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Lower bounds for index of Wente tori

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Dedicated to Katsuhiro Shiohama on the occasion of his sixtieth birthday.

Abstract

We show numerically that any of the constant mean curvature tori first found by Wente must have index at least eight.[1 2]

1 Introduction

The Hopf conjecture asked if all closed surfaces immersed in $\mathbb{R}^3$ with constant mean curvature H must be round spheres. It was proven true when either the surface has genus zero by Hopf himself [H], or the immersion is actually an embedding by Alexandrov [H]. However, it does not hold in general, and the first counterexamples, of genus 1, were found by Wente [We]. Abresch [A] and Walter [Wa] made more explicit descriptions for these surfaces of Wente, which all have one family of planar curvature lines [Sp]. We call these surfaces the *original Wente tori*.

Constant mean curvature surfaces are critical for area, but not necessarily area minimizing, for all compactly supported volume-preserving variations. Hence the index – loosely speaking, the *dimension* of area-reducing volume-preserving variations, to be defined in Section 3 – can be positive. If it is zero, the surface is stable. Do Carmo and Peng [CP] showed that the only complete stable minimal surface is a plane. Fischer-Colbrie [FC] showed that a complete minimal surface in $\mathbb{R}^3$ has finite index if and only if it has finite total curvature, and that the catenoid and Enneper's surface have index 1. Likewise, for surfaces with constant mean curvature $H \neq 0$, Barbosa and Do Carmo [BC] showed that only round spheres are stable, and Lopez and Ros [LR] and Silveira [Si] independently showed that they have finite index if and only if they are compact. This leaves open the question of whether there exist surfaces with constant mean curvature $H \neq 0$ and low positive index, for example with index 1.

The third author [R1], [R2] showed numerically that the most natural candidates for unstable surfaces of constant mean curvature $H \neq 0$ with low index – the original Wente tori – all have index at least 7, and with a numerical experiment suggested that the sharpest lower bound is either 8, 9, or 10, and is most likely 9. This leads one to conjecture that all closed surface with constant mean curvature $H \neq 0$ have index at least 9.

The purpose of this article is to show that the original Wente tori all have index at least 8, improving the lower bound of [R1], [R2]. The final part of our argument relies on numerics.

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2 The original Wente tori

In this section we shall give a brief description of the original Wente tori, based on [Wa]. Later, we shall assume that the mean curvature H is $1/2$, but in this and the next section we shall only assume that H is a nonzero constant.

Let $\mathcal{X} : \mathbb{C}/\Gamma \longrightarrow \mathbb{R}^3$ be a conformal immersion of class C^∞ where $\mathbb{C}/\Gamma$ is a compact 2-dimensional torus determined by the 2-dimensional lattice Γ. Note that (x, y) are then isothermal coordinates on $\mathbb{C}/\Gamma$. The fundamental forms and the Gauss and mean curvature functions are

$$I = E(dx^2 + dy^2),\ II = Ldx^2 + 2M dx\, dy + N dy^2,\ K = \tfrac{LN-M^2}{E^2},\ H = \tfrac{L+N}{2E}.$$

Since H is constant, the Hopf differential $\Phi\, dz^2$ is holomorphic, where $\Phi = \frac{1}{2}(L - N) - iM$ and $z = x + iy$. Thus Φ is constant and $\mathcal{X}$ has no umbilics points. Moreover, by a change of the coordinates (x, y), we may assume $\Phi = 1$ and so $M = 0$, $L = e^F + 1$, $N = e^F - 1$, and (x, y) become curvature line parameters, where $F : \mathbb{C}/\Gamma \longrightarrow \mathbb{R}$ is defined by $HE = e^F$. We have the equations of Gauss and Weingarten:

$$\begin{aligned}
&\mathcal{X}_{xx} = \frac{1}{2}F_x\mathcal{X}_x - \frac{1}{2}F_y\mathcal{X}_y - (e^F + 1)\mathcal{N}\ , \quad \mathcal{X}_{yy} = -\frac{1}{2}F_x\mathcal{X}_x + \frac{1}{2}F_y\mathcal{X}_y - (e^F - 1)\mathcal{N}\ ,\\
&\mathcal{X}_{xy} = \frac{1}{2}F_y\mathcal{X}_x + \frac{1}{2}F_x\mathcal{X}_y\ , \quad \mathcal{N}_x = H(1 + e^{-F})\mathcal{X}_x\ , \quad \mathcal{N}_y = H(1 - e^{-F})\mathcal{X}_y
\end{aligned} \tag{1}$$

$$\Delta F + 4H \sinh F = 0\ , \tag{2}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\mathcal{N} : \mathbb{C}/\Gamma \longrightarrow R^3$ is the unit normal vector field, i.e. the Gauss map. Therefore the problem of finding constant mean curvature immersed tori in $\mathbb{R}^3$ reduces to solving the PDE system (1) and (2) by real analytic functions $F, \mathcal{N}, \mathcal{X}$ defined on $\mathbb{R}^2$ and doubly periodic with respect to some fundamental lattice $\Gamma \subset \mathbb{R}^2$.

In the case of the original Wente tori, in Walter's notation, the solution F of (2) is:

$$\tanh\left(\frac{F}{4}\right) = \gamma \cdot \overline{\gamma} \cdot cn_k\,(\alpha x)\, cn_{\overline{k}}\,(\overline{\alpha} y)\ , \tag{3}$$

where cn_k denotes the Jacobi amplitudinus cosinus function with modulus k, and $k = \sin\,\theta$, $\overline{k} = \sin\,\overline{\theta}$, for $\theta, \overline{\theta} \in (0, \pi/2)$ and $\theta + \overline{\theta} < \pi/2$, and

$$\gamma = \sqrt{\tan\,\theta},\ \overline{\gamma} = \sqrt{\tan\,\overline{\theta}},\ \alpha = \sqrt{4H\tfrac{\sin\,2\overline{\theta}}{\sin\,2(\theta+\overline{\theta})}},\ \overline{\alpha} = \sqrt{4H\tfrac{\sin\,2\theta}{\sin\,2(\theta+\overline{\theta})}} \quad .$$

Lemma 1 ([A], [Wa]). *The set of all original Wente tori are in a one-to-one correspondence with the set of reduced fractions $\ell/n \in (1, 2)$.*

For each ℓ/n, we call the corresponding Wente torus $\mathcal{W}_{\ell/n}$. Following Walter's notation, each $\mathcal{W}_{\ell/n}$ has either one or two planar geodesic loops in the central symmetry plane: two loops if ℓ is odd, and one loop if ℓ is even. Each loop can be partitioned into $2n$ congruent curve segments, and ℓ is the total winding order of the Gauss map along each loop.

