# Closed Weingarten Hypersurfaces in Warped Product Manifolds

Francisco J. Andrade [†*] João L. M. Barbosa [‡†]
Jorge H. S. de Lira[‡]

**Abstract**

Given a compact Riemannian manifold $M$, we consider a warped product $\bar{M} = I \times_h M$ where $I$ is an open interval in $\mathbb{R}$. We suppose that the mean curvature of the fibers do not change sign. Given a positive differentiable function $\psi$ in $\bar{M}$, we find a closed hypersurface $\Sigma$ which is solution of an equation of the form $F(B) = \psi$, where $B$ is the second fundamental form of $\Sigma$ and $F$ is a function satisfying certain structural properties. As examples, we may exhibit examples of hypersurfaces with prescribed higher order mean curvature.

## 1 Introduction

Let $M^n$ be a compact Riemannian manifold and let $I$ be an open interval in $\mathbb{R}$. Given a positive differentiable function $h : I \to \mathbb{R}$ we then consider the product manifold $\bar{M} = I \times M$ endowed with a warped metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 + h^2(t)\,\mathrm{d}\sigma^2, \tag{1}$$

where $\mathrm{d}\sigma^2$ stands for the metric of $M$. We denote the warped metric simply by $\langle \cdot, \cdot \rangle$.

Given a differentiable function $z : M \to I$ its graph is defined as the hypersurface $\Sigma$ whose points are of the form $X(u) = (z(u), u)$ with $u \in M$. This graph is diffeomorphic with $M$ and may be globally oriented by an unit normal vector field $N$ for which it holds that $\langle N, \partial_t \rangle < 0$. With respect to this orientation, let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the vector whose components $\lambda_i$ are the principal curvatures of $\Sigma$, that is, the eigenvalues of the second fundamental form $B = -\langle \mathrm{d}N, \mathrm{d}X \rangle$ in $\Sigma$.

Let $\Gamma$ be an open convex cone with vertex at the origin in $\mathbb{R}^n$ and containing the positive cone. Suppose that $\Gamma$ is symmetric with respect to interchanging coordinates of its points. Let $f$ be a positive differentiable concave function defined in $\Gamma$. In what follows, $f$ is supposed to be symmetric in $\lambda_i$ and it is required that its derivatives satisfy $f_i > 0$ in $\Gamma$.

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We may define a function $F$ in the space of symmetric $n \times n$ matrices $\mathcal{S}$ setting $F(B) = f(\lambda)$ so that it makes sense to write down

$$F(B(z(u))) = f(\lambda(X(u)))$$

when the function $z$ is supposed to be *admissible*, which means that $\lambda(z(u)) \in \Gamma$ for all $u \in M$. Finally, given a positive differentiable function $\psi : \bar{M} \to \mathbb{R}$, it is geometrically relevant to pose the problem of finding an admissible function $z$ which solves the following equation

$$F(B(z(u))) = \psi(z(u), u), \quad u \in M. \tag{2}$$

Since the second fundamental form $B$ may be written in terms of $z$ and its first and second derivatives it happens that in analytical terms this problem is equivalent to prove the existence of solutions for a rather complicated fully nonlinear second order elliptic equation. Naturally, we must impose some additional conditions on the ambient geometry and on the structure of $f$ and $\psi$ in order to provide a solution to (2).

Concerning the ambient geometry, we must suppose that the leaves $M_t = \{(t, u) : u \in M\}$ are mean convex with respect to the inward unit normal vector field $-\partial_t$. This amounts to be equivalent to the condition that

$$\kappa(t) > 0, \quad t \in I, \tag{3}$$

where $\kappa = h'/h$. Let $\delta$ be a strictly increasing and continuous function satisfying $\delta(f) > 0$ whenever $f \geq c_0$ for some positive constant $c_0$. We suppose that

$$\sum_i f_i \geq \delta(f), \quad \sum_i f_i \lambda_i \geq \delta(f) \tag{4}$$

in points of the set

$$\Gamma_{\mu_1,\mu_2} = \{\lambda \in \Gamma : \mu_1 \geq f(\lambda) \geq \mu_2\},$$

where $\mu_1$ and $\mu_2$ are constants with $\mu_2 \geq \mu_1 > 0$. Denoting $\psi_0 = \inf \psi$ we also require that

$$\limsup_{\lambda \to \partial\Gamma} f(\lambda) \leq \bar{\psi}_0, \tag{5}$$

for some constant $\bar{\psi}_0 < \psi_0$. Finally we denote $k = f(\kappa)$. Following this notation, we state our main result.

**Theorem 1** *Let $\bar{M}^{n+1} = I \times M^n$ be endowed with the warped metric given by* (1). *Given $t_-, t_+$ with $t_- < t_+$, consider the region $\bar{M}_{t_-,t_+} = \{(t,p) : t_- \leq t \leq t_+\}$. Suppose that $f$ and $h$ satisfy the conditions* (3)-(5) *and suppose that $\psi$ satisfies*

*a) $\psi(t,p) > k(t)$ for $t \leq t_-$,*

*b) $\psi(t,p) < k(t)$ for $t \geq t_+$,*

*c) $\partial_t\big(h\psi\big) \leq 0$ for $t_- < t < t_+$.*

*Then there exists a differentiable function $z : M^n \to I$ for which*

$$F(B(z(u))) - \psi(z(u), u) = 0 \tag{6}$$

*whose graph $\Sigma$ is contained in the interior of $\bar{M}_{t_-,t_+}$.*

Important particular cases of this theorem concern prescribing the $r$-th mean curvatures $\binom{n}{r} H_r(\lambda) = S_r(\lambda)$, where $S_r$ are the elementary symmetric functions of the principal curvatures which appear in the expansion of the characteristic polynomial of $B$. It may be seen for instance in [13] and [11] that these functions fit in our hypothesis if we consider the suitable Gårding cone. In this sense, the theorem above may be viewed as an extension of existence results found in previous contributions to the subject, notably the works [1], [12], [10], [4], [3], [5], [9] and [7]. In these articles, it is assumed that the variation rate of $\psi$ is controlled in a certain way by the curvature of ambient geodesic spheres. For instance, this hypothesis in [3] is stated in terms of our notation as $\partial_t(t\psi) \leq 0$ in $\bar{M}_{t_-,t_+}$. Here, this hypothesis corresponds to item $(c)$ in the statement of the theorem.

We intend in this paper to show that the powerful elliptic tools presented in the references above are flexible enough to be used in a very general geometrical setting. Warped products constitute a large family of Riemannian manifolds that includes geodesic discs in space forms for suitable choices of $I$ and $h$. Its importance as examples is pervasive in Riemannian Geometry.

The paper is organized as follows. In Section 2, we fix notation and present some geometric and analytic preliminaries, including the detailed description of the problem. In Section 3 we show that under the hypothesis of the theorem, the solutions to the problem remain in the region $\bar{M}_{t_-,t_+}$. In the next section we compute gradient and Hessian of functions which resemble the classical height and support functions. Gradient estimates are obtained in Section 5. The Hessian estimates proved in Section 6 are largely inspired by the technique in [7]. The degree theoretical approach to solving the problem is presented in the last section and it is based on [8], [9] and [7].

# 2 Preliminaries

In the sequel, we use Latin lower case letters $i, j, \ldots$ to refer to indices running from 1 to $n$ and $a, b, \ldots$ to indices from 0 to $n$. The Einstein summation convention is used throughout the paper. Exceptions to these conventions will be explicitly mentioned.

We denote the metric (1) in $\bar{M}$ by $\langle \cdot, \cdot \rangle$. The corresponding Riemannian connection in $\bar{M}$ will be denoted by $\bar{\nabla}$. The usual connection in $M$ will be denoted $\nabla'$. The curvature tensors in $M$ and $\bar{M}$ will be denoted by $R$ and $\bar{R}$, respectively.

Let $e_1, \ldots, e_n$ be an orthonormal frame field in $M$ and let $\theta^1, \ldots, \theta^n$ be the associated dual frame. The connection forms $\theta^i_j$ and curvature forms $\Theta^i_j$ in $M$ satisfy the structural equations

$$\mathrm{d}\theta^i + \theta^i_j \wedge \theta^j = 0, \quad \theta^i_j = -\theta^j_i, \tag{7}$$

$$\mathrm{d}\theta^i_j + \theta^i_k \wedge \theta^k_j = \Theta^i_j. \tag{8}$$

An orthonormal frame in $\bar{M}$ may be defined by $\bar{e}_i = (1/h)e_i$, $1 \leq i \leq n$, and $\bar{e}_0 = \partial/\partial t$. The associated dual frame is then $\bar{\theta}^i = h\theta^i$ for $1 \leq i \leq n$ and $\bar{\theta}^0 = \mathrm{d}t$. A simple computation permits to obtain the connection forms $\bar\theta^a_b$ and the curvature forms $\bar\Theta^a_b$ that are given by

$$\begin{aligned}
\bar\theta^i_j &= \theta^i_j, && (9)\\
\bar\theta^i_0 &= (h'/h)\bar\theta^i, && (10)\\
\bar\Theta^i_j &= \Theta^i_j - (h'^2/h^2)\,\bar\theta^i\wedge\bar\theta_j, && (11)\\
\bar\Theta^i_0 &= (h''/h)\,\bar\theta_0\wedge\bar\theta^i, && (12)
\end{aligned}$$

where $'$ denotes the derivative with respect to $t$. Our convention here is that

$$\bar\theta^j_i = \langle\bar\nabla e_i, e_j\rangle, \quad \bar\Theta^i_j = \langle\bar R(\,\cdot\,,\,\cdot\,)e_j, e_i\rangle.$$

with

$$\bar R(v,w) = \bar\nabla_v\bar\nabla_w - \bar\nabla_w\bar\nabla_v - \bar\nabla_{[v,w]}.$$

