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TOPOLOGICAL TRIVIALITY OF FAMILY OF FUNCTIONS AND SETS

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ABSTRACT. In this article we divulge one of the subjects most interesting in the Theory of Singularities, to know, the topological triviality of families of functions and sets. We present, briefly, definitions and some of the results most important when treating of real singularities.

1. Introduction. Given an object V in the Euclidean space $\mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$, we can look at V as a 1-parameter family of objects in $\mathbf{R}^2$. In fact, let us consider $V_t = \{(x,y) : (x,y) \in V\}$. This is a classic perspective which appears, for example, in the principle of Cavallieri used for the calculation of volumes.

A technique very common in singularity theory explores the inverse way described above. For example, Let $f\colon \mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}$; $f(0 \times \mathbf{R}) = 0$ be a family of functions $f_t\colon \mathbf{R}^2 \to \mathbf{R}$; $f_t(x,y) = f(x,y,t)$. Associated to this family we have the following family of sets $X_t = f^{-1}(0)$ in $\mathbf{R}^2$. Then, from the object $X = f^{-1}(0) \in \mathbf{R}^3$, it is possible to know the local topological property of the family X_t. In this context, Whitney introduced the following concept of regularity of an analytic family of analytic sets X_t in $\mathbf{R}^n$, through $0 \in \mathbf{R}$, with the property that the singular set of $X = \{(x,t) : x \in X_t\}$ is contained in $Y = 0 \times \mathbf{R}$ (t-axis) and $X - Y$ is a smooth analytic subset in $\mathbf{R}^n \times \mathbf{R}$ which is dense in X.

(a) *Condition. X is (a)-regular on the t-axis Y.* If, for each $y \in Y$, the following holds: if (p_i) is a sequence in $X - Y$; $p_i \to y$, and the sequence of tangent planes $T_{p_i}(X - Y)$ converges to a plane τ (in the appropriated Grassmaniann), then $T_y Y \subset \tau$.

(b) *Condition. X is (b)-regular on the t-axis Y.* If, for each $y \in Y$, the following holds: if (p_i) and (y_i) are pairs of sequence of points in $X - Y$ and Y respectively; $p_i \to y$, $y_i \to y$ and the sequence of lines $p_i y_i$

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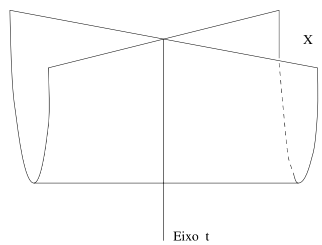
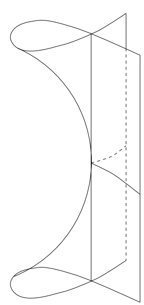
converges to a line λ and the sequence of tangent planes $T_{p_i}(X-Y)$ converges to a plane τ, then $\lambda \subset \tau$.

Example A. Let $X_t = \{(x,y) : y^2 = tx^2\}$. It is easy to see that X is not (a)-regular on the t-axis. We notice that the local topological type of $(X_t, 0)$, i.e., the topological type of X_t in a neighborhood of $0 \in \mathbf{R}^2$, is not constant.

(b)-regularity implies (a)-regularity. Thus, the family X is not (b)-regular.



Example B. Let $X_t = \{(x,y) : y^2 = x^3 + tx^2\}$. Here, X is (a)-regular on the t-axes and, on the other hand, from the following theorem we can conclude that X is not (b)-regular on the t-axis, because the local topological type of X_t, is not constant.



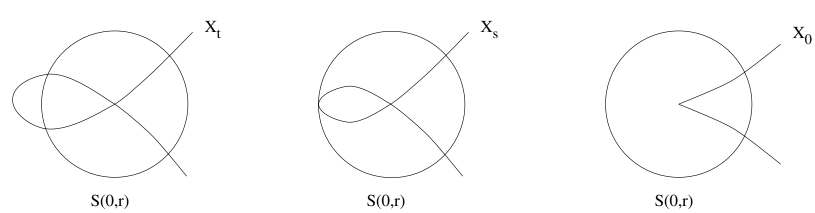
Theorem 1.1 [**5**]. *If X is (b)-regular on the t-axis, then X is topologically trivial on the t-axis, i.e., there exists a local homeomorphism $\phi: U \times \mathbf{R} \to \phi(U \times \mathbf{R}) \subset \mathbf{R}^n \times \mathbf{R}$ of the form $\phi(x,t) = (\varphi(x,t),t)$; $\varphi(0,t) = 0$, $X = \phi(X_{t_0} \times \mathbf{R})$, $U \times \mathbf{R}$ is a neighborhood of t-axis, for some t_0.*

Here it is necessary to define the topological triviality for family of sets.

