An Analytical Closed-Form Lower-Bound on Ergodic Capacity of Correlated Rayleigh-Fading MIMO Channels

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***Abstract*—In this paper, the ergodic capacity of multiple antenna systems over spatially correlated Rayleigh-fading channels is investigated under the assumption that the channel state information (CSI) is unknown at the transmitter and perfectly known at the receiver. We derive a lower-bound expression, in closed form, for the ergodic capacity through the use of majorization theory and the probability density function (PDF) of the sum of Gamma random variables, which is represented by an infinite series. Furthermore, we also obtain other lower-bounds from the truncation of such series, and we associate truncation errors. Finally, the proposal of the paper is compared with a lower-bound reported in the literature.**

I. Introduction

In recent years, the ergodic capacity of single-user multiple-input multiple-output (MIMO) communications over flat-fading wireless channels have been exhaustively explored under different fading conditions and distinct types of spatial correlation [1]–[10]. However, to obtain analytical closed-form expressions for the ergodic capacity, especially in correlated MIMO fading channels, is still a great challenge due to difficulty in manipulating the non-Gaussian joint channel statistics. Thus, one resorts to bounding techniques, whose bounds (lower or upper) should be as close as possible to the empirical ergodic capacity obtained through Monte Carlo methods. In particular, Zhong *et al.* [9], [10], by virtue of some results of majorization theory [11], have obtained upper and lower capacity bounds for Nakagami-m MIMO fading channels.

Rayleigh distribution is a fading model, which is frequently used to model the short-term behaviour of mobile-radio signals [12]. In other words, the envelope of the received complex low-pass signal can be modeled as a random variable with a Rayleigh distribution for non-light-of-sight (NLoS) propagation. Several works, operating in Rayleigh-fading MIMO systems, have been published about analytical closed-form expressions for lower-bounds to the ergodic capacity. In [3], a lower-bound for independent and identically distributed (i.i.d.) flat-fading channels was derived, while [4] has been analyzing the frequency-selective fading case. In [6], tight upper and lower bounds on the ergodic capacity for spatially correlated channels were provided. The spatial double-sided correlation with keyhole has been examined in [5]. Newly, tight bounds for spatially correlated Rician MIMO channels were proposed by [7], [8] at any signal-to-noise ratio (SNR) and for any number of receive and transmit antennas. Moreover, these references devote a significant part to the study of the Rayleigh-fading channels as a particular case.

In this paper, capitalizing on the technique of [9], we analyze the ergodic capacity of spatially correlated Rayleigh-fading MIMO channels based on the Unitary-Independent-Unitary (UIU) formalism [13], [14]. Specifically, through the use of majorization theory and the distribution of the sum of Gamma random variables [15], we derive an analytical closed-form lower-bound for the ergodic capacity assuming that the channel state information (CSI) is available only at the receiver. It is important to mention that our proposal distinguishes from previous results, on Rayleigh-fading MIMO channels, due to the use of majorization theory on the Unitary-Independent-Unitary formulation applied to Kronecker channel model.

The remaining parts of this paper are organized as follows. Section II presents briefly the Rayleigh and Gamma distributions. Furthermore, we list some results on majorization theory. In Section III, we introduce the Rayleigh-fading MIMO channel model. We derive an analytical lower-bound to the ergodic capacity in Section IV. The theoretical and the simulation results are discussed in Section V. Finally, we conclude the paper in Section VI.

