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EXISTENCE OF NONPARAMETRIC SOLUTIONS FOR A CAPILLARY PROBLEM IN WARPED PRODUCTS

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We prove that there exist solutions for a nonparametric capillary problem in a wide class of Riemannian manifolds endowed with a Killing vector field. In other terms, we prove the existence of Killing graphs with prescribed mean curvature and prescribed contact angle along its boundary. These results may be useful for modeling stationary hypersurfaces under the influence of a nonhomogeneous gravitational field defined over an arbitrary Riemannian manifold.

1. Introduction

Let M be an $(n+1)$-dimensional Riemannian manifold endowed with a Killing vector field Y. Suppose that the distribution orthogonal to Y is of constant rank and integrable. Given an integral leaf P of that distribution, let $\Omega \subset P$ be a bounded domain with regular boundary $\Gamma = \partial\Omega$. We suppose for simplicity that Y is complete. In this case, let $\vartheta : \mathbb{R} \times \bar{\Omega} \to M$ be the flow generated by Y with initial values in M. In geometric terms, the ambient manifold is a warped product $M = P \times_{1/\sqrt{\gamma}} \mathbb{R}$, where $\gamma = 1/\|Y\|^2$.

The Killing graph of a differentiable function $u : \bar{\Omega} \to \mathbb{R}$ is the hypersurface $\Sigma \subset M$ parametrized by the map

$$X(x) = \vartheta(u(x), x), \quad x \in \bar{\Omega}.$$

The Killing cylinder K over Γ is in turn defined by

$$K = \{\vartheta(s, x) : s \in \mathbb{R},\ x \in \Gamma\}. \tag{1}$$

The height function with respect to the leaf P is measured by the arc length parameter ς of the flow lines of Y; that is,

$$\varsigma = \frac{1}{\sqrt{\gamma}} s.$$

Fixing these notations, we are able to formulate a capillary problem in this geometric

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context which models stationary graphs under a gravity force whose intensity depends on the point in the space. More precisely, given a *gravitational potential* $\Psi \in C^{1,\alpha}(\overline{\Omega} \times \mathbb{R})$ we define the functional

$$\mathscr{A}[u] = \int_\Sigma \left(1 + \int_0^{u/\sqrt{\gamma}} \Psi(x, s(\varsigma))\, \mathrm{d}\varsigma\right) \mathrm{d}\Sigma. \tag{2}$$

The volume element $\mathrm{d}\Sigma$ of Σ is given by

$$\frac{1}{\sqrt{\gamma}}\sqrt{\gamma + \|\nabla u\|^2}\, \mathrm{d}\sigma,$$

where $\mathrm{d}\sigma$ is the volume element in P. In what follows we denote

$$W = \sqrt{\gamma + \|\nabla u\|^2}.$$

The first variation formula of this functional may be deduced as follows. Given an arbitrary function $v \in C_c^\infty(\Omega)$ we compute

$$\begin{aligned}
\frac{d}{d\tau}\bigg|_{\tau=0} & \mathscr{A}[u+\tau v] \\
&= \int_\Omega \left(\frac{1}{\sqrt{\gamma}} \frac{\langle \nabla u, \nabla v\rangle}{\sqrt{\gamma + \|\nabla u\|^2}} + \frac{1}{\sqrt{\gamma}} \Psi(x, u(x))\, v\right) \sqrt{\sigma}\, \mathrm{d}x \\
&= \int_\Omega \left(\mathrm{div}\left(\frac{1}{\sqrt{\gamma}} \frac{\nabla u}{W} v\right) - \mathrm{div}\left(\frac{1}{\sqrt{\gamma}} \frac{\nabla u}{W}\right) v + \frac{1}{\sqrt{\gamma}} \Psi(x, u(x))\, v\right) \sqrt{\sigma}\, \mathrm{d}x \\
&\quad - \int_\Omega \left(\frac{1}{\sqrt{\gamma}} \mathrm{div}\left(\frac{\nabla u}{W}\right) - \frac{1}{\sqrt{\gamma}} \left\langle \frac{\nabla \gamma}{2\gamma}, \frac{\nabla u}{W}\right\rangle - \frac{1}{\sqrt{\gamma}} \Psi(x, u(x))\right) v\, \sqrt{\sigma}\, \mathrm{d}x,
\end{aligned}$$

where $\sqrt{\sigma}\, \mathrm{d}x$ is the volume element $\mathrm{d}\sigma$ expressed in terms of local coordinates in P. The differential operators div and ∇ are respectively the divergence and gradient in P with respect to the metric induced from M.

We conclude that stationary functions satisfy the capillary-type equation

$$\mathrm{div}\left(\frac{\nabla u}{W}\right) - \left\langle \frac{\nabla \gamma}{2\gamma}, \frac{\nabla u}{W}\right\rangle = \Psi. \tag{3}$$

Notice that a Neumann boundary condition arises naturally from this variational setting: given a $C^{2,\alpha}$ function $\Phi : K \to (-1, 1)$, we impose the prescribed angle condition

$$\langle N, \nu\rangle = \Phi \tag{4}$$

along $\partial\Sigma$, where

$$N = \frac{1}{W}(\gamma Y - \vartheta_* \nabla u) \tag{5}$$

is the unit normal vector field along Σ satisfying $\langle N, Y\rangle > 0$ and ν is the unit normal vector field along K pointing into the Killing cylinder over Ω.

Equation (3) is the prescribed mean curvature equation for Killing graphs. A general existence result for solutions of the Dirichlet problem for this equation may be found in [Dajczer et al. 2008] and [Dajczer and de Lira 2012]. There the authors used local perturbations of the Killing cylinders as barriers for obtaining height and gradient estimates. However this kind of barrier is not suitable to obtain a priori estimates for solutions of Neumann problems. Indeed, these barriers depend on Dirichlet boundary data and do not involve any a priori information about the prescribed contact angle. It turns out that for Dirichlet boundary conditions the slope of the graph along the boundary is controlled in terms of the height of the graph.

For that reason we now consider local perturbations of the graph itself, adapted from the original approach by N. Korevaar [1988] and its extension by M. Calle and L. Shahriyari [2011].