The frame $\bar e_a$ we just defined is adapted to the level hypersurfaces $M_t = \{(t,p);\ p\in M\}$. It follows from (10) that each fiber $M_t$ is umbilical with principal curvatures

$$\kappa(t) = h'(t)/h(t) \tag{13}$$

calculated with respect to the *inward* unit normal $-\bar e_0 = -\partial/\partial t$. Notice that according our convention the Weingarten operator for the leaves with respect to this orientation is defined as

$$\langle\bar\nabla e_0, e_i\rangle = \bar\theta^i_0.$$

Now, consider a smooth function $z: M\to I$. Its graph is the regular hypersurface

$$\Sigma = \{X(u) = (z(u), u):\ u\in M\},$$

whose tangent space is spanned at each point by the vectors

$$X_i = h\,\bar e_i + z_i\,\bar e_0, \tag{14}$$

where $z_i$ are the components of the differential $\mathrm{d}z = z_i\theta^i$. The unit vector field

$$N = \frac{1}{W}\big(\sum_{i=1}^n z^i\bar e_i - h\bar e_0\big) \tag{15}$$

is normal to $\Sigma$, where

$$W = \sqrt{h^2 + |\nabla' z|^2}. \tag{16}$$

Here, $|\nabla' z|^2 = z^i z_i$ is the squared norm of $\nabla' z = z^i e_i$. The induced metric in $\Sigma$ has components

$$g_{ij} = \langle X_i, X_j\rangle = h^2\delta_{ij} + z_i z_j \tag{17}$$

and its inverse has components given by

$$g^{ij} = \frac{1}{h^2}\delta^{ij} - \frac{1}{h^2W^2}z^i z^j. \tag{18}$$

One easily verifies that the second fundamental form $B$ of $\Sigma$ with components $(a_{ij})$ is determined by

$$a_{ij} = \langle\bar\nabla_{X_j}X_i, N\rangle = \frac{1}{W}\big(-hz_{ij} + 2h' z_i z_j + h^2h'\delta_{ij}\big)$$

where $z_{ij}$ are the components of the Hessian $\nabla'^2 z = \nabla' \mathrm{d}z$ of $z$ in $M$. Now, we must compute the components $a^i_j = \sum_k g^{ik} a_{kj}$ of the Weingarten map $A^\Sigma$. To simplify computations, in a fixed point $\bar{u} \in M$ where $\nabla' z \neq 0$, we choose $e_1|_{\bar{u}} = \nabla' z / |\nabla' z|$. We call this frame a *special frame* at $\bar{u}$. For this choice, we obtain $\mathrm{d}z = z_1 \theta^1$ at $\bar{u}$. Since the matrices $g_{ij}|_{\bar{u}}$ and $g^{ij}|_{\bar{u}}$ are diagonal in a special frame, one obtains at $\bar{u}$

$$
\begin{aligned}
a^1_1 &= \frac{1}{W^3}\big(-hz_{11} + 2h' z_1^2 + h^2 h'\big), \\
a^1_i &= -\frac{h}{W^3} z_{1i} \qquad \text{for} \quad 2 \leq i \leq n, \qquad (19)\\
a^i_j &= \frac{1}{h^2 W}\big(-hz_{ij} + h^2 h' \delta_{ij}\big) \qquad \text{for} \quad 2 \leq i, j \leq n.
\end{aligned}
$$

Special frames are quite useful for computing second and third order covariant derivatives of $z$. By definition the Hessian of $z$ is

$$z_{ik}\theta^k = \nabla'^2 z(e_i; \cdot) = \mathrm{d}z_i - \theta^k_i z_k. \tag{20}$$

The third derivative of $z$ is defined by

$$z_{ijk}\theta^k = \nabla'^3(e_i, e_j; \cdot) = \mathrm{d}z_{ij} - \theta^k_i z_{kj} - \theta^k_j z_{ik}. \tag{21}$$

Exterior differentiation of both sides in (20) gives a Ricci identity

$$z_{ijk}\theta^j \wedge \theta^k = \Theta^r_i z_r \tag{22}$$

and in particular (for a special frame)

$$z_{1ii} - z_{ii1} = z_{i1i} - z_{ii1} = K_i z_1, \tag{23}$$

where

$$K_i = \langle R(e_1, e_i)e_i, e_1\rangle. \tag{24}$$

Now, we consider an adapted frame field $E_0 = N, E_1, \ldots, E_n$ in some open set in $\Sigma$. Representing by $\omega^a$ its dual forms, by $\omega^a_b$ its connection forms and by $\bar{\Omega}^a_b$ its curvature forms, we have the following relations:

$$d\omega^i + \omega^i_j \wedge \omega^j = 0, \quad \omega^i_j = -\omega^j_i, \tag{25}$$

$$d\omega^i_j + \omega^i_k \wedge \omega^k_j = \Omega^i_j, \tag{26}$$

where $\Omega^i_j$ are the curvature forms for $\Sigma$. Since $\Sigma$ is a hypersurface of $\bar{M}$ then the Gauss equation reads off as

$$\Omega^i_j = \bar{\Omega}^i_j - \omega^i_0 \wedge \omega^0_j \tag{27}$$

The coefficients $a_{ij}$ of the second fundamental form are given by Weingarten equation

$$\omega^0_i = a_{ij}\,\omega^j. \tag{28}$$

In the sequel, one indicates the covariant derivative in $\Sigma$ by $\nabla$ and by a semi-colon. Remember that

$$\nabla a_{ij} = \mathrm{d}a_{ij} - a_{kj}\,\omega^k_i - a_{ik}\,\omega^k_j = a_{ij;k}\,\omega^k \tag{29}$$

$$
\begin{aligned}
\nabla a_{ij;k} &= \mathrm{d}a_{ij;k} - a_{mj;k}\,\omega^m_i - a_{im;k}\,\omega^m_j - a_{ij;m}\,\omega^m_k \\
&= a_{ij;km}\,\omega^m \qquad (30)
\end{aligned}
$$

The Codazzi equation is a commutation formula for the first derivative of $a_{ij}$ and it is obtained by differentiating (28):

$$a_{ij;k}\omega^j \wedge \omega^k = \bar{\Omega}_i^0. \tag{31}$$

We also prove using the preceding notation a very useful Ricci identity.

**Lemma 2** *Let $\bar{X}$ be a point of $\Sigma$ and $E_0 = N, E_1, \ldots, E_n$ be an adapted frame field such that each $E_i$ is a principal direction and $\omega_i^k = 0$ at $\bar{X}$. Let $(a_{ij})$ be the second quadratic form of $\Sigma$. Then, at the point $\bar{X}$, we have*

$$a_{ii;11} - a_{11;ii} = a_{11}a_{ii}^2 - a_{11}^2 a_{ii} + \bar{R}_{i0i0}\, a_{11} - \bar{R}_{1010}\, a_{ii} + \bar{R}_{i1i0;1} - \bar{R}_{1i10;i}.$$

The frame field $E_a$ may be obtained from the adapted frame field $N, X_1, \ldots, X_n$ by Gram-Schmidt procedure. Since this last frame depends only on $z$ and $\nabla' z$, we may conclude that components of $\bar{R}$ and $\bar{\nabla}\bar{R}$ calculated in terms of the frame $E_a$ depend only on $z$ and $\nabla' z$.

## 2.1 The prescribed curvature equation

Now we formulate the existence problem analytically. We consider $f$ and $\Gamma$ as defined in Section 1. Then, given the second fundamental form $(a_{ij})$ in $\Sigma$ we define

$$F\big((a_{ij})\big) = f(\lambda_1, \ldots, \lambda_n),$$

where $\lambda_i$ are the eigenvalues of $(a_{ij})$ calculated with respect to the induced metric $(g_{ij})$. It is convenient to denote the vector of principal curvatures $(\lambda_1, \ldots, \lambda_n)$ by $\lambda$. Admissible functions are those ones for which $\lambda$ always lies in $\Gamma$. We may consider $F$ as a map from $\mathcal{S} \times \mathbb{R}^n \times \mathbb{R}$ into $\mathbb{R}$ in the variables $z_{ij}$, $z_i$ and $z$.

Thus our problem is to find $\Sigma$, graph of an admissible function, so that

$$F\big(a_{ij}(z(u))\big) = \psi(z(u), u), \quad u \in M,$$

for some prescribed positive function $\psi$. We recall that is required that $f$ satifies

$$f_i = \frac{\partial f}{\partial \lambda_i} > 0 \tag{32}$$

and that $f$ is concave what implies that

$$\sum_i f_i \lambda_i \leq f. \tag{33}$$

We also assume the condition (4) and then we prove using the assumption (5) and following [3] that

$$\sum_i \lambda_i \geq \delta$$

for $\lambda \in \Gamma$ such that $f(\lambda) \geq \psi_0$. In fact, the set

$$\Gamma_\psi = \{\lambda \in \Gamma : f(\lambda) \geq \psi_0\}$$

is closed in $\mathbb{R}^n$, convex and symmetric. Thus the closest point in $\Gamma_\psi$ to the origin is of the form $(\lambda_0, \dots, \lambda_0)$. This geometric fact implies that any $\lambda \in \Gamma_\psi$ is located above the hyperplane

$$H = \Big\{\lambda \in \mathbb{R}^n : \sum_i \lambda_i = n\lambda_0\Big\}. \tag{34}$$

Hence, any $\lambda \in \Gamma_\psi$ is necessarily contained in the convex part of the cone $\Gamma$ which is above $H$. This implies that upper estimates for $\lambda$ imply automatically lower estimates.