Definition 1.2. Let X_t be a family of subsets containing the origin of $\mathbf{R}^n$ and, as usual, let $X = \{(x,t) \in \mathbf{R}^n \times \mathbf{R} \,/\, x \in X_t\}$. X is *topologically trivial along the t-axis* if there exists a local homeomorphism, on its image, $\Phi : U \times \mathbf{R} \to \mathbf{R}^n \times \mathbf{R}$ of type $\Phi(x,t) = (\phi(x,t), t)$ such that $X = \Phi(X_{t_0} \times \mathbf{R})$ in a neighborhood $U \times \mathbf{R}$ of the t-axis, and for some $t_0 \in \mathbf{R}$.

Then, we can review Thom-Mather's theorem saying that (b)-regularity implies the topological triviality. On the other hand, the example B shows that it may be the case that (a)-regularity occurs and the topological triviality does not. The following analysis will be inspired by the phenomenons described above. To be more precise it is natural to ask what condition, preferably minimal, can be added to an (a)-regularity so that we have a topological triviality.

In example B we have that the local topological type of X_t (in a neighborhood of the origin) does not coincide with the topological type of X_0 (in a neighborhood of the origin) for $t \neq 0$, because for a disk $B(0,r)$ centered in the origin of $\mathbf{R}^2$ with radius $r > 0$ sufficiently small, we have $X_t \cap B(0,r)$ has four semi-branches through the origin and $X_0 \cap B(0,r)$ has only two semi-branches through the origin. Observe that, the deformation from X_t to X_0 happens as if we had pulled the loop of a shoe lace. Meanwhile, the tangency between boundary of $B(0,r)$ occurs in the case of the sphere $S(0,r)$ and X_s for some s between 0 and t, as in the figure below:



Uniform Milnor's radius. Let X_t be a family of subsets containing the origin of $\mathbf{R}^n$ and $X = \{(x,t) \in \mathbf{R}^n \times \mathbf{R} \mid x \in X_t\}$. We say that X has *uniform Milnor's radius* if there exist $\varepsilon > 0$ such that $X \overline{\pitchfork} C(0,r)$, for all $0 < r \le \varepsilon$, in which $C(0,r)$ refers to the cylinder $S(0,r) \times \mathbf{R}$.

The name given for the regularity condition above is motivated by a result from Milnor which states the following: if H is a hypersurface in $\mathcal{C}^n$ with isolated singularity in the origin, then there exists $\varepsilon > 0$ such that $H \overline{\pitchfork} S(0,r)$, for all $0 < r \le \varepsilon$, in the $S(0,r)$ refers to the sphere of dimension $2n-1$, in $\mathcal{C}^n$, of radius r and center in the origin.

At the end of the 1980's, Bekka proved the following result.

Theorem 1.3 [**2, 3**]. *Let X_t be a family of subsets containing the origin of $\mathbf{R}^n$ and $X = \{(x,t) \in \mathbf{R}^n \times \mathbf{R} \mid x \in X_t\}$. If X is (a)-regular over the t-axis and has uniform Milnor's radius, then X is topologically trivial along the t-axis.*

From the above result, it is natural to question the independence of the regularity condition admitted in the hypothesis of Bekka's theorem. This is what we will do in the following paragraph.

As seen in example B, X is a family (a)-regular over t axis that does not have uniform Milnor's radius. Therefore (a)-regularity does not cause uniform Milnor's radius. Now consider

$$X_t = \{(x,y) \in \mathbf{R}^2 \mid \cos(t)x - \sin(t)y(x^2+y^2) = 0\}$$

and $X = \{(x,y,t) \in \mathbf{R}^2 \times \mathbf{R} \mid (x,y) \in X_t\}$. We can see, given a sequence $p_i = (x_i, y_i, t_i)$ with $x_i \to 0$, $y_i = 0$ and $t_i = \pi/2$, that X is not (a)-regular on the t-axis and, on the other hand, X has uniform Milnor's radius, cf. [**7**]. However, using the results of King, which will be described next, we can still prove that X is topologically trivial along the t-axis.