Following these two sources we suppose that the data Ψ and Φ satisfy

(i) $|\Psi| + \|\bar\nabla\Psi\| \le C_\Psi$ in $\bar\Omega\times\mathbb{R}$,

(ii) $\langle\bar\nabla\Psi, Y\rangle \ge \beta > 0$ in $\bar\Omega\times\mathbb{R}$,

(iii) $\langle\bar\nabla\Phi, Y\rangle \le 0$,

(iv) $(1-\Phi^2) \ge \beta'$,

(v) $|\Phi| + \|\nabla\Phi\| + \|\nabla^2\Phi\| \le C_\Phi$ in K,

for some positive constants C_Ψ, C_Φ, β and β', where $\bar\nabla$ denotes the Riemannian connection in M. Assumption (ii) is classically referred to as the *positive gravity* condition. Even in the Euclidean space, it seems to be an essential assumption in order to obtain a priori height estimates. A very geometric discussion about this issue may be found in [Concus and Finn 1974]. Condition (iii) is the same as in [Calle and Shahriyari 2011] and [Korevaar 1988] since in those references N is chosen in such a way that $\langle N, Y\rangle > 0$.

The main result in this paper is the following:

Theorem 1. *Let Ω be a bounded $C^{3,\alpha}$ domain in P. Suppose that $\Psi \in C^{1,\alpha}(\bar\Omega\times\mathbb{R})$ and $\Phi\in C^{2,\alpha}(K)$ with $|\Phi|\le 1$ satisfy conditions (i)–(v) above. Then there exists a unique solution $u\in C^{3,\alpha}(\bar\Omega)$ of the capillary problem (3)–(4).*

We observe that $\Psi = nH$, where H is the mean curvature of Σ calculated with respect to N. Therefore Theorem 1 establishes the existence of Killing graphs with prescribed mean curvature Ψ and prescribed contact angle with K along the boundary. Since the Riemannian product $P\times\mathbb{R}$ corresponds to the particular case where $\gamma = 1$, our result extends the main existence theorem in [Calle and Shahriyari 2011]. Space forms constitute other important examples of the kind of warped products we are considering. In particular, we encompass the case of Killing graphs over totally geodesic hypersurfaces in the hyperbolic space $\mathbb{H}^{n+1}$.

In Section 2, we prove a priori height estimates for solutions of (3)–(4) based on the method presented in [Uraltseva 1973]. These height estimates are one of the main steps for using the well-known continuity method in order to prove Theorem 1. At this respect, we refer the reader to the classical references [Concus and Finn 1974], [Gerhardt 1976] and [Simon and Spruck 1976].

Section 3 contains the proof of interior and boundary gradient estimates. There we follow closely a method due to Korevaar [1988] for graphs in the Euclidean spaces and extended by Calle and Shahriyari [2011] for Riemannian products. Finally the classical continuity method is applied to (3)–(4) in Section 4 for proving the existence result.

2. Height estimates

In this section, we use a technique developed by N. Uraltseva [1973] (see also [Ladyzhenskaya and Uraltseva 1964] and [Gilbarg and Trudinger 2001] for classical references on the subject) in order to obtain a height estimate for solutions of the capillary problem (3)–(4). This estimate requires the *positive gravity* assumption (ii) stated in the introduction.

Proposition 2. *Set*

$$\beta = \inf_{\Omega\times\mathbb{R}} \langle \bar{\nabla}\Psi, Y\rangle \quad \textit{and} \quad \mu = \sup_{\Omega} \Psi(x,0). \tag{6}$$

Suppose that $\beta > 0$. Then any solution u of (3)–(4) satisfies

$$|u(x)| \le \frac{\sup_\Omega \|Y\|}{\inf_\Omega \|Y\|}\frac{\mu}{\beta} \tag{7}$$

for all $x \in \bar{\Omega}$.

Proof. Fix an arbitrary real number k with

$$k > \frac{\sup_\Omega \|Y\|}{\inf_\Omega \|Y\|}\frac{\mu}{\beta}. \tag{8}$$

Suppose that the superlevel set

$$\Omega_k = \{x \in \Omega : u(x) > k\}$$

has nonzero Lebesgue measure. Define $u_k : \Omega \to \mathbb{R}$ as

$$u_k(x) = \max\{u(x) - k, 0\}.$$

From the variational formulation we have

$$
\begin{aligned}
0 &= \int_{\Omega_k} \left(\frac{1}{\sqrt{\gamma}} \frac{\langle \nabla u, \nabla u_k \rangle}{\sqrt{\gamma + \|\nabla u\|^2}} + \frac{1}{\sqrt{\gamma}} \Psi(x, u(x))\, u_k \right) \sqrt{\sigma}\, \mathrm{d}x \\
&= \int_{\Omega_k} \left(\frac{1}{\sqrt{\gamma}} \frac{\|\nabla u\|^2}{W} + \frac{1}{\sqrt{\gamma}} \Psi(x, u(x))(u-k) \right) \sqrt{\sigma}\, \mathrm{d}x \\
&= \int_{\Omega_k} \left(\frac{1}{\sqrt{\gamma}} \frac{W^2 - \gamma}{W} + \frac{1}{\sqrt{\gamma}} \Psi(x, u(x))(u-k) \right) \sqrt{\sigma}\, \mathrm{d}x \\
&= \int_{\Omega_k} \left(\frac{W}{\sqrt{\gamma}} - \frac{\sqrt{\gamma}}{W} + \frac{1}{\sqrt{\gamma}} \Psi(x, u(x))(u-k) \right) \sqrt{\sigma}\, \mathrm{d}x.
\end{aligned}
$$

However

$$
\Psi(x, u(x)) = \Psi(x, 0) + \int_0^{u(x)} \frac{\partial \Psi}{\partial s}\, \mathrm{d}s \geq -\mu + \beta u(x).
$$

Since $\sqrt{\gamma}/W \leq 1$ we conclude that

$$
|\Omega_k| - |\Omega_k| - \mu \int_{\Omega_k} \frac{1}{\sqrt{\gamma}}(u-k) + \beta \int_{\Omega_k} \frac{1}{\sqrt{\gamma}} u(u-k) \leq 0,
$$

where $|\Omega_k|$ is the Lebesgue measure of Ω_k. Hence we have

$$
\beta \int_{\Omega_k} \frac{1}{\sqrt{\gamma}} u(u-k) \leq \mu \int_{\Omega_k} \frac{1}{\sqrt{\gamma}}(u-k).
$$

It follows that

$$
\beta k \inf_{\Omega} \|Y\| \int_{\Omega_k} (u-k) \leq \mu \sup_{\Omega} \|Y\| \int_{\Omega_k} (u-k).
$$

Since $|\Omega_k| \neq 0$ we have

$$
k \leq \frac{\sup_\Omega \|Y\|}{\inf_\Omega \|Y\|} \frac{\mu}{\beta},
$$

which contradicts the choice of k. We conclude that

$$
|\Omega_k| = 0 \quad \text{for all } k \geq \frac{\sup_\Omega \|Y\|}{\inf_\Omega \|Y\|} \frac{\mu}{\beta}.
$$

This implies that

$$
u(x) \leq \frac{\sup_\Omega \|Y\|}{\inf_\Omega \|Y\|} \frac{\mu}{\beta}
$$

for all $x \in \overline{\Omega}$. A lower estimate may be deduced in a similar way. □

Remark 3. The construction of geometric barriers similar to those in [Concus and Finn 1974] is also possible at least in the case where P is endowed with a rotationally invariant metric and Ω is contained in a normal neighborhood of a pole of P.