We proceed by stating some useful analytical properties of $F$. Notice that $F$ is differentiable whenever $f$ is. We denote first and second derivatives of $F$ respectively by

$$F^{ij} = \frac{\partial F}{\partial a_{ij}} \quad \text{and} \quad F^{ij,kl} = \frac{\partial^2 F}{\partial a_{ij}\partial a_{kl}}. \tag{35}$$

These derivatives may be easily calculated if we assume that the matrix $(a_{ij})$ is diagonal with respect to $(g_{ij})$, due to the following lemma.

**Lemma 3** *If $(a_{ij})$ is diagonal at $\bar{X}$ then the matrix $(F^{ij})$ is also diagonal with positive eigenvalues $f_i$. Moreover, $F$ is concave and its second derivatives are given by*

$$F^{ij,kl}\eta_{ij}\eta_{kl} = \sum_{k,l} f_{kl}\eta_{kk}\eta_{ll} + \sum_{k\neq l} \frac{f_k - f_l}{\lambda_k - \lambda_l}\eta_{kl}^2. \tag{36}$$

*Finally one has*

$$\frac{f_i - f_j}{\lambda_i - \lambda_j} \leq 0. \tag{37}$$

*These expressions must be interpreted as limits in the case of multiple eigenvalues of $(a_{ij})$.*

The terms $F^{ij}$ are components of a rank two contravariant tensor. Thus one has

$$F^{ij}a_{ij} = F^i_j a^j_i$$

and if the matrix $(g_{ij})$ is assumed to be diagonal at $\bar{X}$, then $(F^i_j)$ is also diagonal at that point.

# 3 Height estimates

Now, we consider, for each $s$, $0 \leq s \leq 1$, the map

$$\Psi(s,t,u) = s\psi(t,u) + (1-s)\phi(t)k(t), \tag{38}$$

where $k(t) = f(\kappa(t))$ and $\phi$ is a positive real function defined in $I$, which satisfies the following conditions:

a) $\phi > 0$,

b) $\phi(t) > 1 \quad \text{for} \quad t \leq t_-$,

c) $\phi(t) < 1 \quad \text{for} \quad t \geq t_+$,

d) $\phi'(t) < 0$.

These conditions imply the existence of a unique point $t_0 \in (t_-, t_+)$ such that $\phi(t_0) = 1$. Combining the conditions above on $\phi$ and the hypothesis $(a)$ and $(b)$ in the statement of the theorem, one proves

**Lemma 4** *For $\psi$ as in Theorem 1, $\phi$ as prescribed above and the function $\Psi$ defined in* (38)*, the following statements are true:*

*i)* $\Psi(1, t, u) = \psi(t, u)$ *and* $\Psi(0, t, u) = \phi(t)k(t)$,

*ii)* $\Psi(s, t, u) > 0$,

*iii)* $\Psi(s, t, u) > k(t)$ *for* $t \leq t_-$,

*iv)* $\Psi(s, t, u) < k(t)$ *for* $t \geq t_+$.

*Furthermore, it is always possible to choose $\phi$ satisfying the prescribed conditions such that:*

*v)* $\frac{\partial}{\partial t}\Psi(s, t, u) + \kappa(t)\Psi(s, t, u) < 0$.

For $0 \leq s \leq 1$, consider the family of equations

$$\Upsilon(s, z) = F(a_{ij}(z)) - \Psi(s, z, u) = 0, \quad z = z(u). \tag{39}$$

Notice that the constant function $t = t_0$ is solution to the problem corresponding to $s = 0$. We denote it by $z_0$.

We are able to prove $C^0$ bounds uniform with respect to the parameter $s$ of this homotopy. More precisely, one proves

**Proposition 5** *Suppose that $\psi$ satisfies the conditions* (a) *and* (b) *in Theorem 1. If $z \in C^2(M)$ is a solution of the equation $\Upsilon(s, z) = 0$ for a given $0 \leq s \leq 1$, then*

$$t_- < z(u) < t_+, \quad u \in M. \tag{40}$$

*Proof:* Let $\bar{u}$ be a point of maximum for the function $z(u)$. This exists by the compactness of $M$. Let's assume that $z(\bar{u}) \geq t_+$. Consider then the leaf $M_{z(\bar{u})}$ and represent by $\Sigma$ the graph of $z$. Observe that $\Sigma$ and $M_{z(\bar{u})}$ are tangent at $(z(\bar{u}), \bar{u})$. Furthermore, with respect to the inwards normal vector common to both hypersurfaces at this point, $\Sigma$ lies above $M_{z(\bar{u})}$. But then the principal curvatures of $\Sigma$ at this point are greater than or equal to $\kappa(z(\bar{u}))$. Thus by the fact that $f$ has positive derivatives one concludes that

$$F(a_{ij}(z)) \geq k(z(\bar{u}))$$

at $(z(\bar{u}), \bar{u})$ what is in contradiction with $(iv)$ of Lemma (4). Hence $z(\bar{u}) < t_+$. Working in a similar way with the minimum $\hat{u}$ of $z(u)$ one concludes that $z(\hat{u}) > t_-$.

Now, we prove the following uniqueness result.

**Proposition 6** *Fixed $s = 0$ there exists an unique admissible solution $z_0$ of the equation $\Upsilon(0, z) = 0$, namely $z_0 = t_0$ where $t_0$ satisfies $\phi(t_0) = 1$.*

*Proof.* That $z_0$ is solution to this problem follows from

$$\Upsilon(0, z_0) = F(a_{ij}(z_0))) - k(t_0) = f(\kappa(t_0)) - k(t_0) = 0.$$

Let $\bar{z}$ be an admissible solution of $\Upsilon(0, z) = 0$. This means that

$$F(a_{ij}(\bar{z})) - \phi(\bar{z})k(\bar{z}) = 0.$$

Now, let $\bar{u} \in M$ a minimum point of $\bar{z}$. At this point, one has $\nabla' \bar{z} = 0$ and $\nabla'^2 \bar{z}$ is positive-definite. Since $\kappa = \frac{h'}{h}$ one computes explicitly at $\bar{u}$

$$a^i_j = g^{ik} a_{kj} = -\frac{1}{h^2} \sigma^{ik} \bar{z}_{kj} + \frac{h'}{h} \delta^i_j$$

Therefore if we consider a local frame around $\bar{u}$ which is orthonormal at $\bar{u}$ and which diagonalizes $\nabla'^2 \bar{z}$ at this point one obtains

$$a^i_j(\bar{z}(\bar{u})) \le \kappa(\bar{z}(\bar{u}))\delta^i_j$$

and since $f$ is increasing with respect to its arguments

$$\phi(\bar{z}(\bar{u}))k(\bar{z}(\bar{u})) = F(a_{ij}(\bar{z}(\bar{u}))) \le f(\kappa(\bar{z}(\bar{u}))) = k(\bar{z}(\bar{u})) = \phi(t_0)k(\bar{z}(\bar{u})).$$

Hence, since $\phi$ is a decreasing function one concludes from the choice of $\bar{u}$ as a minimum point that

$$\bar{z}(u) \ge \bar{z}(\bar{u}) \ge t_0,$$

for all $u \in M$. In a similar way, one proves that

$$\bar{z}(u) \le t_0$$

for all $u \in M$. Thus, one gets $z = z_0$. This finishes the proof.

# 4 Height and support functions

As before, let $\Sigma$ be the graph of $z$. We start by considering the functions $\tau : \Sigma \to \mathbb{R}$ and $\eta : \Sigma \to \mathbb{R}$ given by

$$\tau = -h\langle N, \bar{e}_0\rangle \quad \text{and} \quad \eta = -\int h \, \mathrm{d}t. \tag{41}$$

The following formulae will be useful later.

**Lemma 7** *The gradient vector fields of the functions $\tau$ and $\eta$ are*

$$\nabla\eta = -h\, \bar{e}_0^T, \tag{42}$$
$$\nabla\tau = -A^\Sigma(\nabla\eta), \tag{43}$$

*and its Hessian forms calculated with respect to given vector fields $V, W$ in $\Sigma$ are*

$$\nabla^2\eta(V, W) = \tau B(V, W) - h'\langle V, W\rangle, \tag{44}$$
$$\begin{aligned}\nabla^2\tau(V, W) &= -\langle \nabla_{\nabla\eta} A^\Sigma V, W\rangle - \langle \bar{R}(\nabla\eta, W)V, N\rangle \\ &- \tau\langle A^\Sigma V, A^\Sigma W\rangle + h'\langle A^\Sigma V, W\rangle,\end{aligned} \tag{45}$$

*Here, $\bar{e}_0^T$ denotes the tangential projection of the vector field $\bar{e}_0$.*

*Proof.* To simplify the calculations, we consider a local orthonormal frame $e_a$ around a point $\bar{u}$ of $M$ and the associated adapted frame field $N, E_1, \ldots, E_n$ along $\Sigma$ so that $\bar{\nabla} E_i|_{X(\bar{u})} = 0$. Using (14) one has

$$\mathrm{d}\eta = -h\,\mathrm{d}z = -h\langle \mathrm{d}X, \bar{e}_0\rangle = -h\langle \bar{e}_0^T, \omega^i E_i\rangle$$

and

$$\begin{aligned}
\mathrm{d}\tau &= -\mathrm{d}h\langle N, \bar{e}_0\rangle - h\langle \bar{\nabla} N, \bar{e}_0\rangle - h\langle N, \bar{\nabla}\bar{e}_0\rangle \\
&= -h'\bar{\theta}^0\langle N, \bar{e}_0\rangle + h\langle a_i^j E_j \omega^i, \bar{e}_0\rangle - h\langle N, \bar{\theta}_0^i \bar{e}_i\rangle \\
&= h\langle a_i^j E_j \omega^i, \bar{e}_0\rangle - h'\bar{\theta}^0\langle N, \bar{e}_0\rangle - h'\langle N, \bar{\theta}^i \bar{e}_i\rangle \\
&= h\langle A^\Sigma(E_i), \bar{e}_0^T\rangle \omega^i - h'\langle N, \bar{\theta}^0 \bar{e}_0 + \bar{\theta}^i \bar{e}_i\rangle.
\end{aligned}$$

