

## On a Problem Connected with Navier-stokes Equations in Non Cylindrical Domains

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**Abstract:** In this study we showed the existence of weak solutions of equations that represent flows of a non-homogeneous viscous incompressible fluids in a non cylindrical domain in  $\mathbb{R}^3$ . The classical Navier-stokes equation is a particular case of the equations here considered.

**Key words:** Non Homogeneous Fluids, Navier-stokes, Non Cylindrical Domains

### INTRODUCTION

Let  $T > 0$  be a real number and  $\{\Omega_t\}_{0 \leq t \leq T}$  a family of bounded open subsets of  $\mathbb{R}^n$  with boundary  $\partial \Omega_t = \Gamma_t$ . Let us consider the non cylindrical domain  $\hat{Q} = \bigcup_{0 \leq t \leq T} (\Omega_t \times \{t\})$  whose lateral boundary  $\hat{\Sigma} = \bigcup_{0 \leq t \leq T} (\Gamma_t \times \{t\})$  is assumed to be regular. Consider the flows of viscous, incompressible and nonhomogeneous fluids in  $\hat{Q}$ . The non homogeneity of the fluids means that the density is a non constant function  $\rho = \rho(x,t), (x,t) \in (\Omega_t \times \{t\})$ . These flows are governed by the following system of Navier-stokes types equations.

$$\frac{\partial}{\partial t}(\rho u) + \sum_{j=1}^n \frac{\partial}{\partial x_j} (u_j \rho u) - \mu \Delta u = \rho f - \nabla p \text{ in } \hat{Q} \quad (1)$$

$$\operatorname{div} u = 0 \text{ in } \hat{Q} \quad (2)$$

$$\frac{\partial p}{\partial t} + (u \cdot \nabla) p = 0 \text{ in } \hat{Q} \quad (3)$$

$$u = 0 \text{ on } \hat{\Sigma} \quad (4)$$

$$u(x,0) = u_0(x) \text{ in } \Omega_0 \quad (5)$$

$$p(x,0) = p_0(x) \text{ in } \Omega_0 \quad (6)$$

where,  $u(x,t) = (u_1(x,t), \dots, u_n(x,t))$  is the velocity,  $u = (u_1, \dots, u_n)$ ,  $\nabla u = (\nabla u_1, \dots, \nabla u_n)$ ,  $(u \cdot \nabla) = u_j \frac{\partial}{\partial x_j}$  (with the

summation convention),  $\mu$  is a positive constants,  $p(x,t)$  is the pressure and it is a real valued function,  $\rho(x, t)$  density of the fluid at point  $(x,t) \in (\Omega_t \times \{t\})$  and  $f = f(x,t)$  is the external force vector field. In this study we will consider weak solutions of the system(1-6) on certain non cylindrical domains under standard

hypothesis on  $f$  and  $u_0$  in the dimensions  $n=3$ , we also assume that

$$0 < \rho_0(x) < \infty \quad (7)$$

To define these domains let us consider  $K(t)$  a matrix valued function

$$[0,T] \rightarrow \mathbb{R}^{n^2}$$

$$t \mapsto K(t),$$

and  $\Omega \subset \mathbb{R}^n$  a bounded domain with smooth boundary and containing the origin. Let us define the family of sets

$$\Omega_t = \{x = K(t)y ; y \in \Omega\}.$$

and the respective non cylindrical domains  $\hat{Q} = \bigcup_{0 \leq t \leq T} (\Omega_t \times \{t\})$ . Global existence results for

such nonhomogeneous, incompressible Navier-stokes equation were first obtained by Kazhikov [1], Kazhikov and Smagulov [2], Antonzev and Kajikov [3], Antonzev, *et al.* [4] and Lions [5, 6] – in the case  $0 < \rho_0(x) < \infty$ , that is  $\rho_0$  has a lower bound positive and in context of cylindrical domains. These results were extended by various authors and in particular by Simon [7-9] allowing  $\rho_0$  to vanish. We also observe that Kim [10] has studied problem (1-6) for cylindrical domains under more regularity assumptions on the data  $(u_0, f)$ , thus obtaining considerably more regular solutions. In the bi-dimensional case and still for cylindrical domains the existence and uniqueness of classical solutions, assuming sufficiently regular initial data, were obtained by Ladyshenkaya [11]. The first result for the system (1-6) in non cylindrical domains were obtained by Limaco [12]. Here we are considering the same equations as described by Limaco [12], however, in more general non cylindrical domains. By a

suitable change of variable, we transform the non cylindrical problem (1-6) into a problem defined in the cylinder  $Q = \times (0,T)$ . In  $Q$  we follow the ideas of Lions [5, 6].

**NOTATION AND MAIN RESULTS**

To show our main result we assume the following hypothesis in  $K(t)$ .

(H1)  $K(t) = k(t) M$ , where  $k: [0,T] \rightarrow \mathbb{R}$ ,  $k \in C^1([0,T])$ ,  $k(t) > 0$  and  $M$  is an invertible  $n$  by  $n$  matrix whose entries are real constants.

Consider the notation

$$K(t) = ( \eta_{ij}(t) ) \text{ and } K^{-1}(t) = ( \eta_{ij}^{-1}(t) ).$$

By  $C$  we represent several positive constants. In order to transform the non cylindrical problem (1-6) into a new problem in the cylindrical domain  $Q$ , we introduce the functions

$$\begin{aligned} u(x,t) &= v(K^{-1}(t)x,t), \quad f(x,t) = g(K^{-1}(t)x,t) \\ p(x,t) &= q(K^{-1}(t)x,t), \quad (x,t) = (K^{-1}(t)x,t) \\ u_0(x) &= v_0(K^{-1}(0)x), \quad \phi_0(x) = \phi_0(K^{-1}(0)x). \end{aligned}$$

We have the following identity

$$x_r = \eta_{rj} y_j, \quad y_l = \eta_{lr}^{-1} x_r \text{ and } \frac{\partial y_l}{\partial t} = \beta'_{lr} x_r, \text{ or}$$

$$\frac{\partial y_l}{\partial t} = \beta'_{lr} \alpha_{rj} y_j$$

Since  $\frac{y_l}{x_j} = \eta_{lj}$ , we obtain

$$\frac{\partial u_i}{\partial x_k}(x,t) = \beta_{jk} \frac{\partial v_i}{\partial y_j}(y,t) \tag{8}$$

