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SOBRE HIPERSUPERFÍCIES COM CURVATURA E BORDO PRESCRITOS EM VARIEDADES RIEMANNIANAS

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Orientador: Prof. Dr. Jorge Herbert S. de Lira.

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[...]
Olha lá, que os bravos são
Escravos sãos e salvos de sofrer
Olha lá, quem acha que perder
É ser menor na vida
Olha lá, quem sempre quer vitória
E perde a glória de chorar
Eu que já não quero mais ser um vencedor
Levo a vida devagar pra não faltar amor
[...]

Trecho da música O Vencedor,
da banda Los Hermanos.

RESUMO

Neste trabalhamos invetigamos a existência de hipersuperfícies com curvatura prescrista num contexto amplo. Inicialmente estudamos o problema de Dirichlet para uma equação totalmente não-linear do tipo curvatura, definida em uma variedade Riemanniana. Este problema está intimamente relacionado a existência de hipersuperfícies com curvatura e bordo prescritos. Neste contexto obtemos alguns resultados que extendem para uma varieadade Riemanniana resultados obtidos anteriormente por Caffarelli, Nirenberg, Spruck e Bo Guan para o espao Euclideano.

Investigamos também a existência de hipersuperfícies com curvatura média anisotrópica prescrita. Estabelecemos a solubilidade do problema de Dirichlet relacionado a equação da curvatura média anisotrópica prescrita. Este resultado assegura a existncia de gráficos de Killing com curvatura média anisotrópica e bordo prescritos numa variedade Riemanniana dotada com um campo de Killing sem singularidades. Finalmente, provamos a existência de hiperesferas com curvatura média anisotrópica prescrita no espaço Euclideano, extendendo o resultado obtido Treibergs e Wei para a curvatura média usual.

ABSTRACT

We investigate the existence of hypersurfaces with prescribed curvature in a wide context. First we study the Dirichlet problem for a class of fully nonlinear elliptic equations of curvature type on a Riemannian manifold, which are closely related with the existence of hypersurfaces with prescribed curvature and boundary. In this setting we prove some existence results which extend to a Riemannian manifold previous results by Caffarelli, Nirenberg, Spruck and Bo Guan for the Euclidean space.

We also study the existence of hypersurfaces with prescribed anisotropic mean curvature. We prove existence results for the Dirichlet problem related to the anisotropic mean curvature equation. This ensures the existence of Killing graphs with prescribed anisotropic mean curvature and boundary in a Riemannian manifold endowed with a nonsingular Killing vector field. Finally, we prove the existence of hyperspheres with prescribed anisotropic mean curvature in the Euclidean space, extending a previous result of Treibergs and Wei.

Contents

Chapter 1

Introduction

The problem of existence of hypersurfaces with prescribed curvature is closely related to the theory of nonlinear elliptic equations of second order. This close relationship is due to the success of the search for such hypersurfaces which are globaly graphs over suitable domains. Consequently, the history of the study of hypersurfaces with prescribed curvature is strongly linked to the historical development of the theory of nonlinear elliptic equations. In fact, as is stated for instance in [17], the prescribed mean curvature equation

$$(1+|Du|^2)\Delta u - u^i u^j u_{ij} = nH(1+|Du|^2)^{3/2}$$

was the prototype which led the development of the theory of quasilinear elliptic equations of second order as well as the Monge-Ampère equation

$$\det\left(u_{ij}\right) = \psi(x,u,Du) > 0,$$

which is closely related to the existence of hypersurfaces with prescribed Gauss-Kroneeker curvature. This guided the study of fully nonlinear elliptic equations of second order. We will describe some of the recent advances obtained in the study of this problem.

Using previously contributions of Bernstein, Leray, Jenkins, Finn and others, J. Serrin [36] discovered necessary and sufficient conditions for the solvability of the Dirichlet problem for the quasilinear prescribed mean curvature equation, which depends on the curvature of the boundary of the underlying domains. The corresponding problem for closed hypersurfaces was studied by Aeppli, Aleksandrov, Bakelman, Kantor, Treibergs and Wei. They were able to establish the existence of hyperspheres with prescribed mean curvature in the Euclidean space, see [41]. The Monge-Ampère equations received attention from eminent mathematicians as Pogorelov, Calabi, Nirenberg, Yau, Trudinger, Urbas, Ivochkina and others. The effort employed by these mathematicians culminated in the results obtained by Caffarelli, Nirenberg and Spruck in [7]. These advances allowed to treat the problem of the existence of hypersurface with prescribed mean or Gauss-Kroneeker curvature and boundary in the Euclidean space.

In the closed case, Oliker [34] established an analog of the result obtained by Bakelman, Treibergs *et al*, for the prescribed Gauss-Kroneeker curvature. In the eighties, it began the study of hypersurfaces with prescribed k-th order mean curvature. In a series of papers, Caffarelli, Nirenberg and Spruck studied the Dirichlet problem and the closed problem for a general class of curvature functions, which includes the higher order mean curvature. Independently, Ivochkina established the solvability of the Dirichlet problem for the equation of prescribed k-th order mean curvature under certain curvature conditions on the boundary of the underlying domains. Later, Trudinger, Li and Ivochkina treated the Dirichlet problem for the prescribed curvature quotient equations which do not belong to the class considered by Caffarelli, Nirenberg and Spruck.

In the last decade, many results that were obtained just in the Euclidean space have been extended to more general enviroments such as the space forms, or more generally, warped product manifolds. The existence results corresponding to the Dirichlet problem associated with the prescribed mean curvature equation was extended for a wide class of enviroments, thanks to the efforts of many mathematicians, such as Alias, Dajczer, Hinojosa, Sa Earp, Barbosa, Lira, Oliker, Spruck and others (see, e.g. [15], [39]). The Monge-Ampère equation on Riemannian manifolds was treated independently by Bo Guan and Yan Yan Li in [20] and Atallah and Zuily in [3]. Recently, the closed problem for general curvature function deserved a lot of research efforts. As a consequence of the works of Lira, Barbosa, Oliker, Yan Yan Li and Jin the existence of closed hypersurfaces with prescribed higher order mean curvature was established in space forms. In [2] it is proved the existence of closed hypersurfaces of prescribed general curvature functions in warped product manifolds.

Recently, the techniques presented in [22] and [44] allowed a great development in the study of hypersurfaces with prescribed Gauss-Kroneeker curvature and boundary since these techniques permit the authors to prove the existence of hypersurfaces which are not necessarily global graphs. Guan and Spruck [23] made use of these techniques to extend the existence result obtained in [22] for general curvature functions defined in the positive cone. We must mention also the contributions of Gerhardt to this kind of problem, see e.g. [16]. In fact, the list of contributions to these kind of problem is hardly exhausted in few lines.

In order to present the results of this thesis, we will do a more technical and detailed description of some of the results quoted above. The most important class of fully nonlinear elliptic equation related to differential geometry are implicitly defined equations of the form

$$F(A) = f\big(\lambda(A)\big) = \Psi,$$

where A, for example, is the second fundamental form of a hypersurface, $f(\lambda)$ is a function of the eigenvalues of A and Ψ is a prescribed function of the position. In the nonparametric setting this equation takes the form

$$F[u] = f(\kappa[u]) = \Psi(x, u), \tag{1.1}$$

where $\kappa[u]$ denotes the principal curvatures of the graph of u.

In the first part of this thesis we study the classical Dirichlet problem for this kind of equations, which is named equations of prescribed curvature type. The ambient space will be a complete smooth Riemannian manifold (M, σ). More precisely, we will consider the classical Dirichlet Problem

$$\begin{aligned} F[u] = f(\kappa[u]) &= \Psi \quad \text{in } \Omega \\ u &= \varphi \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where $\kappa[u] = (\kappa_1, \cdots, \kappa_n)$ denotes the principal curvatures of the graph $\Sigma = \{(x, u(x)), x \in \Omega\}$ of a real function u defined in a bounded domain $\Omega \subset M$, Ψ is a prescribed positive function defined on $\overline{\Omega} \times \mathbb{R}$, φ is a function in $C^{2,\alpha}(\bar{\Omega})$ and f is a smooth symmetric function defined in an open, convex, symmetric cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin and containing the positive cone

$$\Gamma^+ = \{\kappa \in \mathbb{R}^n \,:\, \text{each component}\, \kappa_i > 0\}.$$

As we said above, the first breakthroughs in the solvability of the Dirichlet problem (1.2) for general curvature functions f were due to Caffarelli, Nirenberg and Spruck [11]. Under natural geometric conditions, they proved the solvability of the Dirichlet problem (1.2) corresponding to general curvature functions. Their result covered curvature functions f which satisfies the structure conditions

$$f_i = \frac{\partial f}{\partial \kappa_i} > 0 \tag{1.3}$$

and

$$f \text{ is a concave function.} \tag{1.4}$$

In addition, f is assumed to satisfy the following more technical assumptions

$$\sum f_i(\kappa) \geq c_0 > 0 \tag{1.5}$$

$$\sum f_i(\kappa)\kappa_i \geq c_0 > 0 \tag{1.6}$$

$$\limsup_{\kappa \to \partial\Gamma} f(\kappa) \leq \bar{\Psi}_0 < \Psi_0 \tag{1.7}$$

$$f_i(\kappa) \geq \nu_0 > 0 \quad \text{for any } \kappa \in \Gamma \text{ with } \kappa_i < 0 \tag{1.8}$$

$$(f_1 \cdots f_n)^{1/n} \geq \mu_0 \tag{1.9}$$

for $\kappa \in \Gamma_\Psi = \{\kappa \in \Gamma : \Psi_0 \leq f(\kappa) \leq \Psi_1\}$ and constants c_0 and μ_0 depending on Ψ_0 and Ψ_1, where $\Psi_0 = \inf \Psi$ and $\Psi_1 = \sup \Psi$. In this context, a function $u \in C^2(\Omega)$ is called *admissible* if $\kappa[u] \in \Gamma$ at each point of its graph. The result due to Caffarelli, Nirenberg and Spruck is restricted to strictly convex domains and for constant boundary data. More precisely, they proved the following theorem:

Theorem ([11], Theorem 1)**.** *Let f be a curvature function satisfying (1.3)-(1.8). Assume that*

(i) $\Omega \subset \mathbb{R}^n$ is a domain with smooth strictly convex boundary $\partial\Omega$;
(ii) There is an admissible function $\underline{u}$, such that $\underline{u} = 0$ on $\partial\Omega$ and

$$F[\underline{u}] = f(\kappa[\underline{u}]) \geq \Psi \quad \text{in } \overline{\Omega}; \tag{1.10}$$

(iii) For every $C > 0$ and every compact set E in Γ there is a number $R = R(C, E)$ such that

$$f(\kappa_1, \cdots, \kappa_n + R) \geq C, \quad \forall\, \kappa \in E. \tag{1.11}$$

Then there exists a unique admissible smooth solution u to the Dirichlet problem (1.2) with $\varphi = 0$.

The main example of general curvature functions that satisfies (1.3)-(1.9) are the k-th root of the higher order mean curvature functions

$$S_k(\kappa) = \sum_{\kappa_{i_1}<\ldots<i_k} \kappa_{i_1}\cdots\kappa_{i_k}.$$

Despite the cases $f = \big(S_k\big)^{1/k}$ be covered by the generality of f in the above theorem, the condition (1.11) excluded the $(k-l)$-th root of the quotients $S_{k,l} = S_k/S_l$, $0 \le l < k \le n$.

Using a different approach, [26] Ivochkina solved (1.2) for $f = S_k$ and general boundary values. Her approach also allows to weaken the hypothesis about $\partial\Omega$ from strictly convex to k-convexity, adding a suitable type of Serrin condition. The weak or viscosity solution approach of Trudinger [42] allows such a generality about the curvature function f that the cases $f = S_{k,l}$ were included, establishing existence theorems of *Lipschitz* solutions when the domain is f-convex and satisfies a kind of Serrin condition. We note that a domain Ω with boundary $\partial\Omega \in C^2$ is said to be *f-convex (uniformly f-convex)* if the principal curvatures $\kappa' = (\kappa'_1, \cdots, \kappa'_{n-1})$ of $\partial\Omega$ satisfies

$$(\kappa'_1, \cdots, \kappa'_{n-1}, 0) \in \Gamma. \tag{1.12}$$

In the subsequent articles [31] and [27] Ivochkina, Lin and Trudinger adapted the approach used by Ivochkina to solve the Dirichlet problem (1.2) corresponding to the quotients $S_{k,l}$. Their approach make use of highly specific properties of these functions. As our work extends to a general curvature function a result presented in [31] for the quotients we will include here a brief description of it.

Theorem ([31], Theorem 1.1)**.** *Let $0 \le l < k < n$, $0 < \alpha < 1$. Assume that*

> *(i) Ω is a bounded (k-1)-convex domain in $\mathbb{R}^n$ with boundary $\partial\Omega \in C^{4,\alpha}$;*
> *(ii) $\Psi \in C^{2,\alpha}(\Omega \times \mathbb{R})$, $\Psi > 0$, $\frac{\partial\Psi}{\partial z} \ge 0$ in $\overline{\Omega} \times \mathbb{R}$;*
> *(iii) $\Psi(x, 0) \le S_{k,l}(\kappa')$ on $\partial\Omega$, where κ' are the principal curvatures of $\partial\Omega$.*

Then, provided there exists any bounded admissible viscosity subsolution of equation (1.2) in Ω, there exists a unique admissible solution $u \in C^{4,\alpha}(\overline{\Omega})$ of the Dirichlet problem (1.2) for $f = S_{k,l}$ and $\varphi = 0$.

As we could see on the theorems presented above, conditions on the geometry of the boundary $\partial\Omega$ plays a key role in the study of the solvability of the Dirirchlet problem (1.2). Nevertheless, the replacement of the geometric conditions on the boundary by the more general assumption of the existence of a subsolution satisfying the boundary condition had already been done by several authors. We may mention the results presented in the articles [18] and [21] of Guan and Spruck, where the Monge-Ampère equation is treated. In [8] and [19] is shown the existence of a close relationship between the convexity of the boundary and the existence of such subsolutions. This kind of hypothesis is also used in [23], where Guan and Spruck studied the existence of locally strictly convex hypersurfaces with constant prescribed curvature function. There they obtained the following result.

Theorem ([23], Theorem 1.4)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ and f be a curvature function defined on Γ^+ that satisfies the structure conditions (1.3)-(1.7) and (1.11). Assume that*

(i) there exists a locally convex viscosity subsolution $\underline{u} \in C^{0,1}(\overline{\Omega})$ of the equation (1.2) with $\underline{u} = \varphi$ on $\partial\Omega$ and $\underline{u}$ is C^2 and locally strictly convex (up to the boundary) in a neighborhood of $\partial\Omega$;
(ii) Ψ is a positive smooth function defined on $\overline{\Omega} \times \mathbb{R}$ and satisfies $\frac{\partial\Psi}{\partial z} \geq 0$.

Then there exists a unique locally strictly convex solution $u \in C^\infty(\overline{\Omega})$ of the Dirichlet problem (1.2) satisfying $u \geq \underline{u}$ on $\overline{\Omega}$.

We point out that this result extends the theorem of Caffarelli, Nirenberg and Spruck (the first of those presented above) to non-convex domains and general boundary condition, but just for a restricted class of curvature functions f defined in the positive cone Γ^+ and that must be zero on $\partial\Gamma^+$. In [23] this theorem is used to prove the existence of locally convex hypersurfaces with prescribed curvature function f constant and prescribed boundary. However these hypersurfaces are not necessarily global graphs and their boundary may be very complicated.

Our results may be seen as an extension of some of the results cited above for domains Ω contained in a Riemannian manifold. We point out that for equations of Monge-Ampere type ($f = S_n$) this extension was made by Guan and Lin in [20] and by Atallah and Zuily in [3] while the prescribed mean curvature equation ($f = H$) has been studied in [39], as we mentioned above.

Our first result extends the above result of Guan and Spruck to a general Riemannian manifold and for a class of curvature function larger than the one considered in [23] and that is defined in a general cone Γ, not necessarily being the positive cone Γ^+. However, we assume some natural conditions on M, which are naturally satisfied by the Euclidean space.

Theorem 1.1. *Let M be a complete orientable Riemannian manifold with nonnegative Ricci curvature and f be a curvature function that satisfies (1.3)-(1.9). Assume that*

(i) Ω is a bounded domain in M and $\partial\Omega$ has nonnegative mean curvature with respect to inward orientation;
(ii) there exists a viscosity subsolution $\underline{u} \in C^{0,1}(\overline{\Omega})$ of the equation (1.2) with $\underline{u} = \varphi$ on $\partial\Omega$ and $\underline{u}$ is C^2 and locally strictly convex (up to the boundary) in a neighborhood of $\partial\Omega$;
(iii) Ψ is a positive smooth function defined on $\overline{\Omega}\times\mathbb{R}$ and satisfies $\frac{\partial\Psi}{\partial z} \geq 0$;
(iv) there exists a locally strictly convex function in $C^2(\bar{\Omega})$.

Then there exists a unique admissible solution $u \in C^\infty(\overline{\Omega})$ of the Dirichlet problem (1.2) for any smooth boundary data φ.

We note that the condition on the mean curvature of $\partial\Omega$ in (i) and the existence condition in (iv) was used before, for instance in [16], [19] and [24], moreover the Euclidean ambient satisfies all of them. When $M = \mathbb{R}^n$ the above result extend the theorem of Caffarelli, Nirenberg and Spruck presented in [11] to non-convex domains and general boundary values, without the assumption (1.11). We note that by using the techniques discovered in [22] and Theorem 1.1 we may extend the results of [23] to a larger class of curvature functions than the ones covered in [23].

Replacing the assumption about the existence of a subsolution satisfying the boundary condition by geometric conditions on $\partial\Omega$, we obtain an extension of the Theorem 1.1 in [31] (the second one presented above) for a general class of curvature functions and a complete Riemannian manifold.

Theorem 1.2. *Let f be a curvature function that satisfies (1.3)-(1.9) and M a complete orientable Riemannian manifold with nonnegative Ricci curvature. Assume that*

(i) Ω *is a bounded domain in* M *with smooth boundary* $\partial\Omega$;
(ii) $\Psi \in C^{2,\alpha}(\Omega \times \mathbb{R})$, $\Psi > 0$, $\frac{\partial\Psi}{\partial z} \geq 0$ *in* $\overline{\Omega} \times \mathbb{R}$;
(iii) Ω *is* f*-convex and satisfies the Serrin conditions*

$$\begin{aligned} f(\kappa', 0) &\geq \Psi(x, \varphi) \\ f_n(\kappa', 0) &\geq 0 \end{aligned} \tag{1.13}$$

on $\partial\Omega$, *where* κ' *are the principal curvatures of* $\partial\Omega$;
(iv) there exists a locally strictly convex function in $C^2(\bar{\Omega})$.

Then, provided there exists any bounded admissible subsolution $\underline{u}$ *of equation (1.2) in* Ω, *there exists a unique admissible solution* $u \in C^{4,\alpha}(\overline{\Omega})$ *of the Dirichlet problem (1.2) with* φ *constant.*

As is pointed out in [42], [23] and [27], the main difficulty to solve the Dirichlet problem (1.2) lies in the derivation of the second derivative estimates at the boundary for prospective solutions. The essence of our work lies in the derivation of the mixed tangential-normal derivatives by using a barrier function that is a combination of the barrier function used previously by Guan and Ivochkina. To prove the double normal second derivative estimate we adapt the tecnique used in [19].

As we said above, in this thesis we also study the existence of hypersurfaces with prescribed anisotropic mean curvature. The notion of anisotropic mean curvature has drawn attention of many mathematicians. We may cite the recent articles [6], [13], [14], [25], [35], [45] and [46]. Bergner and Dittrich [6] studied the existence of graphs with prescribed anisotropic mean curvature and boundary in the Euclidean space. We are able to obtain an extension of their result to a Riemannian manifold endowed with a nonsingular Killing vector field. More precisely, we obtain a similar result to the one obtained in [15] for the usual mean curvature. We also treat the closed problem for the anisotropic mean curvature in the Euclidean space. More precisely, we prove the analog result to the one obtained in [41] for the usual mean curvature which establishes the existence of hyperspheres with prescribed mean curvature.

Chapter 2

The Dirichlet Problem

In this chapter we fix the notation used in the whole text. It is also proved some useful lemmas and basic facts about curvature functions are established.

2.1 The Geometric Setting

In the sequel, we use Latin lower case letters $i, j, \ldots$ to refer to indices running from 1 to n and greek letters $\alpha, \beta, \ldots$ to indices from 1 to $n-1$. The Einstein summation convention is used throughout the text.

Let (M^n, σ) be a complete Riemannian manifold. We consider the product manifold $\bar{M} = M \times \mathbb{R}$ endowed with the product metric. The Riemannian connections in $\bar{M}$ and M will be denoted respectively by $\bar{\nabla}$ and ∇. The curvature tensors in $\bar{M}$ and M will be represented by $\bar{R}$ and R, respectively. The convention used here for the curvature tensor is

$$R(U,V)W = \nabla_V \nabla_U W - \nabla_U \nabla_V W + \nabla_{[U,V]} W.$$

In terms of a coordinate system (x^i) we write

$$R_{ijkl} = \sigma \left(R \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right).$$

With this convention, the Ricci identity for the derivatives of a smooth function u is given by

$$u_{i;jk} = u_{i;kj} + R_{ilkj} u^l. \tag{2.1}$$

Let Ω be a bounded domain in M. Given a differentiable function $u : \Omega \to \mathbb{R}$, its graph is defined as the hypersurface Σ parameterized by $Y(x) = (x, u(x))$ with $x \in \Omega$. This graph is diffeomorphic with Ω and may be globally oriented by an unit normal vector field N for which it holds that $\langle N, \partial_t \rangle > 0$. With respect to this orientation, the second fundamental form in Σ is by definition the symmetric tensor field $b = -\langle dN, dX \rangle$. We will denote by ∇' the conexion of Σ.

The unit vector field

$$N = \frac{1}{W}\big(\partial_t - \nabla u\big) \tag{2.2}$$

is normal to Σ, where

$$W = \sqrt{1 + |\nabla u|^2}. \tag{2.3}$$

Here, $|\nabla u|^2 = u^i u_i$ is the squared norm of ∇u. The induced metric in Σ has components

$$g_{ij} = \langle Y_i, Y_j \rangle = \sigma_{ij} + u_i u_j \tag{2.4}$$

and its inverse has components given by

$$g^{ij} = \sigma^{ij} - \frac{1}{W^2} u^i u^j. \tag{2.5}$$

We easily verify that the second fundamental form b of Σ with components (a_{ij}) is determined by

$$a_{ij} = \langle \bar{\nabla}_{Y_j} Y_i, N \rangle = \frac{1}{W} u_{i;j}$$

where $u_{i;j}$ are the components of the Hessian $\nabla^2 u$ of u in Ω. Therefore the components a_i^j of the Weingarten map A^Σ of the graph Σ are given by

$$a_i^j = g^{jk} a_{ki} = \frac{1}{W}\left(\sigma^{jk} - \frac{1}{W^2} u^j u^k\right) u_{k;i}. \tag{2.6}$$

For our purposes it is crucial to know the rules of computation involving the covariant derivatives, the second fundamental form of a hypersurface and the curvature of the ambient. In this sense, the Gauss and Codazzi equations will play a fundamental role. They are, respectively,

$$R'_{ijkl} = \bar{R}_{ijkl} + a_{ik} a_{jl} - a_{il} a_{jk} \tag{2.7}$$

$$a_{ij;k} = a_{ik;j} + \bar{R}_{i0jk} \tag{2.8}$$

where the index 0 indicates the normal vector N and R' is the Riemann tensor of Σ. We note that $a_{ij;k}$ indicates the componentes of the tensor $\nabla' b$, obtained by deriving covariantly the second fundamental form b of Σ with respect to the metric g.

The following identities for commuting second derivatives of the second fundamental form will be quite useful. It was first found by Simons in [38].

Proposition 2.1. *The second derivatives of the second fundamental form b satisfies the identity*

$$\begin{aligned} a_{ij;kl} = & a_{kl;ji} + a_{kl} a_i^m a_{jm} - a_{ik} a_j^m a_{lm} + a_{lj} a_i^m a_{km} - a_{ij} a_l^m a_{km} \\ & + \bar{R}_{likm} a_j^m + \bar{R}_{lijm} a_k^m - \bar{R}_{mjik} a_l^m - \bar{R}_{0i0j} a_{kl} + \bar{R}_{0l0k} a_{ij} \\ & - \bar{R}_{mkjl} a_i^m - \bar{\nabla}_l \bar{R}_{0jik} - \bar{\nabla}_i \bar{R}_{0kjl}. \end{aligned} \tag{2.9}$$

Proof. Since (2.9) is a tensorial inequality it is enough to check this formula for a fixed coordinate system. Given $p \in \Sigma$ we fix a geodesic coordinate system centered at p. By the Codazzi equation (2.8) we first get

$$a_{ij;kl} = \nabla'_l \big(a_{kj;i} - \bar{R}_{0jik}\big) = a_{kj;il} - \nabla'_l \big(\bar{R}_{0jik}\big).$$

Then we compute from the definition of a_{ij}

$$\nabla'_l \big(\bar{R}_{0jil}\big) = \bar{\nabla}_l \bar{R}_{0jik} + a_l^m \bar{R}_{mjik} - a_{il} \bar{R}_{0j0l} - a_{kl} \bar{R}_{0ji0}$$

and commute ∇'_i and ∇'_l to derive

$$\begin{aligned} a_{kj;il} = a_{kj;li} + R'_{likm} a_j^m + R'_{lijm} a_k^m - \bar{\nabla}_l \bar{R}_{0jik} \\ - \bar{R}_{mijk} a_l^m + \bar{R}_{0j0k} a_{il} + \bar{R}_{0ji0} a_{kl}. \end{aligned}$$

Then we use the Codazzi equation again to get

$$\begin{aligned} a_{kj;li} &= \nabla' \big(a_{kl;j} - \bar{R}_{0kjl}\big) = a_{kl;ji} - \nabla' \big(R_{0kjl}\big) \\ &= a_{kl;ji} - \bar{\nabla}_i \bar{R}_{0kjl} - \bar{R}_{mkjl} a_i^m + a_{ij} \bar{R}_{0kl0} + a_{il} \bar{R}_{0kj0}. \end{aligned}$$

Employing the Gauss equation (2.7) we finally conclude

$$\begin{aligned} a_{ij;kl} = & a_{kl;ji} + \bar{R}_{likm} a_j^m + \bar{R}_{lijm} a_k^m - \bar{R}_{mjik} a_l^m + \bar{R}_{0j0k} a_{il} + \bar{R}_{0ji0} a_{lk} \\ & - \bar{R}_{mkjl} a_i^m + \bar{R}_{0k0l} a_{ij} + \bar{R}_{0kj0} a_{il} - \bar{\nabla}_l \bar{R}_{0jik} - \bar{\nabla}_i \bar{R}_{0kjl} \\ & + a_{lk} a_i^m a_{jm} - a_{ik} a_l^m a_{mj} + a_{lj} a_i^m a_{mk} - a_{ij} a_k^m a_{ml} \end{aligned}$$

and the conclusion follows from the symmetries of $\bar{R}$. □

2.2 General Curvature Functions

Now we present a brief description of a general curvature function f and we also present some useful properties of these functions. For further details see the reference [16].

Let Γ be an open convex cone with vertex at the origin in $\mathbb{R}^n$ and containing the positive cone $\Gamma_+ = \{\lambda \in \mathbb{R}^n : \lambda_i > 0\}$. Suppose that Γ is symmetric with respect to interchanging coordinates of its points, i.e.,

$$\lambda = (\lambda_i) \in \Gamma \implies (\lambda_{\pi(i)}) \in \Gamma \quad \forall \pi \in \mathcal{P}_n,$$

where $\mathcal{P}_n$ is the set of all permutations of order n. Let f be a positive differentiable function defined in Γ. Suppose that f is symmetric in λ_i, i.e.,

$$f(\lambda_i) = f(\lambda_{\pi(i)}), \quad \forall \pi \in \mathcal{P}_n.$$

Then, f is said to be a *curvature function.* Let $\mathcal{S} \subset T^{0,2}(\Sigma)$ be the space of all symmetric covariant tensors of rank two defined in the Riemannian manifold (Σ, g) and $\mathcal{S}_\Gamma$ be the open subset of those symmetric tensors $a \in \mathcal{S}$ for which the eigenvalues, with respect to the metric g, are contained in Γ. Then we can define the mapping

$$F : \mathcal{S}_\Gamma \longrightarrow \mathbb{R}$$

by setting

$$F(a) = f\big(\lambda(a)\big),$$

where $\lambda(a) = (\lambda_1, \cdots, \lambda_n)$ are the eigenvalues of a with respect to the metric g. It is well known, see e.g. [16], that F is as smooth as f. Furthermore, as is shown in [16], the curvature function F can be viewed as depending solely on the mixed tensor $a^\sharp$, obtained by raising an index of the given symmetric covariant 2-tensor a, as well as depending on the pair of covariant tensors (a, g),

$$F(a^\sharp) = F(a, g).$$

In terms of components, in an arbitrary coordinate system we have

$$F\big(a_i^j\big) = F\big(a_{ij}, g_{ij}\big)$$

with $a_i^j = g^{jk} a_{ki}$. We denote the first derivatives of F by

$$F^{ij} = \frac{\partial F}{\partial a_{ij}} \quad \text{and} \quad F_i^j = \frac{\partial F}{\partial a_j^i},$$

and the second one is indicated by

$$F^{ij,kl} = \frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}}.$$

Hence F^{ij} are the components of a symmetric covariant tensor, while F_i^j define a mixed tensor which is contravariant with respect to the index j and covariant with respect to the index i.

As in [26], we extend the cone Γ to the space of symmetric matrices of order n, which we denote (also) by $\mathcal{S}$. Namely, for $p \in \mathbb{R}^n$, let us define

$$\Gamma(p) = \{r \in \mathcal{S} \,:\, \lambda(p,r) \in \Gamma\},$$

where $\lambda(p,r)$ denotes the eigenvalues of the matrix $A(p,r) = g^{-1}(p)r$ given by

$$A(p,r) = \frac{1}{\sqrt{1+|p|^2}} \left(I - \frac{p \otimes p}{1+|p|^2} \right) r, \tag{2.10}$$

(with eigenvalues calculated with respect to the Euclidean inner product). $A(p,r)$ is obtained from the matrix of the Weingarten map with (p,r) in place of $(\nabla u, \nabla^2 u)$ and δ^{ij} in place of σ^{ij}. We note that the eigenvalues of $A(p,r)$ are the eigenvalues of r (unless the $1/\sqrt{1+|p|^2}$ factor) with respect to the inner product given by the matrix $g = I + p \otimes p$. In this setting it is convenient to introduce the notation (see [9])

$$G(p,r) = F\big(A(p,r)\big) = f\big(\lambda(p,r)\big).$$

Hence, as in [11] and [23] we may write equation (1.1) in the form

$$F[u] = G(\nabla u, \nabla^2 u) = f(\kappa[u]) = \Psi(x,u). \tag{2.11}$$

Now we will calculate the derivatives of F. The computations become simpler if we assume that the matrix $\big(a_{ij}\big)$ is diagonal with respect to the metric $\big(g_{ij}\big)$, as is shown in the following lemma.

Lemma 2.2. *Let $a \in \mathcal{S}_\Gamma$ and $(e_i) \in T_x\Sigma$ be an orthonormal (with respect to the metric (g_{ij})) basis of eigenvectors for $a(x)$ with corresponding eigenvalues λ_i. Then, in terms of this basis, the matrix (F^{ij}) is also diagonal with* *eigenvalues* $f_i = \frac{\partial f}{\partial \lambda_i}$*. Moreover,* F *is concave and its second derivatives are given by*

$$F^{ij,kl}\eta_{ij}\eta_{kl} = \sum_{k,l} f_{kl}\eta_{kk}\eta_{ll} + \sum_{k\neq m} \frac{f_k - f_l}{\lambda_k - \lambda_l}\eta_{kl}^2, \tag{2.12}$$

for any $(\eta_{ij}) \in \mathcal{S}$*. Finally we have*

$$\frac{f_i - f_j}{\lambda_i - \lambda_j} \leq 0. \tag{2.13}$$

These expressions must be interpreted as limits in the case of principal curvatures with multiplicity greater than one.

