Modeling of OFDM Systems With Memory Polynomial Power Amplifiers

C. Alexandre R. Fernandes, João C. M. Mota, Gérard Favier

***Abstract*— Nonlinear power amplifiers (PA) are often modeled as a memoryless polynomial. However, when the input signal bandwidth is large, PA memory effects can not be ignored. The memory polynomial model is a simple but efficient model that captures both memory effects and nonlinear behavior of a PA. In this work, a new theoretical result concerning orthogonal frequency division multiplexing (OFDM) systems with a memory polynomial PA is demonstrated. It states that such an OFDM system can be expressed as a OFDM system with a memoryless polynomial PA whose coefficients vary from one subcarrier to another.**

***Keywords*— OFDM, power amplifier, nonlinear, polynomial, memory polynomial.**

***Resumo*— Amplificadores de potência (PA) não-lineares são freqüentemente modelados como polinômios sem-memória. Entretanto, quando a largura de banda do sinal é grande, os efeitos de memória do PA não podem ser ignorados. O polinômio com memória é um modelo simples e eficiente que consegue captar de forma eficaz os efeitos não-lineares e de memória de um PA. Neste trabalho, um novo resultado teórico sobre sistemas OFDM com PA do tipo polinômio com memória é demonstrado. Este resultado afirma que este tipo de sistema pode ser expresso como um sistema OFDM com PA do tipo polinômio sem-memória com coeficientes que variam de uma sub-portadora para outra.**

***Palavras-Chave*— OFDM, amplificador de potência, não-linear, polinômio, polinômio com memória.**

I. INTRODUCTION

Orthogonal frequency division multiplexing (OFDM) has been considered for a number of applications in the area of wireless communications [1]. A major drawback of OFDM is that the transmitted signals are characterized by a high peak-to-average power ratio (PAPR) [1]–[3]. Due to the presence of nonlinear power amplifiers (PAs), a high PAPR causes the introduction of nonlinear inter-carrier interference (ICI) in the received signals if a high input back-off (IBO) is not used, which can significantly deteriorate the recovery of the information symbols. However, a high IBO results in low-power efficiency of the PA and a lower fade margin.

For bandlimited input signals, nonlinear PAs are often modeled as (memoryless) polynomials with frequency-independent coefficients [2]–[6]. However, when the bandwidth of the input signal is large, PA memory effects can not be ignored. Some models have been proposed for characterizing the behavior of nonlinear PAs with memory. Among these models, one of the most attractive seems to be the memory polynomial, used by a number of authors [2]–[4], [6]–[10]. The memory polynomial, also called Parallel Hammerstein model [3], is a simple but efficient model with frequency-independent coefficients that captures both memory effects and nonlinear behavior of a PA [8]. The memory polynomial model can be viewed as a special case of the Volterra model.

The main contribution of this paper is to develop a new theoretical result concerning OFDM systems with a memory polynomial PA. First, we prove an auxiliary theorem stating that a memory polynomial PA in a OFDM system can be expressed as a memoryless polynomial PA whose coefficients vary from one subcarrier to another. Based on this result, we demonstrate that the frequency domain received signals have very similar expressions for polynomial and memory polynomial PAs, the only difference being the fact that the PA coefficients vary from one subcarrier to another in the case of a memory polynomial PA.

Although theoretical analysis and signal processing techniques for OFDM with nonlinear memoryless PAs have been widely studied in the literature [11]–[14], there is a lack of works on OFDM systems with memory PAs. Using the theoretical result developed in the present paper, we can extend some theoretical results and techniques for ICI canceling that work with polynomial PAs to the case of memory polynomial PAs. As an example of application of this result, we derive new expressions for the theoretical symbol error rate (SER) provided by an OFDM system with a memory polynomial PA, these expressions being derived directly from the polynomial PA case.

In [9], SER expressions for OFDM systems with memory polynomial PAs are developed by expressing the received signal as a scaled version of the data symbol plus uncorrelated Gaussian noise. However, in this paper, the coefficients that multiply the data symbols are not deterministic; they depend on the (stochastic) transmitted signals. Thus, this approach can not be used by signal processing techniques that work with polynomial PAs, such as [14], [15], contrarily to our approach. Moreover, the development of the SER expressions presented in the present paper is much more simple than the one of [9].