Also

$$\frac{\partial^2 u_i}{\partial x_k^2}(x,t) = \beta_{jk} \beta_{rk} \frac{\partial^2 v_i}{\partial y_r \partial y_j}(y,t)$$

Consequently

$$\Delta u_i(x,t) = \beta_{jk} \beta_{rk} \frac{\partial^2 v_i}{\partial y_j \partial y_r}(y,t).$$

We have also

$$\frac{\partial p}{\partial x_i} = \frac{\partial q}{\partial y_j} \beta_{ji}$$

$$\frac{\partial p}{\partial t} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial y_j} \beta'_{ji} x_i$$

and

$$\frac{\partial(\rho u)}{\partial t} = \frac{\partial(\phi v)}{\partial t}(y,t) + \frac{\partial(\phi v)}{\partial y_j} \beta'_{ji} \alpha_{rk} y_k.$$

**Remark 1:** If we have  $\phi_0(x) = \phi_0 = \text{constant}$  say  $\phi_0 = 1$ , then  $\eta = 1$  satisfies (3,4 and 6) and the problem reduces to the classical Navier-stokes situations in the cylindrical domain.

**Remark 2:** since  $\text{div } u = 0$ , (3) is equivalent to  $\frac{\partial p}{\partial t} +$

$$\text{div}(M^{-1}v^t) = 0.$$

Then, from (1-6) it follows that

$$\begin{aligned} \frac{\partial(\phi u)}{\partial t} - \mu \frac{\partial}{\partial y_j} \left( a_{jr}(t) \frac{\partial v}{\partial y_j} \right) + \frac{\partial}{\partial y_j} (v_j \phi v) \beta_{ji} + \frac{\partial(\phi v)}{\partial y_j} \beta'_{ji} \alpha_{rk} y_k \\ = \phi g - \left( \frac{\partial q}{\partial y_j} \beta_{jk}, \dots, \frac{\partial q}{\partial y_j} \beta_{jn} \right)^t \text{ in } Q \end{aligned} \tag{9}$$

$$\text{div}(M^{-1}v^t) = 0 \text{ in } Q \tag{10}$$

$$\frac{\partial \phi}{\partial t} + \frac{\partial(\phi v_i)}{\partial y_j} \beta_{ji} + \frac{\partial \phi}{\partial y_j} \beta'_{ji} \alpha_{rk} y_k = 0 \text{ in } Q \tag{11}$$

$$v = 0 \text{ on } \Sigma \tag{12}$$

$$v(y,0) = v_0(y) \text{ on } \bar{Q} \tag{13}$$

$$\phi(y,0) = \phi_0(y) \text{ on } \bar{Q}. \tag{14}$$

Here  $a_{jr} = \eta_{jk} \eta_{rk}$  and  $v^t$  is the transposed of the row vector  $v = (v_1, \dots, v_n)$  and  $Q = \times (0,T)$ ,  $\Sigma = \times (0,T)$ .

**Remarks 3:** It follows of (8) that

$$\frac{\partial u_i}{\partial x_k}(x,t) = \sum_j \beta_{ji} \frac{\partial v_i}{\partial y_j}(y,t), \quad i \text{ fixed.}$$

$$\text{Therefore, } \text{div } u(x,t) = \sum_{ji} \frac{\partial v_i}{\partial y_j}(y,t).$$

On the other hand  $K(t) = k(t)M$ . Then  $K^{-1}(t) = \frac{1}{k(t)} M^{-1}$ .

Therefore

$$\eta_{ij} = \frac{1}{k} \eta_{ij} \text{ where, } M^{-1} = ( \eta_{ij} ). \text{ Thus,}$$

$$\text{div } u(x,t) = \frac{1}{k(t)} \text{div}(M^{-1} v^t(y,t))$$

and by (2), (10) follows.

**Remark 4:** Let A(t) be the operator

$$A(t)v = -\frac{\partial}{\partial y_r} \left( a_{jr}(t) \frac{\partial v}{\partial y_j} \right), \quad v \in (H_0^1(\Omega))^{n^2} \quad (15)$$

We showed in the Lemma 1 that A(t) is uniformly elliptic in [0, T].

To state the main result we introduce some space. Let

$$V_s(\Omega) = \{ v \in (D(\Omega))^{n^2}; \operatorname{div} v = 0 \}$$

and  $V_s(\bar{\Omega})$  be the closure of  $V_s(\Omega)$  in the space  $(H^s(\bar{\Omega}))^{n^2}$  where, s is a nonnegative real number. We use the particular notation

$$V_1(\bar{\Omega}) = V(\bar{\Omega}) \text{ and } V_0(\bar{\Omega}) = H(\bar{\Omega})$$

The inner product of this spaces are denoted, respectively by  $(u, z)_{H(\Omega)}$  and  $((u, z))_{V(\Omega)}$ . Then for  $u = (u_1, \dots, u_n)$  and  $z = (z_1, \dots, z_n)$  we have

$$(u, z)_{H(\Omega)} = \int_{\Omega} u_i(x) z_i(x) dx,$$

$$((u, z))_{V(\Omega)} = \int_{\Omega} \frac{\partial u_i}{\partial x_j}(x) \frac{\partial z_i}{\partial x_j}(x) dx.$$

Note that  $V_s(\bar{\Omega})$  is continuously embedded in  $(H_0^1(\Omega))^{n^2}$  for  $s \geq n/2$  and

$$V(\bar{\Omega}) = \{ u \in (H_0^1(\Omega))^{n^2}; \operatorname{div} u = 0 \}.$$

For these results by Lions [13]. In similar way we introduce the space  $V_s(\bar{\Omega})$ . In this case  $V_s(\bar{\Omega})$  has the form

$$V_s(\bar{\Omega}) = \{ \phi \in (D(\bar{\Omega}))^{n^2}; \operatorname{div}(M^{-1}\phi) = 0 \}.$$

We consider the particular notations  $V_1(\bar{\Omega}) = V$ ,  $V_0(\bar{\Omega}) = H$  and  $(v, w)_H = (v, w)$ ,  $((u, v))_V = ((v, w))$ ,  $|v|_H = |v|$  and  $\|v\|_V = \|v\|$ .

