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A CAUCHY-CROFTON FORMULA AND MONOTONICITY INEQUALITIES FOR THE BARBOSA-COLARES FUNCTIONALS *

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To Prof. J. L. Barbosa, on the accasion of his sixtieth birthday.

Abstract. We prove a Cauchy-Crofton type formula for a class of geometric functionals, here denoted by $\mathcal{A}_r$, $r = 0, 1, \ldots, n-1$, first considered by L. Barbosa and G. Colares ([BC]) and defined over the space of closed hypersurfaces in a complete simply connected n-dimensional space form. Besides giving an integral-geometric interpretation to these functionals, this formula allows us to prove a monotonicity inequality for the functionals, namely, if $\mathbf{M}_1$ and $\mathbf{M}_2$ are embedded hypersurfaces enclosing convex regions K_1 and K_2, respectively, with $K_1 \subset K_2$, then $\mathcal{A}_r(\mathbf{M}_1) \leq \mathcal{A}_r(\mathbf{M}_2)$ with equality holding if and only if $K_1 = K_2$ (and consequently $\mathbf{M}_1 = \mathbf{M}_2$).

1. Introduction. In this note we consider closed (i.e., compact oriented without boundary) hypersurfaces $\mathbf{M}^{n-1} \subset \mathbb{N}^n(c)$, where $\mathbb{N}^n(c)$ is the complete simply connected space form of constant sectional curvature c, i.e., $\mathbb{N}^n(c)$ is the Euclidean space, the sphere or the hyperbolic space, if $c = 0$, $c > 0$ or $c < 0$, respectively. In [BC], a class of geometric functionals has been defined over the space of such immersions generalizing the classical curvature integrals for hypersurfaces in Euclidean space. More precisely, define

$$\mathcal{A}_r(\mathbf{M}) = \int_{\mathbf{M}} F_r(S_0, S_1, \ldots, S_r)\, d\mathbf{M} \tag{1}$$

where $d\mathbf{M}$ is the area element of $\mathbf{M}$, $S_0, S_1, \ldots, S_{n-1}$ are the elementary symmetric functions of the principal curvatures of $\mathbf{M}$ and the F_r's are defined recursively by

$$\begin{cases} F_0 = 1, \\ F_1 = S_1, \\ F_r = S_r + \frac{c(n-r)}{r-1} F_{r-2}, \quad 2 \leq r \leq n-1. \end{cases} \tag{2}$$

Notice that $\mathcal{A}_0(\mathbf{M})$ is the area of $\mathbf{M}$ and if we are in the Euclidean case, i.e., if $c = 0$, then the functionals above reduce to the classical *curvature integrals* of an immersion $\mathbf{M}^{n-1} \subset \mathbb{R}^n$, namely,

$$\mathcal{A}_r(\mathbf{M}) = \int_{\mathbf{M}} S_r\, d\mathbf{M}. \tag{3}$$

We shall call (1) the *Barbosa-Colares functionals*. The relevance of (1) comes from the fact proved in [BC] that the Euler-Lagrange equation for $\mathcal{A}_r$, under volume preserving variations, is

$$S_{r+1} = \text{const.}, \quad r + 1 \leq n - 1. \tag{4}$$

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Moreover, it is also shown in [BC] that the *stable* critical points for the above volume constrained variational problem defined by $\mathcal{A}_r$ are precisely the geodesic spheres. In other words, the only closed hypersurfaces in $\mathbb{N}^n(c)$ satisfying (4) for a given r and being a local minimum (or maximum) for $\mathcal{A}_r$ under volume preserving variations are the geodesic spheres.

The purpose of this paper is to prove a theorem which provides an integral-geometric interpretation for the Barbosa-Colares functionals, namely, we show that a Cauchy-Crofton type formula holds for these functionals (see Theorem 3.1 below). A few interesting consequences emerge from this result. First of all, restricted to the space $\partial\mathcal{C}^\infty_{n,c}$ of *convex smooth embeddings* $\mathbf{M}^{n-1}\subset\mathbb{N}^n(c)$, these functionals are strictly positive. Moreover, they satisfy the following sharp monotonicity inequalities.

THEOREM 1.1. *Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be embedded hypersurfaces enclosing convex regions K_1 and K_2, respectively, with $K_1\subset K_2$. Then*

$$\mathcal{A}_r(\mathbf{M}_1)\leq\mathcal{A}_r(\mathbf{M}_2), \tag{5}$$

with equality holding if and only if $K_1=K_2$ (and consequently $\mathbf{M}_1=\mathbf{M}_2$).