The rest of the paper is organized as follows. Section II describes the OFDM system model with polynomial PA. Section III describes the OFDM system model with memory polynomial PA and develops the new theoretical result. In Section IV, the result presented in Section III is used to develop new SER expressions for OFDM systems with a memory polynomial PA. In Section V, we evaluate the validity

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of the SER expressions presented in Section IV by means of simulations and the conclusions are drawn in Section VI.

The following notation is used in the paper. Lower-case letters (x) are scalar variables, boldface lower-case letters ($\mathbf{x}$) are vectors and boldface capital letters ($\mathbf{X}$) are matrices. All variables with an overline correspond to frequency-domain variables. Besides, $[\mathbf{x}]_i$ represents the i^{th} element of the vector $\mathbf{x}$, $[\mathbf{X}]_{i,j}$ the $(i,j)^{th}$ element of the matrix $\mathbf{X}$, diag $[\mathbf{x}]$ the diagonal matrix built from the vector $\mathbf{x}$, $\text{diag}_i[\mathbf{X}]$ is the diagonal matrix formed from the i^{th} row of $\mathbf{X}$ and $[\mathbf{X}]_{i,\cdot}$ the i^{th} row of the matrix $\mathbf{X}$. Moreover, $\mathbf{X}^*$ denotes the complex conjugate of $\mathbf{X}$ and the function $\text{cir}(x,N)$, for $-N+1 \leq x \leq N$ is defined as follows: $\text{cir}(x,N) = x$ if $1 \leq x \leq N$ and $\text{cir}(x,N) = x+N$ if $-N+1 \leq x \leq 0$.

II. OFDM System Model with a Polynomial PA

A simplified scheme of the discrete-time equivalent baseband OFDM system used in this work is described in the sequel. Let us denote by N the number of subcarriers, $\bar{s}_n$ the frequency-domain data symbol at the n^{th} subcarrier and $\bar{\mathbf{s}} = [\bar{s}_1 \cdots \bar{s}_N]^T \in \mathbb{C}^{N\times 1}$ the vector containing all the N data symbols of a given symbol period. The frequency-domain data symbols $\bar{s}_n$, for $1 \leq n \leq N$, are assumed to be independent and identically distributed (i.i.d.), with a uniform distribution over a quadrature amplitude modulation (QAM) or a phase-shift keying (PSK) alphabet.

The corresponding time-domain OFDM symbol is obtained by taking the Inverse Fast Fourier Transform (IFFT) of frequency-domain data symbols, that is: $\mathbf{s} = \mathbf{V}\,\bar{\mathbf{s}}$, where the vector $\mathbf{s} = [s_1 \cdots s_N]^T \in \mathbb{C}^{N\times 1}$ contains the time-domain OFDM symbol and $\mathbf{V} \in \mathbb{C}^{N\times N}$ is the IFFT matrix of dimension $N \times N$, with $[\mathbf{V}]_{p,q} = \frac{1}{\sqrt{N}} e^{j2\pi(p-1)(q-1)/N}$, for $1 \leq p, q \leq N$. After the IFFT block, a cyclic prefix (CP) of length M_{cp} is inserted in the time-domain symbol in order to avoid intersymbol interference (ISI) at all the subcarriers, in the following way:

$$\mathbf{s}^{(cp)} = [s_{(N-M_{cp}+1)} \cdots s_N \;\; \mathbf{s}^T]^T \in \mathbb{C}^{(N+M_{cp})\times 1}. \quad (1)$$