The conditions for double periodicity of the position vector function $\mathcal{X}$ are expressed in terms of θ and $\overline{\theta}$. Walter determined that there is exactly one

$$\overline{\theta} \cong 65.354955°$$

that solves one period problem. The other period problem is solved with the correct choice of $\theta \in (0,\ (\pi/2) - \overline{\theta})$, and, for any $\ell/n \in (1, 2)$, this correct choice is the unique solution θ of

$$\int_0^{\pi/2} \frac{1 + \tan\theta\ \tan\overline{\theta}\ \cos^2\varphi}{1 - \tan\theta\ \tan\overline{\theta}\ \cos^2\varphi} \frac{d\varphi}{\sqrt{1 - \sin^2\theta\ \sin^2\varphi}} = \frac{\ell}{n}\frac{\pi}{2}\sqrt{\frac{\sin\, 2\overline{\theta}}{\sin\, 2(\theta + \overline{\theta})}}\,. \tag{4}$$

For any $\ell/n \notin (1, 2)$, there is no solution $\theta \in (0, (\pi/2) - \overline{\theta})$ of (4). In Table 2 we give some values of θ with respect to ℓ/n.

Now, if $x_{\ell n}$ (resp. $y_{\ell n}$) denotes the length of the period of $cn_k(\alpha x)$ (resp. $cn_{\overline{k}}(\overline{\alpha} y)$), then we have the following lemma:

Lemma 2 ([Wa]). *$\mathcal{X} : \mathbb{C}/\Gamma \longrightarrow W_{\ell/n} \subset \mathbb{R}^3$ is a conformal immersion ($W_{\ell/n}$ denotes the image of $\mathcal{X}$), where*

$$\Gamma = span_{\mathbb{Z}}\{(nx_{\ell n}, 0), (0, y_{\ell n})\} \quad \textit{when } \ell \textit{ is odd, and}$$

$$\Gamma = span_{\mathbb{Z}}\left\{\left(\frac{nx_{\ell n}}{2}, \frac{y_{\ell n}}{2}\right), (0, y_{\ell n})\right\} \quad \textit{when } \ell \textit{ is even.}$$

The curves $\{[x_0, y] | x_0 = constant\}$ are mapped by $\mathcal{X}$ to planar curvature lines of $W_{\ell/n}$.

The lengths $x_{\ell n}$ and $y_{\ell n}$ can be computed as follows:

$$x_{\ell n} = \frac{4}{\alpha}\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}}\,, \quad y_{\ell n} = \frac{4}{\overline{\alpha}}\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \overline{k}^2\sin^2\varphi}}\,. \tag{5}$$

3 The definition of index and preliminary results

The Jacobi operator associated to $\mathcal{W}_{\ell/n}$ is $-\Delta_I - |II|^2$ on $\mathbb{C}/\Gamma$, with $|II|^2 = E^{-2}(L^2 + 2M^2 + N^2) = 2H^2(1 + e^{-2F})$ and Δ_I the Laplace-Beltrami operator associated to the metric I. The corresponding quadratic form is

$$Q(u, u) = \int_{\mathbb{C}/\Gamma} u\mathcal{L}(u)\, dxdy, \tag{6}$$

where

$$\mathcal{L}u = -\Delta u - Vu \qquad \text{with} \qquad V = 4H\cosh(F)$$

and Δ the Euclidean Laplacian. Note that in equation (6), we are integrating with respect to the flat metric on $\mathbb{C}/\Gamma$.

Consider a smooth volume-preserving variation $\mathcal{X}_t$ of the immersion $\mathcal{X}$ with parameter t so that $\mathcal{X}_0$ is the surface $\mathcal{W}_{\ell/n}$. By reparametrizing the surfaces of the variation, we may assume that the variation vector field at $t = 0$ is $u\mathcal{N}$ for some $u \in C^\infty(\mathbb{C}/\Gamma)$. Then

$$\tfrac{\partial}{\partial t}\text{area}(\mathcal{X}_t)\Big|_{t=0} = 0 \text{ and } \tfrac{\partial^2}{\partial t^2}\text{area}(\mathcal{X}_t)\Big|_{t=0} = Q(u, u).$$

Furthermore, the volume-preserving condition implies $\int_{\mathbb{C}/\Gamma} u\, dA = 0$. Thus, if

$$\mathcal{V} = \left\{ u \in C^\infty(\mathbb{C}/\Gamma); \int_{\mathbb{C}/\Gamma} u\, dA = 0 \right\},$$

then we can give the following definition (see [BC]):

Definition 1 We define $\mathrm{Ind}(\mathcal{X}(\mathbb{C}/\Gamma))$, the *index* of the immersion $\mathcal{X}$ of $\mathbb{C}/\Gamma$, to be the maximum of the dimensions of the subspaces of $\mathcal{V}$ restricted to which Q is negative definite.

Since the first derivative of area is zero, and the second derivative is $Q(u,u)$, the index in a sense measures the amount of area-reducing volume-preserving variations.

Let $L^2 = L^2(\mathbb{C}/\Gamma) = \{u \in C^\infty(\mathbb{C}/\Gamma) | \int_{\mathbb{C}/\Gamma} u^2 dx\, dy < \infty\}$ provided with the inner product $\langle u, v\rangle_{L^2} = \int_{\mathbb{C}/\Gamma} uv\, dx\, dy$. It follows from the standard spectral theorem that the operator $\mathcal{L} = -\Delta - V$ on $\mathbb{C}/\Gamma$ has a discrete spectrum of eigenvalues

$$\beta_1 < \beta_2 \leq \cdots \nearrow \infty$$

and corresponding eigenfunctions

$$\nu_1, \nu_2 \cdots \in C^\infty(\mathbb{C}/\Gamma),$$

which form an orthonormal basis for L^2. Moreover, we have the following variational characterization for the eigenvalues:

$$\beta_j = \inf_{V_j} \left(\sup_{\phi \in V_j,\ ||\phi||_{L^2}=1} \int_{\mathbb{C}/\Gamma} \phi \mathcal{L} \phi dx\, dy \right),$$

where V_j runs through all j dimensional subspaces of $C^\infty(\mathbb{C}/\Gamma)$.

Lemma 3 ([R1], [R2]). *If $\mathcal{L}$ has k negative eigenvalues, then $\mathrm{Ind}(W_{\ell/n})$ is either k or $k-1$. Furthermore, if there exists a subspace $\mathcal{S} \subset L^2$ such that $\mathcal{S} \subset C^\infty(\mathbb{C}/\Gamma)$ and $\dim(\mathcal{S}) = k$ and Q restricted to $\mathcal{S}$ is negative definite, then $\mathrm{Ind}(W_{\ell/n}) \geq k-1$.*

By Lemma 3, our goal becomes to compute the number of negative eigenvalues of $\mathcal{L}$.

Now, we use a convenient fact: For the flat torus $\mathbb{C}/\Gamma$, with $\Gamma = \mathrm{span}_{\mathbb{Z}}\{(a_1, a_2), (b_1, b_2)\}$, the complete set of eigenvalues of $-\Delta u_i = \alpha_i u_i$ are

$$\tfrac{4\pi^2}{(a_1 b_2 - a_2 b_1)^2}((m_2 b_2 - m_1 a_2)^2 + (m_1 a_1 - m_2 b_1)^2),$$

with corresponding orthonormal eigenfunctions

$$c_{m_1,m_2} \cdot (\sin \text{ or } \cos) \left(\tfrac{2\pi}{a_1 b_2 - a_2 b_1}((m_2 b_2 - m_1 a_2)x + (m_1 a_1 - m_2 b_1)y) \right),$$

where $m_1, m_2 \in \mathbb{Z}$, $c_{m_1,m_2} = \sqrt{2/(a_1 b_2 - a_2 b_1)}$ if $|m_1| + |m_2| > 0$, $c_{0,0} = \sqrt{1/(a_1 b_2 - a_2 b_1)}$. With the aid of Lemma 2 we list 17 of the α_i and u_i in Table 1.