Thus since $A^\Sigma$ is self-adjoint and $\mathrm{d}X = \bar{\theta}^0 \bar{e}_0 + \bar{\theta}^i \bar{e}_i$, one gets

$$\mathrm{d}\tau = h\langle A^\Sigma(\bar{e}_0^T), \omega^i E_i\rangle. \tag{46}$$

Therefore we conclude that

$$\nabla\eta = -h\,\bar{e}_0^T, \quad \nabla\tau = -A^\Sigma(\nabla\eta). \tag{47}$$

Since $\tau_i = h\langle a_i^j E_j, \bar{e}_0^T\rangle$, one computes, using $\nabla_{E_k} E_j|_{X(\bar{u})} = 0$,

$$\begin{aligned}
\tau_{i;k} &= h_k\langle a_i^j E_j, \bar{e}_0\rangle + h\langle a_{i;k}^j E_j, \bar{e}_0\rangle + h\langle a_i^j \bar{\nabla}_{E_k} E_j, \bar{e}_0\rangle \\
&+ h\langle a_i^j E_j, \bar{\nabla}_{E_k}\bar{e}_0\rangle \\
&= h'\langle a_i^j E_j, \bar{\theta}^0(E_k)\bar{e}_0\rangle + h\langle a_{i;k}^j E_j, \bar{e}_0\rangle + h a_i^j a_{kj}\langle N, \bar{e}_0\rangle \\
&+ h'\langle a_i^j E_j, \bar{\theta}^i(E_k)\bar{e}_i\rangle \\
&= h'\langle a_i^j E_j, E_k\rangle + h\langle a_{i;k}^j E_j, \bar{e}_0\rangle + h a_i^j a_{kj}\langle N, \bar{e}_0\rangle \\
&= h' a_{ik} - a_{i;k}^j \eta_j - \tau a_i^j a_{kj}.
\end{aligned}$$

where we used again that $\mathrm{d}X = \bar{\theta}^0 \bar{e}_0 + \bar{\theta}^i \bar{e}_i$ and that $\eta_k = -h\langle \bar{e}_0, E_k\rangle$. Hence, one gets from Codazzi's equation

$$\begin{aligned}
\nabla^2\tau(V, W) &= h'\langle A^\Sigma V, W\rangle - \langle(\nabla_W A^\Sigma)\nabla\eta, V\rangle - \tau\langle A^\Sigma V, A^\Sigma W\rangle \\
&= h'\langle A^\Sigma V, W\rangle - \langle(\nabla_{\nabla\eta} A^\Sigma)W, V\rangle - \langle \bar{R}(\nabla\eta, W)V, N\rangle \\
&- \tau\langle A^\Sigma V, A^\Sigma W\rangle.
\end{aligned}$$

Finally, it follows from the expression $\eta_i = -h\langle E_i, \bar{e}_0\rangle$ that

$$\begin{aligned}
\eta_{i;k} &= -h_k\langle E_i, \bar{e}_0\rangle - h\langle \bar{\nabla}_{E_k} E_i, \bar{e}_0\rangle - h\langle E_i, \bar{\nabla}_{E_k}\bar{e}_0\rangle \\
&= -h'\langle E_i, \bar{\theta}^0(E_k)\bar{e}_0 + \bar{\theta}^i(E_k)\bar{e}_i\rangle - h\,a_{ik}\langle N, \bar{e}_0\rangle \\
&= -h' g_{ik} + \tau a_{ik}.
\end{aligned}$$

Thus we obtain

$$\nabla^2\eta(V, W) = -h'\langle V, W\rangle + \tau\langle A^\Sigma V, W\rangle. \tag{48}$$

This finishes proving the lemma.

One estimates the derivatives of $\eta$ and $\psi$ as follows. In the sequel $\nabla_i$ and $\nabla_{ij}$ denote covariant derivative in $\Sigma$ calculated with respect to a frame adapted to $\Sigma$.

**Lemma 8** *The functions $\eta$ and $\psi$ satisfy the following estimates*

$$|\nabla\eta| \le C, \quad |\nabla\psi| \le C, \quad |\nabla^2\psi| \le C \tag{49}$$

*where $C$ are constants depending on $\psi$, $\nabla'\psi$, $\nabla'^2\psi$ and on $C^0$ and $C^1$ bounds for $z$.*

*Proof.* The first estimate follows from the $C^0$ and $C^1$ estimates for $z$. In fact, one has $\eta_i = -hz_i$. In order to prove the remaining estimates, we observe that

$$\nabla_i\psi = X_i(\psi) = e_i(\psi) + z_i e_0(\psi) =: \psi_i + z_i\psi_z.$$

Thus, using (18) and denoting $\psi_i = e_i(\psi)$ and so on we have

$$\begin{aligned}
|\nabla\psi|^2 &= g^{ij}X_i(\psi)X_j(\psi) = g^{ij}\big(\psi_i + z_i\psi_z\big)\big(\psi_j + z_j\psi_z\big)\\
&= \frac{1}{h^2}\Big(\delta^{ij}\psi_i\psi_j - \frac{z^iz^j}{W^2}\psi_i\psi_j + \delta^{ij}z_i\psi_j\psi_z - \frac{z^iz^j}{W^2}z_i\psi_j\psi_z\\
&+ \delta^{ij}z_j\psi_i\psi_z - \frac{z^iz^j}{W^2}z_j\psi_i\psi_z + \delta^{ij}z_iz_j\psi_z^2 - \frac{z^iz^j}{W^2}z_iz_j\psi_z^2\Big)\\
&= \frac{1}{h^2}\Big(|\nabla'\psi|^2 - \frac{1}{W^2}\langle\nabla'\psi,\nabla'z\rangle^2 + 2\psi_z\langle\nabla'\psi,\nabla'z\rangle\\
&- 2\frac{\psi_z}{W^2}|\nabla'z|^2\langle\nabla'\psi,\nabla'z\rangle + \psi_z^2|\nabla'z|^2 - \frac{\psi_z^2}{W^2}|\nabla'z|^4\Big)\\
&\le C(|z|_1, |\psi|_1, |\psi_z|).
\end{aligned}$$

In a similar way (replacing $\psi$ by $\psi_t = \psi_z$) we prove that

$$|\nabla\psi_z| \le C. \tag{50}$$

One has

$$X_iX_j(\psi) = X_i\big(\psi_j + z_j\psi_z\big) = \psi_{i,j} + z_{i,j}\psi_z + z_j\psi_{zi} + z_i\psi_{zj} + z_iz_j\psi_{zz},$$

where $\psi_{i,j} = e_ie_j(\psi)$ and $z_{i,j} = e_ie_j(z)$. We then choose a geodesic frame $e_a$ around $\bar u \in M$. In this case it holds that $z_{i,j} = \nabla'_{ij}z = z_{ij}$ at $\bar u$. Now using the fact that $\bar\theta^a_b = 0$ at $\bar u$, we obtain

$$\begin{aligned}
\bar\nabla_{X_j}X_i &= \big(\mathrm{d}z_i(X_j) + h'h\theta^i(X_j)\big)\,e_0 + \frac{h'}{h}\,\big(\delta_i^k\theta^0(X_j) + z_i\theta^k(X_j)\big)\,e_k\\
&= \big(z_{ij} + h'h\delta_{ij}\big)\,e_0 + \frac{h'}{h}\,\big(z_je_i + z_ie_j\big)
\end{aligned}$$

which implies that

$$\langle\nabla_{X_j}X_i, \nabla\psi\rangle = \big(z_{ij} + h'h\delta_{ij}\big)\psi_z + \frac{h'}{h}\big(z_j\psi_i + z_i\psi_j\big).$$

Hence, one obtains

$$\begin{aligned}
\nabla_{ij}\psi &= \langle\nabla_{X_j}\nabla\psi, X_j\rangle = \psi_{ij} + z_j\psi_{zi} + z_i\psi_{zj} + z_iz_j\psi_{zz}\\
&- h'h\delta_{ij}\psi_z - \frac{h'}{h}z_j\psi_i - \frac{h'}{h}z_i\psi_j.
\end{aligned}$$

Therefore we conclude that

$$|\nabla^2\psi| \le C(|z|_1, |\psi|_2, |\psi_z|_1).$$

This finishes the proof of the lemma.

# 5 Gradient estimate

In this section, we prove *a priori* global estimate for the first derivatives of $z$.

**Proposition 9** *Under the hypothesis of Theorem 1, if $z(u)$ is a solution of equation* (39) *for some fixed $0 \le s \le 1$, then $|\nabla' z| < C$, where $C$ is a constant that depends only on $t_-$, $t_+$ and $\psi$.*

*Proof.* We present the proof for $s = 1$. There is no essential changes for $0 \le s < 1$.

