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AN EXTRINSIC RIGIDITY THEOREM FOR MINIMAL IMMERSIONS FROM S^2 INTO S^n

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1. Introduction

Let $x: X^2 \to S^n(1)$ be a generalized minimal immersion, where $S^n(1)$ is the unit sphere of the Euclidean space R^{n+1}, and S^2 is the 2-sphere, which will always be considered as having the induced metric. Let $T_k(x)$ be the real osculating space of order k of x. Define the k-normal space $N_k(x)$ associated to x, by taking at each point the orthogonal complement of $T_k(x)$ in the corresponding tangent space of $S^n(1)$. It was shown by Calabi [2] that if W is the subspace of R^{n+1} spanned by $x(S^2)$, then

$$\dim(W) = 2m + 1, \dim T_{k+1} - \dim T_k = 2 \text{ for } 1 \leqslant k \leqslant m.$$

This can also be found in Chern [3] and Barbosa [1]. Thus, for $1 \leqslant k \leqslant m$, N_k is a map from S^2 into $\Lambda^{n-2k}(R^{n+1})$. We denote by (,) the standard inner product in R^{n+1}. This naturally extends to $\Lambda^s(R^{n+1})$. We keep the same notation. The following is the main theorem of this paper.

Theorem. *Let x: $S^2 \to S^n$ be a generalized minimal immersion. Let m be the integer such that $2m + 1$ is the dimension of the subspace of R^{n+1} spanned by $x(S^2)$. If, for an integer k, $1 \leqslant k \leqslant n/2$, there exists a constant decomposable vector $A \in \Lambda^{n-2k}(R^{n+1})$ such that $(A, N_k) > 0$, then $k \geqslant m$. In particular, if $(A, N_1) > 0$ for $A \in \Lambda^{n-2}(R^{n+1})$, then x is the totally geodesic immersion of S^2 into S^n. This theorem answers, for the particular case of S^2, a question posed by S. S. Chern in his Kansas notes* [4]. *Related to this is the following De Giorgi-Simons-Reilly's result* [5], [8], [7]:

Let x: $M^n \to S^{n+p}(1)$ be an isometric minimal immersion of an n-dimensional compact oriented Riemannian manifold into the unit sphere of R^{n+p+1}, and let N: $M \to G(p, n + p + 1)$ be the normal map. If there exists a constant decomposable unit p-vector A, such that

$$(N, A) > \sqrt{(2p - 2)/(3p - 2)}\,,$$

then x is totally geodesic.

Recently, K. Kenmotsu [6] improved this result for the case $n = 2$ and

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$p > 2$ by assuming only that

$$(N, A) > \sqrt{1/2}\ .$$

We should point out that our method differs from those of O'Reilly and Kenmotsu.

Blaine Lawson pointed out that the result established in this paper has the following nice corollary.

Corollary. *Let U be an open set of R^3 and $f\colon U \to R^{n-3}$ be a Lipschitz function whose graph is a weak solution to the minimal surface system. Then f is real analytic and so defines a classical minimal surface.*

We would like to thank Blaine Lawson for having suggested the question we solved in this paper. Recently, Yau [10] proved a particular version of our main theorem, for the case $k = 1$ and $n = 4$.

2. Preliminaries

Let M be an oriented compact differentiable surface, and $x\colon M \to S^n(1)$ a differentiable map into the unit n-sphere of the Euclidean $(n + 1)$-space. The induced metric on M, together with its orientation, defines a covering of M by isothermal coordinates. Relative to a local isothermal parameter z, the metric on M takes the particular form

$$ds^2 = 2F|dz|^2, \tag{2.1}$$

and the area form can then be represented by

$$\omega = iFdz \wedge d\bar{z}. \tag{2.2}$$

When x is an immersion, F is an everywhere positive valued (real analytic) function. Throughout this paper, we will be working with maps that are (minimal) immersions at all but finite many points of M. These will be called generalized (minimal) immersions. In local terms, this means only that we consider F as having at most finitely many zeros.

All higher order derivatives of x with respect to z and $\bar{z}$ will be considered as functions with values in C^{n+1}. The complex osculating space of order m at a point p of M is the pull back of the subspace of C^{n+1} spanned by all the mixed derivatives $\partial^{j+k}x/\partial^j z \partial^k \bar{z}$ with $0 \leqslant j + k \leqslant m$.

In C^{n+1}, the symmetrical product of two vectors $a = (a_0, \cdots, a_n)$ and $b = (b_0, \cdots, b_n)$ is defined by

$$(a, b) = a_0 b_0 + \cdots + a_n b_n,$$

and the Hermitian product of a and b is then defined by

$$(a, \bar{b}).$$

If we set $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial\bar{z}$, we have that

(a) z is an isothermal parameter for the metric on M if and only if

$$(\partial x, \partial x) = 0; \tag{2.3}$$

(b) the function F (obtained in the expression of the induced metric in M) is given by

$$F = (\partial x, \bar{\partial} x); \tag{2.4}$$

(c) the Laplacian operator for the induced metric on M is given by

$$\Delta = \frac{2}{F}\partial\bar{\partial}; \tag{2.5}$$

(d) the Gauss curvature of M is

$$K = -\frac{1}{2}\Delta F = -\frac{1}{F}\partial\bar{\partial}\log F. \tag{2.6}$$

It is known that x is a minimal immersion into S^n if and only if x satisfies the equation

$$\Delta x = \lambda x.$$

(See for example [4, p. 31]). According with our notation, this means

$$\partial\bar{\partial} x = -Fx. \tag{2.7}$$

(See [2], for details). This equation enables us to write any mixed derivative of x, with respect to z and $\bar{z}$, of order $\leqslant k$ in terms of the complex vectors (of C^{n+1})

$$x, \partial x, \cdots, \partial^k x, \bar{\partial} x, \bar{\partial}^2 x, \cdots, \bar{\partial}^k x.$$

Consequently, the complex osculating space of order k at a point p of M is spanned only by these $2k + 1$ vectors evaluated at p.