3. Gradient estimates

Let Ω' be a subset of Ω and define

$$\Sigma' = \{\vartheta(u(x), x) : x \in \Omega'\} \subset \Sigma \tag{9}$$

to be the graph of $u|_{\Omega'}$. Let $\mathcal{O}$ be an open subset in M containing Σ'. We consider a vector field $Z \in \Gamma(TM)$ with bounded C^2 norm and supported in $\mathcal{O}$. Hence there exists $\varepsilon > 0$ such that the local flow $\Xi : (-\varepsilon, \varepsilon) \times \mathcal{O} \to M$ generated by Z is well-defined. We also suppose that

$$\langle Z(y), \nu(y)\rangle = 0 \tag{10}$$

for any $y \in K \cap \mathcal{O}$. This implies that the flow line of Z passing through a point $y \in K \cap \mathcal{O}$ is entirely contained in K.

We define a variation of Σ by a one-parameter family of hypersurfaces Σ_τ, $\tau \in (-\varepsilon, \varepsilon)$, parametrized by $X_\tau : \bar{\Omega} \to M$, where

$$X_\tau(x) = \Xi(\tau, \vartheta(u(x), x)), \quad x \in \bar{\Omega}. \tag{11}$$

It follows from the implicit function theorem that there exist $\Omega_\tau \subset P$ and $u_\tau : \bar{\Omega}_\tau \to \mathbb{R}$ such that Σ_τ is the graph of u_τ. Moreover, $\Omega_\tau \subset \Omega$.

Hence given a point $y \in \Sigma$, denote $y_\tau = \Xi(\tau, y) \in \Sigma_\tau$. It follows that there exists $x_\tau \in \Omega_\tau$ such that $y_\tau = \vartheta(u_\tau(x_\tau), x_\tau)$. Then we denote by $\hat{y}_\tau = \vartheta(u(x_\tau), x_\tau)$ the point in Σ in the flow line of Y passing through y_τ. The vertical separation between y_τ and $\hat{y}_\tau$ is by definition the function $s(y, \tau) = u_\tau(x_\tau) - u(x_\tau)$.

Lemma 4. *For any $\tau \in (-\varepsilon, \varepsilon)$, let A_τ and H_τ be, respectively, the Weingarten map and the mean curvature of the hypersurface Σ_τ calculated with respect to the unit normal vector field N_τ along Σ_τ which satisfies $\langle N_\tau, Y\rangle > 0$. Denote $H = H_0$ and $A = A_0$. If $\zeta \in C^\infty(\mathcal{O})$ and $T \in \Gamma(T\mathcal{O})$ are defined by*

$$Z = \zeta N_\tau + T \tag{12}$$

with $\langle T, N_\tau\rangle = 0$ then

(i) $\partial s/\partial\tau|_{\tau=0} = \langle Z, N\rangle W$,

(ii) $\bar{\nabla}_Z N|_{\tau=0} = -AT - \nabla^\Sigma \zeta$,

(iii) $\partial H/\partial\tau|_{\tau=0} = \Delta_\Sigma \zeta + (\|A\|^2 + \mathrm{Ric}_M(N, N))\zeta + \langle \bar{\nabla}\Psi, Z\rangle$,

where $W = \langle Y, N_\tau\rangle^{-1} = (\gamma + \|\nabla u_\tau\|^2)^{-1/2}$. The operators ∇^Σ and Δ_Σ are, respectively, the intrinsic gradient operator and the Laplace–Beltrami operator in Σ with respect to the induced metric. Moreover, $\bar{\nabla}$ and Ric_M denote, respectively, the Riemannian covariant derivative and the Ricci tensor in M.

Proof. (i) Let $(x^i)_{i=1}^n$ be a set of local coordinates in $\Omega \subset P$. Differentiating (11) with respect to τ we obtain

$$X_{\tau *}\frac{\partial}{\partial \tau} = Z|_{X_\tau} = \zeta N_\tau + T.$$

On the other hand differentiating both sides of

$$X_\tau(x) = \vartheta(u_\tau(x_\tau), x_\tau)$$

with respect to τ we have

$$\begin{aligned} X_{\tau *}\frac{\partial}{\partial \tau} &= \left(\frac{\partial u_\tau}{\partial \tau} + \frac{\partial u_\tau}{\partial x^i}\frac{\partial x^i_\tau}{\partial \tau}\right)\vartheta_* Y + \frac{\partial x^i_\tau}{\partial \tau}\vartheta_* \frac{\partial}{\partial x^i} \\ &= \frac{\partial u_\tau}{\partial \tau}\vartheta_* Y + \frac{\partial x^i_\tau}{\partial \tau}\left(\vartheta_* \frac{\partial}{\partial x^i} + \frac{\partial u_\tau}{\partial x^i}\vartheta_* Y\right). \end{aligned}$$

Since the term between parenthesis after the second equality is a tangent vector field in Σ_τ we conclude that

$$\frac{\partial u_\tau}{\partial \tau}\langle Y, N_\tau\rangle = \left\langle X_{\tau *}\frac{\partial}{\partial \tau}, N_\tau\right\rangle = \zeta,$$

and it follows that

$$\frac{\partial u_\tau}{\partial \tau} = \zeta W$$

and

$$\frac{\partial s}{\partial \tau} = \frac{\partial}{\partial \tau}(u_\tau - u) = \frac{\partial u_\tau}{\partial \tau} = \zeta W.$$

(ii) Now we have

$$\begin{aligned} \langle \bar\nabla_Z N_\tau, X_*\partial_i\rangle &= -\langle N_\tau, \bar\nabla_Z X_*\partial_i\rangle = -\langle N_\tau, \bar\nabla_{X_*\partial_i} Z\rangle = -\langle N_\tau, \bar\nabla_{X_*\partial_i}(\zeta N + T)\rangle \\ &= -\langle N_\tau, \bar\nabla_{X_*\partial_i} T\rangle - \langle N_\tau, \bar\nabla_{X_*\partial_i}\zeta N_\tau\rangle = -\langle A_\tau T, X_*\partial_i\rangle - \langle \nabla^\Sigma \zeta, X_*\partial_i\rangle, \end{aligned}$$

for any $1 \le i \le n$. It follows that

$$\bar\nabla_Z N = -AT - \nabla^\Sigma \zeta.$$

(iii) This is a well-known formula whose proof may be found in a number of references, such as [Gerhardt 2006]. □

