_{L^\infty(Q_r)} + r^{2-\Xi(n,p,q)}\|f\|_{L^{p,q}(Q_r)}\right).$$

Upon appropriate regularity assumption on the boundary and equi-continuity of the data, solutions are pre-compact in the C^0-topology up-to-the-boundary. We state such a result for future references and refer to (CRANDALL *et al.*, 1999, Proposition 4.6) and (CRANDALL, KOCAN, and ŚWIECH, 2000, Lemma 6.3) for a proof.

Proposition 2.4 (Pre-compactness up to the boundary). *Let Ω satisfy a uniform exterior cone condition, $Q := \Omega \times ((-T, 0])$ and $\mathcal{C} \subset C^0(\partial_p Q)$ be compact, $R > 0$ and $\mathfrak{B}_R := \{f \in L^{\mathrm{P}}(Q) : \|f\|_{L^{\mathrm{P}}(Q)} \leq R\}$. Then the set all functions $u \in C^0(\overline{Q})$ such that there exists $\psi \in \mathcal{C}$ and $f \in \mathfrak{B}_R$ for which u is an $L^{\mathrm{P}}-$viscosity solution to*

$$\frac{\partial u}{\partial t} - \mathcal{P}^-_{\lambda,\Lambda}(D^2u) - \Gamma|Du| - f \leq 0 \leq \frac{\partial u}{\partial t} - \mathcal{P}^+_{\lambda,\Lambda}(D^2u) + \Gamma|Du| + f \quad \text{in} \quad Q$$

and $u = \psi$ on $\partial_p Q$ is pre-compact in $C^0(\overline{Q})$.

Another piece of information we need in our approach concerns the stability of the notion of viscosity solutions; that is the limit of a sequence of viscosity solutions turns out to be a viscosity solution of the limiting equation. More precisely, we refer to the following Lemma, whose proof can be found, for instance, in (CRANDALL, KOCAN, and ŚWIECH, 2000, Theorem 6.1).

Lemma 2.5 (Continuity with respect to equation). *Let F_j, F be normalized $(\lambda, \Lambda, \Gamma)$ operators, $\mathrm{P} > \mathfrak{p}_0$, $f, f_j \in L^{\mathrm{P}}(Q_1)$ and u_j be $L^{\mathrm{P}}-$viscosity solutions to*

$$\frac{\partial u_j}{\partial t} - F_j(D^2u_j, Du_j, x, t) = f_j \quad \text{in} \quad Q_1$$

for all $j \in \mathbb{N}$. Assume that $u_j \to u$ locally uniformly as $j \to \infty$. Moreover, for all $Q_r(x_0, t_0) \subset Q_1$ and all $\varphi \in W^{2,1,\mathrm{P}}(Q_r(x_0, t_0))$ (test function), assume that

$$g_j(x, t) := F_j(D^2\varphi(x, t), D\varphi(x, t), x, t) - f_j(x, t)$$

and

$$g(x, t) := F(D^2\varphi(x, t), D\varphi(x, t), x, t) - f(x, t)$$

satisfy

$$\|g - g_j\|_{L^{\mathrm{P}}(Q_r(x_0,t_0))} \to 0 \quad \text{as} \quad j \to \infty. \tag{2.6}$$

Then, u is an $L^{\mathrm{P}}-$viscosity solution to

$$\frac{\partial u}{\partial t} - F(D^2u, Du, x, t) = f \quad \text{in} \quad Q_r(x_0, t_0).$$

Furthermore, if F and f are continuous functions, then u is a C^0-viscosity solution if (2.6) *holds for all $\varphi \in C^{2,1}(Q_1)$ (test function).*

In the sequel, we obtain a Lemma which provides a tangential path toward the regularity theory available for constant coefficient, homogeneous $\mathfrak{F}$-caloric functions.

Lemma 2.6 (**$\mathfrak{F}$-caloric approximation Lemma**)**.** *Let u be an $L^{\mathrm{P}}-$viscosity solution to* (Eq) *with $|u| \leq 1$ and $f \in L^{p,q}(Q_1)$ with* $\mathrm{P} := \min\{p,q\} > \mathfrak{p}_0$. *Define $\mathfrak{h}\colon Q_{1/2} \to \mathbb{R}$ to be the $L^{\mathrm{P}}-$viscosity solution of*

$$\begin{cases} \frac{\partial \mathfrak{h}}{\partial t} - F(D^2\mathfrak{h}, 0, 0, 0) &= 0 \quad in \quad Q_{\frac{1}{2}} \\ \mathfrak{h} &= u \quad on \quad \partial_p Q_{\frac{1}{2}} \end{cases} \tag{2.7}$$

Given $\delta > 0$, there exists $\eta = \eta(\delta, n, \lambda, \Lambda, \mathrm{P}) > 0$ such that if

$$\max\left\{ \left(\fint_{Q_1} \Theta_F^{\mathrm{P}}(x,t) \right)^{\frac{1}{\mathrm{P}}}, \ \|f\|_{L^{p,q}(Q_1)}, \ \Gamma \right\} \leq \eta,$$

then

$$\sup_{Q_{\frac{1}{2}}} |u - \mathfrak{h}| \leq \delta. \tag{2.8}$$

Demonstração. The proof is based on a *Reductio ad absurdum.* Suppose there exists a $\delta_0 > 0$ for which the thesis of the Lemma, namely sentence 2.8, does not hold. That means we could find sequences of functions $(u_j)_{j\geq 1}, (\mathfrak{h}_j)_{j\geq 1} \subset C^0(Q_1)$ with $|u_j| \leq 1$, a sequence of normalized $(\lambda, \Lambda, \Gamma_j)$-operators $F_j\colon Sym(n) \times \mathbb{R}^n \times Q_1 \to \mathbb{R}$ and a sequence of functions $(f_j)_{j\geq 1}$ satisfying

$$\begin{cases} \frac{\partial u_j}{\partial t} - F_j(D^2u_j, Du_j, x, t) &= f_j \quad \text{in} \quad Q_1 \\ \frac{\partial \mathfrak{h}_j}{\partial t} - F_j(D^2\mathfrak{h}_j, 0, 0, 0) &= 0 \quad \text{in} \quad Q_{\frac{1}{2}} \\ \mathfrak{h}_j &= u_j \quad \text{on} \quad \partial_p Q_{\frac{1}{2}} \end{cases} \tag{2.9}$$

in the $L^{\mathrm{P}}-$viscosity sense, with

$$\max\left\{ \left(\fint_{Q_1} \Theta_{F_j}^{\mathrm{P}}(x,t) \right)^{\frac{1}{\mathrm{P}}}, \ \|f_j\|_{L^{p,q}(Q_1)}, \ \Gamma_j \right\} = \mathrm{o}(1) \quad as \quad j \to \infty; \tag{2.10}$$

however

$$\sup_{Q_{\frac{1}{2}}} |u_j - \mathfrak{h}_j| > \delta_0 \quad \text{for all} \quad j \in \mathbb{N}. \tag{2.11}$$

By compactness of the sequences $(u_j)_{j\geq 1}$ and $(\mathfrak{h}_j)_{j\geq 1}$, namely Proposition 2.3 and Proposition 2.4, we may assume, passing to a subsequence if necessary, that $u_j \to u_0$ and $\mathfrak{h}_j \to \mathfrak{h}_0$ uniformly in $\overline{Q_{\frac{1}{2}}}$.

Next we will prove that $u_0 = \mathfrak{h}_0$. The idea is to conclude that both functions solve the same PDE, for which uniqueness is available, and hence this would contradict (2.11) for $j \gg 1$ large enough.

Initially we note that it follows from structural condition imposed on the operators $(F_j)_{j\geq 1}$, namely, (2.1), that, up to a subsequence, can may assume

$$F_j(M,0,0,0) \to \mathfrak{F}_0(M) \tag{2.12}$$

locally uniformly in the space $Sym(n)$, where $\mathfrak{F}_0$ is a $(\lambda, \Lambda, 0)$ operator with constant coefficients – see for instance (CRANDALL, KOCAN, and ŚWIECH, 2000, Section 1 and 6) and (WANG, 1992b, Lemma 1.4). Also, applying Lemma 2.5 along with uniqueness result, for instance (CRANDALL, KOCAN, and ŚWIECH, 2000, Lemma 6.2), we know $\mathfrak{h}_0$ is the unique C^0-viscosity solution to

$$\left\{\begin{array}{rcll} \frac{\partial \mathfrak{h}_0}{\partial t} - \mathfrak{F}_0(D^2\mathfrak{h}_0) & = & 0 & \text{in} \quad Q_{\frac{1}{2}} \\ \mathfrak{h}_0 & = & u_0 & \text{on} \quad \partial_p Q_{\frac{1}{2}}. \end{array}\right. \tag{2.13}$$

To conclude the proof, we will show that u_0 also solves (2.13) in the viscosity sense. For that end, let $\varphi \in C^{2,1}(Q_{\frac{1}{2}})$ be a test function and define

$$\mathfrak{K}_j(\varphi) := \left|F_j\left(D^2\varphi(x,t), D\varphi(x,t), x, t\right) - f_j(x,t) - \mathfrak{F}_0\left(D^2\varphi(x,t)\right)\right|.$$

We estimate

$$\begin{array}{rcl} \mathfrak{K}_j(\varphi) & \leq & \Gamma_j|D\varphi(x,t)| + \Theta_{F_j}(x,t)\left(|D^2\varphi(x,t)|+1\right) + |f_j(x,t)| \\ & + & \left|F_j\left(D^2\varphi(x,t),0,0,0\right) - \mathfrak{F}_0\left(D^2\varphi(x,t)\right)\right|. \end{array} \tag{2.14}$$

Finally, since from (2.10) and (2.12) one has that the $L^{\mathrm{P}}-$norm of RHS of (2.14) goes to zero as $j \to \infty$, we can apply once more Lemma 2.5, which assures u_0 is too a solution of (2.13), and by uniqueness, $u_0 = \mathfrak{h}_0$, which yields a contradiction as indicated before. $\square$

We conclude this section by commenting on reduction processes to be used throughout the proof.

Remark 2.7. [**Preserving ellipticity**] If F is a $(\lambda, \Lambda, \Gamma)$-parabolic operator then

$$\mathcal{G}\left(M, \overrightarrow{Z}, x, t\right) = \kappa_0^2 \cdot F\left(\frac{M}{\kappa_0^2}, \frac{\overrightarrow{Z}}{\kappa_0}, x, t\right)$$

is a $(\lambda, \Lambda, \kappa_0 \cdot \Gamma)$-parabolic operator for any $\kappa_0 > 0$.

Remark 2.8. [**Normalization and scaling**] We can always suppose, without loss of generality, that viscosity solutions of

$$\frac{\partial u}{\partial t}(x,t) - F(D^2u, Du, x, t) = f(x,t) \quad in \quad Q_1$$

satisfy $\|u\|_{L^\infty(Q_1)} \leq 1$. Also given a small number $\varepsilon_0 > 0$, we can also suppose that $\Gamma + \|f\|_{L^{p,q}(Q_1)} < 2\varepsilon_0$. Indeed, for

$$\kappa := \frac{\varepsilon_0}{\varepsilon_0(\|u\|_{L^\infty(Q_1)} + 1) + \|f\|_{L^{p,q}(Q_1)}} \quad \text{and} \quad R > \max\left\{1, \frac{\Gamma}{\varepsilon_0}, \sqrt{\kappa}\right\},$$

we define

$$v(x,t) := \kappa u\left(\frac{1}{R}x, \frac{1}{R^2}t\right).$$

It is easy to verify that

1. $\|v\|_{L^\infty(Q_1)} \leq 1$;
2. $\frac{\partial v}{\partial t}(x,t) - \mathcal{G}(D^2v, Dv, x, t) = g(x,t)$ in Q_1, in the $L^{\mathrm{P}}-$viscosity sense, where

$$\mathcal{G}(M, \overrightarrow{Z}, x, t) = \frac{\kappa}{R^2}F\left(\frac{R^2}{\kappa}M, \frac{R}{\kappa}\overrightarrow{Z}, \frac{1}{R}x, \frac{1}{R^2}t\right) \quad \textit{and} \quad g(x,t) = \frac{\kappa}{R^2}f\left(\frac{1}{R}x, \frac{1}{R^2}t\right);$$

3. $\mathcal{G}$ is a $(\lambda, \Lambda, \Gamma^\sharp)$-parabolic operator, with $\Gamma^\sharp < \varepsilon_0$;
4. $\|g\|_{L^{p,q}(Q_1)} \leq \varepsilon_0$;
5. $\left(\fint_{Q_r} \Theta_{\mathcal{G}}^{\mathrm{P}}(x,t)\right)^{\frac{1}{\mathrm{P}}} \leq \max\left\{1, \frac{\kappa}{R^2}\right\}\left(\fint_{Q_{rR^{-1}}} \Theta_F^{\mathrm{P}}(x,t)\right)^{\frac{1}{\mathrm{P}}}$ (cf. (CRANDALL, KOCAN, and ŚWIECH, 2000, Remark 6.4)).

Once a universal estimate is proven for v, a corresponding one becomes available for the general solution u, properly adjusted by the choices of κ and R.

2.2 Optimal $C^{\alpha,\frac{\alpha}{2}}$ regularity

Our strategy for proving optimal $C^{\alpha,\frac{\alpha}{2}}$ regularity estimates is based on a refined compactness method as in CRANDALL, KOCAN, and ŚWIECH (2000); TEIXEIRA (2006); WANG (1992a,b). It relies on a control of oscillation decay obtained from the regularity theory available for a "better"limiting equation; the realm of the so-called *geometric tangential analysis*. Next lemma is the key access point for the approach, as it provides the first step in the iteration process to be implemented.

Lemma 2.9. *Let u be a normalized $L^{\mathrm{P}}-$viscosity solution for* (Eq)*, that is, $|u| \leq 1$ in Q_1. Given $0 < \gamma < 1$, there exist $\eta(n, \lambda, \Lambda, \gamma) > 0$ and $0 < \rho(n, \lambda, \Lambda, \gamma) \ll \frac{1}{2}$, such that if*

$$\max\left\{\left(\fint_{Q_1} \Theta_F^{\mathrm{P}}(x,t)\right)^{\frac{1}{\mathrm{P}}},\ \|f\|_{L^{p,q}(Q_1)},\ \Gamma\right\} \leq \eta \quad \textit{with} \quad 1 < \Xi(n,p,q) < \frac{n+2}{\mathfrak{p}_0}$$

then, for some $\varsigma \in \mathbb{R}$, *with* $|\varsigma| \leq C(n, \lambda, \Lambda)$ *there holds*

$$\sup_{Q_\rho} |u - \varsigma| \leq \rho^\gamma. \tag{2.15}$$

Demonstração. For a $\delta > 0$ to be chosen *a posteriori*, let $\mathfrak{h}$ be a solution to a homogeneous uniformly parabolic equation with constant coefficients, that is δ-close to u in the L^∞-norm, i.e.,

$$\frac{\partial \mathfrak{h}}{\partial t} - F(D^2\mathfrak{h}) = 0 \quad \text{in} \quad Q_1 \quad \text{and} \quad \sup_{Q_{\frac{1}{2}}} |u - \mathfrak{h}| \leq \delta. \tag{2.16}$$

Lemma 2.6 assures the existence of such a function. Once our choice for δ, to be set of the end of this proof, is universal, then the choice of $\eta(n, \lambda, \Lambda, \delta)$ is universal too. From the regularity theory available for $\mathfrak{h}$, see for instance (CRANDALL, KOCAN, and ŚWIECH, 2000, Section 7) or (WANG, 1992b, Section 1.2), we can estimate

$$|\mathfrak{h}(x,t) - \mathfrak{h}(0,0)| \leq C(n, \lambda, \Lambda) d_{\text{par}}((x,t),(0,0)) \;\text{ for }\; \sqrt{|x|^2 + |t|} < \frac{1}{3}, \tag{2.17}$$

and also,

$$|\mathfrak{h}(0,0)| \leq C(n, \lambda, \Lambda). \tag{2.18}$$

For $\varsigma = \mathfrak{h}(0,0)$ it follows from equations (2.16) and (2.17) via triangular inequality that

$$\sup_{Q_\rho} |u - \varsigma| \leq \delta + C(n, \lambda, \Lambda)\rho. \tag{2.19}$$

We make the following universal selections:

$$\rho := \min\left\{ r_0, \left(\frac{1}{2C}\right)^{\frac{1}{1-\gamma}} \right\} \quad \textit{and} \quad \delta := \frac{1}{2}\rho^\gamma \tag{2.20}$$

where $C > 0$ is a universal constant from equation (2.17) and $0 < r_0 \leq 1$ is a universal constant to appear in the Theorem 2.10. Let us stress that the choices above depend only upon dimension, ellipticity parameters and the fixed exponent. From the above choices we obtain

$$\sup_{Q_\rho} |u - \varsigma| \;\; \leq \rho^\gamma \;\; .$$

and the Lemma is concluded. □

Theorem 2.10. *Let* u *be an* $L^{\mathrm{P}}-$*viscosity solution of* (Eq) *with* $f \in L^{p,q}(Q_1)$ *and*

$$1 < \Xi(n, p, q) < \frac{n+2}{\mathfrak{p}_0}.$$

There exist universal, positive constants r_0 and θ_0 such that if

$$\sup_{0<r\leq r_0}\sup_{(y,\tau)\in Q_{\frac{1}{2}}}\left(\fint_{Q_r(y,\tau)}\Theta_F^{\mathrm{P}}(y,\tau,x,t)\right)^{\frac{1}{\mathrm{P}}}\leq\theta_0,$$

then, for a constant $C>0$ and $\gamma:=2-\Xi(n,p,q)$, there holds

$$\|u\|_{C^{\gamma,\frac{\gamma}{2}}\left(Q_{\frac{1}{2}}\right)}\leq C(n,\lambda,\Lambda,\gamma)\left[\|u\|_{L^\infty(Q_1)}+\|f\|_{L^{p,q}(Q_1)}+1\right].$$

Demonstração. Through normalization and scaling processes, see Remark 2.8, we can suppose without losing generality that $|u|\leq 1$ and $\|f\|_{L^{p,q}(Q_1)}\leq\eta$, where η is the universal constant from Lemma 2.9 when we set $\gamma=\gamma(n,p,q)=2-\Xi(n,p,q)$. Once selected $\theta_0=\eta$ the goal will be to iterate the Lemma 2.9. For a fixed $(y,\tau)\in Q_{\frac{1}{2}}$ we claim that there exists a convergent sequence of real numbers $\{\varsigma_k\}_{k\geq 1}$, such that

$$\sup_{Q_{\rho^k}(y,\tau)}|u-\varsigma_k|\leq\rho^{k.\gamma} \tag{2.21}$$

where the radius $0<\rho\ll\frac{1}{2}$ is given by Lemma 2.9, upon the selection of γ as above.

The proof of (2.21) will follow by induction process. Lemma2.9 gives the first step of induction, $k=1$. Now suppose verified the k^{th} step in (2.21). We define

$$v_k(x,t)=\frac{u(y+\rho^k x,\tau+\rho^{2k}t)-\varsigma_k}{\rho^{k.\gamma}}$$

and

$$F_k(M,Z,x,t):=\rho^{k[2-\gamma]}F\left(\frac{1}{\rho^{k[2-\gamma]}}M,\frac{1}{\rho^{k[1-\gamma]}}Z,y+\rho^k x,\tau+\rho^{2k}t\right).$$

As commented before, see Remark 2.7, F_k is (λ,Λ,Γ)-parabolic operator, moreover by the induction hypothesis, $|v_k|\leq 1$ and

$$\frac{\partial v_k}{\partial t}(x,t)-F_k(D^2v_k,Dv_k,x,t)=\rho^{k.[2-\gamma]}f(y+\rho^k x,\tau+\rho^{2k}t)=:f_k(x,t),$$

in the $L^{\mathrm{P}}-$viscosity sense. One easily computes,

$$\|f_k\|_{L^{p,q}(Q_1)}\;=\;\rho^{k(2-\gamma)}\rho^{-k.\Xi(n,p,q)}\left(\int_{\tau-\rho^{2k}}^{\tau}\left(\int_{B_{\rho^k}(y)}|f(z,s)|^p dz\right)^{\frac{q}{p}}ds\right)^{\frac{1}{q}}.$$

Due to the sharp choice of $\gamma(n,p,q)=2-\Xi(n,p,q)$, we have that

$$\|f_k\|_{L^{p,q}(Q_1)}=\|f\|_{L^{p,q}\left(B_{\rho^k}(y)\times(\tau-\rho^{2k},\tau]\right)}\leq\|f\|_{L^{p,q}(Q_1)}\leq\eta,$$

as well as,

$$\left(\fint_{Q_1} \Theta^{\mathrm{P}}_{F_k}(x,t)\right)^{\frac{1}{\mathrm{P}}} \leq \max\left\{1, \rho^{k(2-\gamma)}\right\} \left(\fint_{Q_{\rho^k}(y,\tau)} \Theta^{\mathrm{P}}_{F}(y,\tau,x,t)\right)^{\frac{1}{\mathrm{P}}} \leq \eta.$$

In conclusion, we are allowed to employ Lemma 2.9 to v_k, which provides the existence of a universally bounded real number $\overline{\varsigma}_k$ with $|\overline{\varsigma}_k| \leq C$, such that

$$\sup_{Q_\rho} |v_k - \overline{\varsigma}_k| \leq \rho^\gamma. \tag{2.22}$$

Finally, if we select

$$\varsigma_{k+1} := \varsigma_k + \rho^{k.\gamma}\overline{\varsigma}_k \tag{2.23}$$

and scaling (2.22) back to the unit domain, we obtain the $(k+1)^{th}$ step in the induction process (2.21). In addition, we have that

$$|\varsigma_{k+1} - \varsigma_k| \leq C\rho^{k.\gamma}, \tag{2.24}$$

and hence the sequence $\{\varsigma_k\}_{k\geq 1}$ is Cauchy, and so it converges. From (2.21) $\varsigma_k \to u(Y,\tau)$. As well as from (2.24) it follows that

$$|u(y,\tau) - \varsigma_k| \leq \frac{C}{1-\rho^\gamma}\rho^{k.\gamma}, \tag{2.25}$$

Finally, for $0 < r < \rho$, let k the smallest integer such that $(x,t) \in Q_{\rho^k}(y,\tau) \setminus Q_{\rho^{k+1}}(y,\tau)$. It follows from (2.21) and (2.25) that

$$\begin{aligned}
\sup_{Q_r(y,\tau)} \frac{|u(x,t)-u(y,\tau)|}{d_{\mathrm{par}}((x,t),(y,\tau))^\gamma} &\leq \sup_{Q_r(y,\tau)} \frac{|u(x,t)-\varsigma_k| + |u(y,\tau)-\varsigma_k|}{d_{\mathrm{par}}((x,t),(y,\tau))^\gamma} \\
&\leq \left(1 + \frac{C}{1-\rho^\gamma}\right) \sup_{Q_r(y,\tau)} \frac{\rho^{k.\gamma}}{d_{\mathrm{par}}((x,t),(y,\tau))^\gamma} \\
&\leq \left(1 + \frac{C}{1-\rho^\gamma}\right) \frac{1}{\rho^\gamma}.
\end{aligned}$$

Last estimate, combined with Remark 2.8 and a standard covering argument provide

$$\|u\|_{C^{\gamma,\frac{\gamma}{2}}\left(Q_{\frac{1}{2}}\right)} \leq C(n,\lambda,\Lambda,\gamma)\left[\|u\|_{L^\infty(Q_1)} + \|f\|_{L^{p,q}(Q_1)} + 1\right].$$

and hence the proof of Theorem is verified. □

Remark 2.11. The exponent of Hölder regularity of our result is sharp. This can be verified through of following example from TEIXEIRA and URBANO (2014): Let $u \in$

$C_{loc}((-1,0]; L^2_{loc}(B_1)) \cap L^2_{loc}((-1,0]; W^{1,2}_{loc}(B_1))$ be a weak solution to

$$\frac{\partial u}{\partial t} - \Delta u = f \quad in \quad Q_1$$

Suppose that $1 < \Xi(n,p,q) < 2$ then for $\gamma := 2 - \Xi(n,p,q)$ we have that $u \in C^{\gamma,\frac{\gamma}{2}}_{loc}(Q_1)$. Remark that in this case $\mathfrak{p}_0 = \frac{n+2}{2}$.

Remark 2.12. Under VMO (Vanishing Mean Oscillation) assumption of the coefficients of the operator F:

$$\fint_{Q_r(y,\tau)} \Theta^{\mathrm{P}}_F(y,\tau,x,t) = \mathrm{o}(1),$$

as $r \to 0$, Theorem 2.10 holds without the smallness oscillation condition, as it can always be assumed upon an appropriate scaling.

Remark 2.13. Under no assumptions on the coefficients, other than ellipticity, adjustments in the proof of previous Theorem yields $C^{\gamma,\frac{\gamma}{2}}_{loc}(Q_1)$ where $\gamma := \min\{\beta^-, 2 - \Xi(n,p,q)\}$ where $0 < \beta < 1$ is the maximal exponent from Proposition 2.3.

Remark 2.14. An adjustment into demonstration of the previous Theorem we obtain that $L^{\mathrm{P}}-$viscosity solutions to (Eq) are $C^{0,\min\left\{\beta^-,2-\left(\frac{n}{p}+\frac{2}{q}\right)\right\}}_{loc}(Q_1)$ where $0 < \beta < 1$ comes from Preposition 2.3. Moreover, we must to interpret this result in following sense

$$\begin{cases} \text{If} \quad 2 - \left(\frac{n}{p} + \frac{2}{q}\right) < \beta \quad \text{then} \quad u \in C^{0,2-\left(\frac{n}{p}+\frac{2}{q}\right)}_{loc}(Q_1) \\ \text{If} \quad 2 - \left(\frac{n}{p} + \frac{2}{q}\right) \geq \beta \quad \text{then} \quad u \in C^{0,\gamma}_{loc}(Q_1), \text{for any } \gamma < \beta \end{cases}$$

2.3 Parabolic Log-Lipschitz type estimates

In this section we address the question of finding the optimal and universal modulus of continuity for solutions of uniformly parabolic equations of the form (Eq) whose right hand side lies in the borderline space $L^{p,q}(Q_1)$, when p and q lie on the critical surface:

$$\Xi(n,p,q) = 1.$$

Such estimate is particularly important to the general theory of fully non-linear parabolic equations. Through a simple analysis one verifies that solutions of (Eq), with sources under the above borderline integrability condition should be asymptotically Lipschitz continuous. Indeed, as $\Xi(n,p,q) \to 1^+$, solutions are Hölder continuous (in the parabolic sense) for every exponent $0 < \alpha < 1$. The key goal in this section is to obtain the sharp, quantitative modulus of continuity for u.

