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SPACELIKE HYPERSURFACES WITH CONSTANT MEAN CURVATURE IN LORENTZ SPACE

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1. Introduction

Hypersurfaces with constant mean curvature in Riemannian manifolds are critical points of the area functional under variations that keep constant a certain volume function. In this context the notion of stability was introduced by Barbosa and do Carmo [BC] for the case of immersions in the Euclidean space. Later Barbosa, do Carmo and Eschenburg [BCE] extended this notion to the case of immersions in Riemannian manifolds. In this work we consider spacelike immersions into Lorentz spaces. In such spaces hypersurfaces with constant mean curvature are also critical points of the area functional with the constraint that the variations leave constant a certain volume function. However, there is a distinct difference between the Riemannian and Lorentz ambient spaces. In contrast to the Riemannian case it is the problem of *maximization* of the area functional that makes sense in Lorentz spaces. Thus, it is necessary to reexamine in this context many of the classical notions. In this paper the concept of stability is introduced and a second variation formula is derived. It follows from this formula that any spacelike immersion with constant mean curvature in a flat Lorentz space is stable. We also consider the case when the ambient manifold is the de Sitter space S_1^{n+1} with constant curvature one and prove that spheres (compact umbilical hypersurfaces) are stable. We observe that spheres are the only spacelike immersions with constant mean curvature of compact manifolds into the de Sitter space. We also observe that complete spacelike

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immersions with constant mean curvature $H^2 < 4(n-1)/n^2$ into the de Sitter space are spheres. This result was obtained earlier by Akutagawa [A] and Montiel [M] and in [O] it was generalized in several directions. For the noncompact case we prove stability of complete spacelike immersions with constant mean curvature H such that either $H^2 \geq 1$ or $H^2 < 4(n-1)/n^2$. In particular, any spacelike immersion with constant mean curvature of a complete surface into S_1^3 is stable, and any spacelike complete maximal immersion (of codimension one) into de Sitter space is also stable.

2. Preliminaries

Let $\bar{M}^{n+1}$ be an orientable Lorentz manifold, i.e., an orientable C^∞ manifold edowed with a pseudo-Riemannian structure $ds^2 = \langle .,. \rangle$ of signature $(1,n)$ called the metric of $\bar{M}$. This metric extends in a natural way to tensors of all orders. In particular, for vector fields tangent to $\bar{M}$,

$$\langle X_1 \wedge \ldots \wedge X_p, Y_1 \wedge \ldots \wedge Y_p \rangle = \det\left(\langle X_i, Y_j \rangle\right) \tag{1}$$

where $(\langle X_i, Y_j \rangle)$ stands for the matrix formed by all the products $\langle X_i, Y_j \rangle$ with $1 \leq i, j \leq p$. A set $V_1, \ldots, V_{n+1}$ of vectors in the tangente space $T_p\bar{M}$ at the point p of $\bar{M}$ is said to be orthonormal if

$$|\langle V_i, V_j \rangle| = \delta_{ij} \quad 1 \leq i, j \leq n+1 \,.$$

An orthonormal frame field is simple a set of $n+1$ vector fields, $e_1, \ldots, e_{n+1}$, defined in an open set of $\bar{M}$ which, at each point, are orthonormal. It is clear that only one of the $\langle e_i, e_i \rangle$ can be equal to -1. We set

$$\varepsilon_i = \langle e_i, e_i \rangle \,. \tag{2}$$

To fix notation let's write down the structure equations on $\bar{M}$. Consider an orthonormal frame field, as defined above, and the coframe $\theta_1, \ldots, \theta_{n+1}$. The connection forms ω_{ij} are defined by the equations:

$$\omega_{ij} = -\omega_{ji} \tag{3}$$

$$d\theta_i = \sum_{j=1}^{n+1} \varepsilon_j \omega_{ij} \wedge \theta_j \tag{4}$$

The curvature form is defined by

$$\bar{\Omega}_{ij} = d\omega_{ij} - \sum_{k=1}^{n+1} \varepsilon_k \omega_{ik} \wedge \omega_{kj} \tag{5}$$

This definition makes true the following equation

$$\sum_{j=1}^{n+1} \varepsilon_j \bar{\Omega}_{ij} \wedge \theta_j = 0 \,.$$

Hence we must have

$$\bar{\Omega}_{ij} = -\frac{1}{2} \sum_{k,l=1}^{n+1} \varepsilon_k \varepsilon_l \bar{R}_{ijkl} \theta_k \wedge \theta_l \tag{6}$$

where $\bar{R}_{ijkl} + \bar{R}_{ijlk} = 0$. If $V = \sum_{i=1}^{n+1} v_i e_i$ then its covariant differential is defined by $\bar{D}V = \sum_{i=1}^{n+1} (\bar{D}v_i) e_i$ where

