UNIVERSIDADE FEDERAL DO CEARÁ
CENTRO DE CIÊNCIAS
DEPARTAMENTO DE MATEMÁTICA
PROGRAMA DE PÓS-GRADUACÃO EM MATEMÁTICA

JOCEL FAUSTINO NORBERTO DE OLIVEIRA

MODIFIED MEAN CURVATURE FLOW OF GRAPHS IN RIEMANNIAN MANIFOLDS

FORTALEZA
2018

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Tese apresentada ao Programa de Pós-graduação em Matemática do Departamento de Matemática da Universidade Federal do Ceará, como parte dos requisitos necessários para a obtenção do título de Doutor em Matemática. Área de concentração: Geometria Diferencial.

Orientador: Prof. Dr. Jorge Herbert Soares de Lira

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Dados Internacionais de Catalogação na Publicação
Universidade Federal do Ceará
Biblioteca Universitária
Gerada automaticamente pelo módulo Catalog, mediante os dados fornecidos pelo(a) autor(a)

O47m Oliveira, Jocel Faustino Norberto de.
Modified Mean Curvature Flow of graphs in Riemannian manifolds / Jocel Faustino Norberto de Oliveira. – 2018.
59 f.

Tese (doutorado) – Universidade Federal do Ceará, Centro de Ciências, Programa de Pós-Graduação em Matemática , Fortaleza, 2018.
Orientação: Prof. Dr. Jorge Herbert Soares de Lira.

1. Modified. 2. Mean curvature flow. 3. Killing field. 4. Graphs. 5. Estimates. I. Título.
CDD 510

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Aprovoda em: 07/12/2018.

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Dedico a meu pai, Eládio Faustino

AGRADECIMENTOS

Agradeço a Deus por me dar saúde e coragem para enfrentar todos os desafios da vida.

A minha amada esposa Anielle Kênia, pelo apoio em todas as horas difíceis nesses últimos anos, onde tive que me ausentar por diversos momentos para me dedicar a este trabalho.

Ao professor Jorge Lira, pela orientação, gentileza e uma enorme paciência.

A minha família, que sempre me apoia quando mais preciso, principalmente os meus pais, Eládio e Maria Ilza e meu irmão, Herivelto.

Aos meus sogros D. Irascilda e José Romão (*In memoriam*) pelo apoio e acolhimento. Agradeço também as minhas cunhadas, Allana e Karyzia pelo carinho e atenção.

Aos colegas da PGMAT pela ajuda direta ou indiretamente, pelo bom convívio e momentos de descontração. Especialmente agradeço a José Tiago, Adam, Amilcar, Fabrício, Emanuel, Halyson, Diego Eloi, Léo Ivo, Damiana e Janielly.

Também gostaria de agradecer aos amigos: Kleber, Edilânio, Joelio, Juscelino, Salete, Aldânia, Valdir, Valdinês e Hugo Lopez.

A todos os professores da Pós-Graduação em Matemática da UFC, em especial aos professores Marcos Melo, Daniel Cibotaru e Diego Moreira, que contribuiram de forma direta para minha formação.

A todos os meus colegas do departamento de Matemática da Universidade Regional do Cariri (URCA) pelo apoio. Destaco aqui meu amigo e colega Flávio França, pelas dicas e sugestões ao longo do período de estudo para formulação desta tese.

Aos funcionários do departamento de Matemática, em especial a Andréa pela prestatividade e agilidade.

A FUNCAP pelo apoio financeiro, o qual foi muito importante.

“Não basta ter razão para estar certo, tem que ter lucidez.”(FERREIRA GULLAR)

RESUMO

Nesta tese provaremos a existência de solução para o problema do fluxo pela curvatura média modificado, com dado inicial suave. Para tal próposito precisaremos de estimativas *a priori* para garantirmos o uso da teoria de equações diferenciais parabólicas, mais precisamente a teoria quasilinear. Nosso resultado principal é obtido para uma variedade suave n-dimensional, completa, não-compacta, com um polo. A variedade ambiente é um *warped-product* dotado de um campo Killing que define a função *warping*. Algumas considerações geométricas são feitas tal que nossa abordagem contemple casos particulares. Equações de evolução são calculadas a fim de serem usadas nas estimativas.

Palavras-chave: Modificado. Fluxo pela curvatura média. Campo de Killing. Gráficos. Estimativas.

ABSTRACT

In this thesis we will prove the existence of solution to the problem of modified mean curvature flow with smooth initial data. For this purpose we will need *a priori* estimates to guarantee the use of the theory of parabolic differential equations, more precisely the quasilinear theory. Our main result is obtained for a smooth, n-dimensional, complete, non-compact manifold with pole. The ambient manifold is a warped-product endowed with a Killing field which defines the warping function. Some geometrical considerations are made in order to encompass particular cases. Evolution equations are calculated in order to be used in the estimates.

Keywords: Modified. Mean curvature flow. Killing field. Graphs. Estimates.

SUMÁRIO

1 INTRODUCTION

In recent years several evolution equations have been studied in Differential Geometry. They lead to interesting systems of nonlinear partial differential equations and provide the appropriate mathematical modelling of physical processes such as material interface propagation and crystal growth. Geometric evolutions as the mean curvature flow are also used in surface processing in Computer Graphics and Signal Processing.

Mean curvature flow (MCF for short) is a term that is used to describe the evolution of a hypersurface whose normal velocity is given by the mean curvature. Let M be an n-dimensional manifold and $\bar{M}$ an $(n+1)$-dimensional Riemannian manifold. A map $\Psi : M \times [0, T) \to \bar{M}$ such that every $\Psi_t := \Psi(\cdot, t) : M \to \bar{M}$ is an immersion, describe the mean curvature flow of Ψ_0 if it is a solution of

$$\frac{\partial \Psi}{\partial t} = n\vec{H}$$

where $\vec{H}(\cdot, t)$ is the mean curvature vector of the immersion Ψ_t.

The papers Ecker and Huisken (1989) and Ecker and Huisken (1991) the mean curvature flow was studied to evolution of non-compact hypersurfaces in Euclidean Space. In Huisken (1989) is studied the case of Dirichlet boundary conditions. In Ecker and Huisken (1989) they show that in the case of Lipschitz initial data with linear growth rate for its height, the flow equation has a smooth solution for all times, moreover they studied the asymptotic behavior of these solutions as $t \to \infty$. In Ecker and Huisken (1991), they obtained some interior estimates and applied them to prove that the hypothesis of linear growth is not necessary, that is: If the initial graph is a locally Lipschitz continuous, then the flow with initial condition Ψ_0 has a smooth solution for all time.

Unterberger (2003) and (1998), studied the MCF in hyperbolic space $\mathbb{H}^{n+1}$ and proved that if the initial surface M_0 has bounded hyperbolic height over $\mathbb{S}^n_+$ (i.e, $\partial M_0 = \partial \mathbb{S}^n_+$) then under the MCF, M_t converges in C^∞ to $\mathbb{S}^n_+$ which is minimal. The hyperbolic case was also studied by Allmann, Lin, and Zhu (2017), where their results are for *modified* mean curvature flow.

There is a vast literature on MCF in different directions, for example Lambert (2014) and Lambert *et al.* (2012) worked mean curvature flow of hypersurfaces in Minkowski space with a perpendicular Neumann boundary condition. He proved that if the boundary manifold is a convex cone made up of timelike rays then any initially spacelike hypersurface satisfying the boundary condition will exist for all time. In Ecker (2012) the notes focussed on the special case where smooth solutions of mean curvature flow develop singularities for the first time. Some important references about geometric flows can be found in Ecker *et al.* (1996),Huisken (1998), Zhu (2002), Ritoré and Sinestrari (2010)among others.

A very interesting case is approached in Borisenko and Miquel (2012), where is proved that if M is completely non-compact with pole and $\bar{M}$ is a warped product, since that M_0 is a smooth graph over M, then there is a solution to MCF for all time, but they do not assume that the initial surface is a graph. Hence they create a set adapted to the immersions and not to the domain, where a diffeomorphism is used. Despite using the methods of Ecker and Huisken (1991) and Unterberger (2003), they not use barriers as done in Unterberger (2003).

This work consists of studying the *modified* mean curvature flow in a warped product structure, where the warping function is the norm of a certain Killing field. Unlike Borisenko and Miquel (2012), here we are taking our initial surface M_0 being a smooth graph and we show that exist a solution to all time. To do this we prove estimates of height, gradient and curvature so we can talk in short time existence and long time existence and thus be supported by the quasilinear parabolic theory. At this point our references are Lieberman (1996), Ecker and Huisken (1991), Ladyzhenskaya (1968), Pulemotov (2013) and also Serrin (1969).

Our height estimate is as in Colding, Minicozzi *et al.* (2003), to prove the gradient estimate were very useful the papers Korevaar (1986) and Ecker and Huisken (1991). To obtain curvature estimates we proceed as Borisenko and Miquel (2012), with this we established our main result:

Theorem 1.1. *Let M be a n-dimensional complete, non-compact Riemannian manifold with a pole o and let $\bar{M}$ be the warped product $M \times_{\varrho} \mathbb{R}$. Suppose that (6), (8) and (9) hold. Given a smooth entire graph M_0 over M, there exists a modified mean curvature flow (2)-(3) defined for all $t \in \mathbb{R}$.*

In order to prove this result we divide this thesis into chapters: In chapter 2 is presented the initial concepts and geometric preliminaries which the work will be developed. In addition, we still display a height estimate. In chapter 3 we prove two types of gradient estimate, but only the second estimate is necessary for our purposes. In chapter 4 we obtain second order bounds, where we deduce evolution equations for the second fundamental form and its squared norm. The main theorem is proved in chapter 5. Some calculations used in the curvature estimate are detailed in the appendix.

2 MODIFIED MEAN CURVATURE FLOW

Let $\bar{M}$ be a $(n+1)$-dimensional complete Riemannian manifold endowed with a complete Killing vector field X. We suppose that X never vanishes and that the orthogonal distribution it determines is integrable. Let M be a given integral leaf of this distribution and let $\Phi : M \times \mathbb{R} \to \bar{M}$ be the flow generated by X with initial values in M. One easily verifies that Φ is a global isometry between $\bar{M}$ and the warped product $M \times_\varrho \mathbb{R}$ endowed with the static warped metric.

$$\varrho^2(x)\,\mathrm{d}s^2 + g,$$

where s is the natural coordinate in $\mathbb{R}$ and g is the induced Riemannian metric in $M \subset \bar{M}$. The coefficient $\varrho \in C^\infty(M)$ is the norm of X which is preserved along the flow lines. In these coordinates, we have $X = \partial_s$ and the leaves $M_s = \Phi_s(M)$ are totally geodesic hypersurfaces in $\bar{M}$.

The Killing graph of a function $u \in C^2(M)$ is by definition the hypersurface in $\bar{M}$ given by

$$\Sigma[u] = \{\Phi(x, u(x)) : x \in M\} \tag{1}$$

A one parameter family of functions $u : M \times [0, T) \to \mathbb{R}$, $T > 0$, defines a *modified* mean curvature flow

$$\Psi(x, t) = \Phi(x, u(x, t)) \tag{2}$$

if and only if

$$\partial_t \Psi = n(H - \sigma)N, \tag{3}$$

where σ is a constant and H is the mean curvature of the Killing graph $\Sigma_t := \Sigma[u(\cdot,\, t)]$ calculated with respect to the orientation given by the unit normal vector field

$$N = N|_{\Psi(\cdot,\, t)} = \frac{1}{W}(\varrho^{-2}X - \Phi_* \nabla^M u) \tag{4}$$

with

$$W = (\varrho^{-2} + |\nabla^M u|^2)^{\frac{1}{2}},$$

where ∇^M denotes the Riemannian connection in (M, g).

We suppose that M is a manifold with a pole. In particular, we consider Gaussian global coordinates $(r, \vartheta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$ defined with respect to a fixed pole $o \in M$. Let $\xi \in C^\infty([0, \infty))$ be a function satisfying the following conditions

$$\begin{aligned} &\xi(r) > 0, \quad \text{for} \quad r > 0, \\ &\xi'(0) = 1, \\ &\xi^{(2k)}(0) = 0, \quad \text{for} \quad k \in \mathbb{N}. \end{aligned} \tag{5}$$

We suppose that the radial sectional curvatures along geodesics rays issuing from o satisfy

$$K(\partial_r \wedge \mathsf{v}) \geq -\frac{\xi''(r)}{\xi(r)} \tag{6}$$

for all $r > 0$, $\mathsf{v} \in TM, \mathsf{v} \perp \partial_r$. In this case, the Hessian comparison theorem in Pigola, Rigoli, and Setti (2005) implies that

$$\nabla^M \nabla^M r \leq \frac{\xi'(r)}{\xi(r)} \left(g - \mathrm{d}r \otimes \mathrm{d}r\right). \tag{7}$$

We also suppose that

$$\left|\frac{\partial_r \varrho}{\varrho}\right| \leq \frac{\xi'(r)}{\xi(r)} \tag{8}$$

and we take

$$\sigma < \frac{1}{n} \inf \left(\frac{|\bar{\nabla} \varrho|}{\varrho} + (n-1)\frac{\xi'(r)}{\xi(r)}\right). \tag{9}$$

Our main result establishes the long-time existence of a modified mean curvature flow of Killing graphs under those conditions on the geometry of the Riemannian warped product $\bar{M} = M \times_\varrho \mathbb{R}$.

2.1 Preliminaries

The induced metric and volume element in $\Sigma_t = \Sigma[u(\cdot, t)]$ are respectively given by

$$g + \varrho^2 \mathrm{d}u \otimes \mathrm{d}u \tag{10}$$

and

$$\mathrm{d}\Sigma_t = \varrho \sqrt{\varrho^{-2} + |\nabla^M u|^2}\, \mathrm{d}M. \tag{11}$$

Given a domain $\Omega \subset M$ and a constant σ we define the constrained area functional

$$\mathcal{A}_\sigma[u] = \int_\Omega \varrho \sqrt{\varrho^{-2} + |\nabla^M u|^2}\, \mathrm{d}M + n\sigma \mathcal{V}[u],$$

where the volume functional $\mathcal{V}$ is defined by

$$\mathcal{V}[u] = \int_\Omega \int_0^{u(x)\varrho(x)} \mathrm{d}M = \int_\Omega \varrho u \, \mathrm{d}M.$$

For an arbitrary compactly supported function $v \in C_0^\infty(\Omega)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{A}_\sigma[u + tv] = -\int_\Omega \left(\mathrm{div}_M \left(\frac{\nabla^M u}{W}\right) + \left\langle \nabla^M \log \varrho, \frac{\nabla^M u}{W}\right\rangle - n\sigma\right) v\varrho\, \mathrm{d}M,$$

where the differential operators ∇^M and div_M are taken with respect to the metric g in M. Then the Euler-Lagrange equation of the functional $\mathcal{A}_\sigma$ is

$$n(H-\sigma) = \operatorname{div}_M\left(\frac{\nabla^M u}{W}\right) + \left\langle \nabla^M \log\varrho, \frac{\nabla^M u}{W}\right\rangle - n\sigma = 0, \tag{12}$$

where H is the mean curvature of the Killing graph of u. However, differentiating (2) with respect to t in gives

$$\partial_t\Psi = \partial_t u\, X.$$

Hence (3) is equivalent to

$$\partial_t u\, X = \left(\operatorname{div}_M\left(\frac{\nabla^M u}{W}\right) + \left\langle \nabla^M \log\varrho, \frac{\nabla^M u}{W}\right\rangle - n\sigma\right) N.$$

Taking the normal projection on both sides yields

$$\partial_t u\langle X, N\rangle = \operatorname{div}_M\left(\frac{\nabla^M u}{W}\right) + \left\langle \nabla^M \log\varrho, \frac{\nabla^M u}{W}\right\rangle - n\sigma.$$

Since $\langle X, N\rangle = 1/W$ we conclude that (2) defines a modified mean curvature flow if and only if $u(\cdot,\, t)$ satisfies the parabolic equation

$$\partial_t u = \mathcal{Q}[u], \tag{13}$$

where

$$\mathcal{Q}[u] = W\left(\operatorname{div}_M\left(\frac{\nabla^M u}{W}\right) + \left\langle \nabla^M \log\varrho, \frac{\nabla^M u}{W}\right\rangle - n\sigma\right). \tag{14}$$

In general, this non-parametric formulation is equivalent to the modified mean curvature flow (3) up to tangential diffeomorphisms of the evolving graphs Σ_t. The equivalence here follows from the fact that we are assuming a fixed *gauge*, namely the choice of coordinates fixed in (2).

Now we deduce some evolution equations that will be useful in the sequel.

Proposition 2.1. *Suppose that (6) holds. The restrictions of the functions r and s to the graphs Σ_t, $t\in[0,T)$, satisfy*

$$(\partial_t-\Delta)r \geq -\frac{\xi'(r)}{\xi(r)}\left(n-|\nabla r|^2\right) - \varrho^2|\nabla s|^2\left(\langle\bar\nabla\log\varrho, \nabla r\rangle - \frac{\xi'(r)}{\xi(r)}\right) - n\sigma\langle\bar\nabla r, N\rangle \tag{15}$$

and

$$(\partial_t-\Delta)s = -(2\langle\bar\nabla\log\varrho, N\rangle + n\sigma)\langle\bar\nabla s, N\rangle. \tag{16}$$

In both expressions, ∇ and Δ are the intrinsic Riemannian connection and Laplacian in Σ_t, respectively, whereas $\bar\nabla$ denotes the Riemannian connection in $\bar M$. Moreover, given *the function*

$$\zeta(\Psi(x,t)) = \int_0^{r(\Psi(t,x))} \xi(\varsigma)\,\mathrm{d}\varsigma \tag{17}$$

we have

$$(\partial_t - \Delta)\zeta \geq -n\xi'(r) - \varrho^2|\nabla s|^2\xi(r)\left(\langle\bar\nabla\log\varrho, \nabla r\rangle - \frac{\xi'(r)}{\xi(r)}\right) - n\sigma\xi(r)\langle\bar\nabla r, N\rangle. \tag{18}$$

Proof: Observe that $\bar\nabla s = \varrho^{-2}X$ and $\nabla s = \varrho^{-2}X^\top$, where $\top$ denotes the tangential projection onto $T\Sigma_t$. Given a local orthonormal tangent frame $\{\mathsf{e}_i\}_{i=1}^n$ in Σ_t, one has

$$\begin{aligned}\Delta s &= \langle\nabla\varrho^{-2}, X^\top\rangle + \varrho^{-2}\sum_{i=1}^n\langle\bar\nabla_{\mathsf{e}_i}X, \mathsf{e}_i\rangle + nH\langle\varrho^{-2}X, N\rangle = \langle\bar\nabla\varrho^{-2}, X^\top\rangle + nH\langle\bar\nabla s, N\rangle\\ &= -\langle\bar\nabla\varrho^{-2}, N\rangle\langle X, N\rangle + nH\langle\bar\nabla s, N\rangle = 2\langle\bar\nabla\log\varrho, N\rangle\langle\bar\nabla s, N\rangle + nH\langle\bar\nabla s, N\rangle.\end{aligned}$$

We also compute

$$\partial_t s = \langle\bar\nabla s, \partial_t\Psi\rangle = n(H-\sigma)\langle\bar\nabla s, N\rangle.$$

Therefore

$$(\partial_t - \Delta)s = -(2\langle\bar\nabla\log\varrho, N\rangle + n\sigma)\langle\bar\nabla s, N\rangle.$$

Now we obtain

$$\langle\bar\nabla_X\bar\nabla r, X\rangle = \langle\bar\nabla_{\bar\nabla r}X, X\rangle = \frac{1}{2}\partial_r|X|^2 = \frac{1}{2}\partial_r\varrho^2 = \varrho\langle\bar\nabla\varrho, \bar\nabla r\rangle.$$

Fixed a local orthonormal tangent frame $\{\mathsf{e}_i\}_{i=1}^n$ in Σ_t, we have

$$\begin{aligned}\Delta r &= \sum_i\langle\nabla_{\mathsf{e}_i}\nabla r, \mathsf{e}_i\rangle = \sum_i\langle\bar\nabla_{\mathsf{e}_i}(\bar\nabla r - \langle\bar\nabla r, N\rangle N), \mathsf{e}_i\rangle\\ &= \sum_i\langle\nabla^M_{\pi_*\mathsf{e}_i}\pi_*\bar\nabla r, \pi_*\mathsf{e}_i\rangle + \frac{1}{\varrho^4}\sum_i\langle\mathsf{e}_i, X\rangle^2\langle\bar\nabla_X\bar\nabla r, X\rangle - \sum_i\langle\bar\nabla r, N\rangle\langle\bar\nabla_{\mathsf{e}_i}N, \mathsf{e}_i\rangle\\ &= \sum_i\langle\nabla^M_{\pi_*\mathsf{e}_i}\pi_*\bar\nabla r, \pi_*\mathsf{e}_i\rangle + \frac{1}{\varrho^4}|X^\top|^2\langle\varrho\bar\nabla\varrho, \nabla r\rangle - \sum_i\langle\bar\nabla r, N\rangle\langle\bar\nabla_{\mathsf{e}_i}N, \mathsf{e}_i\rangle\\ &= \sum_i\langle\nabla^M_{\pi_*\mathsf{e}_i}\nabla^M r, \pi_*\mathsf{e}_i\rangle + |\nabla s|^2\langle\varrho\bar\nabla\varrho, \nabla r\rangle + nH\langle\bar\nabla r, N\rangle\end{aligned}$$

where $\pi : \bar M = M\times\mathbb{R}\to M$ is the projection on the first factor, that is, $\pi(x,s) = x$ for all $(x,s)\in M\times\mathbb{R}$. The Hessian comparison theorem (7) implies that

$$\begin{aligned}\Delta r &\le \frac{\xi'(r)}{\xi(r)}\sum_i\left(|\pi_*\mathsf{e}_i|^2-\langle\mathsf{e}_i,\nabla^P r\rangle^2\right)+|\nabla s|^2\langle\varrho\bar\nabla\varrho,\nabla r\rangle+nH\langle\bar\nabla r,N\rangle\\ &=\frac{\xi'(r)}{\xi(r)}\left(n-\frac{1}{\varrho^2}|X^\top|^2-|\nabla r|^2\right)+\varrho^2|\nabla s|^2\langle\bar\nabla\log\varrho,\nabla r\rangle+nH\langle\bar\nabla r,N\rangle\\ &=\frac{\xi'(r)}{\xi(r)}\left(n-\varrho^2|\nabla s|^2-|\nabla r|^2\right)+\varrho^2|\nabla s|^2\langle\bar\nabla\log\varrho,\nabla r\rangle+nH\langle\bar\nabla r,N\rangle.\end{aligned}\tag{19}$$