Proof. First we calculate by the chain rule,

$$F^{ij} = \sum_k \frac{\partial f}{\partial \lambda_k}\frac{\partial \lambda_k}{\partial a_{ij}} = \sum_k f_k \frac{\partial \lambda_k}{\partial a_{ij}} \tag{2.14}$$

and

$$F^{ij,rs} = \sum_{k,l} f_{kl}\frac{\partial \lambda_k}{\partial a_{ij}}\frac{\partial \lambda_l}{\partial a_{rs}} + \sum_k f_k \frac{\partial^2 \lambda_k}{\partial a_{ij}\partial a_{rs}}. \tag{2.15}$$

Therefore, we must calculate the rate of change of the eigenvalues of the matrix (a_{ij}) with respect to variation of its components. We then define a variation of (a_{ij}) in two parameters by

$$\tilde{a}_{ij} = a_{ij} + tb_{ij} + sc_{ij},$$

for certain matrices (b_{ij}) and (c_{ij}) to be determined later. Therefore, we must expand the characteristic polynomial

$$p(\lambda, t, s) = \det\left(\tilde{a}_{ij} - \lambda\delta_{ij}\right)$$

in powers of t and s. For this, assume that (a_{ij}) is diagonal with

$$(a_{ij}) = (\lambda_1, \ldots, \lambda_n).$$

Suppose further that the eigenvalues of (a_{ij}) are simple. We denote by $\lambda = \lambda(s,t)$ an eigenvalue of $(\tilde{a}_{ij})$, i.e.,

$$0 = p(\lambda, t, s) = \det\left\{\begin{pmatrix} \lambda_1 - \lambda & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_n - \lambda \end{pmatrix} + t(b_{ij}) + s(c_{ij})\right\}.$$

Expanding the determinant, we obtain

$$
\begin{aligned}
0 =&(\lambda_1 - \lambda)\dots(\lambda_n - \lambda) + \sum_i (\lambda_1 - \lambda)\dots(tb_{ii} + sc_{ii})\dots(\lambda_n - \lambda)\\
&+ \sum_{i<j} (\lambda_1 - \lambda)\dots(tb_{ii} + sc_{ii})\dots(tb_{jj} + sc_{jj})\dots(\lambda_n - \lambda)\\
&- \sum_{i<j} (\lambda_1 - \lambda)\dots(tb_{ij} + sc_{ij})\dots(tb_{ji} + sc_{ij})\dots(\lambda_n - \lambda) + O(|(t,s)|^3).
\end{aligned}
$$

Therefore, differentiating with respect to t and evaluating at $t = 0$ we obtain

$$
\begin{aligned}
0 = \frac{\mathrm{d}p}{\mathrm{d}t} =& - \sum_i (\lambda_1 - \lambda)\dots \underbrace{\frac{\mathrm{d}\lambda}{\mathrm{d}t}}_{i} \dots(\lambda_n - \lambda)\\
&+ \sum_i (\lambda_1 - \lambda)\dots b_{ii}\dots(\lambda_n - \lambda)\\
&- \sum_{i\neq j} (\lambda_1 - \lambda)\dots sc_{ii}\dots \underbrace{\frac{\mathrm{d}\lambda}{\mathrm{d}t}}_{j} \dots(\lambda_n - \lambda)\\
&+ \sum_{i<j} (\lambda_1 - \lambda)\dots b_{ii}\dots sc_{jj}\dots(\lambda_n - \lambda)\\
&+ \sum_{i<j} (\lambda_1 - \lambda)\dots sc_{ii}\dots b_{jj}\dots(\lambda_n - \lambda)\\
&- \sum_{i<j}\sum_{l\neq i,j} (\lambda_1 - \lambda)\dots sc_{ii}\dots sc_{jj}\dots \underbrace{\frac{\mathrm{d}\lambda}{\mathrm{d}t}}_{l} \dots(\lambda_n - \lambda)\\
&- \sum_{i<j} (\lambda_1 - \lambda)\dots b_{ij}\dots sc_{ji}\dots(\lambda_n - \lambda)\\
&- \sum_{i<j} (\lambda_1 - \lambda)\dots sc_{ij}\dots b_{ji}\dots(\lambda_n - \lambda)\\
&+ \sum_{i<j}\sum_{l\neq i,j} (\lambda_1 - \lambda)\dots sc_{ij}\dots sc_{ji}\dots \underbrace{\frac{\mathrm{d}\lambda}{\mathrm{d}t}}_{l} \dots(\lambda_n - \lambda) + O(s^2).
\end{aligned}
$$

Now we differentiate with respect to s and evaluating at $s = 0$ to obtain

$$\begin{aligned} 0 = \frac{\mathrm{d}^2 p}{\mathrm{d}t\mathrm{d}s} &= -\sum_i (\lambda_1 - \lambda)\dots \underbrace{\frac{\mathrm{d}^2\lambda}{\mathrm{d}t\mathrm{d}s}}_{i}\dots(\lambda_n - \lambda) \\ &+ \sum_{i\neq j}(\lambda_1 - \lambda)\dots\underbrace{\frac{\mathrm{d}\lambda}{\mathrm{d}t}}_{i}\dots\underbrace{\frac{\mathrm{d}\lambda}{\mathrm{d}s}}_{j}\dots(\lambda_n - \lambda) \\ &- \sum_{i\neq j}(\lambda_1 - \lambda)\dots b_{ii}\dots\underbrace{\frac{\mathrm{d}\lambda}{\mathrm{d}s}}_{j}\dots(\lambda_n - \lambda) \\ &- \sum_{i\neq j}(\lambda_1 - \lambda)\dots c_{ii}\dots\underbrace{\frac{\mathrm{d}\lambda}{\mathrm{d}t}}_{j}\dots(\lambda_n - \lambda) \\ &+ \sum_{i<j}(\lambda_1 - \lambda)\dots b_{ii}\dots c_{jj}\dots(\lambda_n - \lambda) \\ &+ \sum_{i<j}(\lambda_1 - \lambda)\dots c_{ii}\dots b_{jj}\dots(\lambda_n - \lambda) \\ &- \sum_{i<j}(\lambda_1 - \lambda)\dots b_{ij}\dots c_{ji}\dots(\lambda_n - \lambda) \\ &- \sum_{i<j}(\lambda_1 - \lambda)\dots c_{ij}\dots b_{ji}\dots(\lambda_n - \lambda). \end{aligned}$$

Since $\lambda|_t = 0$ is an eigenvalue of (a_{ij}), necessarily $\lambda = \lambda_k$, at $t = 0$, for some k. As the eigenvalues of (a_{ij}) are supposed to be simple, it follows that $(\lambda_i - \lambda) \neq 0$ for $i \neq k$ at $t = 0$. Consequently,

$$\begin{aligned} 0 = &\frac{\mathrm{d}p}{\mathrm{d}t}|_{s,t=0} = -\frac{\mathrm{d}\lambda}{\mathrm{d}t}(\lambda_1 - \lambda)\dots(\widehat{\lambda_k - \lambda})\dots(\lambda_n - \lambda) \\ &+ (\lambda_1 - \lambda)\dots b_{kk}\dots(\lambda_n - \lambda). \end{aligned}$$

If we choose $b_{kk} = 1$ or $b_{kk} = 0$, we get from the last equations, respectively,

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = 1$$

or

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = 0.$$

In particular, the directional derivatives of λ with respect to the paths given by

$$t \mapsto (a_{ij}) + t\mathsf{e}_{kk}$$

and

$$t \mapsto (a_{ij}) + t\mathsf{e}_{lm},$$

where $l \neq k$ or $m \neq k$ and e_{rs} is the matrix with 1 in the entries rs and 0 in all remaining entries, are given respectively by

$$\frac{\partial \lambda_k}{\partial a_{kk}} = 1, \quad \frac{\partial \lambda_k}{\partial a_{lm}} = 0,$$

where $l \neq k$ or $m \neq k$. As these functions take values in the discrete set $\{0, 1\}$, it follows from the continuity that these expressions are valid for all matrices (a_{ij}), with possible multiple eigenvalues.

Now we use the above expansion to obtain informations about the second order derivatives. We have for $b_{kk} = 1$ (the other entries of (b_{ij}) are zero)

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{\partial \lambda_k}{\partial a_{kk}} = 1$$

and

$$\begin{aligned}
0 =& \frac{\mathrm{d}^2 p}{\mathrm{d}t\mathrm{d}s} = -(\lambda_1 - \lambda)\dots \underbrace{\frac{\mathrm{d}^2\lambda}{\mathrm{d}t\mathrm{d}s}}_{k} \dots (\lambda_n - \lambda) \\
&+ \sum_{i\neq k} (\lambda_1 - \lambda)\dots \underbrace{\frac{\mathrm{d}\lambda}{\mathrm{d}t}}_{i} \dots \underbrace{\frac{\mathrm{d}\lambda}{\mathrm{d}s}}_{k} \dots (\lambda_n - \lambda) \\
&- \sum_{i\neq k} (\lambda_1 - \lambda)\dots c_{ii} \dots (\widehat{\lambda_k - \lambda}) \dots (\lambda_n - \lambda) \\
&- \sum_{i\neq k} (\lambda_1 - \lambda)\dots c_{kk} \dots \underbrace{\frac{\mathrm{d}\lambda}{\mathrm{d}t}}_{i} \dots (\lambda_n - \lambda) \\
&+ \sum_{k<i} (\lambda_1 - \lambda)\dots (\widehat{\lambda_k - \lambda}) \dots c_{ii} \dots (\lambda_n - \lambda) \\
&+ \sum_{i<k} (\lambda_1 - \lambda)\dots c_{ii} \dots (\widehat{\lambda_k - \lambda}) \dots (\lambda_n - \lambda),
\end{aligned}$$

which implies

$$
\begin{aligned}
0 = & \frac{\mathrm{d}^2 p}{\mathrm{d}t\mathrm{d}s} = -(\lambda_1 - \lambda)\ldots \underbrace{\frac{\mathrm{d}^2\lambda}{\mathrm{d}t\mathrm{d}s}}_{k} \ldots (\lambda_n - \lambda) \\
& + \sum_{i\neq k} (\lambda_1 - \lambda)\ldots \underbrace{\frac{\mathrm{d}\lambda}{\mathrm{d}t}}_{i} \ldots \underbrace{\frac{\mathrm{d}\lambda}{\mathrm{d}s}}_{k} \ldots (\lambda_n - \lambda) \\
& - \sum_{i\neq k} (\lambda_1 - \lambda)\ldots c_{kk} \ldots \underbrace{\frac{\mathrm{d}\lambda}{\mathrm{d}t}}_{i} \ldots (\lambda_n - \lambda).
\end{aligned}
$$

Thus, if we choose $c_{kk} = 1$ and the other entries equal to zero in (c_{ij}) we get $\frac{\mathrm{d}\lambda}{\mathrm{d}s} = \frac{\partial \lambda_k}{\partial a_{kk}} = 1$ and the last two terms on the right hand side of the last equation cancel out. On the other hand, if we choose $c_{lm} = 1$ for some $l \neq k$ or $m \neq k$ and the other entries (in particular c_{kk}) equal to zero, then $\frac{\mathrm{d}\lambda}{\mathrm{d}s} = \frac{\partial \lambda_k}{\partial a_{lm}} = 0$ and, in this case, these two terms are both zero. Hence, we have

$$
(\lambda_1 - \lambda)\ldots \underbrace{\frac{\mathrm{d}^2\lambda}{\mathrm{d}t\mathrm{d}s}}_{k} \ldots (\lambda_n - \lambda) = 0
$$

and

$$
\frac{\mathrm{d}^2\lambda}{\mathrm{d}t\mathrm{d}s}|_{s,t=0} = \frac{\partial^2 \lambda}{\partial a_{ij}\partial a_{kk}} = 0
$$

for all values of i, j.

Now we consider a variation obtained by taking $b_{lm} = 1$ for $l \neq k$ or $m \neq k$ and putting the other entries equal to zero (including the b_{kk} one). Without loss of generality, we may consider $c_{nr} = 1$ for $n \neq k$ or $r \neq k$ and the other entries equal to zero. As $\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{\partial \lambda_k}{\partial a_{lm}} = 0$ e $\frac{\mathrm{d}\lambda}{\mathrm{d}s} = \frac{\partial \lambda_k}{\partial a_{nr}} = 0$ we have

$$
\begin{aligned}
0 = & \frac{\mathrm{d}^2 p}{\mathrm{d}t\mathrm{d}s} = -(\lambda_1 - \lambda)\ldots \underbrace{\frac{\mathrm{d}^2\lambda}{\mathrm{d}t\mathrm{d}s}}_{k} \ldots (\lambda_n - \lambda) \\
& - \sum_{i<j} (\lambda_1 - \lambda)\ldots b_{ij} \ldots c_{ji} \ldots (\lambda_n - \lambda) \\
& - \sum_{i<j} (\lambda_1 - \lambda)\ldots c_{ij} \ldots b_{ji} \ldots (\lambda_n - \lambda).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(\lambda_1-\lambda)\dots\underbrace{\frac{\mathrm{d}^2\lambda}{\mathrm{d}t\mathrm{d}s}}_{k}\dots(\lambda_n-\lambda) = &-\sum_{k<j}(\lambda_1-\lambda)\dots b_{kj}\dots c_{jk}\dots(\lambda_n-\lambda)\\
&-\sum_{i<k}(\lambda_1-\lambda)\dots b_{ik}\dots c_{ki}\dots(\lambda_n-\lambda)\\
&-\sum_{k<j}(\lambda_1-\lambda)\dots c_{kj}\dots b_{jk}\dots(\lambda_n-\lambda)\\
&-\sum_{i<k}(\lambda_1-\lambda)\dots c_{ik}\dots b_{ki}\dots(\lambda_n-\lambda).
\end{aligned}
$$

So, if we choose $b_{km}=1$ for some $m<k$ and the other entries equal to zero we get

$$
(\lambda_1-\lambda)\dots\underbrace{\frac{\mathrm{d}^2\lambda}{\mathrm{d}t\mathrm{d}s}}_{k}\dots(\lambda_n-\lambda) = -(\lambda_1-\lambda)\dots\underbrace{c_{mk}}_{m}\dots\underbrace{b_{km}}_{k}\dots(\lambda_n-\lambda),
$$

wich implies

$$
\frac{\partial^2\lambda_k}{a_{mk}a_{km}} = \frac{1}{\lambda_k-\lambda_m},
$$

if $k>m$. Choosing $b_{km}=1$ for some $k<m$ and the other entries equal to zero, we obtain

$$
(\lambda_1-\lambda)\dots\underbrace{\frac{\mathrm{d}^2\lambda}{\mathrm{d}t\mathrm{d}s}}_{k}\dots(\lambda_n-\lambda) = -(\lambda_1-\lambda)\dots\underbrace{b_{km}}_{k}\dots\underbrace{c_{mk}}_{m}\dots(\lambda_n-\lambda),
$$

then

$$
\frac{\partial^2\lambda_k}{a_{mk}a_{km}} = -\frac{1}{\lambda_m-\lambda_k}
$$

for $k<m$. By raising indices and permuting the order, we get

$$
\frac{\partial^2\lambda_m}{a_{mk}a_{km}} = -\frac{1}{\lambda_k-\lambda_m}
$$

for $k>m$.

Applying this formula in the expression of the derivative of F above, we conclude that, given an arbitrary symmetric co-vector η_{ij}, we have

$$F^{ij}\eta_{ij} = \sum_i f_i \eta_{ii} \tag{2.16}$$

and

$$\begin{aligned}\eta_{ij}F^{ij,rs}\eta_{rs} &= \sum_{k,l} f_{kl}\frac{\partial\lambda_k}{\partial a_{ij}}\frac{\partial\lambda_l}{\partial a_{rs}}\eta_{ij}\eta_{rs} + \sum_k f_k\eta_{ij}\frac{\partial^2\lambda_k}{\partial a_{ij}\partial a_{rs}}\eta_{rs}\\ &= \sum_{k,l} f_{kl}\eta_{kk}\eta_{ll} + \sum_{k<m} f_k\frac{\partial^2\lambda_k}{\partial a_{km}\partial a_{mk}}\eta_{km}^2 + \sum_{k>m} f_k\frac{\partial^2\lambda_k}{\partial a_{km}\partial a_{mk}}\eta_{km}^2\\ &= \sum_{k,l} f_{kl}\eta_{kk}\eta_{ll} + \sum_{k\neq m}\frac{f_k - f_m}{\lambda_k-\lambda_m}\eta_{km}^2.\end{aligned}$$

This completes the proof of the lemma. □

Since $a^i_j = g^{ik}a_{kj}$, we have

$$F^{ij} = \frac{\partial F}{\partial a_{ij}} = \frac{\partial F}{\partial a^k_l}\frac{\partial a^k_l}{\partial a_{ij}} = F^j_k g^{ki}.$$

Similarly,

$$F^{ij,kl} = \frac{\partial^2 F}{\partial a_{ij}\partial a_{kl}} = \frac{\partial^2 F}{\partial a^p_i \partial a^q_k} g^{pj}g^{ql}.$$

In particular, if we denote

$$G^{ij} = \frac{\partial G}{\partial r_{ij}} \quad \text{and} \quad G^{ij,kl} = \frac{\partial^2 G}{\partial r_{ij} r_{kl}},$$

we obtain

$$G^{ij} = \frac{1}{W}F^{ij}$$

and

$$G^{ij,kl} = \frac{1}{W^2}F^{ij,kl}.$$

Hence, it follows from the above lemma that, under condition (1.3), equation (2.11) is elliptic, i.e., the matrix $G^{ij}(p,r)$ is positive-definite for any $r \in \Gamma(p)$. Moreover, under condition (1.4) the restriction of the function $G(p,\cdot)$ to the open set $\Gamma(p)$ is a concave function. We point out that since $1/W$ and 1 are respectively the lowest and the largest eigenvalues of g^{ij}, we also have

$$\frac{1}{W^3}F_i^j\delta_j^i \leq G^{ij}\delta_{ij} \leq \frac{1}{W}F_i^j\delta_j^i. \tag{2.17}$$

Now we analyze some consequences of the conditions (1.3)-(1.7). First we note that under these conditions f satisfies

$$f(s\kappa) \geq sf(\kappa), \quad 0 < s < 1 \tag{2.18}$$

and

$$\sum_i f_i(\kappa)\kappa_i \leq f. \tag{2.19}$$

In fact, from the concavity condition we have

$$f(s\kappa + (1-s)\varepsilon\kappa) \geq sf(\kappa) + (1-s)f(\varepsilon\kappa) \geq sf(\kappa),$$

for any $0 < \varepsilon < s < 1$. The inequality (2.18) follows by taking $\varepsilon \longrightarrow 0$. To prove (2.19) we note that, for $0 < s < 1$,

$$\frac{f(s\kappa) - f(\kappa)}{s-1} \leq \frac{sf(\kappa) - f(\kappa)}{s-1} = f(\kappa)$$

By taking $s \longrightarrow 1^-$, we get $\frac{\mathrm{d}f(s\kappa)}{\mathrm{d}s}|_{s=1} \leq f(\kappa)$, which proves (2.19). We may also prove that the concavity of f and the condition (1.7) imply that

$$\sum_i \kappa_i \geq \delta > 0 \tag{2.20}$$

for any $\kappa \in \Gamma$ that satisfies $f(\kappa) \geq \Psi_0$. In fact, we note that the set

$$\Gamma_{\Psi_0} = \{\kappa \in \Gamma : f(\kappa) \geq \Psi_0\}$$

is closed, convex and symmetric. The convexity follows from the concavity of f since for any $\kappa, \lambda \in \Gamma_\Psi$ we have

$$f((1-s)\kappa + s\lambda) \geq (1-s)f(\kappa) + sf(\lambda) \geq \Psi_0.$$

The symmetry follows from the symmetry of f.

So, the closest point of Γ_{Ψ_0} to the origin has the form $(\kappa_0, \dots, \kappa_0)$. Otherwise, if this point κ contains two distinct coordinates, say $\kappa_i \neq \kappa_j$, then the point μ obtained from κ by reversing the positions of κ_i and κ_j will also be a minimum, by the definition of distance. Therefore, by the convexity of Γ_Ψ, the line segment whose the endpoints are κ and μ is contained in this set. On the other hand, it is clear that its midpoint is closer to the origin than the extremes points. This contradiction implies that all components of κ are equal. Moreover $\kappa_0 \neq 0$ since $\limsup_{\lambda\to\partial\Gamma} f(\lambda) \leq \bar{\Psi}_0$.

We show thus that every $\kappa \in \Gamma_{\Psi_0}$ lies above the hyperplane

$$H = \Big\{\lambda \in \mathbb{R}^n : \sum_i \lambda_i = n\kappa_0\Big\},$$

which is the support hyperplane of the convex set Γ_{Ψ_0} at the point $(\kappa_0, \ldots, \kappa_0)$. In fact, its normal direction is determined by the segment connecting the origin to the closest point. Thus, every $\kappa \in \Gamma_{\Psi_0}$ is necessarily contained in the convex side of the cone Γ_{Ψ_0} wich lies above H. This geometric fact implies that *upper bounds* for the principal curvatures of the graph of an admissible solution immediately ensure *lower bounds* for these curvatures.

2.3 Some Useful Lemmas

In this section we present some lemmata that will be used in the next chapters. The first one gives an useful formula involving the second and third derivatives of the prospective solutions of the problem (1.2).

Lemma 2.3. *Let u be a solution of equation (2.11). The derivatives of u satisfy the formula*

$$\begin{aligned} G^{ij}u_{k;ij} =& WG^{ij}a^l_j u_{k;i}u_l + WG^{ij}a^l_j u_{k;l}u_i + \frac{1}{W}G^{jl}a_{jl}u^i u_{i;k} \\ &- G^{ij}R_{iljk}u^l + \Psi_k + \Psi_t u_k. \end{aligned} \tag{2.21}$$

Proof. Deriving covariantly equation (2.11) in the k-th direction with respect to the metric σ of M we obtain

$$\Psi_k + \Psi_t u_k = \frac{\partial G}{\partial u_{i;j}} u_{i;jk} + \frac{\partial G}{\partial u_i} u_{i;k} = G^{ij}u_{i;jk} + G^i u_{i;k}. \tag{2.22}$$

From $F(a_i^j[u]) = G(\nabla u, \nabla^2 u)$ we calculate

$$G^i = \frac{\partial G}{\partial u_i} = \frac{\partial F}{\partial a_r^s}\frac{\partial a_r^s}{\partial u_i} = F_s^r \frac{\partial}{\partial u_i}\left(\frac{1}{W} g^{sl} u_{l;r}\right)$$
$$= F_s^r g^{sl} u_{l;r} \frac{\partial}{\partial u_i}\left(\frac{1}{W}\right) + \frac{1}{W} F_s^r \frac{\partial}{\partial u_i}\left(g^{sl}\right) u_{l;r}.$$

We compute

$$F_s^r g^{sl} u_{l;r} \frac{\partial}{\partial u_i}\left(\frac{1}{W}\right) = -\frac{u^i}{W^3} F_s^r g^{sl} u_{l;r} = -\frac{1}{W} G^{rs} a_{rs} u^i$$

and

$$\frac{1}{W} F_s^r \frac{\partial}{\partial u_i}\left(g^{sl}\right) u_{l;r} = G^{rp} g_{sp} \frac{\partial}{\partial u_i}\left(g^{sl}\right) u_{l;r}$$
$$= -G^{rp}(\delta_{ip} g^{sl} u_s + g^{il} u_p) u_{l;r}$$
$$= -W G^{ij} a_j^l u_l - W G^{lj} a_l^i u_j,$$

where we have used

$$g_{sp} \frac{\partial g^{sl}}{\partial u_i} = -g^{sl}(\delta_{is} u_p + u_s \delta_{ip}) = -(\delta_{ip} g^{sl} u_s + g^{il} u_p).$$

It follows that

$$G^i = -\frac{1}{W} G^{rs} a_{rs} u^i - W G^{ij} a_j^l u_l - W G^{lj} a_l^i u_j.$$

Replacing these relations into (2.22) we obtain

$$\Psi_k + \Psi_t u_k = G^{ij} u_{i;jk} - \frac{1}{W} G^{rs} a_{rs} u^i u_{i;k} - W G^{ij} a_j^l u_l u_{i;k} - W G^{lj} a_l^i u_j u_{i;k}.$$

Using the Ricci identity (2.1), equation (2.21) is easily obtained. □

A choice of an appropriate coordinate system simplifies very much the calculation of the components a_i^j of the Weingartein operator. We describe how to obtain such a coordinate system. Fixed a point $x \in M$, we choose a geodesic coordinate system (x^i) of M around x such that the vectors $Y_* \cdot \frac{\partial}{\partial x^i}|_x$ form a basis of principal directions of Σ at $Y(x)$ and $\frac{\partial}{\partial x^i}|_x$ is orthonormal with respect to the inner product given by the matrix $g = I + \nabla u \otimes \nabla u$, i.e., the vectors $Y_* \cdot \frac{\partial}{\partial x^i}|_x$ are orthonormal in $T_{Y(x)}\Sigma$. With this choice we have

$$a_i^j(x) = a_{ij}(x) = \frac{1}{W} u_{i;j}(x)\delta_{ij} = \kappa_i \delta_i^j.$$

and

$$G^{ij} = \frac{1}{W} F_k^i g^{kj} = \frac{1}{W} f_i \delta_k^i \delta^{kj} = \frac{1}{W} f_i \delta_i^j,$$

since (F_i^j) is diagonal whenever (a_i^j) is diagonal and $g^{ij} = \delta^{ij}$ whereas $Y_* \cdot \frac{\partial}{\partial x^i}|_x$ are orthonormal in $T_{Y(x)}\Sigma$. From now on we will call such coordinate system as the *special* coordinate system centered at x.

We note that, at the center of a *special* coordinate system the formula (2.21) takes the more simple form

$$\begin{aligned} \sum_i f_i u_{k;ii} = 2W \sum_i f_i \kappa_i u_i u_{i;k} + \frac{1}{W} \sum_j f_j \kappa_j u^i u_{k;i} \\ - \sum_i f_i R_{ilik} u^l + W(\Psi_k + \Psi_t u_k). \end{aligned} \tag{2.23}$$

Remark 2.4. *Since the principal curvatures $\kappa[u]$ of Σ are the roots κ of the equation*

$$\det(a_{ij} - \kappa g_{ij}) = 0,$$

instead of the Weingarten matrix (a_i^j), some authors, as in [11] and [23], prefer to work with the symmetric matrix given by

$$\tilde{a}_i^j = \gamma_i^k a_{kl} \gamma^{kj}$$

where

$$\gamma^{ij} = \sigma^{ij} - \frac{1}{W(1+W)} u^i u^j$$

and $\gamma_i^j = \sigma_{ik}\gamma^{kj}$. The main feature of this choice is the symmetry of the matrix $\tilde{a}_i^j$.

Following [8], to obtain the *a priori* hessian estimates on the boundary of prospective solutions we will make use of the following technical lemmas.

Lemma 2.5. *Consider a $n \times n$ symmetric matrix*

$$M = \left(\begin{array}{cccc|c} d_1 & & & \bigcirc & a_1 \\ & d_2 & & & \\ & & \ddots & & \\ & \bigcirc & & d_{n-1} & a_{n-1} \\ \hline a_1 & & & a_{n-1} & a \end{array} \right)$$

with $d_1, \dots, d_{n-1}$ fixed, $|a|$ tending to infinity and

$$|a_i| \leq C, \quad i = 1, \dots, n-1.$$

Then the eigenvalues $\lambda_1, \dots, \lambda_n$ behave like

$$\lambda_\alpha = d_\alpha + o(1), \quad 1 \leq \alpha \leq n-1$$
$$\lambda_n = a\Big(1 + O\Big(\frac{1}{a}\Big)\Big),$$

where the $o(1)$ and $O(1/a)$ are uniform – depending only on $d_1, \dots, d_{n-1}$ and C.

Proof. See [8], Lemma 1.2 (p. 272). □

Lemma 2.6. *Let $\Gamma' \subset \mathbb{R}^{n-1}$ be an open, convex, symmetric cone which is not all of $\mathbb{R}^{n-1}$ and contains the positive cone. Suppose that $\tilde{\lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-1}) \in \Gamma'$ and $\tilde{\lambda}_1 \leq \ldots \leq \tilde{\lambda}_{n-1}$. Then the cone Γ' has a plane of support, i.e., there exists $\mu' = (\mu_1, \dots, \mu_{n-1}) \in \mathbb{R}^{n-1}$ such that*

$$\Gamma' \subset \Big\{ \lambda' \in \mathbb{R}^{n-1} \,:\, \lambda' \cdot \mu' = \sum \lambda_\alpha \mu_\alpha > 0 \Big\},$$

with μ' satisfying $\mu_1 \geq \ldots \geq \mu_{n-1} \geq 0$,

$$\sum_\alpha \mu_\alpha = 1 \quad \textit{and} \quad \sum_\alpha \mu_\alpha \tilde{\lambda}_\alpha = \mathit{dist}\big(\tilde{\lambda}, \partial\Gamma'\big).$$

Proof. See [8], Lemma 6.1 (p. 286). □

Lemma 2.7. *Let $A = (a_{ij})$ be a square $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$. Let $\mu_1 \geq \ldots \geq \mu_n \geq 0$ be given numbers. Consider an orthonormal basis of vectors $b^1, \dots, b^n$ and set*

$$a^i = \sqrt{\mu_i} b^i, \quad 1 \leq i \leq n.$$

Then
$$\sum_i \langle Aa^i, a^i\rangle \geq \sum_i \mu_i\lambda_i.$$
In particular, we have
$$\sum_i \mu_i a_{ii} \geq \sum \mu_i\lambda_i.$$

Proof. See [8], Lemma 6.2 (p. 287). □

2.4 The Continuity Method

In this section we apply the continuity method to reduce the problem of existence of solution in Theorems 1.1 and 1.2 to the derivation of a priori estimates for prospective solutions. We are going to include here a succinct description of this method. For a detailed description of the continuity method we refer the reader to [17], chapter 17.

Generally speaking, the continuity method involves the embedding of the given problem in a family of problems indexed by a closed interval, say $[0, 1]$. The subset S of $[0, 1]$ for which the corresponding problems are solvable is shown to be nonempty, closed and open, and hence it coincides with the whole interval. First we present an abstract functional analytic formulation. Let E and F be Banach spaces and T a mapping from an open set $U \subset E$ into F. The mapping T is called Frèchet differentiable at $u \in U$ is there exists a bounded linear mapping $L : E \longrightarrow F$ such that

$$||T[u + h] - T[u] - L[h]||_F/||h||_E \to 0 \tag{2.24}$$

as $h \to 0$ in E. The linear mapping L is called the Frèchet derivative of T at u and will be denoted by T_u. It is evident from the definition that the Frèchet differentibility of T at u implies that T is continuous at u and that the Frèchetderivative T_u is determined uniquely by (2.24). We call T continuously differentiable at u if T is Frèchet differentiable in a neighbourhood of u and the resulting mapping
$$v \longmapsto T_u \in \mathcal{L}(E, F)$$
is continuous at u. Here $\mathcal{L}(E, F)$ denotes the Banach space of bounded linear mappings from E into F with norm given by

$$||L|| = \sup_{v\in E\,;\,v\neq 0} \frac{||Lv||_F}{||v||_E}$$

An implicit function theorem holds for Frèchet differentiable mappings. Suppose that E, F and X are Banach spaces and that $G : E \times X \longrightarrow F$ is Frèchet differentiable at a point $(u,t) \in E \times X$. The partial Frèchet derivatives, $G^1_{(u,t)}, G^2_{(u,t)}$ at (u,t) are the bounded linear mappings from E, X, respectively, into F defined by

$$G_{(u,t)}(h,k) = G^1_{(u,t)}[h] + G^2_{(u,t)}[k]$$

for $(h,k) \in E \times X$. We state the implicit function theorem in the following form.

Theorem ([17], Theorem 17.6)**.** *Let E, F and X be Banach spaces and G a mapping from an open subset of $E \times X$ into F. Let (u_0, t_0) be a point in $E \times X$ satisfying:*
i) $G[u_0, t_0] = 0$;
ii) G is continuously diffrentiable at (u_0, t_0);
iii) the partial Frèchet derivative $L = G^1_{(u_0,t_0)}$ is invertible.
Then there exists a neighbourhood $\mathcal{N}$ of t_0 in X such that the equation $G[u,t] = 0$, is solvable for each $t \in \mathcal{N}$, with solution $u = u_t \in E$.

In order to apply this theorem we suppose that E and F are Banach spaces with T a mapping from an open subset $U \subset E$ into F. Let u_0 be a fixed element in U and define for $u \in U$, $t \in \mathbb{R}$ the mapping $G : U \times \mathbb{R} \longrightarrow F$ by

$$G[u,t] = F[u] - tF[u_0].$$

Let S and R be the subsets of $[0,1]$ and E defined by

$$S = \{t \in [0,1] \,:\, G[u,t] = 0 \text{ for some } u \in U\}$$
$$R = \{u \in U \,:\, G[u,t] = 0 \text{ for some } t \in [0,1]\}$$

Clearly $1 \in S$, $u_0 \in R$ so that S and R are not empty. Let us next suppose that the mapping T is continuosly differentiable on R with invertible Frèchet derivative T_u. It follows then from the implicit function theorem that the set S is open in $[0,1]$. Consequently we obtain the following version of the method of continuity.

Proposition 2.8. *The equation $T[u] = 0$ is solvable for $u \in U$ provided the set S is closed in* $[0,1]$.

Now we are going examine the application of this result to the Dirichlet problem (1.2):

$$\begin{cases} F[u] = \Psi & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

First we need to designate a suitable family of problems. Consider the family of functions

$$\Psi^t = t\Psi + (1-t)\underline{\Psi}, \quad 0 \le t \le 1,$$

where we denotes

$$\underline{\Psi} = F[\underline{u}] = f\big(\kappa[\underline{u}]\big).$$

Now we consider the family of problems

$$\begin{cases} F[u] = \Psi^t & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

We can reduce to the case of zero boundary values by replacing u with $v = u - \varphi$, so this problem is equivalent to

$$\begin{cases} F[v+\varphi] = \Psi^t & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

In order to show the existence of solutions for $t = 1$ we define de (closed) subespaces $E = \{v \in C^{2,\alpha}(\bar{\Omega}) \ : \ v = 0 \text{ on } \partial\Omega\}$ and $F = C^{\alpha}(\bar{\Omega})$, for some $0 < \alpha < 1$. We define the mapping $G : E \times \mathbb{R} \longrightarrow F$ by

$$G[u,t] = F[u+\varphi] - \Psi^t.$$

Let $(u_0, t_0) \in E \times \mathbb{R}$ be a solution of $G[u,t] = 0$. It follows that $u + \varphi$ is an admissible function, so the partial Frèchet derivative $L = G^1_{(u_0,t_0)}$ is invertible (by Schauder theory) since

$$L[h] = G^1_{(u_0,t_0)}[h] = G^{ij}h_{i;j} + G^i h_i + ch$$

where G is the operator given in (2.11) and

$$G^{ij} = \frac{\partial F}{\partial u_{i;j}}\big(\nabla^2(u+\varphi), \nabla(u+\varphi)\big)$$

$$G^{i} = \frac{\partial G}{\partial u_{i}}\big(\nabla^2(u+\varphi), \nabla(u+\varphi)\big)$$

$$c = -\frac{\partial \Psi^{t_0}}{\partial u} = -\frac{\partial \Psi}{\partial u} \le 0.$$

Hence it follows from the above discution (Proposition 2.8) that the existence of solutions of equation $G[u,1]=0$ is reduced to the closedness of the set $S=\{t\in[0,1] : G[u,t]=0 \text{ for some } u\in E\}$. On the other hand, the closedness of S will follows from the $C^{2,\alpha}$ *a priori* estimates for the solutions. In fact, since $C^{2,\alpha}(\bar{\Omega})\hookrightarrow C^2(\bar{\Omega})$ every bounded sequence in $C^{2,\alpha}(\bar{\Omega})$ admits a convergent subsequence in $C^2(\bar{\Omega})$. So, if $t_n\in S$ and $t_n\to t$, the solutions u_n associated with t_n admits a subsequence whose converges to a solution u of the problem $G[u,t]=0$, which implies that $t\in S$. Hence, a existence of solutions is reduced to the $C^{2,\alpha}$ *a priori* estimates.

We note that since G is concave it suffices to establish the C^2 estimates. In fact, the Evans-Krylov $C^{2,\alpha}$ estimates may be applied to improve the estimates. We note that teh boundary $C^{2,\alpha}$ estimates have been simplified by Caffarelli as is pointed out in [28]. For more details about the boundary $C^{2,\alpha}$ estimates we refer the reader to [7], [10] and [17].