$$\bar{D}v_i = dv_i + \varepsilon_i \sum_{j=1}^{n+1} v_j \omega_{ji} \tag{7}$$

It follows that

$$\bar{D}e_i = \sum_{j=1}^{n+1} \varepsilon_j \omega_{ij} e_j \tag{8}$$

$$\omega_{ij} = \langle \bar{D}e_i, e_j \rangle \tag{9}$$

It is useful to use the standard notation $\bar{D}_X Y$ to mean $\bar{D}Y(X)$. It is a simple computation to show that, for vector fields X and Y on $\bar{M}$, and for $1 \leq i, j \leq n+1$

$$d\langle V, W \rangle = \langle \bar{D}V, W \rangle + \langle V, \bar{D}W \rangle \tag{10}$$

$$\bar{\Omega}_{ij}(X,Y) = \langle \bar{D}_X \bar{D}_Y e_i - \bar{D}_Y \bar{D}_X e_i - \bar{D}_{[X,Y]} e_i, e_j \rangle \tag{11}$$

We define, for vector fields X, Y, Z and W on $\bar{M}$,

$$\bar{\Omega}(X,Y,Z,W) = \langle \bar{D}_X \bar{D}_Y Z - \bar{D}_Y \bar{D}_X Z - \bar{D}_{[X,Y]} Z, W \rangle \tag{12}$$

It is easy to see that

$$\bar{\Omega}(X,Y,Z,W) = \sum_{i,j=1} z_i w_j \bar{\Omega}_{ij}(X,Y)$$

$$= -\sum_{i,j,k,l} z_i w_j x_k y_l \varepsilon_k \varepsilon_l \bar{R}_{ijkl}$$

Since $\bar{M}$ is orientable there exists on it a non-zero $(n+1)$-form $d\bar{M}$ (called the volume form) such that if $e_1, \ldots, e_{n+1}$ is any orthonormal basis of $T_p\bar{M}$ then

$$d\bar{M}_p(e_1, \ldots, e_{n+1}) = \pm 1 .$$

If the result is $+1$ the bases is positively oriented. In this case, if $\theta_1, \ldots, \theta_{n+1}$ is its dual bases, then

$$d\bar{M}_p = \theta_1 \wedge \ldots \wedge \theta_{n+1} .$$

Let $x : M^n \to \bar{M}^{n+1}$ be an immersion of a compact, connected, orientable, C^∞ manifold with boundary ∂M (possibly empty). We will consider M endowed with the induced pseudo-metric $d\sigma^2 = x^* ds^2$ so that x becomes an isometry. The immersion x is called *spacelike* when $d\sigma^2$ is a Riemannian metric. From now on we will assume that x is spacelike.

Since M is orientable, there exists a global unit section N of its normal bundle $TM^\perp$. We choose the orientation of M to be compatible with the one of $\bar{M}$. This means to choose the volume form dM on M defined by

$$dM_p(V_1, \ldots, V_n) = d\bar{M}_{x(p)}(dx_p(V_1), \ldots, dx_p(V_n), N(p)) \tag{13}$$

Locally x is an embedding and so, for local matters, we may always consider M as a submanifold of $\bar{M}$ and x as the inclusion map. A frame field $e_1, \cdots, e_{n+1}$ on $\bar{M}$ is adapted to M if, restricted to M, $e_1, \ldots, e_n$ are tangent to M and positively oriented, and furthermore $e_{n+1} = N$. Restricted to the tangent bundle TM of M, $\theta_{n+1} = 0$ and, consequently,

$$\sum_{j=1}^{n} \omega_{n+1j} \wedge \theta_j = 0 .$$

It follows by Cartan's lemma that

$$\omega_{n+1i} = \sum_{j=1}^{n} h_{ij}\theta_j \tag{14}$$

where $h_{ij} = h_{ji}$. The second fundamental form of the immersion x is then defined as

$$II = \sum_{i,j=1}^{n} h_{ij}\theta i\theta j$$

The mean curvature is the trace of II and it is given by

$$H = -\frac{1}{n}\sum_{i=1}^{n} h_{ii} \tag{15}$$

We define the mapping $B : TM \to TM$ by putting $B(V) = \bar{D}_V N$ so that

$$II(V,W) = \langle B(V), W \rangle \tag{16}$$

It follows that

$$\|II\|^2 = \|B\|^2 = \sum_{i,j=1}^{n} h_{ij}^2 .$$

The structure equations for the immersion x are the usual ones:

$$d\theta_i = \sum_{j=1}^{n} \omega_{ij} \wedge \theta_j \tag{17}$$

$$\Omega_{ij} = d\omega_{ij} - \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2}\sum_{k,l=1}^{n} R_{ijkl}\theta_k \wedge \theta_l \tag{18}$$

where $R_{ijkl} + R_{ijlk} = 0$. Using (5), the observation that $\varepsilon_{n+1} = -1$, and the equation (14) one obtains

$$\bar{\Omega}_{ij} - \Omega_{ij} = \sum h_{ik}h_{jl}\theta_k \wedge \theta_l$$

From this one deduces the Gauss equation:

$$\bar{R}_{ijkl} - R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk} \tag{19}$$

3. Variations

A *Variation* of x is a differentiable map $X : M \times (-\varepsilon, \varepsilon) \to \bar{M}$ such that $X_t : M \to \bar{M}$, $t \in (-\varepsilon, \varepsilon)$, defined by $X_t(p) = X(p,t)$, $p \in M$, is a spacelike immersion, $X_0 = x$ and $X_t|_{\partial M} = x|_{\partial M}$, for all t.