The spaces  $L^p(0, T, V_s(\bar{\Omega}))$  are defined in Appendix. Following the ideas of Lions [5,6], we define the notion of weak solutions (for  $n=3$ ) of problem (1-6) and (9-14).

**Non cylindrical Case:** Find  $u(x,t)$  and  $\phi(x,t)$  such that

$$u \in L^2(0, T, V(\bar{\Omega})) \quad L^2(0, T, H(\bar{\Omega})), \quad \phi \in L^\infty(\hat{Q})$$

$$-\int_Q \rho u \frac{\partial \phi}{\partial t} dx dt - \mu \int_Q \nabla u \nabla \phi dx dt - \sum_{i,j=1}^3 \int_Q u_i \rho u_i \frac{\partial \phi}{\partial x_j} dx dt = (16)$$

$$-\int_Q \rho f \phi dx dt + \int_{\Omega_0} \rho_0 u_0 \phi(0) dx$$

$$-\int_Q \rho \frac{\partial \xi}{\partial t} dx dt - \sum_{i=1}^3 \int_Q \rho u_i \frac{\partial \xi}{\partial x_i} dx dt = \int_{\Omega_0} \rho_0 \xi(0) dx \quad (17)$$

for all  $\phi, \xi \in [C^1(\bar{Q})]^3$  with compact support contained in  $\hat{Q} \cup \{0\} \times \{0\}$  and  $\operatorname{div} \phi = 0$ .

**Cylindrical Case:** Find  $u$  and  $\phi$  such that  $u \in L^2(0, T, V)$ ,  $\phi \in L^\infty(0, T, H)$ ,  $\phi \in L^\infty(\bar{\Omega} \times (0, T))$

$$-\int_Q \rho v \frac{\partial \psi}{\partial t} dy dt + \mu \int_Q a_{\mu}(t) \frac{\partial v}{\partial y_j} \frac{\partial \psi}{\partial y_i} dy dt + \int_Q \beta_{\mu} v_i \rho v \frac{\partial \psi}{\partial y_j} dy dt +$$

$$\int_Q \beta'_{\mu} \alpha_{\mu} \rho v \frac{\partial(\psi y_k)}{\partial y_j} dy dt = \int_Q \rho g \psi dy dt - \int_{\Omega} \phi_0 v_0 \psi(0) dy$$

for all  $\psi \in [C^1(\bar{\Omega} \times [0, T])]^3$  with compact support contained in  $\bar{\Omega} \times (0, T)$  and  $\operatorname{div}(M^{-1}\psi) = 0$ .

$$\frac{\partial \phi}{\partial t} + \frac{\partial(\phi v_i)}{\partial y_j} \beta_{ji} + \frac{\partial \phi}{\partial y_j} \beta'_{jl} \alpha_{lk} y_k = 0 \quad \text{in } Q \quad (19)$$

$$(x, 0) = \phi_0(x) \text{ in } \bar{\Omega} \quad (20)$$

Next we shall state the main results of this study. Let  $\hat{Q}$  and  $\hat{\Sigma}$  be as in the section and  $n = 3$ . We have

**Theorem 1:** Assume that hypothesis (H1) is satisfied and that  $\phi_0$  satisfies (7).

If  $f \in L^2(0, T, H(\bar{\Omega}))$ ,  $\phi_0 \in L^\infty(\bar{\Omega})$  and  $u_0 \in H(\bar{\Omega})$ , then there exists a weak solution of the problem (1-6).

The theorem 1 is consequence of the following two results:

**Theorem 2:** If  $g \in L^2(0, T, H)$ ,  $v_0 \in H$  and  $\phi_0 \in L^\infty(\bar{\Omega})$  then there exists a weak solution of the problem (9-14).

**Theorem 3:** Problems (1-6) and (9-14) are equivalent.

**Remark 5:** Uniqueness is an open question. It is still open in the particular case  $\phi_0(x) = \phi_0$  with  $\phi_0$  a constant and in context of cylindrical or noncylindrical domain.

## PROOF OF RESULTS

**Proof of Theorem 2:** We use Semi-Galerkin method. We consider as approximation of (9-14) which is of the Galerkin's type in  $v$  and where in (11) we replace  $v$  by its approximation (hence the terminology of Semi-Galerkin). For this, we may consider a family of internal approximation  $V_m \subset V$  such that  $V_m$  is a subspace of  $V$  dimension  $m$ ,

$\forall v \in V$ , there exist a sequence  $v_m \in V$  such that  $v_m \rightarrow v$  in  $V$  as  $m \rightarrow \infty$ .

We also may assume that all components of

functions  $v \in V_m$  belong to  $C^1(\bar{\Omega})$ . Because this, we can consider a basis  $\{w^i\}$  of  $V$  such that  $w^i \in (C^1(\bar{\Omega}))^3$ . Note that the embedding  $C^1(\bar{\Omega}) \subset V$  is dense, continuous and  $C^1(\bar{\Omega}) \subset V$  is separable. Then there exists  $\{w^i\}$ .