Lemma 2.15. *Let u be a normalized $L^{\mathrm{P}}-$viscosity solution to* (Eq)*. There exist* $\eta(n,\lambda,\Lambda) >$

0 and $0 < \rho(n, \lambda, \Lambda) \ll \frac{1}{2}$, *such that if*

$$\max \left\{ \left(\fint_{Q_1} \Theta_F^{\mathrm{P}}(x,t) \right)^{\frac{1}{\mathrm{P}}}, \ \|f\|_{L^{p,q}(Q_1)}, \ \Gamma \right\} \leq \eta \tag{2.26}$$

under the condition $\Xi(n,p,q) = 1$, *then, we can find an affine function* $\mathfrak{L}(x) := \mathfrak{A} + \langle \mathfrak{B}, x \rangle$, *with universally bounded coefficients,* $|\mathfrak{A}| + |\mathfrak{B}| \leq C(n, \lambda, \Lambda)$, *such that*

$$\sup_{Q_\rho} |u - \mathfrak{L}| \leq \rho. \tag{2.27}$$

Demonstração. For a $\delta > 0$ which will be chosen *a posteriori*, we apply Lemma 2.6 and find a function $\mathfrak{h}\colon Q_{\frac{1}{2}} \to \mathbb{R}$ satisfying

$$\frac{\partial \mathfrak{h}}{\partial t} - F(D^2 \mathfrak{h}) = 0 \quad \text{in} \quad Q_{\frac{1}{2}},$$

in the L^{P}-viscosity sense such that

$$\sup_{Q_{\frac{1}{2}}} |u - \mathfrak{h}| \leq \delta. \tag{2.28}$$

We now define

$$\mathfrak{L}(x) = \mathfrak{h}(0,0) + \langle D\mathfrak{h}(0,0), x \rangle, \tag{2.29}$$

and apply the regularity theory available for $\mathfrak{h}$, see for instance (CRANDALL, KOCAN, and ŚWIECH, 2000, Section 7) or WANG (1992b), as to assure the existence of a universal constants $0 < \alpha_F < 1$ and $C > 0$ such that

$$|\mathfrak{h}(x,t) - \mathfrak{L}(x)| \leq C d_{\mathrm{par}}((x,t),(0,0))^{1+\alpha_F}, \quad \text{for } \sqrt{|x|^2 + |t|} < \frac{1}{3}. \tag{2.30}$$

It is time to make universal choices: we set

$$\rho := \min \left\{ r_0, \left(\frac{1}{2C} \right)^{\frac{1}{\alpha_F}} \right\} < \frac{1}{2} \quad \text{and} \quad \delta := \frac{1}{2}\rho, \tag{2.31}$$

which decides the value of $\eta(n, \lambda, \Lambda) > 0$ through the approximation Lemma 2.6. It the sequel we estimate

$$\sup_{Q_\rho} |u - \mathfrak{L}| \leq \sup_{Q_\rho} |u - \mathfrak{h}| + \sup_{Q_\rho} |\mathfrak{h} - \mathfrak{L}| \leq \rho,$$

and the proof is complete. □

Theorem 2.16. *Let* u *be an* $L^{\mathrm{P}}-$*viscosity solution to* (Eq). *There exists universal cons-*

tants, $r_0 > 0$ *and* $\theta_0 > 0$*, such that if*

$$\sup_{0<r\leq r_0}\sup_{(Y,\tau)\in Q_{\frac{1}{2}}}\left(\fint_{Q_r(y,\tau)}\Theta_F^{\mathrm{P}}(y,\tau,x,t)\right)^{\frac{1}{\mathrm{P}}}\leq\theta_0,$$

then, for a universal constant $C > 0$ *and any* $(x,t),(y,\tau)\in Q_{\frac{1}{2}}$*, there holds*

$$|u(x,t)-u(y,\tau)|\leq C\left[\|u\|_{L^\infty(Q_1)}+\|f\|_{L^{p,q}(Q_1)}+1\right]\cdot\omega(d_{par}((x,t),(y,\tau))),$$

where $\omega(s) := s\log\frac{1}{s}$ *is the Lipschitz logarithmic modulus of continuity.*

Demonstração. We start off the proof by assuming, with no loss of generality, that $|u|\leq 1$ and

$$\|f\|_{L^{p,q}(Q_1)}<\frac{\eta}{4}\quad\text{and}\quad\Gamma<\frac{\eta}{8\max\{1,\mathcal{L}^n(B_1(0))\}},$$

where $\eta=\eta(n,\lambda,\Lambda)$ the largest positive number such that the Lemma 2.15 holds. Choose $\theta_0=\frac{\eta}{8}$. For a fixed $(y,\tau)\in Q_{\frac{1}{2}}$ we will prove the existence of a sequence of affine functions

$$\mathfrak{L}_k(x)=\mathfrak{A}_k+\langle\mathfrak{B}_k,x-y\rangle$$

such that

$$\sup_{B_{\rho^k}(y)\times(\tau-\rho^{2k},\tau]}|u-\mathfrak{L}_k|\leq\rho^k \tag{2.32}$$

and

$$|\mathfrak{A}_{k+1}-\mathfrak{A}_k|\leq C\rho^k\quad\text{and}\quad|\mathfrak{B}_{k+1}-\mathfrak{B}_k|\leq C, \tag{2.33}$$

where $0<\rho\ll\frac{1}{2}$ is the radius given by Lemma 2.15. Notice that the second estimate in (2.33) gives the growing estimate on the linear coefficients of order

$$|\mathfrak{B}_k|\leq Ck. \tag{2.34}$$

We now argue by induction. Lemma 2.15 provides the first step and now we suppose that we have already verified the kth step of (2.32). Define

$$v_k(x,t):=\frac{u(y+\rho^k x,\tau+\rho^{2k}t)-\mathfrak{L}_k(y+\rho^k x)}{\rho^k},$$

which verifies $|v_k|\leq 1$ in Q_1, by the induction condition. Define

$$F_k(M,\overrightarrow{p},x,t):=\rho^k F\left(\frac{M}{\rho^k},\overrightarrow{p},y+\rho^k x,\tau+\rho^{2k}t\right).$$

It is plain to check that F_k is a (λ,Λ,Γ)-parabolic operator and

$$\frac{\partial v_k}{\partial t}(x,t) - F_k(D^2 v_k, Dv_k, x, t) = f_k(x,t) + g_k(x,t) = H_k(x,t)$$

in the $L^{\mathrm{P}}-$viscosity sense, where

$$f_k(x,t) := \rho^k f(y + \rho^k x, \tau + \rho^{2k} t)$$

and

$$g_k(x,t) := F_k(D^2 v_k, Dv_k + B_k, x, t) - F_k(D^2 v_k, Dv_k, x, t).$$

Moreover,

$$\|f_k\|_{L^{p,q}(Q_1)} \quad = \quad \rho^k \rho^{-k.\Xi(n,p,q)} \left(\int_{\tau-\rho^{2k}}^{\tau} \left(\int_{B_{\rho^k}(y)} |f(z,s)|^p dz \right)^{\frac{q}{p}} ds \right)^{\frac{1}{q}}.$$

By the critical condition, $\Xi(n,p,q) = 1$, we verify that

$$\|f_k\|_{L^{p,q}(Q_1)} = \|f\|_{L^{p,q}\left(B_{\rho^k}(y)\times(\tau-\rho^{2k},\tau]\right)} < \frac{\eta}{4}.$$

Moreover, given the smallest regime on Γ, assumption (2.1) and (2.34), we have

$$|g_k(x,t)| \le Ck\rho^k \Gamma < \frac{\eta}{8\max\{1, \mathcal{L}^n(B_1(0))\}}.$$

Thus,

$$\|g_k\|_{L^{p,q}(Q_1)} \le \frac{\eta}{8\max\{1, \mathcal{L}^n(B_1(0))\}} \sqrt[p]{\mathcal{L}^n(B_1(0))} \le \frac{\eta}{8}.$$

Therefore, $\|H_k\|_{L^{p,q}(Q_1)} < \frac{3\eta}{8}$. Furthermore,

$$\left(\fint_{Q_1} \Theta_{F_k}^{\mathrm{P}}(x,t) \right)^{\frac{1}{\mathrm{P}}} \le \max\left\{1, \rho^k\right\} \left(\fint_{Q_{\rho^k}(Y,\tau)} \Theta_F^{\mathrm{P}}(y,\tau,x,t) \right)^{\frac{1}{\mathrm{P}}} \le \frac{\eta}{8}.$$

We have verified that we can apply Lemma 2.15 to the function v_k, assuring the existence of an affine function $\widetilde{\mathfrak{L}}_k(x) = \widetilde{\mathfrak{A}}_k + \langle \widetilde{\mathfrak{B}}_k, x\rangle$ satisfying $|\widetilde{\mathfrak{A}}_k|, |\widetilde{\mathfrak{B}}_k| \le C$, such that

$$\sup_{Q_\rho} |v_k - \widetilde{\mathfrak{L}}_k| \le \rho. \tag{2.35}$$

We now define

$$\mathfrak{A}_{k+1} := \mathfrak{A}_k + \rho^k \widetilde{\mathfrak{A}}_k \quad \text{and} \quad \mathfrak{B}_{k+1} := \mathfrak{B}_k + \widetilde{\mathfrak{B}}_k. \tag{2.36}$$

Re-scaling (2.35) to the unit domain gives the $(k+1)$th induction step. The first estimate in (2.32) assures that the sequence $\{\mathfrak{A}_k\}_{k\ge1}$ converges to $u(y,\tau)$. Also we can estimate, by geometric series,

$$|u(y,\tau) - \mathfrak{A}_k| \leq \frac{C\rho^k}{1-\rho}. \tag{2.37}$$

Finally, for $0 < r < \rho$, let k be the lowest integer such that

$$(x,t) \in Q_{\rho^k}(y,\tau) \setminus Q_{\rho^{k+1}}(y,\tau).$$

It follows by (2.32), (2.34) and (2.37) that

$$\begin{aligned}
\sup_{Q_r(y,\tau)} \frac{|u(x,t) - u(y,\tau)|}{r\log r^{-1}} &\leq \sup_{Q_{\rho^k}(y,\tau)} \frac{|u - \mathfrak{L}_k| + |u(y,\tau) - \mathfrak{A}_k| + |\mathfrak{B}_k|\rho^k}{r\log r^{-1}} \\
&\leq C \sup_{Q_{\rho^k}(y,\tau)} \frac{k\rho^k}{r\log r^{-1}} \\
&\leq C(n,\lambda,\Lambda),
\end{aligned}$$

and the proof of the theorem is concluded. □

Remark 2.17. As a consequence of the estimate given by Theorem 2.16, we are able to derive a precise integral behaviour of the gradient of a solution to (Eq). Indeed, one can derive the following point-wise control, say near $(0,0)$:

$$|Du(x,t)| \lesssim -C\log(|x|^2 + |t|), \text{ for } \sqrt{|x|^2+|t|} \ll \frac{1}{2}$$

Under suitable smallness regime on $f \in L^{p,q}(Q_1)$ and on $\Theta_F \in L^{\mathrm{P}}(Q_1)$, it follows by an adjustment of our arguments, combined with $H^{1,\frac{1}{2},s}$ interior estimates from (CRANDALL, KOCAN, and ŚWIECH, 2000, Theorem 7.3) that one can approximate an $L^{\mathrm{P}}-$viscosity solution of Eq by an $\mathfrak{F}$-caloric function

$$\frac{\partial \mathfrak{h}}{\partial t} - F(D^2\mathfrak{h}, x_0, t_0) = 0 \quad \text{in} \quad Q_{\frac{1}{2}},$$

in the $H^{1,\frac{1}{2},s}\left(Q_{\frac{1}{2}}\right)$ topology. Thus, through an iterative process as indicated in the proof of Theorem 2.16, one can find affine functions $\mathfrak{L}_k$ such that

$$⨍_{Q_{\rho^k}} |D(u - \mathfrak{L}_k)|^s \leq 1.$$

Therefore, it is possible to establish $s-$BMO type of estimates for the gradient in terms of the $L^{p,q}(Q_1)$ norm of f, when the critical condition $\frac{n}{p} + \frac{2}{q} = 1$ is verified. That is,

$$\|Du\|_{s-\mathrm{BMO}(Q_r)} \leq C\left[\|u\|_{L^\infty(Q_1)} + \|f\|_{L^{p,q}(Q_1)} + 1\right], \quad \text{for} \quad 0 < r \ll \frac{1}{4}.$$

Comparing such an estimate with the results from (CRANDALL, KOCAN, and ŚWIECH, 2000, Theorem 7.3), it synthesizes quantitatively the fact of

$$|Du| \in \bigcap_{m\geq 1} L^m_{loc}(Q_1),$$

since $L^{\mathrm{P}}-$viscosity solutions have its gradient in $L^s_{loc}(Q_1)$ for all $s < \frac{n+2}{\Xi(n,p,q)-1}$.

2.4 Optimal $C^{1+\alpha,\frac{1+\alpha}{2}}$ regularity

In this section we obtain asymptotically sharp $C^{1+\sigma,\frac{1+\sigma}{2}}$ interior regularity estimates for solutions of (Eq). Such estimates are already available in the literature, see for instance CRANDALL, KOCAN, and ŚWIECH (2000) and WANG (1992a). We shall only comment on how we can deliver them by means of the arguments designed in Section 2.3.

Initially, we revisit Lemma 2.15 and observe that if $0 < \alpha_F \leq 1$ represents the optimal exponent from the $C^{1+\overline{\alpha},\frac{1+\overline{\alpha}}{2}}$ regularity theory for solutions to homogeneous $(\lambda, \Lambda, \Gamma)$-parabolic operators with constant coefficients, then given

$$\alpha \in (0, \alpha_F) \cap (0, 1 - \Xi(n,p,q)]\,, \tag{2.38}$$

since $\displaystyle\sup_{0<r\leq r_0} \sup_{(y,\tau)\in Q_{\frac{1}{2}}} \left(\fint_{Q_r(y,\tau)} \Theta^{\mathrm{P}}_F(y,\tau,x,t)\right)^{\frac{1}{\mathrm{P}}}$ and $\|f\|_{L^{p,q}}$ are under universal smallest regime assumption, we are able to choose

$$\rho := \min\left\{ r_0, \left(\frac{1}{2C}\right)^{\frac{1}{\alpha_F-\alpha}} \right\} \tag{2.39}$$

such that

$$\sup_{Q_\rho} |u - \mathfrak{L}| \leq \rho^{1+\alpha}, \tag{2.40}$$

where $\mathfrak{L}$ is given by (2.29). This is the first step in our induction process.

Before proceeding with an induction process, we come back to the smallest regime condition, and we will assume with loss of generality that

$$\max\left\{ \left(\fint_{Q_1} \Theta_F(x,t)^P\right)^{\frac{1}{\mathrm{P}}},\ \|f\|_{p,q},\ \Gamma \right\} \leq \Xi, \tag{2.41}$$

where

$$\Xi := \frac{\eta_0}{8\max\left\{1, C\sqrt[p]{\mathcal{L}^n(B_1)}\rho^{1-\alpha}\sum\limits_{j\geq 0}\rho^{j\alpha}\right\}}$$

and $\eta_0 > 0$ is the largest number such that the $\mathfrak{F}$-Caloric Approximation Lemma holds.

In face of previous statements, let us suppose that we have checked the kth step in the induction process, i.e.,

$$\sup_{Q_{\rho^k}} |u - \mathfrak{L}_k| \leq \rho^{k(1+\alpha)} \tag{2.42}$$

with the following order of approximation for the coefficients

$$|\mathfrak{A}_{k+1} - \mathfrak{A}_k| \leq C\rho^{k(1+\alpha)} \quad \text{and} \quad |\mathfrak{B}_{k+1} - \mathfrak{B}_k| \leq C\rho^{k\alpha}. \tag{2.43}$$

We define the re-scaled function

$$v_k(x,t) := \frac{u(y+\rho^k x, \tau + \rho^{2k} t) - \mathfrak{L}_k(y+\rho^k x)}{\rho^{k(1+\alpha)}},$$

which verifies $|v_k| \leq 1$ in Q_1, and satisfies in the $L^{\mathrm{P}}-$viscosity sense

$$\frac{\partial v_k}{\partial t}(x,t) - G_k(D^2 v_k, Dv_k, x, t) = f_k(x,t) + g_k(x,t) = H_k(x,t) \tag{2.44}$$

where

$$G_k(M, \overrightarrow{p}, x, t) = \rho^{k(1-\alpha)} F\left(\frac{1}{\rho^{k(1-\alpha)}} M, \rho^{k\alpha}\overrightarrow{p}, \rho^k x, \rho^{2k} t\right)$$

is a $(\lambda, \Lambda, \Gamma)$-parabolic operator,

$$f_k(x,t) := \rho^{k(1-\alpha)} f(y+\rho^k x, \tau+\rho^{2k} t)$$

and

$$g_k(x,t) := G_k(D^2 v_k, Dv_k + B_k, x, t) - G_k(D^2 v_k, Dv_k, x, t).$$

Now,

$$\|f_k\|_{L^{p,q}(Q_1)} = \omega(\rho^k)\|f\|_{L^{p,q}\left(Q_{\rho^k}(Y,\tau)\right)} < \frac{\eta_0}{4},$$

where $\omega(\rho^k) = \rho^{k[1-\alpha-\Xi(n,p,q)]}$ is computed by change of variables. By the integrability relation and the value of α, we conclude $\omega(\rho^k) \leq 1$ for all integer $k \geq 1$. Also, from Lipschitz structure on F (assumption (2.1)) and (2.43) we get

$$\begin{aligned}
|g_k(x,t)| &\leq \rho^{k(1-\alpha)} \left|G_k(D^2 v_k, Dv_k + B_k, x, t) - G_k(D^2 v_k, Dv_k, x, t)\right| \\
&\leq \rho^{k(1-\alpha)}\Gamma\|B_k\| \\
&\leq \rho^{k(1-\alpha)}\Gamma\mathfrak{C}\sum_{j=0}^{k-1}\rho^{j\alpha} \\
&\leq \mathfrak{C}\Gamma\rho^{k(1-\alpha)}\sum_{j\geq 0}\rho^{j\alpha}.
\end{aligned}$$

Thus, from smallest regime (2.41) we obtain

$$\|g_k\|_{L^{p,q}(Q_1)} \leq \mathfrak{C}\Gamma\rho^{k(1-\alpha)}\sum_{j\geq 0}\rho^{j\alpha}\sqrt[p]{\mathcal{L}^n(B_1(0))} < \frac{\eta_0}{4}.$$

Finally,

$$\left(\fint_{Q_1}\Theta^{\mathrm{P}}_{G_k}(x,t)\right)^{\frac{1}{\mathrm{P}}} \leq \max\left\{1,\rho^{k(1-\alpha)}\right\}\left(\fint_{Q_{\rho^k}(y,\tau)}\Theta^{\mathrm{P}}_{F}(y,\tau,x,t)\right)^{\frac{1}{\mathrm{P}}} \quad \text{and} \quad \|H_k\|_{L^{p,q}(Q_1)} \leq \eta_0.$$

Therefore, we can apply the first induction step, which gives the existence of an affine function $\overline{\mathfrak{L}}_k(x) := \overline{\mathfrak{A}}_k + \langle\overline{\mathfrak{B}}_k, x\rangle$ with $|\overline{\mathfrak{A}}_k|, |\overline{\mathfrak{B}}_k| \leq C(n,\lambda,\Lambda)$ such that

$$\sup_{Q_\rho}|v_k - \overline{\mathfrak{L}}_k| \leq \rho^{1+\alpha}.$$

Rewriting the previous estimate in the unit domain gives

$$\sup_{Q_{\rho^{k+1}}}|u - \mathfrak{L}_{k+1}| \leq \rho^{(k+1)(1+\alpha)},$$

for $\mathfrak{L}_{k+1}(x) := \mathfrak{L}_k(x) + \rho^{k(1+\alpha)}\overline{\mathfrak{L}}_k(\rho^{-k}x)$. The coefficients fulfils

$$|\mathfrak{A}_{k+1} - \mathfrak{A}_k| + \rho^k|\mathfrak{B}_{k+1} - \mathfrak{B}_k| \leq C_0(n,\lambda.\Lambda)\rho^{(1+\alpha)k}. \tag{2.45}$$

Hence, from (2.45), we conclude that $(\mathfrak{A}_k)_{k\geq 1} \subset \mathbb{R}$ and $(\mathfrak{B}_k)_{k\geq 1} \subset \mathbb{R}^n$ are Cauchy sequences, thereby converging to $u(y,\tau)$ and to $Du(y,\tau)$ respectively. Moreover, we have the following rate convergence

$$|u(y,\tau) - \mathfrak{A}_k| \leq C_0\frac{\rho^{k(1+\alpha)}}{1-\rho} \quad \text{and} \quad |Du(y,\tau) - \mathfrak{B}_k| \leq C_0\frac{\rho^{k\alpha}}{1-\rho} \tag{2.46}$$

Finally, given any $0 < r < \rho$, let k be an integer such that $(x,t) \in Q_{\rho^k}(y,\tau) \setminus Q_{\rho^{k+1}}(y,\tau)$. Therefore, we estimate from (2.46) that

$$\sup_{Q_r(y,\tau)}|u(x,t) - [u(y,\tau) + \langle Du(y,\tau), x-y\rangle]| \leq C_0(n,\lambda,\Lambda,\alpha).r^{1+\alpha}$$

and the sketch is finished.

Remark 2.18. We highlight that the previous result must be interpreted in following way

$$\begin{cases} \text{If} \quad 1-\Xi(n,p,q) < \alpha_F \quad \text{then} \quad u \in C^{1+\sigma,\frac{1+\sigma}{2}}_{loc}(Q_1), \quad \text{for} \quad \sigma = 1-\Xi(n,p,q) \\ \text{If} \quad 1-\Xi(n,p,q) \geq \alpha_F \quad \text{then} \quad u \in C^{1+\gamma,\frac{1+\gamma}{2}}_{loc}(Q_1), \quad \text{for any} \quad \gamma < \alpha_F. \end{cases}$$

Remark 2.19. The optimality of previous result can be verified by an example due to Krylov in (KRYLOV, 2008, Page 209).

2.5 Parabolic $C^{1,\text{Log}-\text{Lip}}$ type estimates

In this last section we address the issue of finding the optimal regularity estimate for the limiting upper borderline case $f \in$ BMO, which encompasses the case $f \in L^{\infty,\infty} \simeq L^{\infty}$.

In view of the almost optimal estimates given in the previous section, establishing a quantitative regularity result for solutions to (Eq) with bounded forcing term, requires that $\mathfrak{F}$-harmonic functions are $C^{2+\sigma,\frac{2+\sigma}{2}}$ smooth; otherwise no further information could be revealed from better hypotheses on the source function f. Evans-Krylov's regularity theory EVANS (1982), KRYLOV (1982) and KRYLOV (1983) assures that convex/concave equations do satisfy the $C^{2+\sigma,\frac{2+\sigma}{2}}$ smoothness assumption.

We now state and prove our sharp par $- C^{1,\text{Log-Lip}}$ interior regularity theorem. For simplicity we will work on equations with constant coefficients and with no gradient dependence. Similar result can be easily obtained under continuity condition on the coefficients and Lipschitz control on the gradient dependence.

Theorem 2.20. *Let u be a C^0-viscosity solution to $\frac{\partial u}{\partial t} - F(D^2u) = f(x,t)$ in Q_1. If any solution of $\frac{\partial v}{\partial t} - F(D^2v + M) = K$, where $M \in Sym(n)$ and $K \in \mathbb{R}$ are on the surface $-F(M) = K$, has interior $C^{2+\sigma,\frac{2+\sigma}{2}}$ a priori estimates, i.e.,*

$$\|v\|_{C^{2+\sigma,\frac{2+\sigma}{2}}(Q_r)} \leq \frac{\overline{\Phi}}{r^{2+\sigma}}\|v\|_{L^{\infty}(Q_1)} \tag{2.47}$$

for some $\overline{\Phi}(n,\lambda,\Lambda,K) > 0$ and $\sigma(n,\lambda,\Lambda) \in (0,1)$. Then, for a constant $C(n,\lambda,\Lambda,\sigma,\overline{\Phi}) > 0$, there holds

$$|u(x,t) - [u(0,0) + \langle Du(0,0), x\rangle]| \leq C\left[\|u\|_{L^{\infty}(Q_1)} + \|f\|_{BMO(Q_1)} + 1\right].\omega(d_{par}((x,t),(0,0))) \tag{2.48}$$

where $\omega(r) = r^2 \log r^{-1}$ is the C^1-Log-Lipschitz modulus of continuity.