Hence,

$$\Delta r\le\frac{\xi'(r)}{\xi(r)}\left(n-|\nabla r|^2\right)+\varrho^2|\nabla s|^2\left(\langle\bar\nabla\log\varrho,\nabla r\rangle-\frac{\xi'(r)}{\xi(r)}\right)+nH\langle\bar\nabla r,N\rangle.\tag{20}$$

We also have $\nabla\zeta=\xi(r)\nabla r$ and

$$\Delta\zeta=\xi(r)\Delta r+\xi'(r)|\nabla r|^2.\tag{21}$$

Therefore

$$\Delta\zeta\le n\xi'(r)+\varrho^2|\nabla s|^2\xi(r)\left(\langle\bar\nabla\log\varrho,\nabla r\rangle-\frac{\xi'(r)}{\xi(r)}\right)+nH\xi(r)\langle\bar\nabla r,N\rangle.\tag{22}$$

On the other hand

$$\partial_t r=\left\langle\bar\nabla r,\partial_t\Psi\right\rangle=n(H-\sigma)\langle\bar\nabla r,N\rangle$$

and

$$\partial_t\zeta=n(H-\sigma)\xi(r)\langle\bar\nabla r,N\rangle.$$

We conclude that

$$(\partial_t-\Delta)r\ge-\frac{\xi'(r)}{\xi(r)}\left(n-|\nabla r|^2\right)-\varrho^2|\nabla s|^2\left(\langle\bar\nabla\log\varrho,\nabla r\rangle-\frac{\xi'(r)}{\xi(r)}\right)-n\sigma\langle\bar\nabla r,N\rangle\tag{23}$$

and

$$(\partial_t-\Delta)\zeta\ge-n\xi'(r)-\varrho^2|\nabla s|^2\xi(r)\left(\langle\bar\nabla\log\varrho,\nabla r\rangle-\frac{\xi'(r)}{\xi(r)}\right)-n\sigma\xi(r)\langle\bar\nabla r,N\rangle\tag{24}$$

This finishes the proof of the proposition. □

Proposition 2.2. *If the graphs Σ_t, $t\in[0,T]$, evolve by the modified mean curvature flow (2)-(3), then*

$$(\partial_t-\Delta)\,W=-W(|A|^2+\overline{\mathrm{Ric}}(N,N))-2W^{-1}|\nabla W|^2,\tag{25}$$

where $W=\langle X,N\rangle^{-1}=(\varrho^{-2}+|\nabla^M u)|^2)^{1/2}$ and A is the Weingarten map of Σ_t. If the *ambient Ricci tensor satisfies* $\overline{\mathrm{Ric}} \geq -L$ *for some non-negative constant* L *then*

$$(\partial_t - \Delta)(e^{-Lt}W) \leq 0.$$

In particular, the parabolic maximum principle implies in this case that

$$\sup_{B_R(o)\times[0,T]} W(x,t) \leq e^{LT}\big(\sup_{B_R(o)} W(\cdot,0) + \sup_{\partial B_R(o)\times(0,T]} W\big). \tag{26}$$

Proof: Note that

$$\nabla\langle X, N\rangle = \langle X, N\rangle(\bar\nabla \log\varrho)^\top - \langle\bar\nabla\log\varrho, N\rangle X^\top - AX^\top. \tag{27}$$

Hence,

$$\begin{aligned}&\langle X, N\rangle(\bar\nabla \log\varrho)^\top - \langle\bar\nabla\log\varrho, N\rangle X^\top\\ &\quad= \langle X, N\rangle(\bar\nabla\log\varrho - \langle\bar\nabla\log\varrho, N\rangle N) - \langle\bar\nabla\log\varrho, N\rangle(X - \langle X, N\rangle N)\\ &\quad= \langle X, N\rangle\bar\nabla\log\varrho - \langle\bar\nabla\log\varrho, N\rangle X.\end{aligned}$$

It follows from the second variation formula for the functional $\mathcal{A}_0$ that

$$\Delta\langle X, N\rangle + |A|^2\langle X, N\rangle + \overline{\mathrm{Ric}}(N,N)\langle X, N\rangle = -n\langle\nabla H, X^\top\rangle,$$

where $|A|$ stands for the norm of the Weingarten map of Σ_t and $\top$ denotes the tangencial projection onto $T\Sigma_t$. On the other hand, using that X is a Killing vector field one gets

$$\begin{aligned}\partial_t\langle X, N\rangle &= \langle\bar\nabla_{\partial_t}X, N\rangle + \langle X, \bar\nabla_{\partial_t}N\rangle = n(H-\sigma)\langle\bar\nabla_N X, N\rangle - n\langle X, \nabla(H-\sigma)\rangle\\ &= -n\langle X^\top, \nabla H\rangle,\end{aligned}$$

where $\bar\nabla$ denotes the Riemannian connection in $\bar M$ and so

$$(\partial_t - \Delta)\langle X, N\rangle = |A|^2\langle X, N\rangle + \overline{\mathrm{Ric}}(N,N)\langle X, N\rangle$$

Thus, using that $\langle X, N\rangle = 1/W$ one has

$$\partial_t W = -W^2\partial_t W^{-1} = -W^2\partial_t\langle X, N\rangle$$

and

$$\Delta W - \frac{2}{W}|\nabla W|^2 = -W^2\Delta W^{-1} = -W^2\Delta\langle X, N\rangle. \tag{28}$$

Hence, one concludes that

$$\partial_t W - \Delta W + \frac{2}{W}|\nabla W|^2 = -W^2(\partial_t - \Delta)\langle X, N\rangle = -W(|A|^2 + \overline{\mathrm{Ric}}(N,N))$$

A direct application of the parabolic maximum principle in Mantegazza (2011) finishes the proof of the proposition. □

2.2 The case of rotationally symmetric metrics

Let M_+ be a complete, non-compact, n-dimensional model manifold with respect to a fixed pole $o_+ \in M_+$ in the sense that the Riemannian metric in M_+ can be expressed in Gaussian coordinates $(r,\vartheta) \in \mathbb{R} \times \mathbb{S}^{n-1}$ centered at o_+ as

$$g_+ = \mathrm{d}r^2 + \xi_+^2(r)\,\mathrm{d}\vartheta^2 \tag{29}$$

where $\mathrm{d}\vartheta^2$ denotes the round metric in $\mathbb{S}^{n-1}$ and $\xi_+ \in C^\infty([0,\infty))$ satisfying conditions (5). We also consider a smooth radial function $\varrho_+ \in C^\infty([0,\infty))$ satisfying

$$\begin{aligned} &\varrho_+(r) > 0, \quad \text{for} \quad r > 0,\\ &\varrho_+(0) = 1,\\ &\varrho_+^{(2k+1)} = 0, \quad \text{for} \quad k \in \mathbb{N} \end{aligned} \tag{30}$$

and define the warped metric in $M_+ \times \mathbb{R}$

$$\varrho_+^2(r)\mathrm{d}s^2 + \mathrm{d}r^2 + \xi_+^2(r)\,\mathrm{d}\vartheta^2. \tag{31}$$

We denote

$$A(r) = \varrho_+(\varsigma)\xi_+^{n-1}(\varsigma) \tag{32}$$

and

$$V(r) = \int_0^r \varrho_+(\varsigma)\xi_+^{n-1}(\varsigma)\,\mathrm{d}\varsigma. \tag{33}$$

We also define

$$H(r) = \sigma - \frac{1}{n}\frac{A(r)}{V(r)}. \tag{34}$$

Given $x \in M_+$ we denote $r(x) = \mathrm{dist}(o_+, x)$. Let $B_R(o_+)$ be the closed geodesic ball centered at o_+ with radius $R \in (0,\infty)$. Hence $x \in B_R(o_+)$ if and only if $r(x) \leq R$. The mean curvature of the Killing cylinder over the geodesic sphere $\partial B_r(o_+)$ is given by

$$H_{\mathrm{cyl}}(r) = \frac{1}{n}\left((n-1)\frac{\xi_+'(r)}{\xi_+(r)} + \frac{\varrho_+'(r)}{\varrho_+(r)}\right). \tag{35}$$

Fixed this notation, we are able to state the following result.

Proposition 2.3. *For each $R \in (0,\infty)$, the graph of the function*

$$v_R(x) = \int_R^{r(x)} \frac{n(H(R)-\sigma)V(\varsigma)}{\varrho(\varsigma)(A^2(\varsigma) - n^2(H(R)-\sigma)^2V^2(\varsigma))^{\frac{1}{2}}}\,\mathrm{d}\varsigma \tag{36}$$

defined in $B_R(o_+)$ has constant mean curvature $H(R)-\sigma$ and its boundary is the geodesic sphere $\partial B_R(o_+)$.

Proof. Fixed $R\in(0,\infty)$ let v_R be the (radial) solution of the following Dirichlet problem for the constant mean curvature equation

$$\begin{cases} \operatorname{div}_+\left(\frac{\nabla^+ v_R}{W_+}\right)+g_+\left(\nabla\log\varrho_+,\frac{\nabla^+ v_R}{W_+}\right)=n(H(R)-\sigma) \ \text{ in } \ B_R(o_+), \\ v_R|_{\partial B_R(o_+)}=0, \end{cases} \tag{37}$$

where the differential operators div_+ and ∇^+ are defined with respect to the metric (29) in M_+ and

$$W_+=(\varrho_+^{-2}(r)+v_R'^2(r))^{\frac12},$$

with $'$ denoting derivatives with respect to r. Note that (37) can be written in terms of a weighted divergence as

$$\operatorname{div}_{-\log\varrho_+}\left(\frac{\nabla^+ v_R}{W_+}\right)=\frac{1}{\varrho_+}\operatorname{div}_+\left(\varrho_+\frac{\nabla^+ v_R}{W_+}\right)=n(H(R)-\sigma). \tag{38}$$

Integrating with respect to the density $\varrho_+\,\mathrm{d}M_+$ yields

$$\begin{aligned}\int_{B_r(o)} n(H(R)-\sigma)\,\mathrm{d}M_+ &= \int_{B_r(o)}\operatorname{div}_+\left(\varrho_+\frac{\nabla v_R}{W}\right)\mathrm{d}M_+ \\ &= \int_{\partial B_r(o)} g_+\left(\frac{\nabla v_R}{W_+},\partial_r\right)\varrho_+\,\mathrm{d}\partial B(r),\end{aligned} \tag{39}$$

for $r\le R$. Since v_R is radial (37) becomes

$$\begin{aligned}&\left(\frac{v_R'(r)}{(\varrho_+^{-2}(r)+v_R'^2(r))^{1/2}}\right)'+\frac{v_R'(r)}{(\varrho_+^{-2}(r)+v_R'^2(r))^{1/2}}\left(\frac{\varrho_+'(r)}{\varrho_+(r)}+(n-1)\frac{\xi_+'(r)}{\xi_+(r)}\right)\\ &=n(H(R)-\sigma).\end{aligned} \tag{40}$$

It follows from (39) that v_R is the solution of the first order equation

$$\frac{v_R'(r)}{(\varrho_+^{-2}(r)+v_R'^2(r))^{1/2}}\,\varrho_+(r)\xi_+^{n-1}(r)=\int_0^r n(H(R)-\sigma)\varrho_+(\varsigma)\xi_+^{n-1}(\varsigma)\,\mathrm{d}\varsigma. \tag{41}$$

with initial condition $v_R|_{r=R}=0$. Resolving this expression for v_R', one obtains

$$v_R'(r)=\frac{n(H(R)-\sigma)V(r)}{\varrho_+(r)(A^2(r)-n^2(H(R)-\sigma)^2V^2(r))^{1/2}}. \tag{42}$$

The graph Σ_R of v_R is a rotationally invariant hypersurface which can be parametrized in terms of coordinates (s,r,ϑ) as $\varsigma\mapsto(s(\varsigma),\vartheta,r(\varsigma))$, where ς can be taken as the arc lenght parameter. Denoting by ϕ the angle between the coordinate vector field ∂_r and a given profile curve $\vartheta = \text{constant}$ in Σ_R one has

$$\dot{r} = \cos\phi, \quad \varrho\dot{s} = \sin\phi.$$

Thus, (40) becomes

$$-\frac{\mathrm{d}}{\mathrm{d}\varsigma}(\varrho\dot{s})\frac{\mathrm{d}\varsigma}{\mathrm{d}r} - \varrho\dot{s}\left(\frac{\varrho_+'(r)}{\varrho_+(r)} + (n-1)\frac{\xi_+'(r)}{\xi_+(r)}\right) = n(H(R)-\sigma),$$

that is,

$$\frac{\mathrm{d}\phi}{\mathrm{d}\varsigma} + \sin\phi\left(\frac{\varrho'(r)}{\varrho(r)} + (n-1)\frac{\xi'(r)}{\xi(r)}\right) = -n(H(R)-\sigma).$$

Hence, a profile curve of Σ_R is given by the solution of the first order system

$$\begin{cases} \dot{r} = \cos\phi, \\ \varrho\dot{s} = \sin\phi, \\ \dot{\phi} = -n(H(R)-\sigma) - nH_{\mathrm{cyl}}(r)\sin\phi, \end{cases}$$

with initial conditions $r(0) = R, s(0) = 0, \phi(0) = \frac{\pi}{2}$. In this case (41) can be rewritten as

$$\varrho(r)A(r)\dot{s} = -n(H(R)-\sigma)V(r)$$

where $\cdot$ indicates derivatives with respect to the parameter ς. Hence, it is immediate that when the coordinate r attains its maximum, that is, when $r = R$, we have $\dot{r} = 0$ and $\dot{s} = 1$. This is consistent with the choice of $H(R)$ in (34). We also observe that $\dot{s} \to 0$ and $\dot{r} \to 1$ as $r \to 0^+$. □

For $R \geq r_0$, note that the variable $\mu = R - r_0$ can be considered as the geodesic distance between the geodesic spheres $\partial B_{r_0}(o) = \partial\Sigma_{r_0}$ and $\partial B_R(o) = \partial\Sigma_R$. Hence, $\nabla\mu|_{\partial B_R(o)} = \partial_r|_{r=R}$. Thus, we set a time parameter $t \in [0,\infty)$ given by

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = -n(H(R)-\sigma) = -n(H(\mu+r_0)-\sigma), \tag{43}$$

$$\mu(0) = 0. \tag{44}$$

Hence, $\mu = \mu(t)$ is implicitly defined by

$$\int_{r_0}^{\mu(t)+r_0} \frac{V(\varsigma)}{A(\varsigma)}\,\mathrm{d}\varsigma = t$$

Denote $R(t) = \mu(t) + r_0$. We claim that the one-parameter family of constant mean curvature graphs $\{\Sigma_{R(t)}\}_{t\geq 0}$ evolves by the (negative) mean curvature flow

$$\partial_t \Psi^+ = -n(H(R(t)) - \sigma)N_t, \tag{45}$$

where

$$N_t = \frac{1}{W}(\varrho^{-2}(r)X - v_R'(r)\partial_r) = -\frac{\dot r}{\varrho}X + \varrho \dot s\, \partial_r.$$

This means that $\Sigma_{R(t)} = \Psi_t^+(\Sigma_{r_0})$. In particular, we must have

$$\partial B_{R(t)} = \partial \Sigma_{R(t)} = \Psi_t^+(\partial \Sigma_{r_0}) = \Psi_t^+(\partial B_{r_0}).$$

In other terms, the time parameter t must be chosen in such a way that the geodesic spheres evolve as $\partial B_{R(t)} = \Psi_\tau^+(\partial B_{r_0})$. Since $\dot r = 0$ and $\varrho \dot s = 1$ at $r = R(t)$ it follows from (45) that

$$\begin{aligned}\frac{\mathrm{d}\mu}{\mathrm{d}t} &= \langle \partial_t \Psi^+, \nabla \mu\rangle = \langle \partial_t \Psi^+, \partial_r\rangle = -n(H(R(t)) - \sigma)\langle N_t, \partial_r\rangle_{r=R(t)}\\ &= n(H(r_0 + \mu(t)) - \sigma)\end{aligned}$$

what means that t coincides with the parameter defined in (43) and then satisfying the condition that $\partial B_{R(t)} = \Psi_t^+(\partial B_{r_0})$. Note that $R(t) \geq r_0$ for $t \geq 0$. We conclude that the one-parameter family of functions $u_+(x,t) = v_{R(t)}(r(x))$ defined on the common domain $B_{r_0}(o)$ defines a solution of the geometric flow (45). Hence, we set

$$\Psi^+(x,t) = (x, u_+(x,t)), \quad x \in B_{r_0}(o). \tag{46}$$

It follows that u_+ satisfies the parabolic equation

$$\begin{aligned}\partial_t u_+ = -(\varrho_+^{-2}(r) + |\partial_r u_+|^2)^{1/2}\Bigg(\partial_r\left(\frac{\partial_r u_+}{(\varrho_+^{-2}(r) + |\partial_r u_+|^2)^{1/2}}\right)\\ + \frac{\partial_r u_+}{(\varrho_+^{-2}(r) + |\partial_r u_+|^2)^{1/2}}\left(\frac{\varrho_+'(r)}{\varrho_+(r)} + (n-1)\frac{\xi_+'(r)}{\xi_+(r)}\right) - n\sigma\Bigg).\end{aligned} \tag{47}$$

Besides providing an important example, this flow may be used as a supersolution to the modified mean curvature flow in manifolds whose metrics are not necessarily invariant by rotations. This is the content of the following propositions.

Proposition 2.4. *Suppose that (6) holds with $\xi(r) = \xi_+(r)$ and that $\varrho(x) = \varrho_+(r(x))$. Then, the one-parameter family of functions*

$$u_+(x,t) = v_{R(t)}(r(x)), \quad x \in B_{r_0}(o), \quad t \in [0,\infty) \tag{48}$$

satisfies $\partial_t u_+ + \mathcal{Q}[u_+] \geq 0$ in $B_{r_0}(o) \times [0,T]$ for all $T > 0$.

Proof: Denoting

$$W=(\varrho^{-2}+|\nabla^M u_+|^2)^{1/2}=(\varrho_+^{-2}+u_+'^2(r))^{1/2}$$

we have

$$\begin{aligned}\mathcal{Q}[u_+]-\partial_t u_+ &= W\left(\mathrm{div}_M\left(\frac{\nabla^M u_+}{W}\right)+\left\langle\nabla^M\log\varrho,\frac{\nabla^M u_+}{W}\right\rangle-n\sigma\right)-\partial_t u_+\\ &=(\varrho^{-2}+u_+'^2(r))^{1/2}\Bigg(\partial_r\left(\frac{u_+'(r)}{(\varrho_+^{-2}+u_+'^2(r))^{1/2}}\right)\\ &+\frac{u_+'(r)}{(\varrho_+^{-2}+u_+'^2(r))^{1/2}}\big(\Delta_M r+\langle\nabla^M\log\varrho,\nabla^M r\rangle\big)-n\sigma\Bigg)-\partial_t u_+.\end{aligned}$$

However

$$\langle\nabla\log\varrho,\nabla r\rangle=\frac{\partial_r\varrho}{\varrho}=\frac{\varrho_+'(r)}{\varrho_+(r)}.$$

Moreover (7) implies that

$$\Delta_M r\le(n-1)\frac{\xi_+'(r)}{\xi_+(r)}.$$

Since $u_+'=v_R'\le 0$ we conclude that

$$\begin{aligned}\mathcal{Q}[u_+]+\partial_t u_+ &\ge(\varrho_+^{-2}+u_+'^2(r))^{1/2}\Bigg(\partial_r\left(\frac{u_+'(r)}{(\varrho_+^{-2}+u_+'^2(r))^{1/2}}\right)\\ &+\frac{u_+'(r)}{(\varrho_+^{-2}+u_+'^2(r))^{1/2}}\left(\frac{\varrho_+'(r)}{\varrho_+(r)}+(n-1)\frac{\xi_+'(r)}{\xi_+(r)}\right)-n\sigma\Bigg)+\partial_t u_+=0.\end{aligned}$$

This finishes the proof. □

Proposition 2.5. *Suppose that (6) holds with* $\xi(r)=\xi_+(r)$ *and that* $\varrho(x)=\varrho_+(r(x))$. *Let* u *be a solution of* (2)-(3) *in* $B_R(o)\times[0,T]$ *with Dirichlet boundary condition* $u(x,t)=u(x,0)\in\partial B_{r_0}(o)\times[0,T]$. *Then we have the following height estimate*

$$|u(x,t)|\le\sup_{B_{r_0}(o)}|u|+v_{R(T)}(o)-v_{r_0}(r(x)),\tag{49}$$

that is,

$$\begin{aligned}|u(x,t)|\le\sup_{B_{r_0}(o)}|u(\cdot,0)|&+\int_{R(T)}^0\frac{n(H(R(T))-\sigma)V(\varsigma)}{\varrho(\varsigma)(A^2(\varsigma)-n^2(H(R(T))-\sigma)^2V^2(\varsigma))^{\frac12}}\,\mathrm{d}\varsigma\\ &-\int_{r_0}^{r(x)}\frac{n(H(r_0)-\sigma)V(\varsigma)}{\varrho(\varsigma)(A^2(\varsigma)-n^2(H(r_0)-\sigma)^2V^2(\varsigma))^{\frac12}}\,\mathrm{d}\varsigma.\end{aligned}\tag{50}$$

for $(x,t)\in B_{r_0}(o)\times[0,T]$.