The notations used throughout this paper are as follows. All matrices and vectors will be represented by bold uppercase and lowercase letters, respectively. We use $\mathbf{I}$ or $\mathbf{I}_p$ for the identity matrix of dimension $p \times p$, $\mathbb{C}^{m\times n}$ indicates the $m \times n$ complex vector space and $\mathrm{diag}(\cdot)$ represents a diagonal matrix. The superscripts $(\cdot)^T$ and $(\cdot)^H$ denote the transpose and Hermitian transpose, respectively. The subscript $(\cdot)_i$ is the i-th element of a vector, and $(\cdot)_{ij}$ is the (i, j)-th entry of a matrix. The

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operators $\odot$, $\prec$ and $\mathbb{E}\{\cdot\}$ denote the Schur-Hadamard product, majorization relation, and statistical expectation, respectively. The operator $\det(\cdot)$ stands for the determinant of a square matrix. Finally, the vectors $\mathbf{d}(\cdot)$ and $\boldsymbol{\lambda}(\cdot)$ denote the main diagonal elements and eigenvalues of a Hermitian matrix, respectively.

II. Preliminaries

This section presents the basic notion of majorization theory. There is an extensive list of properties involving the majorization theory, which can be found in the classical reference [11]. However, we selected some important results, which will be used in Section IV. Additionally, in this section, we provide a brief review of the statistical distributions Rayleigh and Gamma, and we present the sum of independent Gamma variables with different parameters.

A. Majorization theory

Definition 1 ([11, 1.A.1]): For any vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^{n\times 1}$, $\mathbf{x}$ is said majorized by $\mathbf{y}$, denoted by, $\mathbf{x} \prec \mathbf{y}$, if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad 1 \leq k \leq n-1 \tag{1}$$

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}. \tag{2}$$

where $x_{[i]}$ and $y_{[i]}$ denote the i-th largest components of $\mathbf{x}$ and $\mathbf{y}$, respectively.

Lemma 1 ([10, Example 2]): Let $\mathbf{x} = (x_1,\ x_2,\ \cdots,\ x_n)$ and $\mathbf{s} = (S,\ 0,\ \cdots,\ 0)$ be vectors in $\mathbb{R}^n$. If $S = \sum_{i=1}^{n} x_i$, then $\mathbf{x} \prec \mathbf{s}$.

Definition 2 ([11, 3.A.1]): A real-valued function $\phi(\cdot)$ on $\mathbb{R}^{n\times 1}$ is said to be Schur-concave if $\phi(\mathbf{x}) \geq \phi(\mathbf{y})$ for any $\mathbf{x} \prec \mathbf{y}$.

Lemma 2 ([11, 3.C.1]): Let $\phi(\cdot)$ be a real-valued function on $\mathbb{R}^{n\times 1}$. If $g : \mathbb{R} \rightarrow \mathbb{R}$ is concave, then $\phi(\cdot)$ defined by $\phi(\mathbf{x}) = \sum_{i=1}^{n} g(x_i)$ is Schur-concave.

Example 1 ([10, Appendix II]): The real-valued function on $\mathbb{R}^{n\times 1}$ $\phi(\cdot)$, defined by $\phi(\mathbf{x}) = \sum_{i=1}^{n} \log_2(1 + \alpha x_i)$, with $\alpha > 0$, is a Schur-concave function.

B. The Rayleigh and Gamma distributions

Definition 3 ([16]): Consider X and Y two independent zero mean Gaussian random variables with same variance σ^2. The envelope $R = \sqrt{X^2 + Y^2}$ is Rayleigh distributed, and the probability density function (PDF) is given by

$$p_R(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) u(r), \tag{3}$$

where $u(\cdot)$ is the unit step function. We will use the short-hand notation $R \sim \text{Rayleigh}\,(\sigma^2)$, to denote that R is Rayleigh distributed with parameter σ^2.

Definition 4 ([16]): A random variable X follows a Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$, denoted by $X \sim \gamma(\alpha, \beta)$, if the PDF of X is given by

$$p_X(x) = \frac{x^{\alpha-1} \exp(-x/\beta)}{\beta^{\alpha}\Gamma(\alpha)} u(x), \tag{4}$$

where $\Gamma(\cdot)$ stands for the gamma function.