Let  $V_m = [w^1, \dots, w^m]$ ,  $v_m = \sum_{i=1}^m g_m(t)w^i$  with  $v_m \in C^1([0, T_m], V_m)$  and

$v_m \in C^1([0, T_m], C^1(\bar{\Omega}))$  satisfying

$$\left(\frac{\partial}{\partial t}(\varphi_m v_m), w^k\right) + \mu \left(a_j(t) \frac{\partial v_m}{\partial y_j}, \frac{\partial w^k}{\partial y_1}\right) + \left(\frac{\partial}{\partial y_j}(\varphi_m v_m) \beta_{ji}, w^k\right) + \quad (21)$$

$$\left(\frac{\partial(\varphi_m v_m)}{\partial y_j} \beta'_{ji} \alpha_{ik} y_k, w^k\right) = (\varphi_m g, w^k), \quad k=1, \dots, m$$

$$\frac{\partial \varphi_m}{\partial t} + \frac{\partial(\varphi_m v_m)}{\partial y_j} \beta_{ji} + \frac{\partial \varphi_m}{\partial y_j} \beta'_{ji} \alpha_{ik} y_k = 0 \quad (22)$$

$$v_m(0) = v_{0m} \quad v_0 \text{ in } V \quad (23)$$

$$(y, 0) = \varphi_m(y) \quad \varphi(y) \text{ in } L^q(\cdot), \quad (24)$$

where,  $1 < q < +\infty$ ,  $\varphi_m < \varphi$

**Local Existence of  $v_m$  and  $\varphi_m$ :** Assuming  $u_m$  to be known, we can express the solution  $v_m(t, y)$  of (22), (24) as follows:

$$v_m(t, y) = \varphi_m(x_m(0, t, y))$$

where,

$x_m(\delta, t, y) = (x_m^1(\delta, t, y), x_m^2(\delta, t, y), x_m^3(\delta, t, y))$  is the solution of

$$\begin{cases} \frac{dx_m^1}{ds}(\delta, t, y) = \beta_{11} v_m(x_m(\delta, t, y), \delta) + \beta'_{11} \alpha_{1k} x_m^k(\delta, t, y) \\ \frac{dx_m^2}{ds}(\delta, t, y) = \beta_{21} v_m(x_m(\delta, t, y), \delta) + \beta'_{21} \alpha_{1k} x_m^k(\delta, t, y) \\ \frac{dx_m^3}{ds}(\delta, t, y) = \beta_{31} v_m(x_m(\delta, t, y), \delta) + \beta'_{31} \alpha_{1k} x_m^k(\delta, t, y) \end{cases}$$

$$x_m(t, y) = y.$$

We multiply (22) by  $v_m w^k$  and we integrate over  $\Omega$  and we add the result to (21), we obtain

$$\int_{\Omega} \left( \varphi_m \frac{\partial v_m}{\partial t} + v_m \varphi_m \frac{\partial v_m}{\partial y_j} \beta_{ji} + \varphi_m \frac{\partial v_m}{\partial y_j} \beta'_{ji} \alpha_{ik} y_k \right) w^k dx + \mu \left( a_j \frac{\partial v_m}{\partial y_j}, \frac{\partial w^k}{\partial y_1} \right) = 0 \quad (25)$$

Since  $v_m(t) = \sum_{r=1}^m g_m(t)w^r$  then (25) has a form

$$Q_{rk}(\varphi_m) \frac{dg_m}{dt} + Pr(g_{1m}, \dots, g_{mm}) = 0 \quad (26)$$

where,  $Q_{rk}(\varphi_m) = \int_{\Omega} \varphi_m(y, t) w^r w^k dy$ ,  $\varphi_m(y, t) = \varphi_m(x_m(0, t, y))$  and  $0 < \varphi_m(y, t) < \varphi$ .

We have that  $\sqrt{\varphi_m} w^i$  are linearly independent. Indeed, if  $\sum_{i=1}^m \lambda_i \sqrt{\varphi_m} w^i = 0$  then  $\sum_{i=1}^m \lambda_i w^i = 0$  and hence  $\lambda_i = 0$ ,  $i = 1, \dots, m$ . Thus  $Q_{rk}(\varphi_m) = (\sqrt{\varphi_m} w^r, \sqrt{\varphi_m} w^k)$  is non singular. Then (26) is equivalent to

$$\frac{dg_m}{dt} = -Q_{rk}^{-1}(\varphi_m) Pr(g_{1m}, \dots, g_{mm}). \quad (27)$$

Where,  $Pr$ ,  $Q_{rk}^{-1}$  are differentiable continuous. Then the system of nonlinear differential equation (27) has a local solution, i.e.,  $v_m$  and  $\varphi_m$  are solutions of (21-24). The standard a priori estimates, which follow prove the global existence of the solution  $v_m, \varphi_m$ . The extension of the solution to the whole interval  $[0, T]$  is consequence of the estimates which follows.

**Estimate I:** If we take  $w^k = 2v_m$  in (21) and if we multiply (22) by  $-|v_m|_{R^n}^2$  (where  $|\cdot|_{R^n}$  denote the usual the norm in  $R^n$ ,  $n = 3$ ), we obtain after adding up

$$\begin{aligned} & 2 \left( \frac{\partial(\varphi_m v_m)}{\partial t}, v_m \right) - \left( \frac{\partial \varphi_m}{\partial t}, |v_m|_{R^n}^2 \right) + \mu \left| \beta_{ji} \frac{\partial v_m}{\partial y_j} \right|_{L^2(\Omega)}^2 + \\ & 2 \left( \frac{\partial}{\partial y_j}(\varphi_m v_m) \beta_{ji}, v_m \right) - \left( \frac{\partial(\varphi_m v_m)}{\partial y_j} \beta_{ji}, |v_m|_{R^n}^2 \right) - \\ & \left( \frac{\partial(\varphi_m)}{\partial y_j} \beta'_{ji} \alpha_{ik} y_k, |v_m|_{R^n}^2 \right) + 2 \left( \frac{\partial(\varphi_m v_m)}{\partial y_j} \beta'_{ji} \alpha_{ik} y_k, v_m \right) = 2(\varphi_m g, v_m). \end{aligned} \quad (28)$$