Proof: By construction, the graph Σ_{r_0} of $u_+(\cdot,0)=v_{r_0}$ is defined in the geodesic ball $B_{r_0}(o)$. Given $T>0$ we have that $\Psi_T^+(\Sigma_{r_0})$ is the graph $\Sigma_{R(T)}$ of $u_+(\cdot,T)=v_{R_T}|_{B_{r_0}(o)}$

with

$$\int_{r_0}^{R(T)} \frac{V(\varsigma)}{A(\varsigma)}\,\mathrm{d}\varsigma = T.$$

Given $\varepsilon > 0$ we have

$$-u_+(x,T) + u_+(o,T) + \sup_{B_{r_0}(o)} u + \varepsilon > u(x,0)$$

for all $x \in B_{r_0}(o)$. We also have

$$v_\varepsilon(x,t) := -u_+(x,T-t) + u_+(o,T) + \sup_{B_{r_0}(o)} u + \varepsilon > u(x,t)$$

for all $(x,t) \in \partial B_{r_0}(o) \times [0,T]$. Proposition 2.4 implies that

$$\partial_t v_\varepsilon - \mathcal{Q}[v_\varepsilon] = \partial_t u_+ + \mathcal{Q}[u_+] \geq 0 \tag{51}$$

in the parabolic cylinder $B_{r_0}(o) \times (0,T)$. Then the parabolic maximum principle implies that

$$u(x,t) \leq v(x,t) \leq v(x,T)$$

in $B_{r_0}(o) \times [0,T]$ where

$$v(x,t) = -u_+(x,T-t) + u_+(o,T) + \sup_{B_{r_0}(o)} u. \tag{52}$$

Hence

$$u(x,t) \leq v(x,T) = u_+(o,T) - u_+(x,0) + \sup_{B_{r_0}(o)} u.$$

Therefore

$$u(x,t) \leq \sup_{B_{r_0}(o)} u + v_{R(T)}(o) - v_{r_0}(r(x)) \tag{53}$$

for $(x,t) \in B_{r_0}(o) \times [0,T]$. We prove in a similar way that

$$u(x,t) \geq w(x,t) \geq w(x,T)$$

in $B_{r_0}(o) \times [0,T]$ where

$$w(x,t) = u_+(x,T-t) - u_+(o,T) + \inf_{B_{r_0}(o)} u. \tag{54}$$

Hence

$$u(x,t) \geq \inf_{B_{r_0}(o)} u - v_{R(T)}(o) + v_{r_0}(r(x)) \tag{55}$$

in $B_{r_0}(o) \times [0,T]$. This finishes the proof.

2.3 The case of the Euclidean space

In the case where $\bar{M}$ is the Euclidean space and X is a parallel vector field with $\varrho = 1$ we have

$$H(r) - \sigma = -\frac{1}{r}$$

and

$$v_R(x) = -\int_R^{r(x)} \frac{R^{-1}\varsigma^n}{(\varsigma^{2(n-1)} - R^{-2}\varsigma^{2n})^{\frac{1}{2}}}\,\mathrm{d}\varsigma = -\int_R^{r(x)} \frac{\varsigma}{(R^2 - \varsigma^2)^{\frac{1}{2}}}\,\mathrm{d}\varsigma = (R^2 - r^2(x))^{1/2}.$$

Therefore the suitable time parameter defined by

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{n}{R(t)} = \frac{n}{\mu(t) + r_0}$$

is given explicitly by

$$R(t) = (r_0^2 + 2nt)^{1/2}, \quad t \in [0, \infty).$$

In this particular case, it is obvious that

$$\sup_{B_{r_0}(o)\times[0,T]} u_+ = \sup_{[0,T]} u_+(o,t) = v_{R(T)}(o) = R(T) \sim T^{1/2}. \tag{56}$$

2.4 The case of the hyperbolic space

Now we discuss in some detail the case where $\bar{M}$ is the hyperbolic space $\mathbb{H}^{n+1}$ which has been already considered in the references Unterberger (2003) and Allmann, Lin, and Zhu (2017). We can define a mean curvature flow ($\sigma = 0$) of geodesic spheres in $\mathbb{H}^{n+1}$ defining a time parameter by the ODE

$$\frac{\mathrm{d}R}{\mathrm{d}t} = n\frac{\cosh(R(t))}{\sinh(R(t))}$$

whose general solution has the form

$$\cosh R(t) = e^{nt}\cosh r_0,$$

where $r_0 > 0$ is the radius of the geodesic sphere at time $t = 0$. In order to give a concrete description of this mean curvature flow, we consider the half-space model

$$\mathbb{H}^{n+1} = \{(x, z) \in \mathbb{R}^n \times (0, +\infty)\}.$$

Let $\mathbb{H}^n$ be the totally geodesic hypersurface $|x|^2 + z^2 = 1$ with $z > 0$ and fix polar coordinates (r, ϑ) centered at $o = (0, 1) \in \mathbb{H}^n$. Hence, we can define cylindrical coordinates (s, r, ϑ) in $\mathbb{H}^{n+1}$ where s is the flow parameter of the Killing field X represented by the Euclidean radial vector field $X(x, z) = (x, z)$. The relation between Euclidean coordinates and hyperbolic cylindrical coordinates is given by

$$e^s = (|x|^2 + z^2)^{\frac{1}{2}} \quad \text{and} \quad \sinh r = \frac{|x|}{z} = \tan\vartheta$$

or, in other terms,

$$|x| = e^s \tanh r \quad \text{and} \quad z = \frac{1}{\cosh r} e^s.$$

Since the mean curvature of the geodesic sphere Σ_t at time t is $H(R(t)) = \coth R(t)$ its Euclidean center and radius are respectively given by

$$C(t) = \left(0, \frac{1}{\cos\vartheta}\right) = \frac{H(R(t))}{(H^2(R(t)) - 1)^{\frac{1}{2}}} \quad \text{and} \quad \rho(t) = \sinh R(t) = \frac{1}{(H^2(R(t)) - 1)^{\frac{1}{2}}}\cdot$$

Hence the Euclidean equation $|x|^2 + z^2 = \rho^2(t)$ of Σ_t can be rewritten as the Killing graph of the radial function $u_+ = u_+(r,t)$ for $r \leq r_0$ given by

$$e^{u_+} = \left(\frac{\cosh^2 R(t)}{\cosh^2 r} - 1\right)^{\frac{1}{2}} + \frac{\cosh R(t)}{\cosh r},$$

that is,

$$e^{u_+(r,t)} = \left(e^{2nt}\frac{\cosh^2 r_0}{\cosh^2 r} - 1\right)^{\frac{1}{2}} + e^{nt}\frac{\cosh r_0}{\cosh r}\cdot$$

In particular

$$\sup_{B_{r_0}(o)\times[0,T]} u_+ = \sup_{[0,T]} u_+(o,t) = (e^{2nT} - 1)^{1/2} + e^{nT} \geq T^{1/2}. \tag{57}$$

3 GRADIENT BOUNDS

In the following, we present two estimates for the gradient. They follow the pattern of estimates made in Colding, Minicozzi *et al.* (2003) and Ecker and Huisken (1991) respectively.

3.1 Gradient estimates depending on the height

Lemma 3.1. *Given $R > 0$ and $T > 0$, consider the restriction of the function*

$$\zeta(\Psi(x,t)) = \int_0^{r(\Psi(x,t))} \xi(\varsigma)\, \mathrm{d}\varsigma \tag{58}$$

to $B_R(o) \times [0,T]$. Let $\eta \in C^\infty(B_R(o) \times [0,T])$ be given by

$$\eta(\Psi(x,t)) = \big(\zeta(R) - \zeta(\Psi(x,t)) - C_R t\big)_+ \tag{59}$$

with

$$C_R = n \sup\nolimits_{B_R(o)} (\xi' + |\sigma|\xi)\,. \tag{60}$$

Suppose that

$$|\partial_r \log \varrho| \leq \frac{\xi'(r)}{\xi(r)}\cdot \tag{61}$$

for $r \in (0,R]$. Then

$$(\partial_t - \Delta)\, \eta \leq 0.$$

Proof: It follows from (18) that

$$(\partial_t - \Delta)\eta \leq n\big(\xi'(r) + \sigma\xi(r)\langle\bar\nabla r, N\rangle\big) + \varrho^2 |\nabla s|^2 \xi(r) \left(\langle \bar\nabla \log \varrho, \bar\nabla r\rangle - \frac{\xi'(r)}{\xi(r)}\right) - C_R.$$

Fixing C_R as in (60) one proves the result. □

Lemma 3.2. *Set $\phi = \eta e^{\frac{z(s)}{t}}$ where $z(s) = as^2$ for some negative constant a and η is defined in (59). Suppose that (60) and (61) hold. Then*

$$\begin{aligned}(\partial_t - \Delta)\, \phi \leq -\frac{\phi}{t^2}\big(z + (\dot z^2 + t\ddot z)|\nabla s|^2 + t\dot z(2\langle\bar\nabla \log\varrho, N\rangle + n\sigma)\langle\bar\nabla s, N\rangle \\ + 2t\dot z\langle\nabla s, \nabla \log \eta\rangle\big),\end{aligned} \tag{62}$$

where $\cdot$ denotes derivatives with respect to s.

Proof: Since

$$(\partial_t - \Delta)\, \phi = \eta(\partial_t - \Delta)e^{\frac{z}{t}} + e^{\frac{z}{t}}(\partial_t - \Delta)\eta - 2\langle \nabla\eta, \nabla e^{\frac{z}{t}}\rangle$$

it follows from Lemma 3.1 that

$$(\partial_t - \Delta)\,\phi \leq \eta(\partial_t - \Delta)e^{\frac{z}{t}} - 2\langle \nabla\eta, \nabla e^{\frac{z}{t}}\rangle = \eta(\partial_t - \Delta)e^{\frac{z}{t}} - 2\frac{\dot{z}}{t}e^{\frac{z}{t}}\langle\nabla\eta, \nabla s\rangle \tag{63}$$

where we used that

$$\nabla e^{\frac{z}{t}} = \frac{\dot{z}}{t}e^{\frac{z}{t}}\nabla s.$$

Since

$$\Delta e^{\frac{z}{t}} = e^{\frac{z}{t}}\left(\frac{\dot{z}}{t}\Delta s + \left(\frac{\dot{z}^2}{t^2} + \frac{\ddot{z}(s)}{t}\right)|\nabla s|^2\right) \tag{64}$$

we also compute

$$\partial_t e^{\frac{z}{t}} = e^{\frac{z}{t}}\left(-\frac{z}{t^2} + \frac{\dot{z}}{t}\partial_t s\right)$$

Using (16) and gathering these terms together, we obtain

$$\begin{aligned}(\partial_t - \Delta)\, e^{\frac{z}{t}} &= -\frac{e^{\frac{z}{t}}}{t^2}\Big(z(s) + (\dot{z}^2(s) + t\ddot{z}(s))|\nabla s|^2 \\ &+ t\dot{z}(s)\big(2\langle\bar{\nabla}\log\varrho, N\rangle + n\sigma\big)\langle\bar{\nabla}s, N\rangle\Big),\end{aligned} \tag{65}$$

The result follows combining this last expression and (63). □

Proposition 3.1. *Let $L \geq 0$ such that $\overline{\mathrm{Ric}}(N,N) \geq -L$. Suppose that (60) and (61) hold. Given $R > 0$, let $T > 0$ such that*

$$T < \frac{\zeta(R)}{C_R} = \frac{\zeta(R)}{n\sup_{B_R(o)}(\xi' + |\sigma|\xi)}. \tag{66}$$

If the graphs Σ_t, $t \in [0,T]$, evolve by the modified mean curvature flow (2)-(3), then

$$W(\Psi(o,t)) \leq \widetilde{C}_{R,T}\,\exp\left(\left(2\left(\frac{|a|}{t}\right)^{\frac{1}{2}}\sup_{B_R(o)\times[0,T]}u + \left(\frac{T}{t}\right)^{\frac{1}{2}}\right)^2 - 1\right), \tag{67}$$

with $|a| = 2\gamma\sup_{B_R(o)}\varrho^2$. Here, $\gamma > 1 + L$ and

$$\widetilde{C}_{R,T} = \frac{\zeta(R)}{\zeta(R) - C_R T}\max\left\{\frac{\sqrt{2}}{\inf_{B_R(o)}\varrho}, \left(\frac{1}{\inf_{B_R(o)}\varrho^2} + \frac{8}{\gamma}\frac{\sup_{B_R(o)}\xi^2}{\sup_{B_R(o)}\varrho^2}\frac{T}{\zeta^2(R)}\right)^{\frac{1}{2}}\right\}. \tag{68}$$

Proof: Replacing u by $u + \sup_{B_R(o)\times[0,T]}|u| + (T/|a|)^{1/2}$ if necessary we assume that

$$T \leq |a|(\inf_{B_R(o)\times[0,T]}u)^2. \tag{69}$$

It follows that we can assume that $s \geq (T/|a|)^{1/2}$ and $t \leq |a|s^2$ in $\Psi(B_R(o)\times[0,T])$. Note that our choice of C_R in (59) implies that $\eta = 0$ in $\Psi(\partial B_R(o)\times[0,T])$. Moreover, (69) implies that $\phi = 0$ in Σ_0, that is, at $t = 0$. At a (interior) maximum point $(x_0, t_0) \in B_R(o) \times (0, T]$ of the function ϕW we have from Proposition 2.2 and Lemma 3.2 that

$$0 = \nabla(\phi W) = W\nabla\phi + \phi\nabla W$$

and

$$\begin{aligned} 0 &\le (\partial_t - \Delta)(\phi W) = W(\partial_t - \Delta)\phi + \phi(\partial_t - \Delta)W - 2\langle\nabla\phi, \nabla W\rangle \\ &= W(\partial_t - \Delta)\phi - \phi W(|A|^2 + \overline{\mathrm{Ric}}(N, N)) \le W(\partial_t - \Delta)\phi - \overline{\mathrm{Ric}}(N, N)\phi W. \end{aligned}$$

at $\Psi(x_0, t_0)$. Note that Lemma (62) implies that

$$\begin{aligned} &\eta z + \eta(\dot z^2 + t_0\ddot z)|\nabla s|^2 + t_0\eta\dot z(2\langle\bar\nabla \log\varrho, N\rangle + n\sigma)\langle\bar\nabla s, N\rangle \\ &\quad + 2t_0\dot z\langle\nabla s, \nabla\eta\rangle + t_0^2\eta\,\overline{\mathrm{Ric}}(N, N) \le 0. \end{aligned} \tag{70}$$

at (x_0, t_0). Suppose that

$$W(\Psi(x_0, t_0)) \ge \frac{1}{\varrho(x_0)}\frac{1}{\varepsilon} \tag{71}$$

for some $\varepsilon \in (1/\sqrt{2}, 1)$. In this case one gets

$$\langle\bar\nabla s, N\rangle = \frac{1}{\varrho^2(x_0)}\langle X, N\rangle = \frac{1}{\varrho^2(x_0)}\frac{1}{W} \le \frac{\varepsilon}{\varrho(x_0)}$$

and

$$|\nabla s|^2 = \frac{1}{\varrho^4}|X^\top|^2 = \frac{1}{\varrho^4}(\varrho^2 - \langle X, N\rangle^2) = \frac{1}{\varrho^2}\left(1 - \frac{1}{\varrho^2 W^2}\right) = \frac{1}{\varrho^2}\frac{|\nabla^M u|^2}{W^2}$$

at (x_0, t_0). Moreover,

$$\langle\nabla s, \nabla\eta\rangle = -\xi(r)\langle\nabla s, \nabla r\rangle = \xi(r)\langle\bar\nabla s, N\rangle\langle\bar\nabla r, N\rangle \le \xi(r)\frac{1}{\varrho^2}\frac{1}{W}$$

at (x_0, t_0). Denote

$$\alpha = \sup_{B_R(o)}\left(2|\bar\nabla \log\varrho| + n\sigma\right). \tag{72}$$

Since $z = as^2$ it follows from the inequality (70) that

$$a\eta s_0^2 + \eta(4a^2s_0^2 + 2at_0)\frac{1}{\varrho_0^2}\frac{|\nabla^M u|^2}{W^2} - 2a\alpha\eta t_0 s_0\frac{\varepsilon}{\varrho_0} - 4at_0s_0\xi\frac{1}{\varrho_0^2}\frac{1}{W} - L\eta t_0^2 \le 0$$

where $s_0 = u(x_0)$ and $\varrho_0 = \varrho(x_0)$. It follows from (69) that given $\gamma > 1 + L$ we have

$$a(1-\gamma)s_0^2 - 2\alpha a s_0 t_0\frac{\varepsilon}{\varrho_0} - Lt_0^2 \ge (\gamma - 1 - L)t_0^2 + 2|a|^{1/2}\alpha\frac{\varepsilon}{\varrho_0}t_0 \ge 0.$$

With this choice we have

$$\gamma\eta a s_0^2 + \eta(4a^2 s_0^2 + 2at_0)\frac{1}{\varrho_0^2}\frac{|\nabla^M u|^2}{W^2} - 4as_0 t_0 \xi \frac{1}{\varrho_0^2}\frac{1}{W} \leq 0.$$

Then

$$\gamma\eta a s_0^2 + \eta(4a^2 s_0^2 + 2at_0)\frac{1}{\varrho_0^2}\frac{|\nabla^M u|^2}{W^2} - 4|as_0| t_0 \xi \frac{1}{\varrho_0^2}\frac{|\nabla^M u|}{W^2} \leq 0.$$

Proceeding as in Colding, Minicozzi *et al.* (2003), either

$$|as_0|\eta(\Psi(x_0,t_0))|\nabla^M u| < 4\xi(r(x_0))T \tag{73}$$

or

$$|as_0|\eta(\Psi(x_0,t_0))|\nabla^M u| \geq 4\xi(r(x_0))T \geq 4\xi(r(x_0))t_0.$$

Now, using (69) one has

$$4a^2 s_0^2 + 2at_0 > 2a^2 s_0^2$$

and one obtains

$$\begin{aligned} 0 &\geq \gamma\eta a s_0^2 + \eta(4a^2 s_0^2 + 2at_0)\frac{1}{\varrho_0^2}\frac{|\nabla^M u|^2}{W^2} - 4|as_0| t_0 \xi(r(x_0)) \frac{1}{\varrho_0^2}\frac{|\nabla^M u|}{W^2} \\ &\geq \gamma\eta a s_0^2 + \eta(4a^2 s_0^2 - a^2 s_0^2 + 2at_0)\frac{1}{\varrho_0^2}\frac{|\nabla^M u|^2}{W^2} = \gamma\eta a s_0^2 + \eta a^2 s_0^2 \frac{1}{\varrho_0^2}\frac{|\nabla^M u|^2}{W^2}. \end{aligned}$$

We conclude that

$$\gamma\eta a s_0^2 + \eta a^2 s_0^2 \frac{1}{\varrho_0^2}\frac{|\nabla^M u|^2}{W^2} \leq 0.$$

Now, we set $|a| = 2\gamma \sup_{B_R(o)} \varrho^2$. Since $\eta \geq 0$ this choice yields

$$W^2(\Psi(x_0,t_0)) \leq \frac{|a|}{\varrho_0^2(|a| - \gamma\varrho_0^2)} \leq \frac{2}{(\inf_{B_R(o)}\varrho)^2} \tag{74}$$

Finally, note that (69) yields $as_0^2/t_0 \leq -1$. We conclude that at an arbitrary point $(x,t) \in B_R(o) \times [0,T]$ one has

$$W(\Psi(x,t)) \leq \frac{1}{\eta(\Psi(x,t))} e^{-\frac{as^2(x)}{t} - 1} \zeta(R) W(x_0,t_0) \leq \sqrt{2}\frac{1}{\eta(\Psi(x,t))} e^{-\frac{as^2(x)}{t} - 1} \frac{\zeta(R)}{\inf_{B_R(o)}\varrho}$$

In particular since $\eta(\Psi(o,t)) = \zeta(R) - C_R t \geq \zeta(R) - C_R T$ we obtain

$$W(\Psi(o,t)) \leq \frac{\sqrt{2}}{\inf_{B_R(o)}\varrho}\frac{\zeta(R)}{\zeta(R) - C_R T} e^{-\frac{a}{t}(u(o) + \sup u + (T/|a|)^{1/2})^2 - 1}. \tag{75}$$