We define the *area* function $A : (-\varepsilon, \varepsilon) \to \mathbb{R}$ by

$$A(t) = \int_M dM_t \tag{20}$$

where dM_t is the volume element of M in the metric induced by X_t, and the volume function $V : (-\varepsilon, \varepsilon) \to I\!R$ by

$$V(t) = \int_{M\times[0,t]} X^* d\bar{M} \tag{21}$$

Let $W(p) = (\partial X/\partial t)|_{t=0}$ be the variation vector field of X and set $f = -\langle W, N\rangle$ so that

$$W = fN + \text{tangent component} \tag{22}$$

Lemma 3.1 *We have*

(i) $(dA/dt)|_{t=0} = -\int_M nHf\,dM$

(ii) $(dV/dt)|_{t=0} = \int_M f\,dM$

where H is the mean curvature of the immersion x.

Proof: The computation of dA/dt is well known and is the same as for the Riemannian case. One ends up with the equation

$$\frac{dA}{dt}(0) = -\int_M n\langle W, \vec{H}\rangle\, dM$$

where $\vec{H} = \frac{1}{n}\sum_{i=1}^n (\bar{D}_{e_i} e_i)^\perp$ is the mean curvature vector. Using (8) one obtains that $\vec{H} = -HN$ and so, using (22) one readly obtains (i). To prove (ii), fix a point $p \in M$ and choose a positively oriented adapted orthonormal moving frame $e_1, \ldots, e_n, e_{n+1} = N$ around $x(p)$. Then

$$X^*(d\bar{M}) = a(p,t)\, dM \wedge dt$$

where

$$a(p,t) = X^* d\bar{M}(e_1, \ldots, e_n, \frac{\partial}{\partial t}) = d\bar{M}(dX_t(e_1), \ldots, dX_t(e_n), \frac{\partial X}{\partial t})$$

Hence, using (22), we obtain

$$a(p,0) = d\bar{M}(dx(e_1), \ldots, dx(e_n), W) =$$

$$f d\bar{M}(dx(e_1), \ldots, dx(e_n), N) = f dM(e_1, \ldots, e_n) = f$$

Since

$$\frac{dV}{dt}(t) = \frac{d}{dt}\int_{M\times[0,t]} a(p,\tau)\, dM \wedge d\tau = \int_M a(p,t)\, dM$$

it follows that

$$\frac{dV}{dt}(0) = \int_M f\, dM$$

as we wished. q.e.d.

A variation is normal if W is parallel to N, and volume-preserving if V(t) = V(0) = 0 for all t.

Lemma 3.2 *Given a smooth function $f : M \to I\!R$ with $f|_{\partial M} = 0$ and $\int_M f\,dM = 0$ there exists a volume preserving normal variation whose variantion vector is fN.*

This lemma was proved as Lemma(2.2) in [BCE]. Although the proof was carried out for the Riemannian case, it works as well for the Lorentzian case.

For a given variation X of an immersion $x : M^n \to \bar{M}^{n+1}$ we set

$$H_o = A^{-1}\int_M H\, dM \quad , \quad A = A(0)$$

and define $J : (-\varepsilon, \varepsilon) \to I\!R$ by

$$J(t) = A(t) + nH_o V(t)\,.$$

Proposition 3.3 *Let $x : M^n \to \bar{M}^{n+1}$ be a spacelike immersion. The following statements are equivalent*

(i) x has constant mean curvature H_o;

(ii) For all volume-preserving variations $A'(0) = 0$;

(iii) For all (arbitrary) variations, $J'(0) = 0$.

The Riemannian version of this proposition was given in [BC]. The same proof works also in the Lorentzian case. Note that the assumption $H_o \neq 0$ made in [BC] is not needed.

To compute the second variation of J, we observe that

$$\frac{dJ}{dt} = \int_M (-nH_t + nH_o) f_t dM_t$$

Here H_t is the mean curvature of X_t, and $f_t = \langle \frac{\partial X}{\partial t}, N_t \rangle$, where N_t is the unit normal vetor of X_t. Thus

$$J''(0) = -\int_M \left(\frac{\partial H_t}{\partial t}\right)(0)\, f\, dM\ . \tag{23}$$

The computation is essentially the same as for the area function in the Riemannian case with a few changes of sign. For the sake of completeness we present its proof the appendix.