Chapter 3

A Priori Estimates

In this chapter we obtain the *a priori* estimates of prospective solutions of the Dirichlet problem (1.2).

3.1 The Height and Boundary Gradient Estimates

Let u be an admissible solution of the Dirichlet problem (1.2). We first consider that the hypotheses in Theorem 1.1 are satisfied. In this context the height estimates for admissible solutions of (1.2) is a direct consequence of the existence of a subsolution $\underline{u}$ satisfying the boundary condition and the inequality

$$\sum \kappa_i \geq \delta > 0,$$

which is satisfied for any $\kappa \in \{\kappa \in \Gamma : f(\kappa) \geq \Psi_0\}$, where $\delta > 0$ is a positive constant which depends only on Γ. In fact, it follows from the comparison principle applied to equation (1.2) that $\underline{u} \leq u$, which yields a lower bound. An upper bound is obtained using the function $\bar{u} = \sup \varphi$ as barrier, which satisfies

$$\begin{aligned} 0 = Q[\bar{u}] \leq Q[u] \quad & \text{in } \Omega \\ \bar{u} \geq u \quad & \text{on } \partial\Omega, \end{aligned}$$

where Q is the mean curvature operator. So, it follows from the comparison principle for quasilinear elliptic equations that $u \leq \bar{u}$. Hence the proof of the height estimate, under the hypotheses of Theorem 1.1, is done. We also note that if φ is constant then $\bar{u}$ also yields an upper barrier for u, which provides the gradient bound on $\partial\Omega$.

To obtain the boundary gradient estimate (under the hypotheses of Theorem 1.1) we use the function $\bar{u}$ as an upper barrier, which satisfies

$$\begin{aligned} 0 = Q[\bar{u}] \leq Q[u] \quad \text{in } \Omega \\ \bar{u} = \varphi \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

which implies that $u \leq \bar{u}$. The hypothesis that the Ricci curvature of M and the mean curvature of $\partial\Omega$ are nonnegative ensure the existence of such a solution (see, e.g. [39]). Since $\underline{u} = u = \bar{u}$ on $\partial\Omega$, the inequality $\underline{u} \leq u \leq \bar{u}$ implies the boundary gradient estimate

$$|\nabla u| < C \text{ on } \partial\Omega. \tag{3.2}$$

This completes the height and boundary gradient estimate in Theorem 1.1.

Now we consider Theorem 1.2. Following the ideas presented in [42], we will use the hypotheses on the boundary geometry to construct a lower barrier function. Let d be the distance function to the boundary $\partial\Omega$. In a small tubular neighborhood $\mathcal{N}$ of $\partial\Omega$ we define the function $w = \varphi - f(d)$, where f is a suitable real function. We choose $\mathcal{N} = \{x \in \Omega : d(x) < 0\}$, where $a > 0$ is chosen sufficiently small to ensure that $d \in C^2(\bar{\mathcal{N}})$ (see [30]). Fixed a point y_0 in $\partial\Omega$. We fix around y_0 Fermi coordinates (y^i) in M along $\mathcal{N}_{d(y_0)} = \{x \in \Omega : d(x) = d(y_0)\}$, such that y^n is the normal coordinate and the tangent coordinate vectors $\{\frac{\partial}{\partial y^\alpha}|_{y_0}\}$, $1 \leq \alpha \leq n-1$, form an orthonormal basis of eigenvectors that diagonalize $\nabla^2 u$ at y_0. Since $\nabla d = \nu$ is the unit normal outward vector along $\mathcal{N}_{d(y_0)}$, we have

$$-\nabla^2 d(y_0) = \operatorname{diag}(\kappa_1'', \kappa_2'', \ldots, \kappa_{n-1}'', 0),$$

where $\kappa'' = (\kappa_1'', \kappa_2'', \ldots, \kappa_{n-1}'')$ denotes the principal curvatures of $\mathcal{N}_{d(y_0)}$ at y_0. Hence, at y_0, $w_i = 0\,(i < n)$, $w_n(y_0) = -f'$ and

$$\nabla^2 w = \operatorname{diag}\big(f'\kappa'', -f''\big),$$

since $d_n = 1$ and $d_i = 0$, $i < n$. Therefore, the matrix of the Weingarten operator of the graph of w at $\big(y_0, w(y_0)\big)$ is

$$\begin{aligned} A[w] =& \Big(g^{ik}(w) a_{jk}(w)\Big) \\ =& \frac{1}{v}\operatorname{diag}\Big(f'\kappa'', -\frac{f''}{v^2}\Big), \end{aligned}$$

where $v = \sqrt{1+f'^2}$. Hence the principal curvatures $\tilde{\kappa} = (\tilde{\kappa}_1, \ldots, \tilde{\kappa}_n)$ of the graph of w at $\big(y_0, w(y_0)\big)$ are

$$\tilde{\kappa}_i = \frac{f'}{v}\kappa''_i \tag{3.3}$$

$$\tilde{\kappa}_n = -\frac{f''}{v^3}. \tag{3.4}$$

To proceed further, we take f of the form

$$f(d) = \frac{1}{\mu}\log(1+kd)$$

for positive constants μ, k to be determined. We have

$$f'(d) = \frac{k}{\mu(1+kd)} \geq \frac{1}{\mu(1+ka)} \tag{3.5}$$

$$f''(d) = -\mu f'(d)^2. \tag{3.6}$$

We may thus estimate

$$\tilde{\kappa_n} \geq \frac{\mu}{2v}$$

provided $v \geq v_0$, $\mu \geq \mu_0$, where μ_0 and v_0 are constants depending on Ω and its boundary. Therefore

$$|\tilde{\kappa}_i - \kappa''_i| \leq \frac{\mu_1}{\mu}\tilde{\kappa}_n, \tag{3.7}$$

for a futher constant μ_1, since the principal curvatures $\tilde{\kappa}_1, \ldots, \tilde{\kappa}_{n-1}$ will differ from $\kappa''_1, \ldots, \kappa''_{n-1}$ by $O\Big(\frac{1}{v}\Big)$ as $v \longrightarrow \infty$. Let $\tilde{y}_0 \in \partial\Omega$ be the closest point of y_0 in $\partial\Omega$, we thus estimate

$$\begin{aligned}\Psi(y_0, w) &\leq \Psi(\tilde{y}_0, \varphi) + |\Psi|_1 d \\ &\leq \Psi(\tilde{y}_0, \varphi) + \frac{|\Psi|_1}{\mu v} \\ &\leq f(\kappa', 0) + \frac{|\Psi|_1}{\mu v},\end{aligned}$$

where we used (3.5) and the hypotheses (ii) and (iii) of Theorem 1.2. Note that κ' denotes the principal curvature of $\partial\Omega$. For $a > 0$ small, we may replace κ_i'' by κ_i' in (3.7). By condition (1.13) there exist positive constants δ_0, t_0 such that

$$f(\tilde{\kappa}) - f(\kappa', 0) \geq \delta_0 t \tilde{\kappa}_n \tag{3.8}$$

whenever $t \leq t_0$, $|\tilde{\kappa}_i - \kappa_i'| \leq t\tilde{\kappa}_n$, $i = 1, \ldots, n-1$. To apply (3.8) we should observe that (1.13) implies $\tilde{\kappa} \in \Gamma$. Then, to deduce our desired inequality, $F[w] \geq \Psi$, we fix μ so that

$$\mu \geq \mu_0, \frac{\mu_1}{t_0} \quad \text{and} \quad \mu^2 \geq \frac{|\Psi|_1}{\delta_0 t_0}.$$

Setting $M = \sup(\varphi - u)$ we then choose k and a so that

$$ka = e^{\mu M} - 1 \quad \text{and} \quad k \geq v_0 \mu e^{\mu M}$$

to ensure $v \geq v_0$, $w \leq u$ on $\partial\mathcal{N}$. Therefore, we find that w is a lower barrier, that is,

$$\begin{aligned} F[w] = f(\tilde{\kappa}[w]) > \Psi \quad & \text{in } \mathcal{N} \\ w \leq u \quad & \text{on } \partial\mathcal{N}, \end{aligned}$$

which implies that $u \geq w$ in $\mathcal{N}$. Since the condition (iii) of Theorem 1.2 implies the mean curvature of $\partial\Omega$ is nonnegative we conclude that there exists a solution $\bar{u}$ of (3.1) which is an upper barrier. This establishes the height and boundary gradiente estimates in Theorem 1.2.

Remark 3.1. *Under the hypotheses of Theorem 1.2 the function w defined above satisfies $w \leq u$ in $\mathcal{N}$ and $\nabla^2 w \in \Gamma(\nabla u)$ on $\mathcal{N}$, for any solution u of (1.2). In fact, since $u = \varphi$ is constant on $\partial\Omega$ we have that the matrix $A(\nabla u, \nabla^2 w)$ defined in (2.10) has the form*

$$A[w] = \frac{1}{\sqrt{1 + f'^2}} diag\Big(f'\kappa', -\frac{f''}{\sqrt{1 + u_n^2}}\Big)$$

on $\partial\Omega$, where $\kappa' \in \Gamma$ denotes the principal curvatures of $\partial\Omega$. Hence, if we choose f as above we have $\nabla^2 w \in \Gamma(\nabla u)$ on $\partial\Omega$. Since $\Gamma(\nabla u)$ is an open set it follows that $\nabla^2 w \in \Gamma(\nabla u)$ in $\mathcal{N}$. We will use w as a lower barrier $\underline{u}$ in the Lemma 3.4 when we consider the conditions of Theorem 1.2.

3.2 A Priori Gradient Estimates

In this section we derive (the interior) *a priori* gradient estimates for an admissible solution u of (1.2).

Proposition 3.2. *Let $u \in C^3(\Omega) \cap C^1(\overline{\Omega})$ be an admissible solution of (1.2). Then, under the conditions (1.3)-(1.8),*

$$|\nabla u| \leq C \quad \text{in } \overline{\Omega}, \tag{3.9}$$

where C depends on $|u|_0$, $|\underline{u}|_1$ and other known data.

Proof. Set $\chi(u) = ve^{Au}$, where $v = |\nabla u|^2 = u^k u_k$ and A is a positive constant to be chosen later. Let x_0 be a point where χ attains its maximum. If $\chi(x_0) = 0$ then $|\nabla u| = 0$ and so the result is trivial. If χ achieves its maximum on $\partial\Omega$, then from the boundary gradient estimate obtained in the last section, we have a bound and we are done. Hence, we are going to assume that $\chi(x_0) > 0$ and $x_0 \in \Omega$. We fix a normal coordinate system (x^i) of M centered at x_0, such that

$$\frac{\partial}{\partial x^1}|_{x_0} = \frac{1}{|\nabla u|(x_0)} \nabla u(x_0).$$

In terms of these coordinates we have $u_1(x_0) = |\nabla u(x_0)| > 0$ and $u_j(x_0) = 0$ for $j > 1$. Since x_0 is a maximum for χ, we have

$$\begin{aligned} 0 = \chi_i(x_0) &= 2Av(x_0)e^{2Au(x_0)}u_i(x_0) + e^{2Au(x_0)}v_i(x_0) \\ &= 2e^{2Au(x_0)}\left(Avu_i(x_0) + u^l u_{l;i}(x_0)\right) \end{aligned}$$

and the matrix $\nabla^2\chi(x_0) = \{\chi_{i;j}(x_0)\}$ is nonpositive. It follows that

$$u^l(x_0)u_{l;i}(x_0) = -Av(x_0)u_i(x_0) \tag{3.10}$$

for every $1 \leq i \leq n$.
From now on all computations will be made at the point x_0. As the matrix $\{G^{i;j}\}$ is positive definite we have

$$G^{ij}\chi_{i;j} \leq 0.$$

We compute

$$\begin{aligned}\chi_{i;j} =&4e^{2Au}(Au^l u_{l;i}u_j + A^2 v u_i u_j)\\ &+ 2e^{2Au}(u^u_{;j}u_{l;i} + u^l u_{l;ij} + 2Au^l u_{l;j}u_i + Avu_{i;j})\\ =&2e^{2Au}\left(u^l u_{l;ij} + u^l_{;i}u_{l;j} + Avu_{i;j} + 2Au^l u_{l;j}u_i\right.\\ &\left.+2Au^l u_{l;j}u_i + 2A^2 v u_i u_j\right).\end{aligned}$$

Hence

$$\begin{aligned}0 \geq \frac{1}{2e^{2Au}}G^{ij}\chi_{i;j} =&G^{ij}u^l u_{l;ij} + G^{ij}u^l_{;i}u_{l;j} + AvG^{ij}u_{i;j}\\ &+ 4AG^{ij}u^l u_{l;j}u_j + 2A^2 vG^{ij}u_i u_j.\end{aligned}$$

It follows from (3.10) that

$$4AG^{ij}u^l u_{l;i}u_j = -4A^2 vG^{ij}u_i u_j,$$

so

$$G^{ij}u^l u_{l;ij} + G^{ij}u^l_{;i}u_{l;j} - 2A^2 vG^{ij}u_i u_j + AvG^{ij}u_{i;j} \leq 0. \tag{3.11}$$

We use the formula (2.21) at the Lemma 2.3 to obtain

$$\begin{aligned}G^{ij}u^l u_{l;ij} =&WG^{ij}a^k_j u^l u_{l;i}u_k + WG^{ij}a^k_j u^l u_{l;k}u_i + \frac{1}{W}G^{ij}a_{ij}u^l u^k u_{l;k}\\ &- G^{ij}R_{iljk}u^l u^k + u^l\Psi_l + \Psi_t v.\end{aligned}$$

Since

$$\begin{aligned}&R_{ijlk}u^l u^k = 0\\ &WG^{ij}a^k_j u^l u_{l;i}u_k = WG^{ij}a^k_j(-Avu_i)u_k = -AvWG^{ij}a^k_j u_i u_k\\ &\frac{1}{W}G^{ij}a_{ij}u^l u^k u_{l;k} = \frac{1}{W}G^{ij}a_{ij}u^k(-Avu_k) = -\frac{1}{W}Av^2 G^{ij}a_{ij},\end{aligned}$$

we get

$$G^{ij}u^l u_{l;ij} = -2AvWG^{ij}a^k_j u_i u_k - \frac{Av^2}{W}G^{ij}a_{ij} + u^l\Psi_l + \Psi_t v.$$

Plugin this expression bock to (3.11) we obtain

$$\begin{aligned}&- 2AvWG^{ij}a^k_j u_i u_k - \frac{Av^2}{W}G^{ij}a_{ij} + u^l\Psi_l + \Psi_t v\\ &+ G^{ij}u^l_{;i}u_{l;j} - 2A^2 vG^{ij}u_i u_j + AvG^{ij}u_{i;j} \leq 0.\end{aligned}$$

Since

$$a_{ij} = \frac{1}{W} u_{i;j}$$

we may rewrite the above inequality as

$$\begin{aligned} &G^{ij} u^l_{;i} u_{l;j} - 2AWvG^{ij} a^k_j u_i u_k - 2A^2 v G^{ij} u_i u_j \\ &+ \left(AvW - \frac{Av^2}{W} \right) G^{ij} a_{ij} + u^l \Psi_l + \Psi_t v \le 0. \end{aligned}$$

Using the hypothesis $\Psi_t \ge 0$ and that

$$AvW - \frac{Av^2}{W} = \frac{Av}{W},$$

we obtain

$$G^{ij} u^l_{;i} u_{l;j} - 2AWvG^{ij} a^k_j u_i u_k - 2A^2 v G^{ij} u_i u_j + \frac{Av}{W} G^{ij} a_{ij} + \Psi_l u^l \le 0. \quad (3.12)$$

From the choice of the coordinate system and (3.10),

$$u_{1;1} = -Av \quad \text{and} \quad u_{1;i} = u_{i;1} = 0 \quad (i > 1).$$

After a rotation of the coordinates $(x^2, \ldots, x^n)$ we may assume that $\nabla^2 u = \{u_{i;j}(x_0)\}$ is diagonal. Since

$$a^j_i = g^{jk} a_{ki} = \frac{1}{W} \left(\sigma^{jk} - \frac{u^j u^k}{W^2} \right) u_{k;i},$$

at x_0 we then have

$$\begin{aligned} a^j_i &= 0 \quad (i \ne j) \\ a^1_1 &= \frac{1}{W^3} u_{1;1} = -\frac{Av}{W^3} < 0 \\ a^i_i &= \frac{1}{W} u_{i;i} \quad (i > 1). \end{aligned}$$

It follows from Lemma 2.2 that the matrix $\{F^j_i\}$ is diagonal. Then the matrix $\{G^{ij}\}$ is also diagonal with

$$\begin{aligned} G^{ii} &= \frac{1}{W} F^i_k g^{ki} = \frac{1}{W} f_i \\ G^{11} &= \frac{1}{W} F^1_k g^{k1} = \frac{1}{W^3} f_1. \end{aligned}$$

Using these relations and discarding the term

$$\frac{Av}{W}G^{ij}a_{ij} = \frac{Av}{W^2}\sum_i f_i\kappa_i \geq 0$$

we get from (3.12) the following inequality

$$G^{ii}u^2_{i;i} - 2AWvG^{11}a^1_1(u_1)^2 - 2A^2vG^{11}(u_1)^2 + \Psi_1 u_1 \leq 0,$$

which may be rewritten as

$$\sum_{\alpha>1} G^{\alpha\alpha}u^2_{\alpha;\alpha} + G^{11}\left(\frac{2A^2v^3}{W^2} - 2A^2v^2 + A^2v^2\right) + \Psi_1\sqrt{v} \leq 0.$$

Since

$$\frac{2A^2v^3}{W^2} - 2A^2v^2 + A^2v^2 = \frac{A^2v^3 - A^2v^2}{(1+v)^2},$$

we have

$$\sum_{\alpha>1} G^{\alpha\alpha}u^2_{\alpha;\alpha} + \frac{A^2v^3 - A^2v^2}{(1+v)^2}G^{11} + \Psi_1\sqrt{v} \leq 0.$$

Then

$$\frac{A^2v^3 - A^2v^2}{(1+v)^2}\frac{1}{W^3}f_1 \leq -\Psi_1\sqrt{v} \leq |D\Psi|\sqrt{v}.$$

Once

$$\kappa_1 = a^1_1 = -\frac{Av}{W^3} < 0,$$

we may apply hypothesis (1.8) to get $f_1 \geq \nu_0 > 0$, which implies that

$$\frac{A^2v^3 - A^2v^2}{W^5\sqrt{v}} \leq \frac{|D\Psi|}{\nu_0}.$$

Now we choose

$$A = \left(\frac{2}{c_0}\sup_{M\times I}|D\Psi|\right)^{1/2},$$

where I is the interval $I = [-C, C]$ with C being a uniform constant that satisfies $|u|_0 < C$. It follows that

$$\frac{(u_1)^3((u_1)^2 - 1)}{(1 + (u_1)^2)^{5/2}} \leq \frac{1}{2}.$$

i.e.,

$$(u_1)^5 - (u_1)^3 - \frac{1}{2}(1 + (u_1)^2)^{5/2} < 0.$$

Since $u_1 > 0$ this yields a bound for u_1 and hence for $\chi(x_0)$, which implies the desired estimate. □

3.3 The Boundary Estimates for Second Derivatives

In this section we establish a *a priori* boundary estimates to the second derivatives of prospective solutions of (1.2). The estimate for pure tangential derivatives follows from the relation $u = \varphi$ on $\partial\Omega$. It remains to estimate the mixed and double normal derivatives.
In order to obtain the mixed and double normal derivatives we use the *barrier method.* The linear operator to be used is given by

$$L = G^{ij} - b^i,$$

where

$$b^i = \frac{1}{W^2} \sum_j f_j \kappa_j u^i.$$

As it was shown in last chapter, it follows from the concavity of f that

$$\sum_j f_j \kappa_j \le f,$$

(see 2.19). Hence, we may conclude from (1.6) and the C^0 estimate that $|b^i| \le C$ for an uniform constant C.

To proceed, we first derive some key preliminary lemmas. Let x_0 be a point on $\partial\Omega$. Let $\rho(x)$ denote the distance from x to x_0,

$$\rho(x) = \mathrm{dist}(x, x_0),$$

and set

$$\Omega_\delta = \{x \in \Omega \,:\, \rho(x) < \delta\}.$$

Since $(\rho^2)_{i;j}(x_0) = 2\sigma_{ij}(x_0)$, by choosing $\delta > 0$ sufficiently small we may assume ρ smooth in Ω_δ and

$$\sigma_{ij} \le (\rho^2)_{i;j} \le 3\sigma_{ij} \quad \text{in } \Omega_\delta. \tag{3.13}$$

Since $\partial\Omega$ is smooth, we may also assume the distance function $d(x)$ to the boundary $\partial\Omega$ is smooth in Ω_δ. In what follows, we denote (also) by φ the extension of the boundary function φ to Ω_δ being constant along of the geodesic normals starting from $\partial\Omega$.

Now we begin the construction of our barrier function. Let ξ be a C^2 arbitrary vector field defined in Ω_δ and χ any extension to Ω_δ of the vector $\nabla u(x_0)$. Consider the function

$$w = \langle \nabla u, \xi\rangle - \langle \nabla\varphi, \xi\rangle - \frac{1}{2}|\nabla u - \chi|^2. \tag{3.14}$$

The function w satisfies a fundamental inequality.

Proposition 3.3. *Assume that f satisfies (1.3)-(1.6). Then the function w satisfies*

$$L[w] \leq C(1 + G^{ij}\sigma_{ij} + G^{ij}w_i w_j) \quad \text{in } \Omega_\delta, \tag{3.15}$$

where C is a uniform positive constant.

Proof. For convenience we denote $\mu = \langle \nabla\varphi, \xi\rangle$. First we calculate the derivatives of w in an arbitrary coordinate system. We have

$$\begin{aligned} w_i &= \langle \nabla_i \nabla u, \xi\rangle + \langle \nabla u, \nabla_i \xi\rangle - \mu_i - \langle \nabla_i \nabla u - \nabla_i \chi, \nabla u - \chi\rangle \\ &= \left(\xi^k + \chi^k - u^k\right) u_{k;i} + \left((\xi^k)_i + (\chi^k)_i\right) u_k - \mu_i - \langle \nabla_i \chi, \chi\rangle \end{aligned}$$

and

$$\begin{aligned} w_{i;j} =& \langle \nabla_j \nabla_i \nabla u, \xi\rangle + \langle \nabla_i \nabla u, \nabla_j \xi\rangle + \langle \nabla_j \nabla u, \nabla_i \xi\rangle + \langle \nabla u, \nabla_j \nabla_i \xi\rangle \\ & - \mu_{i;j} - \langle \nabla_j \nabla_i \nabla u - \nabla_j \nabla_i \chi, \nabla u - \chi\rangle - \langle \nabla_i \nabla u - \nabla_i \chi, \nabla_j \nabla u - \nabla_j \chi\rangle \\ =& \left(\xi^k + K\xi^k - u^k\right) u_{k;ij} + \left((\xi^k)_j + (\chi^k)_j\right) u_{k;i} + \left((\xi^k)_i + (\chi^k)_i\right) u_{k;j} \\ & - u^k_{;i} u_{k;j} + \left((\xi^k)_{i;j} + (\chi^k)_{i;j}\right) u_k - \mu_{i;j} - \langle \nabla_i \chi, \nabla_j \chi\rangle - \langle \nabla_j \nabla_i \chi, \chi\rangle, \end{aligned}$$

where we denote by $\xi^k, (\xi^k)_i$ and $(\xi^k)_{i;j}$ the components of the vectors $\xi, \nabla_i \xi$ and $\nabla_j \nabla_i \xi$, respectively (the same notation is used for χ).
Therefore,

$$\begin{aligned} G^{ij} w_{i;j} =& \left(\xi^k + K\xi^k - u^k\right) G^{ij} u_{k;ij} + 2G^{ij}\left((\xi^k)_j + (\chi^k)_j\right) u_{k;i} - G^{ij} u^k_{;i} u_{k;j} \\ & + G^{ij}\Big(\left((\xi^k)_{i;j} + (\chi^k)_{i;j}\right) u_k - \mu_{i;j} - \langle \nabla_i \chi, \nabla_j \chi\rangle - \langle \nabla_j \nabla_i \chi, \chi\rangle \Big). \end{aligned}$$

Now we use (2.21) to obtain

$$\begin{aligned}\left(\xi^k+\chi^k-u^k\right)G^{ij}u_{k;ij} =&W\left(\xi^k+\chi^k-u^k\right)G^{ij}a_j^l u_{k;i}u_l+W\left(\xi^k+\chi^k-u^k\right)\\ &\times G^{ij}a_j^l u_{k;l}u_i+\frac{1}{W}\left(\xi^k+\chi^k-u^k\right)G^{jl}a_{jl}u^i u_{k;i}\\ &+\left(\xi^k+\chi^k-u^k\right)\left(\Psi_k+\Psi_t u_k-G^{ij}R_{iljk}u^l\right).\end{aligned}$$

On the other hand, it follows from the expression for w_i that

$$\left(\xi^k+\chi^k-u^k\right)u_{i;k}=w_i-\left((\xi^k)_i+(\chi^k)_i\right)u_k+\mu_i+\langle\nabla_i\chi,\chi\rangle.$$

Substituting this equality in the above equations we get

$$\begin{aligned}G^{ij}w_{i;j}=&\,WG^{ij}a_j^l w_i u_l+WG^{ij}a_j^l w_l u_i+\frac{1}{W}G^{jl}a_{jl}u^i w_i-G^{ij}u^k_{;i}u_{k;j}\\ &+2G^{ij}\left((\xi^k)_j+(\chi^k)_j\right)u_{k;i}+WG^{ij}a_j^l u_l\Big(\mu_i-\left((\xi^k)_i+(\chi^k)_i\right)u_k\\ &+\langle\nabla_i\chi,\chi\rangle\Big)+WG^{ij}a_j^l u_i\Big(\mu_l-\left((\xi^k)_l+(\chi^k)_l\right)u_k+\langle\nabla_l\chi,\chi\rangle\Big)+\\ &\frac{1}{W}G^{jl}a_{jl}u^i\Big(\mu_i-\left((\xi^k)_i+(\chi^k)_i\right)u_k+\langle\nabla_i\chi,\chi\rangle\Big)\\ &+G^{ij}\Big(\left((\xi^k)_{i;j}+(\chi^k)_{i;j}\right)u_k-\mu_{i;j}-\langle\nabla_i\chi,\nabla_j\chi\rangle-\langle\nabla_j\nabla_i\chi,\chi\rangle\\ &-\left(\xi^k+\chi^k-u^k\right)R_{iljk}u^l\Big)+\left(\xi^k+\chi^k-u^k\right)\left(\Psi_k+\Psi_t u_k\right).\end{aligned}\tag{3.16}$$

Now note that, since (3.15) does not depend on the coordinate system, i.e., it is a tensorial inequality, it is sufficient to prove it in a fixed coordinate system. Given $x\in\Omega$, let (x^i) be the *special* coordinate system centered at x. In this coordinates, at x, the inequality (3.15) takes the form

$$L[w]=\frac{1}{W}\sum_i f_i w_{i;i}-b^i w_i\le C\left(1+\frac{1}{W}\sum_i f_i\sigma_{ii}+\frac{1}{W}\sum_i f_i w_i^2\right).\tag{3.17}$$

We will prove the above inequality. In what follows all computations are done at the point x.

In these coordinates we have (at x)

$$\kappa_i=a_i^j=a_{ij}=\frac{1}{W}u_{i;j}\delta_{ij}$$

and

$$G^{ij} = \frac{1}{W} f_i \delta_i^j.$$

Since the quantities depending on ∇u, ξ, χ and μ are under control, we get

$$\begin{aligned}
WG^{ij}a_j^l w_i u_l &= \sum_i f_i \kappa_i w_i u_i \leq \varepsilon \sum_i f_i \kappa_i^2 + \frac{1}{\varepsilon} \sum_i f_i w_i^2 u_i^2 \\
&\leq \varepsilon \sum_i f_i \kappa_i^2 + C \sum_i f_i w_i^2 \\
2G^{ij} \left((\xi^k)_j + (\chi^k)_j \right) u_{k;i} &= 2 \left((\xi^i)_i + (\chi^i)_i \right) f_i \kappa_i \leq \varepsilon \sum_i f_i \kappa_i^2 + C \sum_i f_i \\
WG^{ij}a_j^l u_l \mu_i &= \sum_i f_i \kappa_i u_i \mu_i \leq \varepsilon \sum_i f_i \kappa_i^2 + C \sum_i f_i \\
G^{ij} u_{;i}^k u_{k;j} &= G^{ij} \sigma^{kl} u_{l;i} u_{k;j} = W^2 G^{ij} \sigma^{kl} a_{l;i} a_{k;j} = W \sigma^{ii} f_i \kappa_i^2 \\
&\geq C_0 \sum_i f_i \kappa_i^2 \\
\frac{1}{W} G^{jl} a_{jl} u^i \mu_i &= \frac{1}{W^2} \sum_j f_j \kappa_j u^i \mu_i \leq C \sum_j f_j \kappa_j \leq C \\
G^{i;j} \mu_{i;j} &= \frac{1}{W} \sum_i f_i \mu_{i;i} \leq C \sum_i f_i,
\end{aligned}$$

where $\varepsilon > 0$ is any positive number and $C_0 > 0$ depends only on $\sigma|_{\overline{\Omega}}$. Note that to obtain the above inequalities we made use of the ellipticity condition $f_i > 0$. Estimating all the terms in (3.16) as above, we conclude that equality (3.16) implies the inequality

$$\begin{aligned}
G^{ij} w_{i;j} - \frac{1}{W^2} G^{jl} a_{jl} u^i w_i \leq{}& \varepsilon C \sum_i f_i \kappa_i^2 + C \sum_i f_i w_i^2 - C_0 \sum_i f_i \kappa_i^2 \\
&+ C \sum_i f_i + C,
\end{aligned}$$

i.e.

$$L[w] \leq (\varepsilon C - C_0) \sum_i f_i \kappa_i^2 + C \sum_i f_i w_i^2 + C \sum_i f_i + C. \tag{3.18}$$

Choosing $\varepsilon > 0$ sufficiently small such that the first term on the sum above becomes negative we obtain

$$L[w] \leq C\big(1 + \sum_i f_i + \sum_i f_i w_i^2\big).$$

Using that $\sigma_{ii} > C_0 > 0$ in $\overline{\Omega}$ and W is under controll, we get (3.17). $\square$

We note that inequality (3.15) may be simplified further. In fact, since

$$G^{ij}\sigma_{ij} \geq \delta_0 > 0,$$

replacing C to $C/\delta_0 + C$ (we may assume $1 > \delta_0 > 0$) we get

$$L[w] \leq C(G^{ij}\sigma_{ij} + G^{ij}w_i w_j) \quad \text{in } \Omega_\delta. \tag{3.19}$$

Setting

$$\tilde{w} = 1 - e^{-a_0 w} \tag{3.20}$$

for a positive constant a_0, we get

$$\tilde{w}_i = a_0 e^{-a_0 w} w_i$$

and

$$\tilde{w}_{i;j} = a_0 e^{-a_0 w}\left(w_{i;j} - a_0 w_i w_j\right).$$

Therefore,

$$L[\tilde{w}] = G^{ij}\tilde{w}_{ij} - b^i \tilde{w}_i = a_0 e^{-a_0 w}(L[w] - a_0 G^{ij} w_i w_j),$$

if we choose a_0 large such that $a_0 \geq C$, where C is the constant in (3.19),

$$L[w] - a_0 G^{ij} w_i w_j \leq L[w] - C G^{ij} w_i w_j \leq C G^{ij}\sigma_{ij}.$$

Therefore

$$L[\tilde{w}] \leq C G^{ij}\sigma_{ij}. \tag{3.21}$$

The following lemma gives the elements to complete the construction of our barrier function.

