Minimal Ruled Submanifolds in Spaces of Constant Curvature

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1. Let $\bar{M} = \bar{M}^N(c)$ be an N-dimensional Riemannian manifold with constant curvature c, and $M = M^{n+1}$ a submanifold of $\bar{M}$. We say that M is *ruled* if there exists a foliation of M by codimension one totally geodesic submanifolds of $\bar{M}$. We call each leaf of the foliation a *ruling* of M.

One can easily obtain an abundance of examples of ruled submanifolds just by taking, along any differentiable curve, the image of its normal bundle under the exponential map in $\bar{M}$. However, ruled minimal submanifolds are rare. In fact, the classical theorem of Catalan states that the only ruled minimal surface in Euclidean space E^3, other than the plane, is the helicoid. Blair and Vanstone [2] showed that complete ruled minimal hypersurfaces of E^{n+1}, other than hyperplanes, are Riemannian products of E^{n-2} and a helicoid in E^3. More recently, Barbosa and do Carmo [1] exhibited an example of a noncomplete ruled minimal hypersurface in E^4 and announced that, even locally, any other ruled minimal hypersurface in E^{n+1} is contained in a product of either a helicoid or their own example with a fixed Euclidean space E^{n-2} or E^{n-3}. They also constructed an example of a four-dimensional ruled minimal submanifold in E^6. We should also mention that some results and examples of ruled minimal submanifolds of E^N have already appeared in [5] and [6].

Our purpose in this paper is to describe all ruled minimal submanifolds in $\bar{M}^N(c)$. We shall observe that Lawson [4] found all ruled minimal surfaces in the sphere S^3. See also [3] for the hyperbolic space H^3.

All the irreducible examples that we mentioned above turn out to belong to a class of ruled minimal submanifolds of $\bar{M}^N(c)$ which can be described as follows: Let $\alpha = \alpha(s)$ be a curve in $\bar{M}^N(c)$ with constant curvatures and Frenet frame $E_1, \ldots, E_m$. Set $n = [m/2]$ and define the map $X: R^{n+1} \to \bar{M}^N(c)$ by

$$X(s,t_1,\ldots,t_n) = \exp_{\alpha(s)}\left(\sum_{j=1}^{n} t_j E_{2j}(s)\right).$$

We call X, or any reparametrization of X, a *helicoid* and refer to it as the helicoid associated to the curve α. In (3.3) we will show that any helicoid is a minimal submanifold. Using the fact that any curve α with constant curvatures in $\bar{M}^N(c)$ is the orbit of a one-parameter subgroup of rigid motions $A(s)$ of $\bar{M}$, one can also

Indiana University Mathematics Journal ©, Vol. 33, No. 4 (1984)

show that the helicoid associated to α is invariant by $A(s)$.

However, it is not true that any ruled minimal submanifold invariant by a one-parameter group of rigid motions is a helicoid. Consider a totally geodesic submanifold S of $\bar{M}$ orthogonal to the orbits of a one-parameter subgroup of rigid motions A of $\bar{M}$ and set $X(s,P) = A(s)P$ for the action of A at S. Then, whenever X is a minimal immersion, we call it a *generalized helicoid* in $\bar{M}$. The main purpose of this paper is to show: *Even locally, any ruled submanifold of $\bar{M}^N(c)$ is part of a generalized helicoid* (see (4.1)). A complete classification of generalized helicoids when $\bar{M}^N(c)$ is E^N, S^N or H^N is given in (3.9) and (3.10). Several other properties of generalized helicoids are also discussed. In particular, we observe that a generalized helicoid is not necessarily a regular map (cf. (3.9)). In (3.15) through (3.21) we describe the set of singular points and show that in certain cases generalized helicoids are complete and even embedded.

We would like to thank the referee for pointing out a serious gap in an earlier version of this work.

2. Preliminary results. Let $\bar{M}^N(c)$ be a space form and $M^{n+1} \subset \bar{M}$ be an $(n + 1)$-dimensional ruled submanifold of $\bar{M}$. Locally we can always assume that $\bar{M}^N(c)$ is the Euclidean sphere $S^N(c)$, the Euclidean space E^N or the hyperbolic space $H^N(c)$ according to c being positive, zero or negative. For simplicity we take $c = 0$, 1 or -1. We will consider $S^N = S^N(1)$, E^N and $H^N = H^N(1)$, respectively, as hypersurfaces of the *ambient spaces* R^{N+1}, and L^{N+1}, respectively, where L^{N+1} denotes the Lorentz $(N + 1)$-dimensional space with the canonical flat metric

$$ds^2 = -dX_0^2 + \sum_{i=1}^{N} dX_i^2. \tag{2.1}$$

We also will denote the usual derivative of the ambient space by $'$ and the connection on $\bar{M}$ by $\bar{D}$.

Let $\alpha(s)$ be an integral curve of the field of normal directions to the foliation of M which we will assume to be parametrized by arc length. Choose n orthonormal vector fields $e_1(s), \ldots, e_n(s)$ along α that generate the tangent space $T_{\alpha(s)}S_{\alpha(s)}$ of the leaf $S_{\alpha(s)}$ through $\alpha(s)$ for each s. The next lemma shows that we can choose the vector fields $e_1, \ldots, e_n$ such that each one does not move with respect to any other.

(2.2) Lemma. *Given a curve α and orthonormal vector fields $e_1, \ldots, e_n$ along α in a Riemannian manifold $\bar{M}$ with Riemannian connection $\bar{D}$, we can always choose orthonormal vector fields $f_1, \ldots, f_n$ along α such that:*

(a) *the sets of vectors $\{f_j(s) : 1 \le j \le n\}$ and $\{e_j(s) : 1 \le j \le n\}$ generate the same subspace of $T_{\alpha(s)}\bar{M}$.*

(b) *the vector field $(\bar{D}/ds) f_i(s)$ is normal to the subspace of $T_{\alpha(s)}\bar{M}$ spanned by $\{f_j(s) : 1 \le j \le n\}$ for each $1 \le i \le n$.*

Proof. Case $n = 1$ is trivial. Assume $n \ge 2$ and write

$$f_i(s) = \sum_{j=1}^{n} a_{ij}(s)\, e_j(s), \qquad 1 \le i \le n,$$

where s is the parameter of α and $a_{ij}(s)$ are functions to be determined. Condition (b) of the lemma is now equivalent to the following system of differential equations:

$$\frac{d}{ds} a_{ij} + \sum_{k=1}^{n} a_{ik}\left(\frac{D}{ds} e_k, e_j\right) = 0, \qquad 1 \le i,\ j \le n. \tag{2.3}$$

This can always be solved and the functions a_{ij} are always determined up to initial conditions. For a fixed s_0, we set

$$a_{ij}(s_0) = \delta_{ij}. \tag{2.4}$$

Now the vector fields f_j are completely determined and condition (b) implies that (f_j, f_k) are constant functions. From (2.4) we have that

$$(f_j, f_k) = \delta_{jk},$$

as we wished to prove.