REMARK 1.1. *In the spherical case with $c>0$, one should assume further that $\mathbf{M}$ is contained in an open hemisphere, since this is implied by the convexity assumption (see Definition 2.1 below).*

We stress that the positivity of $\mathcal{A}_r$ over $\partial\mathcal{C}^\infty_{n,c}$, which is an imediate consequence of our Cauchy-Crofton formula (see Remark 3.1) is rather surprising since, when evaluated over an arbitrary immersion, or even embedding, these functionals can assume *negative* values. Moreover, in the language of Integral Geometry (see [KR], for example) we deduce from Theorem 1.1 that the Barbosa-Colares functionals are invariant monotone continuous valuations over $\partial\mathcal{C}^\infty_{n,c}$ (see Remark 4.1 for more on this point). In fact, ideas from Integral Geometry play a crucial role in the proof of the above theorem, since a crucial ingredient is a non-euclidean version of the well-known formula relating the measure of planes of a given dimension cutting convex subset $K\subset\mathbb{R}^n$ and the arithmethic mean of the measure of the orthogonal projections of K over planes with the complementary dimension (see (15), (21) and (22) below).

This paper is organized as follows. In Section 2, we recall the basic facts on the Integral Geometry of convex hypersurfaces in space forms which will be used in this work. Emphasis here is given on explicit expressions for the densities dL_r and on describing the quermassintegrals as a suitable arithmetic mean of projected volumes. In Section 3 we prove the Cauchy-Crofton formulas and finally in Section 4 we furnish the proof of Theorem 1.1.

2. Integral geometry of hypersurfaces in space forms. In this section, we collect some basic facts on the integral geometry of convex hypersurfaces in simply connected space forms which will be useful later. Our presentation follows [S] closely and the reader is referred to this marvelous book for further details.

Recall that the isometry group $\mathbf{G}_c$ of $\mathbb{N}^n(c)$ acts transitively on the space $\mathcal{I}_{c,r}$ of totally geodesic submanifolds of $\mathbb{N}^n(c)$ of a given dimension r. An element of $\mathcal{I}_{c,r}$ is called a *r-plane* and is denoted by L_r. If $\mathbf{H}_c$ is the isotropy group of a given L_r with respect to the above action, it follows that $\mathcal{I}_{c,r}$ is naturally identified to the homogeneous space $\mathbf{G}_c/\mathbf{H}_c$. This allows us to define on $\mathcal{I}_{c,r}$ a canonical left invariant measure, or *density*, denoted by dL_r, that is naturally induced from the Haar measure in $\mathbf{G}_c$. The existence of this density permits us to consider the measure of r-planes intersecting a fixed subset of $\mathbf{N}^n(c)$, a procedure which will be crucial in what follows. A variant of the above construction yields the *Grassmanian* $G_{n,r}$, which is obtained by considering the set of all elements of $\mathcal{I}_{c,r}$ passing through a given point of $\mathbb{N}^n(c)$, which is then named the *origin* of $\mathbb{N}^n(c)$. The corresponding density is denoted by $dL_{r[0]}$.

As mentioned in the Introduction, a first step toward the proof of Theorem 1.1 is the identification of the Barbosa-Colares functionals to certain quermassintegrals via a Cauchy-Crofton type formula. For this, we need some basic facts on the geometry of convex bodies in space forms.

DEFINITION 2.1. *A convex body K in $\mathbb{N}^n(c)$ is a compact set with nonempty interior and having the property that, given points p and q in K, the unique minimizing geodesic segment in $\mathbf{N}^n(c)$ joining p to q lies entirely in K.*

REMARK 2.1. *Notice that, in the spherical case when $c > 0$, the definition above implies that K is contained in an open hemisphere.*