The time-domain symbol with the CP is then amplified by a PA that is modeled as a nonlinear function. In this section, we consider that the PA is memoryless and modeled by a polynomial of order $2K+1$ [2]–[4] and, in Section III, we will consider the case of a memory polynomial PA [2]–[4], [6]–[10]. Denoting by $u_n^{(cp)}$ the output of the PA, we then have:

$$u_n^{(cp)} = \sum_{k=0}^{K} f_{2k+1} \psi_{2k+1}(s_n^{(cp)}), \quad (2)$$

for $1 \leq n \leq N + M_{cp}$, where $s_n^{(cp)} = [\mathbf{s}^{(cp)}]_n$, f_{2k+1} $(0 \leq k \leq K)$ are the polynomial equivalent baseband coefficients and the operator $\psi_{2k+1}(\cdot)$ is defined as $\psi_{2k+1}(a) = |a|^{2k}a$. Note that the signal $u_n^{(cp)}$ also have a cyclic prefix, that is, the sequences $\{u_1^{(cp)}, \cdots, u_{M_{cp}}^{(cp)}\}$ and $\{u_{(N-M_{cp}+1)}^{(cp)}, \cdots, u_N^{(cp)}\}$ have the same elements.

The equivalent baseband polynomial model (2) includes only the odd-order power terms with one more non-conjugated term than conjugated terms because the other nonlinear products of input signals correspond to spectral components lying outside the channel bandwidth, and can therefore be eliminated by bandpass filtering [16].

The signal $u_n^{(cp)}$ is transmitted through a frequency-selective fading wireless channel with impulse response denoted by w_m, for $m = 0, 1, ..., M$, where M is the wireless channel delay spread, and additive white Gaussian noise (AWGN) of variance σ^2. At the receiver, the CP is removed from the time-domain received signal $x_n^{(cp)}$ $(1 \leq n \leq N + M_{cp})$. Thus, assuming that length of the cyclic prefix is greater than or equal to the channel delay spread ($M_{cp} \geq M$), the wireless channel can be represented by a circular convolution:

$$x_n = \sum_{m=0}^{M} w_m u_{\text{cir}(n-m,N)} + v_n, \quad (3)$$

for $1 \leq n \leq N$, where x_n and u_n denote respectively the time domain received signal and PA output without cyclic prefix, with $x_n = x_{(n+M_{cp})}^{(cp)}$ and $u_n = u_{(n+M_{cp})}^{(cp)}$, v_n is the corresponding AWGN.

Eq.(3) can also be expressed in a vector form as:

$$\mathbf{x} = \mathbf{W}\mathbf{u} + \mathbf{v}, \quad (4)$$

where $\mathbf{x} = [x_1 \cdots x_N]^T \in \mathbb{C}^{N\times 1}$, $\mathbf{u} = [u_1 \cdots u_N]^T \in \mathbb{C}^{N\times 1}$, $\mathbf{v} = [v_1 \cdots v_N]^T \in \mathbb{C}^{N\times 1}$ and $\mathbf{W} \in \mathbb{C}^{N\times N}$ is the circulant channel matrix. The FFT of the received signals is then calculated as:

$$\bar{\mathbf{x}} = \mathbf{V}^*\mathbf{x} + \bar{\mathbf{v}} = \mathbf{V}^*\mathbf{W}\mathbf{V}\bar{\mathbf{u}} + \bar{\mathbf{v}}. \quad (5)$$

where $\bar{\mathbf{x}} \in \mathbb{C}^{N\times 1}$ is the vector of frequency-domain received signals, $\bar{\mathbf{u}} = \mathbf{V}^*\mathbf{u}$ is the frequency-domain version of $\mathbf{u}$ and $\bar{\mathbf{v}} = \mathbf{V}^*\mathbf{v} \in \mathbb{C}^{N\times 1}$ is the frequency-domain noise vector, which is also white and Gaussian with the same covariance $\sigma^2\mathbf{I}_N$ of $\mathbf{v}(i)$, $\mathbf{I}_N$ being the identity matrix of size N.