Set $\chi(z) = |\nabla' z| e^{Az}$, where $A$ is a positive constant to be chosen later on. Let $\bar{u}$ be a point where $\chi$ attains its maximum. If $\chi(\bar{u}) = 0$ then $|\nabla' z| \equiv 0$ and so the result is trivial. Hence, we are going to assume that $\chi(\bar{u}) > 0$. Thus we may define the function $\ln \chi(z) = \ln |\nabla' z| + Az$ which also attains its maximum at $\bar{u}$. Hence, fixing a special frame in some neighborhood of $\bar{u}$ one has

$$
\begin{aligned}
0 \quad &= \quad \chi_i = \frac{1}{|\nabla' z|} e^{Az} z_{ik} z^k + A e^{Az} |\nabla' z| z_i \\
&= \quad e^{Az} z_{i1} + A e^{Az} z^1 z_i,
\end{aligned}
$$

which implies by the symmetry $z_{i1} = z_{1i}$ of the Hessian form that

$$
z_{11} = -A z_1^2, \quad z_{1i} = 0, \quad i > 1. \tag{51}
$$

where we used the fact that $z_i|_{\bar{u}} = 0$ for $i \ne 1$. Substitution of this into (19) yields $a_{1i} = 0$ for $i > 1$. This implies that the direction $e_1$ at $\bar{u}$ is principal. Then, we may rotate the other vectors $e_2, \ldots, e_n$ so that they are also principal at $\bar{u}$. With this choice we have $a_{ij} = 0$ for $i \ne j$ ate $\bar{u}$. As a consequence of this, one sees from (19) that $z_{ij}(\bar{u}) = 0$ for $i \ne j$. Thus, the Hessian of $z$ is diagonal at $\bar{u}$.

Differentiating again the function $\chi$ at $\bar{u}$, one obtains (no summation over the index $i$)

$$
\begin{aligned}
0 \quad &\ge \quad \chi_{i;i} = A e^{Az} z_i z_{i1} + A^2 e^{Az} z_1 z_i^2 \\
&+ \quad e^{Az} \left( -\frac{1}{z_1} z_{i1}^2 + z_{i1i} + \frac{1}{z_1} z_{ii}^2 + A z_i z_{i1} + A z_1 z_{ii} \right).
\end{aligned}
$$

Hence, one concludes from this inequality that

$$
z_{111} + A^2 z_1^3 + 3A z_1 z_{11} \le 0, \tag{52}
$$

$$
z_{i1i} + \frac{1}{z_1} z_{ii}^2 + A z_1 z_{ii} \le 0. \tag{53}
$$

Combining the first inequality just above and (51) gives

$$
z_{111} - 2A^2 z_1^3 \le 0. \tag{54}
$$

From (53) and (23) one gets

$$
z_{ii1} \le -\frac{z_{ii}^2}{z_1} - A z_1 z_{ii} - K_i z_1 \quad \text{for } i > 1\,. \tag{55}
$$

Now we can start putting all this information together to obtain the desired estimate. We start by taking the derivative of equation (39) with respect to the direction $e_1$. Using the fact that the matrix $(a^i_j)$ is diagonal at $u_0$ and the remarks just after Lemma 3, we obtain:

$$\sum_{i=1}^n F^i_i \frac{\partial a^i_i}{\partial z_1} z_{11} + \sum_{i=1}^n F^i_i \frac{\partial a^i_i}{\partial z} z_1 = \psi_z z_1 - \sum_{i=1}^n F^i_i \frac{\partial a^i_i}{\partial z_{ii}} z_{ii1}. \tag{56}$$

Taking derivatives of $a^i_i$, using (19) we obtain

$$\begin{aligned}
\frac{\partial a^1_1}{\partial z_{11}} &= -\frac{h}{W^3}, \\
\frac{\partial a^1_1}{\partial z_1} &= -\frac{3z_1}{W^2} a^1_1 + \frac{4z_1 h'}{W^3}, \\
\frac{\partial a^1_1}{\partial z} &= \left(\frac{h'}{h} - \frac{3hh'}{W^2}\right) a^1_1 + \frac{2}{hW^3}\left(hh'' - h'^2\right) z_1^2 \\
&+ \frac{1}{W^3}\left(hh' + h^2 h''\right)
\end{aligned}$$

and for $i > 1$

$$\begin{aligned}
\frac{\partial a^i_i}{\partial z_{ii}} &= -\frac{1}{hW}, \\
\frac{\partial a^i_i}{\partial z_1} &= -\frac{z_1}{W^2} a^i_i, \\
\frac{\partial a^i_i}{\partial z} &= -h'\left(\frac{h}{W^2} + \frac{1}{h}\right) a^i_i + \frac{(hh')'}{hW}.
\end{aligned}$$

Replacing this into (56) and using (51) and rearranging terms yields

$$\begin{aligned}
& z_1 \left(\frac{3Az_1^2}{W^2} + \frac{h'}{h} - \frac{3hh'}{W^2}\right) F^1_1 a^1_1 \\
& + z_1 \left(-\frac{4Az_1^2 h'}{W^3} + \frac{2}{hW^3}(hh'' - h'^2) z_1^2 + \frac{1}{W^3}(hh' + h^2 h'')\right) F^1_1 \\
& + z_1 \left(\frac{Az_1^2}{W^2} - h'\left(\frac{h}{W^2} + \frac{1}{h}\right)\right) \sum_{i>1} F^i_i a^i_i + z_1 \frac{(hh')'}{hW} \sum_{i>1} F^i_i \\
& = \psi_z z_1 + F^1_1 \frac{h}{W^3} z_{111} + \sum_{i>1} F^i_i \frac{1}{hW} z_{ii1}.
\end{aligned} \tag{57}$$

Using (54) and (55) we can estimate the right hand side of (57) by

$$\begin{aligned}
& \mathrm{RHS} \le \psi_z z_1 + F^1_1 \frac{2A^2 h z_1^3}{W^3} - A z_1 \sum_{i>1} F^i_i \frac{z_{ii}}{hW} - \frac{K_i z_1}{hW} \sum_{i>1} F^i_i \\
& \le \psi_z z_1 + A z_1 \sum_i F^i_i a^i_i + F^1_1 \left(\frac{A^2 h z_1^3}{W^3} - A z_1 \frac{h'}{W^3}(2z_1^2 + h^2)\right) \\
& - \left(\frac{Ah' z_1}{W} + \frac{K_i z_1}{hW}\right) \sum_{i>1} F^i_i,
\end{aligned} \tag{58}$$

where we used the expressions of $a_1^1$ and $a_i^i$ given in (19) and the fact that $F_i^i > 0$.

Transposing the term in $\sum_{i>1} F_i^i$ from the right hand side in (58) to the left hand side of the equation (57), and adding it with the one that was already there and finally choosing $A$ so that

$$Ah'h + (h'h)' + \min_i K_i > 0 \tag{59}$$

results, by the fact that $h' > 0$, in a positive term that can be discarded. Notice that $K_i = \langle R(e_1, e_i)e_1, e_i\rangle$ does not depend on derivatives of $z$. This and the fact that $h$ and its derivatives are uniformly bounded in the annulus $\bar{M}_{t_-,t_+}$ show that we may choose any $A \geq A_0$ for some $A_0$ which depends only on $t_-, t_+$ and $|z|_0$.

We may estimate the left hand side of the inequality resulting from (57) after these manipulations as

$$\begin{aligned} \mathrm{LHS} \quad &\geq \quad z_1 \left(\frac{Az_1^2}{W^2} - \frac{hh'}{W^2} - \frac{h'}{h}\right) \sum_i F_i^i a_i^i \\ &+ \quad z_1 \left(\frac{2Az_1^2}{W^2} + \frac{2h'}{h} - \frac{2hh'}{W^2}\right) F_1^1 a_1^1 \\ &+ \quad \frac{z_1}{W^3}\bigg( - 4Ah'z_1^2 + \frac{2z_1^2}{h}(hh'' - h'^2) \\ &+ \quad hh' + h^2h''\bigg) F_1^1. \end{aligned} \tag{60}$$

Transpose the term with $F_1^1$ from the right hand side in (58) to the right hand side in (60) and add it to the one that exists there. Transpose the term in $\sum_i F_i^i a_i^i$ from the right hand side in (60) to the right hand side of the inequality (58) obtaining

$$\mathrm{RHS} \leq \psi_z z_1 + Az_1 \sum_i F_i^i a_i^i - z_1 (\frac{Az_1^2}{W^2} - \frac{hh'}{W^2} - \frac{h'}{h}) \sum_i F_i^i a_i^i. \tag{61}$$

For the left hand side we obtain

$$\begin{aligned} \mathrm{LHS} \quad &\geq \quad 2z_1 \left(\frac{Az_1^2}{W^2} + \frac{h'}{h} - \frac{hh'}{W^2}\right) F_1^1 a_1^1 + \frac{z_1}{W^3}\bigg(\frac{2z_1^2}{h}(hh'' - h'^2) \\ &+ \quad hh' + h^2h'' + Ah'(-2z_1^2 + h^2) - A^2 z_1^2 h\bigg) F_1^1. \end{aligned} \tag{62}$$

Thus, replacing in (62) the expression for $a_1^1$ in (19) and gathering the resulting expression to (61), one gets

$$\begin{aligned} &\frac{2z_1}{W^3}\left(\frac{Az_1^2}{W^2} + \frac{h'}{h} - \frac{hh'}{W^2}\right)\left(Ahz_1^2 + 2h'z_1^2 + h^2h'\right) F_1^1 \\ &+ \frac{z_1}{W^3}\left(\frac{2z_1^2}{h}(hh'' - h'^2) + hh' + h^2h'' + Ah'(-2z_1^2 + h^2) - A^2 z_1^2 h\right) F_1^1 \\ &\leq \psi_z z_1 + Az_1 \sum_i F_i^i a_i^i - z_1 (\frac{Az_1^2}{W^2} - \frac{hh'}{W^2} - \frac{h'}{h}) \sum_i F_i^i a_i^i. \end{aligned} \tag{63}$$

Observe that in (63) all coefficients of $F_1^1$ have uniform lower bounds and moreover that the first term in the left hand side of (63) is nonnegative. Thus, it is possibel to consider this inequality as polynomial in $A$ writing it as

$$\begin{aligned} F_1^1\big(aA^2+bA+c\big) &\leq \psi_1+\psi_z z_1 + Az_1\sum_i F_i^i a_i^i \\ &- z_1\sum_i F_i^i a_i^i\big(\frac{Az_1^2}{W^2}-\frac{hh'}{W^2}-\frac{h'}{h}\big), \end{aligned} \tag{64}$$

where $a, b$ e $c$ are coefficients uniformly bounded in terms of the functions $h$, $h'$ e $h''$. Thus, we must consider two cases. First, we suppose that $F_1^1$ is uniformly bounded from zero, i.e., that there exists a constant $C>0$ such that $F_1^1 \geq C$ em $\Sigma$. In this case, the coefficient

$$a=\frac{hz_1^3}{W^5}(z_1^2-h^2)F_1^1 \tag{65}$$

is necessarily nonpositive, since $A$ may be chosen arbitrarily large in (64). Thus, it follows that $z_1(\bar{u}) \leq h(z(\bar{u}))$ and therefore $z_1(\bar{u}) < h(t_+)$.