Let us now consider the case where $M = S^2$. Using (2.7), the previous observation, and the topology of S^2, one can prove that

$$(\partial^j x, \partial^k x) = 0 \text{ for } j + k > 0, \tag{2.8}$$

where our notation was extended by identifying x with $\partial^0 x$. (See Calabi [2] or Barbosa [1]). Geometrically, (2.8) means that the subspace $V(p)$ of C^{n+1} spanned by the vectors ∂x, $\partial^2 x$, $\partial^3 x, \cdots$ at a point p of S^2 is totally isotropic (i.e., perpendicular to its own conjugate) and perpendicular to $x(p)$. Furthermore, if $m = \dim V(p)$, then $V(p)$ is spanned by the vectors ∂x, $\partial^2 x, \cdots, \partial^m x$. The following theorem due to Calabi can then be easily obtained:

(2.9) Theorem. *Let x: $S^2 \to S^n$ be a generalized minimal immersion and W be the subspace of R^{n+1} spanned by $x(S^2)$. Then* $\dim W = 2m + 1$.

This theorem can be extended (see [1]) to generalized minimal immersions of compact surfaces M, provided (2.8) is included as an additional hypothesis.

3. An expression for the normal map in terms of the derictrix curve

Let $x\colon S^2 \to S^{2m}$ be a minimal immersion. Consider S^2 covered by isothermal coordinates as before, and assume that $x(S^2)$ is not contained in any lower dimensional subspace of R^{2m+1}. Construct, in a coordinate neighborhood, the following local vector valued functions:

$$
\begin{aligned}
&G_0 = x,\\
&G_1 = \bar{\partial} x,\\
&G_2 = \bar{\partial}^2 x - a_2^1 G_1,\\
&\cdots\cdots\cdots\cdots\\
&G_k = \bar{\partial}^k x - \sum_{j=1}^{k-1} a_k^j G_j,\\
&\cdots\cdots\cdots\cdots
\end{aligned}
\tag{3.1}
$$

where a_k^j are chosen in such a way that

$$(G_k, \overline{G_j}) = 0. \tag{3.2}$$

Thus we conclude that $G_{m+k} = 0$ for any k, $\{G_1, \cdots, G_m\}$ is an orthonormal basis for V, and $(G_k, G_j) = 0$ if $j + k > 0$. Furthermore, the direction of each G_k (where $G_k \neq 0$) is invariant under change of coordinates. We can then use the G_k's to define functions into the complex projective space CP^{2m}. Those are well defined wherever $G_k \neq 0$. The following lemma, which gives a new proof for [1, 3.12], shows that one can extend them to S^2.

(3.3) Lemma. *For $0 \leqslant k \leqslant m$, each local function G_k is C^∞ and has only isolated zeroes. Furthermore, if z_0 is one such zero, then there exists a positive integer r such that $H_k = (\bar{z} - \bar{z}_0)^{-r} G_k$ is C^∞ and nonzero in a neighborhood of z_0.*

Proof. The proof will be done by induction on k. The lemma is true for $k = 0$. Assume it is true for $j < k$. Then from the definition of G_k it follows that G_k is C^∞, $(H_i, H_j) = 0$ for all $i, j < k$ and $(H_i, \overline{H_j}) = 0$ for $i < j < k$. Therefore $H_0, H_1, \overline{H}_1, \cdots, H_{k-1}, \overline{H}_{k-1}$ are independent in a neighborhood of z_0. Let $e_{2k+2}, \cdots, e_{2m}$ be sections of $x^*T(S^{2m})$ which are independent and orthogonal to H_0, H_1, $\overline{H}_1, \cdots, H_{k-1}$, $\overline{H}_{k-1}$. Then $G_k = \sum a_i e_i$ (mod H's) where the a_i's are C^∞. Now

$$\partial e_i = \sum b_{ij} e_j \quad (\text{mod } H\text{'s}),$$

where the b's are C^∞ and

$$\partial G_k = 0 \quad (\text{mod } H\text{'s}),$$

since $\partial G_k = -|G_k|^2 G_{k-1}/|G_{k-1}|^2$. Thus

$$\partial a_i = \sum b_{ij} a_j,$$

which is an elliptic linear system of equations. We claim that either a solution of such system is identically zero, or at an isolated zero z_0 there exists an r such that $(\bar{z} - \bar{z}_0)^{-r}a_i$ are C^∞ and not all zero. This proves the induction hypothesis for k, modulo the claim. It is obviously equivalent to proving the claim for a system of the following type:

$$\bar{\partial} W = A(z) \cdot W, \tag{3.4}$$

where $A(z)$ is a C^∞ $n \times n$ matrix function and W is a column vector in C^n. In [3, p. 32] Chern shows that there is an r such that if W is a solution of (3.4) then either W is identically zero or $(z - z_0)^{-r}W$ is continuous and nonzero. Let $\tilde{W} = (z - z_0)^{-r}W$. Then $\tilde{W}$ satisfies (3.4) as well except at z_0, and it is easily checked that $\tilde{W}$ is a distribution solution of (3.4) in a neighborhood of z_0, and thus by elliptic regularity W is C^∞.