Lemma 3: If $X \sim \text{Rayleigh}\,(\sigma^2)$, then the random variable $Y = kX^2$ has a Gamma distribution with parameters $\alpha = 1$ and $\beta = 2k\sigma^2$, i.e., $Y \sim \gamma(1, 2k\sigma^2)$.

Now, we present the distribution of the sum of m independent Gamma variables with parameters α_m and β_m. Proposed by Moschopoulos [15], this result is expressed by an infinite series, as it will be shown in the sequel.

Lemma 4 ([15]): Let $\{X_i\}_{i=1}^{m}$ be a set of m independent Gamma random variables such as $X_m \sim \gamma(\alpha_m, \beta_m)$, then the PDF of $Y = \sum_{i=1}^{m} X_i$ is given by

$$p_Y(y) = \eta \sum_{k=0}^{\infty} \frac{\delta_k\ y^{\mu+k-1} \exp(-y/\beta_*)}{\Gamma(\mu+k)\beta_*^{\mu+k}} u(y), \tag{5}$$

where $\beta_* = \min\limits_{1\leq k\leq m} \{\beta_k\}$, $\eta = \prod_{k=1}^{m} \left(\frac{\beta_*}{\beta_k}\right)^{\alpha_k}$ and $\mu = \sum_{k=1}^{m} \alpha_k$. In addition, the coefficients δ_i can be obtained recursively by

$$\begin{cases} \delta_0 = 1, \\ \delta_{k+1} = \dfrac{1}{k+1} \displaystyle\sum_{l=1}^{k+1} \left[\sum_{j=1}^{m} \alpha_j \left(1 - \frac{\beta_*}{\beta_j}\right)^{l}\right] \delta_{k+1-l} \end{cases} \tag{6}$$

where $k = 0, 1, 2, \cdots$.

III. System Model and Channel Capacity

We focus our study on single-user MIMO communications over flat-fading wireless channels with n_T transmit antennas, and n_R receive antennas. The input-output relationship is given by

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}, \tag{7}$$

where $\mathbf{y} \in \mathbb{C}^{n_R\times 1}$ and $\mathbf{x} \in \mathbb{C}^{n_T\times 1}$ are the received and transmitted signal vectors, respectively, while $\mathbf{n} \in \mathbb{C}^{n_R\times 1}$ is the complex additive white Gaussian noise (AWGN) vector with zero mean and covariance matrix $\mathbb{E}\left\{\mathbf{n}\mathbf{n}^H\right\} = N_0\mathbf{I}$. We assume that the transmitted signal vector satisfies the power constraint $\mathbb{E}\left\{\mathbf{x}^H\mathbf{x}\right\} \leq P_T$. In addition, $\mathbf{H} \in \mathbb{C}^{n_R\times n_T}$ is the MIMO channel matrix, whose elements h_{ij} represent the complex fading parameter between the j-th transmit and i-th receive antenna.

The channel gain is considered to undergo Rayleigh-fading with spatial correlation occurring at both ends of the MIMO link, and we also assume that the channel matrix $\mathbf{H}$ is modeled according to the Kronecker model [17] to describe the correlation between the elements. Thus

$$\mathbf{H} = \mathbf{R}_{R_x}^{1/2} \mathbf{H}_w (\mathbf{R}_{T_x}^{1/2})^H, \tag{8}$$

where the entries of the matrix $\mathbf{H}_w = [\widetilde{h}_{ij}]$ are i.i.d. complex Gaussian random variables with zero mean and unit variance. The matrices $\mathbf{R}_{R_x} \in \mathbb{C}^{n_R\times n_R}$ and $\mathbf{R}_{T_x} \in \mathbb{C}^{n_T\times n_T}$ denote the receive and transmit correlation matrices and their respective eigen-decomposition can be given by

$$\mathbf{R}_{R_x} = \mathbf{U}_{R_x}\boldsymbol{\Lambda}_{R_x}\mathbf{U}_{R_x}^H \quad \text{and} \quad \mathbf{R}_{T_x} = \mathbf{U}_{T_x}\boldsymbol{\Lambda}_{T_x}\mathbf{U}_{T_x}^H, \tag{9}$$