Lemma 3.4. *Assume that f satisfies (1.3)-(1.9). There exist some uniform positive constants t, δ, ε sufficiently small and N sufficiently large such that the function*

$$v = (u - \underline{u}) + td - \frac{N}{2}d^2 \tag{3.22}$$

satisfies

$$L[v] \leq -\varepsilon(1 + G^{ij}\sigma_{ij}) \quad \text{in } \Omega_\delta \tag{3.23}$$

and

$$v \geq 0 \quad \text{on } \partial\Omega_\delta.$$

Proof. Since $\underline{u}$ is locally strictly convex in a neighborhood of $\partial\Omega$ we may choose $\delta > 0$ small enough such that the eigenvalues $\lambda(\nabla^2\underline{u}) \in \Gamma^+$ in Ω_δ. In particular, we have that $\nabla^2\underline{u} \in \Gamma(\nabla u)$ in Ω_δ.
Consider the function $v^* = \underline{u} - 3\varepsilon\rho^2$. Since $\Gamma(\nabla u)$ is open and $F[\underline{u}] > 0$, we may choose $\varepsilon > 0$ sufficiently small, such that v^* is admissible and $\nabla^2 v^* \in \Gamma(\nabla u)$ in Ω_δ.
We recall that it follows from the concavity of $G(p, \cdot)$ the inequality

$$G^{ij}(p, r)(r_{ij} - s_{ij}) \leq G(p, r) - G(p, s) \quad \forall\, r, s \in \Gamma(p).$$

Applying this property we get

$$\begin{aligned} L[u - \underline{u}] &= L[u - v^* - 3\varepsilon\rho^2] \\ &= G^{ij}(u_{i;j} - v^*_{i;j}) - b^i(u_i - v^*_i) - 3\varepsilon L[\rho^2] \\ &\leq G(\nabla u, \nabla^2 u) - G(\nabla u, \nabla^2 v^*) - b^i(u_i - v^*_i) \\ &- 3\varepsilon G^{ij}(\rho^2)_{\,i;j} + 6\varepsilon\rho b^i\rho_i. \end{aligned}$$

Since $G(\nabla u, \nabla^2 u) = \Psi$ and $G(\nabla u, \nabla^2 v^*) > 0$, it follows from the C^1 estimate and the boundedness of b^i that

$$L[u - \underline{u}] \leq C_1 - 3\varepsilon G^{ij}(\rho^2)_{i;j}.$$

Hence, we conclude from (3.13)

$$L[u - \underline{u}] \leq C_1 - 3\varepsilon G^{ij}\sigma_{i;j}. \tag{3.24}$$

As in the previous lemma, the inequality proposed is a tensorial one. So, it is enough to prove (3.23) in a fixed coordinate system. Since d is smooth on Ω_δ we may define Fermi coordinates on Ω_δ as follows: we associate to $x \in \Omega_\delta$

coordinates (y^i) such that $x = \exp^M(y^n\nu(y))$, where $y(x) = (y^1, \ldots, y^{n-1})$ is the closest point to x in $\partial\Omega$, $\nu(y)$ is the interior unit normal vector field to $\partial\Omega$ and $y^n = d(x)$. In these coordinates we have $d_\alpha(x) = 0$, $1 \leq \alpha \leq n-1$, and $d_n(x) = 1$. Hence, a direct calculus yields

$$L\left[td - \frac{N}{2}d^2\right] = (t - dN)L[d] - NG^{nn}.$$

Since there exists a uniform positive constant C that satisfies $d_{i;j} \leq C\sigma_{ij}$ in Ω_δ and $|b^i| < C$, we have

$$L\left[td - \frac{N}{2}d^2\right] \leq C_2(t + N\delta)(1 + G^{ij}\sigma_{ij}) - NG^{nn}.$$

This inequality and (3.24) give

$$\begin{aligned} L[v] &\leq L[u - \underline{u}] + L\left[td - \frac{N}{2}d^2\right] \\ &\leq C_1 - 3\varepsilon G^{ij}\sigma_{ij} + C_2(t + N\delta)(1 + G^{ij}\sigma_{ij}) - NG^{nn} \\ &= C_1 + C_2(t + N\delta) + (C_2(t + N\delta) - 3\varepsilon)\, G^{ij}\sigma_{ij} - NG^{nn}. \end{aligned}$$

Now we follow the reasoning presented in [19]. We choose indices such that $f_1 \geq \cdots \geq f_n$. Since the eigenvalues of the matrix G^{ij} are $\frac{1}{W}f_1, \ldots, \frac{1}{W}f_n$, it follows from our choice of indices that

$$G^{nn} \geq \frac{1}{W}f_n \geq c_1 f_n$$

and we also have

$$G^{ij}\sigma_{ij} \geq c_2 \sum_i f_i,$$

where the constants c_i depend only on $|u|_1$, and the metric of M, $\sigma|_{\overline{\Omega}}$. To verify the above inequalities we fix (x^i) the *special* coordinate system centered at the given point $x \in \Omega_\delta$. In terms of these coordinates, the matrices G^{ij} and g^{ij} are diagonal at the given point x, therefore

$$G^{ij}\sigma_{ij} = G^{kl}_x \sigma^x_{rs} \frac{\partial y^i}{\partial x^k}\frac{\partial y^j}{\partial x^l}\frac{\partial x^r}{\partial y^i}\frac{\partial x^s}{\partial y^j} = G^{kl}_x\sigma^x_{kl} = \frac{1}{W}\sum_i f_i\sigma^x_{ii} \geq c_2\sum_i f_i.$$

Similarly,

$$G^{nn} = G_x^{kl}\frac{\partial y^n}{\partial x^k}\frac{\partial y^n}{\partial x^l} = \frac{1}{W}\sum_i f_i\left(\frac{\partial y^n}{\partial x^i}\right)^2 \geq \frac{1}{W}f_n\sum_i\left(\frac{\partial y^n}{\partial x^i}\right)^2$$
$$= \frac{1}{W}f_n g_y^{nn} \geq c_2\sum_i f_i.$$

We use the arithmetic-geometric mean inequality and (1.9) to get

$$\varepsilon G^{ij}\sigma_{ij} + NG^{nn} \geq c_2\sum_i f_i + c_1 N f_n$$
$$\geq cn\varepsilon(Nf_1\cdot\ldots\cdot f_n)^{1/n} =: C_3N^{1/n}.$$

Now we apply this relation into the above inequality to get

$$L[v] \leq C_1 + C_2(t+N\delta) + (C_2(t+N\delta) - 2\varepsilon)G^{ij}\sigma_{ij} - C_3N^{1/n}.$$

Since $\delta^2 \leq t\delta/N$ implies $t\delta - N/2\delta^2 \geq 0$ and $u \geq \underline{u}$, we choose $t = \frac{\varepsilon}{2C_2}$ and $\delta \leq \frac{t}{N}$ to get $v \geq 0$ on $\Omega \cap \partial\Omega_\delta$. With this choice we have

$$L[v] \leq C_1 - \varepsilon G^{ij}\sigma_{ij} - C_3N^{1/n}.$$

By choosing N large such that $C_3N^{1/n} \geq C_1 + 2\varepsilon$ we obtain (3.23). □

Remark 3.5. *Under the hyphoteses of Theorem (1.2) we construct a subsolution w defined in Ω_δ and that is not necessarily strictly convex but satisfies $\nabla^2 w \in \Gamma(\nabla u)$. We replace $\underline{u}$ by w in the proof presented above to get the result. See Remark 3.1.*

Mixed Second Derivative Boundary Estimate

We define the function

$$h = \tilde{w} + b_0\rho^2 + c_0 v, \tag{3.25}$$

where b_0 and c_0 are constants to be chosen later. Assume the vector field ξ is tangent along $\partial\Omega \cap \partial\Omega_\delta$. With this choice we have

$$\tilde{w} = 1 - e^{-a_0 w} = 1 - \exp\left(-a_0\langle\nabla u, \xi\rangle + a_0\langle\nabla\varphi, \xi\rangle + a_0\frac{1}{2}|\nabla u - \chi|^2\right)$$
$$= 1 - \exp\left(a_0\frac{1}{2}|\nabla u - \chi|^2\right)$$

on $\partial\Omega \cap \partial\Omega_\delta$, for $u = \varphi$ on $\partial\Omega$. We also note that, since $\chi(x_0) = \nabla u(x_0)$, for any vector field η tangent along $\partial\Omega \cap \partial\Omega_\delta$, we have

$$\tilde{w}(x_0) = \nabla_\eta \tilde{w}(x_0) = 0.$$

Hence we conclude that $\tilde{w} = O(\rho^2)$ on $\partial\Omega \cap \partial\Omega_\delta$, i.e., there exists a positive constant M such that

$$\tilde{w} \le M\rho^2 \quad \text{on } \partial\Omega \cap \partial\Omega_\delta,$$

if $\delta > 0$ is small enough. Then, since $v \ge 0$ on $\partial\Omega_\delta$, if b_0 is sufficiently large we have $h \ge 0$ on $\partial\Omega_\delta$. On the other hand, it follows from (3.13), (3.21) and (3.23) that

$$\begin{aligned} L[h] &= L[\tilde{w}] + b_0 L[\rho^2] + c_0 L[v] \\ &\le (C_1 + C_2 b_0 - c_0\epsilon)(1 + G^{ij}\sigma_{ij}) + b_0. \end{aligned}$$

Therefore, for $c_0 \gg b_0 \gg 1$ both sufficiently large, we get $L[h] \le 0$ in Ω_δ and $h \ge 0$ on $\partial\Omega_\delta$. It follows from the maximum principle that $h \ge 0$ in Ω_δ. Consequently,

$$\nabla_\nu h(x_0) \ge 0$$

i.e.,

$$\begin{aligned} \nabla_\nu h(x_0) &= a_0 e^{-a_0 w(x_0)} \big(u_{\xi;\nu} + \langle \nabla u, \nabla_\nu \xi\rangle - |\nabla u - \chi| \nabla_\nu |\nabla u - \chi|\big)(x_0) \\ &\quad + 2b_0 \rho \nabla_\nu \rho(x_0) + c_0 \left(\nabla_\nu (u - \underline{u}) + t\nabla_\nu d - Nd\nabla_\nu d\right)(x_0) \\ &= a_0 u_{\xi;\nu}(x_0) + a_0 \langle \nabla u, \nabla_\nu \xi\rangle(x_0) + c_0 (u - \underline{u})_\nu (x_0) + c_0 t \ge 0. \end{aligned}$$

So

$$u_{\xi;\nu}(x_0) \ge -\langle \nabla u, \nabla_\nu \xi\rangle(x_0) - \frac{c_0}{a_0}(u - \underline{u})_\nu(x_0) - \frac{c_0}{a_0} t.$$

Replacing ξ by $-\xi$ at the definition of w we establish a bound for the mixed normal-tangential derivatives on $\partial\Omega$, i.e.,

$$|u_{\xi;\nu}(x_0)| \le C,$$

for any direction tangent ξ to $\partial\Omega$. Since x_0 is arbitrary, we have

$$|u_{\xi;\nu}| < C \qquad \partial\Omega. \tag{3.26}$$

Double Normal Second Derivative Boundary Estimate

For the pure normal second derivative, since $\sum_i \kappa_i[u] \geq \delta_0 > 0$, we need only to derive an upper bound

$$u_{\nu;\nu} \leq C \quad \text{on } \partial\Omega. \tag{3.27}$$

In fact, it follows from the trace invariance that

$$\operatorname{tr}(a_i^j) = \sum_i a_i^i = \sum_i \kappa_i \geq \delta > 0,$$

regardless of the coordinate system chosen. Hence, given an arbitrary point $y \in \partial\Omega$, fixing a coordinate system (y^i) centered at y such that $\frac{\partial}{\partial y^n}|_y = \nu(y)$ and the coordinate vector fields $\{\frac{\partial}{\partial y^i}|_y\}$ are orthogonal with respect to the induced inner product given by $g = Id + \nabla u \otimes \nabla u$, we obtain (at y)

$$\begin{aligned}\sum_i a_i^i &= \frac{1}{W} g^{ki} u_{i;k} = \frac{1}{W} g^{ii} u_{i;i} \\ &= \sum_i \frac{1}{W g_{ii}} u_{i;i} = \sum_i \frac{1}{W(\sigma_{ii} + u_i^2)} u_{i;i}.\end{aligned}$$

In this coordinates we have

$$u_{i;i} = A_i^2 u_{\nu;\nu} + 2A_i B_i^\alpha u_{\nu;\alpha} + B_i^\alpha B_i^\beta u_{\alpha;\beta}, \quad 1 \leq i \leq n-1,$$

where indices α and β denote tangential derivatives. Hence

$$u_{\nu;\nu} \geq \left(\sum_{i=1}^n \frac{A_i^2}{W(\sigma_{ii} + u_i^2)} \right)^{-1} \left(\delta - \sum_{i=1}^n \frac{2A_i B_i^\alpha u_{\nu;\alpha} + B_i^\alpha B_i^\beta u_{\alpha;\beta}}{W(\sigma_{ii} + u_i^2)} \right),$$

since $A_n = 1$ and we already have tangential and tangential-normal estimates, it follows that

$$u_{\nu;\nu} = u_{n;n} \geq C,$$

for a uniform constant C. Therefore, it remains only to prove (3.27).

First we note that the equality $u = \varphi$ on $\partial\Omega$ implies

$$u_{\xi;\eta}(y) = \varphi_{\xi;\eta}(y) - u_\nu(y) \Pi(\xi, \eta)(y), \tag{3.28}$$

for any tangent vectors $\xi, \eta \in T_y(\partial\Omega) \subset T_yM$, $y \in \partial\Omega$, where Π denotes the second fundamental form of $\partial\Omega$. Let T_u be the $(0,2)$ tensor defined on $\partial\Omega$ by

$$T_u = \big(\tilde{\nabla}^2\varphi - u_\nu\Pi\big), \tag{3.29}$$

where $\tilde{\nabla}$ is the induced conexion on $\partial\Omega$. We note that, since $\varphi_\nu = 0$ we have $\nabla^2\varphi = \tilde{\nabla}^2\varphi$ on $T(\partial\Omega)$. Since $a_{\alpha\beta} = \frac{1}{W}u_{\alpha;\beta}$, it follows from the equality (3.28) that the components of T_u in terms of tangent coordinates (y^α) are $Wa_{\alpha\beta}$. We denote by $\tilde{\kappa} = (\tilde{\kappa}_1, \ldots, \tilde{\kappa}_{n-1})$ the eigenvalues of the tensor T_u with respect to the inner product defined on $\partial\Omega$ by the matrix $\tilde{g} = \tilde{\sigma} + \tilde{\nabla}\varphi \otimes \tilde{\nabla}\varphi$, where $\tilde{\sigma}$ is the induced metric on $\partial\Omega$ by σ.
Let Γ' be the projection of Γ on $\mathbb{R}^{n-1}$, i.e., if $\kappa = (\kappa_1, \ldots, \kappa_n) \in \Gamma$ then $\kappa' = (\kappa_1, \ldots, \kappa_{n-1}) \in \Gamma'$. We denote by $d(\kappa')$ the distance from $\kappa' \in \Gamma'$ to $\partial\Gamma'$. We point out that Γ' is also an open convex symmetric cone.

We will analyze the behavior of $d(\kappa'[u])$, for an admissible solution u of (1.2). First we fix Fermi coordinates (y^i) in M along $\partial\Omega$, such that y^n is the normal coordinate and the tangent coordinate vectors $\{\frac{\partial}{\partial y^\alpha}|_{y_0}\}$, $1 \leq \alpha \leq n-1$, is an orthonormal basis of eigenvectors that diagonalize T_u at a given $y_0 \in \partial\Omega$, with respect to the inner product $\tilde{g} = \tilde{\sigma} + \tilde{\nabla}\varphi \otimes \tilde{\nabla}\varphi$. At y_0 the matrix of the second fundamental of Σ, in terms of this coordinate system is given by

$$a_{ij} = \frac{1}{W}\begin{pmatrix} u_{1;1} & 0 & \cdots & u_{1;\nu} \\ 0 & u_{2;2} & \cdots & u_{2;\nu} \\ \vdots & & \ddots & \vdots \\ u_{\nu;1} & u_{\nu;1} & \cdots & u_{\nu;\nu} \end{pmatrix}. \tag{3.30}$$

We note that $\tilde{\kappa} = (u_{1;1}, \ldots, u_{n-1;n-1})$ are (also) the eigenvalues of the tensor T_u defined above. Since the principal curvatures $\kappa[u] = (\kappa_1, \ldots, \kappa_n)$ of Σ at $\big(y_0, u(y_0)\big)$ are the real roots of the equation $\det\big(a_{ij} - \kappa g_{ij}\big) = 0$ and $g_{\alpha\beta}(y_0) = \tilde{g}_{\alpha\beta}(y_0) = \delta_{\alpha\beta}$ for $1 \leq \alpha, \beta \leq n-1$, they satisfy

$$\det\begin{pmatrix} \frac{1}{W}u_{1;1} - \kappa & 0 & \cdots & \frac{1}{W}u_{1;\nu} - g_{1n} \\ 0 & \frac{1}{W}u_{2;2} - \kappa & \cdots & \frac{1}{W}u_{2;\nu} - g_{2n} \\ \vdots & & \ddots & \vdots \\ \frac{1}{W}u_{\nu;1} - g_{1n} & \frac{1}{W}u_{\nu;1} - g_{2n} & \cdots & \frac{1}{W}u_{\nu;\nu} - \kappa g_{n;n} \end{pmatrix} = 0.$$

By Lemma 1.2 of [8], the C^1 and the tangential-normal estimates, the principal curvatures $\kappa[u](y) = (\kappa_1, \ldots, \kappa_n)$ of Σ, at $\big(y_0, u(y_0)\big)$, behave like

$$\kappa_\alpha = \frac{1}{W} u_{\alpha;\alpha} + o(1), \qquad 1 \le \alpha \le n-1, \tag{3.31}$$

$$\kappa_n = \frac{1}{W g_{n;n}} u_{\nu;\nu} \left(1 + O\left(\frac{1}{u_{\nu;\nu}}\right)\right), \tag{3.32}$$

as $|u_{\nu;\nu}| \to \infty$. Since u is admissible, we have $\kappa'[u] = (\kappa_\alpha) \in \Gamma'$, therefore $W\kappa'[u] \in \Gamma'$. Hence, since Γ' is open, for $u_{\nu;\nu}$ large (we may assume $u_{\nu;\nu} \ge 0$ because we already have a lower bound) we have $\tilde\kappa = (u_{1;1}, \ldots, u_{n-1;n-1}) \in \Gamma'$. Since $y_0 \in \partial\Omega$ is arbitrary, it follows from the gradient, the tangent and tangent-normal second estimate that there exists a uniform positive constant $N_0 > 0$ such that the eigenvalues $\tilde\kappa$ of T_u satisfy $\tilde\kappa \in \Gamma'$ when $u_{\nu;\nu} \ge N_0$.

The following lemma is the key ingredient to obtain our estimate. It is an adaption to the case of curvature equations of the technique used by Guan for Hessian equations in [19]. On the other hand, the technique employed by Guan is inspired in the brilliant idea introduced by Trundiger in [43].

Lemma 3.6. *Let N_0 be the constant defined above and suppose that $u_{\nu;\nu} \ge N_0$. Then there exists a uniform constant $c_0 > 0$ such that*

$$d(y) = d(\tilde\kappa[u](y)) \ge c_0 \quad \text{on } \partial\Omega.$$

Proof. Consider a point $y_0 \in \partial\Omega$ where the function $d(y)$ attains its minimum in Ω. It suffices to prove that $d(y_0) \ge c_0 > 0$. As above we fix Fermi coordinates (y^i) in M along $\partial\Omega$, centered at y_0, such that y^n is the normal coordinate and the tangent coordinate vectors $\{\frac{\partial}{\partial y^\alpha}|_{y_0}\}_{\alpha<n}$ that diagonalize T_u at y_0 with respect to the inner product given by $\tilde\sigma + \tilde\nabla\varphi \otimes \tilde\nabla\varphi$. We choose indices such that

$$\tilde\kappa_1(y_0) \le \cdots \le \tilde\kappa_{n-1}(y_0).$$

It follows from (3.28) that the coordinate system (y^α) diagonalizes also the restriction of $\nabla^2 u$ to $T(\partial\Omega)$ at y_0 and

$$\tilde\kappa_\alpha(y_0) = u_{\alpha;\alpha}(y_0) \quad \alpha < n. \tag{3.33}$$

We extend ν to the coordinate neighborhood by taking its parallel transport along normal geodesics departing from $\partial\Omega$ and set

$$b_{\alpha\beta} = \Pi\left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right) = \left\langle \nabla_{\frac{\partial}{\partial y^\alpha}} \frac{\partial}{\partial y^\beta}, \nu \right\rangle.$$

Using Lemma 6.1 of [8], we may find a vector $\mu' = (\mu_1, \dots, \mu_{n-1}) \in \mathbb{R}^{n-1}$ such that

$$\mu_1 \geq \cdots \geq \mu_{n-1} \geq 0, \qquad \sum_{\alpha<n} \mu_\alpha = 1$$

and

$$d(y_0) = \sum_{\alpha<n} \mu_\alpha \tilde{\kappa}_\alpha(y_0) \left(= \sum_{\alpha<n} \mu_\alpha u_{\alpha;\alpha}(y_0) \right). \tag{3.34}$$

Moreover

$$\Gamma' \subset \left\{ \lambda' \in \mathbb{R}^{n-1} \,:\, \mu' \cdot \lambda' > 0 \right\}. \tag{3.35}$$

Now we apply Lemma 6.2 of [8], with $\mu_n = 0$, to obtain, for all $y \in \partial\Omega$ near y_0,

$$\sum_{\alpha<n} \mu_\alpha T_{\alpha\alpha}(y) = \sum_{\alpha<n} \mu_\alpha u_{\alpha;\alpha}(y) \geq \sum_{\alpha<n} \mu_\alpha \tilde{\kappa}_\alpha(y) \geq d(y) \geq d(y_0), \tag{3.36}$$

where we have used (3.35) and $|\mu| \leq 1$ in the second inequality. We differentiate covariantly the equality $u - \varphi = 0$ on $\partial\Omega$ to obtain

$$(u-\varphi)_{\xi;\eta} = -(u-\varphi)_\nu \Pi(\xi, \eta) \quad \text{on } \partial\Omega, \tag{3.37}$$

for any vectors fields ξ and η that are tangent to $\partial\Omega$. Then, for $y \in \partial\Omega$ near y_0, we have

$$u_\nu(y) \sum_{\alpha<n} \mu_\alpha b_{\alpha\alpha}(y) = \sum_{\alpha<n} \mu_\alpha (\varphi - u)_{\alpha;\alpha}(y).$$

Then

$$\begin{aligned} u_\nu(y) \sum_{\alpha<n} \mu_\alpha b_{\alpha\alpha}(y) &= \sum_{\alpha<n} \mu_\alpha \varphi_{\alpha;\alpha}(y) - \sum_{\alpha<n} \mu_\alpha u_{\alpha;\alpha}(y) \\ &\leq \sum_{\alpha<n} \mu_\alpha \varphi_{\alpha;\alpha}(y) - d(y_0), \end{aligned} \tag{3.38}$$

where we used (3.36) in the last inequality.
Since $\underline{u}$ is locally strictly convex in a neighborhood of $\partial\Omega$ it follows that $\kappa'(\underline{u}_{\alpha;\beta}(y_0))$ belongs to Γ' (since $\Gamma^+ \subset \Gamma$). We point out that $\kappa'(\underline{u}_{\alpha;\beta})$ denotes the eigenvalues of $\nabla^2 \underline{u}$, not the principal curvatures of the graph of $\underline{u}$. We may assume

$$d(y_0) < \frac{1}{2} d(\kappa'(\underline{u}_{\alpha;\beta}(y_0))),$$

otherwise we are done.
Now we use the equality $u = \underline{u}$ on $\partial\Omega$ to get

$$(u - \underline{u})_\nu \sum_{\alpha<n} \mu_\alpha b_{\alpha\alpha} = \sum_{\alpha<n} \mu_\alpha (\underline{u} - u)_{\alpha;\alpha},$$

on $\partial\Omega$. Therefore we conclude from (3.37), (3.35) and Lemma 6.2 of [8] that

$$\begin{aligned}(u - \underline{u})_\nu(y_0) \sum_{\alpha<n} \mu_\alpha b_{\alpha\alpha}(y_0) &= \sum_{\alpha<n} \mu_\alpha \underline{u}_{\alpha;\alpha}(y_0) - \sum_{\alpha<n} \mu_\alpha u_{\alpha;\alpha}(y_0) \\ &\geq d\big(\kappa'(\underline{u}_{\alpha;\beta}(y_0))\big) - d(y_0) \\ &> \frac{1}{2} d\big(\underline{u}_{\alpha;\beta}(y_0)\big) > 0.\end{aligned}$$

Since $(u - \underline{u})_\nu \geq 0$ on $\partial\Omega$, we conclude that there exist uniform positive constants $\bar{c}, \bar{\delta} > 0$, such that

$$\sum_{\alpha<n} \mu_\alpha b_{\alpha\alpha}(y) \geq \bar{c} > 0,$$

for every $y \in \Omega$ satisfying $\operatorname{dist}(y, y_0) < \bar{\delta}$. Hence we may define the function

$$\mu(y) = \frac{1}{\sum_{\alpha<n} \mu_\alpha b_{\alpha\alpha}(y)} \left(\sum_{\alpha<n} \mu_\alpha \varphi_{\alpha;\alpha}(y) - d(y_0) \right), \tag{3.39}$$

for $y \in \Omega_{\bar{\delta}} = \{x \in \Omega : \rho(x) = \operatorname{dist}(x, y_0) < \bar{\delta}\}$, where we have extended φ being constant along of the normal geodesics departing from $\partial\Omega$. We obtain from (3.38) that $u_\nu \leq \mu$ on $\partial\Omega \cap \partial\Omega_{\bar{\delta}}$ and from (3.34) and (3.37) that $u_\nu(y_0) = \mu(y_0)$. Now we may proceed as it was done for the mixed normal-tangential derivatives to get the estimate $\nabla_{\nu\nu} u(y_0) \leq C$, for a uniform constant C.

In fact, at the definition of the function w in (3.14) we may choose the vector field ξ as being an extension of ν and change the function μ there by the function μ defined above, at the equation (3.39). Defining $\tilde{w}$ in the same way as in (3.20), the inequality (3.21) remains true, hence the function h defined at equation (3.25) still satisfies $L[h] \leq 0$ in $\Omega_{\bar{\delta}}$ and $h \geq 0$ on $\partial\Omega_{\bar{\delta}} \cap \Omega$, for appropriate constants a_0, b_0, c_0 and $\bar{\delta} > 0$ sufficiently small. To get the inequality $h \geq 0$ on $\partial\Omega_{\bar{\delta}} \cap \partial\Omega$ we must use that $u_\nu \leq \mu$ on $\partial\Omega \cap \partial\Omega_{\bar{\delta}}$ (this is the main point!). Then, like it was done for the mixed normal-tangential derivatives case, we may conclude that

$$u_{\nu;\nu}(y_0) \leq C. \tag{3.40}$$

Therefore $\kappa[u](y_0)$ is contained in an *a priori* bounded subset of Γ. Since

$$F[u] = f(\kappa[u]) = \Psi \geq \Psi_0 = \inf \Psi > 0$$

it follows from (1.7) that

$$\operatorname{dist}(\kappa[u](y_0), \partial\Gamma) \geq c > 0$$

for a uniform constant $c > 0$. This implies $d(y_0) \geq c_0$, for some uniform constant $c_0 > 0$. □

We are now in position to prove (3.27). We may assume that $u_{\nu;\nu} \geq N_0$, where N_0 is the uniform constant defined above (otherwise we are done). By our choice of N_0 we have that $\tilde{\kappa}[u] \in \Gamma'$ on $\partial\Omega$, where $\tilde{\kappa}$ are the eigenvalues of the tensor T_u defined in (3.29). Fixed $y \in \partial\Omega$, we may choose Fermi coordinates centered at y as it was done above to conclude that $\tilde{\kappa}[u](y) = (u_{1;1}, \ldots, u_{n-1;n-1})$ are the eigenvalues of T_u and such that the principal curvatures $\kappa[u](y) = (\kappa_1, \ldots, \kappa_n)$ of Σ, at $(y, u(y))$, behave like

$$\kappa_\alpha = \frac{1}{W} u_{\alpha;\alpha}(y) + o(1) \qquad 1 \leq \alpha \leq n-1 \tag{3.41}$$

$$\kappa_n = \frac{1}{W g_{nn}} u_{\nu;\nu}(y) \left(1 + O\left(\frac{1}{u_{\nu;\nu}(y)}\right)\right) \tag{3.42}$$

as $|u_{\nu;\nu}(y)| \to \infty$. Since $u_{\nu;\nu}(y)$ have a lower bound the module may be removed. Therefore, since $\frac{1}{W}\tilde{\kappa}[u] \in \Gamma'$ and Γ' is open, there exists a uniform constant N_1 such that, if $u_{\nu;\nu}(y) \geq N_1$ then the distance of $\kappa'[u] = \kappa'\big(a_i^j[u]\big)(y)$ to $\partial\Gamma'$ is greater then $c_0/2$, where c_0 is the constant at Lemma 3.6. So we have

$$d\Big(\kappa'[u](y)\Big) \geq \frac{c_0}{2},$$

for $y \in \Lambda = \{y \in \Omega \,:\, u_{\nu;\nu}(y) \geq N_1\}$.
Since there exists a uniform constant $\delta_0 > 0$ such that

$$\lim_{t\to\infty} f(\kappa'[u](y), t) \geq \Psi(x, u) + \delta_0 \tag{3.43}$$

uniformly for $y \in \Lambda$, we have a uniform upper bound $\kappa_n[u](y) \leq C$ for $y \in \Lambda$. This yields a uniform upper bound $\nabla_{\nu\nu} u(y) \leq C$ for $y \in \Lambda$ and thus establishes (3.27).

Remark 3.7. *Under the hyphoteses of Theomem 1.2 we replace the subsolution* $\underline{u}$ *by the function* w *defined in section 1, see Remark 3.1.*

3.4 Global Bounds for The Second Derivatives

This section is devoted to the proof of the global Hessian estimate of solutions u of (1.2). We will show that the terms of the second fundamental form b of the graph of u are bounded by above. Combined with the fact that $\sum \kappa_i \geq \delta > 0$ (see Chapter 1), this provides us with uniform bounds for b. Since we already have the C^1 estimate, then this information allow us to obtain the Hessian estimate.

Proposition 3.8. *Suppose that conditions (1.3)-(1.7) hold and that there exists a locally strictly convex function $\chi \in C^2(\overline{\Omega})$. Let $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ be an admissible solution of (1.2). Then*

$$|\nabla^2 u| \leq C \quad in\ \overline{\Omega}, \tag{3.44}$$

where C depends on $|u|_1$, $\max_{\partial\Omega} |\nabla^2 u|$, $|\underline{u}|_2$ and other known data.

Proof. First we extend the locally strictly convex function $\chi \in C^2(\overline{\Omega})$ to $\overline{\Omega} \times \mathbb{R}$ by setting

$$\chi(x, t) = \chi(x) + t^2.$$

This extension is also locally strictly convex and we will use the same symbol χ to represent it also.

We define the following function on the unit tangent bundle of Σ,

$$\tilde{\zeta}(y, \xi) = b(\xi, \xi) \exp\big(\phi(\tau(y)) + \beta\chi(y)\big),$$

where $y \in \Sigma$, ξ is a unit tangent vector to Σ at y, the function τ is the support function defined on Σ by $\tau = \langle N, \partial_t \rangle$, $\beta > 0$ is a constant to be chosen later and ϕ is a real function defined as follows. By definition the function τ is bounded by constants depending on bound for ∇u. Hence, it is possible to choose $a > 0$ so that $\tau \geq 2a$. Thus, we define

$$\phi(\tau) = -\ln(\tau - a).$$

Hence, differentiating with respect to τ, we conclude that

$$\ddot{\phi} - (1+\epsilon)\dot{\phi}^2 = \frac{1}{(\tau - a)^2} - \frac{1+\epsilon}{(\tau - a)^2} = -\frac{\epsilon}{(\tau - a)^2} < 0, \tag{3.45}$$

for any positive constant $\epsilon > 0$. Notice that, by the choice of a, given an arbitrary positive constant C, we have

$$\begin{aligned} -(1+\dot{\phi}\tau) + C(\ddot{\phi} - (1+\epsilon)\dot{\phi}^2) &= -1 + \frac{\tau}{\tau - a} - \frac{c_1\epsilon}{(\tau-a)^2} \\ &\geq \frac{a^2}{2(\tau-a)^2} \geq \hat{C}, \end{aligned} \tag{3.46}$$

for some positive constant $\hat{C}$ depending on the bound for ∇u.