For further reference, we point out that the comparison principle [Gilbarg and Trudinger 2001] when applied to (3)–(4) may be stated in geometric terms as follows. Fix τ, and let $x \in \bar\Omega'$ be a point of maximal vertical separation $s(\cdot, \tau)$. If x is an interior point we have

$$\nabla u_\tau(x, \tau) - \nabla u(x) = \nabla s(x, \tau) = 0,$$

which implies that the graphs of the functions u_τ and $u+s(x,\tau)$ are tangent at their common point $y_\tau = \vartheta(u_\tau(x), x)$. Since the graph of $u+s(x,\tau)$ is obtained from Σ only by a translation along the flow lines of Y we conclude that the mean curvatures of these two graphs are the same at corresponding points. Since the graph of $u+s(x,\tau)$ is locally above the graph of u_τ we conclude that

$$H(\hat{y}_\tau) \geq H_\tau(y_\tau). \tag{13}$$

If $x \in \partial\Omega \subset \partial\Omega'$, we have

$$\langle \nabla u_\tau, \nu\rangle|_x - \langle \nabla u, \nu\rangle|_x = \langle \nabla s, \nu\rangle \leq 0,$$

since ν points toward Ω. This implies that

$$\langle N, \nu\rangle|_{y_\tau} \geq \langle N, \nu\rangle|_{\hat{y}_\tau}. \tag{14}$$

3.1. ***Interior gradient estimate.***

Proposition 5. *Let $B_R(x_0) \subset \Omega$, where $R < \operatorname{inj} P$. Then there exists a constant $C > 0$ depending on β, C_Ψ, Ω and K such that*

$$\|\nabla u(x)\| \leq C \frac{R^2}{R^2 - d^2(x)}, \tag{15}$$

where $d = \operatorname{dist}(x_0, x)$ in P.

Proof. Fix $\Omega' = B_R(x_0) \subset \Omega$. We consider the vector field Z given by

$$Z = \zeta N, \tag{16}$$

where ζ is a function to be defined later. Fix $\tau \in [0, \varepsilon)$, and let $x \in B_R(x_0)$ be a point where the vertical separation $s(\cdot, \tau)$ attains a maximum value.

If $y = \vartheta(u(x), x)$ it follows that

$$H_\tau(y_\tau) - H_0(y) = \frac{dH_\tau}{d\tau}\bigg|_{\tau=0} \tau + o(\tau). \tag{17}$$

However, the comparison principle implies that $H_0(\hat{y}_\tau) \geq H_\tau(y_\tau)$. By Lemma 1(iii) we conclude that

$$H_0(\hat{y}_\tau) - H_0(y) \geq \frac{dH_\tau}{d\tau}\bigg|_{\tau=0} \tau + o(\tau) = (\Delta_\Sigma \zeta + \|A\|^2\zeta + \operatorname{Ric}_M(N,N)\zeta)\tau + o(\tau).$$

Since $\hat{y}_\tau = \vartheta(-s(y,\tau), y_\tau)$, we have

$$\frac{d\hat{y}_\tau}{d\tau}\bigg|_{\tau=0} = -\frac{ds}{d\tau}\vartheta_* \frac{\partial}{\partial s} + \frac{\partial y^i_\tau}{\partial \tau}\vartheta_* \frac{\partial}{\partial x^i} = -\frac{ds}{d\tau} Y + \frac{dy_\tau}{d\tau}\bigg|_{\tau=0} = -\frac{ds}{d\tau} Y + Z(y). \tag{18}$$

Hence using Lemma 1(i) and (16) we have

$$\left.\frac{d\hat{y}_\tau}{d\tau}\right|_{\tau=0} = -\zeta WY + \zeta N. \tag{19}$$

On the other hand, for each $\tau \in (-\varepsilon, \varepsilon)$ there exists a smooth $\xi : (-\varepsilon, \varepsilon) \to TM$ such that

$$\hat{y}_\tau = \exp_y \xi(\tau).$$

Hence we have

$$\left.\frac{d\hat{y}_\tau}{d\tau}\right|_{\tau=0} = \xi'(0).$$

With a slight abuse of notation we denote $\Psi(s, x)$ by $\Psi(y)$, where $y = \vartheta(s, x)$. It results that

$$\begin{aligned} H_0(\hat{y}_\tau) - H_0(y) &= \Psi(x_\tau, u(x_\tau)) - \Psi(x, u(x)) = \Psi(\exp_y \xi_\tau) - \Psi(y) \\ &= \langle \bar{\nabla}\Psi|_y, \xi'(0)\rangle\tau + o(\tau). \end{aligned}$$

However,

$$\langle \bar{\nabla}\Psi, \xi'(0)\rangle = \zeta\langle \bar{\nabla}\Psi, N - WY\rangle = -\zeta W\frac{\partial\Psi}{\partial s} + \zeta\langle \bar{\nabla}\Psi, N\rangle. \tag{20}$$

We conclude that

$$-\zeta W\frac{\partial\Psi}{\partial s}\tau + \zeta\langle \bar{\nabla}\Psi, N\rangle\tau + o(\tau) \geq (\Delta_\Sigma\zeta + \|A\|^2\zeta + \mathrm{Ric}_M(N, N)\zeta)\tau + o(\tau).$$

Suppose that

$$W(x) > \frac{C + \|\bar{\nabla}\Psi\|}{\beta} \tag{21}$$

for a constant $C > 0$ to be chosen later. Hence we have

$$(\Delta_\Sigma\zeta + \mathrm{Ric}_M(N, N)\zeta)\tau + C\zeta\tau \leq o(\tau).$$

Following [Calle and Shahriyari 2011] and [Korevaar 1988] we choose

$$\zeta = 1 - \frac{d^2}{R^2},$$

where $d = \mathrm{dist}(x_0, \cdot)$. It follows that

$$\nabla^\Sigma\zeta = -\frac{2d}{R^2}\nabla^\Sigma d,$$

and

$$\Delta_\Sigma\zeta = -\frac{2d}{R^2}\Delta_\Sigma d - \frac{2}{R^2}\|\nabla^\Sigma d\|^2.$$

