FOLIATION OF 3-DIMENSIONAL SPACE FORMS BY SURFACES WITH CONSTANT MEAN CURVATURE

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1. Introduction

Let Y represent a 3-dimensional complete simply-connected space form. We study C^2-foliations F of Y by leaves with the same constant mean curvature. We prove that if the curvature of Y is positive such foliations can not exist. When Y is the Euclidean space then such a foliation must consist of parallel planes. When Y is the hyperbolic space, if we further assume that the mean curvature satisfies $H \geq 1$, then F must be a foliation by horospheres. These results are still true if F is a foliation of an open set U of Y and if we further assume that the leaves are complete and orientable.

We observe that on hyperbolic space there are examples of nontrivial foliations of open sets by complete surfaces with the same constant mean curvature $0 < H < 1$. One example can be obtained from the 1-parameter family of catenoids studied by do Carmo and Dajczer [CD] and by Gomes [G].

To prove the results, we consider a codimension-one foliation of an orientable Riemannian manifold, whose leaves are orientable and have the same constant mean curvature, and first show that its leaves are strongly stable in the sense defined in [BCE]. We then apply the classification theorem for complete stable surfaces of a 3-dimensional space form proved in [BCE] and [S].

Since the first version of this work was announced, some progress was made on this subject. W. Meeks [M] showed that the

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only C^2-foliation of the 3-dimensional Eucliean space, for which each leaf L has constant mean curvature H_L, is the foliation by planes. In his proof he uses theorem (3.12) below. Barbosa, Kenmotsu and Oshikiri [BKO] have shown that if M is a compact Riemannian manifold with nonnegative curvature, then any codimension-one C^2-foliation of M, for which each leaf L has constant mean curvature H_L, is a foliation with totally geodesic leaves.

2. Preliminary results

Let M be an orientable n-dimensional manifold and Y be an orientable Riemannian manifold of dimension $n+1$. $\langle .,. \rangle$ will represent the metric on Y. Let $x: M \to Y$ be an immersion. We will consider M endowed with the induced metric so that x becomes an isometry. N will represent a unit normal vector field to M that defines its orientation.

If $\{e_1, \ldots, e_n\}$ is a local orthonormal frame field then the second fundamental form of the immersion is given by

$$(2.1) \qquad B(e_i) = -\nabla_{e_i} N = \sum_{j=1}^{n} h_{ij} e_j,$$

where ∇ is the Riemannian connection of Y. The mean curvature of x is then defined as

$$(2.2) \qquad H = \frac{1}{n} \sum_{i=1}^{n} h_{ij}$$

and the norm of the second fundamental forms is given by

$$(2.3) \qquad \|B\|^2 = \sum_{i,j=1}^{n} h_{ij}^2$$

We will now assume that the immersion x has constant mean curvature. D will represent a relatively compact domain with smooth boundary. We say that D is stable when, for each function $u: D \to R$ such that

(2.4) $$u|_{\partial D} = 0$$

and

(2.5) $$\int_D u\,dM = 0$$

we have

(2.6) $$\int_D \{-u\Delta u - (\|B\|^2 + R)u^2\}\,dM \geq 0.$$

Here Δ represents the Laplacian on M and R represents the Ricci curvature in the normal direction. When condition (2.6) is satisfied, whether if (2.5) occurs or not, we say that D is strongly stable. Observe that if D is strongly stable then it is stable.

We say that the immersion x is stable when each relatively compact domain D on M, with smooth boundary, is stable. We make a similar definition for x strongly stable. For a more detailed discussion of the concept of stability see [BCE].

2.7 Theorem. Let $x_t: M \to Y$ be a 1-parameter family of immersions with constant mean curvature $H(t)$. Then, for each fixed t there is a function $f: M \to R$ satisfying:

(2.8) $$\Delta f + (\|B\|^2 + R)f = n\frac{\partial H}{\partial t}$$

Proof. Let $X: (-\varepsilon,\varepsilon) \times M \to Y$ be the mapping defined by $X(t,p) = x_t(p)$. Set

(2.9) $$\xi = \frac{\partial X}{\partial t}.$$

Let $N(t,p)$ represent the unit vector field normal to $x_t(M)$ at the point p. Define a function $F:(-\varepsilon,\varepsilon)\times M \to R$ by

$$F(t,p) = \langle \xi(t,p), N(t,p) \rangle$$

and, for a fixed value of t, set

(2.10) $$f(p) = F(t, p).$$

We are going to show that the function f satisfies the equation (2.8).