Demonstração. By standard reduction arguments, we can assume that $\|u\|_{L^{\infty}(Q_1)} \leq \frac{1}{2}$ and $\|f\|_{\text{BMO}(Q_1)} \leq \vartheta_0$ for some $\vartheta_0 > 0$ which will be chosen *a posteriori*. Throughout the proof we use the notation

$$[f]_{\text{avg},Q_1} := \fint_{Q_1} f(z,\varsigma)dzd\varsigma.$$

The strategy is to find parabolic quadratic polynomials

$$\mathfrak{P}_k(x,t) := \frac{1}{2}\langle\mathfrak{A}_k x, x\rangle + \mathfrak{B}_k t + \langle\mathfrak{C}_k, x\rangle + \mathfrak{D}_k$$

such that $\mathfrak{P}_0 = \mathfrak{P}_{-1} = \frac{1}{2}\langle Nx, x\rangle$, where $-F(N) = [f]_{\text{avg},Q_1}$ and for all $k \geq 0$,

$$\mathfrak{B}_k - F(\mathfrak{A}_k) = [f]_{\text{avg},Q_1} \quad \textit{and} \quad \sup_{Q_{\rho^k}} |u - \mathfrak{P}_k| \leq \rho^{2k}, \tag{2.49}$$

with

$$\rho^{2(k-1)}(|\mathfrak{A}_k - \mathfrak{A}_{k-1}| + |\mathfrak{B}_k - \mathfrak{B}_{k-1}|) + \rho^{k-1}|\mathfrak{C}_k - \mathfrak{C}_{k-1}| + |\mathfrak{D}_k - \mathfrak{D}_{k-1}| \leq C\rho^{2(k-1)} \tag{2.50}$$

where the radius $0 < \rho \ll \frac{1}{2}$ in (2.49) and (2.50) will also be determined *a posteriori*. We prove the existence of such polynomials by induction process in k. The first step of induction, $k = 0$, it is obviously satisfied. Suppose now that we have verified the thesis of induction for $k = 0, 1, \ldots, i$. Then, defining the re-scaled function $v := Q_1 \to \mathbb{R}$ given by

$$v_k(x,t) = \frac{(u - \mathfrak{P}_k)(\rho^k x, \rho^{2k} t)}{\rho^{2k}},$$

we have, by induction hypothesis, that $|v_k| \leq 1$ and it solves

$$\frac{\partial v_k}{\partial t}(x,t) - F_k(D^2 v_k) = f(\rho^k x, \rho^{2k} t) := f_k(x,t)$$

in the C^0-viscosity sense, where $F_k(M) := F(M + \mathfrak{A}_k) - \mathfrak{B}_k$ which is a $(\lambda, \Lambda, 0)$-parabolic operator with

$$\begin{aligned} \|f_k\|_{\text{BMO}(Q_1)} &:= \sup_{0<r\leq 1} \fint_{Q_r} |f_k(x,t) - [f_k]_{\text{avg},Q_r}|\, dxdt \\ &= \sup_{0<r\leq 1} \fint_{Q_{r\rho}} \left|f(z,\varsigma) - [f]_{\text{avg},Q_{r\rho}}\right| dz d\varsigma \\ &\leq \|f\|_{\text{BMO}(Q_1)} \\ &\leq \vartheta_0. \end{aligned}$$

As in Lemma (2.6), with some slight changes, and, under smallness assumption on $\|f\|_{BMO(Q_1)}$ to be set soon, we can find a C^0-viscosity solution $\mathfrak{h}$ to

$$\frac{\partial \mathfrak{h}}{\partial t} - F(D^2\mathfrak{h} + M_k) = [f]_{\text{avg},Q_1} \quad in \quad Q_1,$$

such that

$$\sup_{Q_{\frac{1}{2}}} |v_k - \mathfrak{h}| \leq \delta,$$

for some $\delta > 0$ which we will choose below. From hypothesis (2.47), $\mathfrak{h}$ is $C^{2+\sigma, \frac{2+\sigma}{2}}$ at the origin with universal bounds. Thus, if we define

$$\mathfrak{P}(x,t) := \frac{1}{2}\langle D^2\mathfrak{h}(0,0)x, x\rangle + \mathfrak{h}_t(0,0)t + \langle D\mathfrak{h}(0,0), x\rangle + \mathfrak{h}(0,0),$$

by the $C^{2+\sigma,\frac{2+\sigma}{2}}$ regularity assumption (2.47), we can estimate

$$|D^2\mathfrak{h}(0,0)| + |\mathfrak{h}_t(0,0)| + |D\mathfrak{h}(0,0)| + |\mathfrak{h}(0,0)| \le C\overline{\Phi}$$

where

$$|(\mathfrak{h} - \mathfrak{P})(x,t)| \le C(n)\overline{\Phi} d_{\text{par}}((x,t),(0,0))^{2+\sigma}.$$

Now, we are able to select

$$\rho := \left(\frac{1}{2C\overline{\Phi}}\right)^{\frac{1}{\sigma}} \quad \text{and} \quad \delta := \frac{1}{2}\rho^2.$$

The choice above for $\rho(\overline{\Phi},\sigma,\Lambda,\lambda,n) \ll \frac{1}{2}$ decides the value for $\delta(\overline{\Phi},\sigma,\Lambda,\lambda,n) > 0$ which determines, by Lemma (2.6), the universal smallness regime given by the constant $\vartheta_0 > 0$. From the previous choices, we readily obtain

$$\sup_{Q_\rho} |v_k - \mathfrak{P}| \le \rho^2. \tag{2.51}$$

Rewriting (3.50) back to the unit domain yields

$$\sup_{Q_{\rho^{k+1}}} \left| u(x,t) - \left[\mathfrak{P}_k(x,t) + \rho^{2k}\mathfrak{P}\left(\frac{x}{\rho^k}, \frac{t}{\rho^{2k}}\right)\right]\right| \le \rho^{2(k+1)}. \tag{2.52}$$

Therefore, defining

$$\mathfrak{P}_{k+1}(x,t) := \mathfrak{P}_k(x,t) + \rho^{2k}\mathfrak{P}\left(\frac{x}{\rho^k}, \frac{t}{\rho^{2k}}\right),$$

we verify the $(k+1)^{\underline{th}}$ step of induction and, clearly, the required conditions (2.49) and (2.50) are satisfied. From (2.50) we conclude that $\mathfrak{D}_k \to u(0,0)$ and $\mathfrak{D}_k \to Du(0.0)$, with the following estimates

$$|u(0,0) - \mathfrak{D}_k| \le \frac{C\rho^{2k}}{1-\rho} \quad \text{and} \quad |Du(0,0) - \mathfrak{C}_k| \le \frac{C\rho^k}{1-\rho}. \tag{2.53}$$

Furthermore, equation (2.50) yields the growth estimates:

$$|\mathfrak{A}_k| \le \sum_{j=1}^{k} |\mathfrak{A}_j - \mathfrak{A}_{j-1}| \le Ck \quad \text{and} \quad |\mathfrak{B}_k| \le \sum_{j=1}^{k} |\mathfrak{B}_j - \mathfrak{B}_{j-1}| \le Ck. \tag{2.54}$$

Finally, given any $0 < r < \rho$, let k be an integer such that

$$(x,t) \in Q_{\rho^k}(y,\tau) \setminus Q_{\rho^{k+1}}(y,\tau)$$

We estimate from equations (2.49), (2.53) and (2.54),

$$\begin{aligned}\sup_{Q_r(0)} |u(x,t) - [u(0,0) + \langle Du(0,0), x\rangle]| \leq & \ \rho^{2k} + |u(0,0) - \mathfrak{D}_k| + \rho^k |Du(0,0) - \mathfrak{C}_k| \\ & + \ \rho^{2k}(|\mathfrak{B}_k| + |\mathfrak{A}_k|) \\ & \leq \ C(n, \lambda, \Lambda, \sigma, \overline{\Phi}).r^2 \log r^{-1},\end{aligned}$$

and the proof of Theorem is finished. □

Remark 2.21. The final estimate says that solutions to (Eq) are asymptotically $C^{2,1}$ in the parabolic sense. Furthermore, adjustments in the previous explanation yield $u_t, D^2 u \in s - BMO\left(Q_{\frac{1}{2}}\right)$, with appropriate *a priori* estimate in terms of the BMO-norm of f in Q_1. Indeed, under appropriate smallness regime on $f \in BMO(Q_1)$ we can approximate u by a solution $\mathfrak{h}$ to

$$\frac{\partial \mathfrak{h}}{\partial t} - F(D^2\mathfrak{h}, x_0, t_0) = [f]_{\text{avg},Q_1} \quad \text{in} \quad Q_{\frac{1}{2}}$$

in the $W^{2,1,s}\left(Q_{\frac{1}{2}}\right)$ topology. Thus, by an iterative process similar to the one used here one finds parabolic quadratic polynomials $\mathfrak{P}_k$ such that

$$\fint_{Q_{\rho^k}} \left(\left| \frac{\partial (u - \mathfrak{P}_k)}{\partial t} \right|^s + |D^2(u - \mathfrak{P}_k)|^s \right) \leq 1$$

Therefore, the previous sentence provides the desired $s-$BMO estimate. In other words,

$$\|u_t\|_{s-BMO(Q_r)} + \|D^2 u\|_{s-BMO(Q_r)} \leq C\{\|u\|_{L^\infty(Q_1)} + \|f\|_{BMO(Q_1)}\}, \quad \text{for} \quad 0 < r \ll 1$$

Remark 2.22. The result proven in this Section can be further applied to equations of the form $\frac{\partial u}{\partial t} - F(D^2u, x, t) = f(u, x, t)$, where f is continuous. It is particularly meaningful to geometric flow problems:

$$\frac{\partial H}{\partial t} - \Delta H - H|A|^2 = 0,$$

where H is the inwards mean curvature vector of the surface at position x and time t and $|A|$ represents the norm of the second fundamental form. This equation describes the mean curvature hyper-surface in the Euclidean space $\mathbb{R}^{n+1}$, see for example SHENG and WANG (2010).

Remark 2.23. As a final remark, we note that the results proven in this chapter can be generalized for a more general class of anisotropic Lebesgue spaces with mixed norms.

Namely, consider $\overrightarrow{p} = (p_1, \ldots, p_n)$. Let $f \in L^{p_1,\ldots,p_n,q}(\hat{Q}_1)$, i.e., $f \in L^{p_1}_{x_1} \ldots L^{p_n}_{x_n} L^q_t$, where $\hat{Q}_1 = \left\{ x \in \mathbb{R}^n : \max_{1\leq j\leq n} \{|x_j|\} = \frac{1}{2} \right\} \times (-1, 0]$. The quantity

$$\Xi(n, p_1, \ldots, p_n, q) := \left(\sum_{i=1}^{n} \frac{1}{p_i} \right) + \frac{2}{q}$$

sets up the following regularity regimes, with universal *a priori* estimates:

- $1 < \Xi(n, p_1, \ldots, p_n, q) < \frac{n+2}{\mathfrak{p}_0} < 2$ for the $C^{\alpha,\frac{\alpha}{2}}$ regularity regime;
- $\Xi(n, p_1, \ldots, p_n, q) = 1$ for the Lipschitz logarithmic type estimates;
- $0 < \Xi(n, p_1, \ldots, p_n, q) < 1$ for the $C^{1+\alpha,\frac{1+\alpha}{2}}$ regularity regime.

3 REGULARITY ESTIMATES OF FLAT SOLUTIONS TO FULLY NON-LINEAR PROBLEMS

In this chapter we are interested in investigating sharp interior and improved regularity estimates for solutions of the class of non-convex fully non-linear parabolic equations of the form

$$\frac{\partial u}{\partial t} - F(x,t,D^2u) = f(x,t). \tag{3.1}$$

Precisely, under differentiability of F with respect to the Hessian argument and suitable continuity assumptions on the coefficients and the source function, we establish *Schauder type estimates* for *flat viscosity solutions*, namely solutions whose oscillation is small enough, $\|u\| \leq \delta \ll 1$, where the flatness degree depends only on universal and structural parameters of the problem. Such regularity issues play an important role in many contexts, such as Mathematical Physics, Stochastic Process, Differential Geometry, Geometric Analysis, Free boundary problems, among others, because they enable us to access high-order estimates, as well as to find the sharp moduli of continuity to highest derivatives of the solutions in terms of the regularity of the data for problems governed by equations of the form (3.1).

In our studies the fully non-linear operator $F\colon Q_1 \times \mathrm{Sym}(n) \to \mathbb{R}$ is assumed to be *uniformly elliptic*, i.e., for any $M, N \in \mathrm{Sym}(n)$ with $M \geq N$ and all $(x,t) \in Q_1$ fixed there holds

$$\mathcal{P}^-_{\lambda,\Lambda}(M-N) \leq F(x,t,M) - F(x,t,N) \leq \mathcal{P}^+_{\lambda,\Lambda}(M-N), \tag{3.2}$$

where

$$\mathcal{P}^+_{\lambda,\Lambda}(M) := \lambda.\sum_{e_i<0} e_i + \Lambda.\sum_{e_i>0} e_i \quad \text{and} \quad \mathcal{P}^-_{\lambda,\Lambda}(M) := \lambda.\sum_{e_i>0} e_i + \Lambda.\sum_{e_i<0} e_i$$

are the well-known *Pucci's extremal operators*, with $\{e_i : 1 \leq i \leq n\}$ being the eigenvalues of M and $\Lambda \geq \lambda > 0$ are the *ellipticity constants* to F. Another fundamental aspect of our approach consists in that we do not assume convexity or concavity on F. For this very reason viscosity solutions may not, in principle, be classical (compare with ZOU and CHEN (2002), TIAN and WANG (2013) and (WANG, 1992b, Section 1.1)).

Therefore, given a flat viscosity solution to (3.1) with f a Dini continuous function, we obtain that $u \in C^{2,1,\psi}_{loc}(Q_1)$ where

$$\psi(s) = \int_0^s \frac{\tau(r)}{r} dr + s\int_s^1 \frac{\tau(r)}{r^2} dr.$$

Such a modulus of continuity is sharp, as well as it revels that the Dini condition is pivotal in our assumption.

This chapter is organized as following: In Section 3.1 we define the statements of this chapter. In Section 3.1 we explain the core of the geometric tangential analysis in order to access the improved estimates for flat solution. In Section 3.2 we shall prove main the result of this chapter, namely a Schauder type estimates for flat solutions. In sequel, Section 3.4 we apply the main result in three consequences, result of Evans-Krylov type, a partial regularity result and high regularity estimates for some geometric flows. To finish, in Section 3.5 we comment how to adjust our technique in order to obtain C^1 Lipschitz logarithmic modulus of continuity for flat solutions when the Dini assumption fails.

3.1 Assumptions and statement of the main result

Let us move towards the hypotheses, set-up and main notations used in this chapter.

The equations and problems studied here are designed in the $\mathbb{R}^n_- := \mathbb{R}^n \times (-\infty, 0]$ space. The semi-open cylinder of radius $r > 0$ centred at the point $(x_0, t_0) \in \mathbb{R}^n_-$, is denoted by $Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0]$. Usually the cylinder of radius r, centred at the origin is written simply as Q_r. The space of $n \times n$ symmetric matrices will be denoted by $\mathrm{Sym}(n)$. By modulus of continuity we mean an increasing and sub-addictive function $\theta\colon [0, +\infty) \to [0, +\infty)$, which is continuous and satisfies $\theta(0) = 0$. Hereafter we shall assume the following conditions on the operator $F\colon Q_1 \times \mathrm{Sym}(n) \to \mathbb{R}$ and source $f\colon Q_1 \to \mathbb{R}$:

(A1) [**Uniform Ellipticity condition**] There exist constants $\Lambda \geq \lambda > 0$ (ellipticity constants) such that for any $M, P \in \mathrm{Sym}(n)$, with $P \geq 0$ and all $(x,t) \in Q_1$, there holds

$$\lambda \|P\| \leq F(x,t,M+P) - F(x,t,M) \leq \Lambda \|P\|. \tag{3.3}$$

(A2) [**Regularity condition**] $F(x,t,M)$ is differentiable with respect to M and for a modulus of continuity ω, there holds

$$\|D_M F(x,t,M_1) - D_M F(x,t,M_2)\| \leq \omega(\|M_1 - M_2\|), \tag{3.4}$$

for all $(x,t,M_i) \in Q_1 \times \mathrm{Sym}(n)$.

(A3) [**Parabolic Dini condition**] For another modulus of continuity τ, there holds

$$\frac{|F(x,t,M) - F(y,s,M)|}{\|M\| + 1} \leq \tau(d((x,t),(y,s))), \quad \text{for } M \in \mathrm{Sym}(n). \tag{3.5}$$

$$|f(x,t) - f(y,s)| \leq \tau(d((x,t),(y,s))), \quad \text{for } (x,t),(y,s) \in Q_1. \tag{3.6}$$

where $d((x,t),(y,s)) = \sqrt{|x-y|^2 + |t-s|}$ is the standard *parabolic distance*. Furthermore, τ satisfies the well-known *Dini continuity condition*.

$$\int_0^1 \frac{\tau(r)}{r} dr < \infty. \tag{3.7}$$

(A4) [**Compatibility conditions**] There exist $0 < \alpha^\sharp(n,\lambda,\Lambda,\omega) < 1$ and $0 < \sigma^\sharp(n,\lambda,\Lambda,\tau) \ll 1$ such that

$$\sup_{\sigma \leq \sigma^\sharp} \frac{\sigma^{\alpha^\sharp}\varphi(\sigma^k)}{\varphi(\sigma^{k+1})} = \mathfrak{c} \leq 1 \ \ \forall\ k \geq 1 \quad \text{and} \quad \lim_{s \to 0^+} \frac{\varphi(s)}{s} = \infty, \tag{3.8}$$

where $\varphi(s) = s\displaystyle\int_s^1 \frac{\tau(r)}{r^2} dr$ is a modulus of continuity.

(A5) [**Reducibility condition**] It will be enforced in this chapter the following reduction condition:

$$F(0,0,0_{n\times n}) = f(0,0) = 0. \tag{3.9}$$

Condition (**A2**) fixes a modulus of continuity ω to the derivative of F. The regularity estimates proven in this chapter depends upon ω. Condition (**A3**) sets the continuity of the medium, as well as it is also important in order to observe that Dini assumption on the source f is a necessary condition for high order derivatives to be continuous, see the classical articles IL'IN (1967) and KRUŽKOV (1967), for the linear case, and also ZOU and CHEN (2002), TIAN and WANG (2013) for surveys on this topic in the fully nonlinear scenery. The compatibility condition (**A4**) is naturally satisfied for the Hölder modulus $\tau(r) = r^{\alpha^\sharp}$, however it covers a wide class of moduli which are not necessarily $\alpha^\sharp$-homogeneous. For example, a modulus of continuity like $\tau(r) = r^\gamma\left(\ln \frac{1}{r}\right)^\beta$ is not Hölder continuous for many values of $\gamma \in [0,\alpha^\sharp]$ and $\beta \in \mathbb{R}$ but it satisfies (3.8), as well as (3.7). Summarily, such a condition arises to fill the lack of $\alpha^\sharp$-homogeneity of τ and φ respectively when compared to standard Schauder approach. Moreover, such an assumption (first sentence in (3.8)) is crucial in our iterative process, because it allow us, at each step, to fall in the flatness approximation hypothesis (cf. Lemma 3.9). The second sentence in (3.8) is imposed for selecting an appropriated radius in the proof of the key Lemma 3.4. Notice that such a φ in (3.8) comes from Dini-Campanato spaces theory (cf. KOVATS (1999))(as well as the classical Potential theory, cf. TIAN and WANG (2013) and it is the one which will imply that flat solutions are embedding in $C^{2,1}$ with an appropriated moduli of continuity for highest derivatives. The last hypothesis (**A5**) is not restrictive, because one can always reduce the problem in order to check it. Finally, assumption (**A1**), which is equivalent to condition (3.2), concerns the notion of uniform ellipticity, i.e., the mapping $M \mapsto F(x,t,M)$ is increasing in the natural ordering on $Sym(n)$.

Remark 3.1. Following classical terminology in the current literature, any constant which depends only on dimension and ellipticity parameters λ and Λ will be denoted *universal*. In the same way, *structural* parameters are the ones that depend upon τ and ω.

Under the structural condition of Uniform Ellipticity the theory of viscosity solutions provides an appropriate notion for weak solutions:

Definition 3.2 (**Viscosity solutions**, CRANDALL, KOCAN, and ŚWIECH (2000) and WANG (1992a))**.** A continuous function $u \in C^0(Q_1)$ is said to be a viscosity sub-solution (resp. super-solution) to (3.1) in Q_1 if whenever one touches the graph of u by above (resp. by bellow) by a function $\phi \in C^{2,1}(Q_1)$ at $(x_0,t_0) \in Q_1$, there holds

$$\frac{\partial \phi}{\partial t} - F(x_0,t_0,D^2\phi(x_0,t_0)) \leq f(x_0,t_0) \quad (\text{resp. } \geq f(x_0,t_0)).$$

Finally, we say u is a viscosity solution to (3.1) if it is simultaneously a viscosity sub-solution and a super-solution of (3.1).

For a fixed $(x_0,t_0) \in Q_1$, we measure the oscillation of the coefficients of F around (x_0,t_0) by

$$\Theta_F(x_0,t_0,x,t) := \sup_{M \in Sym(n)} \frac{|F(x,t,M) - F(x_0,t_0,M)|}{||M||+1}. \tag{3.10}$$

Moreover, by simplicity we write $\Theta_F(0,0,x,t) = \Theta_F(x,t)$.

For a modulus of continuity ζ and $(x_0,t_0) \in Q_1$ we say $u \in C^{2,1,\zeta}(Q_{r_0}(x_0,t_0))$ if

$$[u]_{C^{2,1,\zeta}(Q_{r_0}(x_0,t_0))} := \sup_{0<r\leq r_0} \left(\inf_{\mathfrak{p} \in \mathfrak{Q}_p} \frac{\|u - \mathfrak{p}\|_{L^\infty(Q_r(x_0,t_0))}}{r^2\zeta(r)} \right)$$

is finite, where $0 < r_0 < \min\{1, \operatorname{dist}(x_0, \partial B_1)\}, \operatorname{dist}((x_0,t_0), \partial Q_1)\}$ and $\mathfrak{Q}_p$ denotes the spaces of parabolic polynomials of degree at most 2. Moreover, we can define the following norm

$$\begin{aligned} \|u\|_{C^{2,1,\zeta}(Q_{r_0}(x_0,t_0))} &:= \|u\|_{L^\infty(Q_{r_0}(x_0,t_0))} + \|Du\|_{L^\infty(Q_{r_0}(x_0,t_0))} + \left\|\tfrac{\partial u}{\partial t}\right\|_{L^\infty(Q_{r_0}(x_0,t_0))} \\ &+ \|D^2u\|_{L^\infty(Q_{r_0}(x_0,t_0))} + [u]_{C^{2,1,\zeta}(Q_{r_0}(x_0,t_0))}. \end{aligned}$$

Similarly, we say $u \in \text{par} - C^{1,\zeta}(Q_{r_0}(x_0,t_0))$ if the semi-norm

$$[u]_{\text{par}-C^{1,\zeta}(Q_{r_0}(x_0,t_0))} := \sup_{0<r\leq r_0} \left(\inf_{\mathfrak{l} \in \mathcal{L}} \frac{\|u - \mathfrak{l}\|_{L^\infty(Q_r(x_0,t_0))}}{r\zeta(r)} \right)$$

is finite, where $\mathcal{L}$ denotes the space of affine functions. Therefore, we can define the corresponding norm as follows

$$\|u\|_{\mathrm{par}-C^{1,\varsigma}(Q_{r_0}(x_0,t_0))} := \|u\|_{L^\infty(Q_{r_0}(x_0,t_0))} + \|Du\|_{L^\infty(Q_{r_0}(x_0,t_0))} + [u]_{\mathrm{par}-C^{1,\varsigma}(Q_{r_0}(x_0,t_0))}$$

Modus operandi

We will conclude this part by explaining the ideas and mechanisms behind the proof of our main results (see Theorem 3.3 and Theorem 3.21). In fact, given a fully non-linear operator F (for simplicity's sake assume that F has constant coefficients and $F(0) = 0$), one can associate to it a family of scaling operators as follows

$$\mathcal{G}_\iota(M) := \frac{F(\iota M)}{\iota}, \quad \iota > 0.$$

Notice that $(\mathcal{G}_\iota)_{\iota>0}$ defines a continuous "curve" of operators preserving the ellipticity parameters $\Lambda \geq \lambda > 0$. Now, should F be differentiable at the origin, one verifies that

$$\mathcal{G}_\iota(M) \longrightarrow \frac{\partial F}{M_{ij}}(0) \cdot M_{ij}, \quad as \quad \iota \longrightarrow 0^+. \tag{3.11}$$

Hence, the (linear) second order operator $\mathfrak{L}_F[M] := \frac{\partial F}{M_{ij}}(0) \cdot M_{ij}$ is the limiting (or tangential) equation of $\mathcal{G}_\iota$ as $\iota \longrightarrow 0^+$. On the other hand, if u is a solution to an equation related to F, then $v := \frac{u}{\iota}$ is a solution to a similar equation related to $\mathcal{G}_\iota$, namely

$$\frac{\partial v}{\partial t} - \mathcal{G}_\iota(D^2 v) = g_\iota(x,t) := \frac{1}{\iota} f(x,t). \tag{3.12}$$

In other words, if the norm of u is controlled by ι (the flatness degree) and $|g_\iota| = \mathrm{o}(1)$ as $\iota \ll 1$, then v is a normalized solution to the ι-correspondent equation (3.12). For this reason, we are able to access the available regularity for the (linear) limiting profile via the geometric tangential device from (3.11) combined with standard compactness and stability methods (cf. CRANDALL, KOCAN, and ŚWIECH (2000), DA SILVA and TEIXEIRA (2017) and WANG (1992a,b)). Furthermore, we can recover such fine estimates back to u, via the geometric tangential path used to access the limiting regularity profile. Therefore, we shall interpret the heat equation as the *geometric tangential equation* of the limit formed by the "curve" of fully nonlinear operators $\mathcal{G}_\iota$, providing that the flatness degree of u and data be vanishing (See DOS PRAZERES and TEIXEIRA (2016) for a similar reasoning and insights). Finally, by using such a tangential tool we demonstrate that the graph of a solution to (3.1) can be approximated in Q_σ (for a certain $\sigma \ll 1$) by an appropriated F-caloric quadratic polynomial, whose error is of the order $\sim \mathrm{O}\left(\iota\sigma^2\varphi(\sigma)\right)$, for an specific modulus of continuity φ. Such an estimate will provide that flat solutions are classical with suitable moduli of continuity for highest derivatives.

3.2 Geometric tangential approach

In this section we shall obtain regularity estimates for flat parabolic solutions to the problem (3.1), which will provide us that viscosity flat solutions are classical solutions. Moreover, we will specify the modulus of continuity of $\frac{\partial u}{\partial t}$ and D^2u in accordance to Dini-continuity of source f. In other words, if f satisfies the Dini continuity assumption then $\frac{\partial u}{\partial t}, D^2u \in C^{0,\psi(s)}_{loc}$, where

$$\psi(s) = \int_0^s \frac{\tau(r)}{r}dr + s\int_s^1 \frac{\tau(r)}{r^2}dr. \tag{3.13}$$

More precisely, we will prove in this chapter the following main result:

Theorem 3.3 ($C^{2,1,\psi}_{\mathbf{loc}}$ **parabolic estimates).** *Let $u \in C^0(Q_1)$ be a bounded viscosity solution to* (3.1) *such that* $(A1)-(A5)$ *hold. There exists a* $\overline{\delta} > 0$ *depending only upon* $n, \lambda, \Lambda, \omega, \tau$*, such that if*

$$\sup_{Q_1} |u| \leq \overline{\delta}$$

then $u \in C^{2,1,\psi}(Q_{1/2})$ *and*

$$\|u\|_{C^{2,1,\psi}(Q_{1/2})} \leq C(n, \lambda, \Lambda, \omega) \cdot \overline{\delta}.$$

We obtain such an estimate by adapting some techniques of geometric tangential methods from DOS PRAZERES and TEIXEIRA (2016). The following Lemma is key in the proof of Theorem3.3.