Let's begin with the following result.

Theorem 1.4 [**5**]. *Suppose that $f_t : \mathbf{R}^n \to \mathbf{R}^p$, $n \ge p$, is a continuous family of polynomials and that there exists a neighborhood $V \subset \mathbf{R}^n$, $0 \in V$ such that, if there is $x \in V$ with $\operatorname{rank}(df_t(x)) < p$ for some $t \in \mathbf{R}$, then $x = 0$ (we call this condition good deformation).*

Additionally, if the family of $f_t^{-1}(0)$ has uniform Milnor's radius, then there is a family of homeomorphisms $h_t : \mathbf{R}^n \to \mathbf{R}^n$, with $h_t(0) = 0$, that trivializes the family, that is, $f_t \circ h_t = f_0$ in a neighborhood of the origin. In particular, the family of the zero set is topologically trivial.

As a good example, we can verify in the previous case that a family of functions $f_t(x, y) = \cos(t)x - \sin(t)y(x^2 + y^2)$ has good deformation and that the family $f_t^{-1}(0)$ has uniform Milnor's radius. However, the family is topologically trivial. It is not always easy to verify if the uniform Milnor's radius exists, because this assertion depends on the analysis of the behavior of the matrix $Jf_t(x)$ in relation to the position vector x in the set $f_t^{-1}(0)$. However, if we have good deformation and we manage to decide if the zero sets are homeomorphic for each time t, we still have topological triviality. This is the next result.

Theorem 1.5 [**6**]. *Let $f_t : \mathbf{R}^n, 0 \to \mathbf{R}^p, 0$, $t \in \mathbf{R}$, be a family with good deformation. Suppose that, for each $t \in \mathbf{R}$, $f_t^{-1}(0)$ are homeomorphics. Then, the family f_t is topologically trivial.*

In the next section we approach a class of families that satisfies the hypothesis of Bekka's theorem and also the hypothesis of King's theorem, mentioned above, cf. [**7**]. As we have not presented so far any proof, we will make a demonstration in the next section that exhibits some usual techniques of the singularity theory. The following theorem, due to Buchner and Kucharz, which we will present and prove only in the case of a function of two variables, in fact is still valid for more variables and has analogous formulations for applications of several variables.

2. Theorem on isolated singularity. Let $\alpha = (\alpha_1, \alpha_2)$ be a pair of positive integer numbers and β a positive integer number. A function $P : \mathbf{R}^2 \to \mathbf{R}$ is called weighted homogeneous of type (α, β) if $P(r^{\alpha_1}x_1, r^{\alpha_2}x_2) = r^\beta P(x)$, for all $r \in \mathbf{R}$, $x = (x_1, x_2)$.

Example. $P(x, y) = x^2 - y^3$ is weighted homogeneous of type $((3, 2); 6)$.

Given an analytic function $f : \mathbf{R}^2 \to \mathbf{R}$, we define $w(f,\alpha)$ as the minimum of the numbers $\alpha_1 i_1 + \alpha_2 i_2$ where the monomial $x_1^{i_1} x_2^{i_2}$ appears with coefficient different from zero in the Taylor's expansion of f.

Definition 2.1. We say that a function $f : \mathbf{R}^2 \to \mathbf{R}$ is *strongly not degenerated* if for all $x \in \mathbf{R}^2 \backslash \{0\}$ the differential $d_x f$ is surjective.

Definition 2.2. A function $F : U \times \mathbf{R} \to \mathbf{R}$, where U is a neighborhood of the origin in $\mathbf{R}^2$, is *strongly admissible* by (α, β) if F may be written in the form $F = f + g$ with $f, g : U \times \mathbf{R} \to \mathbf{R}$ satisfying:

(I) f and g are of class C^2. $g, (\partial g/\partial t)$ are of class C^k on the $x-$variable where $k > \max\{(\beta/\alpha_1), (\beta/\alpha_2)\}$;

(II) For each $t \in \mathbf{R}$, $f_t = f(.,t)$ is the restriction to U of a function strongly not degenerated of type (α, β);

(III) $w(T^{k-1} g_t, \alpha) > \beta$ for all $t \in \mathbf{R}$, where $T^k(.)$ denotes the Taylor's polynomial of degree k.