Theorem 3.4 *Let $x : M^n \to \bar{M}$ be a spacelike immersion with constant mean curvature H and let X be a variation of x that keeps the volume fixed. Then $J''(0)$ is given by*

$$J''(0)(f) = \int_M (f\Delta f - \|B\|^2 f^2 + Ricc(N) f^2)\, dM$$

Here Δ is the Laplacian in the induced metric, $\|B\|$ is the norm of the second fundamental form of x, and $Ricc(N)$ is the (normalized) Ricci curvature of $\bar{M}$ in the direction of N.

Definition. Let M^n be compact and $x : M^n \to \bar{M}^{n+1}$ a spacelike immersion with constant mean curvature. The immersion x is stable if $J''(0) \leq 0$ for all volume-preserving variations of x. If M is noncompact, we say that x is stable if for every compact submanifold $\widetilde{M} \subset M$, the restriction $x|_{\widetilde{M}}$ is stable.

Just as in [BC], one can prove, using (3.2) and (3.4) the following criterion for stability. Let $\mathcal{F}$ be the set of differentiable functions $f : M \to I\!R$ with $f|_{\partial M} = 0$ and $\int_M f\, dM = 0$.

Proposition 3.5 *$x : M^n \to \bar{M}^{n+1}$ is stable if and only if $J''(0)(f) \leq 0$ for all $f \in \mathcal{F}$.*

We define a bilinear form $I : \mathcal{F} \to I\!R$ by

$$I(f,g) = \int_M g\{\Delta f - \|B\|^2 f + Ricc(N) f\}\, dM$$

Definition. A normal vector field $V = fN$, $f \in \mathcal{F}$, to an immersion $x : M^n \to \bar{M}^{n+1}$ with constant mean curvature is a Jacobi field if $f \in \operatorname{Ker} I$, i.e., if $I(f,g) = 0$, for all $g \in \mathcal{F}$.

Proposition 3.6 *Let $f \in \mathcal{F}$. Then fN is a Jacobi field if and only if*

$$\Delta f + (Ricc(N) - \|B\|^2) f = const. \tag{24}$$

The proof is simple. If fN is a Jacobi field then the condition $I(f,g) = 0$ implies that $\Delta f + (Ricc(N) - B^2)f$ is orthogonal in the $L^2(M)$ inner product to all nonconstant functions in $L^2(M)$. This implies (24). The converse is obvious. This proposition is analogous to Proposition 2.9 in [BCE].

4. The main results

Let's consider the case when $\bar{M}$ is the Minkowski space $I\!L^{n+1}$, i.e., the vector space $I\!R^{n+1}$ endowed with the pseudo-metric

$$\langle V, W\rangle = -v_0 w_0 + \sum_{j=1}^{n} v_j w_j$$

This is a flat space and therefore, for any spacelike immersion x with constant mean curvature, we have for the second variation:

$$I(f,f) = \int_M (f\Delta f - \|B\|^2 f^2) dM = -\int_M (\|\text{grad } f\|^2 + \|B\|^2 f^2) dM \leq 0$$

Hence x is stable. The same argument can be used to prove the following simple result.

Proposition 4.1 *Let λ_1 be the first eigenvalue of the laplatian of M. Any spacelike immersion with constant mean curvature H of M^n into $\bar{M}^{n+1}$ for which $\lambda_1 + \|B\|^2 - Ricc(N) \geq 0$ is stable. In particular this occurs when $Ricc(N) \leq 0$ along M.*

If M is small, then λ_1 is big and the above inequality always holds in a sufficiently small disk around any point of a spacelike immersion with constant mean curvature. In fact, when the above inequality occurs, we have $A''(0) \leq 0$ for any variation of the immersion x. In the literature this is usually called *strong stability.*

Since any immersion with constant mean curvature in the Minkowski space is stable (in fact strongly stable), we move to the next interesting example of Lorentz space, namely the de Sitter space S_1^{n+1}. This is a quadric in the Minkowski space $\mathbb{L}^{n+2}$ defined by the condition $\langle V, V\rangle = 1$. From this equation one can deduce that $\langle dV, V\rangle = 0$ and, consequently, the position vector V is everywhere normal to the de Sitter space. Hence, if $V = e_0, \ldots, e_n, e_{n+1}$ is an adapted frame field on S_1^{n+1} then we have

$$dV = \sum_{i=1}^{n+1} \theta_i e_i$$

$$de_0 = \sum_{i=1}^{n+1} \varepsilon_i \omega_{0i} e_i$$

Since $V = e_0$ we have

$$\omega_{0i} = \varepsilon_i \theta_i \tag{25}$$

The coefficients of the second fundamental form of S_1^{n+1} are given by

$$\omega_{0i} = \sum_{j=1}^{n+1} \varepsilon_i \bar{h}_{ij} \theta_j.$$

Therefore,

$$\bar{h}_{ij} = \delta_{ij} \tag{26}$$

and the curvature form of S_1^{n+1} is

$$\bar{\Omega}_{ij} = \omega_{i0} \wedge \omega_{0j} = -\varepsilon_i \varepsilon_j \theta_i \wedge \theta_j \tag{27}$$