We denote by $\mathcal{C}_{n,c}$ the set of all convex bodies in $\mathbb{N}^n(c)$. In general, an element in $\mathcal{C}_{n,c}$ has the property that its boundary $\mathbf{M} = \partial K$ is smooth almost everywhere and we reserve the notation $\mathcal{C}^{\infty}_{n,c}$ for the subset of $\mathcal{C}_{n,c}$ formed by convex bodies with *smooth* boundaries. Also, it is convenient to denote by $\partial\mathcal{C}_{n,c}$ the set of boundaries of elements in $\mathcal{C}_{n,c}$ and, similarly, by $\partial\mathcal{C}^{\infty}_{n,c}$ the set of smooth hypersurfaces bounding elements of $\mathcal{C}^{\infty}_{n,c}$. In this latter class, the Barbosa-Colares functionals (1) are well-defined. We shall see, however, that as a consequence of the Cauchy-Crofton formula presented below, these functionals can be naturally extended to $\partial\mathcal{C}_{n,c}$ (see Remark 3.1). We also notice that a convex body $K \in \mathcal{C}_{n,c}$ is always measurable. In particular, the measure of the elements of $\mathcal{I}_{c,r}$ intersecting a given $K \in \mathcal{C}^{\infty}_{n,c}$, namely,

$$\mathcal{Q}_r(K) = \int_{L_r \cap K \neq \phi} dL_r, \tag{6}$$

is well defined and will be referred to as a *quermassintegral* of K. It turns out that it can be explicitly computed in terms of the geometry of $\mathbf{M} = \partial K$, according to the parity of r. For $c \neq 0$, these are given by the expressions

$$b(n,r)\left[c^{r'} O_{n-1} V(K) + \sum_{i=1}^{r'} \tilde{b}(n,r,i) \mathcal{M}_{2i-1}(\partial K)\right], \quad r = 2r', \tag{7}$$

$$d(n,r) \sum_{i=1}^{r'} \tilde{d}(n,r,i) \mathcal{M}_{2i}(\partial K), \quad r = 2r' + 1. \tag{8}$$

Here,

$$\mathcal{M}_r(\partial K) = \begin{pmatrix} n-1 \\ r \end{pmatrix}^{-1} \int_{\mathbf{M}} S_r \, d\mathbf{M}, \tag{9}$$

O_i is the area of the unit Euclidean sphere of dimension i and the universal constants appearing in (7) and (8) are given by

$$b(n,r)=\frac{O_{n-2}\dots O_{n-r}}{O_r\dots O_1}, \tag{10}$$

$$\tilde{b}(n,r,i)=\left(\begin{array}{c} r-1 \\ 2i-1 \end{array}\right)\frac{O_rO_{r-1}O_{n-2i+1}}{O_{2i-1}O_{r-2i}O_{r-2i+1}}c^{r'-i}, \tag{11}$$

$$d(n,r)=\frac{O_{n-2}\dots O_{n-r}}{O_{r-1}\dots O_1}, \tag{12}$$

$$\tilde{d}(n,r,i)=\left(\begin{array}{c} r-1 \\ 2i \end{array}\right)\frac{O_{r-1}O_{n-2i}}{O_{2i}O_{r-2i-1}O_{r-2i}}c^{r'-i}. \tag{13}$$

In case $c=0$, we have

$$\mathcal{Q}_r(K)=a(n,r)\mathcal{M}_{r-1}(\partial K), \tag{14}$$

where

$$a(n,r)=\frac{nO_{n-2}\dots O_{n-r-1}}{(n-r)O_{r-1}\dots O_1O_0}.$$

By convention, $O_0=2$. These formulas follow from the generalized Gauss-Bonnet theorem and the explicit formulas for the densities dL_r. The details can be found in [S], page 310.

It is well-known that in the Euclidean case ($c=0$), the quermassintegrals $\mathcal{Q}_r$ admit a dual description as an arithmetic mean of the $(n-r)$-dimensional volumes of the orthogonal projections of K over $(n-r)$-planes $L_{n-r}\in G_{n,n-r}$. More precisely, one has up to a universal multiplicative constant

$$\mathcal{Q}_r(K)=\int_{G_{n,n-r}}\mathrm{vol}_{n-r}(\pi_{L_{n-r[0]}}(K))\,dL_{n-r[0]}. \tag{15}$$

Here, vol_{n-r} denotes the $(n-r)$-dimensional Hausdorff volume and $\pi_{L_{n-r[0]}}:\mathbb{R}^n\to\mathbb{R}^n$ denotes the orthogonal projection over $L_{n-r[0]}\in G_{n,n-r}$. The rationale behind (15) is as follows. First, we have the disjoint union

$$\{L_r; L_r\cap K\neq\emptyset\}=\bigcup_{L_{n-r[0]}\in G_{n,n-r}}\mathcal{H}(L_{n-r[0]}), \tag{16}$$

where

$$\mathcal{H}(L_{n-r[0]})=\{L_r; L_r\cap K\neq\emptyset, L_r \text{ is perpendicular to } L_{n-r[0]}\}. \tag{17}$$

On the other hand, the density dL_r can be expressed (see [S], page 204) as

$$dL_r=d\sigma_{n-r}\wedge dL_{n-r[0]}, \tag{18}$$

where $d\sigma_{n-r}$ is the volume element of $L_{n-r[0]}$. By integrating this over $\{L_r; L_r\cap K\neq\emptyset\}$ and using (6) and (16), (15) follows straightforwardly.