The other possibility is that $F_1^1$ has no strictly positive lower bound. In this case, it is convenient to write the left hand side in (63) as

$$\begin{aligned} &F_1^1\Big(2\big(A+\frac{h'}{h}\big)(Ah+h')x^5+\big(h''-\frac{h'}{h}-Ah'-A^2h\big)x^3 \\ &+\big(h''+\frac{h'}{h}+Ah'\big)x\Big). \end{aligned} \tag{66}$$

where $x=\frac{z_1}{W}$. Notice that we may suppose without loss of generality that $x=O(1)$. Otherwise, there exists some constant $\alpha<1$ so that $x\leq\alpha$ what implies the estimate

$$(1-\alpha^2)z_1^2 \leq \alpha^2h^2.$$

Thus, fixing $A=A_0$ in (64), the coefficients in $x$ are uniformly bounded for $x=O(1)$. This implies that the the expression in (64) is $O(\varepsilon)$ for some very small $\varepsilon>0$. Thus, we conclude using the inequality $\psi \geq \sum_i F_i^i a_i^i$ that (63) may be written as

$$O(\varepsilon)W^2-\big(\psi_z+\frac{h'}{h}\psi\big)z_1W^2 \leq \psi_1W^2+A_0h^2\psi+h'h\psi. \tag{67}$$

The hypothesis $(c)$ in Theorem 1 may be stated as

$$\psi_z+\frac{h'}{h}\psi \leq 0. \tag{68}$$

Then if we choose

$$\varepsilon \ll \frac{1}{W^2},$$

an estimate for $W|_{\bar{u}}$ follows from (67).

In both cases, by definition of the function $\chi$, a bound for $z_1(\bar{u})$ implies an uniform bound for $\nabla' z$. This completes the proof of the Proposition 9.

# 6 Hessian bounds

This section is devoted to the proof of Hessian estimates. We will show that the terms of the second fundamental form are bounded by above. Since we already have $C^0$ and $C^1$ estimates, then this information allow us to obtain the Hessian estimates.

With this purpose in mind, we define the following function on the unit tangent bundle of $\Sigma$:

$$\tilde{\zeta}(u,\xi) = B(\xi,\xi)\, e^{\varphi(\tau)-\beta\eta}, \tag{69}$$

where $u \in M$, $\xi$ is an unit tangent vector to $\Sigma$ at $(z(u),u)$, the functions $\tau$ e $\eta$ are defined in (41), the constant $\beta > 0$ will be chosen later and the real function $\varphi$ is defined as follows. Notice that by definition the function $\tau$ is bounded by constants depending on bounds for $z$ and $\nabla' z$. Hence, it is possible to choose $a > 0$ so that $\tau \geq 2a$. Thus, we define

$$\varphi(\tau) = -\ln(\tau - a). \tag{70}$$

Hence, one has differentiating with respect to $\tau$

$$\ddot{\varphi} - (1+\epsilon)\dot{\varphi}^2 = \frac{1}{(\tau-a)^2} - \frac{1+\epsilon}{(\tau-a)^2} = -\frac{\epsilon}{(\tau-a)^2} < 0 \tag{71}$$

and by the choice of $a$ given an arbitrary positive constant $C$, one has

$$-(1+\dot{\varphi}\tau) + C(\ddot{\varphi} - (1+\epsilon)\dot{\varphi}^2) \geq \hat{C},$$

for some positive constant $\hat{C}$ depending on bounds for $z$ and $\nabla' z$.

We suppose that the maximum of $\tilde{\zeta}$ is attained at a point $\bar{u}$ and along the direction $\bar{\xi}$ tangent to $\bar{X} = (z(\bar{u}),\bar{u})$. We may choose a geodesic orthonormal reference frame $E_a$ around $\bar{X}$ as defined in Section 2 so that $\omega_i^k|_{\bar{X}} = 0$. One may rotate this frame in such a way that $\bar{\xi} = E_1$ at $\bar{X}$. We then consider the local function $a_{11} = B(E_1,E_1)$. Thus we easily verifies that the function

$$\zeta(p) = a_{11}\, e^{\varphi(\tau)-\beta\eta} \tag{72}$$

attains maximum at $\bar{X}$. Thus, it holds at $\bar{u}$

$$0 = (\ln\zeta)_i = \frac{a_{11;i}}{a_{11}} + \dot{\varphi}\tau_i - \beta\eta_i \tag{73}$$

and the Hessian matrix with components

$$(\ln\zeta)_{i;j} = \frac{a_{11;ij}}{a_{11}} - \frac{a_{11;i}a_{11;j}}{a_{11}^2} + \dot{\varphi}\tau_{i;j} + \ddot{\varphi}\tau_i\tau_j - \beta\eta_{i;j}$$

is negative-definite. Thus

$$\begin{aligned} F^{ij}(\ln\zeta)_{ij} &= \frac{1}{a_{11}}F^{ij}a_{11;ij} - \frac{1}{a_{11}^2}F^{ij}a_{11;i}a_{11;j} + \dot{\varphi}F^{ij}\tau_{i;j} \\ &+ \ddot{\varphi}F^{ij}\tau_i\tau_j - \beta F^{ij}\eta_{ij} \leq 0 \end{aligned} \tag{74}$$

It is clear that $a_{11}$ is the greatest eigenvalue of $B$ and therefore $a_{1i} = 0$ for $i \neq 1$. Thus, we may rotate the orthogonal complement of $E_1$ so that in the resulting frame the matrix $(a_{ij})$ is diagonal at $\bar{X}$. By Lemma 3, it results that $(F^{ij})$ is also diagonal with $F^{ii} = f_i$. We denote $\lambda_i = a_{ii}(\bar{u})$ and choose indices in such a way that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

Moreover, we assume without loss of generality that $\lambda_1 > 1$ at $\bar{u}$. Thus, according Lemma 3, we have

$$f_1 \leq f_2 \leq \cdots \leq f_n.$$

From (74) one then gets

$$\sum_i \Big(\frac{1}{\lambda_1} f_i a_{11;ii} - \frac{1}{\lambda_1^2} f_i |a_{11;i}|^2 + \dot{\varphi} f_i \tau_{i;i} + \ddot{\varphi} f_i |\tau_i|^2 - \beta f_i \eta_{i;i}\Big) \leq 0 \qquad (75)$$

Now, we differentiate covariantly with respect to the metric $(g_{ij})$ in $\Sigma$ the equation (6) in the direction of $E_1$ obtaining

$$F^{ij} a_{ij;1} = \psi_1$$

and differentiating again

$$F^{ij} a_{ij;11} + F^{ij,kl} a_{ij;1} a_{kl;1} = \psi_{1;1}. \qquad (76)$$

From Ricci identity in Lemma 1 and using the fact that $\delta(f) \leq \sum_i f_i \lambda_i \leq f = \psi$ we have

$$\begin{aligned} F^{ij} a_{ij;11} \quad &\leq \quad -\lambda_1^2 \delta + |\bar{R}_{1010}|\psi + \sum_i \big(f_i a_{11;ii} + \lambda_1 f_i \lambda_i^2 + \lambda_1 f_i \bar{R}_{i0i0} \\ &+ \quad f_i \bar{R}_{i0i0;1} - f_i \bar{R}_{1010;i}\big). \end{aligned}$$

Combining this expression and (76) and replacing the resulting expression in (75) one has

$$\begin{aligned} &\frac{\psi_{1;1}}{\lambda_1} + \frac{1}{\lambda_1}\big(\delta \lambda_1^2 - \psi |\bar{R}_{1010}|\big) - \frac{1}{\lambda_1} F^{ij,kl} a_{ij;1} a_{kl;1} - \sum_i f_i \lambda_i^2 \\ &- \sum_i f_i \bar{R}_{i0i0} - \frac{1}{\lambda_1} \sum_i f_i \big(\bar{R}_{i0i0;1} - \bar{R}_{1010;i}\big) + \sum_i \Big( - \frac{1}{\lambda_1^2} f_i |a_{11;i}|^2 \\ &+ \dot{\varphi} f_i \tau_{i;i} + \ddot{\varphi} f_i |\tau_i|^2 - \beta f_i \eta_{i;i}\Big) \leq 0. \end{aligned}$$