Direct calculation shows that:

$$2 \left( \frac{\partial(\varphi_m v_m)}{\partial t}, v_m \right) - \left( \frac{\partial \varphi_m}{\partial t}, |v_m|_{R^n}^2 \right) = \frac{d}{dt} \int_{\Omega} \varphi_m |v_m|_{R^n}^2 dy \quad (29)$$

$$\begin{aligned} & 2 \left( \frac{\partial}{\partial y_j}(\varphi_m v_m) \beta_{ji}, v_m \right) - \left( \frac{\partial(\varphi_m v_m)}{\partial y_j} \beta_{ji}, |v_m|_{R^n}^2 \right) = \\ & \left( \frac{\partial(\varphi_m v_m)}{\partial y_j} \beta_{ji}, |v_m|_{R^n}^2 \right) + 2 \left( v_m \varphi_m \frac{\partial v_m}{\partial y_j} \beta_{ji}, v_m \right) = \\ & \left( \frac{\partial(\varphi_m v_m)}{\partial y_j} \beta_{ji}, |v_m|_{R^n}^2 \right) + \left( v_m \varphi_m \beta_{ji} \frac{\partial}{\partial y_j} |v_m|_{R^n}^2 \right) = \\ & \int_{\Omega} \frac{\partial}{\partial y_j} (v_m \varphi_m |v_m|^2) \beta_{ji} dy = \int_{\Omega} (v_m \beta_{ji} \varphi_m |v_m|^2) \eta_j dy = 0 \end{aligned} \quad (30)$$

by Gauss Theorem and by virtue of  $\text{div } v_m = 0$  (we denote of  $n = (n_1, \dots, n_n)$  the unit outward normal vector to  $\Omega$ ). Also,

$$\begin{aligned} I_1 = & - \left( \frac{\partial \Phi_m}{\partial y_j} \beta'_{ji} \alpha_{ik} y_k, |v_m|_{R^3}^2 \right) + 2 \left( \frac{\partial(\Phi_m v_m)}{\partial y_j} \beta'_{ji} \alpha_{ik} y_k, v_m \right) = \\ & - \left( \frac{\partial \Phi_m}{\partial y_j} \beta'_{ji} \alpha_{ik} y_k, |v_m|_{R^3}^2 \right) + 2 \left( \frac{\partial \Phi_m}{\partial y_j} \beta'_{ji} \alpha_{ik} y_k, |v_m|_{R^3}^2 \right) + \\ & \left( \Phi_m \beta'_{ji} \alpha_{ik} y_k, \frac{\partial}{\partial y_j} |v_m|_{R^3}^2 \right) = \left( \frac{\partial}{\partial y_j} (\Phi_m |v_m|_{R^3}^2), \beta'_{ji} \alpha_{ik} y_k \right) = \\ & - \int_{\Omega} \Phi_m |v_m|_{R^3}^2 \beta'_{ji} \alpha_{ik} \delta_{kj} dy = - \int_{\Omega} \Phi_m |v_m|_{R^3}^2 \beta'_{ki} \alpha_{ik} dy \end{aligned} \tag{31}$$

where,  $\delta_{kj}$  is the Kronecher symbol.

We observe that, in view of  $\Phi_m(y, t) = \Phi_m(0, t, y)$ , (H1) and (7), we obtain

$$|I_1| \leq C \int_{\Omega} \Phi_m |v_m|^2 dy \tag{32}$$

$$0 < \Phi_m(y, t) < \dots \tag{33}$$

In view of (29-33), we obtain of (28) that

$$\frac{d}{dt} \int_{\Omega} \Phi_m |v_m|_{R^3}^2 dy + C_0 \|v_m\|^2 \leq \int_{\Omega} |g|^2 dy + C \int_{\Omega} \Phi_m |v_m|_{R^3}^2 dy$$

By (33) and Gronwall inequality, we have

$$\alpha |v_m|^2 + C \int_0^T \|v_m(s)\|^2 ds \leq C \tag{34}$$

Using (33) and (34) we obtain that  $v_m$  and  $\Phi_m$  are defined to  $[0, T]$  and

$$(v_m) \text{ is bounded in } L^2(0, T, V) \cap L^{\infty}(0, T, H) \tag{35}$$

**Estimate II:** Considering  $v = w^k \in V_m$  in (21) we have

$$\left( \frac{\partial}{\partial t} (\Phi_m v_m), v \right) = (J_m(t), v) \tag{36}$$

where:

$$(J_m(t), v) = -\mu \left( a_j(t) \frac{\partial v_m}{\partial y_j}, \frac{\partial v}{\partial y_1} \right) - \left( \frac{\partial}{\partial y_j} (v_m \Phi_m v_m) \beta_{ji}, v \right) - \tag{37}$$

$$\left( \frac{\partial(\Phi_m v_m)}{\partial y_j} \beta'_{ji} \alpha_{ik} y_k, v \right) + (\Phi_m g, v)$$

Integrating (37) over  $(t, t + \delta)$  with  $t + \delta = T$  and  $v$  fixed we obtain

$$(\Phi_m(t + \delta) v_m(t + \delta) - \Phi_m(t) v_m(t), v) = \left( \int_t^{t+\delta} J_m(s) ds, v \right) \tag{38}$$

We take now  $v = v_m(t + \delta) - v_m(t)$  in (38). Let us set:

$$X_m = (\Phi_m(t + \delta) v_m(t + \delta) - \Phi_m(t) v_m(t), v_m(t + \delta) - v_m(t)) \tag{39}$$

$$Y_m = (\int_t^{t+\delta} J_m(s) ds, v_m(t + \delta) - v_m(t)) \tag{40}$$

Follows of (38) that

$$X_m + Y_m = \left( \int_t^{t+\delta} J_m(s) ds, v_m(t + \delta) - v_m(t) \right) \tag{41}$$

Since  $\Phi_m$  we have

$$X_m = |v_m(t + \delta) - v_m(t)|^2 \tag{42}$$

Let us now transform  $Y_m$ . It follows from (22) that

$$\begin{aligned} ((\Phi_m(t + \delta) - \Phi_m(t)) v_m(t), v) = & - \int_{\Omega} \left[ \int_t^{t+\delta} \frac{\partial(\Phi_m v_m)}{\partial y_j} \beta_{ji} ds \right] v_m v dy - \int_{\Omega} \left[ \int_t^{t+\delta} \frac{\partial \Phi_m}{\partial y_j} \beta'_{ji} \alpha_{ik} y_k ds \right] v_m v dy \\ = & - \int_{\Omega} \frac{\partial}{\partial y_j} \left[ \int_t^{t+\delta} \Phi_m v_m \beta_{ji} ds \right] v_m v dy - \int_{\Omega} \frac{\partial}{\partial y_j} \left[ \int_t^{t+\delta} \Phi_m \beta'_{ji} \alpha_{ik} ds \right] y_k v_m v dy \\ = & \int_{\Omega} \left[ \int_t^{t+\delta} \Phi_m v_m \beta_{ji} ds \right] \left( \frac{\partial v_m}{\partial y_j} v + v_m \frac{\partial v}{\partial y_j} \right) dy + \\ & + \int_{\Omega} \left[ \int_t^{t+\delta} \Phi_m \beta'_{ji} \alpha_{ik} ds \right] \left( \frac{\partial y_k}{\partial y_j} v_m v + y_k \frac{\partial v_m}{\partial y_j} v + y_k v_m \frac{\partial v}{\partial y_j} \right) dy \end{aligned} \tag{43}$$