However, using the fact that P is totally geodesic and that $[Y, \bar{\nabla} d] = 0$, we have

$$\begin{aligned}\Delta_\Sigma d &= \Delta_M d - \langle \bar{\nabla}_N \bar{\nabla} d, N\rangle + nH\langle \bar{\nabla} d, N\rangle \\ &= \Delta_P d - \left\langle \nabla_{\nabla u/W} \nabla d, \frac{\nabla u}{W}\right\rangle - \gamma^2 \langle Y, N\rangle^2 \langle \bar{\nabla}_Y \bar{\nabla} d, Y\rangle + nH\langle \bar{\nabla} d, N\rangle.\end{aligned}$$

Let $\pi : M \to P$ be the projection defined by $\pi(\vartheta(s, x)) = x$. Then

$$\pi_* N = -\frac{\nabla u}{W}.$$

We denote

$$\pi_* N^\perp = \pi_* N - \langle \pi_* N, \nabla d\rangle \nabla d.$$

If $\mathcal{A}_d$ and $\mathcal{H}_d$ denote, respectively, the Weingarten map and the mean curvature of the geodesic ball $B_d(x_0)$ in P we conclude that

$$\Delta_\Sigma d = n\mathcal{H}_d - \langle \mathcal{A}_d(\pi_* N^\perp), \pi_* N^\perp\rangle + \gamma\langle Y, N\rangle^2 \kappa + nH\langle \bar{\nabla} d, N\rangle,$$

where

$$\kappa = -\gamma\langle \bar{\nabla}_Y \bar{\nabla} d, Y\rangle$$

is the principal curvature of the Killing cylinder over $B_d(x_0)$ relative to the principal direction Y. Therefore we have

$$|\Delta_\Sigma d| \le C_1\Big(C_\Psi, \sup_{B_R(x_0)} (\mathcal{H}_d + \kappa), \sup_{B_R(x_0)} \gamma\Big)$$

in $B_R(x_0)$. Hence setting

$$C_2 = \sup_{B_R(x_0)} \mathrm{Ric}_M,$$

we fix

$$C = \max\Big\{2(C_1 + C_2), \sup_{\mathbb{R}\times\Omega} \|\bar{\nabla}\Psi\|\Big\}. \tag{22}$$

With this choice we conclude that

$$C\zeta \le \frac{o(\tau)}{\tau},$$

a contradiction. This implies that

$$W(x) \le \frac{C - \|\bar{\nabla}\Psi\|}{\beta}. \tag{23}$$

However,

$$\zeta(z)W(z) + o(\tau) = s(X(z), \tau) \le s(X(x), \tau) = \zeta(x)W(x) + o(\tau)$$

for any $z \in B_R(x_0)$. It follows that

$$W(z) \le \frac{R^2 - d^2(z)}{R^2 - d^2(x)} W(x) + o(\tau) \le \frac{R^2}{R^2 - d^2(x)} \frac{C - \|\overline{\nabla}\Psi\|}{\beta} + o(\tau) \le \widetilde{C} \frac{R^2}{R^2 - d^2(x)},$$

for very small $\varepsilon > 0$. □

Remark 6. If Ω satisfies the interior sphere condition for a uniform radius $R > 0$, we conclude that

$$W(x) \le \frac{C}{d_\Gamma(x)}, \tag{24}$$

for $x \in \Omega$, where $d_\Gamma(x) = \operatorname{dist}(x, \Gamma)$.

3.2. ***Boundary gradient estimates.*** Now we establish boundary gradient estimates using another local perturbation of the graph, which this time has also tangential components.

Proposition 7. *Let $x_0 \in P$ and $R > 0$ such that $3R < \operatorname{inj} P$. Denote by Ω' the subdomain $\Omega \cap B_{2R}(x_0)$. Then there exists a positive constant C, depending only on R, β, β', C_Ψ, C_Φ, Ω, K, such that*

$$W(x) \le C, \tag{25}$$

for all $x \in \overline{\Omega}'$.

Proof. Now we consider the subdomain $\Omega' = \Omega \cap B_{2R}(x_0)$. We define

$$Z = \eta N + X, \tag{26}$$

where

$$\eta = \alpha_0 v + \alpha_1 d_\Gamma,$$

and α_0 and α_1 are positive constants to be chosen and d_Γ is a smooth extension of the distance function $\operatorname{dist}(\,\cdot\,, \Gamma)$ to Ω' with $\|\nabla d_\Gamma\| \le 1$ and

$$v = 4R^2 - d^2,$$

where $d = \operatorname{dist}(x_0, \cdot\,)$. Moreover,

$$X = \alpha_0 \Phi (v\nu - d_\Gamma \nabla v).$$

In this case we have

$$\zeta = \eta + \langle X, N\rangle = \alpha_0 v + \alpha_1 d_\Gamma + \alpha_0 \Phi (v\langle N, \nu\rangle - d_\Gamma \langle N, \nabla v\rangle).$$

Fix $\tau \in [0, \varepsilon)$, and let $x \in \overline{\Omega}'$ be a point where the maximal vertical separation between Σ and Σ_τ is attained. We first suppose that $x \in \operatorname{int}(\partial\Omega' \cap \partial\Omega)$. In this case, setting $y_\tau = \vartheta(u_\tau(x), x) \in \Sigma_\tau$ and $\hat{y}_\tau = \vartheta(u(x), x) \in \Sigma$, it follows from the comparison principle that

$$\langle N_\tau, \nu\rangle|_{y_\tau} \geq \langle N, \nu\rangle|_{\hat{y}_\tau}. \tag{27}$$

Note that $\hat{y}_\tau \in \partial\Sigma$. Moreover, since $Z|_{K\cap\mathbb{O}}$ is tangent to K there exists $y \in \partial\Sigma$ such that

$$y = \Xi(-\tau, y_\tau).$$

We claim that

$$\left|\left\langle \overline{\nabla}\langle N_\tau, \nu\rangle, \frac{dy_\tau}{d\tau}\Big|_{\tau=0}\right\rangle\right| \leq \alpha_1(1-\Phi^2) + \widetilde{C}\alpha_0, \tag{28}$$

for some positive constant $\widetilde{C} = C(C_\Phi, K, \Omega, R)$.