Now, a crucial ingredient in the proof of Theorem 1.1 is to realize that a formula similar to (15) holds in the non-euclidean setting ($c\neq 0$). In effect, the decomposition (16) stills holds in this case. As for (18), the non-euclidean analogues are (see [S], page 306)

$$dL_r=\cos^r(\sqrt{c}\rho)d\sigma_{n-r}\wedge dL_{n-r[0]},\ c>0, \tag{19}$$

and

$$dL_r = \cosh^r(\sqrt{-c}\rho)d\sigma_{n-r} \wedge dL_{n-r[0]}, \;\; c < 0, \tag{20}$$

where in both cases ρ denotes the geodesic distance of a generic point in $L_{n-r[0]}$ to the origin of $\mathbf{N}^n(c)$. Proceeding exactly as above, one gets

$$\mathcal{Q}_r(K) = \int_{G_{n,n-r}} \left(\int_{\pi_{L_{n-r[0]}}(K)} \cos^r(\sqrt{c}\rho)\, d\sigma_{n-r} \right) dL_{n-r[0]}, \;\; c > 0, \tag{21}$$

and

$$\mathcal{Q}_r(K) = \int_{G_{n,n-r}} \left(\int_{\pi_{L_{n-r[0]}}(K)} \cosh^r(\sqrt{-c}\rho)\, d\sigma_{n-r} \right) dL_{n-r[0]}, \;\; c < 0, \tag{22}$$

where, as before, $\pi_{L_{n-r[0]}} : \mathbb{N}^n(c) \to \mathbb{N}^n(c)$ is the orthogonal projection over $L_{n-r[0]} \in G_{n,n-r}$ along geodesics. Notice that here $\pi_{L_{n-r[0]}}$ *does not* preserve lengths and the correction term involving ρ is certainly expected.

REMARK 2.2. *As it is always the case, one should be careful here in case* $c > 0$. *In effect, in this case, one should restrict to convex bodies inside a given open hemisphere centered at the origin, otherwise the orthogonal projections above are not well defined. Moreover, since* $0 \leq \rho < \pi/2$, *the integrand of the innermost integral in (21) is strictly positive.*

3. The Cauchy-Crofton formula for the Barbosa-Colares functionals. The purpose of this section is to show that, restricted to $\partial\mathcal{C}^{\infty}_{n,c}$, the Barbosa-Colares functional $\mathcal{A}_{r-1}$ coincides with the quermassintegral $\mathcal{Q}_r$ up to a universal constant depending only on n, r, c. From now on, we shall restrict ourselves to the case $c \neq 0$ and, in order to make computations simpler, we introduce the *modified quermassintegrals* for $K \in \mathcal{C}^{\infty}_{n,c}$ by

$$\mathcal{W}_r(K) = b(n,r)^{-1}\mathcal{Q}_r(K), \;\; r = 2r', \tag{23}$$
$$\mathcal{W}_r(K) = d(n,r)^{-1}\mathcal{Q}_r(K), \;\; r = 2r' + 1. \tag{24}$$

Then we have

LEMMA 3.1. *For* $K \in \mathcal{C}^{\infty}_{n,c}$,

$$\mathcal{W}_r(K) = \frac{O_r O_{n-r+1}}{O_0 O_1}\mathcal{M}_{r-1}(\partial K) + c\mathcal{W}_{r-2}(K), \;\; r = 2r', \tag{25}$$
$$\mathcal{W}_r(K) = \frac{O_{n-r+1}}{O_0 O_1}\mathcal{M}_{r-1}(\partial K) + c\frac{r-1}{2\pi}\mathcal{W}_{r-2}(K), \;\; r = 2r' + 1. \tag{26}$$