To simplify the notation we will identify each vector V, tangent to $M_t = \{t\} \times M$, with the vector $dx_t(V)$. $\partial/\partial t$ will denote the unit vector field on $(-\varepsilon, \varepsilon) \times M$ orthogonal to M with respect to the product metric so that

(2.11) $$\xi = dX(\partial/\partial t)$$

Fix $t = t_0$. In a neighborhood of a point p of M_{t_0} choose an orthonormal frame field $\{e_1(t, p), e_2(t, p), \ldots, e_n(t, p)\}$ tangent to M_t and such that

(2.12) $$(\nabla_{e_i} e_j)^T(p) = 0 \qquad i, j = 1, 2, \ldots, n$$

where $(\ )^T(p)$ means the orthogonal projection into T_pM. For this choice of frame it is true that

(2.13) $$[\xi, e_i](p) = 0 \qquad i = 1, 2, \ldots, n$$

and

(2.14) $$\Delta f(p) = \sum_{i=1}^{n} e_i(e_i(f)).$$

Decomposing ξ by its tangential and normal components, $\xi = \xi^T + \xi^N$, we obtain:

(2.15) $$f = \langle \xi^N, N \rangle.$$

Since $\nabla_{e_i} N$ is tangent to M_t, it follows that

$$e_i(f) = \langle \nabla_{e_i} \xi^N, N \rangle$$

and so we have

(2.16) $$e_i(e_i(f)) = \langle \nabla_{e_i} \nabla_{\xi^N} e_i, N \rangle + \langle \nabla_{e_i} [e_i, \xi^N], N \rangle + \langle \nabla_{e_i} \xi^N, \nabla_{e_i} N \rangle.$$

Using that the curvature tensor of Y is given by

$$(2.17)\qquad R(V,W)Z = \nabla_W\nabla_V Z - \nabla_V\nabla_W Z + \nabla_{[V,W]}Z$$

where rewrite (2.16) as

$$(2.18)\qquad e_i(e_i(f)) = -\langle R(e_i,\xi^N)e_i,\ N\rangle + \langle\nabla_{\xi^N}\nabla_{e_i}e_i,\ N\rangle$$

$$+ \langle\nabla_{[e_i,\xi^N]}e_i, N\rangle + \langle\nabla_{e_i}[e_i,\xi^N],\ N\rangle + \langle\nabla_{e_i}\xi^N,\ \nabla_{e_i}N\rangle.$$

Since $\xi^N = fN$ and $\nabla_{e_i}N$ is tangent to M_t we have:

$$(2.19)\qquad \langle\nabla_{e_i}\xi^N, \nabla_{e_i}N\rangle = f|\nabla_{e_i}N|^2 = f\sum_{j=1}^{n} h_{ij}^2$$

From (2.13) we have that

$$[e_i,\ \xi^N](p) = -[e_i,\xi^T](p).$$

Therefore $[e_i,\xi^N](p)$ is tangent to M_{t_0}. From the symmetry of the second fundamental form of x_t it follows that, at p,

$$(2.20)\qquad \langle\nabla_{[e_i,\xi^N]}e_i, N\rangle = \langle\nabla_{e_i}[e_i,\xi^N],\ N\rangle$$

Since $(\nabla_{\xi^N}N)(p)$ is tangent to M_{t_0} and, also making use of (2.12), we obtain:

$$(2.21)\qquad \langle\nabla_{\xi^N}\nabla_{e_i}e_i,\ N\rangle(p) = \xi^N\langle\nabla_{e_i}e_i,\ N\rangle(p)$$

$$= -\xi^N\langle e_i,\ \nabla_{e_i}N\rangle(p) = \xi^N(h_{ij})(p).$$

Using that $[e_i,\xi^N]$ is tangent to M_{t_0} at the point p and also (2.15) we have

$$(2.22)\qquad \langle N, \nabla_{e_i}[e_i,\xi^N]\rangle(p) = -\langle [e_i,\xi^N],\ \nabla_{e_i}N\rangle(p)$$

$$= \langle \nabla_{e_i}\xi^N - \nabla_{\xi^N}e_i,\ \sum_{j=1}^{n} h_{ij}e_j\rangle(p) =$$

$$= -f(p)\sum_{j=1}^{n} h_{ij}(p) - \sum_{j=1}^{n} h_{ij}\langle \nabla_{\xi^N}e_i,\ e_j\rangle(p)\ .$$