where $\mathbf{U}_{R_x}$ and $\mathbf{U}_{T_x}$ are deterministic unitary matrices. In turn, $\mathbf{\Lambda}_{R_x} \triangleq \operatorname{diag}(\boldsymbol{\lambda}_{R_x})$ and $\mathbf{\Lambda}_{T_x} \triangleq \operatorname{diag}(\boldsymbol{\lambda}_{T_x})$ are diagonal matrices containing the non-zero eigenvalues of $\mathbf{R}_{R_x}$ and $\mathbf{R}_{T_x}$, respectively, where

$$\boldsymbol{\lambda}_{R_x} \triangleq \begin{bmatrix} \lambda_1^{R_x} \\ \lambda_2^{R_x} \\ \vdots \\ \lambda_{n_R}^{R_x} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\lambda}_{T_x} \triangleq \begin{bmatrix} \lambda_1^{T_x} \\ \lambda_2^{T_x} \\ \vdots \\ \lambda_{n_T}^{T_x} \end{bmatrix}. \tag{10}$$

A. Unitary-Independent-Unitary model

Substituting the eigenvalues decomposition for $\mathbf{R}_{R_x}$ and $\mathbf{R}_{T_x}$ (see Eq. (9)) in Eq. (8), can to express the MIMO channel matrix $\mathbf{H}$ using the UIU-model [13], [14], [18] as follows

$$\mathbf{H} = \mathbf{U}_{R_x}(\mathbf{G} \odot \mathbf{H}_w)\mathbf{U}_{T_x}^H, \tag{11}$$

where $\mathbf{G}$ is a given (deterministic) matrix coupling matrix, and the operator $\odot$ is the element-wise Schur-Hadamard multiplication. Therefore, for the Kronecker channel model described in Eq. (8), the coupling matrix $\mathbf{G}$ is given by

$$\mathbf{G} = \boldsymbol{\lambda}_{R_x}^{1/2}\left(\boldsymbol{\lambda}_{T_x}^{1/2}\right)^T, \tag{12}$$

where the vectors $\boldsymbol{\lambda}_{R_x}^{1/2}$ and $\boldsymbol{\lambda}_{T_x}^{1/2}$ are the element-wise square root of $\boldsymbol{\lambda}_{R_x}$ and $\boldsymbol{\lambda}_{T_x}$, respectively.

B. MIMO Channel Capacity

In the sequel, we consider that the receiver has perfect channel state information (CSI), and an equal-power allocation across the transmit antennas. In this situation, the ergodic capacity can be expressed as [8]

$$\overline{C} = \mathbb{E}\left\{\log_2\left[\det\left(\mathbf{I} + \frac{\rho}{n_T}\mathbf{\Gamma}\right)\right]\right\}, \tag{13}$$

where $\rho \triangleq \frac{P_T}{N_0}$ is the received signal-to-noise (SNR) ratio and $\mathbf{\Gamma}$ is the Wishart matrix, which is defined as

$$\mathbf{\Gamma} = \begin{cases} \mathbf{H}\mathbf{H}^H, & n_R \le n_T \\ \mathbf{H}^H\mathbf{H}, & n_R > n_T. \end{cases} \tag{14}$$

Now, we define $r = \min\{n_R,\ n_T\}$ and $t = \max\{n_R,\ n_T\}$. Hence, $\mathbf{\Gamma}$ is always a square matrix of order $r \times r$. Moreover, we also assume that the number of receive antennas does not exceed the number of transmit antennas. Thus, the identity $\det(\mathbf{I} + \mathbf{AB}) = \det(\mathbf{I} + \mathbf{BA})$ ensures that all results on the ergodic capacity can be extended to the case $n_R > n_T$. Finally, based on these assumptions, we conclude that the MIMO channel capacity in Eq. (13) can be expressed as

$$\overline{C} = \mathbb{E}\left\{\log_2\left[\det\left(\mathbf{I}_{n_R} + \frac{\rho}{n_T}(\mathbf{G} \odot \mathbf{H}_w)(\mathbf{G} \odot \mathbf{H}_w)^H\right)\right]\right\} \tag{15}$$