Since, in particular, the embedding  $V \subset (L^4(\Omega))^3$  is continuous ( $n = 3$ ),  $0 < \Phi_m < \dots$  and  $i_j, j_j \in C^1([0, T])$  it follows that

$$\begin{aligned} \left| \int_{\Omega} (\Phi_m(t + \delta) - \Phi_m(t)) v_m(t) v dy \right| \leq & \leq C \left( \int_t^{t+\delta} \|v_m(s)\|_{L^4(\Omega)} ds \right) (\|v_m(t)\| \|v\|_{L^4(\Omega)} + \|v_m(t)\|_{L^4(\Omega)} \|v\|) \\ & + C \delta \|v_m(t)\| \|v\| \\ \leq C \left( \int_t^{t+\delta} \|v_m(s)\| ds \right) \|v_m(t)\| \|v\| + C \delta \|v_m(t)\| \|v\| \\ \leq C \sqrt{\delta} \left( \int_t^{t+\delta} \|v_m(s)\|^2 ds \right)^{1/2} \|v_m(t)\| \|v\| + C \delta \|v_m(t)\| \|v\|. \end{aligned} \tag{44}$$

Taking  $v = v_m(t + \delta) - v_m(t)$  in (44) and using (35), we have

$$|Y_m| \leq C \sqrt{\delta} (\|v_m(t)\|^2 + \|v_m(t + \delta)\|^2). \tag{45}$$

We integrate (45) on  $(0, T - \delta)$  and we use again (35), we obtain

$$\int_0^{T-\delta} |Y_m| dt \leq C \sqrt{\delta} \tag{46}$$

We now estimate the term of the right side of (41). We have of (18) that

$$\begin{aligned} |(J_m(t), v)| \leq \mu \left| \left( a_j(t) \frac{\partial v_m}{\partial y_j}, \frac{\partial v}{\partial y_1} \right) \right| + \left| \left( \frac{\partial}{\partial y_j} (v_m \Phi_m v_m), v \right) \right| + \\ \left| \left( \frac{\partial(\Phi_m v_m)}{\partial y_j} \beta'_{ji} \alpha_{ik} y_k, v \right) \right| + |(g, v)| \end{aligned} \tag{47}$$

In analogy of (43,44), we obtain

$$|J_m(t, v)| \leq C |g(t)| \|v\| + C \|v_m\| \|v\| + C \|v_m\|_{(L^q(\Omega))^3} \|v\| \leq C(|g(t)| + \|v_m\| + \|v_m\|^2) \|v\| \quad (48)$$

Then

$$\left| \int_t^{t+\delta} J_m(s) ds, v_m(t+\delta) - v_m(t) \right| \leq C \left( \int_t^{t+\delta} |g(s)| + \|v_m(s)\| + \|v_m(s)\|^2 \right) (\|v_m(t)\| + \|v_m(t+\delta)\|) \quad (49)$$

We integrate (49) on  $(0, T - \delta)$ ; we obtain

$$\int_0^{T-\delta} \left( \int_t^{t+\delta} J_m(s) ds, v_m(t+\delta) - v_m(t) \right) dt \leq C \int_0^{T-\delta} \int_t^{t+\delta} (|g(s)| + \|v_m(s)\| + \|v_m(s)\|^2) (\|v_m(t)\| + \|v_m(t+\delta)\|) ds dt \quad (50)$$

By Fubini Theorem and with the convention that  $v_m = 0$  on  $(-\delta, 0)$ , we have

$$\begin{aligned} & \int_0^{T-\delta} \left( \int_t^{t+\delta} J_m(s) ds, v_m(t+\delta) - v_m(t) \right) dt \leq \\ & C \int_0^T \left( \int_{s-\delta}^s |g(s)| + \|v_m(s)\| + \|v_m(s)\|^2 \right) (\|v_m(t)\| + \|v_m(t+\delta)\|) ds dt \leq \\ & C \int_0^T (|g(s)| + \|v_m(s)\| + \|v_m(s)\|^2) ds \int_{s-\delta}^s (\|v_m(t)\| + \|v_m(t+\delta)\|) dt \leq \\ & C \left( \int_{s-\delta}^s \|v_m(t)\| dt + \int_{s-\delta}^s \|v_m(t+\delta)\| dt \right) \leq \\ & C\sqrt{\delta} \left[ \left( \int_{s-\delta}^s \|v_m(t)\|^2 dt \right)^{1/2} + \left( \int_{s-\delta}^s \|v_m(t+\delta)\|^2 dt \right)^{1/2} \right] \leq C\sqrt{\delta} \end{aligned} \quad (51)$$

as above.

In view of (41,42,46 and 51) it follow that

$$\begin{aligned} \alpha \int_0^{T-\delta} \|v_m(t+\delta) - v_m(t)\|^2 dt & \leq \int_0^{T-\delta} X_m dt \leq \int_0^{T-\delta} |Y_m| dt + \\ \int_0^{T-\delta} \left( \int_t^{t+\delta} J_m(s) ds, v_m(t+\delta) - v_m(t) \right) dt & \leq c\sqrt{\delta} \end{aligned} \quad (52)$$

for all  $\delta > 0, 0 < \delta < T$ .