Lemma 3.4 (Caloric Approximation Lemma). *Let F be satisfying* (**A1**), (**A2**), (**A5**) *and a modulus of continuity* $\hat{\varphi}$ *such that*

$$\frac{\hat{\varphi}(s)}{s} \longrightarrow +\infty \ \textit{as} \ s \longrightarrow 0^+. \tag{3.14}$$

There exists $\eta > 0$ *depending upon* $n, \lambda, \Lambda, \omega$ *and* $\hat{\varphi}$*, such that for* $\mu > 0$ *if* u *is a viscosity solution to*

$$\frac{\partial u}{\partial t} - \mu^{-1}F(x,t,\mu D^2u) = f(x,t) \quad \textit{in} \quad Q_1 \quad \textit{with} \quad \|u\|_{L^\infty(Q_1)} \leq 1$$

and

$$\max\left\{\mu,\ \ \Theta_F(x,t),\ \ \|f\|_{L^\infty(Q_1)}\right\} \leq \eta$$

then we can find a $0 < \sigma < \frac{1}{2}$*, depending only on* n, λ, Λ *and* $\hat{\varphi}$*, and, a quadratic polynomial* $\mathfrak{p}$ *such that*

$$\begin{cases} \frac{\partial \mathfrak{p}}{\partial t} - \mu^{-1} F(0,0,\mu D^2 \mathfrak{p}) & = & 0 \quad in \quad Q_{\frac{1}{2}} \\ \|\mathfrak{p}\|_{L^\infty\left(Q_{\frac{1}{2}}\right)} & \leq & \mathfrak{C}(n,\lambda,\Lambda) \\ \sup_{Q_\sigma} |u - \mathfrak{p}| & \leq & \sigma^2 \hat{\varphi}(\sigma). \end{cases} \tag{3.15}$$

Demonstração. Let us suppose, for the sake of contradiction that the thesis of the Lemma fails. Then for a $0 < \sigma_0 < 1$ to be chosen *a posteriori* we can find sequences $(u_k)_{k\geq 1} \subset C^0(Q_1)$, $(f_k)_{k\geq 1} \subset L^\infty(Q_1)$, $(\mu_k > 0)_{k\geq 1}$ and $(F_k)_{k\geq 1}$ satisfying (**A1**), (**A2**), (**A5**) and linked through

$$\frac{\partial u_k}{\partial t} - \mu_k^{-1} F_k(x,t,\mu_k D^2 u_k) = f_k(x,t) \tag{3.16}$$

in the viscosity sense such that

$$\|u_k\|_{L^\infty(Q_1)} \leq 1 \quad \text{and} \quad \max\left\{\mu_k, \ \ \Theta_{F_k}(x,t), \ \ \|f_k\|_{L^\infty(Q_1)}\right\} \leq \frac{1}{k}. \tag{3.17}$$

However

$$\sup_{Q_{\sigma_0}} |u_k - \mathfrak{p}| > \sigma_0^2 \hat{\varphi}(\sigma_0) \tag{3.18}$$

for all $\mathfrak{p}$ that satisfies in the viscosity sense

$$\frac{\partial \mathfrak{p}}{\partial t} - \mu^{-1} F_k(0,0,\mu D^2 \mathfrak{p}) = 0 \quad \text{in} \quad Q_{\frac{1}{2}}$$

By (**A1**) passing to a subsequence if necessary we have that $F_k(x,t,M) \longrightarrow F_0(x,t,M)$ locally uniform on $\mathrm{Sym}(n)$ for all $(x,t) \in Q_1$ fixed. Moreover, by $C^{\gamma,\frac{\gamma}{2}}$ estimates for equation (3.16) (cf. (CRANDALL, KOCAN, and ŚWIECH, 2000, Section 5) and (WANG, 1992b, Section 4.4)) we have that

$$u_k \longrightarrow u_0 \tag{3.19}$$

locally uniform in Q_1. From assumptions (**A2**), (**A5**) and (3.17) we deduce that

$$\mu_k^{-1} F_k(x,t,\mu_k M) \longrightarrow D_M F_0(0,0,0) M \tag{3.20}$$

locally uniform on $Sym(n)$ for all $(x,t) \in Q_1$ fixed. Indeed, we have

$$F_k(x,t,\mu_k M) = \|\mu_k M\| \frac{F_k(x,t,\mu_k M) - F_k(0,0,\mu_k M)}{\|\mu_k M\|} + \frac{d}{ds}\int_0^{\mu_k} F_k(0,0,sM)ds.$$

Hence

$$F_k(x,t,\mu_k M) = \|\mu_k M\| \frac{F_k(x,t,\mu_k M) - F_k(0,0,\mu_k M)}{\|\mu_k M\|} + \int_0^{\mu_k} D_M F_k(0,0,sM) M ds.$$

Thus by using (**A2**) we have that

$$F_k(x,t,\mu_k M) \geq -\|\mu_k M\| \frac{F_k(x,t,\mu_k M) - F_k(0,0,\mu_k M)}{\|\mu_k M\|} + \mu_k D_M F_k(0,0,0) M - \mu_k \omega(\|\mu_k M\|)$$

Finally, by dividing by μ_k we obtain when $\mu_k \to 0$

$$D_M F_0(0,0,0).M \leq \lim_{k\to\infty} \mu_k^{-1} F_k(x,t,\mu_k M)$$

Analogously, we can obtain that

$$D_M F_0(0,0,0).M \geq \lim_{k\to\infty} \mu_k^{-1} F_k(x,t,\mu_k M)$$

and thus follows the statement. Therefore, from (3.17), (3.19) and (3.20), by using stability results (i.e., continuity with respect to equation), see (CRANDALL, KOCAN, and ŚWIECH, 2000, Section 6) and (WANG, 1992b, Lemma 1.4), we have

$$\frac{\partial u_0}{\partial t} - D_M F_0(0,0,0) D^2 u_0 = 0 \quad \text{in} \quad Q_{\frac{1}{2}}. \tag{3.21}$$

Since u_0 is a solution of a linear parabolic equation with constants coefficients, it is smooth. We define now

$$\mathfrak{p}(x,t) := u_0(0,0) + Du_0(0,0).x + \frac{\partial u_0}{\partial t}(0,0).t + \frac{1}{2} x^T \cdot D^2 u_0(0,0) \cdot x.$$

As $\|u_0\|_{L^\infty} \leq 1$, it follows from parabolic estimates available on u_0 (cf.(KRYLOV, 2008, Theorem 8.4.4)) that

$$\sup_{Q_r} |u_0(x,t) - \mathfrak{p}(x,t)| \leq \mathfrak{C}.r^3 \quad \text{for all } 0 < r < \frac{1}{3} \tag{3.22}$$

where $\mathfrak{C} > 0$ is a universal constant. Now, from (3.14) $\frac{\hat{\varphi}(s)}{s} \to +\infty$ as $s \to 0^+$, then we can select $0 < \sigma_0 \ll \frac{1}{2}$ such that $\sigma_0 \leq \frac{\hat{\varphi}(\sigma_0)}{7\mathfrak{C}}$. Thus, we readily have

$$\sup_{Q_{\sigma_0}} |u_0 - \mathfrak{p}| \leq \frac{1}{7} \sigma_0^2 \hat{\varphi}(\sigma_0). \tag{3.23}$$

Also from equation (3.21)

$$\frac{\partial \mathfrak{p}}{\partial t} - D_M F_0(0,0,0) D^2 \mathfrak{p} = 0 \quad \text{in} \quad Q_{\frac{1}{2}}$$

which implies that
$$\left|\frac{\partial \mathfrak{p}}{\partial t} - \mu_k^{-1} F_k(0,0,\mu_k D^2 \mathfrak{p})\right| = o(1).$$

Now, by doing $a_k := \frac{\partial \mathfrak{p}}{\partial t} - \mu_k^{-1} F_k(0,0,\mu_k D^2\mathfrak{p})$ we have that $|a_k| = o(1)$ and the parabolic quadratic polynomial

$$\mathfrak{p}_k(x,t) := \mathfrak{p}(x,t) - a_k.t \quad \Longrightarrow \quad \sup_{Q_{\sigma_0}} |\mathfrak{p}_k - \mathfrak{p}| \le \frac{1}{10}\sigma_0^2 \hat{\varphi}(\sigma_0) \text{ for } k \gg 1 \tag{3.24}$$

satisfies in the viscosity sense $\frac{\partial \mathfrak{p}_k}{\partial t} - \mu_k^{-1} F(0,0,\mu_k D^2 \mathfrak{p}_k) = 0 \quad \text{in} \quad Q_{\frac{1}{2}}$. Therefore, in Q_{σ_0} by using the previous sentences (3.19), (3.23) and (3.24) we have for k large enough

$$\begin{aligned} \sup_{Q_{\sigma_0}} |u_k - \mathfrak{p}_k| &\le \sup_{Q_{\sigma_0}} |u_k - u_0| + \sup_{Q_{\sigma_0}} |u_0 - \mathfrak{p}| + \sup_{Q_{\sigma_0}} |\mathfrak{p} - \mathfrak{p}_k| \\ &\le \tfrac{1}{5}\sigma_0^2\hat{\varphi}(\sigma_0) + \tfrac{1}{7}\sigma_0^2\hat{\varphi}(\sigma_0) + \tfrac{1}{10}\sigma_0^2\hat{\varphi}(\sigma_0) \\ &\le \sigma_0^2 \bar{\varphi}(\sigma_0). \end{aligned}$$

which contradicts (3.18). □

Our next step is to transport the information obtained in the geometric tangential equation for (3.1) through a universal smallness condition in the L^∞ of the solution.

Lemma 3.5. *Let F be satisfy* (**A1**), (**A2**) *and* (**A5**), *and a modulus of continuity $\hat{\varphi}$ such that*
$$\frac{\hat{\varphi}(s)}{s} \longrightarrow +\infty \text{ as } s \longrightarrow 0^+.$$
There exists a small constant $\delta > 0$ depending only on $n, \lambda, \Lambda, \omega, \hat{\varphi}$ and a constant $0 < \sigma < 1$ depending only on $n, \lambda, \Lambda, \hat{\varphi}$ such that if u is a solution to (3.1) *and*

$$\|u\|_{L^\infty(Q_1)} \le \delta \quad \text{and} \quad \max\left\{\Theta_F(x,t), \quad \|f\|_{L^\infty(Q_1)}\right\} \le \sqrt[6]{\delta^7}, \tag{3.25}$$

then one can find a parabolic quadratic polynomial $\mathfrak{p}$ satisfying

$$\frac{\partial \mathfrak{p}}{\partial t} - F(0,0,D^2\mathfrak{p}) = 0 \quad \text{and} \quad \|\mathfrak{p}\|_{L^\infty(Q_1)} \le \delta \mathfrak{C}(n,\lambda,\Lambda) \tag{3.26}$$

for a universal constant $\mathfrak{C} > 0$, and, δ as in the Lemma 3.5 such that,

$$\sup_{Q_\sigma} |u - \mathfrak{p}| \le \delta.\sigma^2 \hat{\varphi}(\sigma). \tag{3.27}$$

Demonstração. Let us define the normalized function $v(x,t) := \delta^{-1}u(x,t)$, which satisfies

$$\frac{\partial v}{\partial t} - \delta^{-1}F(x,t,\delta D^2 v) = \delta^{-1} f(x,t).$$

in the viscosity sense. Therefore, if η is as in Lemma 3.4, then by selecting $\eta = \sqrt[6]{\delta}$ the Lemma holds. $\square$

3.3 $C^{2,1,\psi}$ parabolic estimates in Dini continuous media

In this Section we will obtain the aimed parabolic $C^{2,1,\psi}(Q_{1/2})$ estimate by iterating, through an inductive process, Lemma 3.5.

Before starting the proof of the iterative Lemma, let us enunciate some important remarks.

Remark 3.6. We must note that τ increasing and by the definition of φ we have

$$\tau(s) \leq \left(\frac{\sqrt{5}}{\sqrt{5}-s}\right)\varphi(s) \leq 2\varphi(s) \quad \forall\ s < \frac{\sqrt{5}}{2}.$$

We refer Kovats KOVATS (1999) for such a property.

Remark 3.7. Let φ be as before, then for any $0 < \sigma \leq e^{-1}$ there holds that

$$\sum_{j=k}^{\infty} \varphi(\sigma^j) \leq \mathfrak{C}_n \psi(\sigma^k). \tag{3.28}$$

In fact, by using integral test, definition of ψ and integration by parts we obtain

$$\sum_{j=k}^{\infty} \varphi(\sigma^j) \leq \varphi(\sigma^k) + \int_1^{\infty} \varphi(\sigma^{y+k-1})dy = \varphi(\sigma^k) + (\log \sigma^{-1})^{-1}.\int_0^{\sigma^k} \frac{\varphi(s)}{s}ds \leq \mathfrak{C}_n\psi(\sigma^k).$$

Remark 3.8. Due to uniform continuity of τ we may define

$$\chi^{\sharp} := \sup\left\{\varsigma \in (0,1] \;\middle|\; \tau(\varsigma \mathrm{dist}((x,t),(0,0))) \leq \frac{\sqrt[6]{\delta^7}}{2(1+\delta\mathfrak{C}_n\mathfrak{C}\psi(1))}\tau(\mathrm{dist}((x,t),(0,0)))\right\},$$

where $\mathfrak{C}_n > 0$ and $\mathfrak{C} > 0$ come from (3.26) and (3.28) respectively. Furthermore, we may suppose, without loss of generality,

$$\left\{\begin{array}{rcl} \tau(\sqrt{5}) & \leq & 2, \\ \Theta_F(x,t) & \leq & \frac{\sqrt[6]{\delta^7}}{2(1+\delta\mathfrak{C}_n\mathfrak{C}\psi(1))}\tau(\mathrm{dist}((x,t),(0,0))), \ , \\ \max\left\{\|f\|_{L^\infty(Q_1)}, [f]_{C^{0,\tau}(Q_1)}\right\} & \leq & \frac{\sqrt[6]{\delta^7}}{2}. \end{array}\right. \tag{3.29}$$

where

$$[f]_{C^{0,\tau}(Q_1)} := \sup_{\substack{(x,t),(y,s)\in Q_1 \\ (x,t)\neq(y,s)}} \frac{|f(x,t)-f(y,s)|}{\tau(d((x,t),(y,s)))}.$$

Indeed, the following reduction

$$v_\kappa(x,t) := \frac{u(\kappa x, \kappa^2 t)}{\mathfrak{K}\kappa^2}$$

with

$$\kappa := \min\left\{1, \frac{\tau^{-1}(2\mathfrak{K})}{\sqrt{5}}, \chi^\sharp\right\} \text{ and } \mathfrak{K} := \max\left\{1, \|u\|_{L^\infty(Q_1)} + \frac{2}{\sqrt[6]{\delta^7}} \max\left\{\|f\|_{L^\infty(Q_1)}, [f]_{C^{0,\tau}(Q_1)}\right\}\right\},$$

it satisfies

$$\frac{\partial v_\kappa}{\partial t} - F_\kappa(x,t,D^2 v) = f_\kappa(x,t),$$

for

$$\begin{cases} F_\kappa(x,t,M) & := & \mathfrak{K}^{-1}F(\kappa x, \kappa^2 t, \mathfrak{K}M) \\ f_\kappa(x,t) & := & \mathfrak{K}^{-1} f(\kappa x, \kappa^2 t) \\ \tau_\kappa(s) & := & \mathfrak{K}^{-1}\tau(\kappa s) \\ \varphi_\kappa(s) & := & s\displaystyle\int_s^1 \frac{\tau_\kappa(r)}{r^2} dr. \end{cases}$$

attending the required hypothesis (3.29), as well as the assumptions **(A1)-(A5)**. Finally, notice that the sentence (3.29) particularly implies that

$$\max\left\{\Theta_F(x,t), \|f\|_{L^\infty(Q_1)}\right\} \leq \sqrt[6]{\delta^7}, \tag{3.30}$$

in other words, the data are in the smallest regime from Lemma 3.5.

Lemma 3.9. *Let F and f be satisfying* (**A1**) $-$ (**A5**). *Then there exists a $\delta = \delta(n, \lambda, \Lambda, \omega, \tau) > 0$, such that if $\sup\limits_{Q_1} |u| \leq \delta$ then $u \in C^{2,1,\psi}$ at the origin, i.e.,*

$$\sup_{Q_r}\left|u(x,t) - \left[u(0,0) + Du(0,0)\cdot x + \frac{\partial u}{\partial t}(0,0).t + \frac{1}{2}x^T \cdot D^2u(0,0)\cdot x\right]\right| \leq \mathfrak{C}.\delta r^2 \psi(r),$$

where $\mathfrak{C} > 0$ depends only upon n, λ, Λ and ω.

Demonstração. We claim that there exists a sequence of parabolic quadratic polynomials

$$\mathfrak{p}_k(x,t) = \frac{1}{2}x^T \cdot \mathfrak{a}_k \cdot x + \mathfrak{b}_k.t + \mathfrak{c}_k \cdot x + \mathfrak{d}_k \quad \text{with} \quad \frac{\partial \mathfrak{p}_k}{\partial t} - F(0,0,D^2\mathfrak{p}_k) = 0, \tag{3.31}$$

that approximates u in the $C^{2,1,\varphi}$ sense, i.e.,

$$\sup_{Q_{\sigma^k}} |u(x,t) - \mathfrak{p}_k(x,t)| \leq \delta\sigma^{2k}\varphi(\sigma^k), \tag{3.32}$$

where $0 < \sigma \le \min\left\{\sigma^\sharp, \frac{1}{e}\right\}$. Furthermore, the oscillation of the coefficients of $\mathfrak{p}_k$ verifies

$$\left\{\begin{array}{rcl} |\mathfrak{a}_k - \mathfrak{a}_{k-1}| & \le & \mathfrak{C}\delta\varphi(\sigma^{(k-1)}) \\ |\mathfrak{b}_k - \mathfrak{b}_{k-1}| & \le & \mathfrak{C}\delta\varphi(\sigma^{(k-1)}) \\ |\mathfrak{c}_k - \mathfrak{c}_{k-1}| & \le & \mathfrak{C}\delta\sigma^{(k-1)}\varphi(\sigma^{(k-1)}) \\ |\mathfrak{d}_k - \mathfrak{d}_{k-1}| & \le & \mathfrak{C}\delta\sigma^{2(k-1)}\varphi(\sigma^{(k-1)}) \end{array}\right. \tag{3.33}$$

where $\mathfrak{C} > 0$ is universal and σ and δ are the parameters from Lemma 3.5. We will prove this by induction process. The case $k = 1$ is precisely the statement of the Lemma 3.5 when we take $\hat{\varphi} = \varphi$. Let us suppose that the statement is true for k, then we define

$$v_k(x,t) := \frac{(u - \mathfrak{p}_k)(\sigma^k x, \sigma^{2k} t)}{\sigma^{2k}\varphi(\sigma^k)} \tag{3.34}$$

and we observe that v_k solves

$$\frac{\partial v_k}{\partial t} - \mathcal{G}_k(x,t,D^2 v_k) = \frac{f(\sigma^k x, \sigma^{2k} t)}{\varphi(\sigma^k)} := f_k(x,t),$$

where

$$\mathcal{G}_k(x,t,M) := \frac{F(\sigma^k x, \sigma^{2k} t, \varphi(\sigma^k).M + \mathfrak{a}_k) - \mathfrak{b}_k}{\varphi(\sigma^k)}. \tag{3.35}$$

Due to induction hypothesis and smallness condition on $\tau(\sqrt{5})$, we obtain

$$\|v_k\|_{L^\infty(Q_1)} \le \delta \quad \text{and} \quad \max\{\Theta_{\mathcal{G}_k}(x,t), \quad \|f_k\|_{L^\infty(Q_1)}\} \le \sqrt[6]{\delta^7}, \tag{3.36}$$

and since $\mathcal{G}_k$ fulfil $(\mathbf{A1}), (\mathbf{A2})$ and $(\mathbf{A5})$, we can use Lemma 3.5, with $\hat{\varphi}(t) = t^{\alpha^\sharp}$, to find a quadratic polynomial $\tilde{\mathfrak{p}}$ with universal bounded coefficients such that

$$\sup_{Q_\sigma} |v_k - \tilde{\mathfrak{p}}| \le \delta.\sigma^{2+\alpha^\sharp}.$$

In the sequel we define

$$\mathfrak{p}_{k+1}(x,t) := \mathfrak{p}_k(x,t) + \sigma^{2k}\varphi(\sigma^k)\tilde{\mathfrak{p}}_k\left(\frac{x}{\sigma^k}, \frac{t}{\sigma^{2k}}\right),$$

then using the definition of v_k and a change of variables we obtain

$$\sup_{Q_{\sigma^{k+1}}} |u - \mathfrak{p}_{k+1}| \le \delta.\sigma^{2(k+1)}.\sigma^{\alpha^\sharp}.\varphi(\sigma^k),$$

so by condition (**A4**) (the first sentence from (3.8)) we have

$$\sup_{Q_{\sigma^{k+1}}} |u - \mathfrak{p}_{k+1}| \leq \delta.\sigma^{2(k+1)}.\varphi(\sigma^{k+1}),$$

and this completes the induction process.

In order to finish the proof, we observe that (3.33) implies that $(\mathfrak{a}_k)_{k\geq 1} \subset \text{Sym}(n)$,$(\mathfrak{b}_k)_{k\geq 1} \subset \mathbb{R}$, $(\mathfrak{c}_k)_{k\geq 1} \subset \mathbb{R}^n$ and $(\mathfrak{d}_k)_{k\geq 1} \subset \mathbb{R}$ are Cauchy sequences. Hence, we can define the limiting quadratic polynomial

$$\mathfrak{p}_\infty(x,t) := \frac{1}{2}x^T \cdot \mathfrak{a}_\infty \cdot x + \mathfrak{b}_\infty.t + \mathfrak{c}_\infty \cdot x + \mathfrak{d}_\infty,$$

where $\mathfrak{a}_k \longrightarrow \mathfrak{a}_\infty$, $\mathfrak{b}_k \longrightarrow \mathfrak{b}_\infty$, $\mathfrak{c}_k \longrightarrow \mathfrak{c}_\infty$ and $\mathfrak{d}_k \longrightarrow \mathfrak{d}_\infty$. Moreover we have, by 3.28, that

$$\begin{cases} |\mathfrak{a}_k - \mathfrak{a}_\infty| & \leq & \mathfrak{C}\delta\psi(\sigma^k) \\ |\mathfrak{b}_k - \mathfrak{b}_\infty| & \leq & \mathfrak{C}\delta\psi(\sigma^k) \\ |\mathfrak{c}_k - \mathfrak{c}_\infty| & \leq & \mathfrak{C}\delta\sigma^k\psi(\sigma^k) \\ |\mathfrak{d}_k - \mathfrak{d}_\infty| & \leq & \mathfrak{C}\delta\sigma^{2k}\psi(\sigma^k) \end{cases} \tag{3.37}$$

Therefore,

$$|\mathfrak{p}_k(x,t) - \mathfrak{p}_\infty(x,t)| \leq \mathfrak{C}\delta\sigma^{2k}\psi(\sigma^k) \tag{3.38}$$

Finally, fixed $0 < r \ll \frac{1}{2}$, we take $k \in \mathbb{N}$ such that $\sigma^{k+1} < r \leq \sigma^k$ and conclude using (3.32) and (3.38), that

$$\sup_{Q_r} |u(x,t) - \mathfrak{p}_\infty(x,t)| \leq \mathfrak{C}\delta\sigma^{2k}\psi(\sigma^k) \leq \mathfrak{C}_\sharp\delta r^2\psi(r),$$

where $\mathfrak{C}_\sharp = \mathfrak{C}_\sharp(n, \lambda, \Lambda, \tau) > 0$. This finishes the proof of the Lemma. □

Proof of the Theorem 1: Invoking the previous Lemma and using a standard covering argument we have that $u \in C^{2,1,\psi}\left(Q_{\frac{1}{2}}\right)$ with the correspondent estimate

$$\|u\|_{C^{2,1,\psi}\left(Q_{\frac{1}{2}}\right)} \leq \mathfrak{C}(n, \lambda, \Lambda, \tau).\delta.$$

□

Corollary 3.10 (**Standard Schauder type estimate**). *Let u be a flat viscosity solution to* (3.1). *Since for $\alpha \in (0,1)$ the assumptions* (**A1**) − (**A5**) *are in force for $\tau(s) = \mathfrak{c}_0.s^\alpha$, then $u \in C^{2+\alpha,\frac{2+\alpha}{2}}\left(Q_{\frac{1}{2}}\right)$.*

In the previous Corollary 3.10 we must do a special remark for the case $\alpha = 0$. In this case, flat solutions are asymptotically $C^{2,\frac{2}{1}}_{\text{loc}}(Q_1)$. In effect, such solutions are locally $C^{1,\text{Log-Lip}}$ in the parabolic sense, i.e., u has $\zeta(r) = r^2 \log r^{-1}$ as modulus of continuity.

Notice that, particularly, such a Corollary is an improved version when compared with its corresponding Schauder type estimates from ZOU and CHEN (2002) and WANG (1992b).

We should point out that our approach of flatness regime explores just the fact that viscosity solutions to (3.1) have oscillation small enough. For this very reason, we can improve our arguments as to prove similar results under the following statement:

Corollary 3.11 (**ϕ-Small oscillation**). *Let u be a bounded viscosity solution to* (3.1)*. Suppose we are under the hypotheses of Lemma 3.5. Then given ϕ a smooth solution to*

$$\frac{\partial \phi}{\partial t} - F(x,t,D^2\phi) = 0 \quad in \quad Q_1$$

there exists a $\delta = \delta(n,\lambda,\Lambda,\omega,\tau) > 0$ such that if u is contained in the δ-tubular neighbourhood of ϕ, i.e.,

$$\sup_{Q_1} |u - \phi| \leq \delta$$

then $u \in C^{2,1,\psi}\left(Q_{\frac{1}{2}}\right)$. Furthermore,

$$\|u - \phi\|_{C^{2,1,\psi}\left(Q_{\frac{1}{2}}\right)} \leq \mathfrak{C}.\delta$$

for a $\mathfrak{C} > 0$ depending only on universal parameters, ω, τ and $\|\phi\|_{C^{2+1,1+\frac{1}{2}}(Q_1)}$.

Demonstração. The result follows by applying Theorem 3.3 to $v = u - \phi$ with the operator

$$\mathcal{G}(x,t,M) := F(x,t,M + D^2\phi) - F(0,0,D^2\phi).$$

□

Remark 3.12 (**General dependence**). We can assure that similar results follow to equations of following form

$$\frac{\partial u}{\partial t} - F(x,t,u,Du,D^2u) = \mathfrak{f}(x,t,u,Du) \quad \text{in} \quad Q_1 \tag{3.39}$$

under suitable assumption on u and Du dependence. Indeed, Lipschitz continuity is enough to our purposes, see CRANDALL, KOCAN, and ŚWIECH (2000), DA SILVA and TEIXEIRA (2017) and WANG (1992a).

Remark 3.13. It is worth point out that Savin's seminal elliptic paper SAVIN (2007) (resp. Wang's parabolic work WANG (2013)), F is only assumed to be uniform elliptic in a neighbourhood of the origin. The same observation holds in our case. Indeed, by revisiting the proof of Lemma 3.4 (resp. Lemma 3.5) we could assume only uniform ellipticity of F close to the origin and the same conclusion holds true. The key observation is that in the Caloric Approximation Lemma 3.4, temporal, first and second spatial derivatives of approximating sequence remain trapped within a certain $\delta-$neighborhood of the origin, which enable us to apply Krylov-Safonov regularity estimates, as well as stability results. Thus, for the purposes of our reasonings, it is like we were treating (everywhere) uniform elliptic case.

3.4 Consequences and application

Throughout this section we will show some consequences and applications to Theorem 3.3.

3.4.1 $C^{2,1}$ implies $C^{2,1,\psi}$

Given a $u \in C^{2,1}$ classical solution to (3.1), Theorem 3.3 allows us determinate the modulus of continuity to D^2u and $\frac{\partial u}{\partial t}$. This result is a sort of extended version of Evans-Krylov Theorem. Over recent years this type of results has been used to derive higher regularity in geometric analysis problems, see SHENG and WANG (2010) and TIAN and WANG (2013) for some enlightening examples.

Theorem 3.14 (Evans-Krylov type Theorem). *Let* $u \in C^{2,1}(Q_1)$ *be a classical solution to* (3.1). *Assume assumptions* (**A1**) – (**A5**) *are in force. Then,* $u \in C^{2,1,\psi}\left(Q_{\frac{1}{2}}\right)$, *and*

$$\|u\|_{C^{2,1,\psi}\left(Q_{\frac{1}{2}}\right)} \leq \mathfrak{C}(n, \lambda, \Lambda, \omega, \tau(\sqrt{5}), \|u\|_{C^{2,1}(Q_1)}).$$

Demonstração. Define for a $0 < \mu \leq 1$ to be chosen *a posteriori* $v : Q_1 \to \mathbb{R}$ by

$$v(x,t) := \frac{u(\mu x, \mu^2 t) - \left[u(0,0) + \mu Du(0,0)\cdot x + \mu^2 2^{-1} x^T \cdot D^2u(0,0)\cdot x + \mu^2 t . \frac{\partial u}{\partial t}(0,0)\right]}{\mu^2}$$

We then have

$$v(0,0) = \|Dv(0,0)\| = 0 \quad \text{and} \quad \left|\frac{\partial v}{\partial t}(x,t)\right|, |D^2v(x,t)| \leq \vartheta(\mu) := \max\{\zeta(\mu\sqrt{5}), \varsigma(\mu\sqrt{5})\},$$

where ζ and ς are the modulus of continuity for $\frac{\partial u}{\partial t}$ and D^2u respectively. We now can choose $\mu \ll 1$ small enough of the following form

$$\mu := \min\left\{1, \frac{\vartheta^{-1}(\delta)}{\sqrt{5}}\right\}$$

where $\delta \ll 1$ is the constant from Theorem 3.3. Upon such a choice, v is under the assumptions of Theorem 3.3 with

$$\mathcal{G}(x,t,M) := F(sx, s^2t, M + D^2u(0,0)) - \frac{\partial u}{\partial t}(0,0) \quad \text{and} \quad g(x,t) := f(sx, s^2t).$$

Therefore, $v \in C^{2,1,\psi}\left(Q_{\frac{1}{2}}\right)$ and consequently $u \in C^{2,1,\psi}\left(Q_{\frac{\mu}{2}}\right)$.

□

Regularity in some problems from Geometry

Over the last decades the study of geometric flows have proved to be extremely effective in solving some of the most important problems in Topology, Differential Geometry and Geometric Analysis. Geometric considerations drive to equations of the form:

$$\frac{\partial u}{\partial t} - F(x,t,u,Du,D^2u) = \mathfrak{f}\left(x,t,u,\int_{B_1} \mathcal{G}(Du, D^2u)dx\right) \quad \text{in} \quad \mathfrak{M} \subset \mathbb{R}^{n+1}, \tag{3.40}$$

where $\mathcal{G} \in C^\infty(\mathbb{R}^n \times Sym(n), \mathbb{R}^m)$ is a vector field. Such an equations appear in many applications of parabolic PDE in curvature and gradient flows. For this very reason, our work have been motivated by studying such equations coming from Differential Geometry and Geometric Analysis in order to establish high order estimates to certain solutions.