IV. Lower-Bound on Ergodic Capacity

In this section, we derive a closed-form expression, in terms of the Meijer G-function [19, Eq. (9.301)], for the lower-bound on the ergodic capacity using some results of majorization theory and the distribution of the sum of independent Gamma random variables. The details are presented in the following theorem.

Theorem 1: The ergodic capacity of spatially correlated Rayleigh MIMO channels is lower bounded by

$$\overline{C} \ge \frac{\eta}{\ln 2}\sum_{k=0}^{\infty}\frac{\delta_k}{\Gamma(rt+k)}G_{3,2}^{1,3}\left(\frac{\rho\beta_*}{n_T}\Big|_{1,0}^{1-rt-k,1,1}\right), \tag{16}$$

where ρ is the SNR, the constant $\beta_* = \min\left\{\lambda_i^{R_x}\lambda_j^{T_x}\right\}$ and $G_{p,q}^{m,n}(\cdot\,|\,\cdot)$ is the Meijer G-function. Moreover, the constant η is given by

$$\eta = \prod_{i=1}^{r}\prod_{j=1}^{t}\frac{\beta_*}{\lambda_i^{R_x}\lambda_j^{T_x}}, \tag{17}$$

and the coefficients δ_k is obtained recursively by

$$\begin{cases} \delta_0 = 1, \\ \delta_{k+1} = \dfrac{1}{k+1}\displaystyle\sum_{l=1}^{k+1}\delta_{k+1-l}\left[\sum_{i=1}^{r}\sum_{j=1}^{t}\left(1 - \frac{\beta_*}{\lambda_i^{R_x}\lambda_j^{T_x}}\right)^l\right] \end{cases} \tag{18}$$

where $k = 0, 1, 2, \cdots$.

Proof: Firstly, for convenience, we define the matrix $\mathbf{W} \triangleq \mathbf{G} \odot \mathbf{H}_w$ and the following vectors in $\mathbb{R}^{r\times 1}$:

$$\mathbf{d}\left(\mathbf{W}\mathbf{W}^H\right) \triangleq (d_1,\ d_2,\ \cdots,\ d_r), \tag{19a}$$

$$\boldsymbol{\lambda}\left(\mathbf{W}\mathbf{W}^H\right) \triangleq (\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_r) \quad \text{and} \tag{19b}$$

$$\mathbf{\Lambda} \triangleq \left(\sum_{i=1}^{r}\lambda_i,\ 0,\ \cdots,\ 0\right), \tag{19c}$$

where d_i corresponds to the i-th diagonal element of the Hermitian matrix $\mathbf{W}\mathbf{W}^H$, and λ_i represents the respective eigenvalue. Now, let be the real-valued function $\phi(\cdot)$ on $\mathbb{R}^{r\times 1}$ defined by

$$\phi(\mathbf{x}) = \sum_{i=1}^{r}\log_2\left(1 + \frac{\rho}{n_T}x_i\right). \tag{20}$$