Remark. $w(T^k g_t, \alpha) > \beta$ because of (I) $w(x_1^{i_1} x_2^{i_2}) > \beta$ if $i_1 + i_2 \geq k$.

Theorem 2.3 [**4**]. *Let $F : U \times \mathbf{R} \to \mathbf{R}$ be a function strongly admissible by (α, β). Then, given $t_0 \in \mathbf{R}$ and a neighborhood V_0 of t_0 in $\mathbf{R}$, there exists a neighborhood U_0 of 0 in $\mathbf{R}^2$ and a continuous map $\sigma : U_0 \times V_0 \to U$ such that, for each $t \in V_0$, σ_t sends 0 in 0 and transforms homeomorphically U_0 in $\sigma_t(U_0)$ and $F_t \circ \sigma_t = F_{t_0}$.*

Proof. Without loss of generality, we can assume that $t_0 = 0$ and that V_0 is the interval $(-L, L)$ for some real positive number L.

Let $\varepsilon > 0$ be such that $\phi : S^1 \times (-\varepsilon, \varepsilon) \to \mathbf{R}^2$ defined by $\phi(x, r) = (r^{\alpha_1} x_1, r^{\alpha_2} x_2)$ has its values in U.

Define $H : S^1 \times (-\varepsilon, \varepsilon) \times \mathbf{R} \to \mathbf{R}$ by $H(x, r, t) = r^{-\beta} g(\phi(x, r), t)$ for $r \neq 0$ and $H(x, 0, t) = 0$.

Let $\pi : S^1 \times (-\varepsilon, \varepsilon) \times \mathbf{R} \to S^1 \times \mathbf{R}$ be the canonical projection $\pi(x, r, t) = (x, t)$.

Claim 1. *The conditions* (I) *and* (III) *imply that* $H, (\partial H/\partial x_i)0$, $(\partial H/\partial t)$ *are at least of class* C^1 *and equal to zero if* $r = 0$.

In fact,

• by Taylor's infinitesimal formula, we can write $g_t(\phi(x,r))$ in the form $g_t(\phi(x,r)) = T^k g_t(\phi(x,r)) + R_k(\phi(x,r))$ where $\lim_{r\to 0} R_k(\phi(x,r))/|\phi(x,r)|^k = 0$. Thus,

$$H(x,r,t) = \frac{T^k g_t(\phi(x,r))}{r^\beta} + \frac{R_k(\phi(x,r))}{r^\beta}.$$

Observe that, as $w(T^k(g_t)) > \beta$, then $\lim_{r\to 0} T^k g_t(\phi(x,r))/r^\beta = 0$.

Observe that

$$\frac{R_k(\phi(x,r))}{r^\beta} = \frac{R_k(\phi(x,r))}{|\phi(x,r)|^k} \cdot \frac{|\phi(x,r)|^k}{r^\beta}$$

and

$$\left(\frac{|\phi(x,r)|^k}{r^\beta}\right)^2 = \left(\frac{r^{2\alpha_1}x_1^2 + r^{2\alpha_2}x_2^2}{r^{2\beta/k}}\right)^k \le \left(\frac{r^{2\alpha'}}{r^{2\beta/k}}\right)^k = r^{2(\alpha' k - \beta)}$$

where $\alpha' = \min\{\alpha_1, \alpha_2\}$. Therefore $\alpha' k - \beta > 0$ and we have that $\lim_{r\to 0} R_k(\phi(x,r))/r^\beta = 0$.

Thus, we have that $\lim_{r\to 0} H(x,r,t) = 0$, and therefore H is continuous.