If the maximum of $\tilde{\zeta}$ is achieved on $\partial\Sigma$, we can estimate it in terms of uniform constants (see the last section) and we are done. Thus, suppose the maximum of $\tilde{\zeta}$ is attained at a point $y_0 = (x_0, u(x_0)) \in \Sigma$, with $x_0 \in \Omega$, and along the direction ξ_0 tangent to Σ at $y_0 = (x_0, u(x_0))$. We fix a normal coordinate system (y^i) of Σ centered at y_0, such that

$$\frac{\partial}{\partial y^1}|_{y_0} = \xi_0.$$

Notice that ξ_0 is a principal direction of Σ at y_0, hence $a_{1i}\,(y_0) = 0$, for any $i > 1$. We then consider the local function $a_{11} = b(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1})$. Thus we easily verify that the function

$$\zeta = a_{11} \exp(\phi(\tau) + \beta\chi) \tag{3.47}$$

attains maximum at $y_0 = (x_0, u(x_0))$. Thus, it holds at y_0

$$0 = (\ln\zeta)_i = \frac{a_{11;i}}{a_{11}} + \dot{\phi}\tau_i + \beta\chi_i \tag{3.48}$$

and the Hessian matrix with components

$$(\ln\zeta)_{i;j} = \frac{a_{11;ij}}{a_{11}} - \frac{a_{11;i}a_{11;j}}{a_{11}^2} + \dot{\phi}\tau_{i;j} + \ddot{\phi}\tau_i\tau_j + \beta\chi_{i;j}$$

is negative-definite. Thus

$$\begin{aligned} G^{ij}(\ln\zeta)_{i;j} =& \frac{1}{a_{11}}G^{ij}a_{11;ij} - \frac{1}{a_{11}^2}G^{ij}a_{11;i}a_{11;j} + \dot{\phi}G^{ij}\tau_{i;j} \\ &+ \ddot{\phi}G^{ij}\tau_i\tau_j + \beta G^{ij}\chi_{i;j} \leq 0. \end{aligned} \tag{3.49}$$

We may rotate the coordinates $(y^2, \ldots, y^n)$ in such a way that the new coordinates diagonalize the matrix $\{a_{ij}(y_0)\}$. By Lemma 2.2 it results that the matrix $\{G^{ij}\}$ is also diagonal with $G^{ii} = \frac{1}{W} f_i$. We denote $\kappa_i = a_{ii}(y_0)$ and choose indices in such a way that

$$\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n.$$

Moreover, we assume without loss of generality that $\kappa_1 > 1$ at y_0. Thus, according to Lemma 2.2, we have

$$f_1 \leq f_2 \leq \cdots \leq f_n.$$

From (3.49) we get

$$\sum_i \Big(\frac{1}{\kappa_1} f_i a_{11;ii} - \frac{1}{\kappa_1^2} f_i |a_{11;i}|^2 + \dot{\phi} f_i \tau_{i;i} + \ddot{\phi} f_i |\tau_i|^2 + \beta f_i \chi_{i;i} \Big) \leq 0. \tag{3.50}$$

Now, we differentiate covariantly with respect to the metric (g_{ij}) in Σ the equation (2.11) in the direction of $\frac{\partial}{\partial y^1}|_{y_0}$ obtaining $F^{ij} a_{ij;1} = \Psi_1$ and differentiating again

$$F^{ij} a_{ij;11} + F^{ij,kl} a_{ij;1} a_{kl;1} = \Psi_{1;1}. \tag{3.51}$$

From the Simons formula (2.9) we have

$$\begin{aligned} F^{ij} a_{ij;11} = F^{ii} a_{ii;11} = &\sum_i \big(f_i a_{11;ii} + \kappa_1 f_i \kappa_i^2 - \kappa_1^2 f_i \kappa_i \\ &+ \kappa_1 f_i \bar{R}_{i0i0} - \bar{R}_{1010} f_i \kappa_i + f_i \bar{R}_{i1i0;1} - f_i \bar{R}_{1i10;i} \big). \end{aligned} \tag{3.52}$$

We use the fact that $c_0 \leq \sum_i f_i \lambda_i \leq f = \Psi$ to get

$$\begin{aligned} F^{ij} a_{ij;11} \leq & - \kappa_1^2 c_0 + |\bar{R}_{1010}| \Psi \\ & + \sum_i \big(f_i a_{11;ii} + \kappa_1 f_i \kappa_i^2 + \kappa_1 f_i \bar{R}_{i0i0} + f_i \bar{R}_{i0i0;1} - f_i \bar{R}_{1010;i} \big). \end{aligned}$$

Combining this expression and (3.51) we obtain

$$\begin{aligned} \sum_i f_i a_{11;ii} \geq & \Psi_{1;1} - F^{ij,kl} a_{ij;1} a_{kl;1} + \kappa_1^2 \delta - |\bar{R}_{1010}| \psi \\ & - \sum_i \big(\lambda_1 f_i \lambda_i^2 - \lambda_1 f_i \bar{R}_{i0i0} - f_i \bar{R}_{i0i0;1} + f_i \bar{R}_{1010;i} \big). \end{aligned}$$

Replacing this into (3.50) we obtain

$$\begin{aligned}
&\frac{1}{\kappa_1}\big(\Psi_{1;1} - F^{ij,kl}a_{ij;1}a_{kl;1} + \kappa_1^2 c_0 - |\bar{R}_{1010}|\Psi\big)\\
&-\frac{1}{\kappa_1}\sum_i \big(\kappa_1 f_i \kappa_i^2 - \kappa_1 f_i \bar{R}_{i0i0} - f_i \bar{R}_{i0i0;1} + f_i \bar{R}_{1010;i}\big)\\
&+\sum_i \Big(\dot{\phi} f_i \tau_{i;i} - \frac{1}{\kappa_1^2} f_i |a_{11;i}|^2 + \ddot{\phi} f_i |\tau_i|^2 + \beta f_i \chi_{i;i}\Big) \le 0.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\frac{\Psi_{1;1}}{\kappa_1} + \frac{1}{\kappa_1}\big(c_0\kappa_1^2 - \Psi|\bar{R}_{1010}|\big) - \frac{1}{\kappa_1}F^{ij,kl}a_{ij;1}a_{kl;1} - \sum_i f_i \kappa_i^2\\
&-\sum_i f_i \bar{R}_{i0i0} + \sum_i \Big(\dot{\phi} f_i \tau_{i;i} - \frac{1}{\kappa_1^2} f_i |a_{11;i}|^2 + \ddot{\phi} f_i |\tau_i|^2 + \beta f_i \chi_{i;i}\Big)\\
&-\frac{1}{\kappa_1}\sum_i f_i\big(\bar{R}_{i0i0;1} - \bar{R}_{1010;i}\big) \le 0.
\end{aligned}$$

It is well known that

$$\begin{aligned}
\tau_i &= -a_i^k \eta_k\\
\tau_{i;j} &= -\eta^k a_{ii;k} - \eta^k \bar{R}_{kij0} - \tau a_i^k a_{kj},
\end{aligned}$$

where η^k are the components of the vector ∂_t^T, i.e.,

$$\partial_t^T = \eta^k \frac{\partial}{\partial y^k}.$$

Notice that ∂_t^T is the projection of ∂_t onto $T\Sigma$. Hence, since $\dot{\phi} < 0$, we have (at y_0)

$$\dot{\phi}\sum_i f_i \tau_{i;i} = -\dot{\phi}\Big(\sum_i \eta^k f_i a_{ii;k} + \sum_i \eta^k \bar{R}_{kii0} f_i\Big) - \dot{\phi}\tau \sum_i f_i \kappa_i^2.$$

Since

$$\sum_i f_i a_{ii;k} = \Psi_k,$$

we have

$$\dot{\phi}\sum_i f_i \tau_{i;i} = -\dot{\phi}\Big(\eta^k \Psi_k + \sum_i \eta^k \bar{R}_{kii0} f_i\Big) - \dot{\phi}\tau \sum_i f_i \kappa_i^2.$$

We denote by $T = \sum_i f_i$. By estimating the ambient curvature terms by a uniform constant $C > 0$, we obtain

$$\sum_i \eta^k \bar{R}_{kii0} f_i \leq CT.$$

Then,

$$-\dot{\phi}\Big(\eta^k \Psi_k + \sum_i \eta^k \bar{R}_{kii0} f_i\Big) \geq -|\dot{\phi}|(C + CT).$$

Therefore, we have

$$\dot{\phi} \sum_i f_i \tau_{i;i} \geq -|\dot{\phi}|(C + CT) - \dot{\phi}\tau \sum_i f_i \kappa_i^2.$$

Now, we suppose without loss of generality that

$$\kappa_1 \geq \frac{1}{C} \sum_i |R_{i0i0;1} - R_{1010;i}|,$$

for some $C > 0$. Moreover, supposing also that $\kappa_1 \geq 1$, we have

$$-\frac{1}{\kappa_1} \Psi |\bar{R}_{1010}| \geq -C$$

and

$$\frac{\Psi_{1;1}}{\kappa_1} \geq -C$$

for some positive constant C. We note that, since

$$\Psi_{1;1} = \Psi_{t;t}(u_1)^2 + \Psi_t u_{1;1} + \Psi_{1;1}$$

the above assumption is allowed. Finally we have

$$-\sum_i f_i \bar{R}_{i0i0} \geq -T \max_i |\bar{R}_{i0i0}| \geq -CT.$$

We then conclude from these inequalities that

$$\begin{aligned} &-C - CT + c_0\kappa_1 - \frac{1}{\kappa_1} F^{ij,kl} a_{ij;1} a_{kl;1} - \sum_i f_i \kappa_i^2 - \frac{1}{\kappa_1^2} \sum_i f_i |a_{11;i}|^2 \\ &\quad -|\dot{\phi}|(C + CT) - \dot{\phi}\tau \sum_i f_i \kappa_i^2 + \ddot{\phi} \sum_i f_i |\tau_i|^2 + \beta \sum_i f_i \chi_{i;i} \leq 0. \end{aligned} \tag{3.53}$$

Now, to proceed further with our analysis, we consider two cases.
Case I: In this case we suppose that $\kappa_n \leq -\theta\kappa_1$ for some positive constant θ to be chosen later.

We have from (3.48) and the Cauchy inequality with ϵ that

$$\frac{1}{\kappa_1^2} f_i |a_{11;i}|^2 = f_i |\dot{\phi}\tau_i + \beta\chi_i|^2 \leq (1+\frac{1}{\epsilon})\beta^2 f_i|\chi_i|^2 + (1+\epsilon)\dot{\phi}^2 f_i|\tau_i|^2, \quad (3.54)$$

for any $\epsilon > 0$ and any $1 \leq i \leq n$. Now we replace the sum of the terms in (3.54) in the inequality (3.53) to obtain

$$c_0\kappa_1 - C(1+|\dot{\phi}|) - CT(1+|\dot{\phi}|) - \frac{1}{\kappa_1} F^{ij,kl} a_{ij;1}a_{kl;1} - (1+\dot{\phi}\tau)\sum_i f_i\kappa_i^2$$
$$-(1+\frac{1}{\epsilon})\beta^2\sum_i f_i|\chi_i|^2 + \big(\ddot{\phi} - (1+\epsilon)\dot{\phi}^2\big)\sum_i f_i|\tau_i|^2 + \beta\sum_i f_i\chi_{i;i} \leq 0.$$

Since $\{a_{ij}\}$ is diagonal at y_0 and ∂_t is known, we have

$$\sum_i f_i|\tau_i|^2 = \sum_i f_i\lambda_i^2|\eta_i|^2 \leq C\sum_i f_i\kappa_i^2,$$

so, it follows from (3.45) that

$$\big(\ddot{\phi} - (1+\epsilon)\dot{\phi}^2\big)\sum_i f_i|\tau_i|^2 \geq \big(\ddot{\phi} - (1+\epsilon)\dot{\phi}^2\big)C\sum_i f_i\kappa_i^2.$$

We also may use that $|D\chi|$ is a known data to get

$$\sum_i f_i|\chi_i|^2 \leq CT.$$

Hence, we obtain

$$\begin{aligned} &c_0\kappa_1 - C(1+|\dot{\phi}|) - \frac{1}{\kappa_1}F^{ij,kl}a_{ij;1}a_{kl;1} - \big(1+|\dot{\phi}| + (1+\frac{1}{\epsilon})\beta^2\big)CT \\ &+\Big(-(1+\dot{\phi}\tau) + C\big(\ddot{\phi} - (1+\epsilon)\dot{\phi}^2\big)\Big)\sum_i f_i\kappa_i^2 + \beta\sum_i f_i\chi_{i;i} \leq 0. \end{aligned} \quad (3.55)$$

Using the concavity of F and the convexity of χ we may discard the third and the last terms in the left-hand side of (3.55) since they are nonnegative, obtaining

$$-C_1(\beta) - C_2(\beta)T + c_0\kappa_1 + \hat{C}\sum_i f_i\kappa_i^2 \leq 0,$$

where C_1 depends linearly on β and C_2 depends quadratically on β. Since $f_n \geq \frac{1}{n}T$, we have

$$\sum_i f_i\kappa_i^2 \geq f_n\kappa_n^2 \geq \frac{1}{n}\theta^2 T\kappa_1^2.$$

Thus it follows that

$$-C_1 - C_2T + c_0\kappa_1 + \hat{C}\frac{1}{n}\theta^2 T\kappa_1^2 \leq 0. \tag{3.56}$$

This inequality shows that κ_1 has a uniform upper bound. In fact, the left-hand side of this inequality may be seen as a polynomial in κ_1 and therefore

$$\kappa_1 \leq \kappa_+,$$

where

$$\kappa_+ = \sup_T \left\{ -\frac{c_0}{2\hat{C}\frac{1}{n}\theta^2 T} + \left(\frac{c_0^2 + 4\hat{C}\frac{1}{n}\theta^2 T(C_1 + C_2T)}{4\hat{C}^2\frac{1}{n^2}\theta^4 T^2}\right)^{1/2}\right\}.$$

We may conclude that (3.56) implies the estimate also in another way. In fact, notice that the coefficients of the terms in T in (3.56) are

$$\hat{C}\frac{1}{n}\theta^2\kappa_1^2 - C_2.$$

Then, if $\kappa_1 \geq \bar{C}$ for a (suitable) uniform constant $\bar{C}$, we have

$$\hat{C}\frac{1}{n}\theta^2\kappa_1^2 - C_2 \geq 0.$$

In this case, since $T = \sum_i f_i \geq 0$, we may discard the terms in T in (3.56) to obtain

$$-C_1 + c_0\kappa_1 \leq 0$$

i.e.,

$$\kappa_1 \leq \frac{C_1}{c_0}.$$

Case II: In this case, we assume that $\kappa_n \geq -\theta\kappa_1$. Hence, $\kappa_i \geq -\theta\kappa_1$. We then group the indices $\{1, ..., n\}$ in two sets

$$I_1 = \{j; f_j \leq 4f_1\},$$
$$I_2 = \{j; f_j > 4f_1\}.$$

Using (3.54), we have for $i \in I_1$

$$\begin{aligned}\frac{1}{\kappa_1^2} f_i |a_{11;i}|^2 &\le (1+\epsilon)\dot{\phi}^2 f_i |\tau_i|^2 + (1+\frac{1}{\epsilon})(\beta)^2 f_i |\chi_i|^2 \\ &\le (1+\epsilon)\dot{\phi}^2 f_i |\tau_i|^2 + C(1+\frac{1}{\epsilon})(\beta)^2 f_1.\end{aligned}$$

Therefore, it follows from (3.54) that

$$\begin{aligned}&- C - CT + c_0\kappa_1 - \frac{1}{\lambda_1} F^{ij,kl} a_{ij;1} a_{kl;1} - \big(1 + \dot{\phi}\tau\big) \sum_i f_i \kappa_i^2 \\ &- \frac{1}{\kappa_1^2} \sum_{j \in I_2} f_j |a_{11;j}|^2 - |\dot{\phi}|(C + CT) + \big(\ddot{\phi} - (1+\epsilon)\dot{\phi}^2\big) \sum_i f_i |\tau_i|^2 \\ &- C(1+\frac{1}{\epsilon})\beta^2 f_1 + \beta \sum_i f_i \chi_{i;i} \le 0.\end{aligned}$$

Notice that we had summed up to the inequality the non-positive terms

$$-(1+\epsilon)|\dot{\phi}|^2 \sum_{i \in I_2} f_i |\tau_i|^2.$$

Using that

$$|\tau_i| = |\kappa_i \eta_i| \le C\kappa_i$$

we may conclude as above that

$$-\big(1 + \dot{\phi}\tau\big) \sum_i f_i \kappa_i^2 + \big(\ddot{\phi} - (1+\epsilon)\dot{\phi}^2\big) \sum_i f_i |\tau_i|^2 \ge \hat{C} \sum_i f_i \kappa_i^2 \tag{3.57}$$

for some positive constant $\hat{C}$. Thus we have

$$\begin{aligned}&- C - CT + c_0\kappa_1 - \frac{1}{\kappa_1} F^{ij,kl} a_{ij;1} a_{kl;1} + \hat{C} \sum_i f_i \kappa_i^2 \\ &- \frac{1}{\kappa_1^2} \sum_{j \in I_2} f_j |a_{11;j}|^2 - |\dot{\phi}|(C + CT) - C\big(1+\frac{1}{\epsilon}\big)\beta^2 f_1 + \beta \sum_i f_i \chi_{i;i} \le 0.\end{aligned} \tag{3.58}$$

Using Codazzi's equation

$$a_{1j;1} = a_{11;j} + \bar{R}_{01j1}$$

and Lemma (2.2) we get

$$
\begin{aligned}
-\frac{1}{\kappa_1}F^{ij,kl}a_{ij;1}a_{kl;1} &= -\frac{1}{\kappa_1}\sum_{k,l} f_{kl}a_{kk;1}a_{ll;1} - \frac{1}{\kappa_1}\sum_{k\neq m}\frac{f_k - f_l}{\kappa_k - \kappa_l}\eta_{kl}^2 \\
&\geq -\frac{1}{\kappa_1}\sum_{k\neq m}\frac{f_k - f_l}{\kappa_k - \kappa_l}\eta_{kl}^2 \\
&\geq -\frac{2}{\kappa_1}\sum_{j\in I_2}\frac{f_1 - f_j}{\kappa_1 - \kappa_j}(a_{1j;1})^2 \\
&= -\frac{2}{\kappa_1}\sum_{j\in I_2}\frac{f_1 - f_j}{\kappa_1 - \kappa_j}\big(a_{11;j} + \bar{R}_{01j1}\big)^2,
\end{aligned}
$$

since $1 \notin I_2$ and $\frac{f_k - f_l}{\kappa_k - \kappa_l} \leq 0$. We claim that for all $j \in I_2$ it holds the inequality

$$
-\frac{2}{\kappa_1}\frac{f_1 - f_j}{\kappa_1 - \kappa_j} \geq \frac{f_j}{\kappa_1^2}. \tag{3.59}
$$

This is equivalent to

$$
2f_1\kappa_1 \leq f_j\kappa_1 + f_j\kappa_j.
$$

It is clear that $j \in I_2$ implies $f_j > 4f_1$. If $\kappa_j \geq 0$, this is obvious. If $\kappa_j < 0$, then $-\theta\kappa_1 \leq \kappa_j < 0$, and then

$$
f_j\kappa_1 + f_j\kappa_j \geq (1-\theta)f_j\kappa_1 \geq 4(1-\theta)f_1\kappa_1 \geq 2f_1\kappa_1
$$

if we choose $\theta = 1/2$. Hence, with this choice, we use (3.59) to obtain

$$
\begin{aligned}
-\frac{1}{\kappa_1}F^{ij,kl}a_{ij;1}a_{kl;1} &\geq \sum_{j\in I_2}\frac{f_j}{\kappa_1^2}\big(a_{11;j} + \bar{R}_{01j1}\big)^2 \\
&= \sum_{j\in I_2}\frac{f_j}{\kappa_1^2}(a_{11;j})^2 + 2\sum_{j\in I_2}\frac{f_j}{\kappa_1^2}a_{11;j}\bar{R}_{01j1} + \sum_{j\in I_2}\frac{f_j}{\kappa_1^2}(\bar{R}_{01j1})^2.
\end{aligned}
$$

Using this inequality in (3.58) and estimating the curvature term $|R_{01j1}|^2$ we obtain

$$
\begin{aligned}
&-C - CT + c_0\kappa_1 + \sum_{j\in I_2}\frac{f_j}{\kappa_1^2}(a_{11;j})^2 + 2\sum_{j\in I_2}\frac{f_j}{\kappa_1^2}a_{11;j}\bar{R}_{01j1} + \hat{C}\sum_i f_i\kappa_i^2 \\
&-\frac{1}{\kappa_1^2}\sum_{j\in I_2} f_j|a_{11;j}|^2 - |\dot{\phi}|(C + CT) - C\big(1 + \frac{1}{\epsilon}\big)\beta^2 f_1 + \beta\sum_i f_i\chi_{i;i} \leq 0.
\end{aligned}
$$

Using (3.48) we get

$$
\begin{aligned}
&-C-CT+c_0\kappa_1-2\sum_{j\in I_2}\frac{f_j}{\kappa_1}(\dot\phi\tau_j+\beta\chi_j)\bar R_{01j1}+\hat C\sum_i f_i\kappa_i^2\\
&-|\dot\phi|(C+CT)-C\big(1+\frac{1}{\epsilon}\big)\beta^2 f_1+\beta\sum_i f_i\chi_{i;i}\le 0.
\end{aligned}
$$

Since $\dot\phi<0$, $\kappa_j\le\kappa_1$ and $-\kappa_j\le\theta\kappa_1<\kappa_1$ we have the estimate

$$
2\frac{f_j}{\kappa_1}(-\dot\phi\tau_j)\bar R_{01j1}=2\frac{f_j}{\kappa_1}\dot\phi\kappa_j\eta_j\bar R_{01j1}\ge 2\frac{f_j}{\kappa_1}\dot\phi|\kappa_j||\eta_j\bar R_{01j1}|\ge 2f_j\dot\phi|\eta_j\bar R_{01j1}|.
$$

We also may suppose, without loss of generality, that

$$
\kappa_1\ge\frac{3|\chi_j\bar R_{01j1}|}{\gamma_0}
$$

for all $j\in I_2$, where γ_0 is a positive constant that satisfies

$$
\chi_{i;i}\ge\gamma_0>0,\qquad \forall\ 1\le i\le n.
$$

Note that this assumption is equivalent to

$$
\frac{\gamma_0}{3}\ge\frac{|\chi_j\bar R_{01j1}|}{\kappa_1},
$$

which implies

$$
\begin{aligned}
-2\sum_{j\in I_2}\frac{f_j}{\kappa_1}\beta\chi_j\bar R_{01j1}\ge&-2\sum_{j\in I_2}\frac{f_j}{\kappa_1}\beta|\chi_j\bar R_{01j1}|\\
\ge&-2\sum_{j\in I_2}\frac{\beta f_j\gamma_0}{3}\ge -2\frac{\beta\gamma_0}{3}T.
\end{aligned}
$$

These inequalities imply that

$$
\begin{aligned}
&-C-CT+c_0\kappa_1+2\sum_{j\in I_2}f_j\dot\phi|\eta_j\bar R_{01j1}|-2\frac{\beta\gamma_0}{3}T\\
&+\hat C\sum_i f_i\kappa_i^2-|\dot\phi|(C+CT)-C\big(1+\frac{1}{\epsilon}\big)\beta^2 f_1+\beta\sum_i f_i\chi_{i;i}\le 0.
\end{aligned}
$$

Since $\sum_{j\in I_2} f_j \le T$, $|\eta_j \bar{R}_{j1}| \le C$ and $\dot{\phi} < 0$ we have

$$-C - \big(C + C|\dot{\phi}| + 2\beta\frac{\gamma_0}{3} - \beta\gamma_0\big)T - C\big(1 + \frac{1}{\epsilon}\big)\beta^2 f_1 + c_0\kappa_1 + \hat{C} f_1 \kappa_1^2 \le 0.$$

Choosing $\beta > 0$ sufficiently large, the term in T is positive and we may discard it, obtaining

$$-C - C_2(\beta) f_1 + c_0\kappa_1 + \hat{C} f_1 \kappa_1^2 \le 0, \tag{3.60}$$

where C_2 depends quadratically on β. Reasoning as above, we conclude that this inequality gives an upper bound for κ_1. This time, we have the following upper bound for κ_1

$$\kappa_1 \le \hat{\kappa}_+,$$

where

$$\hat{\kappa}_+ = \sup_{f_1} \left\{ -\frac{c_0}{2\hat{C} f_1} + \left(\frac{c_0^2 + 4\hat{C} f_1 (C + C_2 f_1)}{4\hat{C}^2 f_1^2} \right)^{1/2} \right\}.$$

Notice that, if $f_1 \longrightarrow 0$, equation (3.60) becomes

$$\epsilon - C + c_0\kappa_1 \le 0,$$

for some $\epsilon \backsim 0$. Since $c_0 > 0$ this inequality implies the desired estimate. □

Remark 3.9. *In the case $M = \mathbb{R}^n$, the assumption about the existence of a strictly convex function χ is not necessary. In fact, in this case, the auxiliar function*

$$\tilde{\zeta}(y, \xi) = \eta^\beta b(\xi, \xi) \exp\big(\phi(\tau(y))\big),$$

works as the function $\tilde{\zeta}$ defined above, with $\eta = t|_\Sigma$ being the height function. This is shown with details in [37].

Chapter 4

Killing Graphs with Prescribed Anisotropic Curvature

In this chapter we study the existence of Killing graphs with prescribed anisotropic mean curvature. Our approach is inspired in the article [15] where the usual mean curvature case is treated.

4.1 Preliminaries

In this section we fix some notations and present the definition of anisotropic mean curvature. For more details about the notion of anisotropic mean curvature we refer the reader to [33]. Let $\bar{M}$ be a complete oriented $(n+1)$-dimensional Riemannian manifold with metric and Riemannian connection denoted by $\bar{g}$ and $\bar{\nabla}$, respectively. We denote by $T\bar{M}$ the tangent bundle of $\bar{M}$ and by π the natural projection of $T\bar{M}$ onto $\bar{M}$. At each point $(y,\eta) \in T\bar{M}$, the projection π defines the subspace

$$\mathcal{V}_{(y,\eta)} = \ker \pi_*|_{(y,\eta)},$$

called the vertical subspace of $T_{(y,\eta)}T\bar{M}$. On the other hand, the connection $\bar{\nabla}$ defines a subspace $\mathcal{H}_{(y,\eta)}$, called the horizontal subspace of $T\bar{M}$ at (y,η), which satisfies

$$T_{(y,\eta)}T\bar{M} = \mathcal{V}_{(y,\eta)} \oplus \mathcal{H}_{(y,\eta)} \tag{4.1}$$

and it is the kernel of the connection map $K : TT\bar{M} \longrightarrow T\bar{M}$ defined by

$$K_{(y,\eta)}(X_*\zeta) = \bar{\nabla}_\zeta X, \quad \zeta \in T_y\bar{M},$$

where $X \in \Gamma(T\bar{M})$ is a vector field in $\bar{M}$ with $X(y) = \eta$. Associated to the decomposition in (4.1) we have the natural projections

$$\pi^{\mathrm{v}} : T_{(y,\eta)}T\bar{M} \longrightarrow \mathcal{V}_{(y,\eta)} \quad \text{and} \quad \pi^{\mathrm{h}} : T_{(y,\eta)}T\bar{M} \longrightarrow \mathcal{H}_{(y,\eta)}.$$

We also denote by X^{v} and X^{h}, respectively, the vertical and horizontal lift of a vector field $X \in \Gamma(T\bar{M})$. In what follows we assume that $T\bar{M}$ is endowed with the Sasaki metric and we will denote by D the associated Levi-Civita connection.

A parametric Lagrangian in $T\bar{M}$ is a smooth function

$$F\colon\ T\bar{M} \setminus \{0\} \longrightarrow \mathbb{R}_+$$

which is positively homogeneous with respect to the second variable, i.e., for any $(y, \eta) \in TM \setminus \{0\}$ we have

$$F(y, t\eta) = tF(y, \eta), \qquad t > 0,$$

and satisfies the following ellipticity condition (see [45])

$$D^2F|_{(y,\eta)}(\zeta, \zeta) = \frac{\partial^2 F}{\partial\eta^\alpha \partial\eta^\beta}(y, \eta)\zeta^\alpha\zeta^\beta > 0,$$

for any vertical vector field $\zeta = \zeta^\alpha \frac{\partial}{\partial\eta^\alpha} \in \Gamma(T\bar{M})$, $\zeta \neq 0$, satisfying

$$\langle \zeta, \eta^{\mathrm{v}} \rangle_{T\bar{M}} = 0.$$

The main example of parametric Lagrangian is given by

$$F(y, \eta) = |\eta|.$$

In this particular case, the hessian of F is given by

$$\frac{\partial^2 F}{\partial\eta^\alpha \partial\eta^\beta}(\eta) = \frac{1}{|\eta|}\bar{g}_{\alpha\beta} - \frac{1}{|\eta|^3}\eta_\alpha\eta_\beta. \tag{4.2}$$

Given an isometric immersion $\psi : \Sigma \longrightarrow \bar{M}$ oriented by a unit normal vector field N, we define the parametric functional

$$\mathcal{F}[\psi] = \int_M F(\psi, N)\mathrm{d}\Sigma,$$

where $\mathrm{d}\Sigma$ is the volume element induced on Σ by ψ. Note that, when $F(y,\eta) = |\eta|$ the functional $\mathcal{F}$ is the classical area functional.
We define along the cross-section

$$x \in \Sigma \longmapsto \varphi(x) = \big(y = \psi(x), \eta = N(\psi(x))\big) \in TM,$$

the vector fields ξ and χ by setting

$$DF|_{\varphi} = \xi^{\mathrm{v}} + \chi^{\mathrm{h}}.$$

It was shown in [33] that, if ψ is a critical point for the functional

$$\psi \longmapsto \mathcal{F}[\psi] + \Lambda\mathcal{V}[\psi],$$

where Λ is a constant and $\mathcal{V}$ denotes the volume functional, then ψ satisfies the Euler-Lagrange equation

$$\mathrm{div}_{\Sigma}\xi + \langle \chi, N\rangle = -\Lambda.$$

From now on we will restrict ourselves to parametric Lagrangians F that are horizontally constant. A parametric Lagrangian F is said to be horizontally constant if

$$\pi_*^{\mathrm{h}}(DF) = 0.$$

In particular, it holds in this case that $\chi = 0$, and the above Euler-Lagrange equation becomes

$$\mathrm{div}_{\Sigma}\xi = -\Lambda.$$

This suggests the definition of the anisotropic mean curvature of Σ associated with the parametric Lagrangian F as

$$nH_F = -\mathrm{div}_{\Sigma}\,\xi. \tag{4.3}$$

In order to obtain a workable expression for H_F we define on Σ the following bilinear form

$$\mathcal{A}_F(X_i, X_j) = \langle D_{X_i^{\mathrm{v}}} DF, X_j^{\mathrm{v}}\rangle_{T\bar{M}} \circ \varphi, \tag{4.4}$$

where $\langle\cdot,\cdot\rangle_{T\bar{M}}$ denotes the Sasaki metric on $T\bar{M}$. Hence,

$$\begin{aligned} nH_F &= -\mathrm{div}_{\Sigma}\,\xi \\ &= -g^{ij}\langle D_{(\bar{\nabla}_{X_i}N)^{\mathrm{v}}} DF, X_j^{\mathrm{v}}\rangle_{T\bar{M}} \circ \varphi = \mathrm{tr}_g\, \mathcal{A}_F \\ &= \mathrm{tr}_g\, \mathcal{A}_{\mathcal{F}}^{*} A, \end{aligned} \tag{4.5}$$

where A is the Weingarten map of Σ, g_{ij} are the components of the metric of Σ and $\mathcal{A}_F^*$ is the linear operator metrically equivalent to the bilinear form $\mathcal{A}_F$. Now we will describe the geometry setting that will be considered.

We assume that $\bar{M}$ is endowed with a nonsingular Killing vector field Y with complete flow lines and such that the orthogonal distribution

$$p \in \bar{M} \mapsto \{v \in T_p\bar{M} : \langle v, Y\rangle = 0\}$$

is integrable. We note that the integral leaves of the distribution are totally geodesic hypersurfaces. Let M be a fixed integral leaf. The flux $\Psi\colon \mathbb{R}\times M \to \bar{M}$ generated by Y takes isometrically $M = M_0$ to the leaf $M_s = \Psi_s(M)$ for any $s \in \mathbb{R}$, where $\Psi_s = \Psi(s, \cdot)$. Given local coordinates $x^1, \ldots, x^n$ for M, then s, $x^1, \ldots, x^n$ are local coordinates for $\bar{M}$ defined by

$$q \in \bar{M} \mapsto (s, x^1, \ldots, x^n) \quad \text{if} \quad q = \Psi(s, p),$$

where $p \in M$ is the point with coordinates $x^1, \ldots, x^n$. The corresponding coordinate vector field along the flux are

$$\partial_0(q) = \frac{\mathrm{d}}{\mathrm{d}s}\,\Psi(s,p) = Y(\Psi(s,p))$$

and

$$\partial_i(q) = \frac{\partial}{\partial x^i}\Psi(s,p) = \Psi_{s_*}(p)\partial_i(p).$$

The ambient metric in terms of these coordinates has components

$$\bar{g}_{00} = \langle \partial_s, \partial s\rangle = \varrho, \quad \bar{g}_{0i} = \langle \partial_s, \partial_i\rangle = 0$$

and

$$\bar{g}_{ij} = \langle \Psi_{s*}\partial_i, \Psi_{s*}\partial_j\rangle = \langle \partial_i, \partial_j\rangle = \sigma_{ij},$$

where σ_{ij} are the components of the metric in M in terms of the coordinates (x^i). Observe that the components of the metric do not depend on s. The gradient of the function s is

$$\bar{\nabla} s = \bar{g}^{00}\,\partial_0 = |Y|^{-2}\,Y =: \gamma\,Y.$$

Fixed coordinates $p \longmapsto (s\,, x^1 \ldots, x^n)$ in $\bar{M}$, a tangent vector $\eta \in T_p\bar{M}$ may be written as

$$\eta = \eta^0\frac{\partial}{\partial s} + \eta^k\frac{\partial}{\partial x^k}, \qquad k = 1, \cdots, n.$$

Then we may define local coordinates on the tangent bundle $T\bar{M}$ setting

$$(p,\eta) \longmapsto (s, x^1, \cdots, x^n, \eta^0, \cdots, \eta^n),$$

the coordinate vector fields associated are

$$\frac{\partial}{\partial s}, \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial \eta^0}, \cdots, \frac{\partial}{\partial \eta^n}.$$

We also assume that the parametric Lagrangian F is invariant under Ψ, i.e.,

$$\frac{\partial F}{\partial s} = 0.$$

Given a bounded $C^{2,\alpha}$ domain Ω in M and a function u on $\bar{\Omega}$, we define the associated *Killing graph* by

$$\Sigma = \{X(p) = \Psi(u(p), p) : p \in \Omega\}.$$

In terms of the coordinates $s, x^1 \ldots, x^n$ defined above, we have the following parametrization of Σ

$$X(p) \in \Sigma \mapsto (u(x^1, \ldots, x^n), x^1, \ldots, x^n).$$

Associated with this parametrization we have the coordinate vectors

$$X_i(q) = u_i(p)\partial_0(q) + \partial_i(q)$$

and the components of the metric in Σ are

$$g_{ij} = \sigma_{ij} + \frac{u_i u_j}{\gamma}.$$

The unit vector field

$$N = \frac{1}{W}\bar{\nabla}\Phi = \frac{1}{W}(\gamma\,\partial_s - \Psi_*\nabla u) \tag{4.6}$$

is normal to Σ, where

$$W^2 = \gamma + |\nabla u|^2.$$

Hence, the local section of $T\bar{M}$ defined by $\varphi(q) = (q, \eta(q)) \in T\bar{M}$, where

$$\eta^i = -\frac{u^i}{W} \quad \text{and} \quad \eta^0 = \frac{\gamma}{W},$$

maps points in Σ to the unit normal field N defined above.

We point out that it follows from the homogeneity of F the Euler relations

$$\gamma \frac{\partial^2 F}{\partial \eta^0 \partial \eta^j}\Big|_\varphi = u^i \frac{\partial^2 F}{\partial \eta^i \partial \eta^j}\Big|_\varphi \tag{4.7}$$

$$\gamma \frac{\partial^2 F}{\partial \eta^0 \partial \eta^0}\Big|_\varphi = u^i \frac{\partial^2 F}{\partial \eta^i \partial \eta^0}\Big|_\varphi \tag{4.8}$$

$$-\frac{\partial F}{\partial \eta^\alpha \eta^\beta}\Big|_\varphi = -u^i \frac{\partial F}{\partial \eta^\alpha \eta^\beta \eta^i}\Big|_\varphi + \gamma \frac{\partial F}{\partial \eta^\alpha \eta^\beta \eta^0}\Big|_\varphi . \tag{4.9}$$

Note that above and throughout this chapter (again) we use the Einstein summation convention, with Latin lower case letters $i, j, \ldots$ to refer to indices running from 1 to n.