From (44) we have at $\bar{X}$

$$\beta \sum_i f_i \eta_{i;i} = \beta \sum_i \big(\tau f_i a_{ii} - h' f_i g_{ii}\big) \leq \beta\big(\tau \psi - h' T\big),$$

where $T = \sum_i f_i$. From (45) and denoting

$$\bar{R}_{ki} := \langle \bar{R}(E_k, E_i) E_i, N \rangle = \bar{\Omega}_i^0(E_k, E_i)$$

and using that $\dot{\varphi} < 0$ it holds at $\bar{X}$ that

$$\begin{aligned} \dot{\varphi} \sum_i f_i \tau_{i;i} \quad &\geq \quad -\dot{\varphi}\Big(\sum_{i,k} \eta^k f_i a_{ii;k} + \sum_{i,k} \eta^k \bar{R}_{ki} f_i\Big) - \dot{\varphi}\tau \sum_i f_i \lambda_i^2 \\ &+ \quad \dot{\varphi} h' \psi \\ &= \quad -\dot{\varphi}\Big(\sum_k \eta^k \psi_k + \sum_{i,k} \eta^k \bar{R}_{ki} f_i\Big) - \dot{\varphi}\tau \sum_i f_i \lambda_i^2 + \dot{\varphi} h' \psi. \end{aligned}$$

Using (49) and estimating the ambient curvature terms by constants $C_k$ terms one obtains from Lemma 8

$$-\sum_k \dot\varphi\big(\eta^k(\psi_k + C_k T)\big) + \dot\varphi h'\psi \geq -|\dot\varphi|(C + CT).$$

Therefore, we have

$$\dot\varphi \sum_i f_i \tau_{i;i} \geq -|\dot\varphi|(C + CT) - \dot\varphi\tau \sum_i f_i \lambda_i^2.$$

Now, we suppose without loss of generality that

$$\lambda_1 \geq \frac{1}{C}\sum_i |R_{i0i0;1} - R_{1010;i}|,$$

for some $C > 0$. Moreover, supposing that $\lambda_1 \geq 1$ one has

$$-\frac{1}{\lambda_1}\psi|\bar R_{1010}| \geq -C \quad \text{and} \quad \frac{\psi_{1;1}}{\lambda_1} \geq -C$$

for some positive constant $C$. Finally one has

$$-\sum_i f_i \bar R_{i0i0} \geq -T \max_i |\bar R_{i0i0}| \geq -CT.$$

We then conclude from these inequalities that

$$\begin{aligned}
&-C - CT + \delta\lambda_1 - \frac{1}{\lambda_1}F^{ij,kl}a_{ij;1}a_{kl;1} - \sum_i f_i\lambda_i^2 \\
&-\frac{1}{\lambda_1^2}\sum_i f_i|a_{11;i}|^2 - |\dot\varphi|(C + CT) - \dot\varphi\tau\sum_i f_i\lambda_i^2 \\
&+\ddot\varphi\sum_i f_i|\tau_i|^2 - \beta\big(\tau\psi - h'T\big) \leq 0.
\end{aligned} \tag{77}$$

Finally, we also have from (73) for any $\epsilon > 0$ the inequality

$$\frac{1}{\lambda_1^2}f_i|a_{11;i}|^2 \leq (1 + \frac{1}{\epsilon})\beta^2 f_i|\eta_i|^2 + (1+\epsilon)\dot\varphi^2 f_i|\tau_i|^2. \tag{78}$$

Now, for proceed in our analysis, we consider two cases.

$1^{st}$ **Case.** In this case, we suppose that $\lambda_n \leq -\theta\lambda_1$ for some positive constant $\theta$ to be chosen later.

Replacing the sum of terms in (78) in the inequality (77) and using Lemma 8 one has after grouping terms in $T$

$$\begin{aligned}
&\delta\lambda_1 - C - C|\dot\varphi| - \frac{1}{\lambda_1}F^{ij,kl}a_{ij;1}a_{kl;1} \\
&-\big(C + C|\varphi'| - h'\beta + C(1 + \frac{1}{\epsilon})\beta^2\big)T \\
&-(1 + \dot\varphi\tau)\sum_i f_i\lambda_i^2 + \big(\ddot\varphi - (1+\epsilon)\dot\varphi^2\big)\sum_i f_i|\tau_i|^2 - \beta\tau\psi \leq 0.
\end{aligned}$$

Using (43) and the fact that $\big(a_{ij}\big)$ is diagonal at $\bar{X}$ and Lemma 8 we calculate

$$\sum_i f_i|\tau_i|^2 = \sum_i f_i\lambda_i^2|\eta_i|^2 \leq C\sum_i f_i\lambda_i^2. \tag{79}$$

Hence, we get

$$\begin{aligned}
&\delta\lambda_1 - C - C|\dot\varphi| - \frac{1}{\lambda_1}F^{ij,kl}a_{ij;1}a_{kl;1} \\
&-\big(C + C|\dot\varphi| - h'\beta + C(1+\frac{1}{\epsilon})\beta^2\big)T \\
&+\big(-(1+\dot\varphi\tau) + C(\ddot\varphi - (1+\epsilon)\dot\varphi^2)\big)\sum_i f_i\lambda_i^2 - \beta\tau\psi \leq 0. \quad (80)
\end{aligned}$$

Now, using the concavity of $F$ we may discard the third term in the left-hand side of (80) since it is non-negative obtaining

$$-C_1(\beta) - C_2(\beta)T + \delta\lambda_1 + \hat{C}\sum_i f_i\lambda_i^2 \leq 0,$$

where $C_1$ depends linearly on $\beta$ and $C_2$ depends quadratically on $\beta$. Since $f_n \geq \frac{1}{n}T$, we have

$$\sum_i f_i\lambda_i^2 \geq f_n\lambda_n^2 \geq \frac{1}{n}\theta^2 T\lambda_1^2.$$

Thus it follows that

$$-C_1 - C_2T + \delta\lambda_1 + \hat{C}\frac{1}{n}\theta^2T\lambda_1^2 \leq 0. \tag{81}$$

This inequality shows that $\lambda_1$ has an upper bound. In fact, if we assume without loss of generality that $\lambda_1 \geq \bar{C}$ for some positive constant $\bar{C}$, the coefficients of the terms in $T$ in (81) have a nonnegative sum. Thus, discarding these terms, one gets

$$\lambda_1 \leq \frac{C_1}{\delta}.$$

$2^{nd}$ **Case:** In this case, we assume that $\lambda_n \geq -\theta\lambda_1$. Hence, $\lambda_i \geq -\theta\lambda_1$. We then group the indices in $\{1,...,n\}$ in two sets $I_1 = \{j; f_j \leq 4f_1\}$ and $I_2 = \{j; f_j > 4f_1\}$. Using (78) we have for $i \in I_1$

$$\frac{1}{\lambda_1^2}f_i|a_{11;i}|^2 \quad \leq \quad (1+\epsilon)\dot\varphi^2 f_i|\tau_i|^2 + C(1+\frac{1}{\epsilon})(\beta)^2 f_1.$$

Therefore, it follows from (77) that

$$\begin{aligned}
&-C - CT + \delta\lambda_1 - \frac{1}{\lambda_1}F^{ij,kl}a_{ij;1}a_{kl;1} - \big(1+\dot\varphi\tau\big)\sum_i f_i\lambda_i^2 \\
&-\frac{1}{\lambda_1^2}\sum_{j\in I_2} f_j|a_{11;j}|^2 - |\dot\varphi|(C+CT) + \big(\ddot\varphi - (1+\epsilon)\dot\varphi^2\big)\sum_i f_i|\tau_i|^2 \\
&-C(1+\frac{1}{\epsilon})\beta^2 f_1 - \beta\big(\tau\psi - h'T\big) \leq 0.
\end{aligned}$$

Notice that we had summed up to the inequality the non-positive terms

$$-(1+\epsilon)|\dot{\varphi}|^2 \sum_{i\in I_2} f_i |\tau_i|^2$$

Using Lemma 8, one has

$$|\tau_i|^2 = |\lambda_i \eta_i|^2 \leq C\lambda_i^2$$

and as we had seen above one may prove that

$$-\big(1+\dot{\varphi}\tau\big)\sum_i f_i\lambda_i^2 + \big(\ddot{\varphi} - (1+\epsilon)\dot{\varphi}^2\big)\sum_i f_i|\tau_i|^2 \geq \hat{C}\sum_i f_i\lambda_i^2 \tag{82}$$

for some positive constant $\hat{C}$. Thus we have

$$\begin{aligned}
&-C - CT + \delta\lambda_1 - \frac{1}{\lambda_1}F^{ij,kl}a_{ij;1}a_{kl;1} + \hat{C}\sum_i f_i\lambda_i^2 \\
&-\frac{1}{\lambda_1^2}\sum_{j\in I_2} f_j|a_{11;j}|^2 - |\dot{\varphi}|(C+CT) - C\big(1+\frac{1}{\epsilon}\big)\beta^2 f_1 \\
&-\beta\big(\tau\psi - h'T\big) \leq 0.
\end{aligned}$$

Denoting $\bar{R}_{j1} = \bar{\Omega}_1^0(E_j, E_1)$ one has by Lemma 3 and the fact that $1 \notin I_2$ and using Codazzi's equation

$$-\frac{1}{\lambda_1}F^{ij,kl}a_{ij;1}a_{kl;1} \geq -\frac{2}{\lambda_1}\sum_{j\in I_2}\frac{f_1 - f_j}{\lambda_1 - \lambda_j}\big(a_{11;j} + \bar{R}_{j1}\big)^2. \tag{83}$$

Following [7], we may verify that choosing $\theta = \frac{1}{2}$ it holds that for all $j \in I_2$ it holds that

$$-\frac{2}{\lambda_1}\frac{f_1 - f_j}{\lambda_1 - \lambda_j} \geq \frac{f_j}{\lambda_1^2}. \tag{84}$$