With the orderings for the eigenvalues as chosen in Table 1, we do not necessarily have $\alpha_i \leq \alpha_j$ for $i \leq j$. However, we still have $\alpha_i \nearrow \infty$ as $i \nearrow \infty$. Choose $\alpha_{\rho_{\ell/n}(1)}, \alpha_{\rho_{\ell/n}(2)}, \cdots$ the complete set of eigenvalues with multiplicity 1 of the operator $-\Delta$ on the flat torus $\mathbb{C}/\Gamma$ reordered by the permutation $\rho_{\ell/n}$ of $\mathbb{N}$ so that $\alpha_{\rho_{\ell/n}(1)} < \alpha_{\rho_{\ell/n}(2)} \leq \cdots \nearrow \infty$.

eigenvalues α_i for ℓ odd	eigenfunctions u_i for ℓ odd	eigenvalues α_i for ℓ even	eigenfunctions u_i for ℓ even
$\alpha_1 = 0$	$u_1 = \frac{1}{\sqrt{nx_{\ell n}y_{\ell n}}}$	$\alpha_1 = 0$	$u_1 = \frac{\sqrt{2}}{\sqrt{nx_{\ell n}y_{\ell n}}}$
$\alpha_2 = \frac{4\pi^2}{n^2x_{\ell n}^2}$	$u_2 = \frac{\sin(2\pi x/(nx_{\ell n}))}{\sqrt{nx_{\ell n}y_{\ell n}/2}}$	$\alpha_2 = \frac{16\pi^2}{n^2x_{\ell n}^2}$	$u_2 = \frac{\sin(4\pi x/(nx_{\ell n}))}{\sqrt{nx_{\ell n}y_{\ell n}/4}}$
$\alpha_3 = \frac{4\pi^2}{n^2x_{\ell n}^2}$	$u_3 = \frac{\cos(2\pi x/(nx_{\ell n}))}{\sqrt{nx_{\ell n}y_{\ell n}/2}}$	$\alpha_3 = \frac{16\pi^2}{n^2x_{\ell n}^2}$	$u_3 = \frac{\cos(4\pi x/(nx_{\ell n}))}{\sqrt{nx_{\ell n}y_{\ell n}/4}}$
$\alpha_4 = \frac{4\pi^2}{y_{\ell n}^2}$	$u_4 = \frac{\sin(2\pi y/y_{\ell n})}{\sqrt{nx_{\ell n}y_{\ell n}/2}}$	$\alpha_4 = \frac{4\pi^2}{n^2x_{\ell n}^2} + \frac{4\pi^2}{y_{\ell n}^2}$	$u_4 = \frac{\sin(\frac{2\pi x}{nx_{\ell n}} + \frac{2\pi y}{y_{\ell n}})}{\sqrt{nx_{\ell n}y_{\ell n}/4}}$
$\alpha_5 = \frac{4\pi^2}{y_{\ell n}^2}$	$u_5 = \frac{\cos(2\pi y/y_{\ell n})}{\sqrt{nx_{\ell n}y_{\ell n}/2}}$	$\alpha_5 = \frac{4\pi^2}{n^2x_{\ell n}^2} + \frac{4\pi^2}{y_{\ell n}^2}$	$u_5 = \frac{\cos(\frac{2\pi x}{nx_{\ell n}} + \frac{2\pi y}{y_{\ell n}})}{\sqrt{nx_{\ell n}y_{\ell n}/4}}$
$\alpha_6 = \frac{16\pi^2}{n^2x_{\ell n}^2}$	$u_6 = \frac{\sin(4\pi x/(nx_{\ell n}))}{\sqrt{nx_{\ell n}y_{\ell n}/2}}$	$\alpha_6 = \frac{4\pi^2}{n^2x_{\ell n}^2} + \frac{4\pi^2}{y_{\ell n}^2}$	$u_6 = \frac{\sin(\frac{2\pi x}{nx_{\ell n}} - \frac{2\pi y}{y_{\ell n}})}{\sqrt{nx_{\ell n}y_{\ell n}/4}}$
$\alpha_7 = \frac{16\pi^2}{n^2x_{\ell n}^2}$	$u_7 = \frac{\cos(4\pi x/(nx_{\ell n}))}{\sqrt{nx_{\ell n}y_{\ell n}/2}}$	$\alpha_7 = \frac{4\pi^2}{n^2x_{\ell n}^2} + \frac{4\pi^2}{y_{\ell n}^2}$	$u_7 = \frac{\cos(\frac{2\pi x}{nx_{\ell n}} - \frac{2\pi y}{y_{\ell n}})}{\sqrt{nx_{\ell n}y_{\ell n}/4}}$
$\alpha_8 = \frac{4\pi^2}{n^2x_{\ell n}^2} + \frac{4\pi^2}{y_{\ell n}^2}$	$u_8 = \frac{\sin(\frac{2\pi x}{nx_{\ell n}} + \frac{2\pi y}{y_{\ell n}})}{\sqrt{nx_{\ell n}y_{\ell n}/2}}$	$\alpha_8 = \frac{16\pi^2}{y_{\ell n}^2}$	$u_8 = \frac{\sin(4\pi y/y_{\ell n})}{\sqrt{nx_{\ell n}y_{\ell n}/4}}$
$\alpha_9 = \frac{4\pi^2}{n^2x_{\ell n}^2} + \frac{4\pi^2}{y_{\ell n}^2}$	$u_9 = \frac{\cos(\frac{2\pi x}{nx_{\ell n}} + \frac{2\pi y}{y_{\ell n}})}{\sqrt{nx_{\ell n}y_{\ell n}/2}}$	$\alpha_9 = \frac{16\pi^2}{y_{\ell n}^2}$	$u_9 = \frac{\cos(4\pi y/y_{\ell n})}{\sqrt{nx_{\ell n}y_{\ell n}/4}}$
$\alpha_{10} = \frac{4\pi^2}{n^2x_{\ell n}^2} + \frac{4\pi^2}{y_{\ell n}^2}$	$u_{10} = \frac{\cos(\frac{2\pi x}{nx_{\ell n}} - \frac{2\pi y}{y_{\ell n}})}{\sqrt{nx_{\ell n}y_{\ell n}/2}}$	$\alpha_{10} = \frac{64\pi^2}{n^2x_{\ell n}^2}$	$u_{10} = \frac{\sin(8\pi x/(nx_{\ell n}))}{\sqrt{nx_{\ell n}y_{\ell n}/4}}$
$\alpha_{11} = \frac{4\pi^2}{n^2x_{\ell n}^2} + \frac{4\pi^2}{y_{\ell n}^2}$	$u_{11} = \frac{\cos(\frac{2\pi x}{nx_{\ell n}} - \frac{2\pi y}{y_{\ell n}})}{\sqrt{nx_{\ell n}y_{\ell n}/2}}$	$\alpha_{11} = \frac{64\pi^2}{n^2x_{\ell n}^2}$	$u_{11} = \frac{\cos(8\pi x/(nx_{\ell n}))}{\sqrt{nx_{\ell n}y_{\ell n}/4}}$
$\alpha_{12} = \frac{16\pi^2}{y_{\ell n}^2}$	$u_{12} = \frac{\sin(4\pi y/y_{\ell n})}{\sqrt{nx_{\ell n}y_{\ell n}/2}}$	$\alpha_{12} = \frac{36\pi^2}{n^2x_{\ell n}^2} + \frac{4\pi^2}{y_{\ell n}^2}$	$u_{12} = \frac{\sin(\frac{6\pi x}{nx_{\ell n}} + \frac{2\pi y}{y_{\ell n}})}{\sqrt{nx_{\ell n}y_{\ell n}/4}}$
$\alpha_{13} = \frac{16\pi^2}{y_{\ell n}^2}$	$u_{13} = \frac{\cos(4\pi y/y_{\ell n})}{\sqrt{nx_{\ell n}y_{\ell n}/2}}$	$\alpha_{13} = \frac{36\pi^2}{n^2x_{\ell n}^2} + \frac{4\pi^2}{y_{\ell n}^2}$	$u_{13} = \frac{\cos(\frac{6\pi x}{nx_{\ell n}} + \frac{2\pi y}{y_{\ell n}})}{\sqrt{nx_{\ell n}y_{\ell n}/4}}$
$\alpha_{14} = \frac{36\pi^2}{n^2x_{\ell n}^2}$	$u_{14} = \frac{\sin(6\pi x/nx_{\ell n})}{\sqrt{nx_{\ell n}y_{\ell n}/2}}$	$\alpha_{14} = \frac{36\pi^2}{n^2x_{\ell n}^2} + \frac{4\pi^2}{y_{\ell n}^2}$	$u_{14} = \frac{\sin(\frac{6\pi x}{nx_{\ell n}} + \frac{2\pi y}{y_{\ell n}})}{\sqrt{nx_{\ell n}y_{\ell n}/4}}$
$\alpha_{15} = \frac{36\pi^2}{n^2x_{\ell n}^2}$	$u_{15} = \frac{\cos(6\pi x/nx_{\ell n})}{\sqrt{nx_{\ell n}y_{\ell n}/2}}$	$\alpha_{15} = \frac{36\pi^2}{n^2x_{\ell n}^2} + \frac{4\pi^2}{y_{\ell n}^2}$	$u_{15} = \frac{\cos(\frac{6\pi x}{nx_{\ell n}} + \frac{2\pi y}{y_{\ell n}})}{\sqrt{nx_{\ell n}y_{\ell n}/4}}$
$\alpha_{16} = \frac{16\pi^2}{n^2x_{\ell n}^2} + \frac{4\pi^2}{y_{\ell n}^2}$	$u_{16} = \frac{\sin(\frac{4\pi x}{nx_{\ell n}} + \frac{2\pi y}{y_{\ell n}})}{\sqrt{nx_{\ell n}y_{\ell n}/2}}$	$\alpha_{16} = \frac{16\pi^2}{n^2x_{\ell n}^2} + \frac{16\pi^2}{y_{\ell n}^2}$	$u_{16} = \frac{\sin(\frac{4\pi x}{nx_{\ell n}} + \frac{4\pi y}{y_{\ell n}})}{\sqrt{nx_{\ell n}y_{\ell n}/4}}$
$\alpha_{17} = \frac{16\pi^2}{n^2x_{\ell n}^2} + \frac{4\pi^2}{y_{\ell n}^2}$	$u_{17} = \frac{\cos(\frac{4\pi x}{nx_{\ell n}} + \frac{2\pi y}{y_{\ell n}})}{\sqrt{nx_{\ell n}y_{\ell n}/2}}$	$\alpha_{17} = \frac{16\pi^2}{n^2x_{\ell n}^2} + \frac{16\pi^2}{y_{\ell n}^2}$	$u_{17} = \frac{\cos(\frac{4\pi x}{nx_{\ell n}} + \frac{4\pi y}{y_{\ell n}})}{\sqrt{nx_{\ell n}y_{\ell n}/4}}$