From this we deduce that

$$\bar{R}_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \tag{28}$$

Therefore S_1^{n+1} has constant sectional curvature $\bar{R}_{ijij} = 1$. Now we consider a spacelike immersion $x : M^n \to S_1^{n+1}$ of an orientable manifold M. As before, we fix a unit vector field N along M and observe that $\langle N, N\rangle = -1$. Let $e_1, \ldots, e_n, e_{n+1} = N$ be an adapted frame field to x. Using (28) we get

$$Ricc(N) = \sum_{i=1}^{n} \bar{R}_{n+1in+1i} = n \tag{29}$$

If x has constant mean curvature then the second variation formula is given by

$$\begin{aligned} I(f,f) &= \int_M (f\Delta f - \|B\|^2 f^2 + nf^2) dM \\ &= -\int_M \{\|\text{grad } f\|^2 + (\|B\|^2 - n) f^2\} dM \end{aligned} \tag{30}$$

Following Montiel [M], we obtain examples of hypersurfaces in S_1^{n+1} with constant mean curvature by considering intersections of S_1^{n+1} with affine hyperplanes of $\mathbb{L}^{n+2}$. Specifically, let $W \in \mathbb{L}^{n+2}$ be a constant vector such that $< W, W >= -1$, and let

$$M = \{V \in \mathbb{L}^{n+2}; \langle V, V\rangle = 1 \text{ and } \langle V, W\rangle = \tau\}$$

where the range for admissible values of τ is determined by the requirement that $V \in S_1^{n+1}$. Note, that the hyperplane $\langle V, W\rangle = \tau$ is spacelike. The hypersurface M is compact, umbilic, $\|B\|^2 = nH^2 = \frac{n\tau^2}{\tau^2+1}$, and its sectional curvature is given by $R_{ijkl} = \frac{1}{\tau^2+1}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$. By analogy with the Riemannian case we call M a *small sphere* of S_1^{n+1} if $\tau \neq 0$. If $\tau = 0$ we call it a *big sphere*. Observe that spheres of S_1^{n+1} with mean curvature H are isometric to Euclidean spheres with possible values for H^2 in the interval $[0, 1)$.

Proposition 4.2 *Spheres of S_1^{n+1} are stable.*

Proof: Consider a sphere M in S_1^{n+1} with constant mean curvature H. Since M is isometric to an Euclidean sphere with sectional curvature $\frac{1}{\tau^2+1}$ then the first eigenvalue of the laplacian of M is $\lambda_1 = \frac{n}{\tau^2+1}$. Using (30) we obtain

$$\begin{aligned} I(f,f) &= -\int_M \{\|\text{grad } f\|^2 + (\|B\|^2 - n) f^2\} dM \\ &\leq -\int_M (\lambda_1 + nH^2 - n) f^2 \, dM \;=\; 0 \end{aligned}$$

Hence the inclusion of M into the de Sitter space is stable. In fact we will prove that the converse of this result is also true. For that we need some lemmas and the Theorem 4.6 below which may be found in the literature (see, for example, [M]; note that the mean curvature in [M] is the negative of the mean curvature here). For the sake of completeness we present here the corresponding results with the proofs.

Lemma 4.3 *Let $x : M^n \to S_1^{n+1}$ be a spacelike immersion of an orientable manifold M and represent by N its unit normal vector field. If x has constant mean curvature then, for each fixed vector $A \in \mathbb{L}^{n+2}$ the Laplacians of the functions $g = \langle x, A\rangle$ and $f = \langle N, A\rangle$ are given by:*

$$\Delta g = nHf - ng \qquad \Delta f = \|B\|^2 f - nHg$$

Proof. Take a local adapted frame field $x = e_0, e_1, \ldots, e_n, e_{n+1} = N$. It follows that, at each point x, we have $dg = \sum_{i=1}^n \langle e_i, A\rangle\theta_i = \sum g_i\theta_i$ and

$$\begin{aligned} D\langle e_i, A\rangle &= d\langle e_i, A\rangle + \sum_{j=1}^n \langle e_j, A\rangle\omega_{ji} \\ &= \sum_{k=0}^{n+1} \epsilon_k\langle e_k, A\rangle\omega_{ik} + \sum_{j=1}^n \langle e_j, A\rangle\omega_{ji} \\ &= \langle e_0, A\rangle\omega_{i0} - \langle e_{n+1}, A\rangle\omega_{in+1} \\ &= -\langle x, A\rangle\theta_i + \langle N, A\rangle\sum_{j=1}^n h_{ij}\theta_j = \sum_{j=1}^n g_{ij}\theta_j \end{aligned}$$