Proof. Let us first consider the case $r = 2r'$. By (7) and (23), we have

$$\mathcal{W}_r(K) = c^{r'} O_{n-1} V(K) + \sum_{i=1}^{r'} \tilde{b}(n,r,i)\mathcal{M}_{2i-1}(\partial K) \tag{27}$$

and then

$$\mathcal{W}_{r-2}(K) = c^{r'-1} O_{n-1} V(K) + \sum_{i=1}^{r'-1} \tilde{b}(n,r-2,i)\mathcal{M}_{2i-1}(\partial K). \tag{28}$$

On the other hand, we can rewrite (27) as

$$\mathcal{W}_r(K) = c^{r'} O_{n-1} V(K) + \tilde{b}(n,r,r')\mathcal{M}_{r-1}(\partial K) + \sum_{i=1}^{r'-1} \tilde{b}(n,r,i)\mathcal{M}_{2i-1}(\partial K)\,. \quad (29)$$

Now, a straightforward computation using (11) and the fact that

$$\frac{O_i}{O_{i-2}} = \frac{2\pi}{i-1} \quad (30)$$

gives us

$$\frac{\tilde{b}(n,r,i)}{\tilde{b}(n,r-2,i)} = c$$

and

$$\tilde{b}(n,r,r') = \frac{O_r O_{n-r-1}}{O_0 O_1}.$$

Comparing (28) and (29) we obtain (25), proving the lemma in this case.

The case $r = 2r' + 1$ is treated similarly. From (8) and (24) we have

$$\mathcal{W}_r(K) = \sum_{i=1}^{r'} \tilde{d}(n,r,i)\mathcal{M}_{2i}(\partial K)\,, \quad (31)$$

and then

$$\mathcal{W}_{r-2}(K) = \sum_{i=1}^{r'} \tilde{d}(n,r-2,i)\mathcal{M}_{2i}(\partial K)\,. \quad (32)$$

We rewrite (31) as

$$\mathcal{W}_r(K) = \tilde{d}(n,r,r')\mathcal{M}_{r-1}(\partial K) + \sum_{i=1}^{r'-1} \tilde{d}(n,r,i)\mathcal{M}_{2i}(\partial K)\,. \quad (33)$$

Again we compute

$$\frac{\tilde{d}(n,r,i)}{\tilde{d}(n,r-2,i)} = \frac{c(r-1)}{2\pi}$$

and

$$\tilde{d}(n,r,i) = \frac{O_{n-r+1}}{O_0 O_1}.$$

Comparing (32) and (33), (26) follows easily. This completes the proof of the lemma.

This lemma, together with (1) and (2), show that the modified quermassintegral $\mathcal{W}_r$ resembles $\mathcal{A}_{r-1}$. In fact, according to the theorem below, they actually coincide up to universal constants. This is the promised *Cauchy-Crofton formula* for the Barbosa-Colares functionals.

THEOREM 3.1. *Let $K \in \mathcal{C}^{\infty}_{n,c}$. Then,*

$$\mathcal{W}_r(K) = \frac{O_r O_{n-r+1}}{O_0 O_1} \begin{pmatrix} n-1 \\ r-1 \end{pmatrix}^{-1} \mathcal{A}_{r-1}(\partial K), \quad r = 2r', \tag{34}$$

$$\mathcal{W}_r(K) = \frac{O_{n-r+1}}{O_0 O_1} \begin{pmatrix} n-1 \\ r-1 \end{pmatrix}^{-1} \mathcal{A}_{r-1}(\partial K), \quad r = 2r'+1. \tag{35}$$

Proof. We use induction on the parity of r and start with the even case $r = 2r'$. From (7) and (23) we have $W_0(K) = V(K)$, and this shows the theorem for $r = 0$. Suppose now by induction that

$$\mathcal{W}_{r-2}(K) = \frac{O_{n-r+3} O_{r-2}}{O_0 O_1 \binom{n-1}{r-3}} \mathcal{A}_{r-3}(\partial K), \quad r \geq 3.$$

By Lemma 3.1 and the induction assumption we find that

$$\begin{aligned} \mathcal{W}_r(K) &= \frac{O_r O_{n-r+1}}{O_0 O_1 \binom{n-1}{r-1}} \int_{\mathbf{M}} S_{r-1} d\mathbf{M} + c\mathcal{W}_{r-2}(K) \\ &= \frac{O_r O_{n-r+1}}{O_0 O_1 \binom{n-1}{r-1}} \left(\int_M S_{r-1} d\mathbf{M} + e(n,r)\mathcal{A}_{r-3}(K) \right), \end{aligned}$$

where

$$e(n,r) = \frac{\binom{n-1}{r-1}}{\binom{n-1}{r-3}} \cdot \frac{O_{r-2} O_{n-r+3}}{O_r O_{n-r+1}} \cdot c = \frac{c(n-r+1)}{r-2},$$

so that finally we get

$$\mathcal{W}_r(K) = \frac{O_r O_{n-r+1}}{O_0 O_1} \binom{n-1}{r-1}^{-1} \mathcal{A}_{r-1}(\partial K), .$$