Table 1: *The first 17 eigenvalues and eigenfunctions of* $-\Delta u_i = \alpha_i u_i$.

The first of the following two lemmas follows from the variational characterization for eigenvalues, and the second follows from Lemma 3, the Courant nodal domain theorem, and geometric properties of the surfaces $W_{\ell/n}$:

Lemma 4 ([R1], [R2]). *Choose* $\mu \in \mathbb{Z}^+$ *so that* $\alpha_{\rho_{\ell/n}(\mu)} < 4H$. *Then* $\mathrm{Ind}(W_{\ell/n}) \geq \mu - 1$.

Lemma 5 ([R1], [R2]). *For all* $n \in \mathbb{Z}^+$, $n \geq 2$ *we have that* $\mathrm{Ind}(W_{\ell/n}) \geq 2n-2$ *if* ℓ *is odd, and* $\mathrm{Ind}(W_{\ell/n}) \geq n-2$ *if* ℓ *is even.*

4 The lower bound 8 for Ind($W_{\ell/n}$)

We now show the following:

Numerical Result: $\mathrm{Ind}(W_{\ell/n}) \geq 8$ for all ℓ/n.

Observe that, although the eigenvalues of $\mathcal{L}$ depend on the choice of H, the number of negative eigenvalues is independent of H. So without loss of generality we fix $H = 1/2$.

By Lemma 5, $\mathrm{Ind}(W_{\ell/n})$ can be less than 8 only if ℓ/n is one of 3/2, 4/3, 5/3, 5/4, 7/4, 6/5, 8/5, 8/7, 10/7, 12/7, 10/9, 14/9, or 16/9. Lemma 4 also gives explicit lower bounds for the index, since we know the values of $x_{\ell n}$ and $y_{\ell n}$ numerically by formula (5), and hence we know the $\alpha_{\rho_{\ell/n}(i)}$ (see Table 1). Lemma 4 implies that the index is at least 8 when ℓ/n is 5/4, 6/5, 8/7, 10/7, or 10/9. Thus we only need to consider the following eight surfaces:

$$W_{3/2},\ W_{4/3},\ W_{5/3},\ W_{7/4},\ W_{8/5},\ W_{12/7},\ W_{14/9},\ \text{and } W_{16/9}.$$

For these surfaces we list, in Table 2, the corresponding θ, $x_{\ell n}$, $y_{\ell n}$ and lower bounds for index. These approximate values for θ, $x_{\ell n}$, and $y_{\ell n}$ were computed numerically using formulas (4) and (5) and the software Mathematica. Recall that always $\overline{\theta} \cong 65.354°$.