It follows now from Lemma (3.3) and [1, Lemma (3.7)] that the function G_m can be extended to a function

$$\xi \colon S^2 \to CP^{2m}$$

which is holomorphic. Such a function is called the directrix curve of the minimal immersion x. For $0 \leqslant k \leqslant m$, its kth derivative is given by

$$\xi^k = \sum_{j=0}^{k-1} A^k_{m-j} G_{m-j} + (-1)^k \frac{1}{|G_{m-k}|^2} G_{m-k},$$

where the coefficients A^i_j are functions of z and $\bar{z}$. From this expression it follows that ξ is totaly isotropic, i.e.,

$$(\xi, \xi) = (\xi', \xi') = \cdots = (\xi^{m-1}, \xi^{m-1}) = 0.$$

(3.5) Proposition. *For $0 < x < m$, let $T_k(x)$ be the real osculating space of order k of x. Then $N_k(x) = T_k(x)^\perp$ can be locally represented in homogeneous coordinates by $\alpha_k \psi_k / |\psi_k|$, when $\alpha_k = \sqrt{(-1)^{m-k}}$, and*

$$\psi_k = \xi \wedge \xi^1 \wedge \cdots \wedge \xi^{m-k-1} \wedge \bar{\xi} \wedge \bar{\xi}^1 \wedge \cdots \wedge \bar{\xi}^{m-k-1}.$$

Proof. First let us observe that $\{\xi, \xi^1, \cdots, \xi^{s-1}\}$ and $\{G_m, G_{m-1}, \cdots, G_{m-s+1}\}$ span the same subspace of C^{2m+1}. Hence ψ_k represents a complex $2(m-k)$-plane whose orthogonal complement is spanned by $\{G_k, G_{k-1}, \cdots, G_1, G_0, \overline{G}_1, \cdots, \overline{G}_{k-1}, \overline{G}_k\}$. But the latter is the same as the complex k-osculating space of x, which is nothing more than the complexification of $T_k(x)$. Since α_k is adjusted so that $\alpha_k \psi_k$ is a real vector, $\alpha_k \psi_k / |\psi_k|$ is a unitary real vector field which represents $N_k(x)$ in homogeneous coordinates.

In the next proposition we will prove that N_k defines a global map from S^2

into $S^{n(k)}$, and that in S^2 with the induced metric, the parameters we are using will still be isotermic.

(3.6) Proposition. *The function* $N_k = \alpha_k \psi_k / |\psi_k|$ *is independent of the particular local coordinates used, and so it defines a global map from* S^2 *into* $S^{n(k)}$, $n(k) = \binom{2m+1}{2m-2k} - 1$. *Furthermore,* $(\partial N_k, \partial N_k) = 0$ *for any local parameter* z.

Proof. If z and w are two local isothermal coordinates in S^2, then

$$\xi^s(w) = \xi^s(z)\left(\frac{dz}{dw}\right)^s + \text{terms in } \xi^j(z) \text{ with } j < s.$$

Thus

$$\psi_k(w) = \psi_k(z)\left|\frac{dz}{dw}\right|^{(1+2+\cdots+(m-z-1))}.$$

Because $\psi_k(w)$ and $\psi_k(z)$ differ only by a real factor, we have

$$\frac{\psi_k(w)}{|\psi_k(w)|} = \frac{\psi_k(z)}{|\psi_k(z)|}.$$

We also have that $\psi_k/|\psi_k|$ Is invariant under change of local representation of ξ. In fact, if $\zeta = \lambda\xi$ is another local representation, then $\psi_\zeta = |\lambda|^{2(m-k)}\psi_\xi$. Consequently $\psi_\zeta/|\psi_\zeta| = \psi_\xi/|\psi_\xi|$. One should notice that ψ_k may have some isolated zeros. But, even at these points, $\psi_k/|\psi_k|$ is well defined. Indeed, if $\psi_k(z_0) = 0$, then $\xi \wedge \xi^1 \wedge \cdots \wedge \xi^{m-k-1}$ has a zero of a certain order r at z_0. We may then factorize ψ_k as

$$\psi_k(z) = |z - z_0|^{2r}\varphi_k(z), \text{ with } \varphi_k(z_0) \neq 0.$$

consequently, the functions $\alpha_k\psi_k(z)/|\psi_k(z)|$ are local expressions for a global function N_k from S^2 into $S^{n(k)}$ where $n(k) = \binom{2m-1}{2m-2k} - 1$.

All that remains to be done to complete the proof of the proposition is to show that $(\partial N_k, \partial N_k) = 0$. In fact, we can prove the following more general fact.