Now we compute the components of $\mathcal{A}_F$ in terms of the local coordinates defined above. We have

$$\begin{aligned}
(\mathcal{A}_F)_{ij} &= \mathcal{A}_F(X_i, X_j) = \langle D_{X_i^{\mathrm{v}}} DF, X_j^{\mathrm{v}} \rangle_{T\bar{M}} \circ \varphi \\
&= \frac{\partial^2 F}{\partial \eta^\alpha \partial \eta^\beta}\Big|_\varphi X_i^\alpha X_j^\beta \\
&= \frac{\partial^2 F}{\partial \eta^0 \partial \eta^0}\Big|_\varphi u_i u_j + \frac{\partial^2 F}{\partial \eta^0 \partial \eta^i}\Big|_\varphi u_i + \frac{\partial^2 F}{\partial \eta^0 \partial \eta^j}\Big|_\varphi u_j + \frac{\partial^2 F}{\partial \eta^i \partial \eta^j}\Big|_\varphi .
\end{aligned}$$

Let b be the second fundamental form of Σ. Since

$$A_F = \mathcal{A}_F^* A = (g^{-1}\mathcal{A}_F) g^{-1} b = (g^{-1}\mathcal{A}_F g^{-1}) b,$$

to calculate H_F is sufficient to compute $g^{-1}\mathcal{A}_F g^{-1}$ and b. We denote

$$F_{\alpha\beta} = D^2 F\Big|_\varphi \Big(\Big(\frac{\partial}{\partial x^\alpha}\Big)^{\mathrm{v}}, \Big(\frac{\partial}{\partial x^\beta}\Big)^{\mathrm{v}} \Big) = \frac{\partial^2 F}{\partial \eta^\alpha \partial \eta^\beta}\Big|_\varphi .$$

Now we compute

$$\begin{aligned}
g_{ik} F^{kl} g_{jl} =& (\sigma_{ik} + \gamma^{-1} u_i u_k)(\sigma^{kp}\sigma^{lq} F_{pq})(\sigma_{jl} + \gamma^{-1} u_j u_l) \\
=& \sigma_{ik}\sigma^{kp}\sigma^{lq} F_{pq}\sigma_{jl} + \sigma_{ik}\sigma^{kp}\sigma^{lq} F_{pq}\gamma^{-1} u_j u_l + \gamma^{-1} u_i u_k \sigma^{kp}\sigma^{lq} F_{pq}\sigma_{jl} \\
& + \gamma^{-1} u_i u_k \sigma^{kp}\sigma^{lq} F_{pq}\gamma^{-1} u_j u_l \\
=& F_{ij} + \gamma^{-1}\delta_i^p u_j u^q F_{pq} + \gamma^{-1}\delta_j^q u_i u^p F_{pq} + \gamma^{-2} u_i u_j u^p u^q F_{pq},
\end{aligned}$$

then it follows from (4.7) and (4.8) that

$$\begin{aligned} g_{ik}F^{kl}g_{jl} &= F_{ij} + u_iF_{0j} + u_iF_{0i} + F_{00}u_iu_j \\ &= (\mathcal{A}_F)_{ij}. \end{aligned}$$

Hence

$$g^{ik}(\mathcal{A}_F)_{kl}g^{lj} = F^{ij}.$$

Therefore the anisotropic mean curvature is given by

$$nH_F = F^{ij}b_{ij}. \tag{4.10}$$

It remains to compute the components of the second fundamental form b of Σ. By definition,

$$b_{ij} = \langle \bar{\nabla}_{X_i}X_j, N\rangle.$$

We compute

$$\begin{aligned} \bar{\nabla}_{X_i}X_j =& \bar{\nabla}_{u_i\partial_0+\partial_i}(u_j\partial_0 + \partial_j) \\ =& u_{ij}\partial_0 + u_j\bar{\nabla}_{\partial_i}\partial_0 + u_i\bar{\nabla}_{\partial_0}\partial_j + u_ju_i\bar{\nabla}_{\partial_0}\partial_0 + \bar{\nabla}_{\partial_i}\partial_j, \end{aligned}$$

from the expression for N given in (4.6) we get

$$\begin{aligned} Wb_{ij} =& \gamma(u_{ij}\langle\partial_0,\partial_0\rangle + u_j\langle\bar{\nabla}_{\partial_i}\partial_0,\partial_0\rangle + u_i\langle\bar{\nabla}_{\partial_0}\partial_j,\partial_0\rangle \\ &+ u_iu_j\langle\bar{\nabla}_{\partial_0}\partial_0,\partial_0\rangle + \langle\bar{\nabla}_{\partial_i}\partial_j,\partial_0\rangle) - u_{ij}\langle\partial_0,\Psi_*\nabla u\rangle \\ &- u_j\langle\bar{\nabla}_{\partial_i}\partial_0,\Psi_*\nabla u\rangle - u_i\langle\bar{\nabla}_{\partial_0}\partial_j,\Psi_*\nabla u\rangle \\ &- u_iu_j\langle\bar{\nabla}_{\partial_0}\partial_0,\Psi_*\nabla u\rangle - \langle\bar{\nabla}_{\partial_i}\partial_j,\Psi_*\nabla u\rangle. \end{aligned} \tag{4.11}$$

As the leaves M_s are totally geodesic, we have

$$\langle\bar{\nabla}_{\partial_i}\partial_j,\partial_0\rangle = \langle\bar{\nabla}_{\partial_i}\partial_0,\Psi_*\nabla u\rangle = \langle\bar{\nabla}_{\partial_0}\partial_j,\Psi_*\nabla u\rangle = 0.$$

Moreover, since Y is a Killing field,

$$\langle\bar{\nabla}_{\partial_0}\partial_0,\partial_0\rangle = 0.$$

On the other hand,

$$\langle\bar{\nabla}_{\partial_0}\partial_0,\partial_i\rangle = -\frac{1}{2}\partial_i(\gamma^{-1}) = -\frac{1}{2}\frac{\gamma_i}{\gamma^2}$$

and

$$\langle \bar{\nabla}_{\partial_0}\partial_0, \nabla u\rangle = -\frac{1}{2}u^k\frac{\gamma_k}{\gamma^2}.$$

Replacing these expressions into (4.11) we obtain

$$\begin{aligned} Wb_{ij} =& \, u_{ij} - \langle \bar{\nabla}_{\partial_i}\partial_j, \nabla u\rangle - \gamma u_i\langle \bar{\nabla}_{\partial_0}\partial_0, \partial_j\rangle \\ & - \gamma u_j\langle \bar{\nabla}_{\partial_0}\partial_0, \partial_i\rangle - u_i u_j\langle \bar{\nabla}_{\partial_0}\partial_0, \nabla u\rangle. \end{aligned}$$

Since $u_{i;j} = u_{ij} - \langle \nabla_{\partial_i}\partial_j, \nabla u\rangle$ are the components of the second covariant derivative of u with respect to the connection of M, we may rewrite the above expression as

$$Wb_{ij} = u_{i;j} - \frac{1}{2}u_i\frac{\gamma_j}{\gamma} - \frac{1}{2}u_j\frac{\gamma_i}{\gamma} - \frac{1}{2}u_iu_ju^k\frac{\gamma_k}{\gamma^2}. \tag{4.12}$$

Therefore the anisotropic mean curvature of Σ is given by

$$nWH_F = F^{ij}(u_{i;j} - \frac{1}{2\gamma}u_j\gamma_i - \frac{1}{2\gamma}u_i\gamma_j - \frac{1}{2\gamma^2}u_ju_iu^k\gamma_k).$$

There is also an useful alternative expression for H_F. Using (4.7) and (4.8) we obtain

$$nWH_F = F^{ij}u_{i;j} - F_0^i\gamma_i - \frac{1}{2}F_{00}\gamma^iu_i,$$

hence, it follows from

$$\langle v, \bar{\nabla}\gamma\rangle = v(\frac{1}{|\partial_0|^2}) = 2\gamma^2\langle \bar{\nabla}_{\partial_0}\partial_0, v\rangle,$$

the expression

$$nWH_F = F^{ij}u_{i;j} - F_0^i\gamma_i - \gamma^2F_{00}\langle \bar{\nabla}_{\partial_0}\partial_0, \nabla u\rangle. \tag{4.13}$$

We conclude that a function $u \in C^{2,\alpha}(M)$ whose Killing graph has prescribed anisotropic mean curvature H_F satisfies the PDE

$$\frac{1}{W}F^{ij}u_{i;j} = \frac{1}{W}F_0^i\gamma_i + \frac{1}{2W}F_{00}\gamma^iu_i + nH_F. \tag{4.14}$$

Denoting

$$a^{ij} = \frac{1}{W}F^{ij}, \quad b = \frac{1}{W}F_0^i\gamma_i + \frac{1}{2W}F_{00}\gamma^iu_i + nH_F,$$

the anisotropic mean curvature equation (4.14) becomes

$$\mathcal{Q}[u] = a^{ij}u_{i;j} - b = 0.$$

We point out that, in the particular case of the area $F = |\eta|$, since

$$F^{ij} = \sigma^{ij} - \frac{u^i u^j}{W^2}\,, \qquad F_0^i = \frac{u^i}{W^2} \quad \text{and} \quad F_{00} = \frac{|\nabla u|^2}{\gamma W^2}\,,$$

the anisotropic mean curvature equation (4.14) is

$$\frac{1}{W}\left(\sigma^{ij} - \frac{u^i u^j}{W^2}\right)u_{i;j} = \frac{u^i}{W^3}\gamma_i + \frac{|\nabla u|^2}{2\gamma W^3}u^i\gamma_i + nH$$

or

$$\frac{1}{W}\left(\sigma^{ij} - \frac{u^i u^j}{W^2}\right)u_{i;j} = \frac{u^i\gamma_i}{2W^3}\gamma_i + \frac{1}{W}\frac{u^i\gamma_i}{2\gamma} + nH. \tag{4.15}$$

This equation agree with the equation of prescribed mean curvature obtained in [15].

Killing Cylinders

We call the *Killing cylinder* over $\Gamma = \partial\Omega$ the submanifold

$$K = \{\Psi(s,p) : s \in \mathbb{R},\ p \in \Gamma\}.$$

ruled by the flow lines of Y. If $s^1, \ldots, s^{n-1}$ are local coordinates for Γ, then $s, s^1, \ldots, s^{n-1}$ are coordinates for K. We denote by $\bar{\partial}_s, \bar{\partial}_1, \ldots, \bar{\partial}_{n-1}$ the corresponding coordinate vector fields. Let ν be the unit normal vector field along Γ as a submanifold of M. We equally denote by ν the unit normal vector field $\Psi_{s*}\nu$ along K. Thus

$$\langle \nu, \partial_s\rangle = 0 = \langle \nu, \partial_i\rangle.$$

Since ν and $\bar{\partial}_i$ are tangent to the totally geodesic leaves $\mathbb{P}_s$ we have

$$\langle \bar{\nabla}_{\partial_i}\partial_s, \nu\rangle = 0.$$

Hence ∂_s is a principal direction of K and the corresponding principal curvature is the geodesic curvature

$$\kappa = \gamma\langle \bar{\nabla}_{\partial_s}\partial_s, \nu\rangle$$

of the flow lines through Γ.

In the sequel, we deduce some useful properties of the distance function $d = \mathrm{dist}(\,\cdot\,, K)$ from K. We denote by Γ_ϵ and K_ϵ the level sets $d = \epsilon$ in M and $\bar{M}$, respectively. Thus Γ_ϵ and K_ϵ are equidistant from Γ and K, respectively. Clearly K_ϵ is the Killing cylinder over Γ_ϵ. Since we assume that Γ is smooth, the function d is also smooth at points of $\Psi(\mathbb{R}\times\Omega_0)$, where $\Omega_0 \subset \Omega$ is the set of points which can be joined to Γ by a *unique* minimizing geodesic. We point out that it was shown in [30] that the function d in Ω_0 has the same regularity as Γ. We may define coordinates on $\Psi(\mathbb{R}\times\Omega_0)$ as follows: for $q \in \Psi(\mathbb{R}\times\Omega_0)$ we associate coordinates (s^i, d) by $q = \exp_p(d\,\nu)$, when $p = p(s, s^1, \ldots, s^{n-1})$ in K. Then

$$\bar{\nabla}_{\partial d}\partial d = 0$$

and

$$|\bar{\nabla} d| = 1.$$

It follows from these relations that

$$d^i d_{i;j} = 0, \tag{4.16}$$

where $d^i = \bar{g}^{ij} d_j$ as usual. We observe that $\bar{\nabla} d|_\epsilon = \partial_d = \nu_\epsilon$ is the unit inward normal field to K_ϵ.

Now we will compute the anisotropic mean curvature $H_F^K(\varepsilon)$ of K_ε. First we note that the components of the metric induced on K_ε are

$$\theta_{ab} = \sigma_{ab}, \quad \theta_{0a} = 0, \quad \text{and} \quad \theta_{00} = \gamma^{-1},$$

with $1 \le a, b \le n-1$. Hence

$$-nH_F^K(\varepsilon) = \mathrm{div}_{K_\varepsilon}\,\xi = \theta^{ab}\langle D_{(\bar{\nabla}_{\partial_a}\bar{\nabla} d)^{\mathrm{v}}} DF, \partial_b^{\mathrm{v}}\rangle_{T\bar{M}}\circ\varphi + \gamma\langle D_{(\bar{\nabla}_{\partial_0}\bar{\nabla} d)^{\mathrm{v}}} DF, \partial_0^{\mathrm{v}}\rangle_{T\bar{M}}\circ\varphi$$

where φ is the local section of the tangent bundle $T\bar{M}$ defined by

$$\varphi(x) = (x, \nu(x)) = (x, \bar{\nabla} d(x)).$$

Since

$$\langle \bar{\nabla}_{\partial_a}\bar{\nabla} d, \partial_0\rangle = \langle \bar{\nabla}_{\partial_0}\bar{\nabla} d, \partial_a\rangle = 0,$$

we have

$$\begin{aligned} -nH_F^K(\varepsilon) =& \sigma^{ij}\sigma^{kl}\langle \bar{\nabla}_{\partial_i}\bar{\nabla} d, \partial_k\rangle\langle D_{\partial_l^{\mathrm{v}}} DF, \partial_j^{\mathrm{v}}\rangle_{TM}\circ\varphi \\ &+ \gamma^2\langle \bar{\nabla}_{\partial_0}\bar{\nabla} d, \partial_0\rangle\langle D_{\partial_0^{\mathrm{v}}} DF, \partial_0^{\mathrm{v}}\rangle_{TM}\circ\varphi \\ =& F^{ij}|_\varphi d_{i;j} + \gamma^2 F_{00}|_\varphi\langle \bar{\nabla}_{\partial_s}\bar{\nabla} d, \partial_s\rangle. \end{aligned}$$

Note that, as the n-th coordinate corresponds to the function d, the sum above may be taken from 1 up to n, since $\theta_{na} = 0$, $\bar{\nabla}_{\partial d}\partial d = 0$ and $\langle \bar{\nabla} d, \partial_a \rangle = 0$, if $1 \leq a \leq n-1$.

Finally, the anisotropic mean curvature of the cylinder K_ε is

$$nH_F^{\mathrm{cyl}}(\varepsilon) = -F^{ij}|_\varphi d_{i;j} + \gamma F_{00}|_\varphi \kappa, \tag{4.17}$$

when the orientation is defined by taking the inward normal. The Weingarten operator of K_ε will be denoted by A_ε. The anisotropic mean curvature of K is denoted just by H_F^{cyl}

If the orientation is defined by choosing the outward normal, the anisotropic mean curvature of K_ε becomes

$$n\bar{H}_F^{\mathrm{cyl}}(\varepsilon) = F^{ij}|_{\bar{\varphi}} d_{i;j} - \gamma F_{00}|_{\bar{\varphi}} \kappa, \tag{4.18}$$

where $\bar{\varphi}$ is the local section of $T\bar{M}$ defined by $\bar{\varphi}(x) = (x, -\nabla d)$. Note that the choice of the orientation is indicated by a bar on H_F^{cyl}.

Following [33] we define the anisotropic Ricci curvature of $\bar{M}$ in a given direction X as the tensor Ric_F given by

$$\mathrm{Ric}_F(X) = \sum_{i=1}^{n} = \mathcal{A}_F\big(\bar{R}(e_i, X)X, e_i\big),$$

where $\bar{R}$ is the curvature tensor in $\bar{M}$ and $e_1, \ldots, e_n, X$ is an orthonormal basis. Our aim in this chapter is the establishment of the following existence result.

Theorem 4.1. *Let $\Omega \subset \bar{M}$ be a bounded domain with $C^{2,\alpha}$ boundary $\Gamma = \partial\Omega$. Suppose that*

$$\inf_{\bar{M}} \mathrm{Ric}_F \geq -n\lambda \inf_{\Gamma} H_{\mathrm{cyl}}^2,$$

where $\lambda > 0$ is the smallest eigenvalue of $\mathcal{A}_F$ and H_{cyl} is the usual mean curvature of Γ. Let $H_F \in C^\alpha(\Omega)$ and $\phi \in C^{2,\alpha}(\Gamma)$ be given. If

$$\bar{H}_F^{\mathrm{cyl}} < H_F < H_F^{\mathrm{cyl}}$$

then there exists a unique function $u \in C^{2,\alpha}(\Omega) \cap C^0(\bar{\Omega})$ satisfying $u|_\Gamma = \phi$ whose Killing graph has anisotropic mean curvature H_F.

To prove this theorem we will use the continuity method which reduces the problem to the establishment of a priori estimates for prospective solutions. In the next sections we will establish such estimates.

4.2 The C^0 Estimates

In this section we will present the C^0 estimates for prospective solutions of the problem

$$\begin{aligned} \mathcal{Q}[u] &= a^{ij}u_{i;j} - b = 0 \\ u_{|\Gamma} &= \phi, \end{aligned} \tag{4.19}$$

where ϕ is a $C^{2,\alpha}$ function defined on Γ. As (4.19) is a quasilnear elliptic PDE, we may apply the maximum and comparison principles. Thus we must construct barriers for the solutions u of (4.19). The barriers will be cylinders, hence we must know how the anisotropic mean curvature of the cylinders K_ε are related.

Lemma 4.2. *Assume that the anisotropic Ricci curvature of $\bar{M}$ satisfies*

$$\mathrm{Ric}_F(\nu,\nu) \geq -n\lambda \inf_{\Gamma} H^2_{\mathrm{cyl}}. \tag{4.20}$$

Let $x \in \Gamma$ be the closest point to a given point $y \in \Gamma_\varepsilon \subset \Omega_0$. Then

$$H_F^{\mathrm{cyl}}(\varepsilon)_{|_y} \geq H_F^{\mathrm{cyl}}{}_{|_x} \tag{4.21}$$

$$\bar{H}_F^{\mathrm{cyl}}(\varepsilon)_{|_y} \leq \bar{H}_F^{\mathrm{cyl}}{}_{|_x}, \tag{4.22}$$

where $H_F^{\mathrm{cyl}}(\varepsilon)$ and $\bar{H}_F^{\mathrm{cyl}}(\varepsilon)$ are the anisotropic mean curvature of Γ_ε with respect to the inward and outward (normal) orientation, respectively.

Proof. Applying the formula for the second variation obtained in [33] we get

$$n\dot{H}_F^{\mathrm{cyl}}(\varepsilon) = \mathrm{tr}(A_F^\varepsilon A^\varepsilon) + \mathrm{Ric}_F(\nu,\nu). \tag{4.23}$$

By the trace invariance, we may suppose that the above matrices are diagonal, say,

$$\mathcal{A}_F^\varepsilon = \mathrm{diag}(\lambda_1, \cdots, \lambda_n) \quad \text{and} \quad A^\varepsilon = \mathrm{diag}(\kappa_1, \cdots, \kappa_n),$$

where $\mathcal{A}_F^\varepsilon$ is the bilinear form defined on K_ε as in (4.4). Hence

$$\mathrm{tr}(A_F^\varepsilon A^\varepsilon) = \sum_i \lambda_i \kappa_i^2 \geq \lambda \sum_i \kappa_i^2,$$

where $\lambda > 0$ is the smallest eigenvalue of $\mathcal{A}_F^\varepsilon$. We apply the Cauchy-Schwarz inequality to obtain

$$nH_{\mathrm{cyl}}(\varepsilon) = \sum_i \kappa_i \le n^{1/2}\big(\sum_i \kappa_i^2\big)^{1/2},$$

where $H_{\mathrm{cyl}}(\varepsilon)$ is the mean curvature of K_ε. Replacing this inequality into (4.23) we obtain

$$n\dot{H}_F^{\mathrm{cyl}}(\varepsilon) \ge n\lambda H_{\mathrm{cyl}}^2(\varepsilon) + \mathrm{Ric}_F(\nu,\nu) \ge n\lambda\Big(H_{\mathrm{cyl}}^2(\varepsilon) - \inf_\Gamma H_{\mathrm{cyl}}^2\Big).$$

Therefore,

$$\dot{H}_F^{\mathrm{cyl}}(d) \ge \lambda\big(H_{\mathrm{cyl}}(d) + \inf_\Gamma H_{\mathrm{cyl}}\big)\big(H_{\mathrm{cyl}}(d) - \inf_\Gamma H_{\mathrm{cyl}}\big).$$

Hence, $\dot{H}_F^{\mathrm{cyl}}(d) \ge c\big(H_{\mathrm{cyl}}(d) - \inf_\Gamma H_{\mathrm{cyl}}\big)$ in some interval $d \in [0, d_0]$ $(d_0 > 0)$ for a constant $c > 0$. It follows that $H_F^{\mathrm{cyl}}(d)$ does not decrease with increasing d, which proves (4.21). The proof of inequality (4.22) is completely analogous. □

Under the condition (4.20) we may construct barriers for solutions of (4.19) setting a function of the form

$$\varphi = \sup_\Gamma \phi + h \circ d,$$

where d is the distance from Γ in M and $h \in C^\infty(\mathbb{R})$ is a real function that will be chosen later.

Proposition 4.3. *Assume that (4.20) holds and*

$$\inf_\Omega H_F > \sup_\Gamma \bar{H}_F^{\mathrm{cyl}}. \tag{4.24}$$

Then, for a suitable choice of h, the function φ satisfies

$$\mathcal{Q}[\varphi] < 0 \quad \text{on } \Omega_0,$$

where $\Omega_0 \subset \Omega$ is the set of points which can be joined to Γ by a unique minimizing geodesic.

Proof. Fixed $x \in \Omega_0$, let K_ε be the cylinder equidistant from K that contains x. In the sequel, all computations are done on x and, for convenience, we omit it. Setting

$$h(d) = \frac{\mathrm{e}^{CA}}{C}(1 - \mathrm{e}^{-Cd}),$$

where $A > \mathrm{diam}(\Omega)$ and $C > 0$ are constants that will be chosen later. By definition, $h' = \mathrm{e}^{C(A-d)}$ and $h'' = -Ch'$. Along Ω_0 we have

$$\varphi_i = h' d_i \quad \text{and} \quad \varphi_{i;j} = h' d_{i;j} + h'' d_i d_j.$$

The unit vector field

$$\eta = \frac{1}{W}(\gamma\, \partial_s - h' \nabla d)$$

is normal (outward) along the Killing graph generated by φ, where $W = \sqrt{\gamma + h'^2}$. Hence,

$$\begin{aligned}
\mathcal{Q}[\varphi] =& \frac{h'}{W} F^{ij}|_\eta d_{i;j} - C\frac{h'}{W} F^{ij}|_\eta d_i d_j \\
& - \frac{h'}{W}\gamma F_{00}|_\eta \kappa_\varepsilon - \frac{1}{W} F^i_0|_\eta \gamma_i - nH_F \\
& = F^{ij}|_{(\frac{\gamma}{h'}, -\nabla d)} d_{i;j} - C F^{ij}|_{(\frac{\gamma}{h'}, -\nabla d)} d_i d_j \\
& - \gamma F_{00}|_{(\frac{\gamma}{h'}, -\nabla d)} \kappa_\varepsilon - \frac{1}{h'} F^i_0|_{(\frac{\gamma}{h'}, -\nabla d)} \gamma_i - nH_F.
\end{aligned}$$

Applying the mean-value theorem to the real function

$$l(\theta) = F^{ij}|_{(\frac{\theta\gamma}{h'}, -\nabla d)} d_{i;j}, \quad 0 \le \theta \le 1,$$

we get

$$F^{ij}|_{(\frac{\gamma}{h'}, -\nabla d)} d_{i;j} = F^{ij}_{(0, -\nabla d)} d_{i;j} + \frac{\gamma}{h'} F^{ij}_0|_{(\frac{\bar\theta\gamma}{h'}, -\nabla d)} d_{i;j}, \quad 0 \le \bar\theta \le 1.$$

Similarly,

$$F_{00}|_{(\frac{\gamma}{h'}, -\nabla d)} = F_{00}|_{(0, -\nabla d)} + \frac{\gamma}{h'} F_{000}|_{(\frac{\tilde\theta\gamma}{h'}, -\nabla d)}, \quad 0 \le \tilde\theta \le 1.$$

Using these relations we get

$$\begin{aligned}
\mathcal{Q}[\varphi] =& F^{ij}|_{(0, -\nabla d)} d_{i;j} - \gamma F_{00}|_{(0, -\nabla d)} \kappa_\varepsilon - C h' F^{ij}|_{(\gamma, -h'\nabla d)} d_i d_j - nH_F \\
& + \frac{1}{h'}\left\{ -F^i_0|_{(\frac{\gamma}{h'}, -\nabla d)} \gamma_i + \gamma F^{ij}_0|_{(\frac{\bar\theta\gamma}{h'}, -\nabla d)} d_{i;j} + \gamma^2 F_{000}|_{(\frac{\tilde\theta\gamma}{h'}, -\nabla d)} \kappa_\varepsilon \right\}.
\end{aligned}$$

From the definition of $\bar{H}_F^{\mathrm{cyl}}$ presented in (4.18),

$$\begin{aligned}\mathcal{Q}[\varphi] =& n\bar{H}_F^{\mathrm{cyl}}(\varepsilon) - Ch'F^{ij}|_{(\gamma,-h'\nabla d)}d_i d_j - nH_F \\ &+ \frac{1}{h'}\left\{-F_0^i|_{(\frac{\gamma}{h'},-\nabla d)}\gamma_i + \gamma F_0^{ij}|_{(\frac{\bar{\theta}\gamma}{h'},-\nabla d)}d_{i;j} + \gamma^2 F_{000}|_{(\frac{\tilde{\theta}\gamma}{h'},-\nabla d)}\kappa_\varepsilon\right\}.\end{aligned}$$

It follows from the ellipticity condition satisfied by F that

$$-Ch'F^{ij}|_{(\gamma,-h'\nabla d)}d_i d_j \leq 0.$$

Hence, by Lemma 4.2 we conclude

$$\begin{aligned}\mathcal{Q}[\varphi] \leq& \frac{1}{h'}\left\{-F_0^i|_{(\frac{\gamma}{h'},-\nabla d)}\gamma_i + \gamma F_0^{ij}|_{(\frac{\bar{\theta}\gamma}{h'},-\nabla d)}d_{i;j} + \gamma^2 F_{000}|_{(\frac{\tilde{\theta}\gamma}{h'},-\nabla d)}\kappa_\varepsilon\right\} \\ &+ n\bar{H}_F^{\mathrm{cyl}} - nH_F.\end{aligned}$$

Now consider the compact subset $\mathcal{S}$ of $T\bar{M}$ defined by

$$\mathcal{S} = \{(x,\eta) \in T\bar{M} : x \in \bar{\Omega}_0 \text{ and } 1 \leq |\eta| \leq 2\}. \tag{4.25}$$

If $h'^2 > \gamma$, we have

$$\left(x, \frac{\bar{\theta}\gamma}{h'}, -\nabla d\right), \left(x, \frac{\tilde{\theta}\gamma}{h'}, -\nabla d\right), \left(x, \frac{\bar{\theta}\gamma}{h'}, -\nabla d\right) \in \mathcal{S}$$

Let $\mu : \mathcal{S} \longrightarrow \mathbb{R}$ be the real function defined by

$$\mu(x,\xi) = -F_0^i|_{(x,\xi)}\gamma_i(x) + \gamma(x)F_0^{ij}|_{(x,\xi)}d_{i;j}(x) + \gamma^2(x)F_{000}|_{(x,\xi)}\kappa_\varepsilon.$$

There exists a uniform constant $R = R(\gamma, \nabla\gamma, D^2F, D^3F)$ such that $\mu \leq R$. We also have from the hypothesis (4.24) that $\bar{H}_F^{\mathrm{cyl}} - H_F < 0$ in $\bar{\Omega}_0$. Therefore, the inequality

$$\mathcal{Q}[u] \leq n(\bar{H}_F^{\mathrm{cyl}} - H_F) + \frac{\mu}{h'}$$

implies that $\mathcal{Q}[\varphi] < 0$, if we choose h' large such that

$$h' > \max\left\{\gamma, \sup_{\bar{\Omega}_0} \frac{R}{n(H_F - \bar{H}_F^{\mathrm{cyl}})}\right\}.$$

Since $h' \longrightarrow +\infty$ as $C \longrightarrow +\infty$, if $C > 0$ is large enough, the above inequality holds. □

To obtain a lower barrier we set the function

$$\tilde{\varphi} = \inf_{\Gamma} \phi - h \circ d.$$

As was done above, we compute

$$\begin{aligned}\mathcal{Q}[\tilde{\varphi}] =& - F^{ij}|_{(0,\nabla d)} d_{i;j} + \gamma F_{00}|_{(0,\nabla d)} \kappa_\varepsilon + Ch' F^{ij}|_{(\gamma, h'\nabla d)} d_i d_j \\ &+ \frac{1}{h'} \left\{ -F_0^i|_{(\frac{\gamma}{h'},\nabla d)} \gamma_i - \gamma F_0^{ij}|_{(\frac{\bar{\theta}\gamma}{h'}, -\nabla d)} d_{i;j} - \gamma^2 F_{000}|_{(\frac{\bar{\theta}\gamma}{h'},\nabla d)} \kappa_\varepsilon \right\} - nH_F \\ \geq& nH_F^{\text{cyl}}(\varepsilon) - nH_F + \frac{\tilde{\mu}}{h'} \\ \geq& nH_F^{\text{cyl}} - nH_F + \frac{\tilde{\mu}}{h'}.\end{aligned}$$

where $\tilde{\mu}$ is a function defined in $\mathcal{S}$ in a similar way to the function μ defined above. Therefore, under the hypothesis $H_F^{\text{cyl}} > H_F$ we have $\mathcal{Q}[\tilde{\varphi}] > 0$, if

$$h' > \max \left\{ \gamma, \sup_{\bar{\Omega}_0} \frac{\tilde{R}}{n(H_F - H_F^{\text{cyl}})} \right\}.$$

These results allow the following conclusion.

Proposition 4.4. *Suppose that the anisotropic Ricci curvature of $\bar{M}$ satisfies (4.20). If*

$$\bar{H}_F^{\text{cyl}} < H_F < H_F^{\text{cyl}} \tag{4.26}$$

(or alternatively, $\inf_\Gamma H_F^{\text{cyl}} > \sup_\Omega H_F$ and $\inf_\Omega H_F > \sup_\Gamma \bar{H}_F^{\text{cyl}}$), then there exists a uniform constant $C = C(F, D^2F, D^3F, H_F, \Omega)$ such that

$$|u|_0 \leq C + |\phi|_0$$

if $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $\mathcal{Q}[u] = 0$ and $u_{|\Gamma} = \phi$.

Proof. As it was explaned above, the functions

$$\varphi = \sup_\Gamma \phi + h \circ d \quad \text{and} \quad \tilde{\varphi} = \inf_\Gamma \phi - h \circ d,$$

satisfy

$$\mathcal{Q}[\varphi] < \mathcal{Q}[u] \quad \text{and} \quad \mathcal{Q}[\tilde{\varphi}] > \mathcal{Q}[u]$$

in Ω_0. We also have

$$\varphi_{|\Gamma} \geq u_{|\Gamma} \quad \text{and} \quad \tilde{\varphi}_{|\Gamma} \leq u_{|\Gamma},$$

since $d = 0$ on Γ and $h(0) = 0$. Therefore, it follows from the comparison principle (see [17], Teorema 10.1) that

$$\tilde{\varphi} \leq u \leq \varphi$$

in Ω_0. To extend these inequalities to $\bar{\Omega}$ we follow [15]. We will prove that $\varphi \geq u$ in $\bar{\Omega}$ by contradiction. Assume that there exist points for which the continuous function $\hat{u} = u - \varphi$ satisfies $\hat{u} > 0$. Hence $m = \hat{u}(y) > 0$ at a maximum point $y \in \bar{\Omega}$ of $\hat{u}$. Choose a minimizing geodesic γ joining y to Γ for which the distance $d = d(y, \Gamma)$ is attained. Thus, $\gamma(t) = \exp_{y_0} t\nu$, $0 \leq t \leq d$, starts from a point $y_0 \in \Gamma$ with unit speed ν. Since γ is minimizing, we have $d(\gamma(t), \Gamma) = t$ and the function φ restricted to γ is differentiable with $\varphi'(\gamma(t)) = \mathrm{e}^{C(A-t)}$. Since the maximum of $\hat{u}$ restricted to γ occurs at $t = d$, i.e., at the point y, we have

$$u'(y(d)) - \varphi'(\gamma(d)) = \hat{u}'(\gamma(d)) \geq 0.$$

This implies that

$$\langle \nabla u(y), \gamma'(d) \rangle \geq \varphi'(\gamma(d)) = \mathrm{e}^{C(A-d)} > 0.$$

In particular $\nabla u(y) \neq 0$, and hence the level hypersurface

$$S = \{x \in \Omega \cap B_r(y) \, : \, u(x) = u(y)\}$$

is regular for small radius r. Along S we have

$$\hat{u}(x) + \varphi(x) = \hat{u}(y) + \varphi(y) \geq \hat{u}(y) + \varphi(y).$$

Thus $\varphi(x) \geq \varphi(y)$. Since φ is an increasing function of $d(\,\cdot\,, \Gamma)$, it follows that $d(x, \Gamma) \geq d(y, \Gamma) = d$. Hence, the points in S are at a distance at least d from Γ.

Since S is C^2 it satisfies the interior sphere condition: there exists a small ball $B_\varepsilon(z)$ touching S at y contained in the side for which $\nabla u(y)$ and $\gamma'(d)$ point. Thus, the points of $B_\varepsilon(z)$ satisfy $u(x) \geq u(y)$, and hence

$$\varphi(x) + m \geq u(x) \geq u(y) = \varphi(y) + m, \quad x \in B_\varepsilon(z),$$

where in the first inequality we used the definition of m. Again because φ is an increasing function of d, we have $d(x,\Gamma) \geq d$ on $B_\varepsilon(z)$, and therefore this ball is contained in the interior of Ω far away from Γ. This allows us to extend the geodesic γ through $B_\varepsilon(z)$. We claim that the center z of the ball is contained in this extension. Otherwise, the broken line consisting of γ and of the radius in $B_\varepsilon(z)$ from z to y has length smaller than the minimizing geodesic joining z to y_0 (for a suitable small ε such a geodesic must cross the level hypersurface S at a point $x \neq y$ at distance to Γ greater than d). Thus, if there exists at least two distinct minimizing geodesics joining y to Γ, then the point z is contained in the extension of both geodesics after its intersection at y. Choosing ε sufficiently small, we see that this configuration is not possible (the construction we made above applies to both geodesics). This contradiction implies that the maximum point y belongs to Ω_0. However, in this case, $\hat{u} \leq 0$, a contradiction. We conclude that $u \leq \varphi$ throughout $\bar{\Omega}$ and therefore φ is a continuous supersolution for the Dirichlet problem (4.19). In a similar way, we may prove that $\tilde{\varphi}$ is a continuous subsolution for (4.19). It is clear that the existence of these barries implies the C^0 a priori estimate estated in the proposition. □

Remark 4.5. *We point out that hypothesis (4.27) in Proposition 4.4 is also used in [6], where the Euclidean case is treated (see [6], Theorem 5).*

4.3 Boundary Gradient Estimates

In this section we will establish the a priori gradient estimates along the boundary for the Dirichlet problem (4.19).