$W_{\ell/n}$	θ	$x_{\ell n}$	$y_{\ell n}$	Lemma 4 lower bound for $\mathrm{Ind}(W_{\ell n})$	Lemma 5 lower bound for $\mathrm{Ind}(W_{\ell n})$
$W_{3/2}$	17.7324°	2.5556	4.2131	2	2
$W_{4/3}$	12.7898°	3.2767	6.3355	6	1
$W_{5/3}$	21.4807°	1.7557	2.6402	2	4
$W_{7/4}$	22.8449°	1.3315	1.9447	2	6
$W_{8/5}$	20.1374°	2.0842	3.2321	2	3
$W_{12/7}$	22.3044°	1.5150	2.2380	2	5
$W_{14/9}$	19.1243°	2.2970	3.6514	4	7
$W_{16/9}$	23.2182°	1.1872	1.7208	2	7

Table 2: *$x_{\ell n}$, $y_{\ell n}$ are computed using the value $H = 1/2$.*

We will find specific spaces on which $\mathcal{L}$ is negative definite, for these eight surfaces.

Let N be an arbitrary positive integer. Consider a finite subset $\{\tilde{u}_1 = u_{i_1}, \ldots, \tilde{u}_N = u_{i_N}\}$ of the eigenfunctions u_i of $-\Delta$ on $\mathbb{C}/\Gamma$, defined in Section 3, with corresponding eigenvalues $\tilde{\alpha}_j = \alpha_{i_j}$, $j = 1, \ldots, N$. If we consider any $u = \sum_{i=1}^N a_i \tilde{u}_i \in \mathrm{span}\{\tilde{u}_1, \ldots, \tilde{u}_N\}$, $a_1, \ldots, a_N \in \mathbb{R}$, then $\int_{\mathbb{C}/\Gamma} u\mathcal{L}(u)dx\,dy = \sum_{i,j=1}^N a_i(\tilde{\alpha}_j\delta_{ij} - \tilde{b}_{ij})a_j$, where $\tilde{b}_{ij} := \int_{\mathbb{C}/\Gamma} V\tilde{u}_i\tilde{u}_j\,dxdy$. So we have $\int_{\mathbb{C}/\Gamma} u\mathcal{L}(u)dx\,dy < 0$ for all nonzero $u \in \mathrm{span}\{\tilde{u}_1, \ldots, \tilde{u}_N\}$ if and only if the matrix $(\tilde{\alpha}_j\delta_{ij} - \tilde{b}_{ij})_{i,j=1,\ldots,N}$ is negative definite. Lemma 3 then implies:

Theorem 1 ([R1], [R2]). *If the $N \times N$ matrix $(\tilde{\alpha}_j\delta_{ij} - \tilde{b}_{ij})_{i,j=1,\ldots,N}$ is negative definite, then* $\mathrm{Ind}(W_{\ell/n}) \geq N - 1$.

$W_{\ell/n}$	$\tilde{u}_1$	$\tilde{u}_2$	$\tilde{u}_3$	$\tilde{u}_4$	$\tilde{u}_5$	$\tilde{u}_6$	$\tilde{u}_7$	$\tilde{u}_8$	$\tilde{u}_9$
$W_{3/2}$	u_1	u_2	u_3	u_4	u_5	u_7	u_8	u_9	u_{17}
$W_{4/3}$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
$W_{5/3}$	u_1	u_2	u_3	u_5	u_6	u_7	u_8	u_9	u_{15}
$W_{7/4}$	u_1	u_2	u_3	u_6	u_7	u_8	u_9	u_{14}	u_{15}
$W_{8/5}$	u_1	u_2	u_3	u_4	u_5	u_{10}	u_{11}	u_{12}	u_{13}
$W_{12/7}$	u_1	u_2	u_3	u_4	u_5	u_{10}	u_{11}	u_{12}	u_{13}
$W_{14/9}$	u_1	u_2	u_3	u_4	u_5	u_{10}	u_{11}	u_{12}	u_{13}
$W_{16/9}$	u_1	u_2	u_3	u_4	u_5	u_{10}	u_{11}	u_{12}	u_{13}

Table 3: *Eigenfunctions of $-\Delta$ producing 9-dimensional spaces on which Q is negative definite.*

Definition 2 Given A, B even integers and $\ell, n \in \mathbb{Z}^+$, we now define the following basic integrals:

$$I_0(\ell, n, A, B) = \frac{1}{nx_{\ell n}y_{\ell n}} \int_0^{x_{\ell n}/4} \int_0^{y_{\ell n}/4} V \left(\cos \frac{2\pi x}{x_{\ell n}}\right)^A \left(\cos \frac{2\pi y}{y_{\ell n}}\right)^B dydx,$$

$$I_1(\ell, n) = \frac{8}{nx_{\ell n}y_{\ell n}} \int_0^{nx_{\ell n}/4} \int_0^{y_{\ell n}/4} V \cos\left(\frac{4\pi x}{nx_{\ell n}}\right) dydx,$$

$$I_2(\ell, n) = \frac{8}{nx_{\ell n}y_{\ell n}} \int_0^{nx_{\ell n}/4} \int_0^{y_{\ell n}/4} V \cos\left(\frac{8\pi x}{nx_{\ell n}}\right) dydx,$$

$$I_3(\ell, n) = \frac{8}{nx_{\ell n}y_{\ell n}} \int_0^{nx_{\ell n}/4} \int_0^{y_{\ell n}/4} V \cos\left(\frac{16\pi x}{nx_{\ell n}y_{\ell n}}\right) dydx,$$

$$I_4(\ell, n) = \frac{8}{nx_{\ell n}y_{\ell n}} \int_0^{nx_{\ell n}/4} \int_0^{y_{\ell n}/4} V \cos\left(\frac{4\pi x}{nx_{\ell n}}\right) \cos\left(\frac{4\pi y}{y_{\ell n}}\right) dydx,$$

$$I_5(\ell, n) = \frac{8}{nx_{\ell n}y_{\ell n}} \int_0^{nx_{\ell n}/4} \int_0^{y_{\ell n}/4} V \cos\left(\frac{4\pi x}{nx_{\ell n}}\right) \cos\left(\frac{8\pi x}{nx_{\ell n}}\right) dydx,$$

$$I_6(\ell, n) = \frac{8}{nx_{\ell n}y_{\ell n}} \int_0^{nx_{\ell n}/4} \int_0^{y_{\ell n}/4} V \cos\left(\frac{8\pi x}{nx_{\ell n}}\right) \cos\left(\frac{4\pi y}{y_{\ell n}}\right) dydx,$$

$$I_7(\ell, n) = \frac{8}{nx_{\ell n}y_{\ell n}} \int_0^{nx_{\ell n}/4} \int_0^{y_{\ell n}/4} V \cos\left(\frac{12\pi x}{nx_{\ell n}}\right) \cos\left(\frac{4\pi y}{y_{\ell n}}\right) dydx.$$