Higher regularity results by estimating the gradient of the mean curvature flow under volume constraint assumption, as well as gradient flow associated with the k-Hessian equations have been obtained in the context of Geometric Analysis, see HuiskenHUISKEN (1987) and Tian and WangTIAN and WANG (2013) for more detail about this topics.

Next, we comment on interior regularity results for general non-linear curvature and gradient flows (3.40), so yielding an interesting application in the geometric setting. We consider $\mathfrak{M}$ to be a closed manifold without boundary under volume constraint assumption, thus interior regularity is sufficient. Therefore, a result as Theorem 3.3 can be proved for flat solutions to equations of form (3.40).

In conclusion, we stress that similar results can be obtained considering $C^{2,1}$ solution to (3.40) on a hyper-surfaces without boundary instead of flat solutions on a manifold. Thus, we are able to apply Theorem 3.14 and prove the corresponding desired estimate. Finally, the result proven in this part can be further applied to equations of the form

$$\frac{\partial \mathfrak{H}}{\partial t} - F(D\mathfrak{H}, D^2\mathfrak{H}) - \mathfrak{H}|A|^2 = 0,$$

where $\mathfrak{H}$ is the inwards mean curvature vector of the surface at position x and time t and $|A|$ represents the norm of the second fundamental form. This equation describes the mean curvature hypersurface in the Euclidean space $\mathbb{R}^{n+1}$, see for example SHENG and WANG (2010).

In order to finish this part let us comment an application of our last result in the context of Differential Geometry (Elliptic setting): Let $u \in C^2(B_1) \cap C^{0,1}(\overline{B_1})$ be a strictly convex solution to

$$F(x, Du, D^2u) = \det(D^2u) - \mathcal{K}(x)\left(1 + |Du|^2\right)^{\frac{n+2}{2}} = 0,$$

where $\mathcal{K}$ is the Gauss curvature of the graph of u at the point $(x, u(x))$. Remember that such an equation is known as *Equation of Prescribed Gauss Curvature.* Finally, if $\mathcal{K}$ is Dini continuous, then according to Theorem 3.14 solutions have a universal modulus of continuity for second derivatives.

3.4.2 Partial regularity results for fully nonlinear parabolic equations

The result of the previous subsection 3.4.1 raises a question: considering viscosity solutions instead of classical solutions to (3.1), what can we say about the size of the eventual singular set? For the class of equations treated here, Theorem 3.3 allows us to answer the question with a partial regularity result.

Remark 3.15. It will be useful from now we introduce the notion of *parabolic Hausdorff dimension* for a set $\Omega \subseteq \mathbb{R}^{n+1}$:

$$\mathcal{H}_{par}(\Omega) := \inf\left\{0 \le s < \infty : \forall \gamma > 0 \; \exists \{Q_{r_j}(x_j, t_j)\}_{j\ge1} \; s.t. \; \Omega \subseteq \bigcup_{j\ge1} Q_{r_j}(x_j, t_j) \text{ and } \sum_{j\ge1} r_j^s < \gamma\right\}$$

Theorem 3.16 (**Parabolic partial regularity result**). *Let u be a viscosity solution to*

$$\frac{\partial u}{\partial t} - F(D^2u) = f(x,t) \quad in \quad Q_1,$$

where $F \in C^1(Sym(n))$ satisfies $\mathfrak{c} \le D_{u_iu_j}F(M) \le \mathfrak{c}^{-1}$ for some constant $\mathfrak{c} > 0$ and a source function f which is Lipschitz continuous. Then, there exist $\epsilon_0 > 0$, depending only on universal parameters, and, a closed set $\Gamma_{Sing} \subset Q_1$, with $\mathcal{H}_{par}(\Gamma_{Sing}) \le n + 2 - \epsilon_0$ such that $u \in C^{2+\alpha, \frac{2+\alpha}{2}}(Q_1 \setminus \Gamma_{Sing})$ for all $\alpha \in (0,1)$.

Remark 3.17. Let f be a Lipschitz continuous function. Then, for every unity vector $\nu \in \mathbb{R}^n$, the function $u_\nu = \nu \cdot Du$ fulfils

$$\frac{\partial u_\nu}{\partial t} - \mathcal{P}^+_{\lambda,\Lambda}(D^2u_\nu) - \mathfrak{L}_0 \le 0 \le \frac{\partial u_\nu}{\partial t} - \mathcal{P}^-_{\lambda,\Lambda}(D^2u_\nu) + \mathfrak{L}_0 \quad \text{in} \quad Q_1$$

in the viscosity sense, where $\mathfrak{L}_0 = \|f\|_{\text{Lip}(Q_1)}$. Thus, by Parabolic $W^{3,\varepsilon}$ estimate from (DANIEL, 2015, Theorem 1.2)

$$\mathcal{L}^{n+1}\left(\left\{(x,t) \in Q_{\frac{1}{2}}; \Xi\left(u, Q_{\frac{3}{4}}\right)(x,t) > \kappa\right\}\right) \le \mathfrak{C}\kappa^{-\varepsilon} \tag{3.41}$$

for a constant $\mathfrak{C} = \mathfrak{C}(n, \lambda, \Lambda, \mathfrak{L}_0, \|u\|_{L^\infty(Q_1)})$ and for all $\kappa > 0$, where

$$\begin{aligned}\Xi(u,\Omega)(x,t) &:= \inf\{\,\mathfrak{A}\geq 0; \exists (b,\overrightarrow{p},M)\in\mathbb{R}\times\mathbb{R}^n\times Sym(n)\ \ s.t.\ \forall (y,s)\in\Omega, s\leq t,\\ &\quad \left|u(y,s)-\left[u(x,t)+\overrightarrow{p}.(y-x)+b(s-t)+\tfrac{1}{2}(y-x)^T.M.(y-x)\right]\right|\\ &\leq \tfrac{1}{6}\mathfrak{A}d^3((x,t),(y,s))\}\end{aligned}$$

In order to prove the partial regularity result we must to use the Main Theorem for determine the relation between Ξ and the local $C^{2+\alpha,\frac{2+\alpha}{2}}$ estimate to solutions after using a covering argument. For this very reason we will proceed in two steps in order to drive the proof more clearly.

Lemma 3.18. *Let u be a viscosity solution to* (3.1) *satisfying $\|u\|_{L^\infty(Q_1)}\leq 1$ and $\alpha\in(0,1)$. There exist a constant $\delta^\sharp=\delta^\sharp(n,\lambda,\Lambda,\tau,\alpha)$ such that for every $(x_0,t_0)\in Q_{\frac{1}{2}}$ and $0<r<\frac{1}{100}$, if*

$$\left\{\Xi\left(u,Q_{\frac{3}{4}}\right)(x,t)\leq r^{-1}\delta^\sharp\right\}\cap Q_r(x_0,t_0)\neq\emptyset \quad then \quad u\in C^{2+\alpha,\frac{2+\alpha}{2}}$$

Demonstração. Let $\delta^\sharp$ to be chosen *a posteriori*, $0<r<\frac{1}{100}$, $(x_0,t_0)\in Q_{\frac{1}{2}}$ and $(z_0,s_0)\in Q_r(x_0,t_0)$ such that

$$\Xi\left(u,Q_{\frac{3}{4}}\right)(z_0,s_0)\leq r^{-1}\delta^\sharp$$

then there exist $b\in\mathbb{R}$, $\overrightarrow{p}\in\mathbb{R}^n$ and $M\in M_n(\mathbb{R})$ such that for any $(x,t)\in Q_{\frac{3}{4}}$ with $t\leq s_0$

$$\left|u(x,t)-\left[u(z_0,s_0)+\overrightarrow{p}.(x-z_0)+b(t-s_0)+\frac{1}{2}(x-z_0)^T.M.(x-z_0)\right]\right|\leq\frac{1}{6r}\delta^\sharp d^3((x,t),(z_0,s_0)). \qquad (3.42)$$

We may assume without loss of generality that $M\in Sym(n)$ since we can replace M by $\frac{M^T+M}{2}$. Moreover, for $(x,t)\in Q_1$ we have $(z_0+4rx, s_0+16r^2t)\in Q_{\frac{3}{4}}$. From now, consider the scaled function

$$v(x,t):=\frac{u(z_0+4rx,s_0+16r^2t)-[u(z_0,s_0)+4r\overrightarrow{p}\cdot x+8r^2x^T\cdot M\cdot x+16r^2bt]}{16r^2},$$

as well as the operator

$$\mathcal{G}(x,t,N):=F(z_0+4rx,s_0+16r^2t,N+M)-F(z_0,s_0,M).$$

According to sentence (3.42), for a universal constant $\mathfrak{c}_0>0$ we have that

$$\|v\|_{L^\infty(Q_1)}\leq\mathfrak{c}_0.\delta^\sharp.$$

Since u is a viscosity solution to (3.1) follows that $b-F(z_0,s_0,M)=f(z_0,s_0)$. Thus, v

satisfies in the viscosity sense

$$\frac{\partial v}{\partial t} - \mathcal{G}(x,t,D^2v) = f(z_0 + 4rx, s_0 + 16r^2t) - f(z_0, s_0) := g(x,t) \quad \text{in} \quad Q_1$$

Moreover, $\mathcal{G}$ and g fulfil the assumptions $(\mathbf{A1}) - (\mathbf{A5})$. From now, let $\delta > 0$ the constant from Theorem 3.3. If we do the choice

$$\delta^\sharp < \mathfrak{c}_0^{-1}.\delta$$

then by applying Theorem 3.3 to v we obtain $v \in C^{2+\alpha,\frac{2+\alpha}{2}}\left(Q_{\frac{1}{2}}\right)$ for all $0 < \alpha < 1$, which we conclude that $u \in C^{2+\alpha,\frac{2+\alpha}{2}}(Q_{2r}(z_0, s_0))$. The proof finishes by noting that $Q_r(x_0, t_0 - r^2) \subset Q_{2r}(z_0, s_0)$. □

Proof of Theorem 3.16: By using a standard covering argument, it is suffices to prove for $\alpha \in (0,1)$ fixed that

$$u \in C^{2+\alpha,\frac{2+\alpha}{2}}(\mathfrak{J} \setminus \Gamma_{\text{Sing}}) \text{ where } \mathfrak{J} := B_{\frac{7}{100}} \times \left(-\frac{1}{2}, -\frac{1}{20000}\right).$$

Now, let $\Gamma_{\text{Sing}} \subset \mathfrak{J}$ be defined by

$$\Gamma_{\text{Sing}} := \left\{(x,t) \in \mathfrak{J} : u \notin C^{2+\alpha,\frac{2+\alpha}{2}}\left(Q_r\left(x, t + \frac{r^2}{2}\right)\right) \ \forall r > 0\right\}.$$

Note that Γ_{Sing} is closed, consequently compact. For $0 < r < \frac{1}{100}$ fixed follows by a Vitali Covering Theorem for parabolic cylinders, see (LIEBERMAN, 1996, Lemma7.8) , there exists a finite family of disjoint parabolic cylinders of radius r centred in $(x_i, t_i) \in \Gamma_{\text{Sing}}$, $\left\{Q_r\left(x_i, t_i + \frac{r^2}{2}\right)\right\}$ such that

$$\Gamma_{\text{Sing}} \subseteq \bigcup_{i=1}^{d} Q_{5r}\left(x_i, t_i + \frac{25r^2}{2}\right).$$

By Lemma 3.18, since $(x_i, t_i) \in \Gamma_{\text{Sing}}$, there exists a $\delta^\sharp > 0$ such that

$$\Xi\left(u, Q_{\frac{3}{4}}\right)(y,s) > r^{-1}\delta^\sharp, \ \forall (y,s) \in \bigcup_{i=1}^{d} Q_r\left(x_i, t_i + \frac{3r^2}{2}\right).$$

It follows by sentence (3.41) applied to $Q_r\left(x_i, t_i + \frac{3r^2}{2}\right) \subseteq Q_{\frac{1}{2}}$ that

$$dr^{n+2} \leq \mathcal{L}^{n+1}\left(\left\{(x,t) \in Q_{\frac{1}{2}}; \Xi\left(u, Q_{\frac{3}{4}}\right)(x,t) > r^{-1}\delta^\sharp\right\}\right) \leq \mathfrak{C}(r^{-1}\delta^\sharp)^{-\varepsilon}$$

for positive universal constants $\mathfrak{C}, \varepsilon$. Therefore,

$$\sum_{i=1}^{d}(5r)^{n+2-\varepsilon} \leq \mathfrak{C}(n, \lambda, \Lambda, \varepsilon, \delta^{\sharp}) < \infty$$

Particularly, we deduce that

$$\mathcal{H}_{par}(\Gamma_{\text{Sing}}) \leq n + 2 - \varepsilon.$$

□

The next result assures that for a restrict class of solutions and non-linearities a similar partial regularity result is verified for problems with Lipschitz data.

Corollary 3.19. *Let u be a viscosity solution to* (3.1) *with $F \in C^1(Sym(n))$ satisfying $\mathfrak{c} \leq D_{u_i u_j} F(M) \leq \mathfrak{c}^{-1}$ for some constant $\mathfrak{c} > 0$ and data Lipschitz continuous. Suppose that $D^2 u : Q_1 \to Sym(n)$ is a bounded map and that there exists $\Theta > 0$ such that*

$$|F(\cdot, D^2 u(x,t) - F(\cdot, D^2 u(y,s))| \leq \Theta . d((x,t),(y,s)). \tag{3.43}$$

Then, the thesis of the Theorem 3.16 holds for a constant $\varepsilon > 0$ depending only on universal parameters, Θ, $\|f\|_{Lip(Q_1)}$ and $\|D^2 u\|_{L^\infty(Q_1)}$.

Demonstração. Notice that we can re-write (3.1) as

$$\frac{\partial u}{\partial t} - F(x_0, t_0, D^2 u(x,t)) = g(x,t) \quad \text{in} \quad Q_1,$$

where $g(x,t) := f(x,t) + F(x,t,D^2 u(x,t)) - F(x_0, t_0, D^2 u(x,t))$. Since the data are Lipschitz, we get that g is Lipschitz as well by using (3.5), (3.6)and (3.43). Moreover,

$$[g]_{\text{Lip}(Q_1)} \leq [f]_{\text{Lip}(Q_1)} + 2\Theta + \left(1 + \|D^2 u\|_{L^\infty(Q_1)}\right).$$

Finally, the result now holds from Theorem 3.16. □

Remark 3.20. The first result in this direction was obtained in DANIEL (2015) and similar results appear in DOS PRAZERES and TEIXEIRA (2016). Recently, it was built up an example in the elliptic case which point out that $\epsilon \leq 2(\frac{\Lambda}{\lambda} + 1)^{-1}$.

3.5 Parabolic Log-Lipschitz type estimates

In this Section, we shall comment on the $C^{1,\text{Log-Lip}}$ parabolic interior estimate which will be obtained by adjustments in the analysis carried out in Section 3.3. This result is enclosed in previous works, DA SILVA and TEIXEIRA (2017) and WANG (2006).

Recall that Log-Lipschitz estimates take place in many contexts in mathematics since Monge-Ampère equation, modeling semi-geostrophic equation in meteorology, as well as in transport-diffusion equations in Besov spaces, just to mention a few. In fact, they are interpreted as qualitative improvement in borderline conditions where the correspondent Lipschitz regularity can not hold, see (DOS PRAZERES and TEIXEIRA, 2016, Section 6), (DA SILVA and TEIXEIRA, 2017, Section 6) and WANG (2006). Furthermore, it is well-known that it is possible to build up a solution u to

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \mathfrak{a}_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x,t) \quad \text{in} \quad Q_1$$

where $\mathfrak{a}_{ij}$ and f are just continuous, but neither $\frac{\partial u}{\partial t}$ or $\partial_{ij}u$ are bounded, see IL'IN (1967) and KRUŽKOV (1967) for enlightening references.

In the next, we shall show that under continuity assumption on the coefficients of F and on the source f, after a suitable scaling argument, solutions are under the smallness regime of Lemma 3.5. For a $\kappa > 0$ to be determined *a posteriori* define

$$v_\kappa(x,t) := \frac{u(\kappa x, \kappa^2 t)}{\kappa^2}$$

Thus we have

$$\frac{\partial v_\kappa}{\partial t} - F_\kappa(x,t,D^2 v) = f_\kappa(x,t) \quad \text{in} \quad Q_1$$

in the viscosity sense, where

$$F_\kappa(x,t,M) := F(\kappa x, \kappa^2 t, M) \quad \text{and} \quad f_\kappa(x,t) := f(\kappa x, \kappa^2 t)$$

Now, if we do the following choice

$$\kappa := \min\left\{1, \frac{\tau^{-1}\left(\sqrt[6]{\delta^7}\right)}{\sqrt{5}}\right\}$$

with the following definition

$$\tau_\kappa(r) := \tau(\kappa r).$$

Now note that

$$\max\left\{|f_\kappa(x,t) - f_\kappa(y,s)|, \frac{|F_\kappa(x,t,M) - F_\kappa(y,s,M)|}{\|M\|+1}\right\} \leq \tau_\kappa(d((x,t),(y,s)))$$

Thus,

$$\max\{\|f_\kappa\|_{L^\infty(Q_1)}, \Theta_{F_\kappa}(x,t)\} \leq \sqrt[6]{\delta^7} \tag{3.44}$$

Finally, if we take

$$\|u\|_{L^\infty(Q_1)} \le \hat{\delta} := \delta\kappa^2,$$

then

$$\|v_\kappa\|_{L^\infty(Q_1)} \le \delta.$$

Therefore, the estimates proven to v drives the desired one to u.

Now, if f is just bounded then we can revisit the proof of Lemma 3.9 and, under the standard smallest assumptions, we can find a universal $0 < \sigma < \frac{1}{2}$ and an F-caloric quadratic function $\mathfrak{p}_1$ such that

$$\sup_{Q_\sigma} |u - \mathfrak{p}_1| \le \delta\sigma^2. \tag{3.45}$$

By proceeding inductively we are able to find a sequence of F-caloric quadratic functions

$$\mathfrak{p}_k(x,t) := \frac{1}{2}x^T \cdot \mathfrak{a}_k \cdot x + \mathfrak{b}_k.t + \mathfrak{c}_k \cdot x + \mathfrak{d}_k$$

such that

$$\mathfrak{b}_k - F(0,0,\mathfrak{a}_k) = 0 \quad \text{and} \quad \sup_{Q_{\sigma^k}} |u - \mathfrak{p}_k| \le \delta\sigma^{2k} \tag{3.46}$$

Moreover, we have the following estimates

$$\begin{cases} |\mathfrak{a}_k - \mathfrak{a}_{k+1}| & \le & \mathfrak{C}\delta \\ |\mathfrak{b}_k - \mathfrak{b}_{k+1}| & \le & \mathfrak{C}\delta \\ |\mathfrak{c}_k - \mathfrak{c}_{k+1}| & \le & \mathfrak{C}\delta\sigma^k \\ |\mathfrak{d}_k - \mathfrak{d}_{k+1}| & \le & \mathfrak{C}\delta\sigma^{2k} \end{cases} \tag{3.47}$$

Hence, the coefficients satisfies the following estimates

$$|\mathfrak{d}_k - u(0,0)| \le \mathfrak{C}\delta\sigma^{2k} \quad \text{and} \quad |\mathfrak{c}_k - Du(0,0)| \le \mathfrak{C}\delta\sigma^k; \tag{3.48}$$

however

$$|\mathfrak{a}_k| \le \mathfrak{C}k\delta \quad \text{and} \quad |\mathfrak{b}_k| \le \mathfrak{C}k\delta \tag{3.49}$$

We prove the existence of such polynomials by induction process in k. The first step of induction, $k = 1$, it is exactly the previous statement (3.45). Suppose now that we have verified the thesis of induction for $k = 1, \ldots, i$. Then, defining the re-scaled function $v := Q_1 \to \mathbb{R}$ given by

$$v_k(x,t) = \frac{(u - \mathfrak{p}_k)(\sigma^k x, \sigma^{2k} t)}{\delta\sigma^{2k}},$$

we have, by induction hypothesis, that $|v_k| \leq 1$ and it solves

$$\frac{\partial v_k}{\partial t} - F_k(x,t,D^2 v_k) = f(\sigma^k x, \sigma^{2k} t) := f_k(x,t)$$

in the viscosity sense, where $F_k(x,t,M) := F(\sigma x, \sigma^2 t, M + \mathfrak{a}_k) - \mathfrak{b}_k$.

Now, given the smallest conditions on the data, namely (3.44), we are able to apply the first step of induction. Thus, we readily obtain for a quadratic polynomial $\mathfrak{p}$ the following

$$\sup_{Q_\sigma} |v_k - \mathfrak{p}| \leq \delta \sigma^2. \tag{3.50}$$

Rewriting (3.50) back to the unit domain yields

$$\sup_{Q_{\sigma^{k+1}}} \left| u(x,t) - \left[\mathfrak{p}_k(x,t) + \delta \sigma^{2k} \mathfrak{p}\left(\frac{x}{\sigma^k}, \frac{t}{\sigma^{2k}} \right) \right] \right| \leq \delta \sigma^{2(k+1)}. \tag{3.51}$$

Therefore, defining

$$\mathfrak{p}_{k+1}(x,t) := \mathfrak{p}_k(x,t) + \delta \sigma^{2k} \mathfrak{p}\left(\frac{x}{\sigma^k}, \frac{t}{\sigma^{2k}} \right),$$

we verify the $(k+1)^{\underline{th}}$ step of induction and, clearly, the required conditions (3.46) and (3.47) are satisfied.

Finally, based on estimates (3.48) and (3.49) we are able to prove that

$$\sup_{Q_r} |u(x,t) - [u(0,0) + Du(0,0) \cdot x]| \leq \mathfrak{C} \delta r^2 \log r^{-1}, \tag{3.52}$$

for a constant $\mathfrak{C} > 0$ that depends only upon n, λ, Λ and τ.

Therefore, this proves the following Theorem:

Theorem 3.21 (**Parabolic $C^{1,\textbf{Log-Lip}}$ estimates**, DA SILVA and TEIXEIRA (2017))**.** *Let $u \in C^0(Q_1)$ be a bounded viscosity solution to* (3.1) *such that* $(A1)-(A2)$ *hold. There exists a $\delta = \delta(n, \lambda, \Lambda, \omega)$ such that if $\sup_{Q_1} \|u\| \leq \delta$, then $u \in C^{1,\mathrm{Log-Lip}}(Q_{1/2})$ and*

$$\|u\|_{C^{1,Log-Lip}(Q_{1/2})} \leq C(n, \lambda, \Lambda, \omega) \cdot \delta$$

Remark 3.22 (L^p-**BMO type estimates on** $\frac{\partial u}{\partial t}$ **and** $D^2 u$ $(p \in (1, \infty))$). The final estimate (3.52) says us that solutions to (3.1) are asymptotically $C^{2,1}$ in the parabolic sense, as well as $\frac{\partial u}{\partial t}$ and $D^2 u$ have a logarithmic behaviour at the origin. Therefore, adjustments on the previous explanation yield $\frac{\partial u}{\partial t}, D^2 u \in p - BMO\left(Q_{\frac{1}{2}}\right)$, with the appropriate *a priori* estimate on the $p - BMO\left(Q_{\frac{1}{2}}\right)$ norm of the temporal derivative and Hessian of u in terms of the L^∞-norm of f in Q_1. Indeed, under appropriate smallness regime on

$f, \Theta_F \in C^0(Q_1) \cap L^\infty(Q_1)$, we can approximate u by a viscosity solution to

$$\frac{\partial \mathfrak{h}}{\partial t} - F(x_0, t_0, D^2\mathfrak{h}) = 0 \quad \text{in} \quad Q_{\frac{1}{2}}$$

in the $W^{2,1}_p\left(Q_{\frac{1}{2}}\right)$ topology: for quadratic polynomial functions

$$\mathfrak{p}_k(x,t) := \frac{1}{2}x^T \cdot \mathfrak{a}_k \cdot x + \mathfrak{b}_k.t + \mathfrak{c}_k \cdot x + \mathfrak{d}_k$$

we have that

$$u_k(x,t) := \frac{(u-\mathfrak{p}_k)(\rho^k x, \rho^{2k} t)}{\rho^{2k}}$$

fulfils $|u_k| \le \delta$ and $\left|\frac{\partial u_k}{\partial t} - F_k(x,t,D^2u_k)\right| \ll 1$ in Q_1. By using interior $W^{2,1}_p$ estimates (cf. CRANDALL, KOCAN, and ŚWIECH (2000) and WANG (1992a)) we obtain

$$\left\|\frac{\partial u_k}{\partial t}\right\|_{L^p\left(Q_{\frac{1}{2}}\right)}, \|D^2 u_k\|_{L^p\left(Q_{\frac{1}{2}}\right)} \le \mathfrak{C}(\delta + \|f\|_{L^\infty(Q_1)}).$$

Consequently,

$$⨍_{Q_{\frac{\rho^k}{2}}} \left(\left|\frac{\partial}{\partial t}(u-\mathfrak{p}_k)\right|^p + |D^2(u-\mathfrak{p}_k)|^p\right) \le \mathfrak{C}(\delta + \|f\|_{L^\infty(Q_1)}) \quad \forall\, k \in \mathbb{N}.$$

Therefore, the previous sentence provides the desired $p-$BMO estimate. In other words,

$$\left\|\frac{\partial u}{\partial t}\right\|_{p-BMO(Q_r)} + \|D^2 u\|_{p-BMO(Q_r)} \le \mathfrak{C}(n,p,\lambda,\Lambda,\tau)(\delta + \|f\|_{L^\infty(Q_1)}), \ \forall\, r \ll 1.$$

4 GLOBAL REGULARITY FOR FULLY NONLINEAR PROBLEMS OF COMBUSTION TYPE

The goal of the present chapter is obtaining optimal regularity estimates up to the boundary for approximating solutions. More precisely, we shall prove a uniform gradient estimate (Lipschitz regularity) up to the boundary for viscosity solutions of the singular perturbation problem

$$\begin{cases} F(x, Du^{\varepsilon}, D^2u^{\epsilon}) &=& \zeta_{\epsilon}(u^{\epsilon}) & \text{in} & \Omega \\ u^{\epsilon}(x) &=& g(x) & \text{on} & \partial\Omega, \end{cases} \tag{E_ϵ}$$

where the singular reaction term $\zeta_{\varepsilon}(s) = \frac{1}{\varepsilon}\zeta\left(\frac{s}{\varepsilon}\right)$ for some non-negative $\zeta \in C_0^{\infty}([0,1])$, a parameter $\varepsilon > 0$, a non-negative $g \in C^{1,\gamma}(\overline{\Omega})$, with $0 < \gamma < 1$, and, a bounded $C^{1,1}$ domain Ω (or $\partial\Omega$ for short). Throughout this chapter we will assume the following bounds: $\|g\|_{C^{1,\gamma}(\overline{\Omega})} \leq \mathcal{A}$ and $\|\zeta\|_{L^{\infty}([0,1])} \leq \mathcal{B}$.

The chapter is organized according the following way: In Section 4.1 we introduce the notation, definitions and statements this chapter. In Section 4.3 we establish the optimal bounds for the gradient to solutions of (E_ϵ). In Section 4.4 we obtain the correspondent limit for free boundary problem, i.e. optimal Lipschitz continuity up to the boundary for solutions. In the end, in Section 4.5 we two technical results that were postponed for the end the chapter by didactic motives.