Hence (4) implies that

$$\langle N, \nu\rangle|_{\hat{y}_\tau} - \langle N, \nu\rangle|_y = \Phi(\hat{y}_\tau) - \Phi(y) = \tau\left\langle \overline{\nabla}\Phi, \frac{d\hat{y}_\tau}{d\tau}\Big|_{\tau=0}\right\rangle + o(\tau).$$

Therefore

$$\langle N, \nu\rangle|_{y_\tau} - \langle N, \nu\rangle|_y \geq \tau\left\langle \overline{\nabla}\Phi, \frac{d\hat{y}_\tau}{d\tau}\Big|_{\tau=0}\right\rangle + o(\tau).$$

On the other hand we have

$$\langle N, \nu\rangle|_{y_\tau} - \langle N, \nu\rangle|_y = \tau\left\langle \overline{\nabla}\langle N, \nu\rangle, \frac{dy_\tau}{d\tau}\Big|_{\tau=0}\right\rangle + o(\tau).$$

We conclude that

$$\tau\left\langle \overline{\nabla}\langle N, \nu\rangle, \frac{dy_\tau}{d\tau}\Big|_{\tau=0}\right\rangle \geq \tau\left\langle \overline{\nabla}\Phi, \frac{d\hat{y}_\tau}{d\tau}\Big|_{\tau=0}\right\rangle + o(\tau).$$

Hence we have

$$\alpha_1(1-\Phi^2)\tau + \widetilde{C}\alpha_0\tau \geq \tau\left\langle \overline{\nabla}\Phi, \frac{d\hat{y}_\tau}{d\tau}\Big|_{\tau=0}\right\rangle + o(\tau).$$

It follows from (28) that

$$\alpha_1(1-\Phi^2) + \widetilde{C}\alpha_0 \geq -\zeta W\langle\overline{\nabla}\Phi, Y\rangle + \zeta\langle\overline{\nabla}\Phi, N\rangle + o(\tau)/\tau.$$

Since

$$\langle\overline{\nabla}\Phi, Y\rangle = \frac{\partial\Phi}{\partial s} \leq 0,$$

we conclude that

$$W(x) \leq C(C_\Phi, \beta', K, \Omega, R). \tag{29}$$

We now prove the claim. For that, observe that Lemma 1(ii) implies that

$$\begin{aligned}\langle N,\nu\rangle|_{y_\tau}-\langle N,\nu\rangle|_{y}&=\tau\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\langle N_\tau,\nu\rangle|_{y_\tau}+o(\tau)\\&=\tau(\langle N,\overline{\nabla}_Z\nu\rangle|_y-\langle AT+\nabla^\Sigma\zeta,\nu\rangle|_y)+o(\tau).\end{aligned}$$

Since $Z|_y\in T_yK$, it follows that

$$\langle N,\nu\rangle|_{y_\tau}-\langle N,\nu\rangle|_{y}=-\tau(\langle A_KZ,N\rangle|_y+\langle AT+\nabla^\Sigma\zeta,\nu\rangle|_y)+o(\tau),$$

where A_K is the Weingarten map of K with respect to ν. We conclude that

$$-\tau(\langle A_KZ,N\rangle|_y+\langle AT+\nabla^\Sigma\zeta,\nu\rangle|_y)\geq\tau\left\langle\overline{\nabla}\Phi,\frac{d\hat{y}_\tau}{d\tau}\bigg|_{\tau=0}\right\rangle+o(\tau),\tag{30}$$

where

$$\nu^T=\nu-\langle N,\nu\rangle N.$$

We have

$$\langle\nabla^\Sigma\zeta+AT,\nu^T\rangle=\alpha_0\langle\nabla v,\nu^T\rangle+\alpha_1\langle\nabla^\Sigma d_\Gamma,\nu^T\rangle+\langle\nabla^\Sigma\langle X,N\rangle,\nu^T\rangle+\langle AT,\nu^T\rangle.$$

We compute

$$\begin{aligned}\langle\nabla^\Sigma\langle X,N\rangle,\nu^T\rangle&=\alpha_0(v\langle N,\nu\rangle-d_\Gamma\langle N,\nabla v\rangle)\langle\overline{\nabla}\Phi,\nu^T\rangle\\&\quad+\alpha_0\Phi\big(\langle\nabla v,\nu^T\rangle\langle N,\nu\rangle+v(\langle\overline{\nabla}_{\nu^T}N,\nu\rangle+\langle N,\overline{\nabla}_{\nu^T}\nu\rangle)\\&\qquad-\langle\nabla d_\Gamma,\nu^T\rangle\langle N,\nabla v\rangle-d_\Gamma(\langle\overline{\nabla}_{\nu^T}N,\nabla v\rangle+\langle N,\overline{\nabla}_{\nu^T}\nabla v\rangle)\big).\end{aligned}$$

Hence we have at y that

$$\begin{aligned}\langle\nabla^\Sigma\langle X,N\rangle,\nu^T\rangle&=\alpha_0(v\Phi-d_\Gamma\langle N,\nabla v\rangle)\langle\overline{\nabla}\Phi,\nu^T\rangle\\&\quad+\alpha_0\Phi\big(\langle\nabla v,\nu^T\rangle\Phi+v(-\langle A\nu^T,\nu^T\rangle+\langle N,\overline{\nabla}_\nu\nu\rangle\\&\qquad-\langle N,\nu\rangle\langle N,\overline{\nabla}_N\nu\rangle)-\langle\nu,\nu^T\rangle\langle N,\nabla v\rangle\\&\qquad\quad-d_\Gamma(-\langle A\nu^T,\nabla v\rangle+\langle N,\overline{\nabla}_\nu\nabla v\rangle-\langle N,\nu\rangle\langle N,\overline{\nabla}_N\nabla v\rangle)\big).\end{aligned}$$

Therefore we have

$$\begin{aligned}\langle\nabla^\Sigma\langle X,N\rangle,\nu^T\rangle&=\alpha_0(v\Phi-d_\Gamma\langle N,\nabla v\rangle)\langle\overline{\nabla}\Phi,\nu^T\rangle\\&\quad+\alpha_0\Phi\big(\langle\nabla v,\nu^T\rangle\Phi-v(\langle A\nu^T,\nu^T\rangle+\langle N,\nu\rangle\langle N,\overline{\nabla}_N\nu\rangle)\\&\qquad-\langle\nu,\nu^T\rangle\langle N,\nabla v\rangle\\&\qquad\quad+d_\Gamma(\langle A\nu^T,\nabla v\rangle-\langle N,\overline{\nabla}_\nu\nabla v\rangle+\langle N,\nu\rangle\langle N,\overline{\nabla}_N\nabla v\rangle)\big).\end{aligned}$$

It follows that

$$\begin{aligned}\langle \nabla^\Sigma \zeta + AT, \nu^T\rangle = \langle AT, \nu^T\rangle &+ \alpha_0\langle \nabla v, \nu^T\rangle + \alpha_1\langle \nu, \nu^T\rangle \\ &+ \alpha_0(v\Phi - d_\Gamma\langle N, \nabla v\rangle)\langle \bar{\nabla}\Phi, \nu^T\rangle \\ &+ \alpha_0\Phi\big(\langle \nabla v, \nu^T\rangle\Phi - v(\langle A\nu^T, \nu^T\rangle + \langle N, \nu\rangle\langle N, \bar{\nabla}_N\nu\rangle) \\ &- \langle \nu, \nu^T\rangle\langle N, \nabla v\rangle \\ &+ d_\Gamma(\langle A\nu^T, \nabla v\rangle - \langle N, \bar{\nabla}_\nu\nabla v\rangle + \langle N, \nu\rangle\langle N, \bar{\nabla}_N\nabla v\rangle)\big).\end{aligned}$$