From Lemma 1, we have that the vector $\mathbf{\Lambda}$ majorizes $\boldsymbol{\lambda}\left(\mathbf{W}\mathbf{W}^H\right)$, i.e., $\boldsymbol{\lambda}\left(\mathbf{W}\mathbf{W}^H\right) \prec \mathbf{\Lambda}$. Since $\phi(\cdot)$ is a Schur-concave function (see Example 1), we obtain the following numerical inequality: $\phi\left(\boldsymbol{\lambda}\left(\mathbf{W}\mathbf{W}^H\right)\right) \ge \phi(\mathbf{\Lambda})$. Now, applying the expectation operator $\mathbb{E}\{\cdot\}$, in this inequality, and observing that the ergodic capacity presented in Eq. (15) is equal to $\mathbb{E}\left\{\phi\left(\boldsymbol{\lambda}\left(\mathbf{W}\mathbf{W}^H\right)\right)\right\}$, that is,

$$\overline{C} = \mathbb{E}\left\{\sum_{i=1}^{r}\log_2\left(1 + \frac{\rho}{n_T}\lambda_i\right)\right\}, \tag{21}$$

we obtain a lower-bound to the ergodic capacity as shown below:

$$\overline{C} \geq \mathcal{C}_{\text{lo}} = \mathbb{E}\left\{\log_2\left(1+\frac{\rho}{n_T}\sum_{j=i}^{r}\lambda_i\right)\right\}. \tag{22}$$

According to the singular value decomposition (SVD) of the matrix $\mathbf{W}\mathbf{W}^H$, we ensure that $\sum_{i=1}^{r}\lambda_i = \sum_{i=1}^{r} d_i$. Thus, the lower-bound $\mathcal{C}_{\text{lo}}$ can be equivalently written as

$$\mathcal{C}_{\text{lo}} = \frac{1}{\ln 2}\int_0^\infty G_{2,2}^{1,2}\left(\frac{\rho}{n_T}s\Big|_{1,0}^{1,1}\right)p_S(s)\,ds, \tag{23}$$

where $p_S(\cdot)$ is the PDF of the random variable $S \triangleq \sum_{i=1}^{r} d_i$ and $\ln(1+ks)$ is equal to $G_{2,2}^{1,2}\left(ks\Big|_{1,0}^{1,1}\right)$ [10]. Now, note that

$$S = \sum_{i=1}^{r}\sum_{j=1}^{t}\lambda_i^{R_x}\lambda_j^{T_x}|\widetilde{h}_{ij}|^2. \tag{24}$$

In other words, the variable S is a sum of independent Gamma variables. Specifically, $\lambda_i^{R_x}\lambda_j^{T_x}|\widetilde{h}_{ij}|^2 \sim \gamma\left(1,\ \lambda_i^{R_x}\lambda_j^{T_x}\right)$. Thus, based on Lemma 4, the PDF $p_S(\cdot)$ is given by

$$p_S(s) = \eta\sum_{k=0}^{\infty}\frac{\delta_k\ s^{rt+k-1}\exp\left(-s/\beta_*\right)}{\Gamma(rt+j)\beta_*^{rt+j}}u(s), \tag{25}$$

where $\beta_* = \min\left\{\lambda_i^{R_x}\lambda_j^{T_x}\right\}$ and the parameters η and δ_i are described in Eq. (17) and (18), respectively. Now, substituting the PDF $p_S(\cdot)$ into Eq. (23) and using the fact [19, Eq. (7.813-1)], [10]

$$\int_0^\infty x^{-\rho}\exp\left(-\beta x\right)G_{p,q}^{m,n}\left(\alpha x\Big|_{b_1,b_2,\cdots,b_q}^{a_1,a_2,\cdots,a_p}\right)dx = \beta^{\rho-1}G_{p+1,q}^{m,n+1}\left(\frac{\alpha}{\beta}\Big|_{b_1,b_2,\cdots,b_q}^{\rho,a_1,a_2,\cdots,a_p}\right), \tag{26}$$

we conclude, after some algebra, that the lower-bound $\mathcal{C}_{\text{lo}}$ is given by Eq. (16). This completes the proof. ■