• By the uniqueness of Taylor's expansion, we have that

$$T^{k-1}\left(\frac{\partial g_t}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\, T^k(g_t)$$

and

$$T^{k-2}\left(\frac{\partial^2 g_t}{\partial x_i \partial x_j}\right) = \frac{\partial^2}{\partial x_i \partial x_j}\,(T^k g_t);$$

therefore, $w(T^{k-1}(\partial g_t/\partial x_i)) > \beta - \alpha_i$ and $w(T^{k-2}(\partial^2 g_t/\partial x_i \partial x_j)) > \beta - \alpha_i - \alpha_j$. Thus, in the same way as above, we can prove that $\lim_{r\to 0} \partial H/\partial x_i = 0$ and $\lim_{r\to 0} \partial^2 H/(\partial x_i \partial x_j) = 0$.

Therefore, H and $\partial H/\partial x_i$ are, at least, C^1.

• It is easy to verify that again we have $\partial H/\partial t$ at least of class C^1. This concludes the claim.

Let

$$A : S^1 \times (-\varepsilon, \varepsilon) \times \mathbf{R} \longrightarrow \mathbf{R}^2$$

the map $A(x, r, t) = \nabla_x(f \circ \pi + H) = (\partial/\partial x_1(f \circ \pi + H), \partial/\partial x_2(f \circ \pi + H))$ and

$$A^* : S^1 \times (-\varepsilon, \varepsilon) \times \mathbf{R} \longrightarrow \mathbf{R}^2$$

the map defined by $A^*(x, r, t) = \nabla_x(f \circ \pi) = (\partial/(\partial x_1)(f \circ \pi), \partial/(\partial x_2)(f \circ \pi))$.

Remark. By (II) we have that $A^*(x, r, t)$ is surjective at each point of $S^1 \times (-\varepsilon, \varepsilon) \times \mathbf{R}$ and therefore A has the same property where ε is chosen sufficiently small and $\mathbf{R}$ is changed by $(-L - \delta, L + \delta)$.

Hence, we can write $(\partial f/\partial t) \circ \pi + (\partial H/\partial t)$ as the linear combination of the coordinates functions of A whose coefficients are functions of class C^1 in $S^1 \times (-\varepsilon, \varepsilon) \times (-L - \delta, L + \delta)$. Thus,

$$\frac{\partial f}{\partial t} \circ \pi + \frac{\partial H}{\partial t} = \langle \nabla_x(f \circ \pi + H), u \rangle \tag{1}$$

where $u(x, r, t) = (u_1(x, r, t), u_2(x, r, t))$ is a map of class C^1 in $S^1 \times (-\varepsilon, \varepsilon) \times (-L - \delta, L + \delta)$.

Remark. Since $\nabla_x(f \circ \pi + H)$ is of class C^1, the implicit function theorem tells us that u is of class C^1.

Multiplying expression (1) by r^β we have that:

$$\frac{\partial F}{\partial t}(\phi(x, r), t) = \langle \nabla_{\phi(x,r)} F(\phi(x, r), t), w(x, r, t) \rangle \tag{2}$$

with $(x, r, t) \in S^1 \times ((-\varepsilon, 0) \cup (0, \varepsilon)) \times (-L - \delta, L + \delta)$ where $w(x, r, t) = (r^{\alpha_1} u_1(x, r, t), r^{\alpha_2} u_2(x, r, t))$ is of class C^1 in $S^1 \times (-\varepsilon, \varepsilon) \times (-L - \delta, L + \delta)$.

Now we may write (2) as

$$\frac{\partial F}{\partial t}(y, t) = \langle \nabla_y F(y, t), w(\phi^{-1}(y), t) \rangle \tag{3}$$

where $y \in \phi(S^1 \times (-\varepsilon, \varepsilon)) \backslash \{0\}$.

Consider the vector field $W(y,t) = \partial/(\partial t) - w(\phi^{-1}(y),t)$ defined in $[\phi(S^1 \times (-\varepsilon,\varepsilon))\backslash\{0\}] \times \mathbf{R}$ which is of class C^1.

Claim 2. $W \perp \nabla F$. *In fact,* $\nabla F = (\nabla_x F, \frac{\partial F}{\partial t})$*; therefore,*

$$\langle W, \nabla F\rangle = \frac{\partial F}{\partial t} - \langle \nabla_y F, w(\phi^{-1}(y),t)\rangle = 0.$$

Remark. Give $y = \phi(x,r) \in \phi(S^1 \times (-\varepsilon,\varepsilon))\backslash\{0\}$, then make $y \to 0$ equivalent to make $r \to 0$.