Now, for each surface $W_{\ell/n}$ given in Table 2, we will fix $N = 9$ and choose the subset $\{\tilde{u}_1, \ldots, \tilde{u}_9\}$ such that the matrix $(\tilde{\alpha}_j \delta_{ij} - \tilde{b}_{ij})_{i,j=1,\ldots,N}$ is negative definite. These choices are given in Table 3. With these choices for $\tilde{u}_i$, we have the following lemma:

Lemma 6 *With the choices given in Table 3, all elements of the eight matrices $\mathcal{M}(\ell, n) := (\tilde{\alpha}_j \delta_{ij} - \tilde{b}_{ij})_{i,j=1,\ldots,9}$ can be expressed in terms of the basic integrals $I_0(\ell, n, A, B)$ and $I_j(\ell, n)$ for A, B even and $j = 1, 2, \ldots, 7$.*

Proof: The symmetries $V(x, y) = V(-x, y) = V(x, -y) = V\left(\frac{x_{\ell n}}{2} - x, y\right) = V\left(x, \frac{y_{\ell n}}{2} - y\right)$ of V and the identities $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$, $\sin(a \pm b) = \sin(a)\cos(b) \pm \sin(b)\cos(a)$ give the relations shown in Table 4, proving the lemma.

For $W_{3/2}$	$M_{1,1} = -32\ I_0(3,2,0,0)$, $M_{4,4} = \alpha_4 - 64(I_0(3,2,0,0) - I_0(3,2,0,2))$ $M_{5,5} = \alpha_5 - 64\ I_0(3,2,0,2)$, $M_{7,7} = \alpha_7 - 64\ I_0(3,2,2,0)$ $M_{17,17} = \alpha_{17} - 64(I_0(3,2,0,0) - I_0(3,2,0,2) - I_0(3,2,2,0) + 2\ I_0(3,2,2,2))$ $M_{i,i} = \alpha_i - 32 I_0(3,2,0,0)$, for $i = 2,3,8,9$ $M_{i,j} = 0$ for $i < j$ with $i,j \in \{1,2,3,4,5,7,8,9,17\}$.
For $W_{4/3}$	$M_{1,1} = -48\ I_0(4,3,0,0)$, $M_{2,2} = \alpha_2 - 48\ I_0(4,3,0,0) + 2\ I_2(4,3)$ $M_{3,3} = \alpha_3 - 48\ I_0(4,3,0,0) - 2\ I_2(4,3)$, $M_{4,4} = \alpha_4 - 48\ I_0(4,3,0,0) + 2\ I_4(4,3)$ $M_{5,5} = \alpha_5 - 48\ I_0(4,3,0,0) - 2\ I_4(4,3)$, $M_{6,6} = \alpha_6 - 48\ I_0(4,3,0,0) + 2\ I_4(4,3)$ $M_{7,7} = \alpha_7 - 48\ I_0(4,3,0,0) - 2\ I_4(4,3)$ $M_{8,8} = \alpha_8 - 384\ (I_0(4,3,0,2) - I_0(4,3,0,4))$ $M_{9,9} = \alpha_9 - 96\ (4\ I_0(4,3,0,4) - 4\ I_0(4,3,0,2) + I_0(4,3,0,0))$ $M_{1,3} = -2\sqrt{2} I_1(4,3)$, $M_{1,9} = -48\sqrt{2}(-I_0(4,3,0,0) + 2 I_0(4,3,0,2))$ $M_{4,6} = -48(-I_0(4,3,0,0) + 2 I_0(4,3,0,2)) + 2 I_1(4,3)$ $M_{5,7} = -48(-I_0(4,3,0,0) + 2 I_0(4,3,0,2)) - 2 I_1(4,3)$ $M_{3,9} = -4\ I_4(4,3)$, all other $M_{i,j}$ with $i < j \leq 9$ are zero.
For $W_{5/3}$	$M_{1,1} = -48\ I_0(5,3,0,0)$, $M_{2,2} = \alpha_2 - 48\ I_0(5,3,0,0) + 2\ I_1(5,3)$ $M_{3,3} = \alpha_3 - 48\ I_0(5,3,0,0) - 2\ I_1(5,3)$, $M_{5,5} = \alpha_5 - 96\ I_0(5,3,0,2)$ $M_{6,6} = \alpha_6 - 48\ I_0(5,3,0,0) + 2\ I_2(5,3)$, $M_{7,7} = \alpha_7 - 48\ I_0(5,3,0,0) - 2\ I_2(5,3)$ $M_{8,8} = \alpha_8 - 48\ I_0(5,3,0,0) + 2\ I_4(5,3)$, $M_{9,9} = \alpha_9 - 48\ I_0(5,3,0,0) - 2\ I_4(5,3)$ $M_{15,15} = \alpha_{15} - 96\ I_0(5,3,2,0)$, $M_{1,7} = -2\sqrt{2} I_1(5,3)$ $M_{3,15} = -2(I_1(5,3) + I_2(5,3))$, all other pertinent $M_{i,j}$ with $i < j$ are zero.
For $W_{7/4}$	$M_{i,i} = \alpha_i - 64\ I_0(7,4,0,0)$ for $i = 1,2,3,6,7,8,9,14,15$ all other pertinent $M_{i,j}$ with $i < j$ are zero.
For $W_{8/5}$, $W_{12/7}$, $W_{14/9}$, $W_{16/9}$	$M_{1,1} = -16n\ I_0(\ell,n,0,0)$, $M_{2,2} = \alpha_2 - 16n\ I_0(\ell,n,0,0) + 2 I_1(\ell,n)$ $M_{3,3} = \alpha_3 - 16n\ I_0(\ell,n,0,0) - 2 I_1(\ell,n)$, $M_{4,4} = \alpha_4 - 16n\ I_0(\ell,n,0,0) + 2 I_4(\ell,n)$ $M_{5,5} = \alpha_5 - 16n\ I_0(\ell,n,0,0) - 2 I_4(\ell,n)$ $M_{10,10} = \alpha_{10} - 16n\ I_0(\ell,n,0,0) + 2 I_3(\ell,n)$ $M_{11,11} = \alpha_{11} - 16n\ I_0(\ell,n,0,0) - 2 I_3(\ell,n)$ $M_{12,12} = \alpha_{12} - 16n\ I_0(\ell,n,0,0) + 2 I_7(\ell,n)$ $M_{13,13} = \alpha_{13} - 16n\ I_0(\ell,n,0,0) - 2 I_7(\ell,n)$, $M_{1,3} = -2\sqrt{2} I_1(\ell,n)$ $M_{1,11} = -2\sqrt{2} I_2(\ell,n)$, $M_{2,10} = -4\ I_1(\ell,n) + 4\ I_5(\ell,n)$ $M_{3,11} = -4\ I_5(\ell,n)$, $M_{4,12} = -2 I_1(\ell,n) + 2 I_6(\ell,n)$ $M_{5,13} = -2 I_1(\ell,n) - 2 I_6(\ell,n)$, all other pertinent $M_{i,j}$ with $i < j$ are zero.