It can be shown that a circulant matrix is diagonalized by a IFFT matrix, i.e. $\Lambda = \mathbf{V}^*\mathbf{W}\mathbf{V}$, where $\Lambda \in \mathbb{C}^{N\times N}$ is a diagonal matrix containing the eigenvalues of $\mathbf{W}$ [17]. The n^{th} eigenvalue of $\mathbf{W}$, denoted by $\lambda_n = [\Lambda]_{n,n}$, represents the channel frequency response (CFR) at the n^{th} subcarrier. Thus, by defining: $\Psi_{2k+1}(\mathbf{a}) = [\psi_{2k+1}(a_1) \cdots \psi_{2k+1}(a_N)]^T \in \mathbb{C}^{N\times 1}$, for $\mathbf{a} = [a_1 \cdots a_N] \in \mathbb{C}^{N\times 1}$, and using (2), we can rewrite (5) as:

$$\bar{\mathbf{x}} = \Lambda \sum_{k=0}^{K} f_{2k+1}\mathbf{V}^*\Psi_{2k+1}(\mathbf{s}) + \bar{\mathbf{v}}. \quad (6)$$

Defining $\bar{\Psi}_{2k+1}(\mathbf{s}) = \mathbf{V}^*\Psi_{2k+1}(\mathbf{s}) \in \mathbb{C}^{N\times 1}$ as the frequency-domain version of $\Psi_{2k+1}(\mathbf{s})$, $\bar{\xi}_{2k+1}(n) = [\bar{\Psi}_{2k+1}(\mathbf{s})]_n$ and $\bar{\boldsymbol{\phi}}_n = [\bar{s}_n \quad \bar{\xi}_3(n) \cdots \bar{\xi}_{2K+1}(n)]^T \in \mathbb{C}^{(K+1)\times 1}$, the frequency-domain PA output $\bar{u}_n = [\bar{\mathbf{u}}]_n$ and received signal $\bar{x}_n = [\bar{\mathbf{x}}]_n$ can be expressed respectively as:

$$\bar{u}_n = f_1\;\bar{s}_n + \sum_{k=1}^{K} f_{2k+1}\;\bar{\xi}_{2k+1}(n), \quad (7)$$

and

$$\bar{x}_n = \lambda_n f_1\;\bar{s}_n + \lambda_n \sum_{k=1}^{K} f_{2k+1}\;\bar{\xi}_{2k+1}(n) + \bar{v}_n, \quad (8)$$

where $\bar{v}_n$ is the corresponding noise component in the frequency domain.

Equation (8) shows that the frequency-domain received signal $\bar{x}_n$ equals a scaled version of the frequency-domain data symbol $\lambda_n f_1 \bar{s}_n$ plus weighted nonlinear ICI terms $\sum_{k=1}^{K} \lambda_n f_{2k+1} \bar{\xi}_{2k+1}(n)$, plus a noise term, with $\bar{\xi}_{2k+1}(n)$ representing the $(2k+1)$th-order ICI at the n^{th} subcarrier. Note also that $\bar{\xi}_1(n) = \bar{s}_n$.

III. OFDM System Model with a Memory Polynomial PA

In this section, we present a OFDM system model considering a memory polynomial PA. It is demonstrated that the received signals can be expressed as in (8), with PA coefficients depending on the subcarrier.

Assuming that the length M_{cp} of the cyclic prefix satisfies: $M_{cp} \geq M + M_{pa}$, where M_{pa} is the PA memory, the output $u_n^{(cp)}$ of the memory polynomial PA is expressed as:

$$u_n^{(cp)} = \sum_{m=0}^{M_{pa}} \sum_{k=0}^{K} f_{2k+1,m} \ \psi_{2k+1} \left(s_{(n-m)}^{(cp)} \right), \tag{9}$$

for $1 + M_{pa} \leq n \leq N + M_{cp}$, where $f_{2k+1,m}$ are the PA coefficients. Note that the signal $u_n^{(cp)}$ have a cyclic block of $(M_{cp} - M_{pa}) > M$ symbols, that is, the sequences $\{u_{(M_{pa}+1)}^{(cp)}, \cdots, u_{M_{cp}}^{(cp)}\}$ and $\{u_{(N+M_{pa}+1)}^{(cp)}, \cdots, u_{(N+M_{cp})}^{(cp)}\}$ have the same elements. As a consequence, eqs. (3), (4) and (5) still hold for a memory polynomial PA, with all the variables being defined in the same way as in Section II.