It follows that $g_{ij} = -g\delta_{ij} + fh_{ij}$ and so

$$\Delta g = \sum_{i=1}^n g_{ii} = -ng + nHf$$

For the case of f we have

$$df = \langle N, A\rangle = \sum_{j=1}^n \langle e_j, A\rangle\omega_{n+1j} = \sum_{ij=1}^n h_{ji}\langle e_j, A\rangle\theta_i = \sum_{i=1}^n f_i\theta_i$$

Hence

$$\begin{aligned} Df_i &= \sum_{j,k=1}^n h_{jik}\langle e_j, A\rangle\theta_k + \sum_{j=1}^n h_{ji}D\langle e_j, A\rangle \\ &= \sum_{j,k=1}^n h_{jik}\langle e_j, A\rangle\theta_k + \sum_{j=1}^n h_{ji}\{\sum_{l=0}^{n+1}\epsilon_l\langle e_l, A\rangle\omega_{jl} + \sum_{k=1}^n \langle e_k, A\rangle\omega_{kj}\} \\ &= \sum_{j,k=1}^n h_{jik}\langle e_j, A\rangle\theta_k + \sum_{j=1}^n h_{ji}\{\langle e_0, A\rangle\omega_{j0} - \langle e_{n+1}, A\rangle\omega_{jn+1}\} \end{aligned}$$

$$\begin{aligned} &= \sum_{j,k=1}^n h_{jik}\langle e_j, A\rangle\theta_k - g\sum_{j=1}^n h_{ji}\theta_j + f\sum_{j,k=1}^n h_{ji}h_{jk}\theta_k \\ &= \sum_{k=1}^n\{\sum_{j=1}^n h_{ijk}\langle e_j, A\rangle - gh_{ki} + f\sum_{j=1}^n h_{ji}h_{jk}\}\theta_k = \sum_{k=1}^n f_{ik}\theta_k \end{aligned}$$

Therefore

$$\Delta f = \sum_{i=1}^n f_{ii} = \sum_j H_j - nHg + \|B\|^2 f$$

Since H is constant, $H_j = 0$ for each j and we arrived at the desired formula. This proves the lemma.

The proof of the following lemma is the same as for the Riemannian case.

Lemma 4.4 *For any spacelike immersion we have $\|B\|^2 - nH^2 \geq 0$. Equality occurs if and only if the immersion is umbilic.*

Lemma 4.5 *Let $x : M^n \to S_1^{n+1}$ be a spacelike isometric immersion of a compact connected oriented Riemannian manifold. If x has constant mean curvature then x is umbilic.*

Proof. We integrate over M the expressions of the Laplacians of the functions f and g, given in (4.3), to obtain

$$H\int_M f\,dM = \int_M g\,dM \quad \text{and} \quad \int_M \|B\|^2 f\,dM = nH\int_M g\,dM$$

Which implies that

$$\int_M (\|B\|^2 - nH^2)f\,dM = 0 \qquad (31)$$

Remember that f depends on the choice a vector A and we choose it so that $\langle A, A\rangle = -1$. Any vector V in $\mathbb{L}^{n+2}$ can be decomposed as $V = vA + \bar{V}$ where $\langle \bar{V}, A\rangle = 0$ and $\langle \bar{V}, \bar{V}\rangle \geq 0$. Using this decomposition for N and the fact that $\langle N, N\rangle = -1$ we write $N = n_o A + \bar{N}$ and obtain

$$-1 = -n_o^2 + \langle \bar{N}, \bar{N}\rangle$$

It follows that

$$\langle N, A\rangle^2 = n_o^2 = 1 + \langle \bar{N}, \bar{N}\rangle \geq 1\,.$$

Since M is connected, we may choose A such that $f = \langle N, A\rangle \geq 1$. Taking such f and using Lemma(4.4) we conclude from equation (31) that $\|B\|^2 = nH^2$ and that x is umbilic. This concludes the proof of the lemma.

Theorem 4.6 *Let $x : M^n \to S_1^{n+1}$ be a spacelike isometric immersion of a compact, connected, orientable Riemannian manifold M into the de Sitter space S_1^{n+1}. If x has constant mean curvature then M is isometric to an Euclidean sphere and x is its inclusion as an sphere of S_1^{n+1}.*

Proof. We already know, from the previous lemma, that x is umbilic and therefore $\|B\|^2 = nH^2$. Using this in Lemma(4.3) we obtain $\Delta(f - Hg) = 0$. Since M is compact we conclude that

$$f - Hg = \text{const.}$$

for any choice of the vector A. This means that, in fact, we have

$$N - Hx = C$$

where C is a constant vector in $\mathbb{L}^{n+2}$. One shall observe that any vector tangent to M is perpendicular to C. Since M is connected it lies in some affine hyperplane normal to C, not necessarily passing through the origin of $\mathbb{L}^{n+2}$. Since M is compact, M is a sphere of S_1^{n+1}. This proves the theorem.