(3.7) Lemma. For each $r > 0$ we have

$$(\partial^r N_k, \partial^r N_k) = 0.$$

Proof. Observe that

$$(\partial^r N_k, \partial^r N_k) = (-1)^{m-k}\sum_{i,j=0}^{r}\binom{r}{i}\binom{r}{j}\partial^{r-i}\left(\frac{1}{|\psi_k|}\right)\partial^{r-j}\left(\frac{1}{|\psi_k|}\right)(\partial^i\psi_k, \partial^j\psi_k),$$

and, if we set $T = \xi \wedge \xi^1 \wedge \cdots \wedge \xi^{m-k-1}$, then

$$\begin{aligned}(\partial^i\psi_k, \partial^j\psi_k) &= (-1)^{m-k}(\partial^i T, \bar{T})(\partial^j T, \bar{T}) \\ &= (-1)^{m-k}\partial^i|T|^2\partial^j|T|^2 = (-1)^{m-k}\partial^i|\psi_k|\partial^j|\psi_k|.\end{aligned} \tag{3.8}$$

Hence

$$(\partial^r N_k, \partial^r N_k) = \left(\sum_{i=0}^{r}\binom{r}{i}\partial^{r-i}\left(\frac{1}{|\psi_k|}\right)\partial^i|\psi_k|\right)^2 = \left(\partial^r\left(\frac{|\psi_k|}{|\psi_k|}\right)\right)^2.$$

Therefore

$$(\partial^r N_k, \partial^r N_k) = 0.$$

(3.9) Corollary. *The complex subspace of $C^{n(k)+1}$ spanned by ∂N_k, $\partial^2 N_k, \cdots, \partial^j N_k, \cdots$ at any fixed point of S^2 is totally isotropic and perpendicular to N.*

Proof. To prove this corollary, we have to show that, for each $r + s > 0$,

$$(g,^r N_k, \partial^s N_k) = 0.$$

But this can be easily proven using induction on $r + s$. (It helps to make a matrix of products $(\partial^r N_k, \partial^s N_k)$, and indicate the ones we are assuming to be zero in each step.) The geometrical consequence of this lemma is that if V is the space generated by the derivatives $\partial N_k, \partial^2 N_k, \cdots$ then V is perpendicular to its own conjugate and also perpendicular to N_k.

We have in mind to compute the mean curvature of N_k: $S^2 \to S^{n(k)}$. To do this, we first set up some machinery. Since

$$\begin{aligned}
\psi_k &= \xi \wedge \cdots \wedge \xi^{m-k-1} \wedge \bar{\xi} \wedge \cdots \wedge \bar{\xi}^{m-k-1},\\
\partial\psi_k &= \xi \wedge \cdots \wedge \xi^{m-k-2} \wedge \xi^{m-k} \wedge \bar{\xi} \wedge \cdots \wedge \bar{\xi}^{m-k-1}, \qquad (3.10)\\
\bar{\partial}\psi_k &= \xi \wedge \cdots \wedge \xi^{m-k-1} \wedge \bar{\xi} \wedge \cdots \wedge \bar{\xi}^{m-k-2} \wedge \bar{\xi}^{m-k},
\end{aligned}$$

by setting $T = \xi \wedge \cdots \wedge \xi^{m-k-1}$, we have the following equalities:

$$\begin{aligned}
(T, T) &= 0,\\
(\psi_k, \psi_k) &= (-1)^{m-k}|T|^4,\\
(\psi_k, \bar{\psi}_k) &= |\psi_k|^2 = |T|^4,\\
(\psi_k, \partial^j\psi_k) &= (-1)^{m-k}|T|^2\partial^j|T|^2,\\
(\psi_k, \bar{\partial}^j\psi_k) &= (-1)^{m-k}|T|^2\bar{\partial}^j|T|^2, \qquad (3.11)\\
(\partial\psi_k, \partial^j\psi_k) &= (-1)^{m-k}\partial|T|^2\partial^j|T|^2,\\
(\bar{\partial}\psi_k, \partial^j\psi_k) &= (-1)^{m-k}|T|^2\bar{\partial}\partial^j|T|^2,\\
(\partial\bar{\partial}\psi_k, \partial^j\psi_k) &= (-1)^{m-k}\partial|T|^2\bar{\partial}\partial^j|T|^2,\\
(\partial\bar{\partial}\psi_k, \partial\bar{\partial}\psi_k) &= (-1)^{m-k}\partial\bar{\partial}|T|^2\partial\bar{\partial}|T|^2, \; k > 0.
\end{aligned}$$

The next proposition gives a criterion for the regularity of the map N_k: $S^2 \to S^{n(k)}$.