Now note that (73) implies that

$$\eta(\Psi(x_0,t_0))|\nabla^M u|(x_0,t_0) \le \frac{4\xi(r(x_0))T}{|as_0|}$$

Hence,

$$\eta W|_{\Psi(x_0,t_0)} \le \left(\frac{\eta^2(\Psi(x_0,t_0))}{\varrho^2(x_0)} + \frac{16\xi^2(r(x_0))T^2}{|as_0|^2}\right)^{\frac{1}{2}} \le \left(\frac{\zeta^2(R)}{\inf_{B_R(o)}\varrho^2} + \frac{16\sup_{B_R(o)}\xi^2 T^2}{|as_0|^2}\right)^{\frac{1}{2}}$$

Translating u downward by $\sup_{B_R(o)\times[0,T]}|u| + (T/|a|)^{1/2}$ and using (69) yields

$$W(\Psi(o,t)) \le e^{-\frac{a}{t}(u(o)+\sup u+(T/|a|)^{1/2})^2-1}\frac{\zeta(R)}{\zeta(R)-C_R T}\left(\frac{1}{\inf_{B_R(o)}\varrho^2} + \frac{16\sup_{B_R(o)}\xi^2 T}{|a|\zeta^2(R)}\right)^{\frac{1}{2}} \tag{76}$$

where we have used that $|as_0| \ge |aT|^{1/2}$. Gathering both cases (75) and (76) in a single expression one gets

$$W(\Psi(o,t)) \le \widetilde{C}_{R,T}\,\exp\left(-\frac{a}{t}\left(2\sup_{B_R(o)\times[0,T]}u + \left(\frac{T}{|a|}\right)^{1/2}\right) - 1\right) \tag{77}$$

with

$$\widetilde{C}_{R,T} = \frac{\zeta(R)}{\zeta(R)-C_R T}\max\left\{\frac{\sqrt{2}}{\inf_{B_R(o)}\varrho}, \left(\frac{1}{\inf_{B_R(o)}\varrho^2} + \frac{8}{\gamma}\frac{\sup_{B_R(o)}\xi^2}{\sup_{B_R(o)}\varrho^2}\frac{T}{\zeta^2(R)}\right)^{\frac{1}{2}}\right\}.$$

This finishes the proof of the proposition. □

3.2 Gradient estimates depending on the initial slope

Proposition 3.2. *Let $o \in M$. Given $R > 0$ and $T > 0$, suppose that*

$$|\partial_r \log\varrho| \le \frac{\xi'(r)}{\xi(r)} \tag{78}$$

in $B_R(o) \subset M$ and that $\overline{\mathrm{Ric}} \ge -L$ for some constant $L \ge 0$ in $B_R(o)\times[0,T]$. Then

$$W(o,t) \le \frac{\exp(\lambda\zeta(R))}{\exp(\lambda\zeta(R)/2)-1}\sup_{B_R(o)}W(\cdot,0) \tag{79}$$

for $0 \le t \le T$, *where*

$$\lambda = 2\beta^2 C_R \sup^2_{B_R(o)} \varrho + \left((2\beta^2 C_R \sup^2_{B_R(o)} \varrho)^2 + 4\beta^2 (L + \widetilde{\delta}) \sup^2_{B_R(o)} \varrho\right)^{1/2} \tag{80}$$

with

$$\beta = \frac{1}{\zeta(R)} \left(\sup_{B_R(o)\times[0,T]} |u| - \sup_{[0,T]} u(o,t)\right)$$

and $\widetilde{\delta} = 2\sup^2_{B_R(o)}(2|\bar\nabla \log \varrho| + 2\xi + n\sigma)$.

Proof: Denote

$$\alpha = \sup_{B_R(o)} \left(2|\bar\nabla \log \varrho| + n\sigma\right) \tag{81}$$

and

$$C_R = n \sup_{B_R(o)} (\xi' + |\sigma|\xi). \tag{82}$$

Given an arbitrary constant $\widetilde{C}_R$ in $[0, C_R)$ we also set

$$\delta = C_R - \widetilde{C}_R. \tag{83}$$

Following Korevaar (1986) and Ecker and Huisken (1991) we define

$$\phi(\tau) = e^{\lambda\tau} - 1, \quad \tau \ge 0, \tag{84}$$

where

$$\lambda = 2\beta^2 \delta \sup^2_{B_R(o)} \varrho + \left((2\beta^2 \delta \sup^2_{B_R(o)} \varrho)^2 + 4\beta^2 (L + \widetilde{\delta}) \sup^2_{B_R(o)} \varrho\right)^{1/2} \tag{85}$$

with $\widetilde{\delta} = 2(\alpha + 2\sup_{B_R(o)} \xi)^2$ and

$$\beta := \frac{1}{\zeta(R')} \left(\sup_{B_R(o)\times[0,T]} |u| - \sup_{[0,T]} u(o,t)\right) \tag{86}$$

for some $R' \in (0, R]$. Given the function ζ defined in (17) one sets

$$\eta(\Psi(x,t)) = \phi\left(\left(\zeta(R) - \zeta(r(x)) + \chi(u(x,t)) - \widetilde{C}_R t\right)_+\right), \tag{87}$$

where $r(x) = \mathrm{dist}(o, x)$ and

$$\chi(s) = \frac{1}{2\beta} \left(s - \sup_{B_R(o)\times[0,T]} |u|\right). \tag{88}$$

Denote

$$\widetilde{\mathcal{C}}_{R,T} = \bigcup_{t\in[0,T]} \left\{\Psi(x,t) \in \bar M : \zeta(r(x)) + \widetilde{C}_R t \le \zeta(R)\right\}. \tag{89}$$

Fixing
$$\widetilde{C}_R \leq \frac{1}{T}(\zeta(R) - \zeta(R'))$$
one obtains the following inclusions
$$B_{R'}(o) \times [0,T] \subset \widetilde{\mathcal{C}}_{R,T} \subset B_R(o) \times [0,T]. \tag{90}$$
Using (25) one computes
$$\begin{aligned}&(\partial_t - \Delta)(\eta W) = W(\partial_t - \Delta)\eta + \eta(\partial_t - \Delta) W - 2\langle \nabla W, \nabla \eta\rangle \\ &= W(\partial_t - \Delta)\eta - (|A|^2 + \overline{\mathrm{Ric}}(N,N))\eta W \leq W(\partial_t - \Delta)\eta + L\eta W.\end{aligned}$$
However
$$\begin{aligned}&(\partial_t - \Delta)\eta = \Big((\partial_t - \Delta)\zeta + \dot\chi(s)(\partial_t - \Delta)s - \ddot\chi(s)|\nabla s|^2 - \widetilde{C}_R\Big)\phi' \\ &-\big(|\nabla\zeta|^2 + \dot\chi^2(s)|\nabla s|^2 - 2\xi(r)\dot\chi(s)\langle\nabla r, \nabla s\rangle\big)\phi''\end{aligned}$$
Since $\phi' > 0$ and using (16), (18) and (78) one gets
$$\begin{aligned}&(\partial_t - \Delta)\eta \leq \phi'\big(n\xi'(r) + n\sigma\xi(r)\langle\bar\nabla r, N\rangle - \widetilde{C}_R - \dot\chi(s)(2\langle\bar\nabla\log\varrho, N\rangle + n\sigma)\langle\bar\nabla s, N\rangle \\ &-\ddot\chi(s)|\nabla s|^2\big) - \phi''\left(|\nabla\zeta|^2 + \dot\chi^2(s)|\nabla s|^2 + 2\xi(r)\dot\chi(s)\langle\bar\nabla r, N\rangle\langle\bar\nabla s, N\rangle\right)\end{aligned}$$
where we used the fact that $\langle\nabla r, \nabla s\rangle = -\langle\bar\nabla r, N\rangle\langle\bar\nabla s, N\rangle$. Note that
$$n\xi'(r) + n\sigma\xi(r)\langle\bar\nabla r, N\rangle - \widetilde{C}_R \leq C_R - \widetilde{C}_R =: \delta. \tag{91}$$
Since $\phi'' \geq 0$ one obtains rearranging terms and discarding a non-negative term that
$$(\partial_t - \Delta)\eta \leq \delta\phi' + \left(\alpha\phi' - 2\xi(r)\langle\bar\nabla r, N\rangle\phi''\right)\dot\chi(s)\langle\bar\nabla s, N\rangle - \left(\ddot\chi(s)\phi' + \dot\chi^2(s)\phi''\right)|\nabla s|^2$$
Note that
$$\langle\bar\nabla s, N\rangle = \frac{1}{\varrho^2}\langle X, N\rangle = \frac{1}{\varrho^2}\frac{1}{W}\cdot \tag{92}$$
and
$$|\nabla s|^2 = |\bar\nabla s|^2 - \langle\bar\nabla s, N\rangle^2 = \frac{1}{\varrho^2} - \frac{1}{\varrho^4}\langle X, N\rangle^2 = \frac{1}{\varrho^2}\left(1 - \frac{1}{\varrho^2}\frac{1}{W^2}\right).$$
Therefore
$$(\partial_t - \Delta)\eta \leq \delta\phi' + (\alpha\phi' + 2\xi(r)\phi'')\frac{\dot\chi}{\varrho}\frac{1}{\varrho W} - \left(\frac{\ddot\chi}{\varrho^2}\phi' + \frac{\dot\chi^2}{\varrho^2}\phi''\right)\left(1 - \frac{1}{\varrho^2}\frac{1}{W^2}\right)$$

Setting $\ddot{\chi} = 0$ we conclude that

$$(\partial_t - \Delta)(\eta W) \leq \left[\delta \frac{\phi'}{\phi} + \left(\alpha \frac{\phi'}{\phi} + 2\xi(r)\frac{\phi''}{\phi}\right)\frac{\dot{\chi}}{\varrho}\frac{1}{\varrho W} - \frac{\dot{\chi}^2}{\varrho^2}\frac{\phi''}{\phi}\left(1 - \frac{1}{\varrho^2}\frac{1}{W^2}\right) + L\right]\eta W.$$

Supposing that $\varrho W \geq \lambda^2/\sqrt{\varepsilon}$ we have

$$\begin{aligned}
&\delta\phi' + (\alpha\phi' + 2\xi(r)\phi'')\frac{\dot{\chi}}{\varrho}\frac{1}{\varrho W} - \frac{\dot{\chi}^2}{\varrho^2}\phi''\left(1 - \frac{1}{\varrho^2}\frac{1}{W^2}\right) + L\phi \\
&\leq e^{\lambda\tau}\left[\left(\alpha\lambda + 2\xi(r)\lambda^2\right)\frac{\dot{\chi}}{\varrho}\frac{1}{\varrho W} - \frac{\dot{\chi}^2}{\varrho^2}\lambda^2\left(1 - \frac{1}{\varrho^2}\frac{1}{W^2}\right) + L + \lambda\delta\right] \\
&\leq e^{\lambda\tau}\left[\left(\alpha\frac{1}{\lambda} + 2\xi(r)\right)\frac{\dot{\chi}}{\varrho}\sqrt{\varepsilon} - \frac{\dot{\chi}^2}{\varrho^2}\left(\lambda^2 - \frac{\varepsilon}{\lambda^2}\right) + L + \lambda\delta\right] \\
&\leq e^{\lambda\tau}\left[\left(\alpha\frac{1}{\lambda} + 2\xi(r)\right)\frac{\dot{\chi}}{\varrho}\sqrt{\varepsilon} + \frac{\dot{\chi}^2}{\varrho^2}\frac{\varepsilon}{\lambda^2} - \frac{\dot{\chi}^2}{\varrho^2}\lambda^2 + \delta\lambda + L\right]
\end{aligned}$$

Fixing $\lambda \geq 1$ choose ε small enough so that

$$\left(\alpha\frac{1}{\lambda} + 2\xi(r)\right)\frac{\dot{\chi}}{\varrho}\sqrt{\varepsilon} + \frac{\dot{\chi}^2}{\varrho^2}\frac{\varepsilon}{\lambda^2} \leq (\alpha + 2\xi)\frac{\dot{\chi}}{\varrho}\sqrt{\varepsilon} + \frac{\dot{\chi}^2}{\varrho^2}\varepsilon \leq \widetilde{\delta} := 2(\alpha + 2\sup\nolimits_{B_R(o)}\xi)^2.$$

Then, fix $\lambda \geq 1$ large enough so that

$$-\frac{\dot{\chi}^2}{\varrho^2}\lambda^2 + \delta\lambda + L + \widetilde{\delta} \leq 0.$$

Explicitly, we may fix

$$\sqrt{\varepsilon} < 2\beta \inf\nolimits_{B_R(o)}\varrho\left(\alpha + 2\sup\nolimits_{B_R(o)}\xi\right)$$

and

$$\lambda \geq 2\beta^2\delta \sup\nolimits^2_{B_R(o)}\varrho + \left((2\beta^2\delta\sup\nolimits^2_{B_R(o)}\varrho)^2 + 4\beta^2(L + \widetilde{\delta})\sup\nolimits^2_{B_R(o)}\varrho\right)^{1/2}.$$

Note that in this case

$$W \geq \frac{1}{\inf_{B_R(o)}\varrho}\frac{\lambda^2}{\sqrt{\varepsilon}} \geq \beta\delta\frac{1}{\alpha + 2\sup_{B_R(o)}\xi}\frac{\sup^2_{B_R(o)}\varrho}{\inf^2_{B_R(o)}\varrho}.$$

With these choices we conclude that

$$(\partial_t - \Delta)(\eta W) \leq 0.$$

In view of (90) the parabolic maximum principle implies that

$$\sup\nolimits_{B_{R'}(o)\times[0,T]}\eta W \leq \sup\nolimits_{B_R(o)\times[0,T]}\eta W \leq \sup\nolimits_{B_R(o)}\eta W(\cdot, 0)$$

and in particular

$$\eta(o,t)W(o,t) \le e^{\lambda\zeta(R)}\text{sup}_{B_R(o)}W(\cdot,0).$$

We conclude that

$$W(o,t) \le \frac{\exp(\lambda\zeta(R))}{\exp(\lambda\zeta(R')/2)-1}\,\text{sup}_{B_R(o)}W(\cdot,0) \tag{93}$$

This finishes the proof. □

4 CURVATURE ESTIMATE

In order to obtain second order bounds we need to deduce evolution equations for the second fundamental form and its squared norm, a variant of the classical Simons' formula.

Lemma 4.1. *The squared norm $|A|^2$ of the second fundamental form of the graphs Σ_t, $t \in [0,T]$, evolve as*

$$\frac{1}{2}(\partial_t - \Delta)|A|^2 + |\nabla A|^2 = -n\sigma a_i^s a_{sj} a^{ij} + |A|^4 + n(H-\sigma) a^{ij} \bar{R}_{i00j} \\ + g^{k\ell} \left(\nabla_i L_{kj\ell} + \nabla_k L_{\ell ij}\right) a^{ij} + g^{k\ell}(a_{is}\bar{R}^s_{kj\ell} + a_{sk}\bar{R}^s_{\ell ij}) a^{ij} \tag{94}$$

where L is the $(0,3)$-tensor in Σ_t defined by $L_{ijk} = \langle \bar{R}(\partial_i, \partial_j)N, \partial_k\rangle$. This expression is rewritten in terms of the ambient curvature tensor as

$$\frac{1}{2}(\partial_t - \Delta)|A|^2 + |\nabla A|^2 = -n\sigma(a_i^s a_{sj} + \bar{R}_{i00j}) a^{ij} + |A|^4 + |A|^2 \overline{\mathrm{Ric}}(N,N) \\ + g^{k\ell}(\bar{\nabla}_i \bar{R}_{kj0\ell} + \bar{\nabla}_k \bar{R}_{\ell i0j}) a^{ij} + 2g^{k\ell}(a_{is}\bar{R}^s_{kj\ell} + a_{sk}\bar{R}^s_{\ell ij}) a^{ij}. \tag{95}$$

Proof: We have

$$\partial_t a_{ij} = n\nabla_i\nabla_j H - n(H-\sigma) a_{is} a^s_j + n(H-\sigma)\bar{R}_{i00j}$$

Since

$$\partial_t g^{ij} = 2n(H-\sigma) a^{ij}$$

we have

$$\tfrac{1}{2}\partial_t |A|^2 = g^{j\ell} a_{ij} a_{k\ell} \partial_t g^{ik} + g^{ik} g^{j\ell} a_{k\ell} \partial_t a_{ij} = 2n(H-\sigma) a^{ik} a_i^\ell a_{k\ell} \\ + a^{ij}(n\nabla_i\nabla_j H - n(H-\sigma) a_{i\ell} a^\ell_j + n(H-\sigma)\bar{R}_{i00j}). \tag{96}$$

We conclude that

$$\frac{1}{2}\partial_t |A|^2 = n(H-\sigma) a^{ik} a_i^\ell a_{k\ell} + n a^{ij}\nabla_i\nabla_j H + n(H-\sigma) a^{ij}\bar{R}_{i00j}. \tag{97}$$

On the other hand

$$\Delta a_{ij} = n\nabla_i\nabla_j H + nH a_i^s a_{sj} - a_{ij}|A|^2 - g^{k\ell}\left(\nabla_i L_{kj\ell} + \nabla_k L_{\ell ij}\right) + g^{k\ell}(\bar{R}^s_{ik\ell} a_{sj} + \bar{R}^s_{ikj} a_{\ell s})$$

and

$$\frac{1}{2}\Delta |A|^2 - |\nabla A|^2 = a^{ij}\Delta a_{ij} = n a^{ij}\nabla_i\nabla_j H + nH a_i^s a_{sj} a^{ij} - |A|^4 \\ - g^{k\ell}\left(\nabla_i L_{kj\ell} + \nabla_k L_{\ell ij}\right) a^{ij} + g^{k\ell}(\bar{R}^s_{ik\ell} a_{sj} + \bar{R}^s_{ikj} a_{\ell s}) a^{ij} \tag{98}$$

Therefore

$$\frac{1}{2}(\partial_t - \Delta)|A|^2 + |\nabla A|^2 = -n\sigma a_i^s a_{sj} a^{ij} + |A|^4 + n(H-\sigma) a^{ij} \bar{R}_{i00j} \\ + g^{k\ell}\left(\nabla_i L_{kj\ell} + \nabla_k L_{\ell ij}\right) a^{ij} - g^{k\ell}(\bar{R}^s_{ik\ell} a_{sj} + \bar{R}^s_{ikj} a_{\ell s}) a^{ij} \tag{99}$$

It is worth to point out that

$$\nabla_i L_{kj\ell} + \nabla_k L_{\ell ij} = \bar{\nabla}_i \bar{R}_{kj0\ell} + \bar{\nabla}_k \bar{R}_{\ell i0j} + a_{ik}\bar{R}_{0j0\ell} + a_{ij}\bar{R}_{k00\ell} + a_{is}\bar{R}^s_{kj\ell} \\ + a_{k\ell}\bar{R}_{0i0j} + a_{ki}\bar{R}_{\ell 00j} + a_{ks}\bar{R}^s_{\ell ij}.$$

Therefore

$$g^{k\ell}(\nabla_i L_{kj\ell} + \nabla_k L_{\ell ij}) a^{ij} = g^{k\ell}(\bar{\nabla}_i \bar{R}_{kj0\ell} + \bar{\nabla}_k \bar{R}_{\ell i0j}) a^{ij} - a_i^\ell a^{ij} \bar{R}_{j00\ell} + |A|^2 \overline{\mathrm{Ric}}(N,N) \\ + a_{is} a^{ij} g^{k\ell} \bar{R}^s_{kj\ell} - nH a^{ij} \bar{R}_{i00j} + a_i^\ell a^{ij} \bar{R}_{\ell 00j} + g^{k\ell} a^{ij} a_{sk} \bar{R}^s_{\ell ij}.$$

Cancelling and grouping some terms one gets

$$g^{k\ell}(\nabla_i L_{kj\ell} + \nabla_k L_{\ell ij}) a^{ij} = g^{k\ell}(\bar{\nabla}_i \bar{R}_{kj0\ell} + \bar{\nabla}_k \bar{R}_{\ell i0j}) a^{ij} + |A|^2 \overline{\mathrm{Ric}}(N,N) \\ + a_{is} a^{ij} g^{k\ell} \bar{R}^s_{kj\ell} - nH a^{ij} \bar{R}_{i00j} + g^{k\ell} a^{ij} a_{sk} \bar{R}^s_{\ell ij}.$$

Since

$$-g^{k\ell}(\bar{R}^s_{ik\ell} a_{sj} + \bar{R}^s_{ikj} a_{\ell s}) a^{ij} = a^{ij} g^{k\ell}(a_{is}\bar{R}^s_{kj\ell} + a_{sk}\bar{R}^s_{\ell ij})$$

we conclude that

$$\frac{1}{2}(\partial_t - \Delta)|A|^2 + |\nabla A|^2 = -n\sigma(a_i^s a_{sj} + \bar{R}_{i00j}) a^{ij} + |A|^4 + |A|^2 \overline{\mathrm{Ric}}(N,N) \\ + g^{k\ell}(\bar{\nabla}_i \bar{R}_{kj0\ell} + \bar{\nabla}_k \bar{R}_{\ell i0j}) a^{ij} + 2g^{k\ell}(a_{is}\bar{R}^s_{kj\ell} + a_{sk}\bar{R}^s_{\ell ij}) a^{ij}. \tag{100}$$

This finishes the proof. □

Given $R > 0$ we are going to estimate $|A|$ in the set

$$\mathcal{C}_{R,T} = \big\{ y = \Psi(x,t) : \zeta(r(\Psi(x,t))) + C_R t \le \zeta(R) \big\}. \tag{101}$$