Theorem 4.7 *Let $x : M^n \to S_1^{n+1}$ be an immersion with constant mean curvature of a complete, connected and orientable Riemannian manifold M into the de Sitter space. Then x is stable, if one of the following conditions hold: a) M is compact; b) $H^2 \geq 1$; c) $H^2 < 4(n-1)/n^2$.*

Proof. If M is compact, the result follows from the previous theorem. If $H^2 \geq 1$ then the result follows from equation (30) and Lemma(4.4). Now assume that $H^2 < 4(n-1)/n^2$. We want to estimate the value of the Ricci curvature at each point of M. Let $V = \sum_{i=1}^n v_i e_i$ be any tangent vetor to M. Since $Ricc(V) = \sum_{i,j,k=1}^n R_{ijkj} v_i v_k$, using (19) and (28) we obtain

$$Ricc(V) = \sum_{i,j,k=1}^n (\delta_{ik}\delta_{jj} - \delta_{ij}\delta_{jk}) v_i v_k - \sum_{i,j,k}^n (h_{ik}h_{jj} - h_{ij}h_{jk}) v_i v_k$$

We may assume that, at the point where we are doing the computation, the second fundamental form of x is diagonalized, and that $k_1, k_2, \ldots, k_n$ are its principal curvatures. Then we obtain

$$Ricc(V) = (n-1)|V|^2 + \sum_{i=1}^n k_i^2 v_i^2 + nH \sum_{i=1}^n k_i v_i^2$$

from where we deduce that

$$\begin{aligned} Ricc(V) &= (n-1)|V|^2 + \sum_{i=1}^n (k_i + nH/2)^2 v_i^2 - \frac{1}{4}(nH)^2 |V|^2 \\ &\geq \{(n-1) - \frac{1}{4}(nH)^2\}|V|^2 \end{aligned}$$

Observe that the lower bound for $Ricc(V)$ is of the form $C|V|^2$ where C is a constant which is positive due to our hypothesis. Hence, by Myers theorem, M is compact and we are back to the case already studied. This proves the theorem.

We observe that in the case of dimension two the hypothesis in the last theorem cover all possibilities and we have the following strong result.

Corollary 4.8 *Let $x : M^2 \to S_1^3$ be an immersion with constant mean curvature of a complete, connected orientable Riemannian manifold M into the de Sitter space, then x is stable.*

5. Appendix

In this appendix we want to prove the formula for the second variation stated in Proposition(3.4) for the case of normal variations $X : M \times (-\varepsilon, \varepsilon) \to \bar{M}$ of a spacelike immersion x. We will use the notation and results of section(). For each $t \in (-\varepsilon, \varepsilon)$, X_t is a spacelike immersion with mean curvature H_t computed with respect to the unit normal vector field N_t. The fact that X is normal means that

$$W = \frac{\partial X}{\partial t}(0) = f N$$

From (23) one can see that in order to compute the second derivative of J, it is sufficient to compute $(\partial H_t/\partial t)(0)$. To do this we need the following two lemmas.

Lemma 5.1 *grad* $f = \bar{D}_W N_t|_{t=0}$

Proof. Consider a set of coordinates $u_1, \dots, u_n$ in a neighborhood of p on M. It follows that $\partial X/\partial u_i$ at the point $(u_1, \dots, u_n, t)$ is tangent to $X_t(M)$ and that

$$\left[\frac{\partial X}{\partial u_i}, \frac{\partial X}{\partial t}\right] = 0$$

Consequently, if $V_i = \frac{\partial X}{\partial u_i}$ and $W = \frac{\partial X}{\partial t}$ we have for each $1 \le i \le n$

$$\bar{D}_W V_i = \bar{D}_{V_i} W \tag{32}$$

Now the lemma is proved by the following chain of equalities

$$\begin{aligned}\langle \bar{D}_W N_t, V_i\rangle|_{t=0} &= -\langle N_t, \bar{D}_W V_i\rangle|_{t=0}\\ &= -\langle N, \bar{D}_{V_i} W|_{t=0}\rangle\\ &= -\langle N, V_i[f]\, N + f\, \bar{D}_{V_i} N\rangle\\ &= df(V_i)\end{aligned}$$

Associated with each X_t we have a second second fundamental form II and a map B defined as in (16).