As for the odd case, we see that if $r = 1$, (35) reduces to

$$\int_{L_1 \cap K \neq \emptyset} dL_1 = \frac{O_n}{4\pi} \mathcal{A}_0(\partial K),$$

which is the well-known non-euclidean analogue of the classical *Cauchy formula* ([S], page 310). Now assume by induction that

$$\mathcal{W}_{r-2}(K) = \frac{O_{n-r+3}}{O_0 O_1} \begin{pmatrix} n-1 \\ r-3 \end{pmatrix}^{-1} \mathcal{A}_{r-3}(\partial K), \ r \geq 3. \tag{36}$$

Then, by Lemma 3.1,

$$\begin{aligned} \mathcal{W}_r(K) &= \frac{O_{n-r-1}}{O_0 O_1 \begin{pmatrix} n-1 \\ r-1 \end{pmatrix}} \int_{\partial K} S_{r-1} \, d\mathbf{M} + \frac{c(r-1) O_{n-r+3}}{2\pi O_0 O_1 \begin{pmatrix} n-1 \\ r-3 \end{pmatrix}} \mathcal{A}_{r-3}(\partial K) \\ &= \frac{O_{n-r-1}}{O_0 O_1 \begin{pmatrix} n-1 \\ r-1 \end{pmatrix}} \left(\int_{\partial K} S_{r-1} \, d\mathbf{M} + e(n,r) \mathcal{A}_{r-3}(\partial K) \right), \end{aligned} \tag{37}$$

where, by (30),

$$e(n,r)=\frac{c(r-1)}{2\pi}\frac{O_{n-r+3}}{O_{n-r+1}}\frac{\begin{pmatrix} n-1 \\ r-1 \end{pmatrix}}{\begin{pmatrix} n-1 \\ r-3 \end{pmatrix}}=\frac{c(n-r+1)}{r-2}.$$

Now, (35) follows easily from (37), (1) and (2), finishing the proof.

Remark 3.1. *Notice that, as an immediate consequence of this theorem, we see that $\mathcal{A}_r$ is strictly positive when restricted to $\partial\mathcal{C}^\infty_{n,c}$. Moreover, the Cauchy-Crofton formulas above allow us to extend $\mathcal{A}_r$ to $\partial\mathcal{C}_{n,c}$ in a natural way so that this positivity property is preserved.*

Remark 3.2. *The same argument as above yields more general formulas than in Theorem 3.1. More precisely, it is possible to show that, when evaluated over a closed embedded hypersurface $\mathbf{M}$, $\mathcal{A}_{r-1}(\mathbf{M})$ equals, up to a universal constant, the integral*

$$\int_{L_r\cap K\neq\emptyset}\chi(L_r\cap K)\,dL_r,$$

where χ denotes the Euler characteristic and K is the (not necessarily convex) region bounded by $\mathbf{M}$. Clearly, if $\mathbf{M}\in\partial\mathcal{C}^\infty_{n,c}$ then $K\in\mathcal{C}^\infty_{n,c}$ and from this we get that $L_r\cap K$ is convex in L_r for any L_r intersecting K. Hence, $\chi(L_r\cap K)=1$ for any such L_r and we recover Theorem 3.1.

4. The proof of Theorem 1.1 and concluding remarks. The proof of Theorem 1.1 above can now be easily carried out. We consider only the spherical case (taking Remark 2.2 into account), the hyperbolic case being similar. Let be given hypersurfaces $\mathbf{M}_1$ and $\mathbf{M}_2$ enclosing convex bodies K_1 and K_2, respectively, with $K_1\subset K_2$. By the Cauchy-Crofton formula (Theorem 3.1), one has to prove that $\mathcal{Q}_r(K_1)\leq\mathcal{Q}_r(K_2)$ with equality holding if and only if $K_1=K_2$. It follows from the assumptions that, for each $L_{n-r[0]}\in G_{n,n-r}$, there holds $\pi_{L_{n-r[0]}}(K_1)\subset\pi_{L_{n-r[0]}}(K_2)$, and hence

$$\int_{\pi_{L_{n-r[0]}}(K_1)}\cos^r(\sqrt{c}\rho)\,d\sigma_{n-r}\leq\int_{\pi_{L_{n-r[0]}}(K_2)}\cos^r(\sqrt{c}\rho)\,d\sigma_{n-r} \tag{38}$$