Since (h_{ij}) is a symmetric matrix, the last term vanishes. Thus

$$(2.23)\qquad \langle N, \nabla_{e_i}[e_i,\xi^N]\rangle(p) = -f(p)\sum_{j=1}^{n} h_{ij}^2(p).$$

Putting together (2.14), (2.18), (2.19), (2.20), (2.21) and (2.23) we obtain

$$(2.24)\qquad \Delta f(p) = -f(p)\sum_{i=1}^{n}\langle R(e_i, N)e_i, N\rangle + \sum_{i=1}^{n}\xi^N(h_{ii}) - f(p)\sum_{i,j=1}^{n} h_{ij}^2$$

Observing that $R = \sum_{i=1}^{n}\langle R(e_i, N)\ e_i, N\rangle$, that $\sum h_{ij}^2 = \|B\|^2$ and that

$$(2.25)\qquad \frac{\partial}{\partial t}(nH) = \xi(nH) = \xi^T(nH) + \xi^N(nH) = \xi^N(nH)$$

(where the last equality was obtained by using that H is constant along each M_t) we may then rewrite (2.24) as

$$(2.26)\qquad (\Delta f + (\|B\|^2 + R)f)(p) = \frac{\partial}{\partial t}(nH)(p).$$

Since p was arbitrary the theorem is proved.

3. Foliations

For basic facts about foliations see [CN]. We start this section by proving the following theorem.

3.1 Theorem. Let Y be a n-dimensional orientable Riemannian manifold and F be a C^2-foliation of Y by orientable hypersurfaces. Assume that each leaf of F has the same constant mean curvature. Then each leaf of F is strongly stable.

Proof. Let L be a leaf of F. By definition of a foliation, for any point p of L, there is a neighborhood D_p of p on L and an embedding

$$\psi_p : D_p \times (-\varepsilon, \varepsilon) \to Y$$

such that

(a) for each t in $(-\varepsilon, \varepsilon)$, $\psi_p(D_p \times \{t\})$ lies in a leaf of F

(b) $\psi_p(q, 0) = q$ for each q in D_p.

If the images of two such maps, say ψ_p and ψ_q, have a common intersection, then $\psi_q^{-1} \circ \psi_p$ is of the form

(3.2) $$h(r, t) = (h_1(r, t), h_2(t)).$$

Represent by $g_p : D_p \to R$ the mapping (associated to ψ_p) defined by

(3.3) $$g_p(r) = <\frac{\partial \psi_p}{\partial t}(r, 0), N(r)>$$

where N is the unit normal field that defines the orientation of L. If g_q is the mapping associated ψ_q then, at the points of $D_p \cap D_q$, we will have

(3.4) $$g_p(r) = \frac{\partial h_2}{\partial t}(0) g_q(r).$$

Hence g_p and g_q differ by multiplication by a nonzero number on each connected component of $D_p \cap D_q$. Consider now the universal covering L' of L and the covering projection $\pi : L' \to L$. We use the projection π to pull back the metric of L and the local function g_p. Since L' is simply connected, we may use the local functions g_p to construct, via a monodromy type reasoning, a nonzero real function g defined on L', that locally will be given by

(3.5) $$g = \langle \xi , N \rangle$$

where

(3.6) $$\xi(q) = \frac{\partial}{\partial t} \psi_p(q, ct)$$

for some p on L and some nonzero constant c. If we compare this local expression of g with the definition given in (2.10) for f, we conclude that g satisfies the equation (2.8). Since we are assuming that the mean curvature, besides being constant on each leaf, does not change from leaf to leaf, then (2.8) simplifies to

(3.7) $$\Delta g + (\|B\|^2 + R)g = 0.$$

We observe that, although g is not well defined on L, the field $X = \nabla g/g$ is well defined on L. Using the equation (3.7) in the computation of $\Delta \log g$, we obtain

(3.8) $$\mathrm{div}_L X = - \|B\|^2 - R - |X|^2 .$$

This equation allows us to prove that L is strongly stable.

Let D represent a locally compact domain on L with smooth boundary and let $u: D \to R$ be any differentiable function which is zero on ∂D. Observe that

(3.9) $$|\nabla u|^2 - (\|B\|^2 + R)u^2 = |\nabla u|^2 + u^2 \mathrm{div}_L X + u^2 |X|^2$$

and that

(3.10) $$\mathrm{div}_L(u^2 X) = 2u\langle \nabla u, X\rangle + u^2 \mathrm{div}_L X.$$

Hence we obtain

(3.11) $$|\nabla u|^2 - (\|B\|^2 + R)u^2 = |\nabla u + uX|^2 + \mathrm{div}_L(u^2 X).$$

It follows that

$$\int_D \{-u\Delta u - (\|B\|^2 + R)u^2\}\, dM = \int_D |\nabla u + uX|^2\, dM \geq 0$$

where we have used twice Stokes' theorem and the fact that u is zero on ∂D. Therefore D is stable. Since D is arbitrary we conclude that L is stable.