However,

$$\langle AT, \nu^T\rangle = \langle A\nu^T, X\rangle = \alpha_0\Phi v\langle A\nu^T, \nu^T\rangle - \alpha_0\Phi d_\Gamma\langle A\nu^T, \nabla v\rangle.$$

Hence we have

$$\begin{aligned}\langle \nabla^\Sigma \zeta + AT, \nu^T\rangle = \alpha_0\langle \nabla v, \nu^T\rangle &+ \alpha_1\langle \nu, \nu^T\rangle + \alpha_0(v\Phi - d_\Gamma\langle N, \nabla v\rangle)\langle \bar{\nabla}\Phi, \nu^T\rangle \\ &+ \alpha_0\Phi\big(\langle \nabla v, \nu^T\rangle\Phi - v\Phi\langle N, \bar{\nabla}_N\nu\rangle - \langle \nu, \nu^T\rangle\langle N, \nabla v\rangle \\ &- d_\Gamma(\langle N, \bar{\nabla}_\nu\nabla v\rangle - \langle N, \nu\rangle\langle N, \bar{\nabla}_N\nabla v\rangle)\big).\end{aligned}$$

Since $d_\Gamma(y) = 0$, we have

$$\begin{aligned}\langle \nabla^\Sigma \zeta + AT, \nu^T\rangle = \alpha_0\langle \nabla v, \nu^T\rangle &+ \alpha_1\langle \nu, \nu^T\rangle + \alpha_0 v\Phi\langle \bar{\nabla}\Phi, \nu^T\rangle \\ &+ \alpha_0\Phi\big(\langle \nabla v, \nu^T\rangle\Phi - v\Phi\langle N, \bar{\nabla}_N\nu\rangle - \langle \nu, \nu^T\rangle\langle N, \nabla v\rangle\big).\end{aligned}$$

Rearranging terms we obtain

$$\begin{aligned}\langle \nabla^\Sigma \zeta + AT, \nu^T\rangle = \alpha_1(1 - \langle N, \nu\rangle^2) &+ \alpha_0\langle \nabla v, \nu^T\rangle(1+\Phi^2) + \alpha_0 v\Phi\langle \bar{\nabla}\Phi, \nu^T\rangle \\ &- \alpha_0\Phi\big(v\Phi\langle N, \bar{\nabla}_N\nu\rangle + (1 - \langle N, \nu\rangle^2)\langle N, \nabla v\rangle\big).\end{aligned}$$

Therefore there exists a constant $C = C(\Phi, K, \Omega, R)$ such that

$$|\langle \nabla^\Sigma \zeta + AT, \nu^T\rangle| \le \alpha_1(1-\Phi^2) + C\alpha_0. \tag{31}$$

Since $d_\Gamma(y) = 0$, it holds that

$$|\langle A_K Z, N\rangle| = \|A_K\|\|Z\| \le \|A_K\|(\eta + \|X\|) \le 4R^2\alpha_0\|A_K\|(1+\Phi),$$

from which we conclude that

$$\left|\left\langle \bar{\nabla}\langle N_\tau, \nu\rangle, \frac{dy_\tau}{d\tau}\Big|_{\tau=0}\right\rangle\right| \le \alpha_1(1-\Phi^2) + \widetilde{C}\alpha_0, \tag{32}$$

for some constant $\widetilde{C}(C_\Phi, K, \Omega, R) > 0$.

Now we suppose that $x \in \overline{\partial\Omega' \cap \Omega}$. In this case we have $v(x) = 0$. Then $\eta = \alpha_1 d_\Gamma$ and

$$X = -\alpha_0\Phi d_\Gamma\nabla v$$

at x. Thus

$$\zeta = \eta + \langle X, N\rangle = \alpha_1 d_\Gamma + 2\alpha_0 \Phi d d_\Gamma \langle \nabla d, N\rangle.$$

Moreover, we have

$$W(x) \le \frac{C}{d_\Gamma(x)}$$

(see Remark 6). It follows that

$$\zeta W \le C(\alpha_1 + 2\alpha_0 \Phi d\langle \nabla d, N\rangle) \le C(\alpha_1 + 4R\alpha_0\Phi). \tag{33}$$

We conclude that

$$W(x) \le C(C_\Phi, K, \Omega, R). \tag{34}$$

Now we consider the case when $x \in \Omega \cap \Omega'$. In this case we have

$$\begin{aligned}\Delta_\Sigma \zeta = \alpha_0 \Delta_\Sigma v + \alpha_1 \Delta_\Sigma d_\Gamma + \alpha_0 \Delta_\Sigma \Phi (v\langle N, \nu\rangle - d_\Gamma \langle N, \nabla v\rangle)\\ + \alpha_0 \Phi\big(\Delta_\Sigma v \langle N, \nu\rangle + v\Delta_\Sigma \langle N, \nu\rangle + 2\langle \nabla^\Sigma v, \nabla^\Sigma \langle N, \nu\rangle\rangle - \Delta_\Sigma d_\Gamma \langle N, \nabla v\rangle\\ - d_\Gamma \Delta_\Sigma \langle N, \nabla v\rangle - 2\langle \nabla^\Sigma d_\Gamma, \nabla^\Sigma \langle N, \nabla v\rangle\rangle\big) + 2\alpha_0 \big\langle \nabla^\Sigma \Phi, \nabla^\Sigma v \langle N, \nu\rangle\\ + v\nabla^\Sigma \langle N, \nu\rangle - \nabla^\Sigma d_\Gamma \langle N, \nabla v\rangle - d_\Gamma \nabla^\Sigma \langle N, \nabla v\rangle\big\rangle.\end{aligned}$$

Notice that given an arbitrary vector field U along Σ, we have

$$\langle \nabla^\Sigma \langle N, U\rangle, V\rangle = -\langle AU^T, V\rangle + \langle N, \overline{\nabla}_V U\rangle$$

for any $V \in \Gamma(T\Sigma)$. Here, U^T denotes the tangential component of U. Hence using Codazzi's equation we obtain

$$\Delta_\Sigma \langle N, U\rangle \le \langle \overline{\nabla}(nH), U^T\rangle + \mathrm{Ric}_M(U^T, N) + C\|A\|,$$

for a constant C depending on $\overline{\nabla} U$ and $\overline{\nabla}^2 U$. Hence using (3) we conclude that

$$\Delta_\Sigma \langle N, U\rangle \le \langle \overline{\nabla}\Psi, U^T\rangle + \widetilde{C}\|A\|, \tag{35}$$

where $\widetilde{C}$ is a positive constant depending on $\overline{\nabla} U$, $\overline{\nabla}^2 U$ and Ric_M.