(3.12) Proposition. *Let ξ_{m-k-1} be the holomorphic $(m-k-1)$-associated curve to ξ. Then*

$$\left(\partial N_k, \bar\partial N_k\right) = \frac{|\xi_{m-k-1} \wedge \xi'_{m-k-1}|^2}{|\xi_{m-k-1}|^4}.$$

Proof. Since $N_k = \alpha_k \psi_k |\psi_k|^{-1}$ when $\alpha_k = \sqrt{(-1)^{m-k}}$, we have that

$$\partial N_k = \alpha_k\left\{\psi_k \partial|\psi_k|^{-1} + |\psi_k|^{-1}\partial\psi_k\right\}. \tag{3.13}$$

It follows that

$$\left(\partial N_k, \bar\partial N_k\right) = (-1)^{m-k}\left\{\partial|\psi_k|^{-1}\bar\partial|\psi_k|^{-1}(\psi_k, \psi_k) + |\psi_k|^{-1}\partial|\psi_k|^{-1}\left(\psi_k, \bar\partial\psi_k\right)\right.$$
$$\left. + |\psi_k|^{-1}\bar\partial|\psi_k|^{-1}(\psi_k, \partial\psi_k) + |\psi_k|^{-1}\left(\partial\psi_k, \bar\partial\psi_k\right)\right\}.$$

By applying this identity to the formulas obtained in (3.11) we see that

$$\left(\partial N_k, \bar\partial N_k\right) = |T|^{-4}\left\{|T|^2|\partial T|^2 - |(\partial T, \bar T)|^2\right\}. \tag{3.14}$$

Using the definition of T, we obtain the desired result.

(3.15) The consequence of this proposition is that N_k and ξ_{m-k-1} are isometric, and therefore N_k will be regular in all points where ξ_{m-k-1} is. Hence N_k will be regular in all but finitely many points.

(3.16) Lemma. $(\partial\bar\partial N_k, \partial^J\psi_k) = |\psi_k|^{-1}\partial^J|\psi_k|(\partial\bar\partial N_k, \psi_k)$, $J > 0$.

Proof. Computing $\bar\partial$ of (3.13), we obtain

$$\partial\bar\partial N_k = \alpha_k\left\{\psi_k\partial\bar\partial|\psi_k|^{-1} + \bar\partial|\psi_k|^{-1}\partial\psi_k + \partial|\psi_k|^{-1}\bar\partial\psi_k + |\psi_k|^{-1}\partial\bar\partial\psi_k\right\}. \tag{3.17}$$

Consequently

$$\left(\partial\bar\partial N_k, \partial^J\psi_k\right) = \alpha_k\left\{\partial\bar\partial|\psi_k|^{-1}(\psi_k, \partial^J\psi_k) + \bar\partial|\psi_k|^{-1}(\partial\psi_k, \partial^J\psi_k)\right.$$
$$\left. + \partial|\psi_k|^{-1}\left(\bar\partial\psi_k, \partial^J\psi_k\right) + |\psi_k|^{-1}\left(\partial\bar\partial\psi_k, \partial^J\psi_k\right)\right\}.$$

The substitution of (3.11) in this expression yields

$$\left(\partial\bar\partial N_k, \partial^J\psi_k\right) = \alpha_k^3 T^{-4}\left(-|T|^2|\partial T|^2 + |(\partial T, \bar T)|^2\right)\partial^J|\psi_k|.$$

Using (3.14) and the fact that $(N_k, N_k) = 1$, we obtain

$$\left(\partial\bar\partial N_k, \partial^J\psi_k\right) = \alpha_k^3\left(\partial\bar\partial N_k, N_k\right)\partial^J|\psi_k|. \tag{3.18}$$

But this is the desired result if we replace N_k by it's local expression $\alpha_k\psi_k|\psi_k|^{-1}$.

(3.19) Proposition. The Laplacian of N_k is perpendicular to the subspace of $C^{n(k)+1}$ spanned by $\partial N_k, \partial^2 N_k, \cdots$, and forms a fixed angle of $\pi/4$ with N_k for $k \geqslant 1$.

Proof. Since

$$\partial^s N_k = \alpha_k \sum_{j=0}^{s} \binom{s}{j} \partial^{s-j} |\psi_k|^{-1} \partial^j \psi_k, \tag{3.20}$$

we have

$$\left(\partial\bar{\partial} N_k,\, \partial^s N_k\right) = \alpha_k \sum_{j=0}^{s} \binom{s}{j} \partial^{s-j} |\psi_k|^{-1} \left(\partial\bar{\partial} N_k,\, \partial^j \psi_k\right).$$

Using the previous lemma we obtain, for $s > 0$,

$$\left(\partial\bar{\partial} N_k,\, \partial^s N_k\right) = \left(\partial\bar{\partial} N_k,\, N_k\right) \left\{ \sum_{j=0}^{s} \binom{s}{j} \partial^{s-j} |\psi_k|^{-1} \partial^j |\psi_k| \right\}.$$

The expression inside the braces is just $\partial^s(1)$ and therefore zero. Since $\Delta = (2/F_k)\partial\bar{\partial}$, where $F_k = (\partial N_k, \bar{\partial} N_k)$, we conclude that $(\Delta N_k, \partial^s N_k) = 0$ for each $s > 0$.

The second part of the proposition follows from the next lemma.

(3.21) Lemma. $(\partial\bar{\partial} N_k, \partial\bar{\partial} N_k) = 2(\partial\bar{\partial} N_k, N_k)^2$, $k > 0$.