In order to do this, we will proceed as in Borisenko and Miquel (2012) studying the evolution of the function

$$f = \psi(W)|A|^2, \tag{102}$$

where

$$\psi(W) = \frac{W^2}{\gamma - \delta W^2} \tag{103}$$

with
$$\gamma = \frac{1}{\sup_{B_R(o)} \varrho^2}$$
and
$$\delta = \frac{1}{2} \frac{\gamma}{\sup_{B_{R'(o)} \times [0,T]} W^2}$$
for $R' \in (0, R)$ such that
$$\zeta(r) + C_R T \leq \zeta(R)/2 \tag{104}$$
for $r < R'$. Therefore
$$\delta\psi(W) \leq \frac{\gamma/2}{\gamma - \gamma/2} = 1.$$
Since $\delta\psi$ is non-decreasing and $W^2 \geq \varrho^{-2}$ we have
$$\delta\psi(W) = \frac{\delta W^2}{\gamma - \delta W^2} \geq \frac{\delta/\varrho^2}{\gamma - \delta/\varrho^2} \geq \frac{\delta}{\gamma \sup_{\mathcal{C}_R} \varrho^2 - \delta} = \frac{\delta}{1-\delta} =: \widetilde{\delta}.$$
We also have
$$-\frac{2}{W} \frac{1}{\psi'(W)} - \frac{\psi''(W)}{\psi'^2(W)} + \frac{3}{2} \frac{1}{\psi(W)} < 0. \tag{105}$$
In fact, we have
$$\begin{aligned} -\frac{2}{W} \frac{1}{\psi'(W)} - \frac{\psi''(W)}{\psi'^2(W)} &= -\frac{(\gamma - \delta W^2)^2}{\gamma W^2} - \frac{(2\gamma^2 + 6\gamma\delta W^2)}{(\gamma - \delta W^2)^3} \frac{(\gamma - \delta W^2)^4}{4\gamma^2 W^2} \\ = -\frac{\gamma - \delta W^2}{\gamma W^2} \left(\gamma - \delta W^2 + \frac{2\gamma^2 + 6\gamma\delta W^2}{4\gamma} \right) &= -\frac{\gamma - \delta W^2}{\gamma W^2} \left(\frac{3}{2}\gamma + \frac{1}{2}\delta W^2 \right). \end{aligned}$$
Therefore
$$\begin{aligned} -\frac{2}{W} \frac{1}{\psi'(W)} - \frac{\psi''(W)}{\psi'^2(W)} + \frac{3}{2} \frac{1}{\psi(W)} &= -\frac{\gamma - \delta W^2}{W^2} \left(\frac{3}{2} + \frac{1}{2} \frac{1}{\gamma} \delta W^2 \right) + \frac{3}{2} \frac{\gamma - \delta W^2}{W^2} \\ = -\frac{\delta}{2\gamma} (\gamma - \delta W^2) \leq 0. \end{aligned}$$

Lemma 4.2. *We have in $\mathcal{C}_{R,T}$ that*
$$(\partial_t - \Delta) f \leq -\frac{1}{\psi} \langle \nabla f, \nabla \psi \rangle - a f^2 + b f + 2(C + n\sigma |\bar{R}(\cdot, N, N, \cdot)|) \sqrt{\psi} \sqrt{f} \tag{106}$$
where $a = 2\delta - \sigma\varepsilon(1-\delta) > 0$ and $b = 2(\widetilde{C} + \widetilde{\delta} L) + \frac{\sigma}{\varepsilon}$. Here, C and $\widetilde{C}$ are non-negative constants depending on ϱ and its derivatives.

Proof: Mimicking Lemma 8 in Borisenko and Miquel (2012), we write the evolution of f is terms of (3) and (4.1) as follows

$$(\partial_t - \Delta) f = |A|^2\psi'(W)\,(\partial_t - \Delta) W - |A|^2\psi''(W)|\nabla W|^2 + \psi(W)\,(\partial_t - \Delta)|A|^2 \\ - 2\langle \nabla\psi, \nabla |A|^2\rangle.$$

However

$$\begin{aligned} 2\langle \nabla\psi, \nabla |A|^2\rangle &= \frac{1}{\psi}\langle \nabla\psi, \nabla f\rangle - \frac{1}{\psi}|\nabla\psi|^2|A|^2 + \langle \nabla\psi, \nabla |A|^2\rangle \\ &\geq \frac{1}{\psi}\langle \nabla\psi, \nabla f\rangle - \frac{1}{\psi}|\nabla\psi|^2|A|^2 - 2|A||\nabla|A|||\nabla\psi| \\ &\geq \frac{1}{\psi}\langle \nabla\psi, \nabla f\rangle - \frac{1}{\psi}|\nabla\psi|^2|A|^2 - 2\frac{1}{\sqrt{2\psi}}|A||\nabla\psi|\sqrt{2\psi}|\nabla|A|| \\ &\geq \frac{1}{\psi}\langle \nabla\psi, \nabla f\rangle - \frac{1}{\psi}|\nabla\psi|^2|A|^2 - \frac{1}{2\psi}|A|^2|\nabla\psi|^2 - 2\psi|\nabla|A||^2. \end{aligned}$$

Using Kato's inequality $|\nabla|A||^2 \leq |\nabla A|^2$ we conclude that

$$-2\langle \nabla\psi, \nabla |A|^2\rangle \leq -\frac{1}{\psi}\langle \nabla f, \nabla\psi\rangle + 2\psi|\nabla A|^2 + \frac{3}{2}\frac{1}{\psi}|A|^2|\nabla\psi|^2.$$

Hence, expressions (25) and (95) yield

$$\begin{aligned} (\partial_t - \Delta) f \leq& -|A|^2\psi'(W)\left(W(|A|^2 + \overline{\mathrm{Ric}}(N,N)) + 2W^{-1}|\nabla W|^2\right) \\ &+ 2\psi(W)\big(- |\nabla A|^2 - n\sigma(a_i^s a_{sj} + \bar{R}_{i00j})a^{ij} + |A|^4 + |A|^2\overline{\mathrm{Ric}}(N,N) \\ &+ g^{k\ell}(\bar{\nabla}_i\bar{R}_{kj0\ell} + \bar{\nabla}_k\bar{R}_{\ell i0j})a^{ij} + 2g^{k\ell}(a_{is}\bar{R}^s_{kj\ell} + a_{sk}\bar{R}^s_{\ell ij})a^{ij}\big) - |A|^2\psi''(W)|\nabla W|^2 \\ &- \frac{1}{\psi}\langle \nabla f, \nabla\psi\rangle + 2\psi(W)|\nabla A|^2 + \frac{3}{2}\frac{1}{\psi}|A|^2|\nabla\psi|^2 \end{aligned}$$

where $'$ denotes derivatives with respect to W. Grouping similar terms, one obtains

$$\begin{aligned} (\partial_t - \Delta) f \leq& (|A|^4 + |A|^2\overline{\mathrm{Ric}}(N,N))(2\psi(W) - \psi'(W)W) \\ &+ 2\psi(W)\big(- n\sigma(a_i^s a_{sj} + \bar{R}_{i00j})a^{ij} + g^{k\ell}(\bar{\nabla}_i\bar{R}_{kj0\ell} + \bar{\nabla}_k\bar{R}_{\ell i0j})a^{ij} \\ &+ 2g^{k\ell}(a_{is}\bar{R}^s_{kj\ell} + a_{sk}\bar{R}^s_{\ell ij})a^{ij}\big) - \frac{1}{\psi}\langle \nabla f, \nabla\psi\rangle \\ &- 2|A|^2\frac{\psi'(W)}{W}|\nabla W|^2 - |A|^2\psi''(W)|\nabla W|^2 + \frac{3}{2}\frac{\psi'^2(W)}{\psi(W)}|A|^2|\nabla W|^2 \end{aligned}$$

A straightforward but lengthy (see Appendix 6) computation allows us to verify that

$$g^{k\ell}(\bar{\nabla}_i\bar{R}_{kj0\ell} + \bar{\nabla}_k\bar{R}_{\ell i0j})a^{ij} + 2g^{k\ell}(a_{is}\bar{R}^s_{kj\ell} + a_{sk}\bar{R}^s_{\ell ij})a^{ij} \leq C|A| + \widetilde{C}|A|^2 \tag{107}$$

where the constants C and $\widetilde{C}$ depends on ϱ and its derivatives, that is, depends only on the geometry of M. Observing that

$$
\begin{aligned}
&-2|A|^2\frac{\psi'(W)}{W}|\nabla W|^2 - |A|^2\psi''(W)|\nabla W|^2 + \frac{3}{2}\frac{\psi'^2(W)}{\psi}|A|^2|\nabla W|^2 \\
&= -\left(2\frac{\psi'(W)}{W} + \psi''(W) - \frac{3}{2}\frac{\psi'^2(W)}{\psi(W)}\right)|A|^2|\nabla W|^2 \le 0
\end{aligned}
$$

one concludes that

$$
\begin{aligned}
&(\partial_t - \Delta) f \le (|A|^4 + |A|^2\overline{\mathrm{Ric}}(N,N))(2\psi(W) - \psi'(W)W) \\
&+ 2\psi(W)(C|A| + \widetilde{C}|A|^2) - 2n\sigma\psi(W)\left(a^{ij}a_i^s a_{sj} + a^{ij}\bar{R}_{i00j}\right) - \frac{1}{\psi}\langle\nabla f, \nabla\psi\rangle.
\end{aligned}
$$

Given a constant $\varepsilon > 0$ to be chosen later one has

$$
2a^{ij}a_i^s a_{sj} \le 2|A|^3 \le \varepsilon|A|^4 + \frac{1}{\varepsilon}|A|^2.
$$

Therefore

$$
\begin{aligned}
&(\partial_t - \Delta) f \le |A|^4((2+\sigma\varepsilon)\psi(W) - \psi'(W)W) + |A|^2\overline{\mathrm{Ric}}(N,N)(2\psi(W) - \psi'(W)W) \\
&+ 2\psi(W)(C|A| + \widetilde{C}|A|^2) + \frac{1}{\varepsilon}\sigma\psi(W)|A|^2 - 2n\sigma\psi(W)a^{ij}\bar{R}_{i00j} - \frac{1}{\psi}\langle\nabla f, \nabla\psi\rangle.
\end{aligned}
$$

Note that (103) implies that

$$
\begin{aligned}
&((2+\sigma\varepsilon)\psi(W) - W\psi'(W))\,|A|^4 = \left(\frac{2+\sigma\varepsilon}{\psi(W)} - W\frac{\psi'(W)}{\psi^2(W)}\right)\psi^2|A|^4 \\
&\left(\sigma\varepsilon\left(\frac{\gamma}{W^2} - \delta\right) - 2\delta\right) f^2 \le (\sigma\varepsilon(\gamma\varrho^2 - \delta) - 2\delta) f^2
\end{aligned}
$$

as well as

$$
\begin{aligned}
&(2\psi(W) - W\psi'(W))\,|A|^2\overline{\mathrm{Ric}}(N,N) = \left(\frac{2}{\psi(W)} - W\frac{\psi'(W)}{\psi^2(W)}\right)\overline{\mathrm{Ric}}(N,N)\,\psi^2|A|^2 \\
&= -2\delta\overline{\mathrm{Ric}}(N,N)\psi f.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&(\partial_t - \Delta) f \le (\sigma\varepsilon(\gamma\varrho^2 - \delta) - 2\delta) f^2 - 2\delta\overline{\mathrm{Ric}}(N,N)\psi f + 2C\psi|A| + 2\widetilde{C} f + \frac{\sigma}{\varepsilon} f \\
&+ 2n\sigma|\bar{R}(\cdot, N, N, \cdot)||A|\psi - \frac{1}{\psi}\langle\nabla f, \nabla\psi\rangle
\end{aligned}
$$

Since $\overline{\mathrm{Ric}} \ge -L$ for some $L \ge 0$ and $\delta\psi \ge \widetilde{\delta}$ we obtain

$$
2\widetilde{C} - 2\delta\psi\,\overline{\mathrm{Ric}}(N,N) + \frac{\sigma}{\varepsilon} \le 2(\widetilde{C} + \widetilde{\delta}L) + \frac{\sigma}{\varepsilon} = b.
$$

Since $\gamma\varrho^2 \leq 1$ denoting $a = 2\delta - \sigma\varepsilon(1-\delta)$ one has

$$(\partial_t - \Delta) f \leq -\frac{1}{\psi}\langle \nabla f, \nabla \psi\rangle - af^2 + bf + 2(C + n\sigma|\bar{R}(\cdot, N, N, \cdot)|)\sqrt{\psi}\sqrt{f}$$

what ends the proof. □

Proposition 4.1. *Let $R' \in (0,R)$ be fixed so that (104) holds. Then the norm of the Weingarten map A and its covariant derivatives are bounded in $B_{R'}(o)\times[0,T]$ by constants that depend on $\sup_{B_R(o)} W^2(\cdot,0)$ and on the geometric data $\sup_{B_R(o)}\varrho$, $\xi(R)$, $\zeta(R)$, C, $\widetilde{C}$ and L.*

Proof: Now, we consider the function

$$\phi(\Psi(x,t)) = (\zeta(R) - \zeta(r(\Psi(x,t))) - C_R t)^2 \tag{108}$$

defined in the set

$$\mathcal{C}_{R,T} = \big\{ y = \Psi(x,t) : \zeta(r(\Psi(x,t))) + C_R t \leq \zeta(R) \big\}. \tag{109}$$

Using (18) and (78) one obtains

$$\begin{aligned}(\partial_t - \Delta)\, \phi &\leq -2(\zeta(R) - \zeta - C_R t)(\partial_t \zeta - \Delta\zeta + C_R) - 2|\nabla\zeta|^2 \\ &\leq 2(\zeta(R) - \zeta - C_R t)(n\xi'(r) + n\sigma\xi(r)\langle \bar{\nabla} r, N\rangle - C_R) - 2|\nabla\zeta|^2\end{aligned}$$

Since

$$C_R = \sup_{B_R(o)}(\xi' + \sigma\xi) \tag{110}$$

we have

$$(\partial_t - \Delta)\, \phi \leq -2|\nabla\zeta|^2.$$

Therefore we compute

$$\begin{aligned}(\partial_t - \Delta)\,(\phi f) &= f\,(\partial_t - \Delta)\,\phi + \phi\,(\partial_t - \Delta)\, f - 2\langle \nabla\phi, \nabla f\rangle \\ &\leq -2f|\nabla\zeta|^2 - \Big\langle \nabla(\phi f) - f\nabla\phi, \frac{\nabla\psi}{\psi}\Big\rangle - a\phi f^2 + b\phi f \\ &\quad + 2\phi(C + |\bar{R}(\cdot, N, N, \cdot)|)\sqrt{\psi}\sqrt{f} - 2\Big\langle \frac{\nabla\phi}{\phi}, \nabla(\phi f) - f\nabla\phi\Big\rangle.\end{aligned}$$

However we have

$$-2|\nabla\zeta|^2 + 2\frac{|\nabla\phi|^2}{\phi} = -2|\nabla\zeta|^2 + 8\frac{(\zeta(R) - \zeta(r) - C_R t)^2}{(\zeta(R) - \zeta(r) - C_R t)^2}|\nabla\zeta|^2 = 6|\nabla\zeta|^2.$$

Hence,

$$\begin{aligned}(\partial_t - \Delta)(\phi f) \leq 6|\nabla\zeta|^2 f - \left\langle \nabla(\phi f), \frac{\nabla\psi}{\psi} + 2\frac{\nabla\phi}{\phi}\right\rangle + \left\langle f\nabla\phi, \frac{\nabla\psi}{\psi}\right\rangle \\ -a\phi f^2 + b\phi f + 2\phi(C + \sigma|\bar{R}(\cdot, N, N, \cdot)|)\sqrt{\psi}\sqrt{f}.\end{aligned}$$

Since that $\nabla\psi = \psi'(W)\nabla W$ and

$$\nabla W^{-1} = \nabla\langle X, N\rangle = \langle X, N\rangle(\bar{\nabla}\log\varrho)^\top - \langle\bar{\nabla}\log\varrho, N\rangle X^\top - AX^\top$$

we have

$$\begin{aligned}\frac{\nabla\psi}{\psi} &= \frac{\psi'(W)}{\psi(W)}\nabla W = \frac{2\gamma}{\gamma - \delta W^2}\frac{\nabla W}{W} = -\frac{2\gamma}{\gamma - \delta W^2} W\nabla W^{-1} \\ &= -\frac{2\gamma}{\gamma - \delta W^2} W\left(\langle X, N\rangle\bar{\nabla}\log\varrho - \langle\bar{\nabla}\log\varrho, N\rangle X - AX^\top\right).\end{aligned}$$

Hence,

$$\begin{aligned}\left|\frac{\nabla\psi}{\psi}\right| &\leq 2\gamma\frac{\psi(W)}{W}\left(2|X||\bar{\nabla}\log\varrho| + |A||X|\right) \leq 4\frac{\gamma}{\delta}|\bar{\nabla}\varrho| + 2\gamma\varrho\frac{\sqrt{\psi(W)}}{W}\sqrt{f} \\ &= \frac{2\gamma}{W^2}\psi(W)|\bar{\nabla}\log\varrho| + \frac{2\gamma}{W}\psi(W)\left(|X||\bar{\nabla}\varrho| + \varrho|A|\right) \\ &\leq 2\gamma\varrho^2\psi(W)|\bar{\nabla}\log\varrho| + 2\gamma\varrho\psi(W)\left(|\bar{\nabla}\varrho| + \varrho|A|\right) \\ &\leq 4\gamma\varrho^2|\bar{\nabla}\log\varrho|\psi(W) + 2\gamma\varrho^2\psi(W)|A| \leq 4\delta^{-1}\sup_{\mathcal{C}_R}|\bar{\nabla}\log\varrho| + 4\sqrt{\psi}\sqrt{f}.\end{aligned}$$

where we used that $\psi(W) \leq \delta^{-1}$. Denoting $c = 4\delta^{-1}\sup_{B_R(o)}|\bar{\nabla}\log\varrho|$ we have

$$\begin{aligned}(\partial_t - \Delta)(\phi f) \leq 6|\nabla\zeta|^2 f - \left\langle \nabla(\phi f), \frac{\nabla\psi}{\psi} + 2\frac{\nabla\phi}{\phi}\right\rangle + |\nabla\phi|\left(c + 4\sqrt{\psi}\sqrt{f}\right)f \\ -a\phi f^2 + b\phi f + 2\phi(C + |\bar{R}(\cdot, N, N, \cdot)|)\sqrt{\psi}\sqrt{f}.\end{aligned}$$

We conclude that at a point where ϕf attains a maximum value in the parabolic cylinder $B_{R'}(o) \times [0, T_R] \subset \mathcal{C}_{R,T}$ it holds (in case $t \neq 0$) that

$$a\phi f^2 \leq 4|\nabla\phi|\sqrt{\psi}\sqrt{f}f + \left(6|\nabla\zeta|^2 + b\phi + c|\nabla\phi|\right)f + 2(C + |\bar{R}(\cdot, N, N, \cdot)|)\sqrt{\phi\psi}\sqrt{\phi f}$$

so multiplying by $\sqrt{\phi}/\sqrt{f}$ and grouping the terms

$$\begin{aligned}a(\sqrt{\phi f})^3 \leq 4|\nabla\phi|\sqrt{\psi}\sqrt{\phi}f + \left(6|\nabla\zeta|^2 + b\phi + c|\nabla\phi|\right)\sqrt{\phi f} \\ +2(C + |\bar{R}(\cdot, N, N, \cdot)|)\phi\sqrt{\psi}\sqrt{\phi}.\end{aligned}$$

Considering that

$$\nabla\phi = -2\sqrt{\phi}\,\nabla\zeta = -2\sqrt{\phi}\,\xi(r)\nabla r \quad \text{and} \quad \phi \leq \zeta^2(R) \quad \text{and} \quad \sqrt{\psi} \leq \frac{1}{\sqrt{\delta}} \tag{111}$$

and using that $|\nabla r| \leq 1$ one concludes that

$$a(\sqrt{\phi f})^3 \leq \frac{8}{\sqrt{\delta}}\xi(r)(\sqrt{\phi f})^2 + \left(6\xi^2(r) + 2c\xi(r)\zeta(R) + b\zeta^2(R)\right)(\sqrt{\phi f}) \\ + \frac{2}{\sqrt{\delta}}(C + |\bar{R}(\cdot, N, N, \cdot)|)\zeta^3(R).$$

Therefore

$$a\left(\frac{\sqrt{\phi f}}{\zeta(R)}\right)^3 - \frac{8}{\sqrt{\delta}}\frac{\xi(R)}{\zeta(R)}\left(\frac{\sqrt{\phi f}}{\zeta(R)}\right)^2 - \left(6\frac{\xi^2(R)}{\zeta^2(R)} + 2c\frac{\xi(R)}{\zeta(R)} + b\right)\left(\frac{\sqrt{\phi f}}{\zeta(R)}\right) \\ - \frac{2}{\sqrt{\delta}}(C + |\bar{R}(\cdot, N, N, \cdot)|) \leq 0.$$

Fix $\varepsilon \leq \frac{1}{2}\frac{\delta}{(1-\delta)}\frac{1}{\sigma}$ so that $a \geq \delta$ and $\frac{\sigma}{\varepsilon} \geq 2\frac{1-\delta}{\delta}\sigma^2$. In this case, either

$$a\frac{\sqrt{\phi f}}{\zeta(R)} - \frac{8}{\sqrt{\delta}}\frac{\xi(R)}{\zeta(R)} \leq \sqrt{\delta}\,\frac{\xi(R)}{\zeta(R)} \tag{112}$$

or

$$\sqrt{\delta}\frac{\xi(R)}{\zeta(R)}\left(\frac{\sqrt{\phi f}}{\zeta(R)}\right)^2 - \left(6\frac{\xi^2(R)}{\zeta^2(R)} + 2c\frac{\xi(R)}{\zeta(R)} + b\right)\left(\frac{\sqrt{\phi f}}{\zeta(R)}\right) - \frac{2}{\sqrt{\delta}}(C + |\bar{R}(\cdot, N, N, \cdot)|) \leq 0.$$

We conclude that

$$\frac{\sqrt{\phi f}}{\zeta(R)} \leq C_1 := \max\left\{\left(\frac{1}{\sqrt{\delta}} + \frac{8}{\delta^{3/2}}\right)\frac{\xi(R)}{\zeta(R)}, \frac{1}{2\sqrt{\delta}}\left(6\frac{\xi(R)}{\zeta(R)} + 2c + b\frac{\zeta(R)}{\xi(R)}\right)\right. \\ \left. + \frac{1}{2\sqrt{\delta}}\left(\left(6\frac{\xi(R)}{\zeta(R)} + 2c + b\frac{\zeta(R)}{\xi(R)}\right)^2 + 8\frac{\zeta(R)}{\xi(R)}\left(C + \sup_{B_R(o)}|\bar{R}|\right)\right)^{\frac{1}{2}}\right\}.$$

Therefore

$$\inf_{B_{R'}(o)\times[0,T]}\left(1 - \frac{\zeta(r)}{\zeta(R)} - \frac{C_R}{\zeta(R)}t\right)\sqrt{f} \leq C_1.$$

Since $f = \psi(W)|A|^2$ the choice of $R' \in (0, R)$ in (104) implies that

$$\inf_{B_{R'}(o)\times[0,T]}\left(1 - \frac{\zeta(r)}{\zeta(R)} - \frac{C_R}{\zeta(R)}t\right) \geq \frac{1}{2}$$

we have

$$\sup_{B_{R'}(o)\times[0,T]}|A| \leq 2\sqrt{1-\delta}\,C_1 \leq 2C_1. \tag{113}$$

This shows that $|A|$ is bounded in $B_{R'}(o) \times [0,T]$ by some constant that depends on $\sup_{B_R(o)} W^2(\cdot,0)$ and on the geometric data $\sup_{B_R(o)} \varrho$, $\xi(R)$, $\zeta(R)$, C, $\widetilde{C}$ and L.