From now on we assume the $\{e_j\}$ are chosen satisfying conditions (a) and (b) of Lemma (2.2). If we represent by $\langle\ ,\ \rangle$ the inner product of the ambient space, this amounts to saying that for all $1 \le i,\ j \le n$, we have that

$$\langle e_i', e_j\rangle = 0. \tag{2.5}$$

We consider the parametrization of M given by

$$X(s, t_1, \ldots, t_n) = \exp_{\alpha(s)}\left(\sum_{j=1}^{n} t_j e_j(s)\right), \tag{2.6}$$

where exp stands for the exponential map of the space form $\bar{M}$. Since $\bar{M}$ is S^N, E^N or H^N, then we know that the exponential map has the following simple expression:

$$\exp_p(rv) = f(r)p + g(r)v, \tag{2.7}$$

where $p \in \bar{M}(c)$ and $v \in T_p\bar{M}$, $|v| = 1$, are considered to be vectors in the ambient space. The functions f, g are given by

$$\begin{cases} f(r) = 1 & g(r) = r, \quad \text{if } c = 0 \\ f(r) = \cos r & g(r) = \sin r, \quad \text{if } c = 1 \\ f(r) = \cosh r & g(r) = \sinh r, \quad \text{if } c = -1 \end{cases}. \tag{2.8}$$

It is now simple to verify that the induced metric in M is given by

$$d\sigma^2 = |X_s|^2\, ds^2 + d\theta^2, \tag{2.9}$$

where X_s means derivative of X in the ambient space and $d\theta^2$ is the metric on the totally geodesic submanifold $S_{\alpha(s)}$ of $\bar{M}$. Since $d\sigma^2$ has this particularly simple form we may extend the vector fields $e_1, \ldots, e_n, \alpha'$ along α to orthonormal vector fields $f_1, \ldots, f_{n+1}$ in a neighborhood of α in M in such a way that

$$f_{n+1} = X_s/|X_s| \tag{2.10}$$

and $f_1, \ldots, f_n$ are tangent to S_α. Since S_α is totally geodesic the second fundamental form of M *restricted* to S_α is identically zero. The mean curvature vector of M then becomes:

$$H = \frac{1}{(n+1)|X_s|^2}(X_{ss})^\perp, \tag{2.11}$$

where $(\ \ )^\perp$ means projection on the normal space of M as a submanifold of $\bar{M}(c)$.

Assume $c \neq 0$. Since X and X_{t_i}, $1 \leq i \leq n$, are given by a linear combination of α and e_j, $1 \leq j \leq n$, it follows that the subspace of the ambient space spanned by $X, X_s, X_{t_1}, \ldots, X_{t_n}$ is the same as the one spanned by $\alpha, X_s, e_1, \ldots, e_n$. Then, if $c \neq 0$, we obtain

$$(n+1)|X_s|^2 \alpha \wedge e_1 \wedge \ldots \wedge e_n \wedge X_s \wedge H = \alpha \wedge e_1 \wedge \ldots \wedge e_n \wedge X_s \wedge X_{ss}. \tag{2.12}$$

If $c = 0$, the same argument shows that

$$(n+1)|X_s|^2 e_1 \wedge \ldots \wedge e_n \wedge X_s \wedge H = e_1 \wedge \ldots \wedge e_n \wedge X_s \wedge X_{ss}. \tag{2.13}$$

It follows that:

$$\begin{aligned} (n+1)|X_s|^3\,|H| &= |\alpha \wedge e_1 \ldots \wedge e_n \wedge X_s \wedge X_{ss}|, \quad \text{if } c \neq 0 \\ (n+1)|X_s|^3\,|H| &= |e_1 \wedge \ldots \wedge e_n \wedge X_s \wedge X_{ss}|, \text{ if } c = 0. \end{aligned} \tag{2.14}$$

As a simple application of this formula consider the case when $c = 0$ and $|H| = h(s) \neq 0$, that is, $|H|$ is a non-zero constant along the rulings. Squaring equation (2.14) and considering $t = t_1$, $t_2 = \ldots = t_n = 0$, one reduces (2.14) to an equation of the type

$$(n+1)^2 h^2 \left(\sum_{k=0}^{2} a_k t^k\right)^3 = \sum_{j=0}^{4} A_j t^j,$$

where $a_2 = |e_1'|^2$ and A_j, a_k do not depend on t. It follows that $|e_1'| = 0$ and so $e_1' = 0$. The same kind of argument shows that $|e_i'| = 0$, $1 \leq i \leq n$. Therefore the n-plane spanned by $e_1, e_2, \ldots, e_n$ does not depend on s and hence it is constant along the curve α. Thus X splits as a product:

$$X: U \times I \subset R^n \times R \to E^n \times E^{N-n}$$

$$(p,s) \to (p, \alpha(s))$$

and so M is a piece of a generalized cylinder. We just proved:

(2.15) Theorem. *If M is a ruled submanifold of E^N whose mean curvature is a non-zero constant along the rulings then M is a piece of generalized cylinder.*

3. Helicoids. Let $\alpha : R \to \bar{M}^N(c)$ be a curve with constant curvatures parametrized by arc length. Denote by $E_1, \ldots, E_m$ the Frenet frame associated to the curve α. This means

$$E_1 = \frac{d\alpha}{ds} \tag{3.1}$$

and

$$\frac{\bar{D}}{ds} E_j = -K_{j-1} E_{j-1} + K_j E_{j+1} \qquad 1 \leq j \leq m,$$

where $K_1, \ldots, K_{m-1}$ are the curvatures of α and $K_0 = K_m = 0$. Set $n = [m/2]$. The map $X : R^{n+1} \to \bar{M}^N(c)$ given by

$$X(s, t_1, \ldots, t_n) = \exp_{\alpha(s)} \left(\sum_{j=1}^{n} t_j E_{2j}(s) \right), \tag{3.2}$$

defines, wherever it is regular, a ruled immersion on $\bar{M}$. We call X, or any reparametrization of X, a *helicoid* and refer to X as the helicoid associated to the curve α.