From (5), it can be concluded that the noiseless part of the frequency domain received signal $\bar{x}_n$ depends on the CFR λ_n and the frequency domain output of the PA $\bar{u}_n = [\bar{\mathbf{u}}]_n$. The next theorem demonstrates that, when the PA is represented by a memory polynomial model, the signal $\bar{u}_n$ can be written as the frequency domain output of a memoryless polynomial PA, the coefficients of which vary from one subcarrier to another.

Theorem 1: Let $\bar{u}_n$ $(1 \leq n \leq N)$ be the frequency domain output of a memory polynomial PA with coefficients denoted by $f_{2k+1,m}$, for $0 \leq k \leq K$ and $0 \leq m \leq M_{pa}$. Then, $\bar{u}_n$ can be expressed as the frequency domain output of a memoryless polynomial PA:

$$\bar{u}_n = \sum_{k=0}^{K} \tilde{f}_{2k+1}(n) \ \bar{\xi}_{2k+1}(n), \tag{10}$$

with subcarrier dependent coefficients given by:

$$\tilde{f}_{2k+1}(n) = \sum_{m=0}^{M_{pa}} f_{2k+1,m} \ e^{-\jmath 2\pi m(n-1)/N}. \tag{11}$$

Proof:
From (9), we can write:

$$\mathbf{u} = \sum_{m=0}^{M_{pa}} \sum_{k=0}^{K} f_{2k+1,m} \ \Psi_{2k+1} \left(\mathbf{s}^{(cir)}(m) \right), \tag{12}$$

with

$$\mathbf{s}^{(cir)}(m) = [s_{\text{cir}(1-m,N)}^{(cir)} \ \cdots \ s_{\text{cir}(N-m,N)}^{(cir)}]^T \in \mathbb{C}^{N \times 1}. \tag{13}$$

Note that the vector $\mathbf{s}^{(cir)}(m)$ can be expressed as:

$$\begin{aligned} \mathbf{s}^{(cir)}(m) &= \mathbf{U}\mathbf{s}^{(cir)}(m-1) \\ &= \mathbf{U}^m \mathbf{s}^{(cir)}(0) = \mathbf{U}^m \mathbf{s}, \end{aligned} \tag{14}$$

where $\mathbf{U} \in \mathbb{C}^{N \times N}$ is the circulant lower shift matrix of order N defined as:

$$\mathbf{U} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{15}$$

Thus, substituting (14) into (12), we get:

$$\begin{aligned} \mathbf{u} &= \sum_{m=0}^{M_{pa}} \sum_{k=0}^{K} f_{2k+1,m} \ \Psi_{2k+1}(\mathbf{U}^m \mathbf{s}) \\ &= \sum_{m=0}^{M_{pa}} \sum_{k=0}^{K} f_{2k+1,m} \ \mathbf{U}^m \Psi_{2k+1}(\mathbf{s}) \end{aligned} \tag{16}$$

Calculating the FFT of both sides of (16), we obtain:

$$\bar{\mathbf{u}} = \sum_{m=0}^{M_{pa}} \sum_{k=0}^{K} f_{2k+1,m} \mathbf{V}^* \mathbf{U}^m \Psi_{2k+1}(\mathbf{s}). \tag{17}$$

Using the Lemma in Appendix I, we can rewrite (17) as:

$$\bar{\mathbf{u}} = \sum_{m=0}^{M_{pa}} \sum_{k=0}^{K} f_{2k+1,m} \ \sqrt{N} \ \text{diag}_{(m+1)}[\mathbf{V}^*] \mathbf{V}^* \Psi_{2k+1}(\mathbf{s}). \tag{18}$$

On the other hand, by defining $\tilde{\mathbf{f}}_{2k+1} = [\tilde{f}_{2k+1}(1) \cdots \tilde{f}_{2k+1}(N)]^T \in \mathbb{C}^{N \times 1}$, we can write from (11):

$$\text{diag}[\tilde{\mathbf{f}}_{2k+1}] = \sqrt{N} \sum_{m=0}^{M_{pa}} f_{2k+1,m} \ \text{diag}_{(m+1)}[\mathbf{V}^*]. \tag{19}$$

Substituting (19) into (18), we get:

$$\bar{\mathbf{u}} = \sum_{k=0}^{K} \text{diag}[\tilde{\mathbf{f}}_{2k+1}] \bar{\Psi}_{2k+1}(\mathbf{s}), \tag{20}$$

which corresponds to the desired result.