4.1 Notations and statements

Hereafter in this chapter, $F\colon \Omega \times \mathbb{R}^n \times \mathrm{Sym}(n) \to \mathbb{R}$ is a fully nonlinear uniformly elliptic operator, i.e, there exist constants $\Lambda \geq \lambda > 0$ such that

$$\lambda\|N\| \leq F(x, \overrightarrow{p}, M+N) - F(x, \overrightarrow{p}, M) \leq \Lambda\|N\|, \tag{Unif. Ellip.}$$

for all $M, N \in \mathrm{Sym}(n), N \geq 0, \overrightarrow{p} \in \mathbb{R}^n$ *and* $x \in \Omega$. As usual $\mathrm{Sym}(n)$ denotes the set of all $n \times n$ symmetric matrices. Moreover, we must to observe the mapping $M \mapsto F(x, \overrightarrow{p}, M)$ is monotone increasing in the natural order on $Sym(n)$ and Lipschitz. Under such a structural condition, the theory of viscosity solutions provides a suitable notion for weak solutions.

Definition 4.1 (Viscosity solution). For an operator $F\colon \Omega \times \mathbb{R}^n \times \mathrm{Sym}(n) \to \mathbb{R}$, we say a function $u \in C^0(\Omega)$ is a viscosity super-solution (resp. sub-solution) to

$$F(x, Du, D^2u) = f(x) \quad \text{in} \quad \Omega,$$

if whenever we touch the graph of u by below (resp. by above) at a point $y \in \Omega$ by a smooth function ϕ, there holds

$$F(y, D\phi(y), D^2\phi(y)) \leq f(y) \quad (\text{resp.} \geq f(y)).$$

Finally, we say u is a viscosity solution if it is simultaneously a viscosity super-solution and sub-solution.

Remark 4.2. All functions considered in the chapter will be assumed continuous in $\overline{\Omega}$, namely C-viscosity solutions, see Caffarelli-Cabré CAFFARELLI and CABRÉ (1995) and Teixeira TEIXEIRA (2006). However, we also can to consider L^p-viscosity notion for such solutions, see for example Winter WINTER (2009).

Theorem 4.3 (Global uniform Lipschitz estimate). *Let u^ϵ be a viscosity solution to the singular perturbation problem (E_ϵ). Then under the assumptions (F1)-(F2) there exists a constant $C = C(n, \lambda, \Lambda, b, \mathcal{A}, \mathcal{B}) > 0$ independent of ϵ, such that*

$$\|Du^\epsilon\|_{L^\infty(\overline{\Omega})} \leq C.$$

Our new estimate allows us to obtain existence for corresponding free boundary problem (1.8), keeping the prescribed boundary value data, see Theorem 4.17. Finally, we should emphasize our estimate generalizes the local gradient bound proven in TEIXEIRA (2006), see also RICARTE and TEIXEIRA (2011) for a rather complete local analysis of such a free boundary problem.

With regard to existence of solutions throughout this chapter we will deal with Perron's type solutions to the problem (E_ϵ), i.e., solutions u^ϵ derived by least super-solution method

$$u^\epsilon(x) := \inf\{v(x) : v \text{ is supersolution to } (E_\epsilon), u_* \leq u^\epsilon \leq u^*\}$$

where u_* and u^* are respectively fixed sub-solution and super-solution to (E_ϵ) satisfying $u_* \leq u^*$ in Ω and $u_* = u^* = g$ on $\partial\Omega$. Therefore, for each $\varepsilon > 0$ fixed, the existence of such a Perron's solution is ensured by usual methods of sub and super solutions.

Although we have chosen to carry out the global analysis for the homogeneous case, the results presented in this chapter can be adapted, under some natural adjustments, for the non-homogeneous case,

$$\begin{cases} F(x, Du^\varepsilon, D^2u^\epsilon) &= \zeta_\epsilon(u^\epsilon) + f_\varepsilon(x) \quad \text{in} \quad \Omega \\ u^\epsilon(x) &= g(x) \quad\quad\quad\quad \text{on} \quad \partial\Omega, \end{cases}$$

with $0 < c \leq f_\varepsilon \leq c^{-1}$.

We shall introduce some notations and structural assumptions which we will use throughout this chapter.

✓ n indicates the dimension of the Euclidean space.

✓ $\mathcal{H}_+$ is the half-space $\{x_n > 0\}$.

✓ $\mathcal{H} := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\}$ indicates the hyperplane.

✓ $\hat{x}$ is the vertical projection of X on $\mathcal{H}$.

✓ $\mathcal{C}_x := \left\{y \in H_+ : |y - \hat{y}| \geq \frac{1}{2}|y - x|\right\}$ is the cone with vertex at point $x \in \mathcal{H}$.

✓ $B_r(x)$ is the ball with center at X and radius r, and, B_r the ball $B_r(0)$.

✓ $B_r^+(x) := B_r(x) \cap \mathcal{H}_+$.

✓ $B_r'(x)$ is the ball with center at x and radius r in $\mathcal{H}$.

Remark 4.4. Throughout this chapter *Universal constants* are the ones depending only on the dimension, ellipticity and structural properties of F, i. e., n, λ, Λ and b.

Also, following classical notation, for constants $\Lambda \geq \lambda > 0$ we denote by

$$\mathcal{P}^+_{\lambda,\Lambda}(M) := \lambda \sum_{e_i<0} e_i + \Lambda \sum_{e_i>0} e_i \quad \text{and} \quad \mathcal{P}^-_{\lambda,\Lambda}(M) := \lambda \cdot \sum_{e_i>0} e_i + \Lambda \cdot \sum_{e_i<0} e_i$$

the *Pucci's extremal operators*, where $e_i = e_i(M)$ are the eigenvalues of $M \in Sym(n)$.

We shall introduce structural conditions that will be frequently used throughout of this chapter:

(F1) (**Ellipticity and Lipschitz regularity condition**) For all $M, N \in \mathrm{Sym}(n)$, $\overrightarrow{p}, \overrightarrow{q} \in \mathbb{R}^n$, $x \in \Omega$

$$\mathcal{P}^-_{\lambda,\Lambda}(M-N) - b|\overrightarrow{p} - \overrightarrow{q}| \leq F(x, \overrightarrow{p}, M) - F(x, \overrightarrow{q}, N) \leq \mathcal{P}^+_{\lambda,\Lambda}(M-N) + b|\overrightarrow{p} - \overrightarrow{q}|.$$

(F2) (**Normalization condition**) We shall suppose that,

$$F(x, 0, 0) = 0.$$

(F3) (**Small oscillation condition**) We shall assume that

$$\sup_{x_0 \in \Omega} \Theta_F(x, x_0) < \theta(n, \lambda, \Lambda) \ll 1,$$

where

$$\Theta_F(x, x_0) := \sup_{M \in Sym(n) \setminus \{0\}} \frac{|F(x, 0, M) - F(x_0, 0, M)|}{\|M\|}$$

Remark 4.5. Assumption (**F1**) is equivalent to notion of uniform ellipticity Unif. Ellip. when $\overrightarrow{p} = \overrightarrow{q}$. The assumption (**F2**) is not restrictive, since we can always redefine the operator in order to check it. The smallest regime on oscillation of F, namely condition

(**F3**), depends only on universal parameters, see WINTER (2009).

Example 4.6 (**Isaacs type operators).** An example which we must have in mind are the Isaacs' operators from stochastic game theory

$$F(x, \overrightarrow{p}, M) = \sup_{\alpha \in \mathfrak{A}} \inf_{\beta \in \mathfrak{B}} \left(-\mathrm{Tr} \left[A^{\alpha,\beta}(x) \cdot M \right] + \left\langle B^{\alpha,\beta}(x), \overrightarrow{p} \right\rangle \right) \left(\text{resp.} \inf_{\mathfrak{A}} \sup_{\mathfrak{B}} (\cdots) \right), \tag{4.1}$$

where $A^{\alpha,\beta}$ is a family of measurable $n \times n$ real symmetric matrices with small oscillation satisfying

$$\lambda \|\xi\|^2 \leq \xi^T A^{\alpha,\beta}(x)\xi \leq \Lambda \|\xi\|^2, \ \forall \ \xi \in \mathbb{R}^n \quad \text{and} \quad \|B^{\alpha,\beta}\|_{L^\infty(\Omega)} \leq b.$$

4.2 Existence of solutions

In this Section we shall comment on the existence of appropriated viscosity solutions to the singularly perturbed problem (E_ϵ). Such solutions are labeled by *Perron's type solutions.*

Theorem 4.7 (**Perron's type method**, RICARTE and TEIXEIRA (2011))**.** *Let* $f \in C^{0,1}([0, \infty))$ *be a bounded function. Suppose that there exist a viscosity sub-solution* $\underline{u} \in C(\overline{\Omega}) \cap C^{0,1}(\Omega)$ *(respectively super-solution* $\overline{u} \in C(\overline{\Omega}) \cap C^{0,1}(\Omega)$*) to* $F(x, \nabla w, D^2 w) = f(w)$ *satisfying* $\underline{u} = \overline{u} = g \in C(\partial\Omega)$*. Define the set of functions*

$$\mathcal{S} := \left\{ v \in C(\overline{\Omega}) \;\middle|\; \begin{array}{c} v \textit{ is a viscosity super-solution to} \\ F(x, Dw, D^2 w) = f(w(x)) \textit{ such that } \underline{u} \leq v \leq \overline{u} \end{array} \right\}.$$

Then,

$$u(x) := \inf_{v \in \mathcal{S}} v(x), \ \textit{for } x \in \Omega \tag{4.2}$$

is a continuous viscosity solution to $F(x, \nabla w, D^2 w) = f(w(x))$ *in* Ω *with* $u = g$ *continuously on* $\partial\Omega$*.*

Existence of Perron's type solution to (E_ϵ) will follow by choosing $\underline{u} := \underline{u}^\varepsilon$ and $\overline{u} := \overline{u}^\varepsilon$ as solutions to the boundary value problems:

$$\begin{cases} F(x, D\underline{u}^\varepsilon, D^2 \underline{u}^\varepsilon) &= \sup_{x \in [0,\infty)} \zeta_\varepsilon(u^\varepsilon(x)) & \text{in} & \Omega \\ \underline{u}^\varepsilon(x) &= g(x) & \text{on} & \partial\Omega \end{cases}$$

and

$$\begin{cases} F(x, D\overline{u}^\varepsilon, D^2 \overline{u}^\varepsilon) &= 0 & \text{in} & \Omega \\ \overline{u}^\varepsilon(x) &= g(x) & \text{on} & \partial\Omega, \end{cases}$$

We must note that for each $\varepsilon > 0$ fixed, existence of such a $\underline{u}^\varepsilon$ and $\overline{u}^\varepsilon$ follows as consequence of standard methods of sub and super solutions. Moreover, we have that $\underline{u} \in C(\overline{\Omega}) \cap C^{0,1}(\Omega)$ and $\overline{u} \in C(\overline{\Omega}) \cap C^{0,1}(\Omega)$ are viscosity sub-solution and super-solution to (E_ϵ) respectively. Finally, as consequence of the Theorem 4.7 we have the following existence Theorem:

Theorem 4.8 (Existence of Perron's type solutions, RICARTE and TEIXEIRA (2011)). *Given $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $g \in C(\partial\Omega)$ be a nonnegative boundary datum. There exists for each $\varepsilon > 0$ fixed, a nonnegative Perron's type viscosity solution $u^\varepsilon \in C(\overline{\Omega})$ to (E_ϵ).*

Finally, minimal type solutions are the corresponding non variational counterpart to minimizers of Euler-Lagrange functionals from variational theory. Therefore, unless otherwise specified, viscosity solutions to (E_ϵ), will mean minimal type solutions according to Theorem 4.8.

4.3 Optimal Lipschitz regularity

In this section, we shall present the proof of Theorem 4.3. Thus let us assume the assumptions of problem (E_ϵ).

We make a pause as to discuss some remarks which will be important throughout this chapter.

Remark 4.9. Firstly, it is important to highlight that is always possible to perform a change of variables to flatten the boundary. Indeed, if $\partial\Omega$ is a $C^{1,1}$ set, the part of Ω near $\partial\Omega$ can be covered with a finite collection of regions that can be mapped onto half-balls by diffeomorphisms (with portions of $\partial\Omega$ being mapped onto the "flat" parts of the boundaries of the half-balls). Hence, we can use a smooth mapping, reducing this way the general case to that one on B_1^+, and, the boundary data would be given on $B_1 \cap \{x_n = 0\}$.

In effect, consider $x_0 \in \partial\Omega$. Since $\partial\Omega \in C^{1,1}$ there exists a neighborhood of x_0, namely $\mathcal{V}(x_0)$ and a $C^{1,1}-$diffeomorfism $\Phi : \mathcal{V}(x_0) \to B_1(0)$ such that

$$\Phi(x_0) = 0 \quad \text{and} \quad \Phi(\Omega \cap \mathcal{V}(x_0)) = B_1^+$$

Therefore, if u is a solution to

$$F(x, Du, D^2u) = f(u(x)) \quad \text{in} \quad \Omega$$

then $v(x) := u(\Phi^{-1}(x))$ is a solution in $B_1^+(0)$ to

$$G(x, Dv, D^2v) = f(v(x))$$

where

$$F(\Phi^{-1}(x), \overrightarrow{p} D\Phi(\Phi^{-1}(x)), D\Phi^T(\Phi^{-1}(x)) M D\Phi(\Phi^{-1}(x)) + \overrightarrow{p} D^2\Phi(\Phi^{-1}(x))).$$

Before starting the proof of the global Lipschitz estimative, we need to assure the non-negativity of solutions to (E_ϵ). Such a result will be used several times throughout this chapter. This statement is a consequence of the Aleksandrov-Bakelman-Pucci estimate, see (CAFFARELLI and CABRÉ, 1995, Ch. 3) for more details.

Lemma 4.10 (**Nonnegativity and bounds**, RICARTE and TEIXEIRA (2011) and TEIXEIRA (2006))**.** *Let u^ε be a viscosity solution to (E_ϵ). Then there exists a universal constant $C > 0$ such that*

$$0 \leq u^\varepsilon(x) \leq C\|g\|_{L^\infty(\overline{\Omega})} \quad in \quad \Omega.$$

Demonstração. Define $v^\varepsilon(x) := u^\varepsilon(x) - \|g\|_\infty$. Notice that

$$\begin{cases} F(x, Dv^\varepsilon, D^2v^\varepsilon) = F(x, Du^\varepsilon, D^2u^\varepsilon) & = & \zeta_\epsilon(u^\epsilon) \geq 0 \quad \text{in} \quad \Omega \\ v^\epsilon & \leq & 0 \qquad\qquad \text{on} \quad \partial\Omega, \end{cases}$$

Estimate by above follows as an immediate application of Aleksandrov-Bakelman-Pucci estimate, see CAFFARELLI and CABRÉ (1995).

Let us prove the non-negativity of u^ε. Suppose, for sake of contradiction that the region

$$\mathcal{U}_\varepsilon := \{x \in \Omega : u^\varepsilon(x) < 0\} \neq \varnothing.$$

Since $\text{supp}(\zeta_\varepsilon) = [0, \varepsilon]$, we have

$$F(x, Du^\varepsilon, D^2u^\varepsilon) = 0 \quad \text{in} \ \mathcal{U}_\varepsilon.$$

Applying, one more time the Aleksandrov-Bakelman-Pucci estimate we conclude $u^\varepsilon \geq 0$ in $\mathcal{U}_\varepsilon$, leading us to a contradiction. □

We will now establish a universal bound for the Lipschitz norm of u^ε up to the boundary. The proof will be divided in two cases.

Case 1: Global Lipschitz regularity in the region $\{0 \leq u^\varepsilon \leq \varepsilon\}$.

Theorem 4.11. *Let u^ϵ be a viscosity solution to (E_ϵ). For $x \in \{0 \leq u^\varepsilon \leq \varepsilon\} \cap B^+_{\frac{1}{2}}$ there* *exists a universal constant* $C_1 > 0$ *independent of* ε *such that*

$$|Du^{\epsilon}(x)| \leq C_1.$$

Demonstração. We denote by

$$\delta(x) := \operatorname{dist}(x, \mathcal{H})$$

the vertical distance. If $\delta(x) \geq \epsilon$, then $B_{\varepsilon}(x) \subset B_1^+$ for $\varepsilon \ll 1$. Therefore, from local gradient bounds RICARTE and TEIXEIRA (2011); TEIXEIRA (2006), there exists a universal constant $C_0 > 0$ independent of ε, such that

$$|Du^{\epsilon}(x)| \leq C_0.$$

On the other hand, if $\delta(x) < \epsilon$, then it is sufficient to prove that there exists a universal constant $C_0 > 0$ independent of ε, such that

$$u^{\epsilon}(\hat{x}) \leq C_0\epsilon. \tag{4.3}$$

Indeed, suppose that (4.3) holds. Consider $h\colon \overline{B}_1^+ \to \mathbb{R}$ to be the viscosity solution to the Dirichlet problem

$$\begin{cases} F(y, Dh, D^2h) &= 0 \quad \text{in} \quad B_1^+ \\ h &= u^{\epsilon} \quad \text{on} \quad \partial B_1^+. \end{cases}$$

From $C^{1,\alpha}$ regularity estimates up to the boundary (see for instance (WINTER, 2009, Theorem 3.1)), we know that $h \in C^{1,\alpha}\left(\overline{B}^+_{\frac{3}{4}}\right)$ with the following estimate

$$|Dh(y)| \leq c\left(\|h\|_{L^{\infty}(B_1^+)} + \|g\|_{C^{1,\gamma}(B_1')}\right) \leq C \quad \text{in} \quad B^+_{\frac{3}{4}}$$

and by Comparison Principle we have

$$u^{\epsilon} \leq h \quad \text{in} \quad B_1^+.$$

Hence, it follows from assumption (4.3) that

$$u^{\epsilon}(y) \leq h(y) \leq h(\hat{x}) + C|y - \hat{x}| \leq C\epsilon \quad \text{if} \quad y \in B^+_{2\epsilon}(\hat{x})$$

Then, again applying $C^{1,\alpha}$ regularity estimates from (WINTER, 2009, Theorem 3.1), we obtain

$$|Du^{\varepsilon}(x)| \leq C_0(n, \lambda, \Lambda, b, \mathcal{B}).$$

In order to prove (4.3) suppose, by purpose of contradiction, there exists $\epsilon > 0$ such that

$$u^{\epsilon}(\hat{x}) \geq k\epsilon \quad \text{for} \quad k \gg 1.$$

We shall denote

$$r_0 := \mathrm{dist}(\hat{x}, \{0 \le u^\varepsilon \le \varepsilon\}).$$

Consider $x_0 \in \{0 \le u^\varepsilon \le \varepsilon\} \cap \partial B^+_{r_0}(\hat{x})$ a point to which the distance is achieved, i.e.,

$$r_0 = |x_0 - \hat{x}|.$$

Thereafter, let $\mathcal{C}_{\hat{x}}$ be the cone with vertex at $\hat{x} \in \mathcal{H}$. Suppose initially that $x_0 \in \mathcal{C}_{\hat{x}}$ then $B_{\frac{r_0}{2}}(x_0) \subset B^+_1$.

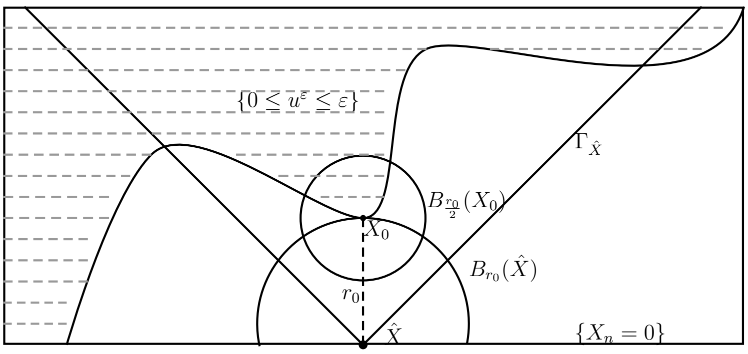

Figura 2: Geometric argument for the case $X_0 \in \mathcal{C}_{\hat{X}}$.

Now, let us define, $v^\epsilon : B_1 \to \mathbb{R}$ by

$$v^\epsilon(y) := \frac{u^\epsilon(x_0 + (r_0/2)y)}{\epsilon}.$$

Therefore, v^ϵ satisfies in the viscosity sense

$$F_\varepsilon(y, Dv^\varepsilon, D^2 v^\epsilon) = \frac{1}{\epsilon^2}\left(\frac{r_0}{2}\right)^2 \zeta(v^\epsilon) := \mathfrak{g}(y),$$

where

$$F_\varepsilon(y, \overrightarrow{p}, M) := \frac{1}{\varepsilon}\left(\frac{r_0}{2}\right)^2 F\left(x_0 + \frac{r_0}{2}y, \frac{2\varepsilon}{r_0}\cdot p, \varepsilon\left(\frac{2}{r_0}\right)^2 M\right). \tag{4.4}$$

Now note that $\mathfrak{g} \in L^\infty(B_1)$, since $r_0 < \epsilon$ and F_ε satisfies $(\mathbf{F1}) - (\mathbf{F3})$ with constant $\tilde{b} = \frac{r_0}{2} \cdot b$. Moreover, since $v^\epsilon(0) \le 1$ it follows from Harnack inequality (cf. (CAFFARELLI and CABRÉ, 1995, Ch. 4)) that

$$v^\epsilon(y) \le c \quad \text{for} \quad y \in B_{\frac{1}{2}},$$

i.e.,

$$u^\epsilon(x) \le c\epsilon, \quad x \in B_{\frac{r_0}{4}}(x_0).$$

Consider now $z \in B'_{r_0}(\hat{x})$. It follows that

$$g(z) \geq g(\hat{x}) - \mathcal{A} \cdot |z - \hat{x}| \geq k\epsilon - r_0 \cdot \mathcal{A} \geq (k - \mathcal{A})\epsilon,$$

since $r_0 < \epsilon$. Define the scaled function $w^\epsilon : B_1^+ \to \mathbb{R}$,

$$w^\epsilon(y) := \frac{u^\epsilon(\hat{x} + r_0 y)}{\epsilon}.$$

It readily follows that

$$\begin{cases} F_\varepsilon(y, Dw^\varepsilon, D^2 w^\epsilon) = 0 & \text{in} \quad B_1^+ \\ w^\epsilon(y) \geq k - \mathcal{A} & \text{on} \quad B'_1, \end{cases}$$

where F_ε is as in (4.4). Therefore according to Lemma 4.19,

$$w^\epsilon(y) \geq c(k - \mathcal{A}) \quad \text{in} \quad B^+_{\frac{3}{4}}.$$

In other words, we have reached that

$$u^\epsilon(x) \geq c\epsilon(k - \mathcal{A}) \quad \text{in} \quad B^+_{\frac{3r_0}{4}}(\hat{x}).$$

Hence

$$c\epsilon(k - \mathcal{A}) \leq u^\epsilon(z_0) \leq c\epsilon, \quad \forall \;\; z_0 \in \partial B^+_{\frac{3r_0}{4}}(\hat{x}) \cap \partial B_{\frac{r_0}{4}}(x_0),$$

which leads to a contradiction for $k \gg 1$.

On the one hand, if $x_0 \notin \mathcal{C}_{\hat{x}}$, choose $x_1 \in \{0 \leq u^\varepsilon \leq \varepsilon\}$ such that

$$r_1 := \text{dist}(\hat{x}_0, \{0 \leq u^\varepsilon \leq \varepsilon\}) = |\hat{x}_0 - x_1|.$$

From triangular inequality and the fact that $r_1 \leq \frac{r_0}{2}$ we have

$$|x_1 - \hat{x}| \leq |x_1 - \hat{x}_0| + |\hat{x}_0 - \hat{x}| \leq r_1 + r_0 \leq \frac{r_0}{2} + r_0.$$

If $x_1 \in \mathcal{C}_{\hat{x}_0}$ the result follows from previous analysis. Otherwise, let x_2 be such that

$$r_2 := \text{dist}(\hat{x}_1, \{0 \leq u^\varepsilon \leq \varepsilon\}) = |\hat{x}_1 - x_2|.$$

As before we have

$$|x_2 - \hat{x}| \leq |\hat{x}_1 - x_2| + |\hat{x}_1 - \hat{x}| \leq \frac{r_0}{4} + \frac{r_0}{2} + r_0,$$

since $r_2 \leq \frac{r_1}{2} \leq \frac{r_0}{4}$. Observe that this process must finish up within a finite number of steps. Indeed, suppose that we have a sequence of points $x_j \in \partial\{0 \leq u^\varepsilon \leq \varepsilon\}, \quad x_{j+1} \notin$

$\mathcal{C}_{\hat{x}_j}$ $(j = 1, 2, \ldots)$ satisfying,

$$r_{j+1} := \text{dist}(\hat{x}_j, \{0 \leq u^\varepsilon \leq \varepsilon\}) = |x_{j+1} - \hat{x}_j|$$

and

$$r_{j+1} \leq \frac{r_j}{2} \leq \frac{r_0}{2^{j+1}}. \tag{4.5}$$

Thus, it follows from (4.5) that

$$|x_j - \hat{x}| \leq r_0 + r_0 \sum_{i=1}^{j} \frac{1}{2^i} \leq 2r_0.$$

Therefore, up to a subsequence, $x_j \to x_\infty \in B'_{2r_0}(\hat{x})$ with $g(x_\infty) = \varepsilon$. However,

$$g(x_\infty) \geq g(\hat{x}) - \mathcal{A} \cdot |\hat{x} - x_\infty| \geq \varepsilon(k - 2\mathcal{A}) \gg \varepsilon$$

for $k \gg 1$ which drives us to a contradiction, and, hence the assertion (4.3) is proved. □

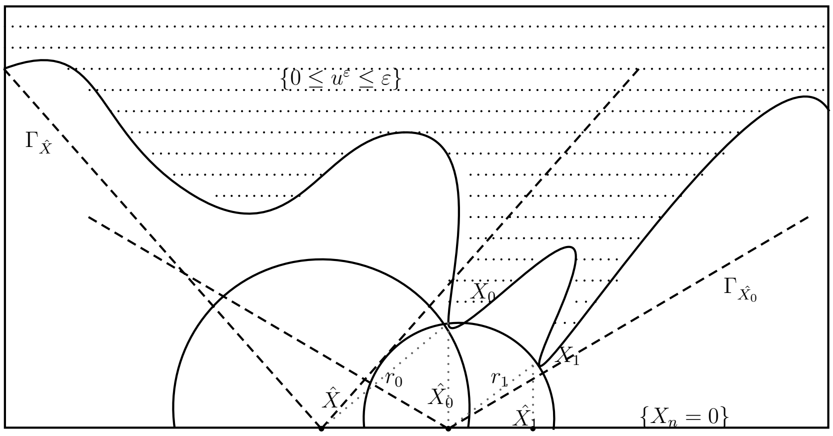

Figura 3: Geometric argument for the inductive process.

Case 2: Lipschitz regularity in the region $B^+_{1/8} \setminus \{0 \leq u^\varepsilon \leq \varepsilon\}$.

Theorem 4.12. *Let u^ε be a viscosity solution to (E_ϵ). If $x \in B^+_{\frac{1}{8}} \cap \{u^\varepsilon > \varepsilon\}$, then there exists a constant $C_0 = C_0(n, \lambda, \Lambda, b, \mathcal{A}) > 0$ such that*

$$|Du^\varepsilon(x)| \leq C_0.$$

The proof of the theorem consists in analysing three possible cases (Lemmas 4.14, 4.15, 4.16 below). Henceforth we shall use the following notation

$$\delta_\varepsilon(x) := \text{dist}(x, \{0 \leq u^\varepsilon \leq \varepsilon\}) \quad \text{and} \quad \delta(x) := \text{dist}(x, \mathcal{H}).$$

The next result is decisive in our approach.