Though the lower-bound obtained can be expressed in an analytical closed-form expression, and can be evaluated very efficiently using standard softwares like MAPLE and MATHEMATICA, for practical numerical evaluations, we consider a truncated version of the infinite series in Eq. (25), and we associate it an approximation error of the area under the PDF $p_S(\cdot)$. Specifically, we define the truncated version of the infinite series in Eq. (25), with an arbitrary truncation parameter L, as follows:

$$p_S(s,L) = \eta\sum_{k=0}^{L}\frac{\delta_k\ s^{rt+k-1}\exp\left(-s/\beta_*\right)}{\Gamma(rt+j)\beta_*^{rt+j}}u(s). \tag{27}$$

Then, repeating the procedures given in Theorem 1, we have the following lower-bound to the ergodic capacity:

$$\mathcal{C}_{\text{lo}}(L) = \frac{\eta}{\ln 2}\sum_{k=0}^{L}\frac{\delta_k}{\Gamma(rt+k)}G_{3,2}^{1,3}\left(\frac{\rho\beta_*}{n_T}\Big|_{1,0}^{1-rt-k,1,1}\right). \tag{28}$$

Now, in order to find a criterion that allows to identify the values of L, which provides a good truncation factor of the infinite series described in (25), we analyze the approximation error of the area under the PDF $p_S(\cdot)$ from the following function $\mathcal{E}(\cdot)$:

$$\mathcal{E}(L) \triangleq \int_0^\infty p_S(s)\,ds - \int_0^\infty p_S(s,L)\,ds. \tag{29}$$

Note that,

$$\begin{aligned}\mathcal{E}(L) &= 1-\eta\sum_{k=0}^{L}\frac{\delta_k}{\Gamma(rt+k)\beta_*^{rt+k}}\\ &\quad\times\int_0^\infty s^{rt+k-1}\exp\left(-s/\beta_*\right)u(s)\,ds.\end{aligned} \tag{30}$$

Applying the integration result [19, Eq. (8.312-2)] in Eq. (30)

$$\int_0^\infty t^{z-1}\exp\left(-kt\right)dt = \frac{\Gamma(z)}{k^z}, \tag{31}$$

we obtain

$$\mathcal{E}(L) = 1-\eta\sum_{k=0}^{L}\delta_k. \tag{32}$$

Hence, we conclude that, if the approximation error $\mathcal{E}(\cdot)$ is sufficiently small, then the area under the curve given by $\mathcal{C}_{\text{lo}}(\cdot)$ is sufficiently close to the area under lower-bound $\mathcal{C}_{\text{lo}}$. Consequently, we obtain a good approximation to the ergodic capacity. The numerical details about truncation factor L and the approximation error $\mathcal{E}(\cdot)$ are described in Section V.

V. NUMERICAL RESULTS

In order to illustrate the theory described in Section IV, we evaluate in this section the ergodic capacity lower-bounds for a number of different cases, assuming the exponential correlation model [8] with receive and transmit correlation coefficients equal to $\delta_{R_x} = 0.3$ and $\delta_{T_x} = 0.5$, respectively. Moreover, we compare our lower-bound with McKay and Collings [7, Section IV] whose reference investigates the Rayleigh-fading channels as a particular case of Rice-fading channels.

Fig. 1 gives the truncation errors of the area under the PDF $p_S(\cdot)$ obtained in Eq. (25). Consequently, the convergence speed of the simulated curves. Thus, from the choice of a specific error factor, we have an effective criterion for determination of the associated truncation factor. Thus, for an error $\mathcal{E} = 1-10^{-3}$, we obtain different truncation factors as summarized in Table I.

TABLE I
TRUNCATION FACTORS FOR DIFFERENT SCENARIOS

Channel	1×1	1×2	1×4	2×2	2×4	3×3
Trunc. factor	0	17	37	38	85	93

Now, in Fig. 2 and Fig. 3, we compare the simulated ergodic capacity (through Monte Carlo methods), and the analytical closed-form expression lower-bound, obtained in Eq. (28), using the truncation factors described in Table I. Fig. 2 depicts, over multiple-input single-output (MISO) systems, that all lower-bounds are equally tight at any SNR values, while for MIMO channels, can be observed, in Fig. 3, that our result is much tighter than specified by [7], in low-SNR regimes.