We also have

$$\begin{aligned}\Delta_\Sigma d_\Gamma &= \Delta_P d_\Gamma + \gamma \langle \overline{\nabla}_Y \overline{\nabla} d, Y\rangle - \langle \overline{\nabla}_N \overline{\nabla} d_\Gamma, N\rangle + nH\langle \overline{\nabla} d_\Gamma, N\rangle\\ &\le C_0 \Psi + C_1,\end{aligned}$$

where C_0 and C_1 are positive constants depending on the second fundamental form of the Killing cylinders over the equidistant sets $d_\Gamma = \delta$ for small values of δ. Similar estimates also hold for $\Delta_\Sigma d$ and then for $\Delta_\Sigma v$.

We conclude that

$$\Delta_\Sigma \zeta \ge -\widetilde{C}_0 - \widetilde{C}_1 \|A\|, \tag{36}$$

where $\widetilde{C}_0$ and $\widetilde{C}_1$ are positive constants depending only on Ω, K, Ric_M, and $|\Phi| + \|\nabla\Phi\| + \|\nabla^2\Phi\|$.

Now proceeding similarly as in the proof of Proposition 5, we observe that Lemma 1(iii) and the comparison principle yield

$$\begin{aligned} H_0(\hat{y}_\tau) - H_0(y) &\geq \frac{dH_\tau}{d\tau}\bigg|_{\tau=0} \tau + o(\tau) \\ &= (\Delta_\Sigma \zeta + \|A\|^2\zeta + \mathrm{Ric}_M(N,N)\zeta)\tau + \tau\langle\bar{\nabla}\Psi, T\rangle + o(\tau). \end{aligned}$$

However,

$$H_0(\hat{y}_\tau) - H_0(y) = \langle \bar{\nabla}\Psi|_y, \xi'(0)\rangle\tau + o(\tau).$$

Using (18) we have

$$\langle\bar{\nabla}\Psi, \xi'(0)\rangle = \langle\bar{\nabla}\Psi, Z - \zeta WY\rangle = \langle\bar{\nabla}\Psi, Z\rangle - \zeta W\frac{\partial\Psi}{\partial s}.$$

We conclude that

$$-\zeta W\frac{\partial\Psi}{\partial s}\tau + \zeta\langle\bar{\nabla}\Psi, N\rangle\tau + o(\tau) \geq (\Delta_\Sigma\zeta + \|A\|^2\zeta + \mathrm{Ric}_M(N,N)\zeta)\tau + o(\tau).$$

Suppose that

$$W > \frac{C + \|\bar{\nabla}\Psi\|}{\beta}, \tag{37}$$

for a constant $C > 0$ as in (22). Hence we have

$$(\Delta_\Sigma\zeta + |A|^2\zeta + \mathrm{Ric}_M(N,N)\zeta)\tau + C\zeta\tau \leq o(\tau).$$

We conclude that

$$-C_0 - C_1|A| + C_2\|A\|^2 + C \leq \frac{o(\tau)}{\tau},$$

a contradiction. It follows from this contradiction that

$$W(x) \leq \frac{C + \|\bar{\nabla}\Psi\|}{\beta}. \tag{38}$$

Now, proceeding as in the end of the proof of Proposition 5, we use the estimate for $W(x)$ in each one of the three cases for obtaining a estimate for W in Ω'. $\square$

4. Proof of Theorem 1

We use the classical continuity method for proving Theorem 1. For details, we refer the reader to [Gerhardt 1976] and [Ladyzhenskaya and Uraltseva 1964]. For any

$\tau \in [0, 1]$, we consider the Neumann boundary problem $\mathcal{N}_\tau$ of finding $u \in C^{3,\alpha}(\bar{\Omega})$ such that

$$\mathscr{F}[\tau, x, u, \nabla u, \nabla^2 u] = 0, \tag{39}$$

$$\left\langle \frac{\nabla u}{W}, \nu \right\rangle + \tau \Phi = 0, \tag{40}$$

where $\mathscr{F}$ is the quasilinear elliptic operator defined by

$$\mathscr{F}[x, u, \nabla u, \nabla^2 u] = \operatorname{div}\left(\frac{\nabla u}{W}\right) - \left\langle \frac{\nabla \gamma}{2\gamma}, \frac{\nabla u}{W} \right\rangle - \tau \Psi. \tag{41}$$

Since the coefficients of the first and second order terms do not depend on u, it follows that

$$\frac{\partial \mathscr{F}}{\partial u} = -\tau \frac{\partial \Psi}{\partial u} \le -\tau\beta < 0. \tag{42}$$

We define $\mathcal{I} \subset [0, 1]$ as the subset of values of $\tau \in [0, 1]$ for which the Neumann boundary problem $\mathcal{N}_\tau$ has a solution. Since $u = 0$ is a solution for $\mathcal{N}_0$, it follows that $\mathcal{I} \neq \varnothing$. Moreover, the implicit function theorem (see [Gilbarg and Trudinger 2001, Chapter 17]) implies that $\mathcal{I}$ is open in view of (42). Finally, the height and gradient a priori estimates we obtained in Sections 2 and 3 are independent of $\tau \in [0, 1]$. This implies that (3) is uniformly elliptic. Moreover, we may ensure the existence of some $\alpha_0 \in (0, 1)$ for which there exists a constant $C > 0$ independent of τ such that

$$|u_\tau|_{1,\alpha_0,\bar{\Omega}} \le C.$$

Redefine $\alpha = \alpha_0$. Thus, this fact, Schauder elliptic estimates and the compactness of $C^{3,\alpha_0}(\bar{\Omega})$ in $C^3(\bar{\Omega})$ imply that $\mathcal{I}$ is closed. It follows that $\mathcal{I} = [0, 1]$.

The uniqueness follows from the comparison principle for elliptic PDEs. We point out that a more general uniqueness statement — comparing a nonparametric solution with a general hypersurface with the same mean curvature and contact angle at corresponding points — is also valid. It is a consequence of a flux formula coming from the existence of a Killing vector field in M. We refer the reader to [Dajczer et al. 2008] for further details.

This finishes the proof of the Theorem 1.

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