We proceed in a similiar way to the last section. We will use barriers of the form $\varphi = h \circ d + \phi$, where $d = \mathrm{dist}(\,\cdot\,,\Gamma)$, h is a real function to be chosen later and we denote (also) by ϕ the extension of the boundary function ϕ to a tubular neighborhood Ω_ε, by setting $\phi(s^i, d) = \phi(s^i)$.

Using the same notation used in Proposition 4.4 we compute

$$\begin{aligned}
\mathcal{Q}[\varphi] =& F^{ij}|_{(\gamma,-h'\nabla d-\nabla\phi)}(h''d_id_j + h'd_{i;j} + \phi_{i;j}) - F_0^i|_{(\gamma,-h'\nabla d-\nabla\phi)}\gamma_i \\
&- \gamma^2 F_{00}|_{(\gamma,-h'\nabla d-\nabla\phi)}\langle\bar\nabla_{\partial_0}\partial_0, h'\nabla d+\nabla\phi\rangle - nH_F \\
=& F^{ij}|_{(\frac{\gamma}{h'},-\frac{\nabla\phi}{h'}-\nabla d)}d_{i;j} + \frac{h''}{h'}F^{ij}|_{(\frac{\gamma}{h'},-\frac{\nabla\phi}{h'}-\nabla d)}d_id_j + \frac{1}{h'}F^{ij}|_{(\frac{\gamma}{h'},-\frac{\nabla\phi}{h'}-\nabla d)}\phi_{i;j} \\
&- \frac{1}{h'}F_0^i|_{(\frac{\gamma}{h'},-\frac{\nabla\phi}{h'}-\nabla d)}\gamma_i - \gamma F_{00}|_{(\frac{\gamma}{h'},-\frac{\nabla\phi}{h'}-\nabla d)}\kappa_\varepsilon \\
&- \frac{\gamma^2}{h'}F_{00}|_{(\frac{\gamma}{h'},-\frac{\nabla\phi}{h'}-\nabla d)}\langle\bar\nabla_{\partial_0}\partial_0,\nabla\phi\rangle - nH_F.
\end{aligned}$$

As we have done in the last section, we apply the mean-value theorem to the real function

$$l(\theta) = F^{ij}|_{(\frac{\theta\gamma}{h'},-\frac{\theta}{h'}\nabla\phi-\nabla d)}d_{i;j}, \qquad 0\le\theta\le1,$$

to obtain

$$\begin{aligned}
F^{ij}_{(\frac{\gamma}{h'},-\frac{\gamma}{h'}\nabla\phi-\nabla d)}d_{i;j} =& F^{ij}|_{(0,-\nabla d)}d_{i;j} + \frac{\gamma}{h'}F_0^{ij}|_{(\frac{\bar\theta\gamma}{h'},-\frac{\bar\theta\theta}{h'}\nabla\phi-\nabla d)}d_{i;j} \\
&- \frac{\phi^k}{h'}F_k^{ij}|_{(\frac{\bar\theta\gamma}{h'},-\frac{\bar\theta\gamma}{h'}\nabla\phi-\nabla d)}d_{i;j}.
\end{aligned}$$

In a similar way we get

$$\begin{aligned}
F_{00}|_{(\frac{\gamma}{h'},-\frac{\gamma}{h'}\nabla\phi-\nabla d)} =& F_{00}|_{(0,-\nabla d)} + F_{000}|_{(\frac{\tilde\theta\gamma}{h'},-\frac{\tilde\theta\gamma}{h'}\nabla\phi-\nabla d)}\frac{\gamma}{h'} \\
&- \frac{\phi^k}{h'}F_{00k}|_{(\frac{\tilde\theta\gamma}{h'},-\frac{\tilde\theta\gamma}{h'}\nabla\phi-\nabla d)}.
\end{aligned}$$

For sake of convennience we will denote

$$V_\theta = (\frac{\theta\gamma}{h'},-\frac{\theta\gamma}{h'}\nabla\phi-\nabla d), \qquad 0\le\theta\le1.$$

We may rewrite the expression obtained above for $\mathcal{Q}[\varphi]$ as

$$\begin{aligned}
\mathcal{Q}[\varphi] =& F^{ij}|_{(0,-\nabla d)}d_{i;j} - \gamma F_{00}|_{(0,-\nabla d)}\kappa_\varepsilon - nH_F + \frac{h''}{h'}F^{ij}|_{V_1}d_id_j \\
&+ \frac{1}{h'}\left\{F^{ij}|_{V_1}\phi_{i;j} - F_0^i|_{V_1}\gamma_i - \gamma^2F_{00}|_{V_1}\langle\bar\nabla_{\partial_0}\partial_0,\nabla\phi\rangle + \gamma F_0^{ij}|_{V_{\bar\theta}}d_{i;j}\right. \\
&\left.- \phi^kF_k^{ij}|_{V_{\bar\theta}}d_{i;j} - \gamma^2F_{000}|_{V_{\tilde\theta}}\kappa_\varepsilon + \gamma\phi^kF_{00k}|_{V_{\tilde\theta}}\kappa_\varepsilon\right\}.
\end{aligned}$$

Let $\mathcal{S}$ be the set defined in (4.25). Note that, if $h'^2 > \gamma^2 + \sup_{\bar{\Omega}_0} |\nabla\phi|^2$, the points of the form (x, V_θ) belong to $\mathcal{S}$. Consider the function $\mu : \mathcal{S} \longmapsto \mathbb{R}$ defined by

$$\begin{aligned}\mu(x,\xi) :=& F^{ij}|_\xi \phi_{i;j} - F_0^i|_\xi \gamma_i - \gamma^2 F_{00}|_\xi \langle \bar{\nabla}_{\partial_0}\partial_0, \nabla\phi\rangle + \gamma F_0^{ij}|_\xi d_{i;j}\phi^k F_k^{ij}|_\xi d_{i;j} \\ &- \gamma^2 F_{000}|_\xi \kappa_\varepsilon + \gamma\phi^k F_{00k}|_\xi \kappa_\varepsilon.\end{aligned}$$

There exists a uniform constant $R = R(\gamma, \nabla\gamma, D^2F, D^3F)$ such that $\mu \leq R$. On the other hand, it follows from the definition of $\bar{H}_F^{\text{cyl}}$ that

$$\mathcal{Q}[\varphi] = n\bar{H}_F^{\text{cyl}}(d) - nH_F + \frac{h''}{h'}F^{ij}(V_1)d_i d_j + \frac{\mu}{h'}.$$

We define

$$h(d) = \mu \ln(1 + Kd)$$

for certain positive constants μ and K to be chosen later. Then

$$h' = \frac{\mu K}{1 + Kd} \quad \text{and} \quad h'' = -\frac{1}{\mu}(h')^2.$$

We choose μ in such a way that $\mu \longrightarrow 0$ as $K \longrightarrow \infty$. It suffices to take

$$\mu = \frac{C}{\ln(1 + K)}$$

for some constant $C > 0$ to be chosen later. In this case, as $K \longrightarrow \infty$ we have

$$h'(0) = \frac{CK}{\ln(1 + K)} \longrightarrow +\infty.$$

It also holds that $\frac{h'}{W} \sim 1$ as $K \longrightarrow \infty$.
It follows from the expression of h that

$$\begin{aligned}\mathcal{Q}[\varphi] =& n\bar{H}_F^{\text{cyl}}(d) - nH_F - \frac{1}{\mu}h'F^{ij}(V_1)d_i d_j + \frac{\mu}{h'} \\ \leq& n(\bar{H}_F^{\text{cyl}} - H_F) + \frac{\mu}{h'},\end{aligned}$$

where we have also used the ellipticity condition on F and Lemma 4.2 to get the last inequality. Hence, if

$$\bar{H}_F^{\text{cyl}} < H_F,$$

and

$$h' > \max\left\{(\gamma^2 + \sup_{\bar{\Omega}_0} |\nabla\phi|^2)^{\frac{1}{2}}, \sup_{\bar{\Omega}_0} \frac{R^2}{n(H_F - H_F^{\mathrm{cyl}})}\right\},$$

we obtain $\mathcal{Q}[\varphi] < 0$ in a small tubular neighborhood Ω_ε of Γ. For K and C large enough we also have that $\varphi \geq u$ on both components of $\partial\Omega_\varepsilon$. Similarly, under the hypothesis $H_F < \bar{H}_F^{\mathrm{cyl}}$ we obtain $\mathcal{Q}[\tilde{\varphi}] > 0$, where $\tilde{\varphi} = -h \circ d + \phi$ and $\tilde{\varphi} \leq u$ on $\partial\Omega_\varepsilon$. Thus we have the following result.

Proposition 4.6. *Suppose that the anisotropic Ricci curvature of $\bar{M}$ satisfies (4.20). Assume that*

$$\bar{H}_F^{\mathrm{cyl}} < H_F < H_F^{\mathrm{cyl}} \tag{4.27}$$

(or alternatively, $\inf_\Gamma H_F^{\mathrm{cyl}} > \sup_\Omega H_F$ *and* $\inf_\Omega H_F > \sup_\Gamma \bar{H}_F^{\mathrm{cyl}}$*), then there exists a uniform constant* $C = C(F, D^2F, D^3F, H_F, \phi, \Omega, n)$ *such that*

$$\sup_\Gamma |\nabla u| \leq C,$$

if $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *satisfies* $\mathcal{Q}[u] = 0$ *and* $u_{|\Gamma} = \phi$.

Proof. As we show above, we may define barriers φ and $\tilde{\varphi}$ such that the function u satisfies

$$\begin{cases} \mathcal{Q}[\varphi] < \mathcal{Q}[u] & \text{in} \quad \Omega_\varepsilon \\ u \leq \varphi & \text{on} \quad \partial\Omega_\varepsilon, \end{cases} \qquad \begin{cases} \mathcal{Q}[\tilde{\varphi}] > \mathcal{Q}[u] & \text{in} \quad \Omega_\varepsilon \\ u \geq \tilde{\varphi} & \text{on} \quad \partial\Omega_\varepsilon. \end{cases}$$

Hence we conclude from the comparison principle that

$$\varphi \leq u \leq \tilde{\varphi},$$

in Ω_ε. Since $\varphi = u = \tilde{\varphi}$ on Γ, the above inequalities imply

$$\frac{\partial\varphi}{\partial\nu} \leq \frac{\partial u}{\partial\nu} \leq \frac{\partial\tilde{\varphi}}{\partial\nu},$$

where ν is the unit normal inward vector along Γ. These inequalities and the equality $u = \phi$ on Γ produce the desired estimate. □

4.4 Interior Gradient Estimates

The last step in providing a priori estimates for (4.19) is the interior gradient estimate of prospective solutions.

Using a suitable test function and the Ricci identities allows us to eliminate third derivatives and to obtain global estimates for $|\nabla u|$ in terms of the height and the boundary C^1 estimates.

We assume that $u \in C^3(\Omega)$ and satisfies $Q[u] = 0$ in Ω, with $u|_\Gamma = \phi$. Consider the test function

$$\chi = \frac{1}{u} e^{\frac{v}{A}},$$

where $v = |\nabla u| = (u^k u_k)^{1/2}$ and $A > 0$ is a constant to be chosen later (if $u = 0$ at some point, we replace u by $u + C_0$, where $C_0 > 0$ is a uniform constant that satisfies $C_0 > u$ in Ω). Let $x_0 \in \bar{\Omega}$ be a point where the function χ achieves its maximum. If $x_0 \in \Gamma$ we have a uniform bound for $|\nabla u|$ in Ω as desired. Hence, we may assume that $x_0 \in \Omega$ is an interior point. We may also assume that $v > 1$ at x_0 (otherwise we are done). We fix local coordinates around x_0 such that $\nabla u = u^1 \partial_1$. Since

$$\chi_i = \left(-\frac{u_i}{u^2} + \frac{u^k u_{k;i}}{Avu} \right) e^{v/A},$$

it follows from $\chi_i(x_0) = 0$ that

$$u_{1;1} = \frac{Av}{u} =: Kv \quad \text{and} \quad u_{1;i} = 0 \, (i > 1). \tag{4.28}$$

We may rotate the coordinates $x^2, \cdots, x^n$ in such a way that $(u_{i;j})$ is diagonal at x_0. We note that the matrix $\{\chi_{i;j}\}$ is negative-definite. We compute

$$\begin{aligned} e^{-v/A} \chi_{i;j} = &- \frac{u_{i;j}}{u^2} + \frac{u^k_{;i} u_{k;j}}{Auv} + \frac{u^k u_{k;ij}}{Auv} - \frac{u^k u^l u_{k:i} u_{k;j}}{Auv^3} + 2\frac{u_i u_j}{u^3} \\ &+ \frac{u^k u^l u_{k;i} u_{l;j}}{A^2 uv^2} - \frac{u^k u_{k;i} u_j}{Au^2 v} - \frac{u^k u_{k;j} u_i}{Au^2 v}. \end{aligned} \tag{4.29}$$

To proceed we differentiate the equation $Q[u] = 0$ in the direction of ∂_1. We have

$$-\frac{u^l_1}{W^2} F^{ij}_l u_{i;j} + \frac{\gamma_1}{W^2} F^{ij}_0 u_{i;j} + \frac{1}{W} F^{ij} u_{i;j1} = b_1.$$

Using (4.28), we get

$$-\frac{Kv}{W^2}F_1^{ii}u_{i;i}+\frac{\gamma_1}{W^2}F_0^{ii}u_{i;i}+\frac{1}{W}F^{ij}u_{i;j1}=b_1. \tag{4.30}$$

Now we use the Ricci equation

$$u_{i;j1}=u_{i;1j}+R_{ikj1}u^k,$$

and the inequality $F^{ij}\chi_{i;j}\leq 0$ to obtain the expression

$$\begin{aligned}\frac{1}{W}F^{ij}u_{i;j1}\leq&-\frac{1}{W}F^{ij}R_{ijk1}u^k+\frac{K}{W}F^{ij}u_{i;j}-\frac{1}{vW}F^{ij}u^k_{;i}u_{k;j}\\&+\frac{K^2v}{W}F^{11}-\frac{Kv^2}{uW}F^{11}.\end{aligned} \tag{4.31}$$

Replacing (5.13) into (4.30), we deduce

$$\begin{aligned}&\frac{Kv^2}{uW}F^{11}-\frac{K^2v}{W}F^{11}-\frac{1}{W^2}(\gamma_1F_0^{ii}-KvF_1^{ii})u_{i;i}\\&+\frac{v}{W}F^{ij}R_{i1j1}+\frac{1}{vW}F^{ii}u^2_{i;i}-Kb+b_1\leq 0.\end{aligned} \tag{4.32}$$

Applying the Cauchy inequality with epsilon we get

$$\frac{1}{W^2}|\gamma_1F_0^{ii}-KvF_1^{ii}||u_{i;i}|\leq\frac{1}{vW}F^{ii}u^2_{i;i}+\frac{v}{F^{ii}W^3}(\gamma_1F_0^{ii}-KvF_1^{ii})^2,$$

so

$$\begin{aligned}-\frac{1}{W^2}(\gamma_1F_0^{ii}-KvF_1^{ii})u_{i;i}\geq&-\frac{1}{W^2}|\gamma_1F_0^{ii}-KvF_1^{ii}||u_{i;i}|\\\geq&-\frac{1}{vW}F^{ii}u^2_{i;i}-\frac{v}{F^{ii}W^3}(\gamma_1F_0^{ii}-KvF_1^{ii})^2.\end{aligned}$$

Replacing this inequality into (4.32), we obtain

$$\begin{aligned}0\geq&\frac{Kv^2}{uW}F^{11}-\frac{K^2v}{W}F^{11}-\frac{v}{F^{ii}W^3}(\gamma_1F_0^{ii}-KvF_1^{ii})^2\\&+\frac{v}{W}F^{ij}R_{i1j1}-Kb+b_1.\end{aligned} \tag{4.33}$$

Now we will analyze the term

$$b_1=b_{x^1}+b_{u^l}u^l_1=b_{x^1}+b_{u^1}Kv.$$

By the definition of $b = b(x, \nabla u)$ we have

$$\begin{aligned} b =& \frac{1}{W} F_0^i \gamma_i + \frac{1}{2W} F_{00} \gamma^i u_i + n H_F, \\ b_{x^1} =& \frac{\gamma_{i;1}}{W} F_0^i + \frac{\gamma_i \gamma_1}{W^2} F_{00}^i + \frac{\gamma_1^2 v}{2W^2} F_{000} + \frac{\gamma_{;1}^1 v}{2W} F_{00} + n(H_F)_1, \\ b_{u^1} =& -\frac{\gamma_i}{W^2} F_{01}^i - \frac{\gamma^1 v}{2W^2} F_{001} + \frac{\gamma^1}{2W} F_{00}. \end{aligned}$$

Hence we conclude that there exist uniform constants

$$A_i = A_i(n, H_F, \gamma, \bar{\nabla}\gamma, \bar{\nabla} H_F, F, D^2F, D^3F, K)$$

that satisfy

$$|b| \leq A_1 \quad \text{and} \quad |b_1| \leq A_2.$$

Therefore we obtain from (4.33) an inequality of the form

$$C_1 v - C_2 \leq 0,$$

where $C_1 = \frac{K}{u} F^{11} > c_0 > 0$ is a positive constant and

$$C_2 = C_2(n, H_F, \gamma, \bar{\nabla}\gamma, \bar{\nabla} H_F, F, D^2F, D^3F, K, Ric_M).$$

This yelds the desired estimate and we have the following result.

Proposition 4.7. *Assume that $u \in C^3(\Omega) \cap C^1(\bar{\Omega})$ satisfies $Q[u] = 0$ in Ω and $u|_\Gamma = \phi$. If u is bounded in Ω and $|\nabla u|$ is bounded in Γ, then $|\nabla u|$ is bounded in Ω by a uniform constant that depends only on $n, |u|_0, \sup_\Gamma |\nabla u|$, $H_F, \gamma, \bar{\nabla}\gamma, \bar{\nabla} H_F, F, D^2F, D^3F, K$ and Ric_M.*

The usual elliptic regularity results guarantee that the above estimate is also true for a $C^{2,\alpha}$ function (see [17]).

Remark 4.8. *To obtain a priori estimates presented above we have to deal with the third derivatives of the function F. We will present the expression of this derivative for some particular cases. We believe this will help to understand the computations and arguments presented above.*

When $F(x, \eta) = |\eta|$ we have

$$\frac{\partial^3 F}{\partial \eta^\alpha \partial \eta^\beta \partial \eta^\theta} = \frac{3}{|\eta|^5} \eta_\alpha \eta_\beta \eta_\theta - \frac{1}{|\eta|^3} (\bar{g}_{\alpha\beta} \eta_\theta + \bar{g}_{\alpha\theta} \eta_\beta + \bar{g}_{\theta\beta} \eta_\alpha). \tag{4.34}$$

In particular, if η *is the normal to the graph of* u *we get*

$$F_{000} = 3\frac{\gamma^2 - \gamma|\nabla u|^2 - 1}{\gamma(\gamma + |\nabla u|^2)^{3/2}},$$
$$F_{0ij} = \frac{3u^i u^j}{W^3} - \frac{\sigma_{ij}}{W},$$
$$F_{00i} = -\frac{3\gamma^2 u_i}{W^3} + \frac{3}{\gamma W} u_i.$$

Another example is obtained setting

$$F = |\eta| f(\langle Y, \frac{\eta}{|\eta|}\rangle_{TM}),$$

where f *is a suitable positive real smooth function. In this case we have*

$$\begin{aligned}\frac{\partial^3 F}{\partial\eta^\alpha\partial\eta^\beta\partial^\theta} =& \frac{1}{|\eta|^5}(3f - 15f' - \Theta^3 f''')\eta_\alpha\eta_\beta\eta_\theta + \frac{\Theta}{|\eta|^2} f''' a_\alpha a_\beta a_\theta \\ &- \frac{\Theta}{|\eta|^3} f''' \sum_\sigma a_\sigma a_\sigma \eta_\sigma + \frac{1}{|\eta|^5}(4\Theta f' - 2\Theta^2 f'') \sum_\sigma \eta_\sigma\eta_\sigma\eta_\sigma \\ &+ \frac{1}{|\eta|^4}(\Theta^2 f''' + 3\Theta f'') \sum_\sigma a_\sigma \eta_\sigma \eta_\sigma \\ &+ \frac{1}{|\eta|^3}(\Theta f' - f - \Theta^2 |\eta|^2 f'') \sum_\sigma \bar{g}_{\sigma\sigma}\eta_\sigma,\end{aligned}$$

where $\Theta = \langle Y, \frac{\eta}{|\eta|}\rangle$ *and* $Y = a^\alpha \partial_\alpha$.

4.5 The Existence of Solutions

The existence of solutions is obtained by way of the well-known continuity method to the family of Dirichlet problems

$$\mathcal{Q}_\sigma[u] = 0, \quad u|_\Gamma = \sigma\phi,$$

where $\sigma \in [0, 1]$ and

$$\mathcal{Q}_\sigma[u] = a^{ij} u_{i;j} - b_\sigma,$$

where

$$b_\sigma = \frac{1}{W} F_0^i \gamma_i + \frac{1}{2W} F_{00} \gamma^i u_i + n\sigma H_F.$$

The subset I of $[0, 1]$ consisting of values of σ for which the above Dirichlet problems has a $C^{2,\alpha}$ solution is nonempty, since $0 \in I$. The openness of I is a direct consequence of a standard application of the implicit function theorem, since the derivative of $\mathcal{Q}$ is a linear homeomorphism. The closedness of I follows from the a priori estimates we had proved and from the linear elliptic PDE theory. Thus, the continuity method assures that $1 \in I$. This establishs the proof of Theorem 4.1.

Chapter 5

Hyperspheres with Prescribed Anisotropic Curvature

In this chapter we study the existence of hyperspheres in the Euclidean space with prescribed anisotropic mean curvature, extending a result of Treibergs and Wei [41].

5.1 Preliminaries

As we observed in the last chapter, the notion of anisotropic mean curvature arises naturally in the study of variational problems as a generalization of the usual mean curvature. In the Euclidean space this curvature has a natural geometric interpretation also. Our line of explanation will follow that one presented in [46], [35] and [25].

We consider parametric functional of the form

$$\mathcal{F}(X) = \int_M F(N)\,\mathrm{d}M,$$

where the integrand $F \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ is a positive Lagrangian satisfying the homogeneity condition

$$F(tz) = tF(z), \quad \text{for all } z \in \mathbb{S}^n,\ t > 0.$$

Here, $X : M \longmapsto \mathbb{R}^{n+1}$ is an immersed closed and oriented hypersurface with Gauss mapping N and induced volume element $\mathrm{d}M$. Moreover, F is always assumed to be elliptic, i.e.,

$$D^2F(z) = \Big(\frac{\partial^2 F}{\partial z^i \partial z^j}(z)\Big)_{i,j=1,\dots,n+1} : z^\perp \longmapsto z^\perp \tag{5.1}$$

is a positive definite endomorphism for all $z \in \mathbb{S}^n$, or equivalently

$$\lambda = \lambda(F) = \inf_{z\in\mathbb{S}^n,\, v\in z^\perp,\, |v|=1} \langle D^2F(z)\cdot v, v\rangle > 0.$$

Clearly, $\mathcal{F}$ generalizes the area functional

$$\mathcal{A}(X) = \int_M \mathrm{d}M,$$

which is obtained when $F(z) = |z|$ is the *area integrand.* Geometrically, the ellipticity condition (5.1) implies that F is the support function of some convex body

$$\bigcap_{z\in\mathbb{S}^n} \{y \in \mathbb{R}^{n+1} \ : \ \langle y, z\rangle \le F(z)\},$$

the boundary $\mathcal{W}_F$ of which is the convex hypersurfaces parametrized by

$$\Phi : \mathbb{S}^n \longmapsto \mathcal{W}_F, \quad \Phi(z) = DF(z).$$

In the terminology of Taylor [40], $\mathcal{W}_F = \Phi(\mathbb{S}^n)$ is called the *Wulff shape.*

Let us now consider an arbitrary variation X_ε of $X = X_0$ with variation vector field $Y = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}(X_\varepsilon)|_{\varepsilon=0}$. Decomposing $Y = \varphi N +$ tangential terms, it is well known (see [35], [46] and [14]) that the first variation of $\mathcal{F}$ is given by

$$\delta\mathcal{F}(X,Y) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}(X_\varepsilon)|_{\varepsilon=0} = -\int_M H_F\varphi \mathrm{d}M,$$

where H_F is the anisotropic mean curvature of X which is defined as follows. Let

$$N_F : M \longmapsto \mathcal{W}_F, \quad N_F = \Phi\circ N,$$

denotes the generalized Gauss mapping into the Wulff shape. The operator $S_F = -\mathrm{d}X^{-1}\circ \mathrm{d}N_F$ is named the anisotropic Weingarten operator. We note that

$$S_F = A_F \circ A,$$

where $A = -\mathrm{d}X^{-1} \circ \mathrm{d}N$ is the classical Weingarten operator of X and A_F is the symmetric positive definite $(1,1)$-tensor given by

$$A_F = \mathrm{d}X^{-1} \circ \mathrm{d}\Phi \circ \mathrm{d}X = -\mathrm{d}X^{-1} \circ D^2F(N) \circ \mathrm{d}X.$$

Finally, the anisotropic mean curvature of X is defined by

$$H_F = \mathrm{tr}(S_F).$$

For instance, the anisotropic mean curvature of the sphere $\mathbb{S}^n(r)$ of radius r is

$$H_F = \Delta F(-z), \qquad z \in \mathbb{S}^n(r). \tag{5.2}$$

In fact, the unit normal vector of $\mathbb{S}^n(r)$ at a point z is $N = -\frac{1}{r}z$ and its Weingarten operator is $A = \frac{1}{r}I$. Hence,

$$H_F(z) = \frac{1}{r}\mathrm{tr}\Big(D^2F\Big(-\frac{1}{r}z\Big)\Big) = \Delta F(-z),$$

since D^2F is homogeneous of degree -1 and $D^2F|_N(N,N) = 0$.

Although the anisotropic Weingarten operator is not necessarily symmetric, it has n real eigenvalues (see e.g. [25]). In fact, to see this we define the abstract metric

$$g_F(v,w) = \langle A_F^{-1}v, w\rangle, \qquad v, w \in TM.$$

Note that the operator A_F is positive definite, hence it is invertible and its inverse is also positive. We have

$$g_F(S_Fv, w) = \langle A_F^{-1}(A_FA)v, w\rangle = \langle Av, w\rangle = \langle v, Aw\rangle = g_F(v, S_Fw)$$

for all $v, w \in TM$, which gives that S_F is symmetric with respect to this inner product. Thus there exists an orthonormal basis (with respect to the metric g_F) that diagonalize S_F. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of S_F are called the anisotropic principal curvatures of X. Obviously, H_F is the sum of these curvatures. We point out that these definitions coincide with their classical counterparts in case $F(z) = |z|$ is the area-integrand.

Here we are interested on the existence of closed hypersurfaces with prescribed anisotropic mean curvature. Treibergs and Wei have studied this problem for the classical mean curvature in [41]. More precisely, they considered the following problem raised by Yau: is there an embedding $Y : \mathbb{S}^n \longmapsto \mathbb{R}^{n+1}$ of the n-dimensional sphere into Euclidean $(n+1)$-space, whose mean curvature is a preassigned sufficiently smooth function H defined on $\mathbb{R}^{n+1}$? A theorem of Bakelman and Kantor in [4] together with the results obtained in [41] asserts the existence of such hypersurfaces assuming only the natural condition that H decay faster than the mean curvature of concentric spheres. Specifically, they proved that, if H is a C^1 positive function defined on the closure of the annular region $U = \{z \in \mathbb{R}^{n+1} : r_1 < |z| < r_2\}$, where $0 < r_1 \leq 1 \leq r_2$, and satisfies

$$\frac{\partial}{\partial \rho}\rho H(\rho z) \leq 0, \quad \text{for all } \rho z \in U \tag{5.3}$$

and

$$\begin{aligned} H(z) &> |z|^{-1}, \quad \text{for } |z| = r_1, \\ H(z) &< |z|^{-1}, \quad \text{for } |z| = r_2, \end{aligned} \tag{5.4}$$

then, for some $0 < \alpha < 1$, there exists an embedded hypersphere $Y \in C^{2,\alpha}(\mathbb{S}^n)$ with mean curvature H which is also a graph over the unit sphere and also satisfies $r_1 \leq |Y| \leq r_2$.

We obtain an extension of this result for the anisotropic mean curvature under similar hypothesis. Our result is:

Theorem 5.1. *Suppose the function $H \in C^1(\bar{U})$ satisfies condition (5.3) in the annular region U defined above and*

$$\begin{aligned} H(z) &> \Delta F(-z), \quad \text{for } |z| \leq r_1, \\ H(z) &< \Delta F(-z), \quad \text{for } |z| \geq r_2. \end{aligned} \tag{5.5}$$

Then there exists a function $u \in C^2(\mathbb{S}^n)$ whose radial graph is contained in U and has prescribed anisotropic curvature $H_F = H$. Moreover, if there is a second function $v \in C^2(\mathbb{S}^n)$ that also satisfies the above conditions, then

$$v = (1 + t_0)u$$

for some $t_0 > -1$, and all intermediate homotheties $v_t = (1+t)u$ has anisotropic mean curvature H.

To prove this theorem we use again the PDE elliptic theory and the continuity method.

5.2 The Anisotropic Mean Curvature

In this section we will derive a suitable expression for the anisotropic mean curvature of a radial graph. First we calculate the second fundamental form of the graph using moving frames. In this chapter we adopt the convention that lower case indices $i, j, k, \dots$ are summed from 1 to n and $a, b, c, \dots$ from 1 to $n+1$.

Let $\{e_1, \cdots, e_{n+1}\}$ be a local orthonormal frame field defined in $\mathbb{R}^{n+1}$ such that e_{n+1} is the outward radial direction. Let $\{\theta^a\}$ denote the dual coframe field. The connection forms are defined as the skew symmetric matrix $\{\theta_a^b\}$ such that

$$\mathrm{d}\theta^a = \theta^b \wedge \theta_b^a.$$

The covariant differentiation on $\mathbb{R}^{n+1}$ is given by

$$\mathrm{d}e_a = \theta_a^b e_b.$$

For a hypersphere $\mathbb{S}^n(r)$ of constant radius r, the position vector is $X = re_{n+1}$. Hence $\{e_i\}$ provide an orthonormal frame on X and we have $\mathrm{d}X = \theta^i e_i$, which implies that

$$\theta^i = r\theta_{n+1}^i. \tag{5.6}$$

Let u be a smooth function defined on the sphere $\mathbb{S}^n$. We denote by ∇ the connection of $\mathbb{S}^n$. The graph Y is conveniently represented by $Y = e^u e_{n+1}$. If u is extended to $\mathbb{R}^{n+1} \setminus \{0\}$ as a constant along radii, the gradient and the Hessian of u, given by

$$\mathrm{d}u = u_i\theta^i, \qquad u_{ab}\theta^b = \mathrm{d}u_a - u_b\theta_a^b,$$

are homogeneous of degrees -1 and -2 respectively, since u is homogeneous of degree 0. Using (5.6) we get

$$u_{n+1i} = u_{n+1b}\theta^b(e_i) = du_{n+1}(e_i) - u_b\theta_{n+1}^b(e_i) = -e^{-u}u_j\theta^j(e_i) = -e^{-u}u_i.$$

Hence, restricting to Y we get the following Hessian formula

$$u_{ij}\theta^j = du_i - u_j\theta_i^j + e^{-u}u_i\theta^{n+1}.$$

The vector fields $E_i = e_i + e^u u_i e_{n+1}$ form a basis to the tangent space at Y. In terms of this basis, the induced metric of Y has components

$$g_{ij} = \langle E_i, E_j \rangle = \delta_{ij} + e^{2u}u_i u_j.$$

Hence its inverse matrix is given by $g^{ij} = \delta_{ij} - f^2 e^{2u} u_i u_j$, where

$$f = (1 + e^{2u}|\nabla u|^2)^{-1/2}.$$

The unit normal vector to Y is

$$N = f(e^u u_i e_i - e_{n+1}).$$

Therefore,

$$\begin{aligned} -\mathrm{d}N =& \mathrm{d}f(e_{n+1} - e^u u_i e_i) + f(\mathrm{d}e_{n+1} - e^u \mathrm{d}u u_i e_i - e^u \mathrm{d}u_i e_i - e^u u_i de_i) \\ =& \mathrm{d}(\log f)N + f\big(\theta^i_{n+1} e_i - e^u u_j \theta^j u_i e_i - e^u e_i (u_{ij}\theta^j + u_j \theta^j_i - e^{-u} u_i \theta^{n+1})\big) \\ &- f e^u u_i \theta^a_i e_a \\ =& \mathrm{d}(\log f)N + f(e^{-u}\delta_{ij} - e^u u_i u_j - e^u u_{ij})\theta^j e_i + f u_i \theta^{n+1} e_i + f u_i \theta^i e_{n+1}. \end{aligned}$$

Hence the components of the second fundamental form b of Y are

$$b_{ij} = -\langle dN(E_i), E_j\rangle = f e^{-u}(\delta_{ij} + e^{2u} u_i u_j - e^{2u} u_{ij}).$$

By the homogeneity of the derivatives of u, we can equate their values on Y and $\mathbb{S}^n$. Pulling back, we conclude that on $\mathbb{S}^n$

$$b_{ij} = (1 + |\nabla u|^2)^{-1/2} e^{-u}(\delta_{ij} + u_i u_j - u_{ij}). \tag{5.7}$$

On the other hand, the components of the bilinear form $\mathcal{A}_F$ metrically equivalent to the operator A_F are

$$\begin{aligned} (\mathcal{A}_F)_{ij} =& \mathcal{A}_F(E_i, E_j) = \langle A_F(E_i), E_j\rangle \\ =& F_{ab} E^a_i E^b_j \\ =& e^{2u} F_{n+1n+1} u_i u_j + e^u F_{n+1i} u_i + e^u F_{n+1j} u_j + F_{ij}, \end{aligned}$$

where F_{ab} denote the components of the Hessian of F in terms of the frame field $\{e_a\}$. Note that the above derivatives of F are calculated in N. In terms of matrices,

$$S_F = A_F A = (g^{-1}\mathcal{A}_F) g^{-1} b = (g^{-1}\mathcal{A}_F g^{-1}) b.$$

On the other hand, decomposing the Hessian matrix of F as

$$D^2 F = \begin{pmatrix} \hat{F} & F_{in+1} \\ F_{in+1} & F_{n+1n+1} \end{pmatrix},$$

we get from the Euler relation $F_{ab}(z)z^b = 0$ that

$$\begin{aligned}(g\hat{F}g)_{ij} =& g_{ik}F_{kl}g_{jl}\\ =&(\delta_{ik} + e^{2u}u_iu_k)(F_{kl})(\delta_{jl} + e^{2u}u_ju_l)\\ =&F_{ij} + e^{2u}u_ju_kF_{ik} + e^{2u}u_iu_kF_{kj} + e^{4u}u_iu_ju_ku_lF_{kl}\\ =&F_{ij} + e^{u}u_iF_{n+1j} + e^{u}u_iF_{n+1i} + e^{2u}F_{n+1n+1}u_iu_j\\ =&(\mathcal{A}_F)_{ij}.\end{aligned}$$

Then, in terms of matrices,

$$S_F = \hat{F}b.$$

We denote $S_F(E_i) = \sum_j s_{ij}E_j$. So

$$s_{ij} = \sum_k F_{ik}(N)b_{kj}, \tag{5.8}$$

which implies that

$$H_F = \sum_{i,j} F_{ij}(N)b_{ij}. \tag{5.9}$$

Hence the anisotropic mean curvature of the graph of u is given by

$$e^uWH_F = F_{ij}(N)(\delta_{ij} + u_iu_j - u_{ij}). \tag{5.10}$$

Thus, the radial graph of a function u has prescribed anisotropic mean curvature H if and only if u is a solution of the quasilinear elliptic equation

$$Q[x, u, u_i, u_{ij}] - H = 0,$$

where

$$Q[x, u, u_i, u_{ij}] = e^{-u}W^{-1}F_{ij}(N)(\delta_{ij} + u_iu_j - u_{ij}).$$

The second fundamental form of a Euclidean graph $(x, v(x)) \in \mathbb{R}^{n+1}$, of a smooth function v defined in a domain $\Omega \subset \mathbb{R}^n$, has components

$$b_{ij} = -\frac{v_{ij}}{\sqrt{1 + |Dv|^2}}.$$

Hence, as it was done above, we conclude that the anisotropic mean curvature of the graph of v is

$$WH_F = -F_{ij}(N)v_{ij},$$

where $W = \sqrt{1 + |Dv|^2}$.