(3.3) Proposition. *The helicoid X associated to a curve $\alpha : R \to \bar{M}^N(c)$, wherever it is regular, describes a minimal immersion in $\bar{M}$.*

To prove the minimallity of X we start with a neighborhood of a regular point. Locally we may assume that $\bar{M}^N(c)$ is S^N, E^N or H^N considered respectively as hypersurfaces of E^{N+1} or L^{N+1}, as we did in Section 2. Making use of (2.7) we obtain that $(\bar{D}X_s/ds)$, as a vector in the ambient space, is a linear combination of $\bar{D}E_1/ds$ and $\bar{D}^2E_{2i}/ds^2$, $1 \leq i \leq n$. Now, using (3.1) plus the fact that K_i, $0 \leq i \leq n$, are constant, it comes out that

$$\frac{\bar{D}^2 E_{2i}}{ds^2} = K_{2i-2}K_{2i-1}E_{2i-2} - (K_{2i-1}^2 + K_{2i}^2)E_{2i} + K_{2i}K_{2i+1}E_{2i+2}. \tag{3.4}$$

Therefore $\bar{D}X_s/ds$ is a linear combination of E_{2i}, $1 \leq i \leq n$ and from (2.14) it follows that the mean curvature H of X vanishes.

The following proposition is certainly well known, but we could not find it in the literature. So we state it and give a sketch of the proof.

(3.5) Proposition. *Let $\bar{M}^N(c)$ represent E^N, S^N or H^N and let α be a regular curve defined on R with values in $\bar{M}$. The curve α has constant curvatures if and only if there is a one-parameter subgroup of rigid motions $A(s)$ of $\bar{M}$ such that $A(s)\alpha(t) = \alpha(s + t)$ for all $s, t \in R$.*

It is a simple consequence of this that $A(s)$, considered as a one-parameter subgroup of rigid motions of the ambient space, preserves the Frenet frame of α in the sense that $A(s)E_j(t) = E_j(s + t)$ for all $s, t \in R$.

We sketch the proof for the case of $S^N \subset E^{N+1}$. Given $s \in R$ and $\alpha : R \to S^N$

a curve with constant curvatures parametrized by arc length, we define a linear map $A(s): E^{N+1} \to E^{N+1}$ by $A(s)\alpha(0) = \alpha(s)$ and $A(s)E_j(0) = E_j(s)$. As before, $E_1, \ldots, E_N$ stands for the Frenet frame of α in S^N. Since $\alpha, E_1, \ldots, E_N$ are orthonormal then $A(s)$ defines a linear isometry of R^{N+1}. It follows that $A(s)$, restricted to S^N, is also an isometry. Now we observe that

$$\{A(s)\alpha(t), A(s)E_1(t), \ldots, A(s)E_N(t)\}$$

and

$$\{\alpha(s + t), E_1(s + t), \ldots, E_N(s + t)\}$$

are both solutions for the linear system

$$\psi_0' = \psi_1$$

$$\frac{\bar{D}\psi_j}{dt} = -K_{j-1}\psi_{j-1} + K_j\psi_{j+1}.$$

Since both solutions agree for $t = 0$, then they agree everywhere. Therefore $A(s)\alpha(t) = \alpha(s + t)$ and $A(s)E_j(t) = E_j(s + t)$. Now it is easy to show that $G = \{A(s), s \in R\}$ is a one-parameter subgroup of $O(N + 1; R)$.

The proof of the converse is quite simple and can be done by taking derivatives with respect to t on both sides of the equation $A(s)\alpha(t) = \alpha(s + t)$.

The proof when $\bar{M}^N$ is H^N is exactly the same changing only R^{n+1} to L^{n+1} and $O(n + 1; R)$ to $O(n, 1; R)$. The case where $\bar{M}^N$ is E^N is treated similarly. In this case we obtain $\alpha(s)$ by integration of $E_1(t)$. If N is odd, then $A(s)$ keeps a line invariant and we need to compose $A(s)$ with the translations in the direction of this line. Therefore we have the following result.

(3.6) Proposition. *Let $\bar{M}^N(c)$ represent E^N, S^N or H^N, and let X be a helicoid associated to a curve α in $\bar{M}$. Then there is a one-parameter subgroup $A(s)$ of rigid motions of $\bar{M}$ such that*

$$A(s)X(t, t_1, \ldots, t_n) = X(s + t, t_1, \ldots, t_n)$$

for all $s, t, t_1, \ldots, t_n \in R$.

It is not true that any ruled minimal immersion invariant under a one-parameter subgroup $A(s)$ of rigid motions of $\bar{M}$ is a helicoid. A simple example can be obtained as follows. Consider the action of a one-parameter subgroup of $O(4; R)$ in the two-plane of R^4 spanned by (1,0,0,0) and (0,0,1,0), given by

(3.7) $$X(s, t_1, t_2) = (t_1 \cos s, t_1 \sin s, t_2 \cos s, t_2 \sin s).$$

For $(t_1, t_2) \neq (0,0)$ this clearly describes a ruled immersion that one can easily show to be minimal. By fixing t_1 and t_2 we obtain plane curves which are normal to the leaves of X. We conclude that X cannot be a helicoid otherwise it would be contained in a three-dimensional subspace of R^4 and this is not the case.

Let $A(s)$ be a one-parameter subgroup of rigid motions of $\bar{M}^N(c)$ and S an n-dimensional complete totally geodesic submanifold of $\bar{M}$ orthogonal to the orbits of $A(s)$. Then the map $X : R \times S \to \bar{M}$ defined by $X(s,p) = A(s)p$, wherever it is regular, describes a ruled immersion. These maps that are minimal wherever they are regular will be called *generalized helicoids*.

From (3.6) we have that helicoids are particular examples of generalized helicoids. The next proposition gives an idea of how generalized helicoids look.

(3.8) Proposition. *Let $A(s)$ be a one-parameter subgroup of rigid motions of $\bar{M}^N(c)$ and $S \subset \bar{M}$ be a complete n-dimensional totally geodesic submanifold of $\bar{M}$ orthogonal to the orbits of $A(s)$. The map $X : R \times S \to \bar{M}$ defined by $X(s,p) = A(s)p$ is a generalized helicoid if and only if the covariant derivative $\bar{D}\alpha'/ds$ of the velocity vector of $\alpha(s) = X(s,p)$ lies in the tangent of $A(s)S$ at $X(s,p)$ for each $s \in R$ and $p \in S$.*

Proof. It is sufficient to consider the case when $\bar{M}^N(c)$ is E^N, S^N or H^N and S is E^n, S^n or H^n respectively. If $p \in S$, then the curve $\alpha(s) = A(s)p$ is, by hypothesis, orthogonal to S. If α is not constant then it is parametrized by a multiple of arc length and is orthogonal to all leaves $A(s)S$. Then the mean curvature vector H of X is given by (2.11). Now observe that

$$(X_{ss})^{\perp} = \left(\frac{\bar{D}X_s}{ds}\right)^{\perp},$$

hence, $H \equiv 0$ if and only if $\bar{D}X_s/ds$ is tangent to the immersion. This occurs if and only if the curve $\alpha(s) = A(s)p$, considered as a curve in $\bar{M}$, verifies that $(\bar{D}/ds)\alpha'$ is tangent to the leaves $A(s)S$.