Lemma 4.13. *Let u^ϵ be a viscosity solution to (E_ϵ) with $g \in C^{1,\gamma}(B'_1)$. Then, for all $x \in B'_{\frac{1}{4}} \cap \{u^\varepsilon > \varepsilon\}$, there exists a constant $c_0 = c_0(n, \lambda, \Lambda, b) > 0$ such that*

$$g(x) \leq \epsilon + c_0 \cdot \delta_\varepsilon(x).$$

Demonstração. Let us suppose for sake of contradiction that there exists an $\epsilon > 0$ and $x_0 \in B'_{\frac{1}{4}} \setminus \{0 \leq u^\varepsilon \leq \varepsilon\}$ such that

$$g(x_0) \geq \epsilon + k \cdot \delta_\varepsilon(x_0)$$

holds for $k \gg 1$, large enough. Let $z = z_\epsilon \in \partial\{0 \leq u^\varepsilon \leq \varepsilon\}$ be a point to which the distance is achieved, i.e.,

$$\delta_\varepsilon := \delta_\epsilon(x_0) = |x_0 - z|.$$

We have two cases to analyse: If $x \in \mathcal{C}_{x_0}$, then the normalized function $v^\epsilon \colon B_1^+ \to \mathbb{R}$ given by

$$v^\epsilon(y) := \frac{u^\epsilon(x_0 + \delta_\epsilon y) - \varepsilon}{\delta_\epsilon}$$

satisfies

$$F_\varepsilon(y, Dv^\varepsilon, D^2 v^\epsilon) = 0 \quad \text{in} \quad B_1^+$$

in the viscosity sense, where

$$F_\varepsilon(y, \overrightarrow{p}, M) := \delta_\varepsilon F\left(x_0 + \delta_\varepsilon y, \overrightarrow{p}, \frac{1}{\delta_\varepsilon} M\right).$$

As in Theorem 4.11, F_ε satisfies $(\mathbf{F1}) - (\mathbf{F3})$ with constant $\tilde{b} = \delta_\varepsilon b$. Moreover,

$$v^\epsilon(y) \geq 0 \quad \text{in} \quad B_1^+.$$

Now, for any $x \in B'_{\delta_\epsilon}(x_0)$ we should have for $k \gg 1$,

$$\begin{aligned} g(x) &\geq g(x_0) - \mathcal{A}\delta_\varepsilon \geq \varepsilon + k\delta_\varepsilon - \mathcal{A}\delta_\varepsilon \\ &\geq \varepsilon + \frac{k}{2}\delta_\varepsilon, \end{aligned}$$

i.e.,

$$\frac{g(x_0 + \delta_\varepsilon y) - \varepsilon}{\delta_\varepsilon} \geq \frac{k}{2} \quad \text{in } B'_1.$$

In other words,

$$v^\epsilon(y) \geq ck \quad \forall\, y \in B'_1.$$

Hence, from Lemma 4.19 we have that

$$v^{\epsilon} \geq ck \quad \text{in} \quad B^{+}_{\frac{3}{4}}.$$

In a more precise manner,

$$u^{\epsilon}(x) \geq \epsilon + Ck\delta_{\epsilon}, \quad x \in B^{+}_{\frac{3\delta_{\epsilon}}{4}}(x_0). \tag{4.6}$$

From now on, let us consider $\tilde{B} := B_{\frac{\delta_{\epsilon}}{4}}(\mathbf{P})$, where $\mathbf{P} = \mathbf{P}_{\epsilon} := z + \frac{x_0 - z}{4}$. If we define $\omega^{\varepsilon} := u^{\epsilon} - \epsilon$, then since $z \in \partial\tilde{B}$, it follows that

$$F_{\epsilon}(x, D\omega^{\varepsilon}, D^2\omega^{\epsilon}) = 0 \quad \text{in} \quad \tilde{B}, \tag{4.7}$$

$$\omega^{\epsilon}(z) = u^{\varepsilon}(z) - \varepsilon = 0, \tag{4.8}$$

$$\frac{\partial\omega^{\varepsilon}}{\partial\nu}(z) \leq |D\omega^{\varepsilon}(z)| \leq C. \tag{4.9}$$

Therefore, from (4.7)-(4.9) we can apply Lemma 4.20, which gives

$$\omega^{\varepsilon}(\mathbf{P}) \leq C_0 \cdot \delta_{\varepsilon},$$

i.e.,

$$u^{\epsilon}(\mathbf{P}) \leq \epsilon + C\delta_{\epsilon}. \tag{4.10}$$

At a point $\mathbf{P}$ on $\partial B^{+}_{\frac{3\delta_{\epsilon}}{4}}(x_0)$ we have (according to (5.5) and (4.10))

$$\epsilon + kc.\delta_{\epsilon} \leq u^{\epsilon}(\mathbf{P}) \leq \varepsilon + C_0.\delta_{\epsilon}$$

which gives a contradiction if k has been chosen large enough.

The second case, namely $z \notin \mathcal{C}_{x_0}$, it is treated similarly as in Theorem 4.11 and for this reason we omit the details here. □

Lemma 4.14. *Let u^{ε} be a viscosity solution to (E_{ϵ}) and $x \in B^{+}_{\frac{1}{8}} \cap \{u^{\varepsilon} > \varepsilon\}$ such that $\delta_{\varepsilon}(x) \leq \delta(x)$. Then there exists a universal constant $C_0 > 0$, such that*

$$|Du^{\epsilon}(x)| \leq C_0.$$

Demonstração. We may assume with no loss of generality that $\delta_{\epsilon}(x) \leq \frac{1}{8}$. Otherwise, if we suppose that $\delta_{\epsilon}(x) > \frac{1}{8}$, then the result would follow from RICARTE and TEIXEIRA (2011); TEIXEIRA (2006). From now on, we select $x_{\epsilon} \in \partial\{0 \leq u^{\varepsilon} \leq \varepsilon\}$ a point which achieves the distance, i.e.,

$$\delta_{\epsilon} := \delta_{\varepsilon}(x) = |x - x_{\epsilon}|.$$

Since

$$|x_\epsilon| \le |x| + \delta_\epsilon \le \frac{1}{4},$$

we must have that $x_\varepsilon \in B^+_{\frac{1}{4}} \cap \{0 \le u^\varepsilon \le \varepsilon\}$. This way, by applying Theorem 4.11, there exists a constant $C_1 = C(n, \lambda, \Lambda, b, \mathcal{A}, \mathcal{B}) > 0$ such that

$$|Du^\epsilon(x_\epsilon)| \le C_1.$$

By defining the re-normalized function $v^\epsilon : B_1 \to \mathbb{R}$ as

$$v^\epsilon(y) := \frac{u^\epsilon(x + \delta_\epsilon y) - \epsilon}{\delta_\epsilon}.$$

Then, as before v^ϵ satisfies

$$F_\epsilon(y, Dv^\varepsilon, D^2 v^\epsilon) = 0 \quad \text{in} \quad B_1, \tag{4.11}$$

$$v^\epsilon(y_\epsilon) = 0, \tag{4.12}$$

$$|Dv^\epsilon(y_\epsilon)| \le C_1, \tag{4.13}$$

$$v^\epsilon(y) \ge 0 \ \text{ for } \ y \in B_1, \tag{4.14}$$

where

$$F_\epsilon(y, \overrightarrow{p}, M) := \delta_\epsilon F\left(x + \delta_\varepsilon y, \overrightarrow{p}, \frac{1}{\delta_\epsilon} M\right) \quad \text{and} \quad y_\epsilon := \frac{x_\epsilon - x}{\delta_\epsilon} \in \partial B_1.$$

From (4.11)-(4.14) we are able to apply Lemma 4.20 and conclude that there exists a universal constant $c > 0$ such that

$$v^\varepsilon(0) \le c.$$

Moreover, from Harnack inequality (cf. (CAFFARELLI and CABRÉ, 1995, Ch. 4))

$$v^\varepsilon \le C_0 \quad \text{in} \quad B_{1/2}.$$

Therefore, by $C^{1,\alpha}$ regularity estimates (see for example (CAFFARELLI and CABRÉ, 1995, Ch. 8 §2)) we must have that

$$|Du^\varepsilon(x)| = |Dv^\varepsilon(0)| \le \frac{1}{\delta_\varepsilon} \|u^\varepsilon - \varepsilon\| \le C_0,$$

and the Lemma is proved. □

Lemma 4.15. *For $x \in B^+_{\frac{1}{8}} \cap \{u^\varepsilon > \varepsilon\}$ such that $\delta(x) < \delta_\varepsilon(x) \le 4\delta(X)$, we have*

$$|Du^\epsilon(x)| \le C_0$$

for some constant $C_0 = C_0(n, \lambda, \Lambda, b, \mathcal{A}, \mathcal{B}) > 0$.

Demonstração. Similarly to Lemma 4.14, we may assume that $\delta_\epsilon \leq \frac{1}{8}$, otherwise, as in Lemma 4.14 the boundedness of the gradient holds from local estimates RICARTE and TEIXEIRA (2011); TEIXEIRA (2006). Define the scaled function $v^\epsilon \colon B_1 \to \mathbb{R}$ by

$$v^\epsilon(y) := \frac{u^\epsilon(x+\delta y) - \epsilon}{\delta},$$

where $\delta = \delta(x)$. Clearly

$$F_\delta(y, Dv^\varepsilon, D^2 v^\epsilon) = 0 \quad \text{in} \quad B_1$$

in the viscosity sense, and, from Harnack inequality (see (CAFFARELLI and CABRÉ, 1995, Ch. 4))

$$v^\varepsilon \leq C v^\varepsilon(0) \sim \frac{1}{\delta} \quad \text{in } B_{\frac{1}{2}}.$$

By applying once more $C^{1,\alpha}$ regularity estimates ((CAFFARELLI and CABRÉ, 1995, Ch. 8 §2)), we obtain

$$|Du^\varepsilon(x)| = |Dv^\varepsilon(0)| \leq \frac{C}{\delta}. \tag{4.15}$$

Therefore, the idea is to find an estimate for $u^\varepsilon - \varepsilon$ in terms of the vertical distance $\delta(x)$. To this end, consider h the viscosity solution to the Dirichlet problem

$$\begin{cases} F(x, Dh, D^2 h) &= 0 \quad \text{in} \quad B_1^+ \\ h &= u^\epsilon \quad \text{on} \quad \partial B_1^+. \end{cases} \tag{4.16}$$

Since $0 \leq u^\epsilon \leq C(n, \lambda, \Lambda, b, \mathcal{B})$, it follows from $C^{1,\alpha}$ estimate up to boundary (WINTER, 2009, Theorem 3.1) that $h \in C^{1,\alpha}\left(\overline{B}^+_{\frac{3}{4}}\right)$. Moreover

$$|Dh(x)| \leq \overline{C}\left(\|h\|_{L^\infty(B_1^+)} + \|g\|_{C^{1,\gamma}(B_1')}\right) \leq \overline{C}(C + \mathcal{A}) := \mathcal{C}^*.$$

From Comparison Principle, we have that

$$u^\epsilon \leq h \quad \text{in} \quad B_1^+.$$

Hence,

$$u^\epsilon(x) \leq h(x) \leq h(\hat{x}) + \mathcal{C}^*|x - \hat{x}| \leq g(\hat{x}) + \mathcal{C}^*\delta. \tag{4.17}$$

Now, we have that $|\hat{x}| \leq |x| + \delta \leq \frac{1}{4}$, and, consequently we are able to apply Lemma 4.13 which gives

$$g(\hat{x}) \leq \epsilon + c_0 \cdot \operatorname{dist}(\hat{x}, \{0 \leq u^\varepsilon \leq \varepsilon\}) \leq \epsilon + c_0(\delta_\epsilon + \delta) \leq \epsilon + 5c_0\delta. \tag{4.18}$$

Thus, it follows from (4.17) and (4.18) that

$$u^\epsilon(x) - \varepsilon \leq C_0\delta,$$

where $C_0 := C(5c_0 + \mathcal{C}^*)$. Finally, if we apply $C^{1,\alpha}$ estimate (cf. (CAFFARELLI and CABRÉ, 1995, Ch. 8 §2)), Harnack inequality (cf. (CAFFARELLI and CABRÉ, 1995, Ch. 4)) and estimate (4.15), respectively, we end up with

$$|Du^\varepsilon(x)| = |Dv^\varepsilon(0)| \leq \frac{1}{\delta}\|u^\varepsilon - \varepsilon\|_{L^\infty\left(B_{\frac{1}{2}}\right)} \leq C_0$$

which concludes the proof. □

Lemma 4.16. *If $x \in B^+_{\frac{1}{8}} \cap \{u^\varepsilon > \varepsilon\}$ and $\delta(x) < \frac{1}{4}\delta_\varepsilon(x)$, then there exists a constant $C_0 = C_0(n, \lambda, \Lambda, b, \mathcal{A}, \mathcal{B}) > 0$ such that*

$$|Du^\epsilon(x)| \leq C_0.$$

Demonstração. Initially we will consider the case when $\delta_\epsilon \leq \frac{1}{8}$. The following inclusion holds true: $B^+_{\frac{\delta_\epsilon}{2}}(\hat{x}) \subset B^+_{\frac{1}{4}} \setminus \{0 \leq u^\varepsilon \leq \varepsilon\}$. In fact, if $y \in B^+_{\frac{\delta_\varepsilon}{2}}(\hat{x})$ then

$$|y| \leq |y - x| + |x| \leq 2\frac{\delta_\epsilon}{2} + |x| \leq \frac{1}{4}.$$

Now, using the same argument as in Lemma 4.15 (see (4.16)) we are able to estimate u^ϵ in $B^+_{\frac{\delta_\epsilon}{2}}(\hat{x})$ as follows

$$u^\epsilon(y) \leq u^\epsilon(\hat{y}) + \mathcal{C}^*\frac{\delta_\epsilon}{2} \leq \epsilon + c_0 \cdot \mathrm{dist}(\hat{y}, \{0 \leq u^\varepsilon \leq \varepsilon\}) + \mathcal{C}^*\frac{\delta_\epsilon}{2}.$$

Since the distance function is Lipschitz continuous with Lipschitz constant 1, we have

$$\mathrm{dist}(\hat{y}, \{0 \leq u^\varepsilon \leq \varepsilon\}) \leq \delta_\epsilon + |\hat{y} - x| \leq 2\delta_\epsilon.$$

Therefore,

$$u^\epsilon(y) \leq \epsilon + \left(2c_0 + \frac{\mathcal{C}^*}{2}\right)\delta_\epsilon = \epsilon + c\delta_\epsilon.$$

By considering the function $v^\epsilon(y) = u^\epsilon(y) - \epsilon$ in $B^+_{\frac{\delta_\epsilon}{2}}(\hat{x})$, we have that

$$F(y, Dv^\varepsilon, D^2v^\epsilon) = 0 \quad \text{in} \quad B^+_{\frac{\delta_\epsilon}{2}}(\hat{x})$$

in the viscosity sense. From $C^{1,\alpha}$ estimate up to boundary (cf. (WINTER, 2009, Theorem

3.1)) and Lemma 4.10, we have

$$|Du^\epsilon(x)| = |Dv^\epsilon(x)| \leq C(c + \mathcal{A}).$$

On the other hand, for the case $\delta_\epsilon \geq \frac{1}{8}$ we have the following inclusion $B^+_{\frac{1}{16}}(\hat{x}) \subset B_1 \setminus \{0 \leq u^\varepsilon \leq \varepsilon\}$. In this situation, since $\text{supp}(\zeta_\epsilon) = [0, \epsilon]$,

$$\begin{cases} F(x, Du^\varepsilon, D^2u^\epsilon) = 0 & \text{in} \quad B^+_{\frac{1}{16}}(\hat{x}) \\ 0 \leq u^\epsilon = g \leq C & \text{on} \quad B'_{\frac{1}{16}}(\hat{x}), \end{cases}$$

and consequently the estimate will follow from $C^{1,\alpha}$ estimates up to the boundary (cf. (WINTER, 2009, Theorem 3.1)). □

4.4 Limiting free boundary problem

An immediate consequence of Theorem 4.3 is the existence of solutions via compactness in the Lip-Topology for any family $(u^\varepsilon)_{\varepsilon>0}$ of viscosity solutions to singular perturbation problem (E_ϵ). We consequently obtain

Theorem 4.17 (Limiting free boundary problem). *Let $(u^\varepsilon)_{\varepsilon>0}$ be a family of solutions to (E_ϵ). For every $\varepsilon_k \to 0$ there are a subsequence $\varepsilon_{k_j} \to 0$ and $u_0 \in Lip(\overline{\Omega})$ such that*

(1) $u^{\varepsilon_{k_j}} \to u_0$ uniformly in $\overline{\Omega}$,

(2) $F(x, Du_0, D^2u_0) = 0$ in $\Omega \cap \{u_0 > 0\}$ in the viscosity sense.

Demonstração. (1) it is a consequence of the uniform Lipschitz continuity up-to boundary, Proposition 4.10 and the Arzelá-Ascoli theorem. For (2), since $u^{\varepsilon_{k_j}} \to u_0$ uniformly in $\overline{\Omega}$, for every $x_0 \in \{u_0 > 0\}$, there exists a neighbourhood $\mathcal{U}(x_0)$ such that $u^{\varepsilon_{k_j}} \geq \frac{u_0(x_0)}{2} > 0$ in $\mathcal{U}(x_0)$. Thus, since the reaction term ζ_ε is supported on $[0, \varepsilon]$, if we choose $\varepsilon_{k_j} \leq \frac{1}{4}u_0(x_0)$, solutions to (E_ϵ) solves

$$F(x, Du^\varepsilon, D^2u^\varepsilon) = 0 \quad \text{in } \mathcal{U}(x_0),$$

and the same holds for the limit u_0 by arguments of stability. □

Remark 4.18. If we consider a ball $B_\rho(x_0) \subset \Omega$, where $x_0 \in \partial\{u^\varepsilon > \varepsilon\}$ and $\rho \ll 1$, it is possible to establish some geometric properties of *least super-solutions* u^ε to (E_ϵ). For example, the linear growth property which says that the solution grows at least at a linear rate with respect to the distance to their free boundaries, i.e., there holds

$$u^\varepsilon(x_0) \geq c.\text{dist}(x_0, \{u^\varepsilon \leq \varepsilon\}).$$

The proof shall be based on appropriate barrier functions. As immediate consequence, it is possible prove that minimal solutions are strongly non-degenerate near ε-level sets. It means that the maximum of u^ε on the boundary of a ball B_ρ centred in $\{u^\varepsilon > \varepsilon\}$ is of the order of ρ. More precisely,

$$c\rho \leq \sup_{B_\rho(x_0)} u^\varepsilon \leq c^{-1}(\rho + u^\varepsilon(x_0)).$$

With these results, we point out that as a direct consequence, we can prove the uniform non-degeneracy property and stronger non-degeneracy for u_0. This results was recently establish in RICARTE and TEIXEIRA (2011).

4.5 Auxiliary results

In this final section we are going to give the proof of some results, which were temporarily omitted.

Lemma 4.19 (Boundary estimates propagation). *Suppose that $u \geq 0$ is a viscosity solution to*

$$\begin{cases} F(x, Du, D^2u) = 0 & \textit{in} \quad B_1^+ \\ u \geq \sigma > 0 & \textit{on} \quad B_1'. \end{cases}$$

Then there exists a universal constant $C = C(n, \lambda, \Lambda, b) > 0$ such that

$$u(x) \geq C\sigma, \quad x \in B^+_{\frac{3}{4}}.$$

Demonstração. First of all consider the following Dirichlet problem

$$\begin{cases} F(x, Dw, D^2w) & = & 0 & \text{in} & B_1^+ \\ w & = & \sigma & \text{on} & B_1' \\ w & = & 0 & \text{on} & \partial B_1 \cap \{x_n > 0\}. \end{cases} \tag{4.19}$$

From $C^{1,\alpha}$ regularity estimate, (WINTER, 2009, Theorem 3.1) we have $w \in C^{1,\alpha}\left(\overline{B}^+_{\frac{3}{4}}\right)$, and, by the Comparison Principle

$$0 \leq w \leq \sigma \quad \text{in} \ B_1^+. \tag{4.20}$$

From now on, it is appropriate we define the following reflection $\mathfrak{U}\colon B_1 \to \mathbb{R}$,

$$\mathfrak{U}(x) := \begin{cases} w(x) & \text{if} \quad x \in B_1^+ \cup B_1' \\ 2\sigma - w(x_1, \ldots, x_{n-1}, -x_n) & \text{if} \quad x \in B_1 \cap \{x_n < 0\}. \end{cases} \tag{4.21}$$

We observe that $\mathfrak{U}$ is a viscosity solution to

$$\mathcal{G}(x, D\mathfrak{U}, D^2\mathfrak{U}) = 0 \quad \text{in} \quad B_1,$$

where

$$\mathcal{G}(x, \overrightarrow{p}, M) := \begin{cases} F(x, \overrightarrow{p}, M) & \text{if} \quad x_n \geq 0 \\ -F(\widetilde{x}, \overrightarrow{\widetilde{p}}, \widetilde{M}) & \text{if} \quad x_n < 0, \end{cases}$$

with

$$\begin{aligned} \widetilde{x} &:= (x_1, \ldots, x_{n-1}, -x_n), \\ \widetilde{p} &:= (-p_1, \ldots, -p_{n-1}, p_n), \\ \widetilde{M} &:= \begin{cases} -M_{ij} & \text{if} \quad 1 \leq i, j \leq n-1 \text{ or } i = j = n \\ M_{ij} & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, from (4.20),

$$\sigma \leq \mathfrak{U} \leq 2\sigma \quad \text{in} \quad B_1^-$$

Hence,

$$0 \leq \mathfrak{U} \leq 2\sigma \quad \text{in} \quad B_1.$$

Moreover, from Harnack inequality (cf. (CAFFARELLI and CABRÉ, 1995, Ch. 4)) we have that

$$\sup_{B_{3/4}} \mathfrak{U} \leq c_0 \inf_{B_{3/4}} \mathfrak{U}.$$

Particularly,

$$w(x) \geq c_0^{-1}\sigma \quad in \quad B^+_{\frac{3}{4}}.$$

Therefore, the proof follows through the previous inequality combined with the Comparison Principle. □

Lemma 4.20 (Hopf type boundary principle). *Let u be a viscosity solution to*

$$\begin{cases} F(x, Du, D^2u) = 0 & in \quad B_r(Z) \\ u \geq 0 & in \quad B_r(Z). \end{cases}$$

with $r \leq 1$. Assume that for some $x_0 \in \partial B_r(Z)$,

$$u(x_0) = 0 \quad and \quad \frac{\partial u}{\partial \nu}(x_0) \leq \theta,$$

where ν is the inward normal direction at x_0. Then there exists a universal constant $C > 0$ such that

$$u(Z) \leq C\theta r.$$

Demonstração. By using a scaling argument, we may assume $r = 1$. Indeed, it is sufficient to consider the scaled function $v : B_1 \to \mathbb{R}$

$$v_r(Y) = \frac{u(Z + rY)}{r}.$$

As before, v_r is a viscosity solution of

$$F_r(Y, Dv_r, D^2 v_r) = 0 \quad \text{in} \quad B_1,$$

with

$$F_r(Y, \overrightarrow{p}, M) := rF\left(Z + rY, \overrightarrow{p}, \frac{1}{r}M\right)$$

Let $\mathfrak{A} := B_1 \setminus B_{\frac{1}{2}}$ be an annular region and define $\omega\colon \overline{\mathfrak{A}} \to \mathbb{R}$ by

$$\omega(Y) := \mu\left(e^{-\delta|Y|^2} - e^{-\delta}\right)$$

where the positive constants μ and δ will be chosen *a posteriori*. One can computer the gradient and Hessian of ω in $\mathfrak{A}$ as follows

$$\begin{aligned} \partial_i \omega(Y) &= -2\mu\delta Y_i e^{-\delta|Y|^2}, \\ \partial_{ij}\omega(Y) &= 4\mu\delta^2 Y_i Y_j e^{-\delta|Y|^2} - 2\mu\delta e^{-\delta|Y|^2}\delta_{ij}, \\ |\nabla\omega(Y)| &= 2\mu\delta e^{-\delta|Y|^2}|Y|. \end{aligned}$$

In particular, for every $M \in \mathcal{A}_{\lambda,\Lambda} := \left\{A \in \mathrm{Sym}(n) \;\middle|\; \lambda\|\xi\|^2 \le \sum\limits_{i,j=1}^{n} A_{ij}\xi_i\xi_j \le \Lambda\|\xi\|^2, \forall\, \xi \in \mathbb{R}^n\right\}$ we have

$$\begin{aligned} \mathrm{Tr}\left(M \cdot D^2\omega\right) - b|D\omega| &= \sum_{i,j=1}^{n} m_{ij}\partial_{ij}\omega - b \cdot \sqrt{\sum_{i=1}^{n}(\partial_i\omega)^2} \\ &= 4\mu\delta^2 e^{-\delta|Y|^2}\mathrm{Tr}(M \cdot Y \otimes Y) - 2\delta\mu\mathrm{Tr}(M)e^{-\delta|Y|^2} - 2\mu\delta b|Y|e^{-\delta|Y|^2} \\ &\ge 4\mu\delta^2\lambda|Y|^2 e^{-\mu|Y|^2} - 2\delta\mu n\Lambda e^{-\delta|Y|^2} - 2\mu\delta b|Y|e^{-\delta|Y|^2} \\ &= 2\mu\delta(2\delta\lambda|Y|^2 - b|Y| - n\Lambda)e^{-\delta|Y|^2} \\ &\ge 2\mu\delta\left(\frac{\delta\lambda}{2} - b - n\Lambda\right)e^{-\delta|Y|^2} \quad \text{in} \quad \mathfrak{A}, \end{aligned}$$

where $\xi \otimes \xi = (\xi_i\xi_j)_{i,j}$. Choose and fix $\delta \ge \frac{2}{\lambda}(b + n\Lambda)$. Then, it follows readily that

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2\omega) - b|D\omega| \ge 0 \quad \text{in} \quad \mathfrak{A}.$$

Therefore, since $r \le 1$, if $\delta \in \left[\frac{2}{\lambda}(\tilde{b} + n\Lambda), +\infty\right)$, with $\tilde{b} = rb$, we have

$$F_r(Y, \nabla\omega(Y), D^2\omega(Y)) \ge 0 \quad \text{in} \quad \mathfrak{A}.$$

Now by Harnack inequality (cf. (CAFFARELLI and CABRÉ, 1995, Ch. 4))

$$v_r(0) \leq \sup_{B_{1/2}} v_r \leq c_0 \inf_{B_{1/2}} v_r,$$

Hence,

$$v_r(Y) \geq c_o^{-1} v_r(0) \quad \text{in} \quad B_{\frac{1}{2}}.$$

By choosing $\mu = \frac{v_r(0)}{c_0\left(e^{-\frac{\delta}{4}} - e^{-\delta}\right)}$ we have

$$\omega \leq v_r \quad \text{on} \quad \partial\mathfrak{A}$$

and Comparison Principle gives that

$$\omega \leq v_r \quad \text{in} \quad \mathfrak{A}$$

Thus, if we label $Y_0 := \frac{X_0 - Z}{r}$ then

$$\mu\delta e^{-\delta} \leq \frac{\partial\omega}{\partial\nu}(Y_0) \leq \frac{\partial v_r}{\partial\nu}(Y_0) \leq \theta.$$

Therefore,

$$v_r(0) \leq \theta\delta^{-1} c_0 \left(e^{\frac{3\delta}{4}} - 1\right),$$

and by returning to the original sentence we can conclude that

$$u(Z) \leq c\theta r.$$

□

5 AN ENLIGHTENING EXAMPLE FOR GEOMETRIC TANGENTIAL ANALYSIS

The objective this Appendix is to clarify the *geometric tangential method* by deriving *a priori* $C^{2,\alpha}$ estimates to viscosity solutions of second order equations of the form

$$F(D^2u) = 0 \quad \text{in} \quad B_1 \subset \mathbb{R}^n, \tag{5.1}$$

under appropriate smallness condition on $\mathfrak{e} := 1 - \frac{\lambda}{\Lambda}$, which measures the aperture of the ellipticity of the operator F.