In virtue of the a priori estimate and by a compactness result (Appendix), Lemma 2, we can extract subsequences, still denoted by  $v_m$  and  $\Phi_m$  such that

$$v_m \rightharpoonup^* u \quad \text{weakly in } L^2(0, T, V) \quad (53)$$

$$v_m \rightharpoonup u \quad \text{weakly-star in } L^2(0, T, H) \quad (54)$$

$$v_m \rightarrow u \quad \text{strongly in } L^p(0, T, (L^q(\Omega))^3) \quad (55)$$

$$\Phi_m \rightharpoonup^* \Phi \quad \text{weak-star in } L^2(Q) \quad (56)$$

where,  $p \in (2, +\infty)$ ,  $q \in (2, 6)$  and  $\frac{1}{q} + \frac{3}{2q} > \frac{3}{4}$

We also know that

$$(y, t) < \dots \quad (57)$$

It follows from (55) and (56) that

$$v_m \rightharpoonup^* v \quad \text{weakly in } L^2(0, T, L^2(\Omega)) \quad (58)$$

and then

$$\beta_{ji} \frac{\partial}{\partial y_j} (u_m, \Phi_m) \rightharpoonup^* \beta_{ji} \frac{\partial}{\partial y_j} (v, \Phi) \quad \text{weak-star in } L^2(0, T, H^1(\Omega)) \quad (59)$$

By analogy

$$\frac{\partial \Phi_m}{\partial y_j} \beta'_{jl} \alpha_{lk} y_k \rightharpoonup^* \frac{\partial \Phi}{\partial y_j} \beta'_{jl} \alpha_{lk} y_k \quad \text{weak-star in } L^2(0, T, H^1(\Omega)) \quad (60)$$

Thus of (59), (60) and (22) we obtain

$$\frac{\partial \Phi_m}{\partial t} \rightharpoonup^* \frac{\partial \Phi}{\partial t} \quad \text{weak-star in } L^2(0, T, H^1(\Omega)) \quad (61)$$

The equation (22) gives in the limit

$$\frac{\partial \Phi}{\partial t} + \frac{\partial(\Phi v_i)}{\partial y_i} \beta_{ji} + \frac{\partial \Phi}{\partial y_j} \beta'_{jl} \alpha_{lk} y_k = 0$$

in the sense of  $L^2(0, T, H^1(\Omega))$ .

It follows from (56) and (61) that in particular  $\Phi_m(y, 0) \rightarrow \Phi(y, 0)$  in  $H^1(\Omega)$

and therefore,  $\Phi(y, 0) = \Phi_0(y)$

Taking  $p = q = 3$  in (55) we deduce that

$$v_m, \Phi_m \rightarrow v, \Phi \quad \text{strongly in } L^3(0, T, L^3(\Omega))$$

Also we deduce of (55), with  $p = q = 2$  and (56) that

$$\Phi_m v_m, \beta'_{jl} \alpha_{lk} \rightarrow \Phi v, \beta'_{jl} \alpha_{lk} \quad \text{weakly in } L^2(0, T, L^2(\Omega)).$$

These convergence and density argument permit to pass to the limit in (21) and (18) is verified. This conclude the proof of Theorem 2.

**Proof of Theorem 3:** Recall that  $x = K(t)y$ ,  $y = K^{-1}(t)x$ ,  $x_r = \sum_j y_j$ ,  $y_l = \sum_r x_r$ . We have established that  $u(x, t) = v(K^{-1}(t)x, t)$  and  $(x, t) = (K^{-1}(t)x, t)$ . Let us consider  $\phi \in (C^1(\bar{\Omega}))^3$  with compact support in  $\hat{Q} \cup \{x \times \{0\}\}$  and  $\text{div} \phi = 0$ . We define.

$$(y, t) = |\det K(t)| \phi(K(t)y, t) \quad \text{or equivalently}$$

$$\phi(x,t) = |\det K^{-1}(t)| (K^{-1}(t)x,t)$$

where,  $\det K$  denotes the determinant of the matrix  $K$ .

It is easy see that  $\psi \in (C^1(\bar{\Omega} \times (0,T)))^3$  with compact support in  $\Omega \times [0,T]$  and  $\text{div}(M^{-1} \psi) = 0$  (Remark 3).

We have the following identity

$$\frac{\partial \phi_i}{\partial t}(x,t) = |\det K^{-1}(t)| \frac{\partial \psi_i}{\partial t}(y,t) + |\det K^{-1}(t)|' \psi_i(y,t) + |\det K^{-1}(t)| \frac{\partial \psi_i}{\partial y_j} \beta_{ji} \alpha_{ik} y_k \quad (62)$$

We have  $\det K(t) = k(t)^n \det M$  and hence

$$|\det K^{-1}(t)|' = -n \frac{k'(t)}{k(t)} |\det K^{-1}(t)| \quad (63)$$

On the other hand,

$$\beta'_{ji} \alpha_{ij} = \text{tr} \left( (K^{-1}(t))' K(t) \right) = \text{tr} \left( -\frac{k'(t)}{k(t)} I \right) = -n \frac{k'(t)}{k(t)} \quad (64)$$

where,  $\text{tr} K$  denote the trace of the matrix  $K$  and  $I$  the identity matrix.

Combining (62), (63) and (64), we get

$$\frac{\partial \phi_i}{\partial t}(x,t) = |\det K^{-1}(t)| \left( \frac{\partial \psi_i}{\partial t} + \beta'_{ji} \alpha_{ik} \frac{\partial (\psi_j y_k)}{\partial y_j} \right)$$

Then

$$-\int_0^T \int_{\Omega} \rho v \left( \frac{\partial \psi}{\partial t} + \beta'_{ji} \alpha_{ik} \frac{\partial (\psi_j y_k)}{\partial y_j} \right) dy dt = -\int_{\hat{Q}} \rho u \frac{\partial \phi}{\partial t} dx dt \quad (65)$$

Also we get

$$\int_0^T \int_{\Omega} a_{ji}(t) \frac{\partial v}{\partial y_j} \frac{\partial \psi}{\partial y_i} dy dt = \int_{\hat{Q}} \nabla u \nabla \phi dx dt \quad (66)$$

$$\int_0^T \int_{\Omega} \beta_{ji} v_i \rho v \frac{\partial \psi}{\partial y_j} dy dt = \int_{\hat{Q}} u_j \rho u_i \frac{\partial \phi_i}{\partial x_j} dx dt \quad (67)$$

$$\int_0^T \int_{\Omega} \rho g \psi dy dt = \int_{\hat{Q}} \rho f \phi dx dt \quad (68)$$

$$\int_{\Omega} \rho_0 v_0 \psi(0) dy dt = \int_{\Omega_0} \rho_0 u_0 \phi(0) dx dt \quad (69)$$

and in view of (65-69) and (18) we obtain (16).