We now extend the definition of generalized helicoids to include the case in which $\bar{M}^N(c)$ is the Lorentz space L^N and S is a Lorentzian totally geodesic submanifold of L^N. In this case (3.8) holds.

(3.9) Proposition. *A generalized helicoid $X : R \times S^n \to S^N$ is the restriction to $R \times S^n$ of a generalized helicoid $\tilde{X} : R \times E^{n+1} \to E^{N+1}$. A generalized helicoid $X : R \times H^n \to H^N$ is the restriction to $R \times H^N$ of a generalized helicoid $\tilde{X} : R \times L^{n+1} \to L^{N+1}$. In both cases $\tilde{X}(s,0) \equiv 0$. The converse is true for generalized helicoids that satisfy this additional property.*

Proof. First of all observe that, if $A(s)$ is a one-parameter subgroup of rigid motions of S^N then, $A(s)$ is a one-parameter subgroup of rigid motions of E^{N+1} that leaves S^N invariant. Now if S is an n-dimensional totally geodesic submanifold of S^N then S is the intersection of an $(n+1)$-dimensional subspace V of R^{n+1} with S^N. Furthermore, if S is orthogonal to the orbits of $A(s)$ then V also has the same property. Consider the map $X : R \times S \to S^N$, defined by $X(s,p) = A(s)p$ and let $\tilde{X} : R \times V \to E^{N+1}$ be defined by $\tilde{X}(s,q) = A(s)q$. It is clear that X is the restriction of $\tilde{X}$ to $R \times S$ and the $\tilde{X}(s,0) = 0$. Since we have, for $q \neq 0$, that

$$\tilde{X}(s,q) = |q|\, X(s,q/|q|)$$

then

$$\tilde{X}_s(s,q) = |q|\, X_s(s,q/|q|) \quad \text{and} \quad \tilde{X}_{ss}(s,q) = |q|\, X_{ss}(s,q/|q|).$$

It follows from (2.9) that $\tilde{X}$ will be singular at q if and only if X is singular at $q/|q|$. At regular points we have from (2.11) that the mean curvatures H of X and $\tilde{H}$ of $\tilde{X}$ are related by $\tilde{H}(s,q) = H(s,q/|q|)/|q|$. Hence X is minimal if and only if $\tilde{X}$ is minimal. The second part of the proposition is proved similarly.

From the above proposition it follows that to classify all generalized helicoids of E^N, S^N or H^N it is sufficient to classify just the ones in E^{N+1} and L^{N+1}.

(3.10) Theorem. *Let X be an $(n + 1)$-dimensional generalized helicoid in E^{N+1} or L^{N+1}. Then, up to rigid motion of the ambient space, we have the following expressions:*

(a) $X(s,t_1,\ldots,t_n) = \sum_{i=1}^{k} t_i e_i(s) + \sum_{i=1}^{n-k} t_{k+i} V_{2k+i} + sbV_{n+k+1}$ *for generalized helicoids X in E^{N+1} and*

(b) $X(s,t_1,\ldots,t_n) = \sum_{i=1}^{k} t_i f_i(s) + \sum_{i=1}^{n-k} t_{k+i} V_{2K+i} + sbV_{n+k+1}$,

or

$$X(s,t_1,\ldots,t_n) = \sum_{i=1}^{n-k} t_i V_i + \sum_{i=1}^{k} t_{n-k+i} f_{n-k+i} + sbV_{n+k+1}$$

for generalized helicoids X in L^{N+1}. Here the vectors $V_1, \ldots, V_{N+1}$ are any special orthonormal bases of E^{N+1} or L^{N+1} respectively and in this least case $\langle V_1,V_1\rangle = -1$. The fields $e_i(s)$ and $f_i(s)$ are defined by

$$e_i(s) = \cos a_i s V_{2i-1} + \sin a_i s V_{2i}, \qquad I = 1, \ldots, n,$$

$$f_1(s) = \cosh a_1 s V_1 + \sinh a_1 s V_2,$$

$$f_i(s) = e_i(s), \qquad i = 2, 3, \ldots, n,$$

where $a_1, \ldots, a_n, b$, are real numbers and $s, t_1, \ldots, t_n \in R$.

Proof. Let $A(s)$ be a one-parameter subgroup of rigid motions of E^{N+1}, $S \subset E^{N+1}$, be an n-plane orthogonal to the orbits of $A(s)$ and $X : R \times S \to E^{N+1}$, defined by $X(s,p) = A(s)p$, be a generalized helicoid.

Take $p \in S$ and $e_1, \ldots, e_n \in T_pS$. Define $\alpha(s) = A(s)p$ and $e_i(s) = A(s)e_i$, $1 \le i \le n$. Then S is parametrized by

$$q = p + \sum_{j=1}^{n} t_j e_j, \qquad t_j \in R,$$

and

$$X(s,q) = \alpha(s) + \sum_{j=1}^{n} t_j e_j(s).$$

Since for each fixed q the curve $A(s)q$ is orthogonal to S, we obtain that $\alpha'(0)$, $e_1'(0), \ldots, e_n'(0)$ are normal to S and consequently $\alpha'(s)$, $e_1'(s), \ldots, e_n'(s)$ are normal to $A(s)S$.

From (3.8) we know that $\alpha''(s)$, $e_1''(s), \ldots, e_n''(s)$ belong to $T_{\alpha(s)}(A(s)S)$ for each value of s. It follows that $\langle e_i''(s), e_j'(s)\rangle = 0$, $1 \le i, j \le n$, and so $\langle e_i'(s), e_j'(s)\rangle$ are constant functions.

Observe that the subspace generated by $\alpha(s)$, $e_j(s)$ and $e_j'(s)$ is independent of

s and, by simplicity, we can assume that its dimension is $N + 1$.

Set $a_{ij} = \langle e_i''(s), e_j(s)\rangle = -\langle e_i'(s), e_j'(s)\rangle$. It is now clear that (a_{ij}) is a constant symmetric matrix. Hence, the second derivative $('')$ is a self-adjoint linear map from T_pS into T_pS. Therefore there exists an orthonormal basis of T_pS relative to which the matrix of $('')$ is diagonal. We still represent such a basis by $e_1, \ldots, e_n$. For such a choice we obtain $e_i''(s) = \lambda_i e_i(s)$ with $\lambda_i = -\langle e_i'(s), e_i'(s)\rangle$. Hence, for each i, $1 \le i \le n$, we have two possibilities:

(3.11)(a) *$e_i(s)$ is a constant function,*
or
(3.11)(b) *$e_i(s)$ describes a circle in a two-dimensional subspace V_i of E^{N+1}. In this case V_i is an invariant subspace of $A(s)$.*

For the case of L^{N+1}, the same kind of argument can be applied to obtain:

(3.11)′(a) *$e_i(s)$ is a constant function,*
or
(3.11)′(b) *$e_i(s)$ is either a circle in a two-dimensional Riemannian subspace V_i of L^{N+1}, or a circle (H^1) in a Lorentz two-dimensional subspace V_i of L^{N+1}. In any case V_i is an invariant subspace of $A(s)$.*

Since $A(s)$ is a one-parameter subgroup of rigid motions of R^N or L^N, then we know that the invariant subspaces of $A(s)$ do not depend on s and have dimension one or two. It is a fact that $A(s)$ decomposes as a product of simple one-parameter subgroups in the following way

$$A(s) = T(s) \circ \ldots \circ A_m(s), \tag{3.12}$$

where $T(s)$ is a translation subgroup (or the identity) of a one-dimensional subspace V_0 and $A_j(s)$ is a one-parameter subgroup of "rotations" of a two-dimensional subspace V_j, $1 \le j \le m$. The subspaces V_j are mutually orthogonal and with respect to an orthonormal base of each V_j, $0 \le j \le m$, $A_j(s)$ has one of the following forms:

$$\text{a)} \begin{pmatrix} \cos a_j s & -\sin a_j s \\ \sin a_j s & \cos a_j s \end{pmatrix} \qquad \text{b)} \begin{pmatrix} \cosh a_j s & \sinh a_j s \\ \sinh a_j s & \cosh a_j s \end{pmatrix}, \tag{3.13}$$

where the a_j are real numbers. Case (a) occurs when V_j is metrically E^2 and case (b) when V_j is metrically L^2. We may then decompose R^{N+1} or L^{N+1} as a direct sum

$$V_0 \oplus V_1 \oplus \ldots \oplus V_m \oplus V_{m+1}, \tag{3.14}$$

where V_{m+1} is a subspace kept pointwise invariant by all $A_j(s)$, $0 \le j \le m$, and by $T(s)$.

The last ingredient we need to complete our proof is the observation that, from the beginning, we may have chosen the point p in such way that $\alpha(s) = A(s)p$ is either a point or a line. To prove that, we use the fact that for each value of $s, \alpha''(s)$ belongs to $T_{\alpha(s)}(A(s)S)$ and therefore $\alpha'' = \sum_{j=1}^n a_j e_j$. Observe that

$\langle \alpha'', e_j \rangle = -\langle \alpha', e_j' \rangle$ are constant functions and remember that $e_i''(s) = \lambda_i e_i(s)$ with $\lambda_i = -\langle e_i'(s), e_i'(s) \rangle$. Assume $\lambda_i \neq 0$ for $1 \leq i \leq k$, and $\lambda_i = 0$ for $i > k$. Then we may write

$$\alpha'' = \sum_{j=1}^{k} a_j e_j = \sum_{j=1}^{k} \frac{a_j}{\lambda_j} e_j''.$$

Hence, if $q = p + \sum_{j=1}^{k} t_j e_j$ where $t_j = -a_j/\lambda_j$, $1 \leq j \leq k$, then the curve $\beta(s) = A(s)q$ satisfies $\beta''(s) \equiv 0$. This fact will be extended in this section to all generalized helicoids.

We now choose $p \in S$ from the beginning such that $\alpha(s) = A(s)p$ is either a point or a line. From (3.11) and (3.11)′ it follows that we may renumber the e_j in such way that $e_j \in V_j$, $1 \leq j \leq k$, and $e_{k+1}, \ldots, e_p \in V_{p+1}$, where k and p are two integers such that $k \leq m$, $k + p = n$ and $p \leq \dim V_{m+1}$. By changing the basis in E^{N+1} or L^{N+1} we obtain the desired result.

A natural question to ask at this point is: how large is the set of points where a generalized helicoid is singular? To answer this question it is convenient to say that a generalized helicoid of E^N described by (13.10) is of type (n,k).

(3.15) Proposition. *Let X be a generalized helicoid of type (n,k) of E^N generated by a one-parameter subgroup $A(s)$ of rigid motions of E^N. The canonical form of $A(s)$ given by (3.12) includes translations if and only if X is regular everywhere. If X is singular, then the set of singular points is the $(n - k + 1)$-plane defined by $t_1 = \ldots = t_k = 0$.*

A similar proposition can be proved for L^N. In that case it is convenient to say that the first generalized helicoid described in (3.10)(b) is of type $(n,k,-1)$ and the second of type $(n,k,1)$.

(3.16) Proposition. *Let X be a generalized helicoid of L^N generated by a one-parameter subgroup $A(s)$ of rigid motions of L^N. If the canonical form of $A(s)$ given by (3.12) includes translations or if X is of type $(n,k,-1)$ then X is everywhere regular. If X is singular, then the set of singular points is the $(n - k + 1)$-plane defined by $t_1 = \ldots = t_k = 0$.*

If X is a generalized helicoid of S^N or H^N, we may associate with X to a generalized helicoid $\tilde{X}$ as was done in (3.9). In the case of S^N, we will say that X is of type (n,k) if $\tilde{X}$ is of type $(n + 1, k + 1)$. For the case of H^N we will say that X is of type $(n,k,-1)$ or $(n,k,1)$ if $\tilde{X}$ is, respectively, of type $(n + 1, k + 1, -1)$ or $(n + 1, k + 1, 1)$.

(3.17) Proposition. *Let X be a generalized helicoid of H^N. If $k = n$ or if X is of type $(n,k,-1)$ then X is everywhere regular. If X is of type $(n,k,1)$ and $k < n$ then X is singular on a totally geodesic H^{n-k-1} of each leaf defined by $t_1 = \ldots = t_{k+1} = 0$.*

(3.18) Proposition. *Let X be a generalized helicoid of S^N of type (n,k). If $k = n$ then X is everywhere regular. If $k < n$, then X is singular on a totally geodesic S^{n-k-1} of each leaf, defined by $t_1 = \ldots = t_{k+1} = 0$.*

On (3.17) H^0 means just one point of H^N. On (3.18) S^0 means the set of two antipodal points of S^N.

To prove (3.15), (3.16), (3.17) and (3.18) we must observe that, according to (2.9), X is regular at a point if and only if $|X_s| \neq 0$ in that point. If X is a helicoid of type (n,k) of R^N then

$$X_s = \sum_{j=1}^{k} t_j e_j'(s) + bV_{n+k+1}.$$

Hence, if $b \neq 0$ ($A(s)$ contains translation) then X is everywhere regular. If $b = 0$ then $t_1 = \ldots = t_k = 0$ is the solution of $X_s(s,t_1,\ldots,t_n) = 0$. This proves (3.15). The other propositions can be proved similarly.