Table 4: *Elements $M_{i,j}$ of the symmetric matrices $\mathcal{M}(\ell,n)$ expressed in terms of the basic integrals. We have chosen here to index the $M_{i,j}$ using the counters associated to α_j and u_j, rather than $\tilde{\alpha}_j$ and $\tilde{u}_j$.*

By numerical methods, we can estimate that all of the relevant $I_j(\ell,n)$ for $j \geq 1$ are approximately zero, and that

$$I_0(3,2,0,0) \cong 0.2968,\ I_0(3,2,2,0) \cong 0.2304,\ I_0(3,2,0,2) \cong 0.2408,\ I_0(3,2,2,2) \cong 0.1947,$$

$$I_0(4,3,0,0) \cong 0.1077,\ I_0(4,3,0,2) \cong 0.0776,\ I_0(4,3,0,4) \cong 0.0667,\ I_0(5,3,0,0) \cong 0.4532,$$

$$I_0(5,3,2,0) \cong 0.3910,\ I_0(5,3,0,2) \cong 0.4046,\ I_0(7,4,0,0) \cong 0.6072,\ I_0(8,5,0,0) \cong 0.1878,$$

$$I_0(12,7,0,0) \cong 0.2652,\ I_0(14,9,0,0) \cong 0.0841,\ I_0(16,9,0,0) \cong 0.3419.$$

These values were computed with a Mathematica program using the NIntegrate and JacobiCN commands, and the program is available at the web site of the third author. One note of warning is that Mathematica has different conventions than Walter's paper, and hence cn_k in [Wa] is equivalent to cn_{k^2} in Mathematica. We include a sample of our code in the Appendix.

Now we can make approximations for the eight matrices $\mathcal{M}(\ell, n)$.

The matrix $\mathcal{M}(3,2)$ is approximately

$$\mathcal{M}(3,2) \approx \begin{pmatrix} -9.50 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -7.99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -7.99 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.36 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -13.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -8.70 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5.76 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.76 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.50 \end{pmatrix},$$

and all nondiagonal terms are known to be zero by rigorous mathematical computation, and all nonzero entries have been computed only numerically.

$\mathcal{M}(4,3)$ is approximately the nondiagonal matrix

$$\mathcal{M}(4,3) \approx \begin{pmatrix} -5.17 & 0 & \mathcal{O} & 0 & 0 & 0 & 0 & 0 & -3.23 \\ 0 & -3.53 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathcal{O} & 0 & -3.53 & 0 & 0 & 0 & 0 & 0 & \mathcal{O} \\ 0 & 0 & 0 & -3.78 & 0 & -2.29 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.78 & 0 & -2.29 & 0 & 0 \\ 0 & 0 & 0 & -2.29 & 0 & -3.78 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.29 & 0 & -3.78 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.25 & 0 \\ -3.23 & 0 & \mathcal{O} & 0 & 0 & 0 & 0 & 0 & -2.21 \end{pmatrix},$$

and again here all entries that are 0 have been computed mathematically rigorously, and all nonzero entries have been computed only numerically. The symbol $\mathcal{O}$ denotes an entry that has been computed numerically to be approximately zero, but not mathematically rigorously. We shall continue to use these conventions in all remaining matrices.

$\mathcal{M}(5,3)$ is approximately the diagonal matrix

$$\mathcal{M}(5,3) \approx \begin{pmatrix} -21.8 & 0 & 0 & 0 & 0 & \mathcal{O} & 0 & 0 & 0 \\ 0 & -20.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -20.3 & 0 & 0 & 0 & 0 & 0 & \mathcal{O} \\ 0 & 0 & 0 & -33.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -16.1 & 0 & 0 & 0 & 0 \\ \mathcal{O} & 0 & 0 & 0 & 0 & -16.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -14.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -14.7 & 0 \\ 0 & 0 & \mathcal{O} & 0 & 0 & 0 & 0 & 0 & -24.7 \end{pmatrix}.$$

$\mathcal{M}(7,4)$ is approximately the diagonal matrix

$$\mathcal{M}(7,4) \approx \begin{pmatrix} -38.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -37.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -37.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -33.3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -33.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -27.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -27.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -26.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -26.3 \end{pmatrix}.$$

$\mathcal{M}(8,5)$ is approximately the diagonal matrix

$$\mathcal{M}(8,5) \approx \begin{pmatrix} -15.0 & 0 & \mathcal{O} & 0 & 0 & 0 & \mathcal{O} & 0 & 0 \\ 0 & -13.6 & 0 & 0 & 0 & \mathcal{O} & 0 & 0 & 0 \\ \mathcal{O} & 0 & -13.6 & 0 & 0 & 0 & \mathcal{O} & 0 & 0 \\ 0 & 0 & 0 & -10.9 & 0 & 0 & 0 & \mathcal{O} & 0 \\ 0 & 0 & 0 & 0 & -10.9 & 0 & 0 & 0 & \mathcal{O} \\ 0 & \mathcal{O} & 0 & 0 & 0 & -9.2 & 0 & 0 & 0 \\ \mathcal{O} & 0 & \mathcal{O} & 0 & 0 & 0 & -9.2 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{O} & 0 & 0 & 0 & -8.0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{O} & 0 & 0 & 0 & -8.0 \end{pmatrix}.$$

$\mathcal{M}(12,7)$ is approximately the diagonal matrix

$$\mathcal{M}(12,7) \approx \begin{pmatrix} -29.7 & 0 & \mathcal{O} & 0 & 0 & 0 & \mathcal{O} & 0 & 0 \\ 0 & -28.3 & 0 & 0 & 0 & \mathcal{O} & 0 & 0 & 0 \\ \mathcal{O} & 0 & -28.3 & 0 & 0 & 0 & \mathcal{O} & 0 & 0 \\ 0 & 0 & 0 & -21.5 & 0 & 0 & 0 & \mathcal{O} & 0 \\ 0 & 0 & 0 & 0 & -21.5 & 0 & 0 & 0 & \mathcal{O} \\ 0 & \mathcal{O} & 0 & 0 & 0 & -24.1 & 0 & 0 & 0 \\ \mathcal{O} & 0 & \mathcal{O} & 0 & 0 & 0 & -24.1 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{O} & 0 & 0 & 0 & -18.7 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{O} & 0 & 0 & 0 & -18.7 \end{pmatrix}.$$

$\mathcal{M}(14,9)$ is approximately the diagonal matrix

$$\mathcal{M}(14,9) \approx \begin{pmatrix} -12.1 & 0 & \mathcal{O} & 0 & 0 & 0 & \mathcal{O} & 0 & 0 \\ 0 & -11.7 & 0 & 0 & 0 & \mathcal{O} & 0 & 0 & 0 \\ \mathcal{O} & 0 & -11.7 & 0 & 0 & 0 & \mathcal{O} & 0 & 0 \\ 0 & 0 & 0 & -9.1 & 0 & 0 & 0 & \mathcal{O} & 0 \\ 0 & 0 & 0 & 0 & -9.1 & 0 & 0 & 0 & \mathcal{O} \\ 0 & \mathcal{O} & 0 & 0 & 0 & -10.6 & 0 & 0 & 0 \\ \mathcal{O} & 0 & \mathcal{O} & 0 & 0 & 0 & -10.6 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{O} & 0 & 0 & 0 & -8.3 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{O} & 0 & 0 & 0 & -8.3 \end{pmatrix}.$$