Proof. From (3.17) we have that

$$\begin{aligned}\left(\partial\bar{\partial} N_k,\, \partial\bar{\partial} N_k\right) = \alpha_k \big\{ &\partial\bar{\partial} |\psi_k|^{-1} \left(\partial\bar{\partial} N_k,\, \psi_k\right) + \partial |\psi_k|^{-1} \left(\partial\bar{\partial} N_k,\, \bar{\partial}\psi_k\right) \\ &+ \bar{\partial} |\psi_k|^{-1} \left(\partial\bar{\partial} N_k,\, \partial\psi_k\right) + |\psi_k|^{-1} \left(\partial\bar{\partial} N_k,\, \partial\bar{\partial}\psi_k\right) \big\},\end{aligned}$$

which can be simplified, in consequence of (3.6), to

$$\left(\partial\bar{\partial} N_k,\, \partial\bar{\partial} N_k\right) = -|\psi_k|^{-1} \partial\bar{\partial} |\psi_k| \left(\partial\bar{\partial} N_k,\, N_k\right) + \alpha_k |\psi_k|^{-1} \left(\partial\bar{\partial} N_k,\, \partial\bar{\partial}\psi_k\right). \tag{3.22}$$

In order to compute the value of $(\partial\bar{\partial} N_k, \partial\bar{\partial}\psi_k)$, we use (3.17) to obtain

$$\begin{aligned}\left(\partial\bar{\partial} N_k,\, \partial\bar{\partial}\psi_k\right) = \alpha_k \big\{ &\partial\bar{\partial} |\psi_k|^{-1} \left(\psi_k,\, \partial\bar{\partial}\psi_k\right) + \partial |\psi_k|^{-1} \left(\bar{\partial}\psi_k,\, \partial\bar{\partial}\psi_k\right) \\ &+ \bar{\partial} |\psi_k|^{-1} \left(\partial\psi_k,\, \partial\bar{\partial}\psi_k\right) + |\psi_k|^{-1} \left(\partial\bar{\partial}\psi_k,\, \partial\bar{\partial}\psi_k\right) \big\}.\end{aligned}$$

Using (3.16) we may simplify this to

$$\begin{aligned}\left(\partial\bar{\partial} N_k,\, \partial\bar{\partial}\psi_k\right) = \alpha_k^3 \big\{ &-3 |\psi_k|^{-2} \partial |\psi_k| \bar{\partial} |\psi_k| \partial\bar{\partial} |\psi_k| \\ &+ 2 |\psi_k|^{-3} (\partial |\psi_k|)^2 \left(\bar{\partial} |\psi_k|\right)^2 + |\psi_k|^{-1} \left(\partial\bar{\partial} |\psi_k|\right)^2 \big\}.\end{aligned} \tag{3.23}$$

Now substitution of (3.23) and (3.14) in (3.22) yields, after simplification,

$$\left(\partial\bar{\partial} N_k,\, \partial\bar{\partial} N_k\right) = 2 |\psi_k|^{-4} \left(|\psi_k| \partial\bar{\partial} |\psi_k| - \partial |\psi_k| \bar{\partial} |\psi_k| \right)^2 = 2\left(\partial\bar{\partial} \log |\psi_k|\right)^2.$$

Since $|\psi_k| = |T|^2$, using (3.14) we obtain

$$\left(\partial\bar{\partial} N_k,\, \partial\bar{\partial} N_k\right) = 2\left(\partial N_k,\, \bar{\partial} N_k\right)^2 = 2\left(\partial\bar{\partial} N_k,\, N_k\right)^2. \quad \text{q.e.d.}$$

The following proposition due to Kenmotsu [6] is now obtained as a consequence of the previous proposition.

(3.24) Proposition. *Let $x\colon S^2 \to S^{2m}$ be a minimal immersion. If there exist a fixed vector $A \in S^{n(k)}(1)$ such that $(N_k, A) > \frac{1}{2}\sqrt{2}$, then x is totally geodesic.*

Proof. If $(N_k, A) > \frac{1}{2}\sqrt{2}$, then the angle between A and N is less than $\pi/4$, and so is the angle between ΔN_k and A from Proposition (3.19). Hence $(\Delta N_k, A) > 0$, and so (N_k, A) is a subharmonic function globally defined on S^2 and is therefore constant. To show that N_k itself is constant just notice that the same reasoning can be carried out for all points A' in a neighborhood of A on $S^{n(k)}$. N_k is constant, x is a totally geodesic immersion.

(3.25) Proposition. *For each $k > 0$, $N_k\colon S^2 \to S^{n(k)}$ has mean curvature with constant length.*

Proof. Propositions (3.6) and (3.12) show that the metric induced on S^2 by N_k is given by

$$ds_k^2 = 2\, F_k |dz|^2,$$

where $F_k = (\partial N_k, \bar{\partial} N_k) = \partial\bar{\partial} \log|\xi_{m-k-1}|^2$, so that its mean curvature in $R^{n(k)+1}$ is given by

$$\tilde{H}_k = \frac{2}{F_k}\, \partial\bar{\partial} N_k.$$

Therefore by (3.21), $|\tilde{H}_k|^2 = 8$, and the mean curvature of N_k in $S^{n(k)}$ is

$$H_k = \frac{2}{F_k}\, \partial\bar{\partial} N_k - \left(\frac{2}{F_k}\, \partial\bar{\partial} N_k,\, N_k\right) N_k,$$

whose length is 2.

4. The main theorem

Let $x\colon S^2 \to S^n(1)$ be a generalized minimal immersion, and W be the subspace of R^{n+1} spanned by $x(S^2)$. From (2.9) we know that W has dimension $2m + 1$, and so x can be considered as a minimal immersion of S^2 into $S^{2m} = W \cap S^n$.