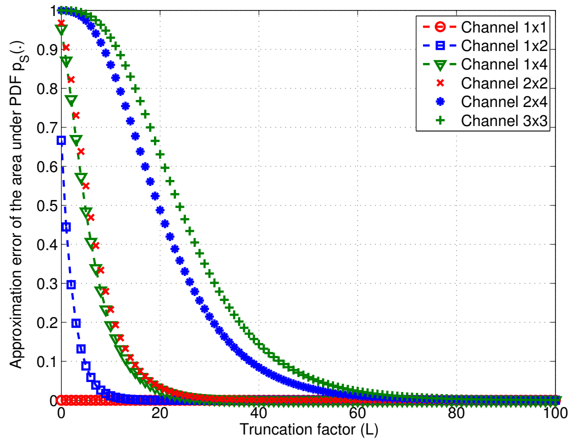

Fig. 1. Approximation error of the area under the PDF $p_S(\cdot)$.

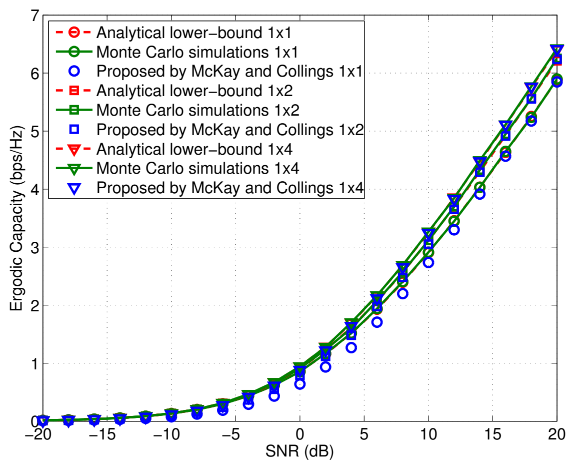

Fig. 2. Comparison of the empirical ergodic capacity and analytical lower-bounds to the ergodic capacity for 1×1, 1×2 and 1×4 correlated Rayleigh-fading channels.

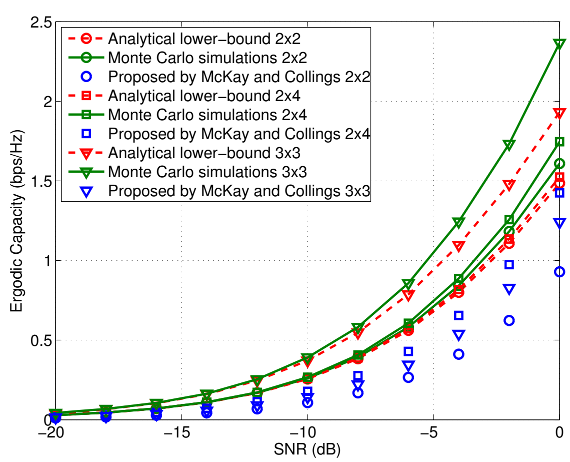

Fig. 3. Comparison of the empirical ergodic capacity and analytical lower-bounds to the ergodic capacity for 2×2, 2×4 and 3×3 correlated Rayleigh-fading channels.

VI. Conclusions

We have investigated the ergodic capacity of MIMO systems operating on spatially correlated Rayleigh-fading channels, assuming the UIU-decomposition. From some results on majorization theory, we have derived an analytical closed-form lower-bound to ergodic capacity. Additionally, we have derived other lower-bounds from the truncated version of the infinite series described by the PDF of the sum of Gamma random variables. In addition, from the analytical lower-bound obtained, we have associated an approximation error in terms of the area under of such PDF. Finally, we have verified that our lower-bound is tighter than previously known analytical lower-bound, in the low-SNRs, while for MISO systems, all results are equally tight.

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