Considering the inequalities (83) and (84) and using (82) one has

$$\begin{aligned}
&-C - CT + \delta\lambda_1 + \sum_{j\in I_2}\frac{f_j}{\lambda_1^2}a_{11;j}^2 + 2\sum_{j\in I_2}\frac{f_j}{\lambda_1^2}a_{11;j}\bar{R}_{j1} \\
&+\hat{C}\sum_i f_i\lambda_i^2 - \sum_{j\in I_2}\frac{f_j}{\lambda_1^2}a_{11;j}^2 - |\dot{\varphi}|(C+CT) \\
&-C(1+\frac{1}{\epsilon})\beta^2 f_1 - \beta\big(\tau\psi - h'T\big) \leq 0.
\end{aligned}$$

Hence one obtains

$$\begin{aligned}
&-C - CT + \delta\lambda_1 + 2\sum_{j\in I_2}\frac{f_j}{\lambda_1}(-\dot{\varphi}\tau_j + \beta\eta_j)\bar{R}_{j1} \\
&+\hat{C}\sum_i f_i\lambda_i^2 - |\dot{\varphi}|(C+CT) - C(1+\frac{1}{\epsilon})\beta^2 f_1 \\
&-\beta\big(\tau\psi - h'T\big) \leq 0.
\end{aligned}$$

We now estimate using that $\dot{\varphi} < 0$ and that $\lambda_j \leq \lambda_1$ and $-\lambda_j \leq \theta\lambda_1 < \lambda_1$

$$2\frac{f_j}{\lambda_1}(-\dot{\varphi}\tau_j)\bar{R}_{j1} \geq 2\frac{f_j}{\lambda_1}\dot{\varphi}|\lambda_j||\eta_j\bar{R}_{j1}| \geq 2f_j\dot{\varphi}|\eta_j\bar{R}_{j1}|.$$

We also may suppose without loss of generality that it holds that

$$\lambda_1 \geq \frac{3|\eta_j \bar{R}_{j1}|}{h'}$$

for all $j \in I_2$. Thus, these inequalities imply that

$$\begin{aligned}
&-C - CT + \delta\lambda_1 + 2\sum_{j\in I_2} f_j \dot{\varphi}|\eta_j \bar{R}_{j1}| - 2\frac{\beta h'}{3}T \\
&+\hat{C}\sum_i f_i \lambda_i^2 - |\dot{\varphi}|(C + CT) - C\big(1 + \frac{1}{\epsilon}\big)\beta^2 f_1 \\
&-\beta\big(\tau\psi - h'T\big) \leq 0.
\end{aligned}$$

Since $\sum_{j\in I_2} f_j \leq T$, $|\eta_j \bar{R}_{j1}| \leq C$, $\dot{\varphi} < 0$ one has

$$-C - \big(C + C|\dot{\varphi}| + 2\beta\frac{h'}{3} - \beta h'\big)T - C\big(1 + \frac{1}{\epsilon}\big)\beta^2 f_1 + \delta\lambda_1 + \hat{C} f_1 \lambda_1^2 \leq 0.$$

Choosing $\beta > 0$ sufficiently large the term in $T$ is positive and we may discard it obtaining

$$-C - C_2(\beta) f_1 + \delta\lambda_1 + \hat{C} f_1 \lambda_1^2 \leq 0, \tag{85}$$

where $C_2$ depends quadratically on $\beta$. Reasoning as above, one concludes that this inequality gives an upper bound for $\lambda_1$.

# 7 The proof of the Theorem

To prove the theorem we are going to use the degree theory for nonlinear elliptic partial differential equations developed by Yan Yan Li. We refer the reader to [8].

In Sections 3, 5 and 6 above, it is proved that admissible $C^4$ function $z$ which solve the equation $\Upsilon(s, z) = 0$ for some $0 \leq s \leq 1$ satisfy the following bounds

$$t_- < z(u) < t_+, \quad u \in M \tag{86}$$

and

$$|z|_2 \leq C \tag{87}$$

for some positive constant $C$ which depends on $n, t_-, t_+$ and $\psi$. Then the $C^{4,\alpha}$ estimate for some $\alpha \in [0, 1]$ follows from (87) and from the results of L. C. Evans e N. V. Krylov as stated in Theorem 17.16 in [6]. One has

$$|z|_{4,\alpha} < C \tag{88}$$

for some constant $C > 0$.

Fixed that $\alpha$ we denote by $C_a^{4,\alpha}(M)$ the subset of $C^{4,\alpha}(M)$ consisting of admissible functions for $F$ and define as in Section 2 the homotopy

$$\Upsilon(s, \cdot\,) : C_a^{4,\alpha}(M) \to C^{2,\alpha}(M), \quad 0 \leq s \leq 1 \tag{89}$$

and we consider the family of equations $\Upsilon(s, z) = 0$. In order to apply degree theory, we need to prove certain assertions which are intermediate steps in the method.

It is easy to see in view of the $C^0$ and $C^1$ estimates that there exists $\hat{C} > 0$ for which

$$\hat{C} \leq \Psi(s, z(u), u) \leq \frac{1}{\hat{C}}, \quad u \in M, \tag{90}$$

for $0 \leq s \leq 1$ and any $z \in C^{4,\alpha}(M)$ satisfying (86) and (88). Now, if $z \in C_a^{4,\alpha}(M)$ solves $\Upsilon(s, z) = 0$ for some $0 \leq s \leq 1$, then

$$F(a_{ij}(z)) = \Psi(s, z(u), u)$$

and obviously

$$\hat{C} \leq F(a_{ij}(z(u))) \leq \frac{1}{\hat{C}}, \quad u \in M. \tag{91}$$

However, we may verify that there exists some open bounded set $V \subset \Gamma$ with $\bar{V} \subset \Gamma$ such that if

$$\hat{C} \leq f(\lambda_1(z(u)), \ldots, \lambda_n(z(u))) \leq \frac{1}{\hat{C}}$$

then

$$\lambda(z(u)) \in V. \tag{92}$$

In particular, by (91) we conclude that the matrix $(a_{ij}(z))$ satisfies

$$\lambda(a_{ij}(z)) \in V. \tag{93}$$

We then define the open set $\mathcal{O}$ in $C_a^{4,\alpha}(M)$ consisting of the admissible functions satisfying (86), (88) and (93). Thus, our reasoning above shows that any admissible solution $z$ of $\Upsilon(s, z) = 0$ for some $0 \leq s \leq 1$ is contained in $\mathcal{O}$. In particular, we conclude that

$$\Upsilon(s, \cdot)^{-1}(0) \cap \partial\mathcal{O} = \emptyset, \quad 0 \leq s \leq 1. \tag{94}$$

Thus, according to Definition 2.2 in [8] the degree $\deg(\Upsilon(s, \cdot), \mathcal{O}, 0)$ is well-defined for all $0 \leq s \leq 1$.

Proposition 6 shows that $z_0 = t_0$ is the unique admissible solution to $\Upsilon(0, z) = 0$ in $C_a^{4,\alpha}(M)$. We must prove that the Frechét derivative $\Upsilon_z(0, z_0)$ calculated around $z_0$ is an invertible operator from $C^{4,\alpha}(M)$ to $C^{2,\alpha}(M)$. One computes

$$\Upsilon(0, \rho z_0) = F(a_{ij}(\rho z_0)) - \phi(\rho t_0)k(\rho t_0) = k(\rho t_0) - \phi(\rho t_0)k(\rho t_0)$$

and using the fact that $\phi(t_0) = 1$ and that $\phi'(t_0) < 0$

$$\Upsilon_z(0, z_0) \cdot z_0 = \frac{\mathrm{d}}{\mathrm{d}\rho}\Upsilon(0, \rho z_0)|_{\rho=1} = -\phi'(t_0)k(t_0) > 0$$

On the other hand, since obviously $\nabla' z_0 = 0$ and $\nabla'^2 z_0 = 0$, then $\Upsilon_z(0, z_0) \cdot z_0$ is just a multiple of the zeroth order term in $\Upsilon_z(0, z_0)$. We conclude that $\Upsilon_z(0, z_0)$ is an invertible negatively elliptic operator.

We finally calculate $\deg(\Upsilon(1, \cdot), \mathcal{O}, 0)$. From Proposition 2.2 in [8], it follows that $\deg(\Upsilon(s, \cdot), O, 0)$ is independent from $s$. In particular,

$$\deg(\Upsilon(1, \cdot), \mathcal{O}, 0) = \deg(\Upsilon(0, \cdot), \mathcal{O}, 0).$$

On the other hand, we had just proved that the equation $\Upsilon(0, z) = 0$ has an unique admissible solution $z_0$ and that the linearized operator $\Upsilon_z(0, z_0)$ is invertible. Thus, by Proposition 2.3 in [8] one gets

$$\deg(\Upsilon(0, \cdot), \mathcal{O}, 0) = \deg(\Upsilon_z(0, z_0), \mathcal{O}, 0) = \pm 1.$$

Therefore,

$$\deg(\Upsilon(1, \cdot), \mathcal{O}, 0) \neq 0.$$

Thus, the equation $\Upsilon(1, z) = 0$ has at least one solution $z \in O$. This completes the proof of the theorem.

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Francisco J. de Andrade
Universidade Federal de Campina Grande
Centro de Formação de Professores
Campus de Cajazeiras
Cajazeiras – Paraíba
58.900-000 – Brazil

João Lucas M. Barbosa and Jorge H. S. de Lira
Departamento de Matemática
Universidade Federal do Ceará
Bloco 914 – Campus do Pici
Fortaleza – Ceará
60455-760 – Brazil
joaolucasbarbosa@gmail.com
jorge.lira@pq.cnpq.br