(3.19) Proposition. *Let X be a generalized helicoid of $\bar{M}^N(c)$. If X is everywhere regular then the metric induced on its domain* ($R \times E^n, R \times S^n$ or $R \times H^n$), *is complete.*

Proof. It is sufficient to prove this when $\bar{M}^N(c)$ is R^N, S^N or H^N. The metric induced by X in $R \times \bar{M}^n(c)$ is $d\sigma^2 = |X_s|^2\, ds^2 + d\theta^2$, where $d\theta^2$ is the standard metric of $\bar{M}^n(c)$. Since X is everywhere regular then $|X_s|^2$ is a positive differentiable function defined on $R \times \bar{M}^n(c)$. It is easy to see that $|X_s|^2$ depends on the variable $q \in \bar{M}^n(c)$ but not on s. Let $\gamma(u) = (s(u),q(u))$ be a curve in $R \times \bar{M}^n(c)$. If $q(u)$ is itself a divergent path on $R \times \bar{M}^n(c)$ then

$$\int_\gamma d\sigma \geq \int_q d\theta = \infty.$$

If $q(u)$ is not a divergent path, then $q([a,\infty))$ is bounded. We may then find $\varepsilon > 0$ such that

$$|X_s(s(u),q(u))|^2 \geq \varepsilon \qquad \text{on } \gamma([a,\infty)).$$

Since γ is divergent and $q([a,\infty))$ is bounded, then $\lim_{u\to\infty} s(u) = \infty$. It follows that

$$\int_\gamma d\sigma > \int_{s(a)}^{\infty} \varepsilon\, ds = \infty.$$

This shows that the length of any divergent path on $R \times \bar{M}^n(c)$, endowed with the metric $d\sigma^2$ is infinite and therefore $(R \times \bar{M}^n(c), d\sigma^2)$ is complete.

It is well known that the helicoid of R^3 has a line as its striction curve. Next proposition shows this to be a general fact for generalized helicoids without singular points. One also may consider that a generalized helicoid with a singular point has this point as a striction curve.

(3.20) Proposition. *Let $\bar{M}^N(c)$ represent E^N, S^N or H^N and let $X : \bar{M}^{n+1}(s) \to \bar{M}^N(c)$ be an everywhere regular generalized helicoid. Then, at least one of the curves $\beta(s) = X(s,q)$ is a geodesic of $\bar{M}^N(c)$.*

Proof. If $X : R^{n+1} \to E^N$ and X is everywhere regular, then

$$X(s,t_1,\ldots,t_n) = \tilde{A}(s)\left(\sum_{j=1}^{n} t_j e_j\right) + bse_0, \qquad b \neq 0$$

where $\tilde{A}(s)$ is a one-parameter subgroup of rigid motions of the subspace of E^N orthogonal to e_0. Hence

$$X_{ss} = \sum_{j=1}^{n} t_j \lambda_j A(s) e_j$$

and the equation $X_{ss} = 0$ gives us $t_1 = \ldots = t_n = 0$. Therefore $\beta(s) = X(s,0,\ldots,0)$ is a geodesic of E^N.

When $c > 0$, it is sufficient to consider the case $X: R \times S^{n-1} \to S^N$. If X is everywhere regular then

$$X(s,t_1,\ldots,t_n) = A(s) \sum_{j=1}^{n} t_j e_j,$$

where $\sum t_j^2 = 1$ and $A(s)$ moves each e_j. We then obtain

$$X_{ss} = \sum t_j \lambda_j A(s) e_j,$$

where $\lambda_1, \ldots, \lambda_n$ are nonzero constants. It is simple to verify that $(\bar{D}/ds)X_s = X_{ss} - \langle X_{ss}, X\rangle X$ and that $(\bar{D}/ds)X_s = 0$ is now equivalent to the linear system.

$$t_j \lambda_j = \left(\sum_{k=1}^{n} t_k^2 \lambda_k\right) t_j \qquad 1 \leq j \leq n.$$

Since $\sum t_j^2 = 1$, this system has at least the following solutions for

$$(t_1,\ldots,t_n): (1,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,\ldots,0,1).$$

By taking q to be any one of such solutions we obtain that $\beta(s) = X(s,q)$ is a geodesic of S^N.

When $c < 0$, it is sufficient to consider the case $X: R \times H^{k+p-1} \to H^N$. If X is everywhere regular then we must have

$$X(s,t_1,\ldots,t_{k+p}) = \sum_{j=1}^{k} t_j A(s) e_j + \sum_{j=k+1}^{k+p} t_j e_j,$$

where $e_1, \ldots, e_{k+p}$ are orthonormal in L^N and $\langle e_{i_0}, e_{i_0}\rangle = -1$ for some i_0, $1 \leq i_0 \leq k$. Assuming $i_0 = 0$, we obtain $X_{ss} = \sum_{j=1}^{n} t_j \lambda_j A(s) e_j$ where $\lambda_1, \ldots, \lambda_k$ are nonzero constants. It is simple to verify that $(\bar{D}/ds)X_s = X_{ss} + \langle X_{ss}, X\rangle X$ and that $(\bar{D}/ds)X_s = 0$ is now equivalent to the linear system

$$t_i \lambda_i + \left(-t_1^2 \lambda_1 + \sum_{j=2}^{k} t_j^2 \lambda_j\right) t_i = 0 \qquad 1 \leq i \leq k + p.$$

Then $t_1 = 1$, $t_2 = \ldots = t_{k+p} = 0$ is a solution for this system and hence $\beta(s) = X(s,1,0,\ldots,0)$ is a geodesic on H^N.

(3.21) ***Proposition.*** *Let $X : R \times E^n \to E^N$ or $X : R \times H^n \to H^N$ be an everywhere regular generalized helicoid. Then X is an embedding.*

Proof. We will show that X is 1-1 and proper. Let $\beta(s) = X(s,q_0)$ be the geodesic of E^N or H^N whose existence was assured in Proposition (3.20). This geodesic is normal to all leaves of X. Two such leaves cannot intersect otherwise we would have a geodesic triangle with two right angles on E^N or H^N. Hence X is 1-1.

Let d(,) represent the intrinsic distance on $X(R \times H^n)$, and $\bar{\mathrm{d}}$(,) be the distance on E^N or H^N. Let $A = \beta(s_0)$, $B = \beta(s_1)$ and $C = X(s_1,q)$, $q \neq q_0$. For the geodesic triangle ABC (with right angle on B) on E^N or H^N, we have $\bar{\mathrm{d}}(A,C) \geq \bar{\mathrm{d}}(A,B)$, $\bar{\mathrm{d}}(A,C) \geq \bar{\mathrm{d}}(B,C)$ and then it follows that $\mathrm{d}(A,C) \leq \mathrm{d}(A,B) + \mathrm{d}(B,C) = \bar{\mathrm{d}}(A,B) + \bar{\mathrm{d}}(B,C) \leq 2\bar{\mathrm{d}}(A,C)$.This is sufficient to show that X is proper.

A full classification of generalized helicoids is directly obtained from (3.10). In the next section we will show that generalized helicoids are the most general examples of minimal ruled submanifolds of space forms.