From this estimate we can conclude that the covariant derivatives of A are also bounded. Indeed, proceeding inductively as in Ecker and Huisken (1991) and Borisenko and Miquel (2012) one supposes that for each $k = 0, 1, \ldots, \ell - 1$ there exists a constant $C_k = C_k(R,T)$ so that

$$|\nabla^k A| \leq C_k \quad \text{for} \quad k = 0, 1, \cdots, \ell - 1$$

where C_k depends on the bounds of $|\nabla^m A|$ and on the tensors $\bar{\nabla}^m \bar{R}$ for $0 \leq m \leq k-1$ in $B_R(o) \times [0,T]$ and on the same sort of geometric data as above.

As in Ecker and Huisken (1991) and Borisenko and Miquel (2012) we are going to use variants of the Simons' inequality for higher order covariant derivatives of A which have the form

$$\frac{1}{2}(\partial_t - \Delta)|\nabla^\ell A|^2 + |\nabla^{\ell+1} A|^2 \leq D_\ell(|\nabla^\ell A|^2 + 1) \tag{114}$$

where the constant D_ℓ depends on the bounds of $|\nabla^k A|$ and on the tensors $\bar{\nabla}^k \bar{R}$ for $0 \leq k \leq \ell - 1$ in $B_R(o) \times [0,T]$. Now, we define

$$h = |\nabla^\ell A|^2 + \lambda |\nabla^{\ell-1} A|^2$$

where λ is a constant to be specified later. Setting $\lambda \geq 2D_\ell$ one obtains

$$\begin{aligned}
&\frac{1}{2}\partial_t h \leq \frac{1}{2}\Delta|\nabla^\ell A|^2 - |\nabla^{\ell+1}A|^2 + D_\ell(|\nabla^\ell A|^2 + 1) + \frac{\lambda}{2}\partial_t|\nabla^{\ell-1}A|^2 \\
&\leq \frac{1}{2}\Delta|\nabla^\ell A|^2 + \frac{\lambda}{2}\Delta|\nabla^{\ell-1}A|^2 - \lambda|\nabla^\ell A|^2 + D_\ell(|\nabla^\ell A|^2 + 1) + D_{\ell-1}(|\nabla^{\ell-1}A|^2 + 1) \\
&\leq \frac{1}{2}\Delta h - \frac{\lambda}{2}|\nabla^\ell A|^2 + D_{\ell-1}|\nabla^{\ell-1}A|^2 + D_\ell + \lambda D_{\ell-1} \\
&\leq \frac{1}{2}\Delta h - \frac{\lambda}{2}h + \frac{\lambda^2}{2}|\nabla^{\ell-1}A|^2 + D_{\ell-1}|\nabla^{\ell-1}A|^2 + D_\ell + \lambda D_{\ell-1}
\end{aligned}$$

Choosing $\lambda^2 \geq 2D_{\ell-1}$ we conclude that

$$(\partial_t - \Delta)h \leq -\lambda h + \lambda^2 \widetilde{C}_\ell + \widetilde{D}_\ell, \tag{115}$$

where $\widetilde{D}_\ell = 2D_\ell + 2\lambda D_{\ell-1}$ and $\widetilde{C}_\ell = 2|\nabla^{\ell-1}A|^2$. Proceeding similarly as above one computes

$$\begin{aligned}
(\partial_t - \Delta)(\phi h) &= h(\partial_t - \Delta)\phi + \phi(\partial_t - \Delta)h - 2\langle \nabla\phi, \nabla h\rangle \\
&\leq -2h|\nabla\zeta|^2 + (-\lambda h + \lambda^2\widetilde{C}_\ell + \widetilde{D}_\ell)\phi - 2\left\langle \phi^{-1}\nabla\phi, \nabla(\phi h) - h\nabla\phi\right\rangle.
\end{aligned}$$

Therefore

$$(\partial_t - \Delta)(\phi h) + 2\left\langle \phi^{-1}\nabla\phi, \nabla(\phi h)\right\rangle \le -2h|\nabla\zeta|^2 + 2\phi^{-1}|\nabla\phi|^2 h + (-\lambda h + \lambda^2\widetilde{C}_\ell + \widetilde{D}_\ell)\phi.$$

Using again that

$$-2|\nabla\zeta|^2 + 2\phi^{-1}|\nabla\phi|^2 = 6|\nabla\zeta|^2$$

one concludes that

$$(\partial_t - \Delta)(\phi h) + 2\left\langle \phi^{-1}\nabla\phi, \nabla(\phi h)\right\rangle \le 6|\nabla\zeta|^2 h - \lambda\phi h + (\lambda^2\widetilde{C}_\ell + \widetilde{D}_\ell)\phi.$$

We have at a maximum point of ϕh that

$$(\lambda\phi - 6\xi^2(r))h \le (\lambda^2\widetilde{C}_\ell + \widetilde{D}_\ell)\zeta^2(R)$$

Since that $\phi \ge \zeta^2(R)/4$ in $B_{R'}(o) \times [0,T]$ we have

$$\left(\frac{\lambda}{4} - 6\frac{\xi^2(R)}{\zeta^2(R)}\right) h \le \lambda^2\widetilde{C}_\ell + \widetilde{D}_\ell$$

Setting

$$\lambda \ge \max\left\{2D_\ell, (2D_{\ell-1})^{1/2}, 48\frac{\xi^2(R)}{\zeta^2(R)}\right\}$$

one obtains

$$h \le \frac{1}{6}\frac{\zeta^2(R)}{\xi^2(R)}\big(\lambda^2\widetilde{C}_\ell + \widetilde{D}_\ell\big).$$

We conclude that

$$|\nabla^\ell A|^2 \le \frac{1}{6}\frac{\zeta^2(R)}{\xi^2(R)}\big(2\lambda^2|\nabla^{\ell-1}A|^2 + \widetilde{D}_\ell\big) - \lambda|\nabla^{\ell-1}A|^2 \tag{116}$$

A suitable choice of a large enough λ yields the desired estimate. This finishes the proof.

5 EXISTENCE OF SOLUTION

In this section we assemble the results obtained previously to guarantee the existence theorem of our equation (3), when M is a complete, non-compact, n-dimensional smooth manifold. More precisely,

Theorem 5.1. *Let M be a n-dimensional complete, non-compact Riemannian manifold with a pole o and let $\bar{M}$ be the warped product $M \times_{\varrho} \mathbb{R}$. Suppose that (6), (8) and (9) hold. Given a smooth entire graph M_0 over M, there exists a modified mean curvature flow (2)-(3) defined for all $t \in \mathbb{R}$.*

Proof: The problem (2)-(3) may be rewritten with initial value problem as follows

$$\begin{cases} \partial_t u = \mathcal{Q}[u] & \text{in} \quad M \times [0,T] \\ u(\cdot, 0) = u_0 & \text{in} \ \ M \end{cases} \tag{117}$$

where $\mathcal{Q}$ is given for (14) and with a smooth initial datum $u_0 : M \to \mathbb{R}$. Given $R > 0$, by the theory of parabolic equations there is a unique solution u_R of the Dirichlet problem

$$\begin{cases} u_t = \left(g^{ij} - \frac{u^i u^j}{W^2}\right) u_{i;j} + \left(1 + \frac{1}{\varrho^2 W^2}\right) (\log \varrho)^i u_i - n\sigma W & B_R(o) \times [0,T] \\ u(\cdot, 0) = u_0(\cdot) & B_R(o) \times \{0\} \\ u(x,t) = u_0(x) & \partial B_R(o) \times [0,T] \end{cases} \tag{118}$$

which is a reformulation of (117) on $B_R(o) \times [0,T]$. The result follows if we have a smooth solution for all time. We have from the estimates 3.2 together with (113) and (116) that we can take constants such that

$$|\nabla^\ell u_R| \leq C_\ell$$

with C_ℓ as in the previous section. Furthermore we have from proposition 2.5 that $|u_R|$ is bounded.

So proceeding as in Borisenko and Miquel (2012), for every $R_0 > 0$, the family of smooth functions $(u_R)_{R \geq R_0}$ converges to a smooth function u_{R_0} on $B_{R_0}(o)$. It is worth noting that u_{R_0} is at least C^1 on t and that it is a solution of (117) on $B_{R_0}(o) \times [0,T]$, then the parabolic theory guarantees that u_{R_0} is also C^∞ on t. Let be a sequence

$$R_0^1 < R_0^2 < \cdots < R_0^i < .. \to \infty$$

For $j > i$ the families $(u_R)_{R \geq R_0^i}$ and $(u_R)_{R \geq R_0^j}$ coincide for $R \geq R_0^i$, this way their limits u_{R^i} satisfy

$$u_{R^j}|_{B_{R^i}(o)} = u_{R^i}$$

that defines a smooth function u on M which is the C^∞ limit on the compacts the family u_R when $R \to \infty$. Now, since that we obtained the previous bounds, we have, on each compact in balls a well-defined limit of M_t when $t \to T$, a standard argument shows, in this case, that the T is infinite, otherwise we could restart the flow from the smooth surface to get as a bound. □

Remark 5.1. *For the case Lipschitz, maybe it's possible to obtain existence of solution to* (3) *as Borisenko and Miquel (2012) and Ecker and Huisken (1991) where M_0 is a graph over M given by a Lipschitz continuous function. If we proceed as in the standard is to be expected that the result follow if we do an approximation of M_0 by a sequence of smooth graphs and applying the existence theorem* 5.1 *to these approximations. In addition, if M is complete and the $\bar{M}$ it is a Cartan-Hadamard manifold such that is possible to define a asymptotic boundary, we can obtain a result as Allmann, Lin, and Zhu (2017), which prescribe the asymptotic data.*

6 APPENDIX

6.1 Curvature tensor in warped space

In that follows we will make some computations in order to obtain the inequality (107). The Riemann curvature tensor in the warped space $M \times_\varrho \mathbb{R}$ has components

$$\begin{cases} \bar{R}(U,V)X = 0 \\ \bar{R}(U,X)V = \dfrac{1}{\varrho}\langle \bar{\nabla}_U \bar{\nabla}\varrho, V\rangle X \\ \bar{R}(U,X)X = -\varrho \bar{\nabla}_U \bar{\nabla}\varrho \\ \bar{R}(U,V)W = R^P(U,V)W \end{cases} \tag{119}$$

for all $U, V, W \perp X$. Given u, v tangent to Σ we have

$$\begin{aligned} \bar{R}(u,v)X &= u^0 \bar{R}(X, v^\perp)X + v^0 \bar{R}(u^\perp, X)X + \bar{R}(u^\perp, v^\perp)X \\ &= u^0 \varrho \bar{\nabla}_{v^\perp}\bar{\nabla}\varrho - v^0 \varrho \bar{\nabla}_{u^\perp}\bar{\nabla}\varrho \\ &= \langle X, u\rangle \frac{1}{\varrho}\bar{\nabla}_{v^\perp}\bar{\nabla}\varrho - \langle X, v\rangle \frac{1}{\varrho}\bar{\nabla}_{u^\perp}\bar{\nabla}\varrho \end{aligned}$$

Now, given $W \perp X$ we have

$$\begin{aligned} \bar{R}(u,v)W &= u^0 \bar{R}(X, v^\perp)W + v^0 \bar{R}(u^\perp, X)W + \bar{R}(u^\perp, v^\perp)W \\ &= -u^0 \frac{1}{\varrho}\langle \bar{\nabla}_{v^\perp}\bar{\nabla}\varrho, W\rangle X + v^0 \frac{1}{\varrho}\langle \bar{\nabla}_{u^\perp}\bar{\nabla}\varrho, W\rangle X + \bar{R}^P(u^\perp, v^\perp)W \\ &= -\frac{1}{\varrho^3}\langle X, u\rangle \langle \bar{\nabla}_{v^\perp}\bar{\nabla}\varrho, W\rangle X + \frac{1}{\varrho^3}\langle X, v\rangle \langle \bar{\nabla}_{u^\perp}\bar{\nabla}\varrho, W\rangle X + \bar{R}^P(u^\perp, v^\perp)W \end{aligned}$$

Hence

$$\begin{aligned} \bar{R}(u,v)N &= \frac{1}{\varrho^3}\langle X, u\rangle\langle X, N\rangle \bar{\nabla}_{v^\perp}\bar{\nabla}\varrho - \langle X, v\rangle\langle X, N\rangle \frac{1}{\varrho^3}\bar{\nabla}_{u^\perp}\bar{\nabla}\varrho - \frac{1}{\varrho^3}\langle X, u\rangle\langle \bar{\nabla}_{v^\perp}\bar{\nabla}\varrho, N^\perp\rangle X \\ &+ \frac{1}{\varrho^3}\langle X, v\rangle \langle \bar{\nabla}_{u^\perp}\bar{\nabla}\varrho, N^\perp\rangle X + \bar{R}^P(u^\perp, v^\perp)N^\perp \end{aligned}$$

Since that $\left[X, \bar{\nabla}\varrho\right] = 0$ we have

$$\begin{aligned} \langle \bar{\nabla}_u \bar{\nabla}\varrho, v\rangle &= u^0 v^0 \langle \bar{\nabla}_X \bar{\nabla}\varrho, X\rangle + u^0 \langle \bar{\nabla}_X \bar{\nabla}\varrho, v^\perp\rangle + v^0 \langle \bar{\nabla}_{u^\perp}\bar{\nabla}\varrho, X\rangle + \langle \bar{\nabla}_{u^\perp}\bar{\nabla}\varrho, v^\perp\rangle \\ &= u^0 v^0 \langle \bar{\nabla}_{\bar{\nabla}\varrho} X, X\rangle + u^0 \langle \bar{\nabla}_{\bar{\nabla}\varrho} X, v^\perp\rangle + \langle \bar{\nabla}_{u^\perp}\bar{\nabla}\varrho, v^\perp\rangle \\ &= \frac{1}{2} u^0 v^0 \langle \bar{\nabla}\varrho, \bar{\nabla}\varrho^2\rangle + \langle \bar{\nabla}_{u^\perp}\bar{\nabla}\varrho, v^\perp\rangle \\ &= u^0 v^0 \varrho |\bar{\nabla}\varrho|^2 + \langle \bar{\nabla}_{u^\perp}\bar{\nabla}\varrho, v^\perp\rangle \\ &= \frac{1}{\varrho^3}\langle X, u\rangle\langle X, v\rangle |\bar{\nabla}\varrho|^2 + \langle \bar{\nabla}_{u^\perp}\bar{\nabla}\varrho, v^\perp\rangle \end{aligned}$$

Therefore given a tangent $w \in T\Sigma$ we have

$$\begin{aligned}
\langle \bar{R}(u,v)N, w\rangle &= \frac{1}{\varrho^3}\langle X,u\rangle\langle X,N\rangle\left(\langle \bar{\nabla}_v\bar{\nabla}\varrho, w\rangle - \langle X,v\rangle\langle X,w\rangle\frac{1}{\varrho^3}|\bar{\nabla}\varrho|^2\right)\\
&- \frac{1}{\varrho^3}\langle X,v\rangle\langle X,N\rangle\left(\langle \bar{\nabla}_u\bar{\nabla}\varrho, w\rangle - \langle X,u\rangle\langle X,w\rangle\frac{1}{\varrho^3}|\bar{\nabla}\varrho|^2\right)\\
&- \frac{1}{\varrho^3}\langle X,u\rangle\langle X,w\rangle\left(\langle \bar{\nabla}_v\bar{\nabla}\varrho, N\rangle - \langle X,v\rangle\langle X,N\rangle\frac{1}{\varrho^3}|\bar{\nabla}\varrho|^2\right)\\
&+ \frac{1}{\varrho^3}\langle X,v\rangle\langle X,w\rangle\left(\langle \bar{\nabla}_u\bar{\nabla}\varrho, N\rangle - \langle X,u\rangle\langle X,N\rangle\frac{1}{\varrho^3}|\bar{\nabla}\varrho|^2\right)\\
&+ \langle \bar{R}^P(u^\perp,v^\perp)N^\perp, w^\perp\rangle
\end{aligned}$$

thereafter

$$\begin{aligned}
\langle \bar{R}(u,v)N, w\rangle &= \frac{1}{\varrho^3}\langle X,u\rangle\langle X,N\rangle\langle \bar{\nabla}_v\bar{\nabla}\varrho, w\rangle - \frac{1}{\varrho^3}\langle X,v\rangle\langle X,N\rangle\langle \bar{\nabla}_u\bar{\nabla}\varrho, w\rangle\\
&- \frac{1}{\varrho^3}\langle X,u\rangle\langle X,w\rangle\langle \bar{\nabla}_v\bar{\nabla}\varrho, N\rangle + \frac{1}{\varrho^3}\langle X,v\rangle\langle X,w\rangle\langle \bar{\nabla}_u\bar{\nabla}\varrho, N\rangle\\
&+ \langle \bar{R}^P(u^\perp,v^\perp)N^\perp, w^\perp\rangle
\end{aligned}$$

If we take $\partial_k, \partial_j, \partial_l$, for the tensor

$$L := L_{kjl} = \langle \bar{R}(\partial_k,\partial_j)N, \partial_l\rangle$$

in terms of the warped product we have

$$L_{kjl} = \langle \bar{R}(\partial_k,\partial_j)N, \partial_l\rangle = \frac{1}{\varrho^3}\langle X,\partial_k\rangle\langle X,N\rangle\langle \bar{\nabla}_{\partial_j}\bar{\nabla}\varrho, \partial_l\rangle - \frac{1}{\varrho^3}\langle X,\partial_j\rangle\langle X,N\rangle\langle \bar{\nabla}_{\partial_k}\bar{\nabla}\varrho, \partial_l\rangle$$
$$-\frac{1}{\varrho^3}\langle X,\partial_k\rangle\langle X,\partial_l\rangle\langle \bar{\nabla}_{\partial_j}\bar{\nabla}\varrho, N\rangle + \frac{1}{\varrho^3}\langle X,\partial_j\rangle\langle X,\partial_l\rangle\langle \bar{\nabla}_{\partial_k}\bar{\nabla}\varrho, N\rangle + \langle R^P(\partial_k^\perp,\partial_j^\perp)N^\perp, \partial_l^\perp\rangle$$

To obtain $\nabla_i L_{kjl}$ we will calculate all terms apart. Firstly, we consider

$$\theta(\partial_k) = \langle X,\partial_k\rangle$$

where θ is the 1-form in M metrically equivalent to X. So we have

$$\begin{aligned}
\left(\nabla_{\partial_i}\theta\right)(\partial_k) &= \nabla_{\partial_i}\left(\theta(\partial_k)\right) - \theta\left(\nabla_{\partial_i}\partial_k\right)\\
&= \partial_i\langle X,\partial_k\rangle - \Gamma_{ik}^m\theta(\partial_m)\\
&= \langle \bar{\nabla}_{\partial_i}X,\partial_k\rangle + \Gamma_{ik}^m\langle X,\partial_m\rangle + \Gamma_{ik}^0\langle X,N\rangle - \Gamma_{ik}^m\langle X,\partial_m\rangle\\
&= \langle \bar{\nabla}_{\partial_i}X,\partial_k\rangle + \bar{\Gamma}_{ik}^0\langle X,N\rangle
\end{aligned}$$