Integrating this over $G_{n,n-r}$ and recalling (21), we conclude $\mathcal{Q}_r(K_1)\leq\mathcal{Q}_r(K_2)$, as desired. On the other hand, if equality holds, one has equality in (38) for any $L_{n-r[0]}\in G_{n,n-r}$. Since, as remarked before, $\pi_{L_{n-r[0]}}(K_1)\subset\pi_{L_{n-r[0]}}(K_2)$ and $\cos^r(\sqrt{c}\rho)$ is always positive, we have

$$\mathrm{vol}_{n-r}\left(\pi_{L_{n-r[0]}}(K_1)\right)=\mathrm{vol}_{n-r}\left(\pi_{L_{n-r[0]}}(K_2)\right),$$

from which we find that

$$\pi_{L_{n-r[0]}}(K_1)=\pi_{L_{n-r[0]}}(K_2). \tag{39}$$

We see then that K_1 and K_2 are such that (39) holds for any $L_{n-r[0]}\in G_{n,n-r}$ and this clearly implies that $K_1=K_2$.

Remark 4.1. *According to [KR], the fundational result in Euclidean Integral Geometry is Hadwiger's characterization theorem. This states that any continuous* *and invariant valuation on* $\mathcal{C}_{n,c}$ *for* $c = 0$ *is a linear combination of* $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ *(here, we define* $\mathcal{A}_n(K) = V(K)$*, the volume of* K*) and the Euler characteristic* χ*. It is very likely that a similar result holds for the remaining simply connected spaces forms. More precisely, it seems natural to investigate whether the Barbosa-Colares functionals, together with the volume and the Euler characteristic, constitute a basis for the space of such valuations over* $\mathcal{C}_{n,c}$*.*

REMARK 4.2. *There is still another interesting consequence of the Cauchy-Crofton formula. Given* $V > 0$*, let* $\partial\mathcal{C}^{\infty}_{n,c,V}$ *be the space of all smooth convex hypersurfaces in* $\mathbb{N}^n(c)$ *enclosing a convex body with volume* V *(naturally, in the spherical case, one should assume that* V *is less the volume of a hemisphere). By using standard arguments based on the direct method of the calculus of variations and Blaschke selection theorem (see [BZ], for example), one can show the existence of a convex body* K_0 *satisfying* $\mathcal{A}_r(K) \geq \mathcal{A}_r(K_0)$ *for any* $K \in \mathcal{C}^{\infty}_{n,c,V}$*. In other words, the variational problem of minimizing* $\mathcal{A}_r$ *over* $\partial\mathcal{C}^{\infty}_{n,c,V}$ *has a solution, namaly,* K_0*. In a companion paper [LL], we prove that* K_0 *is actually the geodesic sphere in* $\mathbb{N}^n(c)$ *with volume* V*. This is not only in conformity with the stability result in [BC] but also yields sharp isoperimetric inequalities for the Barbosa-Colares functionals which generalize the well-known isoperimetric inequalities for the classical curvature integrals (3) (see [BZ]).*

REFERENCES

[BC] BARBOSA, J. L. & COLARES, A. G., *Stability of Hypersurfaces with Contant r-Mean Curvature*, Ann. Global Anal. and Geo., 15(1997), pp. 177–197.

[BZ] BURAGO, Y. & ZALGALLER, V., *Geometric Inequalities*, Gundlehren der Mathematische Wissenschaften, Springer Verlag, Heidelberg, 1980.

[KR] KLAIN, D. & ROTA, G-C., *Introduction to Geometric Probability*, Lezioni Lincee, Cambridge University Press, New York, 1998.

[LL] LÁZARO, I. C. & DE LIMA, L. L., *Sharp isoperimetric inequalities for the Barbosa-Colares functionals*, in preparation.

[S] SANTALÒ, L., *Integral Geometry and Geometric Probability*, Encyclopedia of Mathematics and its Applications, Addison-Wesley, Reading, Massachussets, 1976