In what follows we will represent by $Q^3(a)$ a 3-dimensional orientable complete Riemannian manifold with constant sectional curvature a.

3.12 Theorem. Let $U \subset Q^3(a)$ be an open set and F be a C^2-foliation of U whose leaves are complete orientable surfaces with the same constant mean curvature. Then

(i) $a \leq 0$.

(ii) If $a = 0$ the leaves of F are totally geodesic submanifolds of $Q^3(a)$.

(iii) If $a < 0$ and the mean curvature of the leaves is greater than or equal to $(-a)^{1/2}$, we have $H = (-a)^{1/2}$ and all the leaves flat

Proof. Let Y be the universal covering space of $Q^3(a)$, and U' be the open set of Y that covers U. Consider Y with the induced metric so that it is locally isometric to $Q^3(a)$. We may then lift the foliation F of U to a foliation F' of U' with the same properties as F. It follows from the previous theorem that each leaf of F' is strongly stable and, in particular, is stable.

Barbosa, do Carmo and Eschenburg [BCE] have shown that stable compact surfaces with constant mean curvature in Y are geodesic spheres, and they are not strongly stable. This shows that no leaf of F' can be compact.

Stable immersions of complete noncompact surfaces in Y have been studied by A. Silveira [S]. He showed that there are no such immersions when Y is the sphere $S^3(a)$, that the images of such immersion must be planes when Y is R^3 and must be horospheres when Y is the hyperbolic space and $H \geq (-a)^{1/2}$. It follows from this result that $a \leq 0$ and the leaves of F must be images of planes (when $a = 0$) or of horospheres (when $a < 0$) through the covering projection π. Since π is a local isometry the theorem is now a consequence of the properties of planes in R^3 and horospheres in the hyperbolic space.

The next theorem is a corollary of (3.12).

3.13 Theorem. (a) There is no C^2-foliation of the Euclidean sphere $S^3(a)$ by surfaces with the same constant mean curvature.

(b) The only C^2-foliation of the Euclidean space R^3 by surfaces with the same constant mean curvature is foliation by planes.

(c) The only C^2-foliation of the 3-dimensional hyperbolic space by surfaces with the same constant mean curvature $H \geq 1$ is the foliation by horospheres.

Proof. Let Y be the sphere or the Euclidean space or the hyperbolic space. Let F be a C^2-foliation of Y by surfaces. Since Y is complete, each leaf of F is complete. Since Y is simply connected and orientable, the leaves are orientable. Now this theorem follows from the proof of the previous one.

The fact that we do not need the hypothesis of completeness for the ambient manifold Y in theorem (3.1), makes this theorem useful for understanding certain properties of stability on some families of surfaces with constant mean curvature. Consider for example the case of the family of minimal surfaces in the 3-dimensional hyperbolic space invariant under the group $O(2)$ of isometries. (see [G] and [CD] for details). Each surface in this family is called a Catenoid. The asymptotic boundary of a catenoid consists of two circles in the ideal boundary S_∞ of the hyperbolic space. Moreover, it is known (see [G]) that given two circles in S_∞, there can exist at most two catenoids having those circles as asymptotic boundary.

3.14 Theorem. Given two catenoids with the same asymptotic boundary, one of them is strongly stable.

Proof: Fix a rotation axis (that is invariant under the action of O(2)). Given a positive real number d, the family F of all catenoids is divided into two families, F_d^+ and F_d^-, where the first one consists of the catenoids whose distance to the rotation axis is larger than d, and the other one contains the remaining catenoids. In [G] it is shown that there is a number $d_0 > 0$ such that $F_{d_0}^+$ foliates an open set of the hyperbolic space, namely, the complement of the locus of all catenoids whose distance to the rotation axis is less than or equal to d_0. From (3.1), each catenoid on $F_{d_0}^+$ is strongly stable. In [G], it is also shown that, if two catenoids have the same ideal boundary, then one of them belongs to $F_{d_0}^+$ and the other one to $F_{d_0}^-$. This proves the theorem.

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