Then $\lim_{y\to 0} W(y,t) = \partial/(\partial t) - \lim_{r\to 0} w(x,r,t) = \partial/\partial t$, and therefore the vector field W has a continuous extension to $(0,t)$ with $W(0,t) = \partial/\partial t$.

Consider now the initial value problem

$$\begin{cases} (\partial\tau/\partial t)(y,t) = W(\tau(y,t)) \\ \tau(y,0) = (y,0) \end{cases} \tag{4}$$

Let us suppose that there exists ε', δ' and τ such that $0 < \varepsilon' \le \varepsilon$, $0 < \delta' \le \delta$ and τ is a map of

$$[\phi(S^1 \times (-\varepsilon',\varepsilon'))\backslash\{0\}] \times (-L-\delta', L+\delta') \longrightarrow \phi(S^1 \times (-\varepsilon,\varepsilon))\backslash\{0\}$$

satisfying (4).

Remark. As the vector field W is constant along the parameter space, it is easy to see that $\tau(y,t) = (\bar\tau(y,t),t)$.

Claim 3. $(\partial/\partial t)(F(\tau(y,t))) = 0$. *In fact,*

$$\frac{\partial}{\partial t}(F(\tau(y,t))) = \left\langle \nabla F(\tau(y,t)), \frac{\partial\tau}{\partial t}(y,t)\right\rangle = \langle \nabla F, W\rangle = 0.$$

Hence, $F(\tau(y,t)) = c(y)$ is a "constant depending on y." As $F(\tau(y,0)) = F(y,0) = c(y)$, we have that $F(\bar\tau(y,t),t) = F(y,0)$, i.e., $F_t \circ \bar\tau_t = F_0$.

We know by theorems of differential equations that there exists a unique solution

$$\bar{\tau}_t : \phi(S^1 \times (-\varepsilon', \varepsilon'))\backslash\{0\} \to \phi(S^1 \times (-\varepsilon, \varepsilon))\backslash\{0\}$$

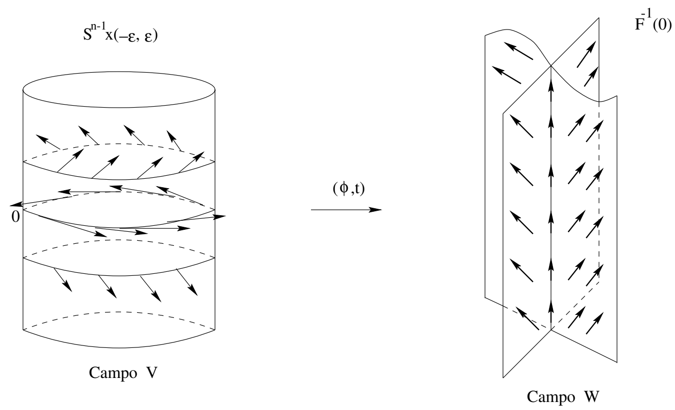
of class C^1.

Claim 4. *Continuing with the construction of $\bar{\tau}(y,t)$, we will show that if we define $\bar{\tau}_t(0) = 0$, i.e., $\tau_t(0,t) = (0,t)$, then $\bar{\tau}_t : \phi(S^1 \times (-\varepsilon', \varepsilon')) \to \phi(S^1 \times (-\varepsilon, \varepsilon))$ is a homeomorphism.*

In fact, let us consider the lift of the vector field $w_t \circ \phi^{-1}$ to $S^1 \times [(-\varepsilon, 0) \cup (0, \varepsilon)]$, i.e., consider the vector field

$$V_t : S^1 \times [(-\varepsilon, 0) \cup (0, \varepsilon)] \longrightarrow T(S^1 \times [(-\varepsilon, 0) \cup (0, \varepsilon)])$$

such that $d_{(x,r)}\phi.V_t = w_t(x,r)$.