Let $N_k(x)$ and $N_k'(x)$ be the k-normal maps associated with x when it's image is considered in S^n and $S^{2m} \subset W$ respectively.

(4.1) Lemma. *If there exists a decomposable vector A belonging to $\Lambda^{n-2k}(R^{n+1})$ such that $(A, N_k) > 0$, then there also exists a decomposable vector $A' \in \Lambda^{2m-2k}(W)$ such that $(A', N_k') > 0$.*

Proof. Choose an orthonormal basis $a_1, \cdots, a_{n-2k}$ for A. Let d be such that $a_1, \cdots, a_d \in W$ and $a_{d+1}, \cdots, a_{n-2k} \in W^{\perp}$, where $W^{\perp}$ stands for the orthogonal complement of W in R^{n+1}. We then have

(4.2) $$A = a_1 \wedge \cdots \wedge a_{n-2k}, \quad 2m - 2k \leqslant d \leqslant 2m + 1.$$

Let $x, e_1, e_2, \cdots, e_n$ be an orthonormal frame field for S^2 around the point x chosen in such a way that

$$e_1, \cdots, e_{n-2k} \in N_k(x),$$

and $e_{2m-2k+1}, \cdots, e_{n-2k}$ are constant vectors belonging to $W^\perp$. We then have $N_k = e_1 \wedge \cdots \wedge e_{n-k}$ and, by hypothesis,

(4.3) $$\det((e_i, a_j)) = (N_k, A) > 0 \; (1 \leqslant i, j \leqslant n - 2k).$$

Under these choices, the maximal possible value for the rank of the above matrix is $(n - 2k) - (d - 2m + 2k)$. From (4.3) this rank must be $n - 2k$. Therefore $d = 2m - 2k$ and

$$(e_1 \wedge \cdots \wedge e_{2m-2k}, a_1 \wedge \cdots \wedge a_{2m-2k}) \neq 0.$$

By changing the sign of some a_j, if necessary, we may assume this product to be positive, and if

$$A' = a_1 \wedge \cdots \wedge a_{2m-2k},$$

we have

(4.4) $$(N_k', A') > 0.$$

(4.5) Theorem. *Let x: $S^2 \to S^n$ be a generalized minimal immersion, and m the integer such that $2m + 1$ is the dimension of the subspace W of R^{n+1} spanned by $x(S^2)$. If for an integer k, $1 \leqslant k \leqslant n/2$, there exists a constant decomposable vector $A \in \wedge^{n-2k}(R^{n+1})$ such that $(A, N_k) > 0$, then $k \geqslant m$. In particular, if $(A, N_1) > 0$ for $A \in \Lambda^{n-2}(R^{n+1})$, then x is the totally geodesic immersion of S^2 into S^n.*

Proof. We will show that for each k, $1 \leqslant k < m$, and any $A \in \wedge^{n-2k}(R^{n+1})$, the function (N_k, A) has zeros. By the previous lemma it is enough to prove this for the case $W = R^{n+1}$, that is, when $n = 2m$ and $X(S^2)$ is not contained in any lower dimensional subspace of R^{2m+1}. Under such hypothesis we are in a position to apply the results obtained in the previous chapter. The proof will depend on the following lemma.

(4.6) Lemma. *The function* $\log(N_k, A)$ *is superharmonic whenever* (N_k, A) *is nonzero.*

Let us postpone the proof of the lemma and proceed with the proof of the theorem. If (N_k, A) is positive over all of S^2, then the function $\log(N_k, A)$ is globally defined, superharmonic in S^2, and therefore constant. Hence (N_k, A) is also constant. We wish to conclude that N_k itself is constant. To this end we start by observing that either $N_k = A$ or $(N_k, A) = c$ with $0 < c < 1$. In the last case there is a neighborhood ν of A such that, for any B belonging to ν we have $(B, N_k) > 0$. Since $A \in G(2m-2k, 2m+1)$, $u = \nu \cap G(2m-2k, 2m+1)$ is a neighborhood of A in $G(2m-2k, 2m+1)$. We may always choose $n(k)+1$ linearly independent vectors $A^1, \ldots, A^{n(k)}$ of $R^{n(k)+1}$ belonging to u. Such choices are possible because $G(2m-2k, 2m+1)$ is real analytic and does not lie in any lower dimensional subspace of $R^{n(k)+1}$. For each one of the A^j, we can repeat the previous argument and conclude that (N_k, A^j) is constant. Therefore N_k is constant.

Now if N_k is constant, it follows that $F_k = 0$, and, by (3.7), ξ^{m-k} must be a linear combination of $\xi, \cdots, \xi^{m-k-1}$. Thus the subspace generated by $\xi, \cdots, \xi^{m-1}$ in C^{2m+1} has at most dimension $m-k$. Hence the subspace spanned by $G_1, \cdots, G_m$ has also dimension less than or equal to $m-k$. But this is a contradiction, since the dimension of this subspace is m and $k \geqslant 1$.