Now we assume (19) and prove (17). Indeed, let  $\psi \in C^1(\bar{Q})$  with compact support in  $\hat{Q} \cup \{0\} \times \{0\}$ . Let  $h(y,t) = |\det K(t)| (K(t)y,t)$ . Then  $h \in C^1(\bar{\Omega} \times [0,T])$  and  $h$  has compact support in  $\Omega \times [0,T]$ . We note that

the equality (19) is in sense of  $L^2(0,T,H^1(\cdot))$ . We multiply (19) by  $h$  and we integrate in  $\Omega \times [0,T]$ ; we obtain

$$-\int_0^T \int_{\Omega} \rho \frac{\partial h}{\partial t} dy dt + \int_0^T \int_{\Omega} \rho v_i \beta_{ij} \frac{\partial h}{\partial y_j} dy dt + \int_0^T \int_{\Omega} \rho \beta'_{ij} \alpha_{jk} \frac{\partial (h y_k)}{\partial y_j} dy dt = 0$$

Similarly as in Theorem 2 we obtain (17). Moreover

$$u \in L^2(0,T,V(\cdot)) \subset L^2(0,T,H(\cdot)) \text{ and } \psi \in L^2(\hat{Q})$$

**Appendix:** For the sake of completeness we will show some auxiliary results and also we will define some spaces used in our work. In order, let  $u(x,t)$  and  $\psi(y,t)$  be vector real functions related by:

$$u(x,t) = |\det K^{-1}(t)| (K^{-1}(t)x,t). \quad (70)$$

We have

$$\|u(t)\|_{V(\Omega_t)}^2 = \sum_{i,j=1}^n \int_{\Omega} |\det K^{-1}(t)| \left( \beta_{ij}(t) \frac{\partial \xi_i}{\partial y_j} \right)^2 dy. \quad (71)$$

This implies

$$c_1 \|\xi(t)\|_V \leq \|u(t)\|_{V(\Omega_t)} \leq c_2 \|\xi(t)\|_V, \quad (72)$$

where,  $c_1$  and  $c_2$  are constants independents of  $u$  and  $\psi$ . Motivate by (71) and (72) we define  $L^p(0,T,V(\cdot))$ , ( $1 < p < \infty$ ) as the space of (classes of) functions  $u: \hat{Q} \rightarrow \mathbb{R}^n$  such that there exists  $\psi \in L^p(0,T,V(\cdot))$  verifying (70) equipped with the norm

$$\|u\|_{L^p(0,T,V(\Omega_t))} = \left( \int_0^T \|u\|_{L^p(0,T,V(\Omega_t))}^p dt \right)^{1/p}, \quad 1 < p < \infty \quad (73)$$

$$\|u\|_{L^\infty(0,T,V(\Omega_t))} = \text{ess sup}_{t \in [0,T]} \|u\|_{V(\Omega_t)} \quad (74)$$

In a similar way we define the space  $L^p(0,T,V_2(\cdot))$ . Let  $u(x,t)$  and  $w(y,t)$  be vector real functions such that

$$u(x,t) = w(K^{-1}(t)x,t). \quad (75)$$

Then

$$\frac{\partial u(x,t)}{\partial t} = \beta'_{ij} \alpha_{ij} y_j \frac{\partial w(y,t)}{\partial y_i} + \frac{\partial w(y,t)}{\partial t}. \quad (76)$$

Take  $u \in L^2(0,T,V_2(\cdot))$ . Then  $w$  verifying (75) belongs to  $L^2(0,T,V)$ . Motivated by (76) we say that  $u \in L^2(0,T,H(\cdot))$  if  $w \in L^2(0,T,H)$ . In a similar way we say that  $u \in L^2(0,T,V_2(\cdot))$  if  $w \in L^2(0,T,V)$ . In this case  $u \in L^2(0,T,V_2(\cdot))$ .

Concerning the operator defined by (15), we have

**Lemma 1:** Let  $A(t)$  the operator defined by (15) and  $a(t,v,w)$  the bilinear form defined by:

$$a(t, v, w) = \int_{\Omega} a_{ij}(t) \frac{\partial v_i}{\partial y_j} \frac{\partial w_i}{\partial y_r} dy$$

Then, we have

- (i)  $\langle A(t)v, w \rangle = a(t, v, w), \forall v, w \in V$ , where  $\langle \cdot, \cdot \rangle$  is the duality pairing between  $V$  and its topological dual  $V'$ .
- (ii)  $a(t, v, v) = a_0 \|v\|^2, \forall v \in V$ ,  $a_0$  positive constant
- (iii)  $|a(t, v, w)| \leq C \|v\| \|w\|, \forall v, w \in V$

**Proof:** The same given in Limaco-Miranda [14].

**Lemma 2:** If  $n=3$  and let us consider

$$V = \{v \in L^2(0, T, V) \mid v|_{(0, T, H)} = 0\}$$

$$\int_0^{T-\delta} |v(t+\delta) - v(t)|^2 dt \leq C \delta^{1/2}, \forall \delta \in [0, T]$$

with  $\|v\|_p = \|v\|_{L^p(0, T, V)} + \|v\|_{L^\infty(0, T, H)}$ . Then, the embedding  $V \subset L^p(0, T, L^q(\cdot))$  is compact when  $p \in (2, +\infty)$ ,  $q \in [2, 6)$  and  $\frac{1}{p} + \frac{3}{2q} > \frac{3}{4}$

**Proof:** By Lions [5], Lemma 5.1 pp: 298.

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