4. The main result.

In this section we prove the following result.

(4.1) ***Theorem.*** *Let M^{n+1} be a minimal ruled submanifold of $\bar{M}^N(c)$, where $\bar{M}^N(c)$ is E^N, S^N or H^N. Then there is a generalized helicoid $X : R \times \bar{M}^n(c) \to \bar{M}^N(c)$ and an open set $U \subset R \times \bar{M}^n(c)$ such that X restricted to U parametrizes M.*

Proof. As we did in Section 2, we parametrize M by a map X described in (2.6) where the domain of X can be taken to be an open set of the product $R \times \bar{M}^n(c)$.

Let us assume $c \neq 0$. From (2.16) we see that X is minimal if and only if

$$\alpha \wedge e_1 \wedge \ldots \wedge e_n \wedge X_s \wedge X_{ss} = 0, \tag{4.2}$$

where

$$X(s,t_1,\ldots,t_n) = f(r)\alpha(s) + \frac{g(r)}{r}\sum_{j=1}^{n} t_j e_j(s)$$

for $r = \left(\sum_{j=1}^{n} t_j^2\right)^{1/2}$ and f and g defined as on (2.8). When $t_1 = \ldots = t_n = 0$ this reduces to $X(s,0,\ldots,0) = \alpha(s)$ and (4.2) becomes

$$\alpha \wedge e_1 \wedge \ldots \wedge e_n \wedge \alpha' \wedge \alpha'' = 0.$$

Since α, α', e_1, $\ldots$, e_n are orthonormal vector fields in the ambient space and α'' is normal to α', this implies that α'' belongs to the subspace generated by α, e_1, $\ldots$, e_n. Consequently

$$\frac{\bar{D}\alpha'}{ds} \in T_\alpha S_\alpha.$$

Since $X_{ss} = f(r)\,\alpha'' + (g(r)/r)\sum_{j=1}^{n} t_j e_j''$ then (4.2) implies, for $r \neq 0$, that

$$\alpha \wedge e_1 \wedge \ldots \wedge e_n \wedge X_s \wedge \sum_{j=1}^{n} t_j e_j'' = 0$$

or equivalently

$$f(r)\sum_{j=1}^{n} \frac{t_j}{r}\,\alpha \wedge e_1 \wedge \ldots \wedge e_n \wedge \alpha' \wedge e_j''$$

$$+ g(r)\sum_{i,j=1}^{n} \frac{t_i t_j}{r^2}\,\alpha \wedge e_1 \wedge \ldots \wedge e_n \wedge e_i' \wedge e_h'' = 0.$$

By setting $\xi_i = t_i/r$, $a_i = \alpha \wedge e_1 \wedge \ldots \wedge e_n \wedge \alpha' \wedge e_i''$ and $b_{ij} = \alpha \wedge e_1 \wedge \ldots \wedge e_n \wedge e_i \wedge e_j''$ we may rewrite this equation as

$$f(r)\sum_{j=1}^{n} \xi_i a_i + g(r)\sum_{i \le j} \xi_i \xi_j (b_{ij} + b_{ji}) = 0.$$

Because $f(r)$ and $g(r)$ are independent functions it follows that

$$\sum_{j=1}^{n} \xi_i a_i = 0 \quad \text{and} \quad \sum_{i \le j} \xi_i \xi_j (b_{ij} + b_{ji}) = 0$$

for $(\xi_1, \ldots, \xi_n)$, a variable point of $S^{n-1} \subset R^n$. By choosing special values for $(\xi_1, \ldots, \xi_n)$ we may conclude that $a_i = 0$ and $b_{ij} + b_{ji} = 0$, $1 \le i, j \le n$. That is,

(a) $$\alpha \wedge e_1 \wedge \ldots \wedge e_n \wedge \alpha' \wedge e_j'' = 0, \qquad 1 \le j \le n$$

and

(b) $$\alpha \wedge e_1 \wedge \ldots \wedge e_n \wedge e_i' \wedge e_j'' + e_1 \wedge \ldots \wedge e_n \wedge e_j' \wedge e_i'' = 0 \qquad 1 \le i, j \le n.$$

In particular, if $i = j$ in equation (b), one obtains

(b′) $$\alpha \wedge e_1 \wedge \ldots e_n \wedge e_j' \wedge e_j'' = 0 \qquad 1 \le j \le n.$$

Hence e_j'' belongs to the intersection of the $(n+2)$-planes generated by $\alpha, e_1, \ldots, e_n, \alpha'$ and $\alpha, e_1, \ldots, e_n, e_j'$. Assume there exists an index i such that $\langle e_i'', \alpha' \rangle \neq 0$. Then α' is parallel to e_i' and consequently, by (a)

$$\alpha \wedge e_1 \wedge \ldots \wedge e_n \wedge e_i' \wedge e_j'' = 0 \qquad 1 \le j \le n.$$

Now from (b) we obtain

$$\alpha \wedge e_1 \wedge \ldots \wedge e_n \wedge e_j' \wedge e_i'' = 0 \qquad 1 \le j \le n.$$

Therefore e_j' is parallel to α' for all j, $1 \le j \le n$. Consequently either α' is parallel to e_i' for all i, $1 \le i \le n$, or e_j'' belongs to the subspace spanned by $\alpha, e_1, \ldots, e_n$. If the first possibility occurs the $(n+2)$-plane generated by $\alpha, e_1, \ldots, e_n, \alpha'$ is a fixed $(n + 2)$-plane of the ambient space. Since this space is the same as the one generated by $X, X_{t_1}, \ldots, X_{t_n}, X_s$, it follows that the normal space of X in $\bar{M}$ is constant in the ambient space and therefore, X describes a totally geodesic immersion in $\bar{M}$. The same kind of argument can be carried out when $c = 0$. We summarize the results obtained so far in the following statement:

(4.3) *Under the hypothesis of (4.2) either M is totally geodesic or $\bar{D}\alpha'/ds$ and De_i'/ds, $1 \leq i \leq n$, belong to $T_\alpha S_\alpha$.*

One shall observe that e_i', $1 \leq i \leq n$, are tangent vectors to X and so it makes sense to write $\bar{D}e_i'/ds$. This is clear when $c = 0$. If $c \neq 0$, since $\langle\alpha,e_i\rangle = 0$ and $\langle\alpha',e_i\rangle = 0$ then $\langle\alpha,e_i'\rangle = \langle\alpha,e_i\rangle' - \langle\alpha',e_i\rangle = 0$ and so e_i' is perpendicular to α and hence belongs to $T_\alpha\bar{M}$.