□

By comparing the expressions for the PA frequency domain output in (7) and (10), it can be concluded that, with respect to the signal $\bar{u}_n$, the memory polynomial model is equivalent to a subcarrier dependent memoryless polynomial model, the relationship between the parameters of these two models being given by (11). Moreover, substituting (20) into (5), we get:

$$\bar{\mathbf{x}} = \Lambda \sum_{k=0}^{K} \text{diag}[\tilde{\mathbf{f}}_{2k+1}] \bar{\Psi}_{2k+1}(\mathbf{s}) + \bar{\mathbf{v}}, \tag{21}$$

or, equivalently,

$$\bar{x}_n = \lambda_n \sum_{k=0}^{K} \tilde{f}_{2k+1}(n) \ \bar{\xi}_{2k+1}(n) + \bar{v}_n, \tag{22}$$

Comparing (8) and (22), it can be viewed that the frequency domain received signals $\bar{x}_n$ have very similar expressions for polynomial and memory polynomial PAs, the only difference being the fact that the PA coefficients f_{2k+1} in (8) are the same for all the subcarriers, while the PA coefficients $\tilde{f}_{2k+1}(n)$ in (22) are dependent on the subcarrier.

An important consequence of the above developments is that some theoretical results and signal processing techniques that work with polynomial PAs can be extended straightforwardly to the case of a memory polynomial PA. In the sequel, we show how an important theoretical result developed for a polynomial PA can be easily extended to a memory polynomial PA using the above results.

IV. Theoretical Error Probability for Nonlinear OFDM Systems

In this section, we develop new expressions for theoretical symbol error rate (SER) provided by an OFDM system with a third-order memory polynomial PA and circular M-QAM or M-PSK modulations. These expressions are developed directly from the theoretical SER obtained for the case of an OFDM system with a polynomial PA.

A. Third-Order Polynomial PA Case

Using the extension of Bussgang's theorem to bandpass memoryless nonlinearities with complex Gaussian inputs, one can derive theoretical SER expressions for an OFDM system with a memoryless PA [12]. In the sequel we derive such expressions for the particular case of a third-order polynomial PA and circular circular M-QAM or M-PSK information signals, based on the results of [12]. Thus, assuming that the transmitted signals s_n are complex Gaussian distributed, eq. (8) can be rewritten as:

$$\bar{x}_n = \lambda_n \left[\alpha_n \bar{s}_n + f_3 \bar{\zeta}_3(n)\right] + \bar{v}_n. \tag{23}$$

with

$$\alpha_n = f_1 + f_3 \frac{r_{\bar{s}\bar{\xi}}}{\sigma_{\bar{s}}^2} = f_1 \left(1 + 2\sigma_{\bar{s}}^2 \frac{f_3}{f_1}\right), \tag{24}$$

where $\sigma_{\bar{s}}^2$ is the variance of $\bar{s}_n$ and $r_{\bar{s}\bar{\xi}} = \mathbb{E}[\bar{s}_n \bar{\xi}_3^*(n)] = 2\sigma_{\bar{s}}^4$ the correlation between $\bar{s}_n$ and $\bar{\xi}_3(n)$, and:

$$\bar{\zeta}_3(n) = \bar{\xi}_3(n) - 2\sigma_{\bar{s}}^2 \bar{s}_n \tag{25}$$

has a complex Gaussian distribution and is uncorrelated with $\bar{s}_{i,n}$. Thus, $\bar{x}_{i,n}$ can be viewed a scaled version of the data symbol $\bar{s}_n$ plus an additive uncorrelated noise of variance:

$$\sigma_T^2(n) = |\lambda_n|^2 |f_3|^2 \sigma_{\bar{\zeta}}^2(n) + \sigma_{\bar{v}}^2, \tag{26}$$

where $\sigma_{\bar{v}}^2$ is the AWGN variance and $\sigma_{\bar{\zeta}}^2(n)$ is given by:

$$\sigma_{\bar{\zeta}}^2(n) = \sigma_{\bar{\xi}}^2 - \frac{r_{\bar{s}\bar{\xi}}^2}{\sigma_{\bar{s}}^2} = \sigma_{\bar{\xi}}^2 - 6\sigma_{\bar{s}}^4, \tag{27}$$

$\sigma_{\bar{\xi}}^2 = 6\sigma_{\bar{s}}^6$ being the variance of $\bar{\xi}_3(n)$.

Thus, considering $\bar{\zeta}_3(i,n)$ as a noise component, the signal to noise ratio (SNR) on the n^{th} subcarrier is given by:

$$SNR_n = \frac{|\lambda_n \alpha_n|^2 \sigma_{\bar{s}}^2}{\sigma_T^2(n)}, \tag{28}$$

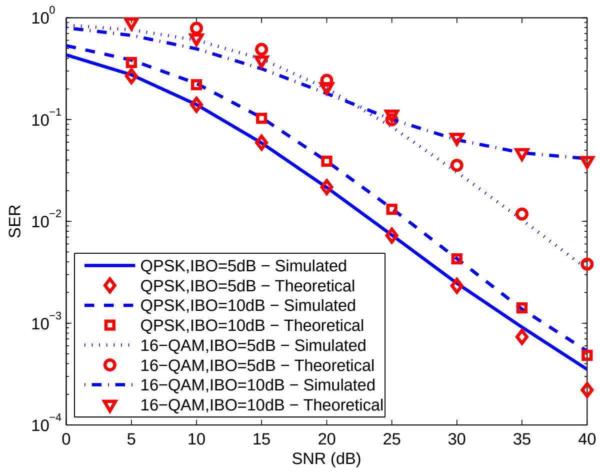

Fig. 1. Theoretical and Simulated SER versus mean SNR.

On the other hand, the theoretical SERs for uniform circular i.i.d. M-PSK and M-QAM signals, for $M \geq 4$, are respectively given by:

$$SER_n = \operatorname{erfc}\left(\sqrt{SNR_n}\ \sin\left(\frac{\pi}{M}\right)\right), \tag{29}$$

and

$$SER_n = 2\left(1 - \frac{1}{\sqrt{M}}\right) \operatorname{erfc}\left(\sqrt{\frac{3SNR_n}{2(M-1)}}\right), \tag{30}$$

where erfc $(\cdot)$ is the Gauss error function. The theoretical SER at the n^{th} subcarrier provided by an OFDM system with a third-order polynomial PA can then be computed from (28)-(30).

B. Third-Order Memory Polynomial PA Case

From (22), the frequency-domain received-signal $\bar{x}_{i,n}$ in an OFDM system with a third-order memory polynomial PA can be expressed as:

$$\bar{x}_n = \lambda_n \left[\tilde{\alpha}_n \bar{s}_n + \tilde{f}_3(n) \bar{\zeta}_3(n)\right] + \bar{v}_n. \tag{31}$$

where

$$\tilde{\alpha}_n = \tilde{f}_1(n) \left(1 + 2\sigma_{\bar{s}}^2 \frac{\tilde{f}_3(n)}{\tilde{f}_1(n)}\right). \tag{32}$$

As $\bar{\zeta}_3(n)$ is uncorrelated with $\bar{s}_{i,n}$, the SNR on the n^{th} subcarrier is given by:

$$SNR_n = \frac{|\lambda_n \tilde{\alpha}_n|^2 \sigma_{\bar{s}}^2}{\tilde{\sigma}_T^2(n)}, \tag{33}$$

where

$$\tilde{\sigma}_T^2(n) = |\lambda_n|^2 |\tilde{f}_3(n)|^2 \sigma_{\bar{\zeta}}^2(n) + \sigma_{\bar{v}}^2, \tag{34}$$

The theoretical SER at the n^{th} subcarrier follows from (33) and (29)-(30).