It has been known that viscosity solutions to fully nonlinear elliptic equations (5.1) are locally $C^{1,\alpha}$, for a constant $\alpha \in (0,1]$ that depends only on dimension and ellipticity constants, CAFFARELLI (1989) and (CAFFARELLI and CABRÉ, 1995, Ch. 5). Through the journey of finding classical, i.e. C^2, solutions to fully nonlinear equations, the result of Evans EVANS (1982) and Krylov KRYLOV (1983) is groundbreaking. It states that under concavity or convexity assumption on F, solutions of (5.1) are $C^{2,\alpha}(B_1)$, for some $0 < \alpha < 1$ () see also (CAFFARELLI and CABRÉ, 1995, Ch. 6). The quest on whether *any* fully nonlinear elliptic operator would enjoy a C^2 *a priori* regularity theory eluded the community for three decades. The counterexamples to $C^{1,1}$ regularity due to Nadirashvili and Vlăduţ, NADIRASHIVILI and VLĂDUŢ (2007) and NADIRASHIVILI and VLĂDUŢ (2008), close the case; however, on the other hand, it opens up an even broader line of investigation. Indeed, in view of the impossibility of a general existence theory for classical solutions to fully nonlinear equations, it becomes a central theme of research to obtain additional conditions on F and on u as to establish C^2 estimates.

Here we paraphrase Cabré and Caffarelli in (CABRÉ and CAFFARELLI, 2003, page 2):

> "Which assumptions on F, in between convexity of F and no assumptions, and perhaps depending on the dimension n, guarantee that solutions of (5.1) are classical?"

Therefore, we will give a theoretical contribution on this line of research.

Theorem 5.1. *Let $u \in C^0(B_1)$ be a bounded viscosity solution to* (5.1). *Given* $0 < \alpha < 1$ *there exists $\epsilon_0 = \epsilon_0(n, \alpha) > 0$ such that, if*

$$1 - \frac{\lambda}{\Lambda} < \epsilon_0,$$

then u is $C^{2,\alpha}$ at the origin, i.e.,

$$\left| u(x) - \left[u(0) + Du(0) \cdot x + \frac{1}{2} x^T \cdot D^2 u(0) \cdot x \right] \right| \leq \mathfrak{C}.\|u\|_{L^\infty(B_1)}.|x|^{2+\alpha} \quad (5.2)$$

for a constant $\mathfrak{C}$ depending upon universal parameters and α.

Once established such an estimate, it then follows by Caffarelli's Theorem CAFFARELLI (1989) (see also CAFFARELLI and CABRÉ (1995)) that the same class of regularity estimates can be established to non-homogeneous equations with Hölder continuous coefficients. Of particular interest, Theorem 5.1 covers *Isaac's type equations*, which appear in stochastic control and in the theory differential games:

$$F(x, D^2 u) = \sup_{\beta \in \mathcal{B}} \inf_{\gamma \in \mathcal{A}} (L_{\gamma\beta} u(x) - f_{\gamma\beta}(x)) = 0, \quad (5.3)$$

where $f_{\gamma\beta} \colon B_1 \to \mathbb{R}$ are Hölder continuous and $L_{\gamma\beta} u = a^{ij}_{\gamma\beta}(x)\partial_{ij} u$ is a family of elliptic operators with Hölder continuous coefficients and ellipticity constants λ and Λ satisfying $1 - \frac{\lambda}{\Lambda} < \varepsilon_0$, where ε_0 is given by Theorem 5.1.

It states that if the operator F is near the "Laplace operator", and u is a normalized viscosity solution to $F(D^2 u) = 0$, then we can find a harmonic function close to u in a inner sub-domain.

Lemma 5.2 (Approximation Lemma). *Let $u \in C^0(B_1)$ be a viscosity solution for (5.1) with $|u| \leq 1$. Given $\alpha \in (0,1)$ there exists universal constants $\varepsilon_0 > 0$ and $0 < \rho < 1$ such that if*

$$1 - \frac{\lambda}{\Lambda} < \varepsilon_0, \quad (5.4)$$

then we can find a quadratic polynomial

$$\mathfrak{p}(x) = \frac{1}{2} x^T \cdot \mathfrak{a} \cdot x + \mathfrak{b} \cdot x + \mathfrak{c}$$

satisfying

$$F(D^2 \mathfrak{p}) = 0 \quad (5.5)$$

$$|\mathfrak{a}| + |\mathfrak{b}| + |\mathfrak{c}| \leq \mathfrak{C}(n, \lambda, \Lambda) \quad (5.6)$$

such that

$$\sup_{B_\rho} |u(x) - \mathfrak{p}(x)| \leq \rho^{2+\alpha}.$$

Demonstração. Let us suppose, for the sake of contradiction that the Lemma is not valid.

That means that there exists a sequences, F_k, λ_k, Λ_k and u_k, satisfying

$$F_k(M) \text{ is } (\lambda_k, \Lambda_k)\text{-elliptic}, \tag{5.7}$$

$$\mathfrak{e}_k := 1 - \frac{\lambda_k}{\Lambda_k} = \mathrm{o}(1), \tag{5.8}$$

$$\|u_k\|_{L^\infty(B_1)} \leq 1 \text{ and } F_k(D^2 u_k) = 0, \tag{5.9}$$

in the viscosity sense, however

$$\sup_{B_{1/2}} |u_k(x) - \mathfrak{p}(x)| > \rho_0^{2+\alpha} \tag{5.10}$$

for some $0 < \rho_0 < 1$ which will be determined *a posteriori* and all quadratic polynomials $\mathfrak{p}$ that satisfies

$$F_k(D^2\mathfrak{p}) = 0.$$

The proof will be divided in three cases:
Case 1. There exist constants λ_0 and Λ_0 such that

$$0 < \lambda_0 \leq \lambda_k \leq \Lambda_k \leq \Lambda_0 < \infty; \tag{5.11}$$

In this case, all F_k are $(\lambda_0, \Lambda_0)-$elliptic; thus from Krylov-Safonov and Caffarelli Hölder regularity, up to a subsequence, we can assume $u_k \to u_\infty$ local uniformly. Also, up to a subsequence, $\lambda_k \to \lambda_\infty$ and $\Lambda_k \to \Lambda_\infty$. However, from (5.8), we have the equality

$$\lambda_\infty = \Lambda_\infty = \mu_0. \tag{5.12}$$

Moreover, by ellipticity, passing once more to a subsequence, if necessary, $F_k \to F_\infty$ locally uniformly in $\mathrm{Sym}(n)$. Clearly, from (5.12), we have

$$F_\infty(M) = \mu_0.\mathrm{tr}(M),$$

By stability of viscosity solutions, we conclude

$$\Delta u_\infty = 0 \quad \text{in } B_{\frac{1}{2}}.$$

As u_∞ is smooth, we define

$$\mathfrak{P}(x) := \frac{1}{2} x^T \cdot D^2 u_\infty(0) \cdot x + Du_\infty(0) \cdot x + u_\infty(0).$$

Since $\|u_\infty\| \leq 1$, it follows from C^3 estimates on u_∞ that

$$\sup_{B_r} |u_\infty(x) - \mathfrak{P}(x)| \leq \mathfrak{c}_0 r^3,$$

for a constant $\mathfrak{c}_0 = \mathfrak{c}_0(n, \lambda_0, \Lambda_0)$. Thus, if we select

$$\rho_0 := \sqrt[1-\alpha]{\frac{1}{4c_0}},$$

a choice that depends only on n, λ, Λ and α, we have

$$\sup_{B_{\rho_0}} |u_\infty(x) - \mathfrak{P}(x)| \leq \frac{1}{4}\rho_0^{2+\alpha}.$$

Now, since F_k are (λ_k, Λ_k)-elliptic and $F_k(0) = 0$ and $\Delta\mathfrak{P} = 0$, it is possible to find a sequence of real number $(\delta_k) \subset \mathbb{R}$ with $\delta_k = o(\mathfrak{e}_k)$, for which the quadratic polynomial

$$\mathfrak{P}_k(x) := \mathfrak{P}(x) + \frac{1}{2}\delta_k|x|^2 \quad \text{satisfies} \quad F_k(D^2\mathfrak{P}_k) = 0.$$

Finally we have, for any point in B_{ρ_0} and $k \gg 1$,

$$\begin{aligned} |u_k(x) - \mathfrak{P}_k(x)| &\leq |u_k(x) - u_\infty(x)| + |u_\infty(x) - \mathfrak{P}(x)| + |\mathfrak{P}_k(x) - \mathfrak{P}(x)| \\ &\leq \frac{1}{4}\rho_0^{2+\alpha} + \frac{1}{4}\rho_0^{2+\alpha} + \frac{1}{2}\delta_k \\ &\leq \rho_0^{2+\alpha}, \end{aligned}$$

for $k \gg 1$, which contradicts (5.10).

Case 2. The sequence $\lambda_k \to 0$.

In this case we define $\mathcal{G}_k := \frac{1}{\lambda_k} \cdot F_k$. Clearly $\mathcal{G}_k$ is $(1, \Lambda_k/\lambda_k)$-elliptic. We repeat the same proof as in the case 1.

Case 3. The sequence $\Lambda_k \to \infty$.

In this case, we define $\mathcal{G}_k := \frac{1}{\Lambda_k} \cdot F_k$. It is easily verified that $\mathcal{G}_k$ is $(\lambda_k/\Lambda_k\, 1)$-elliptic and we return to case 1. □

In the sequel, we shall iterate Lemma 5.2 in appropriate dyadic balls as to obtain the precise oscillation decay of the difference between u and a convergent sequence of quadratic polynomials $\mathfrak{p}_k$.

Lemma 5.3 (Iterative process). *Under the hypotheses of Lemma* (5.2), *we can find a sequence of quadratic polynomials*

$$\mathfrak{p}_k(x) = \frac{1}{2}x^T \cdot \mathfrak{a}_k \cdot x + \mathfrak{b}_k \cdot x + \mathfrak{c}_k$$

satisfying

$$F(\mathfrak{a}_k) = 0 \tag{5.13}$$
$$|\mathfrak{a}_k| + |\mathfrak{b}_k| + |\mathfrak{c}_k| \leq \mathfrak{C}(n, \lambda, \Lambda) \tag{5.14}$$
$$\rho_0^{2k}|\mathfrak{a}_{k+1} - \mathfrak{a}_k| + \rho_0^k|\mathfrak{b}_{k+1} - \mathfrak{b}_k| + |\mathfrak{c}_{k+1} - \mathfrak{c}_k| \leq \mathfrak{C}\rho^{k(2+\alpha)} \tag{5.15}$$

such that

$$\sup_{B_{\rho^k}} |u(x) - \mathfrak{p}_k(x)| \leq \rho^{(2+\alpha)k}. \tag{5.16}$$

Demonstração. The proof is given by induction process. The case $k = 1$ is precisely the statement of Lemma 5.4. Suppose now we have verified the k^{th} step of induction, i.e., by there exists a quadratic polynomial $\mathfrak{p}_k$ satisfying, (5.13), (5.14), (5.15) and (5.16). We define,

$$F_k(M) := \frac{F(\rho^{\alpha k}M + \mathfrak{a}_k)}{\rho^{\alpha k}} \quad \text{and} \quad v_k(x) := \frac{(u - \mathfrak{p}_k)(\rho^k x)}{\rho^{(2+\alpha)k}}.$$

By the induction hypothesis, $|v_k| \leq 1$. It is easy to verify that F_k is (λ, Λ)-elliptic and

$$F_k(D^2 v_k) = 0 \quad \text{in} \quad B_1.$$

in the viscosity sense, as well as $F_k(0) = 0$. Thus, we can apply Lemma (5.3) to v_k and obtain a quadratic polynomial

$$\widetilde{\mathfrak{p}}_k(x) = \frac{1}{2}x^T \cdot \widetilde{\mathfrak{a}}_k \cdot x + \widetilde{\mathfrak{b}}_k \cdot x + \widetilde{\mathfrak{c}}_k \tag{5.17}$$
$$F_k(D^2\widetilde{\mathfrak{p}}_k) = 0 \tag{5.18}$$

with $|\widetilde{\mathfrak{a}}_k| + |\widetilde{\mathfrak{b}}_k| + |\widetilde{\mathfrak{c}}_k| \leq \mathfrak{C}$, for which

$$\sup_{B_{\rho_0}} |v_k(x) - \widetilde{\mathfrak{p}}_k(x)| \leq \rho_0^{2+\alpha}. \tag{5.19}$$

We define,

$$\mathfrak{p}_{k+1}(x) := \mathfrak{p}_k(x) + \rho_0^{k(2+\alpha)}\widetilde{\mathfrak{p}}_k(\rho_0^{-k}x)$$

$$\mathfrak{a}_{k+1} := \mathfrak{a}_k + \rho^{k\alpha}\widetilde{\mathfrak{a}}_k, \quad \mathfrak{b}_{k+1} := \mathfrak{b}_k + \rho^{k(1+\alpha)}\widetilde{\mathfrak{b}}_k \quad \text{and} \quad \mathfrak{c}_{k+1} := \mathfrak{c}_k + \rho^{k(2+\alpha)}\widetilde{\mathfrak{c}}_k. \tag{5.20}$$

Re-scaling (5.19) back with

$$\mathfrak{p}_{k+1}(x) := \frac{1}{2}x^T\mathfrak{a}_{k+1} \cdot x + \mathfrak{b}_{k+1} \cdot x + \mathfrak{c}_{k+1}$$

such that,

$$\sup_{B_{\rho^{k+1}}} |u(x) - \mathfrak{p}_{k+1}(x)| \leq \rho^{(k+1)(2+\alpha)}$$

and by (5.18) we have that $F(\mathfrak{a}_{k+1}) = 0$ and the proof of the Lemma is complete. □

From the estimate obtained in Lemma 5.3, the sequence $\mathfrak{p}_k$ converges to a quadratic polynomial

$$\mathfrak{p}_\infty(x) = \frac{1}{2}x^T \cdot \mathfrak{a}_\infty \cdot x + \mathfrak{b}_\infty \cdot x + \mathfrak{c}_\infty.$$

In fact, from (5.15), there exists a universal constant $\mathfrak{C} > 0$ such that

$$|\mathfrak{a}_{k+1} - \mathfrak{a}_k| \leq \mathfrak{C}\rho^{k\alpha}, \quad |\mathfrak{b}_{k+1} - \mathfrak{b}_k| \leq \mathfrak{C}\rho^{k(1+\alpha)} \quad \text{and} \quad |\mathfrak{c}_{k+1} - \mathfrak{c}_k| \leq \mathfrak{C}\rho^{k(2+\alpha)}. \qquad (5.21)$$

Thus, these are Cauchy sequences and hence

$$\mathfrak{a}_\infty = \lim_{k\to\infty} \mathfrak{a}_k, \quad \mathfrak{b}_\infty = \lim_{k\to\infty} \mathfrak{b}_k, \quad \mathfrak{c}_\infty = \lim_{k\to\infty} \mathfrak{c}_k. \qquad (5.22)$$

In addition we obtain the following controls:

$$|\mathfrak{c}_\infty - \mathfrak{c}_k| \leq \frac{\mathfrak{C}\rho^{k(2+\alpha)}}{1-\rho^{2+\alpha}}, \quad |\mathfrak{b}_\infty - \mathfrak{b}_k| \leq \frac{\mathfrak{C}\rho^{k(1+\alpha)}}{1-\rho^{1+\alpha}} \quad \text{and} \quad |\mathfrak{a}_\infty - \mathfrak{a}_k| \leq \frac{\mathfrak{C}\rho^{k\alpha}}{1-\rho^{\alpha}}. \qquad (5.23)$$

Finally, given $0 < r < \rho$, let k be the an integer such that $\rho_0^{k+1} < r \leq \rho^k$. We estimate

$$\begin{aligned}
\sup_{B_r} |u(x) - \mathfrak{p}_\infty(x)| &\leq \sup_{B_{\rho^k}} |u(x) - \mathfrak{p}_\infty(x)| \\
&\leq \sup_{B_{\rho^k}} |u(x) - \mathfrak{p}_k(x)| + \sup_{B_{\rho^k}} |\mathfrak{p}_k(x) - \mathfrak{p}_\infty(x)| \\
&\leq \mathfrak{C}\rho^{k(2+\alpha)} \\
&\leq \mathfrak{C}_0 r^{2+\alpha}.
\end{aligned}$$

and the proof of Theorem 5.1 is complete.

6 CONCLUSION

The results of the Chapter 2 were obtained in a joint work with my PhD. Advisor, Eduardo V. Teixeira, and they were submitted for publication in DA SILVA and TEIXEIRA (2017). This article classifies the universal moduli of continuity of viscosity solutions to certain class of fully nonlinear parabolic equations in terms of the weak integrability properties of the forcing term (the source f), as well as available *a priori* estimates for the associated homogeneous problem with "frozen" coefficients.

We remind the structure of our problem: $u \in C^0(Q_1)$ is a viscosity solution to

$$\frac{\partial u}{\partial t} - F(x, t, D^2 u) = f(x, t) \quad \text{in} \quad Q_1$$

with F a uniformly elliptic operator and $f \in L^{p,q}(Q_1)$. Moreover,

$$0 \leq \frac{n}{p} + \frac{2}{q} < 2.$$

We must highlight that all the regularity estimates obtained in such a chapter are sharp. Such conclusions can be made through examples or by scaling properties of the solutions/equations.

Apparently just three cases were not analysed in a precise way:

1. When $2 - \left(\frac{n}{p} + \frac{2}{q}\right) = \alpha_0$.
   In this context $0 < \alpha_0 < 1$ is the exponent of Hölder continuity comes from Krylov-Safonov's Harnack inequality. In this setting we conjecture that viscosity solutions have $\omega(s) = s^{\alpha_0} \log s^{-1}$ as universal modulus of continuity. Moreover, we should have the following estimate

$$|u(x.t) - u(0,0)| \leq -C(\|u\|_{L^\infty(Q_1)} + \|f\|_{L^{p,q}(Q_1)})\omega(d((x,t),(0,0))).$$

   This should be a quantitative improvement of the fact that solutions are $C^{\mu,\frac{\mu}{2}}_{loc}(Q_1)$ for all $\mu < \alpha_0$.
2. When $\frac{n}{p} + \frac{2}{q} = 2$.
   In this case, there is a lack of compactness to viscosity solutions due to fact that the Harnack inequality fails in such a scenery. Indeed, according to the previous case, i.e., when $1 < \frac{n}{p} + \frac{2}{q} < 2$, solutions are $C^{\alpha,\frac{\alpha}{2}}_{loc}(Q_1)$ with $\alpha \approx 2 - \left(\frac{n}{p} + \frac{2}{q}\right)$. Therefore, we lose the notion of L^p-viscosity solutions, as well as the Aleksandrov-Bakelman-Pucci-Krylov-Tso Maximum principle, the Harnack inequality and consequently the universal Hölder continuity of solutions as $\frac{n}{p} + \frac{2}{q} \to 2^-$. Therefore, for such problems we should define solutions in a very weak sense. The latter is being handled in a joint project with Disson S. dos Prazeres entitled *"Regularity theory for very weak*

solutions to fully nonlinear equations". Therefore, we conjecture that very weak solutions are BMO (Bounded Mean Oscillation) functions. Moreover, we should have the following estimate

$$\sup_{0<r\leq 1} \fint_{Q_r} \left| u - \fint_{Q_r} u \right| \leq \left(\|u\|_{L^{\infty}(Q_1)} + \|f\|_{L^{p,q}(Q_1)} \right).$$

3. When $1 - \left(\frac{n}{p} + \frac{2}{q}\right) = \alpha_{\text{Hom}}$
In this scenery $0 < \alpha_{\text{Hom}} < 1$ is the Hölder gradient, *a priori* estimates available for homogeneous equation with frozen coefficients, namely $C^{1+\alpha_{\text{Hom}}, \frac{1+\alpha_{\text{Hom}}}{2}}$. As previously, we conjecture that viscosity solutions have $\omega_1(s) = s^{1+\alpha_{\text{Hom}}} \log s^{-1}$ as universal modulus of continuity. Moreover, we must obtain the following estimate

$$\sup_{Q_s} |u(x,t) - [u(0,0) + Du(0,0)]| \leq C \left(\|u\|_{L^{\infty}(Q_1)} + \|f\|_{L^{p,q}(Q_1)} \right) \omega_1(s).$$

All the results in the Chapter 3 are part of the paper DA SILVA and DOS PRAZERES (2019) for which we developed a Schauder type estimate for non-convex fully nonlinear parabolic equations as follows

$$\frac{\partial u}{\partial t} - F(x,t,D^2u) = f(x,t) \quad \text{in} \quad Q_1$$

with appropriate assumption on F, as well as Dini continuity condition on source, medium and smallest regime on u, i.e., $\|u\| \ll 1$ is small enough (flat) for a constant that depends only on the data of the problem.

We can assure our regularity estimates are optimal, and, in particular generalize the Classical Schauder type estimates, see ZOU and CHEN (2002), WANG (1992b), TIAN and WANG (2012) and WANG (2006), when the modulus of continuity of source is a Hölder continuous function.

Regarding the degree of flatness on u, i.e., a plausible estimate for this quantity, is a hard open problem yet. In fact, there is no exist an estimate for showing it, since such a factor arises from a compactness argument.

Concerning to partial regularity, Theorem 3.16, we expect to improve it for sources not necessarily being Lipschitz function. Moreover, this assumption can not be removed by technical reasons. However, with a modern machinery from "calculus without derivatives" or geometric measure theory, for example, we expect to obtain a general result under Dini assumption on f. Another open problem in this direction would be to determine precisely the quantity ε. In ARMSTRONG, SILVESTRE, and SMART (2012) was given a quote for such a quantity in the elliptic case.

The Lipschitz logarithmic estimates that we obtained are optimal. They also are in accordance with orevious results obtained in (DA SILVA and TEIXEIRA, 2017, Section 6).

As another applications for the conception of flatness of solutions, we could develop a result in the context of unique continuation.

The results of the Chapter 4 were a joint work with Gleydson C. Ricarte. Such results can be found in RICARTE and DA SILVA (2015). In this paper we study regularity up to the boundary for fully nonlinear problems associated to high activation models, in other words, viscosity solution to singularly perturbed equations of the form

$$\begin{cases} F(x, Du^{\varepsilon}, D^2u^{\epsilon}) & = & \zeta_{\epsilon}(u^{\epsilon}) & \text{in} & \Omega \\ u^{\epsilon} & = & g & \text{on} & \partial\Omega, \end{cases}$$

under suitable assumptions on the domain Ω and the boundary data g.

It is worth pointing out that the obtained gradient regularity estimate is optimal, since we should not expect better than Lipschitz regularity for solutions (the gradient in general is discontinuous across the free boundary), see MOREIRA and WANG (2014a), RICARTE and TEIXEIRA (2011) and TEIXEIRA (2015) for the local approach to this subject.

In future works we expect to prove the following geometric properties for this problem

1. Linear Growth.
2. Non-degeneracy.
3. Hausdorff estimates.

   The local estimates for Linear Growth and Non-degeneracy for example, intrinsically depend on the distance of a compact subset up to the border of the corresponding set. Moreover, these estimates blow up when this distance goes to zero, see MOREIRA and WANG (2014b) and MOREIRA and WANG (2014a) for details.
4. Free boundary condition.

   The corresponding free boundary condition for this problem is an open problem, because the gradient dependence imposes an extra difficult in the "linearization process of operator". Therefore, we conjecture the following result

   Theorem 6.1 (Free boundary condition). *Let $x_0 \in F(u_0) := \Omega \cap \partial\{u_0 > 0\}$ be a regular point and ν the corresponding inward normal to $F(u_0)$ in measure theoretic sense at x_0, then*

$$u_0(x) = \sqrt{\frac{2\int_{\mathbb{R}}\zeta}{F^*(x_0, \nu, \nu\otimes\nu)}}\langle x - x_0, \nu\rangle^+ + o(|x - x_0|),$$

   where F^* is the Recession operator associated to F, see RICARTE and TEIXEIRA (2011) for more details.

The results from Section 5 are part of a Lecture Notes joint with Damião G. Araújo and Gleydson C. Ricarte, ARAÚJO, RICARTE, and DA SILVA (2013). It provides *a priori* $C^{2,\alpha}$ estimates for fully nonlinear elliptic equations

$$F(D^2u) = 0 \quad \text{in} \quad B_1. \tag{6.1}$$

under a small ellipticity aperture assumption, i.e., $\mathfrak{e} := 1 - \frac{\lambda}{\Lambda}$ is small enough.

This final part of the thesis consists of exemplifying the technique of *Geometric Tangential Analysis* (in short GTA) whose core was inspirited in the seminal Caffarelli's work CAFFARELLI (1989) (see also CAFFARELLI and CABRÉ (1995)) and has been systematically remodeled by Eduardo V. Teixeira (cf. DA SILVA and TEIXEIRA (2017); DOS PRAZERES and TEIXEIRA (2016); TEIXEIRA (2006, 2014, 2015) and TEIXEIRA and URBANO (2014) for enlightening articles about this subject) and consists of using compactness and stability methods to approximate viscosity solutions by solutions of operators which regularity we know *a priori*.

In effect, we can think such a result as a "Cordes-Nirenberg type estimates" for second order fully nonlinear equations. Moreover, we have proved just a qualitative result in the sense that we can not say explicitly how small the universal quantity $\mathfrak{e} := 1 - \frac{\lambda}{\Lambda}$ must be. For this very reason, there are some questions which we can not answer yet.

1. Is it possible to find explicitly the universal constant $\mathfrak{e}$? We already know that in low dimensions solutions to (6.1) are classical (cf. NADIRASHIVILI and VLĂDUŢ (2007), NADIRASHIVILI and VLĂDUŢ (2008), NADIRASHIVILI and VLĂDUŢ (2011) and NADIRASHIVILI and VLĂDUŢ (2013)).
2. There is a critical $\epsilon^* > 0$ such that for all $\mathfrak{e} > \epsilon^*$ solutions to (6.1) are not classical due to the counterexamples due to Nadirashvili and Vlăduţ NADIRASHIVILI and VLĂDUŢ (2007), NADIRASHIVILI and VLĂDUŢ (2008), NADIRASHIVILI and VLĂDUŢ (2011) and NADIRASHIVILI and VLĂDUŢ (2013). Can we determinate precisely such an ϵ^*?
3. What can we expected by iterating the process in the *a priori* $C^{2,\alpha}$ estimates? Can we guess an upper bound for the $C^{1,\alpha}$ estimate?

The key point for answering these questions can to be in an appropriate version of the Maximum Principle or in analysing the remarkable examples due to Nadirashvili and Vlăduţ NADIRASHIVILI and VLĂDUŢ (2007), NADIRASHIVILI and VLĂDUŢ (2008), NADIRASHIVILI and VLĂDUŢ (2011) and NADIRASHIVILI and VLĂDUŢ (2013). There are few results concerning quantitative feature in fully non-linear theory. This is due the fact that many proofs require a compactness argument which yields abstract universal constants.

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