However

$$\bar\nabla_{\partial_i} X = \frac{1}{\varrho^2}\langle \partial_i, X\rangle \bar\nabla_X X + \bar\nabla_{\partial_i^\perp} X$$

and taking the product

$$\begin{aligned}
\langle \partial_k, \bar\nabla_{\partial_i} X\rangle &= \langle \frac{1}{\varrho^2}\langle \partial_i, X\rangle \bar\nabla_X X + \bar\nabla_{\partial_i^\perp} X, \frac{1}{\varrho^2}\langle \partial_k, X\rangle X + \partial_k^\perp\rangle \\
&= \frac{1}{\varrho^2}\langle \partial_i, X\rangle\langle \bar\nabla_X X, \partial_k^\perp\rangle + \frac{1}{\varrho^2}\langle \partial_k, X\rangle\langle \bar\nabla_{\partial_i^\perp} X, X\rangle \\
&= -\frac{1}{\varrho}\langle \partial_i, X\rangle\langle \bar\nabla \varrho, \partial_k\rangle + \frac{1}{\varrho}\langle \partial_k, X\rangle\langle \bar\nabla \varrho, \partial_i\rangle
\end{aligned}$$

It worth observe that

$$\langle \nabla_{u^\perp} X, X\rangle = \frac{1}{2}u^\perp(\varrho^2) = \varrho\langle \bar\nabla\varrho, u\rangle \quad \text{and} \quad \langle \nabla_X X, u^\perp\rangle = -\langle \nabla_{u^\perp} X, X\rangle$$

Now, taking $\theta_i = \langle X, \partial_i\rangle$ and also $\varrho_k = \langle \bar\nabla\varrho, \partial_k\rangle$, and we get

$$\langle \bar\nabla_{\partial_i} X, \partial_k\rangle = -\theta_i\frac{\varrho_k}{\varrho} + \theta_k\frac{\varrho_i}{\varrho}$$

as well as

$$(\nabla_{\partial_i}\theta)(\partial_k) = -\theta_i\frac{\varrho_k}{\varrho} + \theta_k\frac{\varrho_i}{\varrho} + a_{ik}\langle X, N\rangle$$

Let's now consider the account for the tensor $\varrho_{jl} = \langle \bar\nabla_{\partial_j}\bar\nabla\varrho, \partial_l\rangle$. We have like this

$$\begin{aligned}
(\nabla_{\partial_i}\varrho)(\partial_j, \partial_l) &= \partial_i(\varrho(\partial_j, \partial_l)) - \Gamma_{ij}^m\varrho_{ml} - \Gamma_{il}^m\varrho_{jm} \\
&= (\bar\nabla_{\partial_i}\varrho)(\partial_j, \partial_l) + \bar\Gamma_{ij}^m\varrho_{ml} + \bar\Gamma_{ij}^0\varrho_{0l} + \bar\Gamma_{il}^m\varrho_{jm} + \Gamma_{il}^0\varrho_{j0} - \Gamma_{ij}^m\varrho_{ml} - \Gamma_{il}^m\varrho_{jm} \\
&= (\bar\nabla_{\partial_i}\varrho)(\partial_j, \partial_l) + a_{ij}\varrho_{0l} + a_{il}\varrho_{j0}
\end{aligned}$$

In this way we have

$$\nabla_{\partial_i}\varrho_{jl} = \nabla_{\partial_i}\langle \bar\nabla_{\partial_j}\bar\nabla\varrho, \partial_l\rangle = \bar\nabla_{\partial_i}\bar\nabla_{\partial_j}\bar\nabla_{\partial_l}\varrho + a_{ij}\langle \bar\nabla_{\partial_l}\bar\nabla\varrho, N\rangle + a_{il}\langle \bar\nabla_{\partial_j}\bar\nabla\varrho, N\rangle$$

such that the derivative of the tensor L is

$$\begin{aligned}
\nabla_i\left(\theta_k\frac{1}{\varrho^3 W}\varrho_{jl}\right) &= \frac{1}{\varrho^3 W}\varrho_{jl}\left(-\theta_i\frac{\varrho_k}{\varrho} + \theta_k\frac{\varrho_i}{\varrho} + \frac{a_{ij}}{W}\right) + +\theta_k\varrho_{jl}\partial_i\left(\frac{1}{\varrho^3 W}\right) \\
&+ \theta_k\frac{1}{\varrho^3 W}\left[\bar\nabla_{\partial_i}\bar\nabla_{\partial_j}\bar\nabla_{\partial_l}\varrho + a_{ij}\langle \bar\nabla_{\partial_l}\bar\nabla\varrho, N\rangle + a_{il}\langle \bar\nabla_{\partial_j}\bar\nabla\varrho, N\rangle\right]
\end{aligned}$$

since that

$$\nabla_i\left(\theta_k\frac{1}{\varrho^3 W}\varrho_{jl}\right) = \nabla_i\left(\langle X, \partial_k\rangle\langle X, N\rangle\frac{1}{\varrho^3}\langle \bar\nabla_{\partial_j}\bar\nabla\varrho, \partial_l\rangle\right)$$

Similarly we take the tensor $S_j = \langle \bar\nabla_{\partial_j} \bar\nabla \varrho, N\rangle$ and we proceed like previously now as

$$\begin{aligned}\nabla_i S_j &= (\nabla_i S)(\partial_j) = \partial_i(S\partial_j) - S(\nabla_{\partial_i}\partial_j)\\ &= (\bar\nabla_i S)(\partial_j) + S(\bar\nabla_{\partial_i}\partial_j) - S(\nabla_{\partial_i}\partial_j)\\ &= (\bar\nabla_i S)(\partial_j) + \bar\Gamma^m_{ij} S_m - \Gamma^m_{ij} S_m + \Gamma^0_{ij} S_0\end{aligned}$$

In our case, if $N = \partial_0$ we write

$$\nabla_i S_j = \nabla \langle \bar\nabla_{\partial_j} \bar\nabla \varrho, N\rangle = \bar\nabla_i \bar\nabla_j \bar\nabla_0 \varrho + a_{ij} \langle \bar\nabla_N \bar\nabla \varrho, N\rangle$$

let us now see the terms of the form

$$\begin{aligned}\nabla_i\left(\frac{1}{\varrho^3}\theta_k\theta_l S_j\right) &= \frac{1}{\varrho^3}\left(-\theta_i\frac{\varrho_k}{\varrho} + \theta_k\frac{\varrho_i}{\varrho} + \frac{a_{ik}}{W}\right)\theta_l S_j + \frac{1}{\varrho^3}\theta_k\left(-\theta_i\frac{\varrho_l}{\varrho} + \theta_l\frac{\varrho_i}{\varrho} + \frac{a_{il}}{W}\right) S_j\\ &+ \frac{1}{\varrho^3}\theta_k\theta_l\left[\bar\nabla_i\bar\nabla_j\bar\nabla_N\varrho + a_{ij}\langle\bar\nabla_N\bar\nabla\varrho, N\rangle\right] + \partial_i\left(\frac{1}{\varrho^3}\right)\theta_k\theta_l S_j\end{aligned}$$

and also of

$$P(\partial_k, \partial_j, \partial_l) := \langle R^P(\partial_k^\perp, \partial_j^\perp)N^\perp, \partial_l^\perp\rangle$$

It follows that

$$\begin{aligned}\nabla_i P(\partial_k, \partial_j, \partial_l) &= \left(\bar\nabla_i R^P\right)(\partial_k, \partial_j, N, \partial_l) + a_{ik} R^P(N, \partial_j, N, \partial_l) + a_{ij} R^P(\partial_k, N, N, \partial_l)\\ &+ a_{il} R^P(\partial_k, \partial_j, N, N)\end{aligned}$$

Notice that

$$\begin{aligned}\bar R_{kils} &= \langle \bar R(\partial_k, \partial_i)\partial_l, \partial_s\rangle = \theta_k\theta_l\frac{1}{\varrho^3}\langle\bar\nabla_{\partial_i}\bar\nabla\varrho, \partial_s\rangle - \theta_i\theta_l\frac{1}{\varrho^3}\langle\bar\nabla_{\partial_k}\bar\nabla\varrho, \partial_s\rangle - \theta_k\theta_s\frac{1}{\varrho^3}\langle\bar\nabla_{\partial_i}\bar\nabla\varrho, \partial_l\rangle\\ &+ \theta_i\theta_s\frac{1}{\varrho^3}\langle\bar\nabla_{\partial_k}\bar\nabla\varrho, \partial_l\rangle + \langle R^P(\partial_k^\perp, \partial_i^\perp)\partial_l^\perp, \partial_s^\perp\rangle\end{aligned}$$

and now, whereas $\bar R^s_{kil} a_{sj} = \bar R_{kils} a^s_j$ we obtain

$$\begin{aligned}\bar R_{kils} a^s_j = \langle \bar R(\partial_k, \partial_i)\partial_l, \partial_s\rangle a^s_j = \theta_k\theta_l\frac{1}{\varrho^3}\langle\bar\nabla_{\partial_i}\bar\nabla\varrho, \partial_s\rangle a^s_j - \theta_i\theta_l\frac{1}{\varrho^3}\langle\bar\nabla_{\partial_k}\bar\nabla\varrho, \partial_s\rangle a^s_j\\ -\theta_k\theta_s\frac{1}{\varrho^3}\langle\bar\nabla_{\partial_i}\bar\nabla\varrho, \partial_l\rangle a^s_j + \theta_i\theta_s\frac{1}{\varrho^3}\langle\bar\nabla_{\partial_k}\bar\nabla\varrho, \partial_l\rangle a^s_j + \langle R^P(\partial_k^\perp, \partial_i^\perp)\partial_l^\perp, \partial_s^\perp\rangle a^s_j\end{aligned} \tag{120}$$

Let us now express the terms of the tensor derivatives

$$
\begin{aligned}
\nabla_i L_{kjl} &= \frac{1}{\varrho^3 W}\varrho_{jl}\left(-\theta_i\frac{\varrho_k}{\varrho}+\theta_k\frac{\varrho_i}{\varrho}+\frac{a_{ik}}{W}\right)+\theta_k\frac{1}{\varrho^3 W}\left[\bar\nabla_{\partial_i}\bar\nabla_{\partial_j}\bar\nabla_{\partial_l}\varrho+a_{ij}S_l+a_{il}S_j\right]\\
&+ \theta_k\varrho_{jl}\frac{1}{W}\partial_i\left(\frac{1}{\varrho^3}\right)+\theta_k\varrho_{jl}\frac{1}{\varrho^3}\partial_i\left(\frac{1}{W}\right)-\theta_j\varrho_{kl}\frac{1}{W}\partial_i\left(\frac{1}{\varrho^3}\right)-\theta_j\varrho_{kl}\frac{1}{\varrho^3}\partial_i\left(\frac{1}{W}\right)\\
&- \frac{1}{\varrho^3 W}\varrho_{kl}\left(-\theta_i\frac{\varrho_j}{\varrho}+\theta_j\frac{\varrho_i}{\varrho}+\frac{a_{ij}}{W}\right)-\theta_j\frac{1}{\varrho^3 W}\left[\bar\nabla_{\partial_i}\bar\nabla_{\partial_k}\bar\nabla_{\partial_l}\varrho+a_{ik}S_l+a_{il}S_k\right]\\
&- \frac{1}{\varrho^3}\left[-\theta_i\frac{\varrho_k}{\varrho}+\theta_k\frac{\varrho_i}{\varrho}+\frac{a_{ik}}{W}\right]\theta_l S_j-\frac{1}{\varrho^3}\theta_k\left[-\theta_i\frac{\varrho_l}{\varrho}+\theta_l\frac{\varrho_i}{\varrho}+\frac{a_{il}}{W}\right]S_j\\
&- \frac{1}{\varrho^3}\theta_k\theta_l\left[\bar\nabla_i\bar\nabla_j\bar\nabla_N\varrho+a_{ij}\langle\bar\nabla_N\bar\nabla\varrho,N\rangle\right]+\partial_i\left(\frac{1}{\varrho^3}\right)\theta_j\theta_l S_k-\partial_i\left(\frac{1}{\varrho^3}\right)\theta_k\theta_l S_j\\
&+ \frac{1}{\varrho^3}\left[-\theta_i\frac{\varrho_j}{\varrho}+\theta_j\frac{\varrho_i}{\varrho}+\frac{a_{ij}}{W}\right]\theta_l S_k+\frac{1}{\varrho^3}\theta_j\left[-\theta_i\frac{\varrho_l}{\varrho}+\theta_l\frac{\varrho_i}{\varrho}+\frac{a_{il}}{W}\right]S_k\\
&+ \frac{1}{\varrho^3}\theta_j\theta_l\left[\bar\nabla_i\bar\nabla_k\bar\nabla_N\varrho+a_{ik}\langle\bar\nabla_N\bar\nabla\varrho,N\rangle\right]+\nabla_i\langle R^P(\partial_k^\perp\partial_j^\perp)N^\perp,\partial_l^\perp\rangle
\end{aligned}
$$

where also we have

$$
\frac{1}{\varrho^3}\theta_k\varrho_{jl}\partial_i\left(\frac{1}{W}\right)=-\theta_i\theta_k\frac{1}{\varrho^4}\varrho_{jl}\langle\bar\nabla\varrho,N\rangle+\frac{1}{W}\frac{\theta_k}{\varrho^4}\varrho_{jl}\langle\partial_i,\bar\nabla\varrho\rangle-\frac{\theta_k}{\varrho^3}\varrho_{jl}\langle A\partial_i,X\rangle
$$

the same applies to similar terms. Hence, we obtain

$$
\begin{aligned}
\nabla_k L_{lij} &= \frac{1}{\varrho^3 W}\varrho_{ij}\left(-\theta_k\frac{\varrho_l}{\varrho}+\theta_l\frac{\varrho_k}{\varrho}+\frac{a_{kl}}{W}\right)+\theta_l\frac{1}{\varrho^3 W}\left[\bar\nabla_{\partial_k}\bar\nabla_{\partial_i}\bar\nabla_{\partial_j}\varrho+a_{ki}S_j+a_{kj}S_i\right]\\
&+ \theta_l\varrho_{ij}\frac{1}{W}\partial_k\left(\frac{1}{\varrho^3}\right)+\theta_l\varrho_{ij}\frac{1}{\varrho^3}\partial_k\left(\frac{1}{W}\right)-\theta_i\varrho_{lj}\frac{1}{W}\partial_k\left(\frac{1}{\varrho^3}\right)-\theta_i\varrho_{lj}\frac{1}{\varrho^3}\partial_k\left(\frac{1}{W}\right)\\
&- \theta_i\frac{1}{\varrho^3 W}\left[\bar\nabla_{\partial_k}\bar\nabla_{\partial_l}\bar\nabla_{\partial_j}\varrho+a_{kl}S_j+a_{kj}S_l\right]-\frac{1}{\varrho^3 W}\varrho_{lj}\left(-\theta_k\frac{\varrho_i}{\varrho}+\theta_i\frac{\varrho_k}{\varrho}+\frac{a_{ki}}{W}\right)\\
&- \frac{1}{\varrho^3}\left[-\theta_k\frac{\varrho_j}{\varrho}+\theta_j\frac{\varrho_k}{\varrho}+\frac{a_{kj}}{W}\right]\theta_l S_i-\frac{1}{\varrho^3}\theta_j\left[-\theta_k\frac{\varrho_l}{\varrho}+\theta_l\frac{\varrho_k}{\varrho}+\frac{a_{kl}}{W}\right]S_i\\
&- \frac{1}{\varrho^3}\theta_j\theta_l\left[\bar\nabla_k\bar\nabla_i\bar\nabla_N\varrho+a_{ki}\langle\bar\nabla_N\bar\nabla\varrho,N\rangle\right]+\partial_k\left(\frac{1}{\varrho^3}\right)\theta_i\theta_j S_l-\partial_k\left(\frac{1}{\varrho^3}\right)\theta_l\theta_j S_i\\
&+ \frac{1}{\varrho^3}\left[-\theta_k\frac{\varrho_j}{\varrho}+\theta_j\frac{\varrho_k}{\varrho}+\frac{a_{kj}}{W}\right]\theta_i S_l+\frac{1}{\varrho^3}\theta_j\left[-\theta_k\frac{\varrho_i}{\varrho}+\theta_i\frac{\varrho_k}{\varrho}+\frac{a_{ki}}{W}\right]S_l\\
&+ \frac{1}{\varrho^3}\theta_i\theta_j\left[\bar\nabla_k\bar\nabla_j\bar\nabla_N\varrho+a_{kl}\langle\bar\nabla_N\bar\nabla\varrho,N\rangle\right]+\nabla_k\langle R^P(\partial_l^\perp\partial_i^\perp)N^\perp,\partial_j^\perp\rangle
\end{aligned}
$$

and now we join all terms

$$
\begin{aligned}
g^{kl}(\nabla_i L_{kjl}+\nabla_k L_{lij})a^{ij} = & \frac{1}{\varrho^3}a^{ij}\left[-\theta_i\frac{\varrho_k}{\varrho}+\theta_k\frac{\varrho_i}{\varrho}+\frac{a_{ik}}{W}\right]g^{kl}\left(\frac{\varrho_{jl}}{W}-\theta_l S_j\right)\\
&+\frac{1}{\varrho^3}a^{ij}\left[-\theta_i\frac{\varrho_j}{\varrho}+\theta_j\frac{\varrho_i}{\varrho}+\frac{a_{ij}}{W}\right]g^{kl}\left(\theta_l S_k-\frac{\varrho_{jl}}{W}\right)\\
&+\frac{1}{\varrho^3}a^{ij}\left[-\theta_k\frac{\varrho_l}{\varrho}+\theta_l\frac{\varrho_k}{\varrho}+\frac{a_{kl}}{W}\right]g^{kl}\left(\frac{\varrho_{ij}}{W}-\theta_j S_i\right)\\
&+\frac{1}{\varrho^3}a^{ij}\left[-\theta_k\frac{\varrho_i}{\varrho}+\theta_i\frac{\varrho_k}{\varrho}+\frac{a_{ki}}{W}\right]g^{kl}\left(\theta_j S_l-\frac{\varrho_{jl}}{W}\right)\\
&+\frac{1}{\varrho^3}a^{ij}\left[-\theta_i\frac{\varrho_l}{\varrho}+\theta_l\frac{\varrho_i}{\varrho}+\frac{a_{il}}{W}\right]g^{kl}\left(\theta_j S_k-\theta_k S_j\right)\\
&+\frac{1}{\varrho^3}a^{ij}\left[-\theta_k\frac{\varrho_j}{\varrho}+\theta_j\frac{\varrho_k}{\varrho}+\frac{a_{jk}}{W}\right]g^{kl}\left(\theta_i S_l-\theta_l S_i\right)\\
&+\frac{a_{ij}}{W}g^{kl}\left(\theta_k\varrho_{jl}-\theta_j\varrho_{kl}\right)\partial_i\left(\frac{1}{\varrho^3}\right)+\frac{a_{ij}}{W}g^{kl}\left(\theta_l\varrho_{ij}-\theta_i\varrho_{lj}\right)\partial_k\left(\frac{1}{\varrho^3}\right)\\
&+a^{ij}g^{kl}\left(\theta_j\theta_l S_k-\theta_k\theta_l S_j\right)\partial_i\left(\frac{1}{\varrho^3}\right)+a^{ij}g^{kl}\left(\theta_i\theta_j S_l-\theta_l\theta_j S_i\right)\partial_k\left(\frac{1}{\varrho^3}\right)\\
&+\frac{1}{\varrho^3}g^{kl}\varrho_{jl}\left(-\theta_i\frac{\theta_k}{\varrho}\langle\bar\nabla\varrho,N\rangle+\frac{\theta_k}{\varrho W}\langle\bar\nabla\varrho,\partial_i\rangle-\theta_k a_i^s\theta_s\right)a^{ij}\\
&+\frac{1}{\varrho^3}g^{kl}\varrho_{kl}\left(-\theta_i\frac{\theta_k}{\varrho}\langle\bar\nabla\varrho,N\rangle-\frac{\theta_j}{\varrho W}\langle\bar\nabla\varrho,\partial_i\rangle-\theta_j a_i^s\theta_s\right)a^{ij}\\
&+\frac{1}{\varrho^3}g^{kl}\varrho_{ij}\left(-\theta_l\frac{\theta_k}{\varrho}\langle\bar\nabla\varrho,N\rangle+\frac{\theta_l}{\varrho W}\langle\bar\nabla\varrho,\partial_k\rangle-\theta_l a_k^s\theta_s\right)a^{ij}\\
&+\frac{1}{\varrho^3}g^{kl}\varrho_{lj}\left(-\theta_i\frac{\theta_k}{\varrho}\langle\bar\nabla\varrho,N\rangle-\frac{\theta_i}{\varrho W}\langle\bar\nabla\varrho,\partial_k\rangle+\theta_i a_i^s\theta_s\right)a^{ij}\\
&+\frac{\theta_l}{\varrho^3 W}g^{kl}\left[\bar\nabla_k\bar\nabla_i\bar\nabla_j\varrho+a_{ik}S_j+a_{jk}S_i\right]a^{ij}\\
&-\frac{\theta_i}{\varrho^3 W}g^{kl}\left[\bar\nabla_k\bar\nabla_l\bar\nabla_j\varrho+a_{kl}S_j+a_{jk}S_l\right]a^{ij}\\
&+\frac{\theta_k}{\varrho^3 W}g^{kl}\left[\bar\nabla_i\bar\nabla_j\bar\nabla_l\varrho+a_{ij}S_l+a_{il}S_j\right]a^{ij}\\
&-\frac{\theta_j}{\varrho^3 W}g^{kl}\left[\bar\nabla_i\bar\nabla_k\bar\nabla_l\varrho+a_{ik}S_l+a_{il}S_k\right]a^{ij}\\
&+\frac{1}{\varrho^3}g^{kl}\theta_j\theta_l\left(\bar\nabla_i\bar\nabla_k\bar\nabla_N\varrho+a_{ik}\langle\bar\nabla_N\bar\nabla\varrho,N\rangle\right)a^{ij}\\
&-\frac{1}{\varrho^3}g^{kl}\theta_k\theta_l\left(\bar\nabla_i\bar\nabla_j\bar\nabla_N\varrho+a_{ij}\langle\bar\nabla_N\bar\nabla\varrho,N\rangle\right)a^{ij}\\
&+\frac{1}{\varrho^3}g^{kl}\theta_i\theta_j\left(\bar\nabla_k\bar\nabla_j\bar\nabla_N\varrho+a_{kl}\langle\bar\nabla_N\bar\nabla\varrho,N\rangle\right)a^{ij}\\
&-\frac{1}{\varrho^3}g^{kl}\theta_j\theta_l\left(\bar\nabla_k\bar\nabla_i\bar\nabla_N\varrho+a_{ik}\langle\bar\nabla_N\bar\nabla\varrho,N\rangle\right)a^{ij}\\
&+g^{kl}\nabla_i\langle R^P(\partial_k^\perp,\partial_j^\perp)N^\perp,\partial_l^\perp\rangle a^{ij}+g^{kl}\nabla_k\langle R^P(\partial_l^\perp,\partial_i^\perp)N^\perp,\partial_j^\perp\rangle a^{ij}
\end{aligned}
$$