We finalize this section with a maximum principle for graphs with prescribed anisotropic mean curvature.

Proposition 5.2. *Suppose the radial graph Y has prescribed anisotropic mean curvature H and the function $H \in C^1(\mathbb{R}^{n+1} \setminus \{0\})$ satisfies the conditions (5.3) and (5.5). Then $r_1 < |Y| < r_2$.*

Proof. Let u be the function whose radial graph is Σ. By contradiction assume that $R = \sup e^u = e^u(x_0) \geq r_2$. Let $\mathcal{S}$ be the sphere of radius R centered at the origin. Observe that Σ and $\mathcal{S}$ are tangent at the point $Y(x_0) = e^{u(x_0)}x_0$. Furthermore, with respect to the inwards normal vector common to both hypersurfaces at this point, Σ lies above $\mathcal{S}$. Then the principal curvatures κ_i of Σ at this point are greater than or equal to $\frac{1}{R}$. Since the unit normal of Σ at $Y(x_0)$ is

$$N = \frac{1}{\sqrt{1+e^{2u}|\nabla u|^2}}(\nabla u - Y) = -\frac{1}{R}Y,$$

we conclude that

$$\begin{aligned} H =& H_F = \operatorname{tr}\Big(S_F\Big) = \sum_i \langle A_F A(e_i), e_i\rangle \\ =& \sum_i \kappa_i \langle A_F(e_i), e_i\rangle \geq \frac{1}{R}\sum_i \langle A_F(e_i), e_i\rangle \\ =& \frac{1}{R}\Delta F\Big(-\frac{1}{R}Y(x_0)\Big) = \Delta F\big(-Y(x_0)\big), \end{aligned}$$

where $\{e_i\}$ is an orthonormal basis of $(T_{x_0}\Sigma, \langle\cdot,\cdot\rangle)$ formed by eigenvectors of A. But the above inequality contradicts (5.5). Hence $u \leq r_2$. Proceeding in a similar way with the minimum of u we conclude that $u \geq r_1$. □

5.3 The Gradient Estimates

In this section, we prove a priori global estimate for gradient of prospective solutions of equation (5.10). To prove this estimate we follow the technique presented in [9].

Let $u \in C^3(\mathbb{S}^n)$ be a solution of the anisotropic mean curvature equation $H_F = H$. To estimate $|\nabla u|$ we will obtain a uniform positive constant $a = a(n, H, F, \sup|u|)$ that satisfies

$$\langle Y, N\rangle^2 \geq a > 0, \tag{5.11}$$

where N denotes the unit normal vector along $\Sigma = \mathrm{graph}|_u$ and $Y(x) = e^{u(x)}x$, is the position vector. This inequality implies the estimate of the gradient of u. In fact, since

$$N(Y(x)) = \frac{1}{e^u\sqrt{1+|\nabla u|^2}}\big(\nabla u - e^u x\big), \quad x \in \mathbb{S}^n,$$

we have

$$\langle N, Y\rangle^2 = \frac{e^{2u}}{1+|\nabla u|^2},$$

which implies

$$\langle N, Y\rangle^2 \geq a \quad \Leftrightarrow \quad |\nabla u|^2 \leq \frac{e^{2u}}{a} - 1.$$

The estimate (5.11) will be obtained by estimating the maximum of the function φ defined on $\mathbb{S}^n$ by

$$\varphi(x) = \frac{1}{|Y|^2}\exp\left(\frac{1}{A\langle Y, N\rangle^2}\right) = g\exp(f),$$

where A is a positive constant to be chosen later. Clearly, an upper bound for φ implies the estimate (5.11). We may assume (unless a rotation in the $\mathbb{R}^{n+1}$) that φ achieves its maximum at the north pole $q = (0, \cdots, 0, 1) \in \mathbb{S}^n$. In a small neighborhood of $Y(q)$ in Σ we may then use a local Cartesian representation for Σ, i.e., there exists a function $v \in C^3(U)$, such that $Y = (z, v(z)) \in \mathbb{R}^{n+1}$, $z \in U$, where $U \subset \mathbb{R}^n \times \{0\} \equiv \mathbb{R}^n \subset \mathbb{R}^{n+1}$ contains the origin and $(0, v(0)) = Y(q)$. In terms of v, the unit normal vector and the second fundamental form of Σ are given by

$$N = \left(\frac{Dv}{W}, -\frac{1}{W}\right), \quad b_{ij} = -\frac{v_{ij}}{W},$$

where $W^2 = 1 + |Dv|^2$. Near q we may write φ as

$$\varphi(z) = \frac{1}{|z|^2 + v^2}\exp\left(\frac{1+|Dv|^2}{A(z^k v_k - v)^2}\right) = g\exp(f), \qquad z \in U.$$

In particular,

$$\varphi(0) = \frac{1}{v^2}\exp\left(\frac{1+|Dv|^2}{Av^2}\right).$$

Hence, the maximum value of φ, which is $\varphi(0)$, is controlled by $|Dv(0)|$. Therefore, it is sufficient to obtain a uniform constant $C = C(n, H, F, \sup|u|)$ that satisfies $|Dv(0)| \le C$.

We may assume that $|Dv(0)| > 1$, otherwise we are done. After a rotation of the coordinates of $\mathbb{R}^n \subset \mathbb{R}^{n+1}$, if necessary, we have

$$Dv(0) = (v_1, 0, \ldots, 0) \in \mathbb{R}^n.$$

Since $z = 0$ is a maximum point of φ, we have $D\varphi(0) = 0$ and also $\big(\varphi_{ij}(0)\big)$ is a negative definite matrix.

We compute

$$D\varphi = e^f(Dg + gDf),$$

so $D\varphi(0) = 0$ implies

$$Dg(0) = -g(0)Df(0) \quad \Rightarrow \quad g_i(0) = -gf_i(0), \quad i = 1, \ldots, n.$$

It follows that the expression

$$\varphi_{ij}(0) = e^f(g_{ij} + g_i f_j + g_j f_i + g f_i f_j + f_{ij})(0)$$

takes the form

$$\varphi_{ij}(0) = (g_{ij} + g f_{ij} - g f_i f_j)e^f(0). \tag{5.12}$$

Now we compute the derivatives of the functions

$$g(z) = \frac{1}{|z|^2 + v^2} \quad \text{and} \quad f(z) = \frac{1 + |\nabla v|^2}{A(z^k v_k - v)^2}, \quad z \in U.$$

We have

$$\begin{aligned} g_i(z) =& -2\frac{z^i + vv_i}{(|z|^2 + v^2)^2}, \\ g_{ij}(z) =& 8\frac{(z^i + vv_i)(z^j + vv_j)}{(|z|^2 + v^2)^3} - 2\frac{\delta_{ij} + v_i v_j + vv_{ij}}{(|z|^2 + v^2)^2} \end{aligned}$$

and

$$\begin{aligned} f_i(z) =& \frac{2}{A}\left\{\frac{v^k v_{ki}}{(z^k v_k - v)^2} + \frac{z^k v_{ki}(1 + |\nabla v|^2)}{(z^k v_k - v)^3}\right\}, \\ f_{ij}(z) =& \frac{2}{A}\frac{v_i^k v_{kj} + v^k v_{kij}}{(z^k v_k - v)^2} + \frac{8}{A}\frac{v^k z^l v_{ki} v_{lj}}{(x^k v_k - v)^3} + \frac{2}{A}\frac{v_{ij} + z^k v_{kij}(1 + |\nabla v|^2)}{(z^k v_k - v)^3} \\ &+ \frac{6}{A}\frac{z^k z^l v_{ki} v_{lj}(1 + |\nabla v|^2)}{(z^k v_k - v)^4}. \end{aligned}$$

In particular, at the origin we have

$$g_i = -\frac{2}{v^3}v_i, \quad g_{ij} = \frac{8}{v^4}v_i v_j - \frac{2}{v^4}(\delta_{ij} + v_i v_j + v v_{ij}) \tag{5.13}$$

and

$$f_i = \frac{2}{Av^2}v^k v_{ki}, \quad f_{ij} = \frac{2}{Av^2}(v_i^k v_{kj} + v^k v_{kij}) + \frac{2}{Av^3}W^2 v_{ij}. \tag{5.14}$$

As we showed above, the anisotropic mean curvature of a Euclidean graph is given by

$$WH_F = -F^{ij}(N)v_{ij},$$

where, for sake of convenience, we use the notation $F^{ij} = \frac{\partial^2 F}{\partial z^i \partial z^j}$, with $(z^1, \ldots, z^n)$ being the Cartesian coordinates of $\mathbb{R}^n \subset \mathbb{R}^{n+1}$. We derive the equation $H_F = H$ with respect to z^k to obtain

$$\frac{v^l v_{lk}}{W^3}F^{ij}v_{ij} - F_\alpha^{ij}N_k^\alpha v_{ij} - F^{ij}v_{ijk} = H_k + H_{n+1}v_k. \tag{5.15}$$

Since

$$N_k^l = \frac{v_k^l}{W} - \frac{v^l v^p v_{pk}}{W^3}, \qquad N_k^{n+1} = -\frac{v^l v_{lk}}{W^3},$$

for $1 \le l \le n$, we have

$$\begin{aligned} H_k + H_{n+1}v_k = & \frac{v^l v_{lk}}{W^3}F^{ij}v_{ij} - \frac{1}{W^2}F_l^{ij}v_k^l v_{ij} \\ & + \frac{v^p v_{pk}}{W^3}\left(\frac{v^l}{W}F_l^{ij} - \frac{1}{W}F_{n+1}^{ij}\right)v_{ij} - \frac{1}{W}F^{ij}v_{ij}. \end{aligned}$$

Applying the Euler relation

$$F_\alpha^{ij}(X)X^\alpha = -F^{ij}(X), \quad \alpha = 1, \ldots, n+1, \tag{5.16}$$

we get

$$\frac{v^l}{W}F_l^{ij} - \frac{1}{W}F_{n+1}^{ij} = -F^{ij}.$$

Replacing this into the above equation,

$$-\frac{1}{W^2}F_l^{ij}v_k^l v_{ij} - \frac{1}{W}F^{ij}v_{ijk} = H_k + H_{n+1}v_k.$$

As $g_i = -gf_i$ at the origin, it follows from (5.13) and (5.14) that $v^k v_{ki} = Avv_i$. In particular,

$$v_{11} = Av \quad \text{and} \quad v_{1i} = 0, \ (i > 1).$$

Thus, contracting equation (5.15) with v^k, we obtain (at the origin)

$$-\frac{v_1}{W^2} F_1^{ij} v_{11} v_{ij} - \frac{v_1}{W} F^{ij} v_{ij1} = H_1 v_1 + H_{n+1} v_1^2. \tag{5.17}$$

We use the Euler relation (5.16) again to get

$$-\frac{v_1}{W} F_1^{ij} = -\frac{1}{W} F_{n+1}^{ij} + F^{ij}.$$

Hence, equation (5.17) becomes

$$\frac{v_{11}}{W} F^{ij} v_{ij} - \frac{v_{11}}{W^2} F_{n+1}^{ij} v_{ij} - \frac{v_1}{W} F^{ij} v_{ij1} = H_1 v_1 + H_{n+1} v_1^2. \tag{5.18}$$

Using again that $WH_F = -F^{ij} v_{ij} = WH$,

$$-\frac{v_{11}}{W^2} F_{n+1}^{ij} v_{ij} - \frac{v_1}{W} F^{ij} v_{ij1} = Hv_{11} + H_1 v_1 + H_{n+1} v_1^2. \tag{5.19}$$

Now we will eliminate from equation (5.19) the first and the second derivatives of v. To proceed, we note that $F^{ij}\varphi_{ij} \leq 0$, since the matrix $\big(F^{ij}\big)$ is positive definite and $\big(\varphi_{ij}\big)$ is negative. Thus, it follows from (5.12) that

$$F^{ij} g_{ij} + gF^{ij} f_{ij} - gF^{ij} f_i f_j \leq 0.$$

Using (5.13) and (5.14) the above inequality becomes

$$\begin{aligned} 0 \geq & \frac{8}{v^4} F^{ij} v_i v_j - \frac{2}{v^4} F^{ij} (\delta_{ij} + v_i v_j + vv_{ij}) - \frac{4}{A^2 v^6} F^{ij} v^k v^l v_{ki} v_{lj} \\ & + \frac{2}{Av^4} F^{ij} (v_i^k v_{kj} + v^k v_{kij}) + \frac{2W^2}{Av^5} F^{ij} v_{ij}. \end{aligned}$$

Dividing this inequality by $\frac{v^4}{2}$ we get

$$\begin{aligned} -v^k F^{ij} v_{kij} \geq & 4AF^{ij} v_i v_j - AF^{ij} (\delta_{ij} + v_i v_j + vv_{ij}) - \frac{2}{Av^2} F^{ij} v^k v^l v_{ki} v_{lj} \\ & + F^{ij} v_i^k v_{kj} + \frac{W^2}{v} F^{ij} v_{ij}. \end{aligned}$$

Since $WH_F = -F^{ij}v_{ij} = WH$ and $v_i = 0$, $(i > 1)$, $v_{11} = Av$, we have

$$-v_1 F^{ij} v_{1ij} \geq AF^{11}v_1^2 + F^{ij}v_i^k v_{kj} - AF^{ij}\delta_{ij} + AWvH - \frac{W^3}{v}H. \tag{5.20}$$

After rotation of the coordinates $(z^2, \dots, z^n)$ we may assume that $\big(v_{ij}(0)\big)$ is diagonal. Hence,

$$-\frac{v_1}{W}F^{ij}v_{1ij} \geq \frac{A}{W}F^{11}v_1^2 + \frac{1}{W}F^{ii}v_{ii}^2 - \frac{A}{W}F^{ij}\delta_{ij} + AvH - \frac{W^2}{v}H. \tag{5.21}$$

Since $v_{1ij} = v_{ij1}$, we may apply inequality (5.21) to obtain from (5.19) that

$$\begin{aligned} Hv_{11} + H_1 v_1 + H_{n+1}v_1^2 \geq &-\frac{v_{11}}{W^2}F^{ii}_{n+1}v_{ii} + \frac{A}{W}F^{11}v_1^2 + \frac{1}{W}F^{ii}v_{ii}^2 \\ &-\frac{A}{W}F^{ij}\delta_{ij} + AvH - \frac{W^2}{v}H. \end{aligned} \tag{5.22}$$

Note that we eliminate the third derivatives of v on the last equation. To do the same with the second derivatives we first note that $F^{ii} > 0$, for any $i = 1, \cdots n$. In fact,

$$F^{ii} = \text{Hess}(F)_{|N}(e_i, e_i) = \text{Hess}(F)_{|N}(e_i^T, e_i^T) \geq \lambda |e_i^T|^2 > 0,$$

since the tangent component e_i^T of the vector e_i do not vanish whereas N is not multiple of e_i. Thus we may apply the Cauchy inequality with epsilon,

$$ab \leq \varepsilon a^2 + \frac{1}{\varepsilon}b^2,$$

with $a = |v_{ii}|$, $b = |F^{ii}_{n+1}|$ and $\varepsilon = \frac{WF^{ii}}{v_{11}} > 0$, for each $1 \leq i \leq n$ fixed. Then

$$v_{11}|F^{ii}_{n+1}v_{ii}| \leq WF^{ii}v_{ii}^2 + v_{11}^2\frac{(F^{ii}_{n+1})^2}{WF^{ii}} \leq WF^{ii}v_{ii}^2 + A^2v^2\frac{(F^{ii}_{n+1})^2}{W\lambda}.$$

Adding on i we get

$$v_{11}|F^{ii}_{n+1}v_{ii}| \leq WF^{ii}v_{ii}^2 + \frac{A^2B}{W},$$

where

$$B = v(0)^2 \sup_{\mathbb{S}^n} \frac{(F^{ii}_{n+1})^2}{\lambda} > 0.$$

Hence,

$$\frac{v_{11}}{W^2}F^{ii}_{n+1}v_{ii} \geq -\frac{v_{11}}{W}|F^{ii}_{n+1}v_{ii}| \geq -\frac{1}{W}F^{ii}v_{ii}^2 - \frac{A^2B}{W^3} \geq -\frac{1}{W}F^{ii}v_{ii}^2 - A^2B.$$

Replacing the last inequality into (5.22) we obtain

$$\begin{aligned} Hv_{11} + H_1v_1 + H_{n+1}v_1^2 \geq &- A^2B + \frac{A}{W}F^{11}v_1^2 - \frac{A}{W}F^{ij}\delta_{ij} \\ &+ AvH - \frac{W^2}{v}H. \end{aligned} \tag{5.23}$$

As we have $v_{11} = Av$ e $W^2 = 1 + v_1^2$ (at the origin), the above equation may be rewritten as

$$H_1v_1 + v_1^2(H_{n+1} + \frac{H}{v}) + \frac{H}{v} \geq -A^2B + \frac{A}{W}F^{11}v_1^2 - \frac{A}{W}F^{ij}\delta_{ij}.$$

It follows from hypothesis (5.3) that

$$H_{n+1} + \frac{H}{v} \leq 0.$$

In fact,

$$0 \geq \frac{\partial}{\partial\rho}\left(\rho H(\rho(0, v(0))\right)_{|_{\rho=1}} = H(0, v(0)) + v(0)H_{n+1}(0, v(0)).$$

Hence, we conclude from (5.23) that

$$H_1v_1 + \frac{H}{v} \geq -A^2B + \frac{A}{W}F^{11}v_1^2 - \frac{A}{W}F^{ij}\delta_{ij}. \tag{5.24}$$

Since $v_1 > 1$ we have $\frac{v_1^2}{W} \geq \frac{v_1}{\sqrt{2}}$, so

$$\frac{v_1^2}{W}F^{11} \geq \frac{v_1}{\sqrt{2}}F^{11} \geq \frac{v_1}{\sqrt{2}}\lambda.$$

Therefore,

$$H_1v_1 + \frac{H}{v} \geq -A^2B + A\frac{v_1}{\sqrt{2}}\lambda - \frac{A}{W}F^{ij}\delta_{ij}. \tag{5.25}$$

Since

$$\frac{1}{W}F^{ij}\delta_{ij} \leq n\Lambda,$$

where Λ is the largest eigenvalue of D^2F, then it follows from (5.25) that

$$v_1\Big(\frac{A\lambda}{\sqrt{2}} - H_1\Big) \leq \frac{H}{v} + A^2B + nA\Lambda.$$

Thus, if we choose the constant $A > 0$ large such that $A > \frac{\sqrt{2}}{\lambda}\sup|DH|$, we obtain

$$v_1 \leq \frac{H/v + A^2B + nA\Lambda}{\frac{A\lambda}{\sqrt{2}} - H_1}.$$

So, denoting

$$\bar{C} = \frac{\frac{H}{v}(0) + A^2B + nA\Lambda}{\frac{A\lambda}{\sqrt{2}} - H_1(0)},$$

we obtain $|Dv(0)| \leq \bar{C}$, with $\bar{C} = \bar{C}(n, H, F, \sup|u|)$, which proves the following theorem.

Theorem 5.3. *Under the conditions of Theorem 5.1, if $u \in C^3(\mathbb{S}^n)$ is a solution of the prescribed anisotropic mean curvature equation $H_F = H$, then there exists a uniform constant $C = C(n, H, F, \sup|u|)$ such that*

$$|\nabla u| \leq C.$$

5.4 Proof of the Theorem

To prove Theorem 5.1 we use the degree theory for nonlinear elliptic partial differential equations developed by Yan Yan Li. We refer the reader to [29] for more details.

We consider for each t, $0 \leq t \leq 1$, the map

$$H_t(z) = tH(z) + (1-t)\phi(|z|)\Delta F(-z), \quad z \in U, \tag{5.26}$$

where ϕ is a positive real function defined in $\mathbb{R}_+$ which satisfies the following conditions

$$\begin{aligned} \phi(t) > 1 \quad &\text{for} \quad t \leq r_1, \\ \phi(t) < 1 \quad &\text{for} \quad t \geq r_2 \end{aligned} \tag{5.27}$$

and $\phi' < 0$. Note that these conditions imply the existence of a unique point $r_0 \in (r_1, r_2)$ such that $\phi(r_0) = 1$. We point out that, with this choice of the function ϕ, H_t also satisfies the conditions in Theorem 5.1. In fact, it follows from (5.5) that

$$\begin{aligned} H_t(z) =& tH(z) + (1-t)\phi(|z|)\Delta F(-z) \\ >& (t + (1-t)\phi(|z|))\Delta F(-z) \geq \Delta F(-z) \end{aligned}$$

for $|z| \leq r_1$. Similarly, we verify that $H_t(z) < \Delta F(-z)$ for $|z| \geq r_2$. To prove condition (5.3) we compute

$$\begin{aligned} \frac{\partial}{\partial \rho}\Big(\rho H_t(\rho z)\Big) =& \frac{\partial}{\partial \rho}\Big(t\rho H(\rho z) + \rho(1-t)\phi(\rho|z|)\Delta F(-\rho z)\Big) \\ =& t\frac{\partial}{\partial \rho}\Big(\rho H(\rho z)\Big) + (1-t)|z|\phi'(|z|)\Delta F(-z) \\ \leq& t\frac{\partial}{\partial \rho}\Big(\rho H(\rho z)\Big) \leq 0, \end{aligned}$$

where we use that ΔF is homogeneous of degree -1 and is a positive function.

Now we consider the family of equations

$$\Upsilon(t,u) = H_F\big(Y\big) - H_t(Y) = 0, \quad Y = e^{u(x)}x, \ x \in \mathbb{S}^n, \tag{5.28}$$

where H_F is the anisotropic mean curvature of the radial graph defined by $u \in C^2(\mathbb{S}^n)$. It follows from the expression obtained above to H_F that we may write (5.28) in the form

$$\Upsilon(t, x, u, \nabla u, \nabla^2 u) = 0, \quad x \in \mathbb{S}^n. \tag{5.29}$$

Notice that the constant function $u = \ln r_0$ is a solution to the problem corresponding to $t = 0$. We denote it by u_0. The following result ensures the uniqueness of u_0.

Lemma 5.4. *Fixed $t = 0$ there exists a unique solution u_0 of the equation $\Upsilon(t, u(x)) = 0$, namely $u_0 = \ln r_0$, where r_0 satisfies $\phi(r_0) = 1$.*

Proof. That u_0 is a solution to the problem it follows from (5.2) and

$$\begin{aligned} \Upsilon(0, u_0) =& H_F\big(Y\big) - \phi\big(|Y|\big)\Delta F\big(-Y\big) \\ =& \Delta F\big(-Y\big) - \Delta F\big(-Y\big) = 0, \end{aligned}$$

where $Y(x) = e^{u_0}x = r_0 x, \ x \in \mathbb{S}^n$. Let $\bar{u}$ be a solution of $\Upsilon(0, u(x)) = 0$. This means that

$$H_F\big(\bar{Y}\big) - \phi\big(|\bar{Y}|\big)\Delta F\big(-\bar{Y}\big) = 0, \quad \bar{Y}(x) = e^{\bar{u}(x)}x, \ x \in \mathbb{S}^n.$$

Now, let $x_0 \in \mathbb{S}^n$ be a minimum point of $\bar{u}$. At this point, we have $\nabla\bar{u} = 0$ and $\nabla^2\bar{u}$ is positive-definite. We compute explicity at $\bar{Y}(x_0)$

$$b_{ij} = e^{-\bar{u}}(\delta_{ij} - \bar{u}_{ij}).$$

Therefore, if we consider a local frame $\{e_i\}$ around x_0 which is orthonormal at x_0 and which diagonalizes $\nabla^2\bar{u}$ at this point, we obtain

$$\kappa_i \leq e^{-\bar{u}},$$

where κ_i are the principal curvature of the radial graph defined by $\bar{u}$. Hence, since at $\bar{Y}(x_0)$ the unit normal of the graph $\bar{Y}$ is

$$\bar{N} = -\frac{1}{|\bar{Y}|}\bar{Y} = -e^{-\bar{u}}\bar{Y},$$

the anisotropic mean curvature of $\bar{Y}$ satisfies

$$H_F\big(\bar{Y}(x_0)\big) = \sum_i \kappa_i\langle A_F e_i, e_i\rangle \leq e^{-\bar{u}}\Delta F\big(N(x_0)\big) = \Delta F\big(-\bar{Y}(x_0)\big).$$

Therefore, at x_0,

$$\phi\big(|\bar{Y}|\big)\Delta F\big(-\bar{Y}\big) = H_F\big(\bar{Y}\big) \leq \Delta F\big(-\bar{Y}\big) = \phi\big(|Y|\big)\Delta F\big(-\bar{Y}\big).$$

Hence, since ϕ is a decreasing function we conclude from the choice of x_0 as a minimum point that

$$\bar{u}(x) \geq \bar{u}(x_0) \geq u_0,$$

for all $x \in \mathbb{S}^n$. In a similar way, we prove that

$$\bar{u}(x) \leq u_0$$

for all $x \in \mathbb{S}^n$. Thus, we get $\bar{u} = u_0$. This finishes the proof. □

In the two last sections we proved that a differentiable function u which solves the equations $\Upsilon(t, u) = 0$ for some $0 \leq t \leq 1$ satisfies the following bounds

$$r_1 \leq u \leq r_2 \tag{5.30}$$

and

$$|u|_1 \leq C, \tag{5.31}$$

for some positive constant C which depends on n, r_1, r_2, H and F. The standard elliptic regularity theory then provides $C^{2,\alpha}$ estimates. If we suppose that H is a $C^{2,\alpha}$ data, then the regularity of the solution may be improved for $C^{4,\alpha}$. Thus, we obtain a bound

$$|u|_{4,\alpha} < \hat{C} \tag{5.32}$$

for some constant $\hat{C} > 0$.

We then denote by $\mathcal{O}$ the open ball in $C^{4,\alpha}(\mathbb{S}^n)$ with radius $\hat{C}$. Thus, our reasoning above shows that any solution u of $\Upsilon(t,u) = 0$ for some $0 \leq t \leq 1$ is contained in $\mathcal{O}$. In particular, if we consider the restriction

$$\Upsilon : \bar{\mathcal{O}} \subset C^{4,\alpha}(\mathbb{S}^n) \longmapsto C^{2,\alpha}(\mathbb{S}^n),$$

then we conclude that

$$\Upsilon(t,\,\cdot\,)^{-1}(0) \cap \partial\mathcal{O} = \emptyset, \qquad 0 \leq t \leq 1.$$

Thus, according to Definition 2.2 in [29] the degree $\deg(\Upsilon(t,\,\cdot\,), \mathcal{O}, 0)$ is well-defined for all $0 \leq t \leq 1$.

Since Lemma 5.4 assures that $u_0 = \ln r_0$ is the unique solution to $\Upsilon(0,u) = 0$ in $C^{4,\alpha}(\mathbb{S}^n)$, we must prove that the Frechét derivative $\Upsilon_u(0,u_0)$ calculated around u_0 is an invertible operator from $C^{4,\alpha}(\mathbb{S}^n)$ to $C^{2,\alpha}(\mathbb{S}^n)$. We compute

$$\begin{aligned}\Upsilon(0,\rho u_0) =& H_F\big(Y_\rho\big) - \phi\big(|Y_\rho|\big)\Delta F\big(-Y_\rho\big)\\ =& \Delta F\big(-Y_\rho\big) - \phi\big(|Y_\rho|\big)\Delta F\big(-Y_\rho\big),\end{aligned}$$

where $Y_\rho(x) = e^{\rho u_0}x$, $x \in \mathbb{S}^n$. Using the fact that $\phi(r_0) = 1$ and that $\phi'(r_0) < 0$ we get

$$\Upsilon_u(0,u_0)\cdot u_0 = \frac{\mathrm{d}}{\mathrm{d}\rho}\Upsilon(0,\rho u_0)|_{\rho=1} = -\phi'(r_0)\Delta F\big(-Y_1\big) > 0.$$

On the other hand, since obviously $\nabla u_0 = 0$ and $\nabla^2 u_0 = 0$, then $\Upsilon_u(0,u_0)\cdot u_0$ is just a multiple of the zeroth order term in $\Upsilon_u(0,u_0)$. We conclude that $\Upsilon_u(0,u_0)$ is an invertible elliptic operator.

We finally calculate $\deg(\Upsilon(1,\,\cdot\,), \mathcal{O}, 0)$. From Proposition 2.2 in [29], it follows that $\deg(\Upsilon(t,\,\cdot\,), \mathcal{O}, 0)$ does not depend on t. In particular,

$$\deg(\Upsilon(1,\,\cdot\,), \mathcal{O}, 0) = \deg(\Upsilon(0,\,\cdot\,), \mathcal{O}, 0).$$

On the other hand, we had just proved that the equation $\Upsilon(0,u)=0$ has a unique solution u_0 and that the linearized operator $\Upsilon_u(0,u_0)$ is invertible. Thus, by Proposition 2.3 in [29] we get

$$\deg(\Upsilon(0,\,\cdot\,),\mathcal{O},0)=\deg(\Upsilon_u(0,u_0),\mathcal{O},0)=\pm 1,$$

and, therefore,

$$\deg(\Upsilon(1,\,\cdot\,),\mathcal{O},0)=\pm 1.$$

Thus, the equation $\Upsilon(1,u)=0$ has at least one solution $u\in\mathcal{O}$. This completes the proof of the existence in Theorem 5.1. To obtain the uniqueness result we follow the idea presented in [41]. First we extend the prescribed function H to $\mathbb{R}^{n+1}\setminus\{0\}$ on such a way that (5.3) remains true. Let $Y^i(x)=e^{u^i}x$, $i=1,2$, solutions of the prescribed anisotropic mean curvature equation. It follows from Proposition 5.2 that $r_1<|Y_i|<r_2$. Suppose that $u^1>u^2$ at some point. Let $t>1$ such that the radial graph

$$\tilde{Y}^2:=te^{u^2}=e^{\tilde{u}^2}$$

satisfies $|Y^2|\geq|Y^1|$ and $\tilde{Y}^2(x_0)=Y^1(x_0)$ for some point $x_0\in\mathbb{S}^n$. Let H^i_F and $\tilde{H}^2_F$ be the anisotropic mean curvature of Y^i and $\tilde{Y}^2$, respectively. We have

$$\tilde{H}^2_F(\tilde{Y}^2)=\frac{1}{t}H^2_F(Y^2)=\frac{1}{t}H(Y^2).$$

On the other hand, since the function $\psi(\rho)=\rho H(\rho z)$ is decreasing we have

$$\frac{1}{t}H(Y^2)=\frac{1}{t}H(\frac{1}{t}\tilde{Y}^2)\geq H(\tilde{Y}^2). \tag{5.33}$$

Hence

$$\tilde{H}^2_F(\tilde{Y}^2)\geq H(\tilde{Y}^2),$$

which implies that

$$-Q[\tilde{u}^2]+H(\tilde{Y}^2)\leq 0.$$

As

$$-Q[u^1]+H(Y^1)=0,$$

$u^1\leq\tilde{u}^2$, and $u_1(x_0)=\tilde{u}^2(x_0)$, we may apply the maximum principle to obtain (see e.g., [9]) that $\tilde{u}^2=u^1$. In particular, $\tilde{Y}^2=Y^1$ is a solution of the anisotropic mean curvature equation, hence equality (5.33) holds. Using condition (5.3) we may conclude from (5.33) that

$$\frac{1}{s}H(\frac{1}{s}Y^1) = H(Y^1), \quad 1 \leq s \leq t.$$

Thus, since $H_F(sY) = \frac{1}{s}H_F(Y)$, each radial graph $\tilde{Y} = sY^1$, $1 \leq s \leq t$, is a solution. In fact,

$$H_F(\tilde{Y}) = \frac{1}{s}H_F(Y^1) = \frac{1}{s}H(Y^1) = \frac{1}{s}H(\frac{1}{s}sY^1) = H(sY^1) = H(\tilde{Y}).$$

This completes the proof of Theorem 5.1.

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