Claim 4.1. *V_t has a C^1 extension to $S^1 \times (-\varepsilon, \varepsilon)$ that is tangent to $S^1 \times \{0\}$ at all points of $S^1 \times \{0\}$. In fact, if we identify the tangent bundle if S^1 is a subbundle of the trivial bundle $S^1 \times \mathbf{R}^2$, then the equation of V_t becomes*:

$$\begin{bmatrix} r^{\alpha_1} & 0 & \alpha_1 x_1 r^{\alpha_1 - 1} \\ 0 & r^{\alpha_2} & \alpha_2 x_2 r^{\alpha_2 - 1} \\ x_1 & x_2 & 0 \end{bmatrix} \begin{bmatrix} V_{1t} \\ V_{2t} \\ V_{3t} \end{bmatrix} = \begin{bmatrix} r^{\alpha_1} u_1(x,r,t) \\ r^{\alpha_2} u_2(x,r,t) \\ 0 \end{bmatrix}. \tag{5}$$

Remark. The equation of V_t above may be deduced by analyzing the action of $d_{(x,r)}\phi.v$, $v \in T(S^1 \times (-\varepsilon, \varepsilon))$, that is, let $\gamma(s) = (x(s), r(s))$ be a curve with $\gamma'(0) = (x'(0), r'(0)) = v = (v_1, v_2, v_3)$ and $\gamma(0) = (x, r)$

$$\begin{aligned}\therefore\ \frac{\partial}{\partial s}\,(\phi\circ\gamma)_{|s=0} &= \frac{\partial}{\partial s}\,(r(s)^{\alpha_1}x_1(s), r(s)^{\alpha_2}x_2(s))_{|s=0} \\ &= (\alpha_1 x_1 r^{\alpha_1-1}v_3 + r^{\alpha_1}v_1, \alpha_2 x_2 r^{\alpha_2-1}v_3 + r^{\alpha_2}v_2) \\ &= \begin{bmatrix} r^{\alpha_1} & 0 & \alpha_1 x_1 r^{\alpha_1-1} \\ 0 & r^{\alpha_2} & \alpha_2 x_2 r^{\alpha_2-1} \\ x_1 & x_2 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.\end{aligned}$$

Solving equation (5) by Cramer's rule, we have that $V_{it} = \Delta_i/\Delta$, $i = 1, 2, 3$ where

$$\Delta = \begin{vmatrix} r^{\alpha_1} & 0 & \alpha_1 x_1 r^{\alpha_1-1} \\ 0 & r^{\alpha_2} & \alpha_2 x_2 r^{\alpha_2-1} \\ x_1 & x_2 & 0 \end{vmatrix}$$

and Δ_i is the determinant of the matrix obtained by replacing the i^{ht} column by the vector

$$\begin{bmatrix} r^{\alpha_1}u_1 \\ r^{\alpha_2}u_2 \\ 0 \end{bmatrix}.$$

Looking at the above determinants, it is not difficult to see that Δ is the product of $r^{(\sum_i \alpha_i)-1}$ and a nonvanishing term, Δ_i has a factor of $r^{(\sum_i \alpha_i)-1}$ with $i = 1, 2$, and Δ_3 has a factor of $r^{\sum_i \alpha_i}$. Now we have proved Claim 4.1.

Continuing with the proof of Claim 4, consider the initial value problem

$$\text{(6)}\qquad \begin{cases} (\partial\eta/\partial t)(x, r, t) = -V_t(\eta(x, r, t)) \\ \eta(x, r, 0) = (x, r). \end{cases}$$

This has a C^1 solution in $S^1 \times (-\varepsilon', \varepsilon') \times (-L - \delta', L + \delta')$ where $0 < \varepsilon' \le \varepsilon$ and $0 < \delta' \le \delta$.

Claim 4.2. η_t is a C^1-embedding of $S^1 \times (-\varepsilon', \varepsilon') \to S^1 \times (-\varepsilon, \varepsilon)$ such that

$$\eta_t(S^1 \times \{0\}) = S^1 \times \{0\}, \quad \eta_t(S^1 \times (-\varepsilon', 0)) \subset S^1 \times (-\varepsilon, 0)$$

and

$$\eta_t(S^1 \times (0, \varepsilon')) \subset S^1 \times (0, \varepsilon).$$

Now we define ρ_t by $\rho_t = \phi \circ \eta_t \circ \phi^{-1}$ in $\phi(S^1 \times (0, \varepsilon'))$. Then ρ_t satisfies (4). However $\rho = (\rho_t, t)$ has a continuous extension to the t-axis, since $\lim_{(y,t)\to(0,t_0)} \rho_t = 0$. The same occurs with ρ^{-1}. □

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