expressing (120) in terms of derivatives of ϱ, it follow that

$$\bar{R}_{kils}a_j^s = \frac{1}{\varrho^3}\theta_k\theta_l\varrho_{is}a_j^s - \frac{1}{\varrho^3}\theta_i\theta_l\varrho_{ks}a_j^s - \frac{1}{\varrho^3}\theta_k\theta_s\varrho_{il}a_j^s + \frac{1}{\varrho^3}\theta_i\theta_s\varrho_{kl}a_j^s + \langle R^P(\partial_k^\perp, \partial_i^\perp)\partial_l^\perp, \partial_s^\perp\rangle a_j^s$$

and also

$$\bar{R}_{kijs}a_l^s = \frac{1}{\varrho^3}\theta_k\theta_j\varrho_{is}a_l^s - \frac{1}{\varrho^3}\theta_i\theta_j\varrho_{ks}a_l^s - \frac{1}{\varrho^3}\theta_k\theta_s\varrho_{ij}a_l^s + \frac{1}{\varrho^3}\theta_i\theta_s\varrho_{kj}a_l^s + \langle R^P(\partial_k^\perp, \partial_i^\perp)\partial_j^\perp, \partial_s^\perp\rangle a_l^s$$

we have

$$\begin{aligned} g^{kl}\left(\bar{R}_{kil}^s a_{sj} + \bar{R}_{kij}^s a_{sl}\right) &= \frac{1}{\varrho^3}g^{kl}\theta_k\theta_l\varrho_{is}a_j^s - \frac{1}{\varrho^3}g^{kl}\theta_i\theta_l\varrho_{ks}a_j^s - \frac{1}{\varrho^3}g^{kl}\theta_k\theta_s\varrho_{il}a_j^s + \frac{1}{\varrho^3}g^{kl}\theta_i\theta_s\varrho_{kl}a_j^s \\ &+ \frac{1}{\varrho^3}g^{kl}\theta_k\theta_j\varrho_{is}a_l^s - \frac{1}{\varrho^3}g^{kl}\theta_i\theta_j\varrho_{ks}a_l^s - \frac{1}{\varrho^3}g^{kl}\theta_k\theta_s\varrho_{ij}a_l^s + \frac{1}{\varrho^3}g^{kl}\theta_i\theta_s\varrho_{kj}a_l^s \\ &+ g^{kl}\langle R^P(\partial_k^\perp, \partial_i^\perp)\partial_l^\perp, \partial_s^\perp\rangle a_j^s + g^{kl}\langle R^P(\partial_k^\perp, \partial_i^\perp)\partial_j^\perp, \partial_s^\perp\rangle a_l^s \end{aligned}$$

and finally

$$\begin{aligned} g^{kl}\left(\bar{R}_{kil}^s a_{sj} + \bar{R}_{kij}^s a_{sl}\right)a^{ij} &= \frac{1}{\varrho^3}g^{kl}\theta_k\theta_l\varrho_{is}a_j^s a^{ij} - \frac{1}{\varrho^3}g^{kl}\theta_i\theta_l\varrho_{ks}a_j^s a^{ij} - \frac{1}{\varrho^3}g^{kl}\theta_k\theta_s\varrho_{il}a_j^s a^{ij} \\ &+ \frac{1}{\varrho^3}g^{kl}\theta_i\theta_s\varrho_{kl}a_j^s a^{ij} + \frac{1}{\varrho^3}g^{kl}\theta_k\theta_j\varrho_{is}a_l^s a^{ij} - \frac{1}{\varrho^3}g^{kl}\theta_i\theta_j\varrho_{ks}a_l^s a^{ij} \\ &- \frac{1}{\varrho^3}g^{kl}\theta_k\theta_s\varrho_{ij}a_l^s a^{ij} + \frac{1}{\varrho^3}g^{kl}\theta_i\theta_s\varrho_{kj}a_l^s a^{ij} + g^{kl}\langle R^P(\partial_k^\perp, \partial_i^\perp)\partial_l^\perp, \partial_s^\perp\rangle a_j^s a^{ij} \\ &+ g^{kl}\langle R^P(\partial_k^\perp, \partial_i^\perp)\partial_j^\perp, \partial_s^\perp\rangle a_l^s a^{ij} \end{aligned}$$

and now grouping the terms

$$g^{kl}(\nabla_i L_{kjl} + \nabla_k L_{lij})a^{ij} + g^{kl}\left(\bar{R}_{kil}^s a_{sj} + \bar{R}_{kij}^s a_{sl}\right)a^{ij}$$

and estimating we obtain only expressions that appear $|A|$ and $|A|^2$ such that

$$|g^{kl}(\nabla_i L_{kjl} + \nabla_k L_{lij})a^{ij} + g^{kl}\left(\bar{R}_{kil}^s a_{sj} + \bar{R}_{kij}^s a_{sl}\right)a^{ij}| \leq C|A| + \widetilde{C}|A|^2$$

and the coefficients $C, \widetilde{C}$ depends on ϱ and terms that involve their derivatives.

6.2 Simons' formula

We have the following commutation rule

$$
\begin{aligned}
&\nabla_k(\partial_t - \Delta)b_{ij} = \partial_t \nabla_k b_{ij} - g^{rs}\nabla_k\nabla_r\nabla_s b_{ij} \\
&= \partial_t \nabla_k b_{ij} - g^{rs}\big(\nabla_r\nabla_k\nabla_s b_{ij} - R^m_{krs}\nabla_m b_{ij} - R^m_{kri}\nabla_s b_{mj} - R^m_{krj}\nabla_s b_{im}\big) \\
&= \partial_t \nabla_k b_{ij} - g^{rs}\nabla_r\big(\nabla_s\nabla_k b_{ij} - R^m_{ksi}b_{mj} - R^m_{ksj}b_{im}\big) + g^{rs}\big(R^m_{krs}\nabla_m b_{ij} + R^m_{kri}\nabla_s b_{mj} + R^m_{krj}\nabla_s b_{im}\big) \\
&= (\partial_t - \Delta)\nabla_k b_{ij} + g^{rs}\nabla_r\big(R^m_{ksi}b_{mj} + R^m_{ksj}b_{im}\big) + g^{rs}\big(R^m_{krs}\nabla_m b_{ij} + R^m_{kri}\nabla_s b_{mj} + R^m_{krj}\nabla_s b_{im}\big)
\end{aligned}
$$

Hence

$$
\begin{aligned}
(\partial_t - \Delta)\nabla_k b_{ij} &= \nabla_k(\partial_t - \Delta)b_{ij} - g^{rs}\nabla_r\big(R^m_{ksi}b_{mj} + R^m_{ksj}b_{im}\big) \\
&- g^{rs}\big(R^m_{krs}\nabla_m b_{ij} + R^m_{kri}\nabla_s b_{mj} + R^m_{krj}\nabla_s b_{im}\big)
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\frac{1}{2}(\partial_t - \Delta)|\nabla B|^2 = \frac{1}{2}g^{k\ell}(\partial_t - \Delta)\nabla_k b_{ij}\nabla_\ell b^{ij} = g^{k\ell}\nabla_k b^{ij}(\partial_t - \Delta)\nabla_\ell b_{ij} - g^{rs}g^{k\ell}\nabla_r\nabla_k b_{ij}\nabla_s\nabla_\ell b^{ij} \\
&= g^{k\ell}\nabla_\ell b^{ij}(\partial_t - \Delta)\nabla_k b_{ij} - |\nabla^2 B|^2
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\frac{1}{2}(\partial_t - \Delta)|\nabla B|^2 + |\nabla^2 B|^2 = g^{k\ell}\nabla_\ell b^{ij}\nabla_k(\partial_t - \Delta)b_{ij} - g^{k\ell}g^{rs}\nabla_\ell b^{ij}\nabla_r\big(R^m_{ksi}b_{mj} + R^m_{ksj}b_{im}\big) \\
&- g^{k\ell}g^{rs}\big(R^m_{krs}\nabla_m b_{ij}\nabla_\ell b^{ij} + R^m_{kri}\nabla_s b_{mj}\nabla_\ell b^{ij} + R^m_{krj}\nabla_s b_{im}\nabla_\ell b^{ij}\big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{1}{2}(\partial_t - \Delta)|\nabla B|^2 + |\nabla^2 B|^2 = g^{k\ell}\nabla_\ell b^{ij}\nabla_k(\partial_t - \Delta)b_{ij} - g^{k\ell}g^{rs}\nabla_\ell b^{ij}\big(b_{mj}\nabla_r R^m_{ksi} + b_{im}\nabla_r R^m_{ksj}\big) \\
&- g^{k\ell}g^{rs}\big(R^m_{ksi}\nabla_r b_{mj}\nabla_\ell b^{ij} + R^m_{ksj}\nabla_r b_{im}\nabla_\ell b^{ij}\big) \\
&- g^{k\ell}g^{rs}\big(R^m_{krs}\nabla_m b_{ij}\nabla_\ell b^{ij} + R^m_{kri}\nabla_s b_{mj}\nabla_\ell b^{ij} + R^m_{krj}\nabla_s b_{im}\nabla_\ell b^{ij}\big).
\end{aligned}
$$

Using Gauss equation gives

$$
\begin{aligned}
&\frac{1}{2}(\partial_t - \Delta)|\nabla B|^2 + |\nabla^2 B|^2 = g^{k\ell}\nabla_\ell b^{ij}\nabla_k(\partial_t - \Delta)b_{ij} - g^{k\ell}g^{rs}\nabla_\ell b^{ij}\big(b_{mj}\nabla_r R^m_{ksi} + b_{im}\nabla_r R^m_{ksj}\big) \\
&- g^{k\ell}g^{rs}\big((\bar{R}^m_{ksi} + a^m_k a_{is} - a_{ik}a^m_s)\nabla_r b_{mj}\nabla_\ell b^{ij} + (\bar{R}^m_{ksj} + a^m_k a_{js} - a_{jk}a^m_s)\nabla_r b_{im}\nabla_\ell b^{ij}\big) \\
&- g^{k\ell}g^{rs}\big((\bar{R}^m_{krs} + a^m_k a_{rs} - a_{ks}a^m_r)\nabla_m b_{ij}\nabla_\ell b^{ij} + (\bar{R}^m_{kri} + a^m_k a_{ir} - a_{ik}a^m_r)\nabla_s b_{mj}\nabla_\ell b^{ij} \\
&+ (\bar{R}^m_{krj} + a^m_k a_{jr} - a_{jk}a^m_r)\nabla_s b_{im}\nabla_\ell b^{ij}\big).
\end{aligned}
$$

Note that

$$\begin{aligned}
&\nabla_r R^m_{ksi} = \partial_r R^m_{ksi} - \Gamma^\ell_{rk} R^m_{\ell si} - \Gamma^\ell_{rs} R^m_{k\ell i} - \Gamma^\ell_{ri} R^m_{ks\ell} + \Gamma^m_{r\ell} R^\ell_{ksi} \\
&= \partial_r R^m_{ksi} - \bar\Gamma^\ell_{rk} R^m_{\ell si} - \bar\Gamma^\ell_{rs} R^m_{k\ell i} - \bar\Gamma^\ell_{ri} R^m_{ks\ell} + \bar\Gamma^m_{r\ell} R^\ell_{ksi} \\
&= \partial_r (\bar R^m_{ksi} + a^m_k a_{si} - a^m_s a_{ki}) - \bar\Gamma^\ell_{rk} (\bar R^m_{\ell si} + a^m_\ell a_{si} - a^m_s a_{\ell i}) \\
&- \bar\Gamma^\ell_{rs} (\bar R^m_{k\ell i} + a^m_k a_{\ell i} - a^m_\ell a_{ki}) - \bar\Gamma^\ell_{ri} (\bar R^m_{ks\ell} + a^m_k a_{s\ell} - a^m_s a_{k\ell}) + \bar\Gamma^m_{r\ell} (\bar R^\ell_{ksi} + a^\ell_k a_{si} - a^\ell_s a_{ki}) \\
&= \bar\nabla_r \bar R^m_{ksi} + \bar\Gamma^0_{rk} \bar R^m_{0si} + \bar\Gamma^0_{rs} \bar R^m_{k0i} + \bar\Gamma^0_{ri} \bar R^m_{ks0} - \bar\Gamma^m_{r0} \bar R^0_{ksi} + \nabla_r (a^m_k a_{si} - a^m_s a_{ki}) \\
&= \bar\nabla_r \bar R^m_{ksi} + a_{rk} \bar R^m_{0si} + a_{rs} \bar R^m_{k0i} + a_{ri} \bar R^m_{ks0} - a^m_r \bar R^0_{ksi} + \nabla_r (a^m_k a_{si} - a^m_s a_{ki})
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{2}(\partial_t - \Delta)|\nabla B|^2 + |\nabla^2 B|^2 = g^{k\ell} \nabla_\ell b^{ij} \nabla_k (\partial_t - \Delta) b_{ij} \\
&- g^{k\ell} g^{rs} b_{mj} \nabla_\ell b^{ij} \big(\bar\nabla_r \bar R^m_{ksi} + a_{rk} \bar R^m_{0si} + a_{rs} \bar R^m_{k0i} + a_{ri} \bar R^m_{ks0} - a^m_r \bar R^0_{ksi} + \nabla_r (a^m_k a_{si} - a^m_s a_{ki}) \big) \\
&- g^{k\ell} g^{rs} b_{im} \nabla_\ell b^{ij} \big(\bar\nabla_r \bar R^m_{ksj} + a_{rk} \bar R^m_{0sj} + a_{rs} \bar R^m_{k0j} + a_{rj} \bar R^m_{ks0} - a^m_r \bar R^0_{ksj} + \nabla_r (a^m_k a_{sj} - a^m_s a_{kj}) \big) \\
&- g^{k\ell} g^{rs} \big((\bar R^m_{ksi} + a^m_k a_{is} - a_{ik} a^m_s) \nabla_r b_{mj} \nabla_\ell b^{ij} + (\bar R^m_{ksj} + a^m_k a_{js} - a_{jk} a^m_s) \nabla_r b_{im} \nabla_\ell b^{ij} \big) \\
&- g^{k\ell} g^{rs} \big((\bar R^m_{krs} + a^m_k a_{rs} - a_{ks} a^m_r) \nabla_m b_{ij} \nabla_\ell b^{ij} + (\bar R^m_{kri} + a^m_k a_{ir} - a_{ik} a^m_r) \nabla_s b_{mj} \nabla_\ell b^{ij} \\
&+ (\bar R^m_{krj} + a^m_k a_{jr} - a_{jk} a^m_r) \nabla_s b_{im} \nabla_\ell b^{ij} \big).
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{2}(\partial_t - \Delta)|\nabla B|^2 + |\nabla^2 B|^2 \le \langle \nabla B, \nabla(\partial_t - \Delta) B \rangle \\
&+ \big(C_1 + C_0 |A| + |A||\nabla A| \big) |B||\nabla B| + \big(C_0 + |A|^2 \big) |\nabla B|^2
\end{aligned}$$

where C_0 depends on $\bar\nabla \bar R$ and C_1 depends on $\bar R$. In particular, if $B = A$ we have

$$\begin{aligned}
&g^{k\ell} \nabla_\ell b^{ij} \nabla_k (\partial_t - \Delta) b_{ij} = g^{k\ell} \nabla_\ell a^{ij} \nabla_k \big(-(2H - \sigma) n a_{is} a^s_j + n(H - \sigma) \bar R_{i00j} \\
&+ a_{ij} |A|^2 + g^{k\ell} (\bar\nabla_i \bar R_{kj0\ell} + \bar\nabla_k \bar R_{\ell i0j} + a_{ik} \bar R_{0j0\ell} + a_{ij} \bar R_{k00\ell} + a_{is} \bar R^s_{kj\ell} + a_{k\ell} \bar R_{0i0j} + a_{ki} \bar R_{\ell 00j} + a_{ks} \bar R^s_{\ell ij}) \\
&- g^{k\ell} (\bar R^s_{ik\ell} a_{sl} + \bar R^s_{ikj} a_{\ell s}) \big) \le |\nabla A| \big(|A|^2 |\nabla A| + C_0 |A| + C_0 |\nabla A| + C_1 |A| + C_1 |A|^2 + C_2 + C_2 |A| \big),
\end{aligned}$$

where we have used expressions of the following form

$$\begin{aligned}
&\nabla_k \bar R_{i00j} = \partial_k \bar R_{i00j} - \Gamma^m_{ki} \bar R_{m00j} - \Gamma^m_{kj} \bar R_{i00m} = \partial_k \bar R_{i00j} - \bar\Gamma^m_{ki} \bar R_{m00j} - \bar\Gamma^m_{kj} \bar R_{i00m} \\
&= \bar\nabla_k \bar R_{i00j} + \bar\Gamma^0_{ki} \bar R_{000j} + \bar\Gamma^0_{kj} \bar R_{i000} + \bar\Gamma^m_{k0} \bar R_{im0j} + \bar\Gamma^m_{k0} \bar R_{i0mj} \\
&= \bar\nabla_k \bar R_{i00j} + a_{ki} \bar R_{000j} + a_{kj} \bar R_{i000} + a^m_k \bar R_{im0j} + a^m_k \bar R_{i0mj}.
\end{aligned}$$

Note that some terms are missing in the expression above due to the fact that $\bar\Gamma^0_{k0} = 0$.

In sum, when $B = A$ we have

$$\frac{1}{2}(\partial_t - \Delta)|\nabla A|^2 + |\nabla^2 A|^2 \leq \big(C_2 + C_1 + C_0 + C_0|A| + |A||\nabla A|\big)|A||\nabla A| + C_0|\nabla A|^2$$
$$+ (C_1|A| + C_1|A|^2 + C_2)|\nabla A|.$$

In other terms,

$$\frac{1}{2}(\partial_t - \Delta)|\nabla A|^2 + |\nabla^2 A|^2 \leq |A|^2|\nabla A|^2 + C_0|\nabla A|^2 + \big(C_2 + |A|(C_1 + C_0)\big)\big(1 + |A|\big)|\nabla A|.$$

We conclude that

$$\frac{1}{2}(\partial_t - \Delta)|\nabla A|^2 + |\nabla^2 A|^2 \leq \widetilde{C}_1(|\nabla A|^2 + |\nabla A| + 1). \tag{121}$$

where $\widetilde{C}_1$ is a non-negative constant depending on $|A|$ and on $\bar{R}$, $\bar{\nabla}\bar{R}$ and $\bar{\nabla}\bar{\nabla}\bar{R}$. Our induction hypothesis is that

$$\frac{1}{2}(\partial_t - \Delta)|\nabla^\ell A|^2 + |\nabla^{\ell+1} A|^2 \leq \widetilde{C}_\ell(|\nabla^\ell A|^2 + |\nabla^\ell A| + 1). \tag{122}$$

where $\widetilde{C}_\ell$ is a non-negative constant depending on $|\nabla^k A|$ and on $\bar{\nabla}^k\bar{R}$ for all $k = 0, \ldots, \ell - 1$. Note that the trivial inequality

$$|\nabla^\ell A| \leq \frac{1}{2}|\nabla^\ell A|^2 + \frac{1}{2}$$

allows us to write

$$\frac{1}{2}(\partial_t - \Delta)|\nabla^\ell A|^2 + |\nabla^{\ell+1} A|^2 \leq \widetilde{C}_\ell(|\nabla^\ell A|^2 + 1). \tag{123}$$

7 CONCLUSION

The results we get give us geometric gains such as dealing with spaces that have curvature not necessarily compared to constant. We add a force term in the flow equation such that if it were possible to speak of convergence, our case contemplates the convergence for a constant mean curvature surface σ while Borisenko and Miquel (2012) would obtain a minimal surface. Moreover, they deals with spaces of curvature compared with a constant, in this case, hyperbolic. The existence of solution is obtained with estimates on simpler sets.

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