V. Simulation Results

In this section, the validity of the SER expressions developed in Section IV-B is evaluated by means of simulations. An OFDM system with a third-order memory polynomial PA with coefficients equal to $f_{1,0} = 1.9702 + 0.1931\jmath$; $f_{1,1} = -0.9606 + 0.0036\jmath$; $f_{1,2} = 0$; $f_{1,3} = 0.1591 - 0.0132\jmath$; $f_{3,0} = -0.5934 - 0.1174\jmath$; $f_{3,1} = 0.2300 + 0.0560\jmath$; $f_{3,2} = 0$ and $f_{3,3} = -0.0112 - 0.0094\jmath$ [10] , and a wireless link with frequency selective fading due to multipath propagation has been considered for the simulations. The length of the wireless channel impulse response is 3 and the length of the cyclic prefix is 6. The results were obtained with $N = 64$ subcarriers and QPSK (quadrature PSK) or 16-QAM transmitted signals, via Monte Carlo simulations using 2000 independent data realizations.

Fig. 1 shows the BER versus mean SNR at the channel output obtained by means of simulations and using the theoretical expressions of Section IV-B, with QPSK and 16-QAM signals, and input back-off (IBO) values of 5dB and 10dB. The IBO is defined as the ratio between the input power corresponding to the maximum output power and the mean power of the signal at the input of the PA. It can be viewed that the theoretical and simulated curves are very close. The small differences between the simulated and theoretical SERs are due to the Gaussian approximation of the transmitted signals.

VI. Conclusion

In this work, a new theoretical result concerning OFDM systems with memory polynomial PAs has been demonstrated. It states that a OFDM system with a memory polynomial PA can be expressed as a OFDM system with memoryless polynomial PA with coefficients that vary from one subcarrier to another. Consequently, some theoretical results and signal processing techniques that work for a memoryless polynomial PA can be extended to the case of a memory polynomial PA. As an example of application of this result, new expressions for theoretical SER provided by an OFDM system with a memory polynomial PA have been derived directly from the polynomial PA case.

In a future work, we will use this theoretical result to develop signal processing techniques for ICI canceling in OFDM systems with memory polynomial PAs.

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Appendix I
Lemma

Let $\mathbf{V} \in \mathbb{C}^{N\times N}$ be the FFT matrix of order N and $\mathbf{U} \in \mathbb{C}^{N\times N}$ be a circulant lower shift matrix of order N defined in (15). Then, for $0 \leq i \leq N-1$, we have:

$$\mathbf{V}^*\mathbf{U}^i = \sqrt{N}\ \mathrm{diag}_{(i+1)}[\ \mathbf{V}^*\ ]\mathbf{V}^*. \qquad (35)$$

Proof:
Post-multiplying a matrix by $\mathbf{U}^i$ is equivalent to circularly shifting its columns to the left i times. Thus, by defining $\bar{\mathbf{V}}(i) = \mathbf{V}^*\mathbf{U}^i$, we can write:

$$\bar{\mathbf{V}}(i) = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega^{-i} & \omega^{-(i+1)} & \cdots & \omega^{-(i+N-1)} \\ \omega^{-2i} & \omega^{-2(i+1)} & \cdots & \omega^{-2(i+N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^{-(N-1)i} & \omega^{-(N-1)(i+1)} & \cdots & \omega^{-(N-1)(i+N-1)} \end{pmatrix}. \qquad (36)$$

Eq. (36) can be expressed as:

$$\bar{\mathbf{V}}(i) = \frac{1}{\sqrt{N}}\mathrm{diag}\begin{bmatrix} 1 \\ \omega^{-i} \\ \vdots \\ \omega^{-(N-1)i} \end{bmatrix}\mathbf{V}^*. \qquad (37)$$

That completes the proof. □