Lemma 5.2 *If Y and Z are vectors in T_pM then*

$$\langle(\tfrac{\partial}{\partial t}B)Y, Z\rangle = \langle \bar{D}_Y grad(f), Z\rangle + f\bar{\Omega}(N, Y, N, Z) - f\langle A^2 Y, Z\rangle \tag{33}$$

Proof. We will compute the value of the derivative of II in two different ways. First we will use that $II(Y,Z) = \langle \bar{D}_Y N, Z\rangle$ and, as before, we set $W = \partial X/\partial t$. We may extend Y and Z to tangent vector fields on M (also called Y and Z) such that $[Y, W] = [Z, W] = 0$ at the point p.

$$\begin{aligned}\frac{\partial}{\partial t} II(Y,Z) = {}& \langle \bar{D}_W \bar{D}_Y N, Z\rangle + \langle \bar{D}_Y N, \bar{D}_W Z\rangle =\\ & \langle \bar{D}_Y \bar{D}_W N, Z\rangle + \langle \bar{D}_{[W,Y]} N, W\rangle +\\ & \bar{\Omega}(W, Y, N, Z) + \langle \bar{D}_Y N, \bar{D}_Z W\rangle\end{aligned}$$

Setting $t = 0$, using Lemma 5.1, the expression of W and $[Y, W]_p = 0$ rewrite this as:

$$\begin{aligned}\frac{\partial}{\partial t} II(Y,Z) = {}& \langle \bar{D}_Y grad(f), Z\rangle +\\ & f\bar{\Omega}(N, Y, N, Z) + f\langle \bar{D}_Y N, \bar{D}_Z N\rangle\end{aligned} \tag{34}$$

Observe that the last term is equal to $f\langle BY, BZ\rangle = f\langle B^2 Y, Z\rangle$. Now we start the second calculation observing that $II(Y,Z) = \langle BY, Z\rangle$.

$$\begin{aligned}\frac{\partial}{\partial t} II(Y,Z) &= \langle \bar{D}_W(BY), Z\rangle + \langle BY, \bar{D}_W Z\rangle\\ &= \langle(\tfrac{\partial}{\partial t}B)Y + B(\bar{D}_W^{\top} Y), Z\rangle + \langle BY, \bar{D}_Z W\rangle\\ &= \langle(\tfrac{\partial}{\partial t}B)Y, Z\rangle + \langle B(\bar{D}_Y^{\top} W), Z\rangle + f\langle BY, \bar{D}_Z N\rangle\\ &= \langle(\tfrac{\partial}{\partial t}B)Y, Z\rangle + f\langle B(\bar{D}_Y^{\top} N), Z\rangle + f\langle BY, \bar{D}_Z N\rangle\\ &= \langle(\tfrac{\partial}{\partial t}B)Y, Z\rangle + 2f\langle BY, BZ\rangle\end{aligned} \tag{35}$$

Using (34) and (35) one obtains the result. q.e.d.

Proposition 5.3 *For normal variations*

$$-n\frac{\partial H_t}{\partial t}(0) = \Delta f + f\, Ricc(N) - f\|B\|^2$$

Proof. Take a frame fied $e_1, \dots, e_n, e_{n+1} = N$ adapted to the immersion x for which X is a variation. Then it is clear that

$$\sum_{i=1}^{n} \langle B^2 e_i, e_i\rangle = \sum_{i=1}^{n} \langle Be_i, Be_i\rangle = \sum_{i,j=1}^{n} h_{ij}^2 = \|B\|^2 \tag{36}$$

$$\sum_{i=1}^{n} \langle \bar{D}_{e_i} grad(f), e_i\rangle = \Delta f \tag{37}$$

and, by definition,

$$Ricc(N) = \sum_{i=1}^{n} \bar{\Omega}(N, e_i, N, e_i) = -\sum_{i=1}^{n} \varepsilon_i \varepsilon_{n+1} \bar{R}_{n+1in+1i} = \sum_{i=1}^{n} \bar{R}_{n+1in+1i} \tag{38}$$

Observe that

$$-n\frac{\partial H}{\partial t} = \sum_{i=1}^{n}\frac{\partial h_{ii}}{\partial t} = \sum_{i=1}^{n}\langle(\tfrac{\partial B}{\partial t})e_i, e_i\rangle \tag{39}$$

Now we apply Lemma(5.2) and use equations (36), (37) and (38) to obtain the assertion of the proposition. q.e.d.

To conclude, we observe that Theorem(3.4) follows from the above proposition.

References

[A] Akutagawa, K., *On Spacelike Hypersurfaces with Constant Mean Curvature in the de Sitter Space*, Math.Z. **196**, 13-19 (1987).

[BC] Barbosa J.L. and do Carmo M., *Stability of Hypersurfaces with Constant Mean Curvature*, Math.Z. **185**, 339-353 (1984).

[BCE] Barbosa J.L., do Carmo M. and Eschenburg J., *Stability of Hypersurfaces of Constant Mean Curvature in Riemannian Manifolds*, Math.Z. **197**, 123-138 (1988).

[M] Montiel S., *An Integral Inequality for Compact Spacelike Hypersurfaces in de Sitter Space and Applications to the Case of Constant Mean Curvature*, Indiana Univ. Math. J. **37**, 909-917 (1988).

[O] Oliker V., *A Priori Estimates of the Principal Curvatures of Spacelike Hypersurfaces in de Sitter Space with Applications to Hypersurfaces in Hyperbolic Space*, American J. of Math. **114**, 605-626 (1992).

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