Proof (of Lemma 4.6). Let $a_1, a_2, \cdots, a_{2m-2n}$ be a basis for A. We may form the complex vectors $b_{j-1} = 1/\sqrt{2}\,(a_j + ia_{j+m-k})$, $1 \leqslant j \leqslant m-k$. Now $b_0, \cdots, b_{m-k-1}, \bar{b}_0, \cdots, \bar{b}_{m-k-1}$ is a basis for the complex subspace B of C^{2m+1} generated by A. Then A can be represented by

$$B = b_0 \wedge \cdots \wedge b_{m-k-1} \wedge \bar{b}_0 \wedge \cdots \wedge \bar{b}_{m-k-1} = \alpha_k A,$$

and, locally, $(N_k, A) = |\psi_k|^{-1}(\psi_k, B)$. Hence

$$\partial\bar{\partial} \log(N_k, A) = -\partial\bar{\partial} \log|\psi_k| + \partial\bar{\partial} \log(\psi_k, B). \tag{4.7}$$

Since $\partial\bar{\partial} \log|\psi_k| = |\xi_{m-k-1} \wedge \xi'_{m-k-1}|^2/|\xi_{m-k-1}|^4$, we can reduce the proof of the lemma to showing that $\partial\bar{\partial} \log(\psi_k, B) \leqslant 0$. We have that

$$\partial\bar{\partial} \log(\psi_k, B) = \frac{1}{(\psi, B)^2}\left\{(\psi_k, B)\big(\partial\bar{\partial}\psi_k, B\big) - (\partial\psi_k, B)\big(\bar{\partial}\psi_k, B\big)\right\}, \tag{4.8}$$

where

$$\begin{aligned}(\psi_k, B) = \big(&\xi \wedge \bar{\xi} \wedge \xi^1 \wedge \bar{\xi}^1 \wedge \cdots \wedge \xi^{m-k-1} \wedge \bar{\xi}^{m-k-1},\\ &b_0 \wedge \bar{b}_0 \wedge \cdots \wedge b_{m-k-1} \wedge \bar{b}_{m-k-1}\big).\end{aligned} \tag{4.9}$$

Let $v_0, v_1, \ldots, v_{2m-2k+1}$ be vectors in C^{2m-2k} defined by

$$\begin{aligned} v_{2j} &= \big((\xi^j, b_0), (\xi^j, \bar{b}_0), \cdots, (\xi^j, b_{m-k-1}), (\xi^j, \bar{b}_{m-k-1})\big),\\ v_{2j+1} &= \big((\bar{\xi}^j, b_0), (\bar{\xi}^j, \bar{b}_0), \cdots, (\bar{\xi}^j, b_{m-k-1}), (\bar{\xi}^j, \bar{b}_{m-k-1})\big).\end{aligned} \tag{4.10}$$

Then we have

$$\begin{aligned}(\psi_k, B)(\partial\bar\partial\psi_k, B) - (\partial\psi_k, B)(\bar\partial\psi_k, B) \\ = (v_0 \wedge \cdots \wedge v_{2m-2k-1}, v_0 \wedge \cdots \wedge v_{2m-2k-3} \wedge v_{2m-2k} \wedge v_{2m-2k+1}) \\ - (v_0 \wedge \cdots \wedge v_{2m-2k-3} \wedge v_{2m-2k} \wedge v_{2m-2k-1}, \\ v_0 \wedge \cdots \wedge v_{2m-2k-3} \wedge v_{2m-2k-2} \wedge v_{2m-2k+1}).\end{aligned} \tag{4.11}$$

Using Sylvester's theorem for determinants (see [9, p. 78]) we obtain

$$\begin{aligned}\partial\bar\partial \log(\psi_k, B) \\ = \frac{(-1)}{(\psi_k, B)^2}(v_0 \wedge \cdots \wedge v_{2m-2k-3} \wedge v_{2m-2k-1} \wedge v_{2m-2k+1}, \\ v_0 \wedge \cdots \wedge v_{2m-2k-3} \wedge v_{2m-2k} \wedge v_{2m-2k-2}).\end{aligned} \tag{4.12}$$

To simplify this expression, we consider the linear map $J: C^{2m-2k} \to C^{2m-2k}$ defined by

$$J(z_0, w_0, z_1, w_1, \cdots, z_{m-k-1}, w_{m-k-1}) = (w_0, z_0, w_1, z_1, \cdots, w_{m-k-1}, z_{m-k-1}).$$

We have $Jv_{2j} = \bar v_{2j+1}$ and $Jv_{2j+1} = \bar v_{2j}$. Thus

$$\begin{aligned}v_0 \wedge \cdots \wedge v_{2m-2k-3} \wedge v_{2m-2k-1} \wedge v_{2m-2k+1} \\ = (-1)^{m-k-1}\overline{J(v_0)} \wedge \cdots \wedge \overline{J(v_{2m-2k-3})} \\ \wedge \overline{J(v_{2m-2k-2})} \wedge \overline{J(v_{2m-2k})} \\ = (-1)^{m-k-1}\overline{(\det J)}\, \bar v_0 \wedge \cdots \wedge \bar v_{2m-2k-3} \wedge \bar v_{2m-2k-2} \wedge \bar v_{2m-2k}.\end{aligned} \tag{4.13}$$

Since $\det J = (-1)^{m-k}$, (4.12) and (4.14) give

$$\partial\bar\partial \log(\psi_k, B) = \frac{(-1)}{(\psi_k, B)^2}|v_0 \wedge \cdots \wedge v_{2m-2k-2} \wedge v_{2m-2k}|^2.$$

Therefore $\partial\bar\partial \log(\psi_k, B) \leqslant 0$, and the proof of the lemma is complete.

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