If X is a totally geodesic immersion then it is a generalized helicoid and the remainder of the theorem is trivially true. Thus we assume that X is not totally geodesic. Under this hypothesis we will prove the following claim:

(4.4) *The curve α has constant curvatures.*

We start by observing that since $\bar{D}\alpha'/ds$ and De_i'/ds belong to $T_\alpha S_\alpha$ then

$$\langle e_i',e_j'\rangle \quad \text{and} \quad \langle\alpha',e_j'\rangle \quad \text{are constant along } \alpha. \tag{4.5}$$

Let $E_1, E_2, \ldots, E_m$ be the Frenet frame associated to the curve α. Then $E_1 = \alpha'$ and we have the Frenet formulas

$$\frac{\bar{D}}{ds}E_j = -k_{j-1}E_{j-1} + k_jE_{j+1} \qquad 1 \leq j \leq m, \tag{4.6}$$

where $k_1, \ldots, k_{m-1}$ are the curvatures of α and $k_0 = k_m = 0$. Since $E_1 = \alpha'$, we have

$$\frac{\bar{D}\alpha'}{ds} = k_1E_2.$$

Using (4.3) and (4.5) we can write

$$\frac{\bar{D}\alpha'}{ds} = \sum_{j=1}^{m} a_je_j,$$

where the a_j are constant. Then K_1 is constant, E_1 belongs to $(T_\alpha S_\alpha)^\perp$ and E_2 belongs to $T_\alpha S_\alpha$. By (4.6) we have

$$\frac{\bar{D}E_2}{ds} = -k_1E_1 + k_2E_3.$$

Hence

$$\frac{\bar{D}E_2}{ds} = \frac{1}{k_1}\sum_{j=1}^{m} a_je_j'$$

and

$$\frac{\bar{D}^2 E_2}{ds^2} = \frac{1}{k_1} \sum_{j=1}^{m} a_j \frac{\bar{D} e'_j}{ds} = -\frac{1}{k_1} \sum_{j,k=1}^{m} a_j c_{jk} e_k,$$

where $c_{jk} = -\langle e'_j, e'_k \rangle$.

By (4.5) c_{jk}, $|DE_2/ds|$ and $|\bar{D}^2 E_2/ds^2|$ are constant. Therefore k_2 and k_3 are constant, E_3 belongs to $(T_\alpha S_\alpha)^\perp$, E_4 belongs to $T_\alpha S$ and $E_4 = \sum a_{4j} e_j$ where the a_{4j} are constant. Proceeding in this way we conclude that α has constant curvatures, E_{2j} belongs to $T_\alpha S_\alpha$ and E_{2j+1} belongs to $(T_\alpha S_\alpha)^\perp$. Furthermore,

$$E_{2j} = \sum_{i=1}^{n} a_{ji} e_i \quad \text{and} \quad E_{2j-1} = \sum_{i=1}^{n} b_{ji} e'_i, \tag{4.7}$$

where the a_{4j} and b_{ji} are constant. This proves (4.4).

In fact, if we just repeat the argument above for $e_j(s)$ we conclude that

(4.8) *Each $e_j(s)$ describes, in the ambient space, a curve with constant curvatures.*

Let $R(\alpha)$ be the subspace of the ambient space spanned by α, $E_1, \ldots, E_m$. It is clear that $R(\alpha)$ does not depend on s. We may then choose for $R(\alpha) \cap T_\alpha S_\alpha$ the basis $\{E_{2i} : 1 \leq i \leq [m/2]\}$.

Since $R(\alpha)$ is constant, so is its orthogonal complement $\bar{R}(\alpha)$. We can apply Lemma (2.2) to any basis of $\bar{R}(\alpha) \cap T_\alpha S_\alpha$ to obtain a new basis for such a space that has the same properties as the one we have been working with so far. In particular they will satisfy (4.3) and (4.8). We will then have

$$\frac{\bar{D} e'_i}{ds} = \sum_{j > [m/2]}^{n} c_{ij} e_j, \qquad [m/2] < i \leq n,$$

where (c_{ij}) is a constant symmetric matrix. Since (c_{ij}) is symmetric we may change basis again, in such way as to diagonalize (c_{ij}). Since the change of basis will be done through constant matrices, it will keep all properties of e_j, $[m/2] < j \leq n$.

For this last basis we will have that each e_j, $[m/2] < j \leq n$ will describe a curve with constant curvatures contained in a Euclidean plane V_j of the ambient space. Each plane V_j is invariant with s and is generated by e_j and e'_j. They are mutually orthogonal as one can see by observing that, if $i \neq j$ then

$$\langle e_i, e_j \rangle = \langle e_i, e'_j \rangle = \langle e_j, e'_i \rangle = 0$$

and

$$\langle e'_i, e'_j \rangle = -\langle e_i, e''_j \rangle = -\langle e_i, \lambda_j e_j \rangle = 0.$$

Each $e_j(s)$ is then an orbit of a one-parameter subgroup of rigid motions $A_j(s)$ of V_j, $[m/2] < j \leq n$, described by (4.8). Then there is a one-parameter family of rigid motions $\tilde{A}(s)$ satisfying (3.5). Furthermore, as we observe after Proposition (3.5) we have $\tilde{A}(s) E_j(t) = E_j(s + t)$, $1 \leq j \leq m$. If we now take $A =$

$\tilde{A} \circ A_{[m/2]+1} \circ \ldots \circ A_n$, then we obtain the decomposition (3.14) and letting $t_i \in R$ we see that X is the restriction of a generalized helicoid. This proves the theorem.

Added in proof. After this paper had been completed, our attention was called to the work of Günter Aumann (*Die Minimalhyperregelflächen,* Manuscripta Math. **34** (1981), 293–304) in which he treats the particular case of minimal ruled hypersurfaces in E^n.

REFERENCES

1. J. L. BARBOSA & M. DO CARMO, *Helicoids, catenoids, and minimal hypersurfaces of R^n invariant by an ℓ-parameter group of motions,* An. Acad. Brasil. Ciênc. **53** (1981), 403–408.
2. D. E. BLAIR & J. R. VANSTONE, *A generalization of the helicoid* in "Minimal Submanifolds and Geodesics," (pp. 13–16), Kaigai Publications, Tokyo, 1978.
3. M. DO CARMO & M. DAJCZER, *Rotation hypersurfaces in spaces of constant curvature,* Trans. Amer. Math. Soc. **277** (1983), 685–709.
4. H. B. LAWSON, Jr., *Complete minimal surfaces in S^3*, Ann. of Math. **92** (1970), 335–374.
5. JU. G. LUMISTE, *Die n-dimensionalen Minimalflächen mit einer $(n-1)$-dimensionalen asymptotischen Richtung in jedem Punkte* (Russian, Estonian & German summaries), Tartu Riikl Ül. Toimetised **62** (1958), 117–141.
6. C. THAS, *Minimal monosystems,* Yokohama Math. J. **26** (1978), 157–167.

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Received August 9, 1982