$\mathcal{M}(16,9)$ is approximately the diagonal matrix

$$\mathcal{M}(16,9) \approx \begin{pmatrix} -49.2 & 0 & \mathcal{O} & 0 & 0 & 0 & \mathcal{O} & 0 & 0 \\ 0 & -47.9 & 0 & 0 & 0 & \mathcal{O} & 0 & 0 & 0 \\ \mathcal{O} & 0 & -47.9 & 0 & 0 & 0 & \mathcal{O} & 0 & 0 \\ 0 & 0 & 0 & -35.6 & 0 & 0 & 0 & \mathcal{O} & 0 \\ 0 & 0 & 0 & 0 & -35.6 & 0 & 0 & 0 & \mathcal{O} \\ 0 & \mathcal{O} & 0 & 0 & 0 & -43.7 & 0 & 0 & 0 \\ \mathcal{O} & 0 & \mathcal{O} & 0 & 0 & 0 & -43.7 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{O} & 0 & 0 & 0 & -32.8 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{O} & 0 & 0 & 0 & -32.8 \end{pmatrix}.$$

All eight of these matrices are 9×9 and negative definite. Hence Theorem 1 implies the numerical result.

5 Appendix: the Mathematica code

The following is a Mathematica code for computing the values $I_0(4,3,0,0)$, $I_0(4,3,0,2)$, $I_0(4,3,0,4)$, $I_1(4,3)$, $I_2(4,3)$, $I_4(4,3)$, and the elements of the matrix $\mathcal{M}(4,3)$. The seven other needed codes for different ℓ and n were written similarly.

```
H = 1/2;     k1 = Sin[theta1];     k2 = Sin[theta2];
gamma1 = Sqrt[Tan[theta1]];     gamma2 = Sqrt[Tan[theta2]];
alpha1 = Sqrt[4 H Sin[2theta2]/Sin[2(theta1 + theta2)]];
alpha2 = Sqrt[4 H Sin[2theta1]/Sin[2(theta1 + theta2)]];
F = 4ArcTanh[gamma1 gamma2 JacobiCN[alpha1 x, k1^2] JacobiCN[alpha2 y, k2^2]];
V = 4 H Cosh[F];

ell = 4;     n = 3;
theta1 = 2 Pi (12.7898/360);     theta2 = 2 Pi (65.354955354/360);
x0 = 3.2767;     y0 = 6.3355;

Print["I_0(4,3,0,0) is ",I0x4c3c0c0x = (1/(n x0 y0)) NIntegrate[
   V , {x, 0, x0/4}, {y, 0, y0/4}]];

Print["I_0(4,3,0,2) is ",I0x4c3c0c2x = (1/(n x0 y0)) NIntegrate[
   V (Cos[2 Pi x/x0])^0(Cos[2 Pi y/y0])^2,{x, 0, x0/4}, {y, 0, y0/4}]];

Print["I_0(4,3,0,4) is ",I0x4c3c0c4x = (1/(n x0 y0)) NIntegrate[
   V (Cos[2 Pi x/x0])^0(Cos[2 Pi y/y0])^4,{x, 0, x0/4}, {y, 0, y0/4}]];

Print["I_1(4,3) is ",I1x4c3x = (8/(n x0 y0)) (NIntegrate[
   V (Cos[4 Pi x/(n x0)]),{x, 0, x0/4}, {y, 0, y0/4}] + NIntegrate[
   V (Cos[4 Pi x/(n x0)]),{x, x0/4, 2 x0/4}, {y, 0, y0/4}] + NIntegrate[
   V (Cos[4 Pi x/(n x0)]),{x, 2 x0/4, n x0/4}, {y, 0, y0/4}])];

Print["I_2(4,3) is ",I2x4c3x = (8/(n x0 y0)) (NIntegrate[
   V (Cos[8 Pi x/(n x0)]),{x, 0, x0/4}, {y, 0, y0/4}] + NIntegrate[
   V (Cos[8 Pi x/(n x0)]),{x, x0/4, 2 x0/4}, {y, 0, y0/4}] + NIntegrate[
   V (Cos[8 Pi x/(n x0)]),{x, 2 x0/4, n x0/4}, {y, 0, y0/4}])];
```

```
Print["I_4(4,3) is ",I4x4c3x = (8/(n x0 y0)) (NIntegrate[
  V (Cos[4 Pi x/(n x0)]) (Cos[4 Pi y/y0]),{x,0,x0/4},{y,0,y0/4}]+NIntegrate[
  V (Cos[4 Pi x/(n x0)]) (Cos[4 Pi y/y0]),{x,x0/4,2 x0/4},{y,0,y0/4}]+NIntegrate[
  V (Cos[4 Pi x/(n x0)]) (Cos[4 Pi y/y0]),{x, 2 x0/4, n x0/4}, {y, 0, y0/4}])];

aa = 0;  bb = 0;     alpha1 = aa(4 N[Pi^2]/(n^2 x0^2)) + bb (4 N[Pi^2]/(y0^2));

aa = 4;  bb = 0;     alpha2 = aa(4 N[Pi^2]/(n^2 x0^2)) + bb (4 N[Pi^2]/(y0^2));
alpha3 = alpha2;

aa = 1;  bb = 1;     alpha4 = aa(4 N[Pi^2]/(n^2 x0^2)) + bb (4 N[Pi^2]/(y0^2));
alpha5 = alpha4;     alpha6 = alpha4;     alpha7 = alpha4;

aa = 0;  bb = 4;     alpha8 = aa(4 N[Pi^2]/(n^2 x0^2)) + bb (4 N[Pi^2]/(y0^2));
alpha9 = alpha8;

Print["M(1,1) is ", -48 I0x4c3c0c0x];
Print["M(2,2) is ", alpha2 - 48 I0x4c3c0c0x];
Print["M(3,3) is ", alpha3 - 48 I0x4c3c0c0x];
Print["M(4,4) is ", alpha4 - 48 I0x4c3c0c0x];
Print["M(5,5) is ", alpha5 - 48 I0x4c3c0c0x];
Print["M(6,6) is ", alpha6 - 48 I0x4c3c0c0x];
Print["M(7,7) is ", alpha7 - 48 I0x4c3c0c0x];
Print["M(8,8) is ", alpha8 - 384 (I0x4c3c0c2x - I0x4c3c0c4x)];
Print["M(9,9) is ", alpha9 - 96 (I0x4c3c0c0x + 4 I0x4c3c0c4x - 4 I0x4c3c0c2x)];
Print["M(1,9) is ", -48 N[Sqrt[2]] (-I0x4c3c0c0x + 2 I0x4c3c0c2x)];
Print["M(4,6) is ", -48 (-I0x4c3c0c0x + 2 I0x4c3c0c2x)];
Print["M(5,7) is ", -48 (-I0x4c3c0c0x + 2 I0x4c3c0c2x)];
```

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