A Priori Estimates for Starshaped Compact Hypersurfaces With Prescribed m-th Curvature Function in Space Forms

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To Professor Olga A. Ladyzhenskaya on her 80-th birthday

1 Introduction

Let $\mathcal{R}^{n+1}(K)$, $n \geq 2$, be a space form of sectional curvature $K = -1, 0$, or $+1$ and m an integer, $1 \leq m \leq n$. In this paper we establish several a priori bounds for solutions of the following geometric problem: under what conditions a given function $\psi : \mathcal{R}^{n+1}(K) \to (0, \infty)$ is the m-th mean curvature H_m of a hypersurface M embedded in $\mathcal{R}^{n+1}(K)$ as a graph over a sphere?

Let us formulate the problem more precisely. First we describe the space $\mathcal{R}^{n+1}(K)$ in a form convenient for our purposes. In Euclidean space R^{n+1} fix the origin O and a unit sphere S^n centered at O. Denote by u a point on S^n and let (u, ρ) be the spherical coordinates in R^{n+1}. The standard metric on S^n induced from R^{n+1} we denote by e. Let $a = const,\ 0 < a \leq \infty,\ \ I = [0, a)$, and $f(\rho)$ a positive C^∞ function on I such that $f(0) = 0$. Introduce in R^{n+1} a new metric

$$h = d\rho^2 + f(\rho)e. \tag{1}$$

When $a = \infty$ and $f(\rho) = \rho^2$ the space (R^{n+1}, h) is the Euclidean space $\equiv R^{n+1}$. When $a = \infty$ and $f(\rho) = \sinh^2 \rho$ the space $(R^{n+1}, h) = \mathcal{R}^{n+1}(-1)$ is the hyperbolic space H^{n+1} with sectional curvature -1 and when $a = \pi/2,\ f(\rho) = \sin^2 \rho$, $(R^{n+1}, h) = \mathcal{R}^{n+1}(1)$ is the elliptic space S^{n+1}_+ with sectional curvature $+1$. By the m-th mean curvature, H_m, we understand here the normalized elementary symmetric function of order m of principal curvatures $\lambda_1, ..., \lambda_n$ of M, that is,

$$H_m = \frac{1}{\binom{n}{m}} \sum_{i_1 < ... < i_n} \lambda_{i_1} \cdots \lambda_{i_m}.$$

The problem stated in the beginning can now be formulated as follows. Let $\psi(u, \rho),\ \ u \in S^n,\ \ \rho \in I$, be a given positive function. Under what conditions on ψ there exists a smooth hypersurface M given as $(u, z(u)),\ \ u \in S^n,\ \ z > 0$, for which

$$H_m(u) = \psi(u, z(u)) \quad \text{on} \quad M? \tag{2}$$

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In Euclidean space $R^{n+1}(=\mathcal{R}^{n+1}(0))$ such conditions were found by I. Bakelman and B. Kantor [2, 3] and A. Treibergs and S.W. Wei [13] when $m = 1$ (the mean curvature case), by V. Oliker [11] when $m = n$ (the Gauss curvature case), and by L. Caffarelli, L. Nirenberg and J. Spruck [6] when $1 < m < n$. Other forms of such conditions for the Gauss curvature case in R^{n+1} were investigated by P. Delanoë [7], Yan Yan Li [10] and others. In our paper [12] we investigated the Gauss curvature case for hypersurfaces in $\mathcal{R}^{n+1}(-1)$ and $\mathcal{R}^{n+1}(1)$. Special curvature functions for convex hypersurfaces in Riemannian manifolds have been considered recently by C. Gerhardt [8] (see also other references there).

In all investigations of (2) in Euclidean space a priori C^0, C^1 and C^2 estimates for solutions of (2) play a central role in the proofs of existence. However, except for the C^0 estimates, obtaining these a priori estimates for hypersurfaces in the hyperbolic space $\mathcal{R}^{n+1}(-1)$ and elliptic space $\mathcal{R}^{n+1}(1)$ is not straight forward and requires new efforts. The approach of this paper allows us to obtain C^1 a priori bounds in $\mathcal{R}^{n+1}(K)$ for any $1 \le m \le n$ and $K = -1, 0, 1$. When $K = 1$ we obtain also the C^2 a priori estimates. Essentially the same proof of the C^2 estimate works also in case when $K = 0$ treated earlier in [11] for $m = n$ and in [6] for $1 \le m \le n$.

2 Preliminaries

2.1 Local formulas

Unless explicitly stated otherwise, all latin indices are in the range $1, ..., n$, the sums are over this range and summation over repeated lower and upper indices is assumed. Also, since most of our considerations apply to space forms $\mathcal{R}^{n+1}(K)$, where K can be $-1, 0$ or 1, we will discuss the general case, indicating explicitly the restriction on K only where necessary.

We consider hypersurfaces in $\mathcal{R}^{n+1}(K)$ which are graphs over S^n. Thus, for a given smooth positive function $z(u)$, $u \in S^n$, we denote by $r(u) = (u, z(u))$ the graph M of this function. Throughout the paper we will have to use covariant differentiation on the sphere S^n and on the hypersurface M. We fix our notation here. First we do it for S^n. Let $u^1, ..., u^n$ be some smooth local coordinates in a coordinate neigborhood $U \subset S^n$. Let $\partial_i = \partial/\partial u^i$, $i = 1, 2, ..., n$, be the corresponding local frame of tangent vectors on U so that $e(\partial_i, \partial_j) = e_{ij}$. For a smooth function v on U the first covariant derivative $v_i \equiv \nabla'_i v = \partial v/\partial u^i$. Put $\nabla' v = e^{ij} v_j \partial_i$, where $e^{ij} = (e_{ij})^{-1}$. For the covariant derivative of $\nabla' v$ we have

$$\nabla'_{\partial_s} \nabla' v = v_{sj} e^{ji} \partial_i + v_j \nabla'_{\partial_s}(e^{ij}\partial_i) = \nabla'_{sj} v e^{jk} \partial_k \quad \left(v_{sj} = \frac{\partial^2 v}{\partial u^s \partial u^j} \right),$$

or, equivalently,

$$\nabla'_{sj} v = v_{sj} - \Gamma'^i_{sj} v_i,$$

where Γ'^i_{sj} are the Christoffel symbols of the second kind of the metric e. This differentiation is extended to vector-valued functions by differentiating each of the components.

Similarly, if T is a smooth symmetric $(0, 2)-$ tensor on U with components T_{ij} relative to the dual coframe then the components of its first covariant derivatives on S^n are given by

$$\nabla'_l T_{ij} = \frac{\partial T_{ij}}{\partial u^l} - h_{kj} \Gamma'^k_{lj} - T_{ki} \Gamma'^k_{il}.$$

When M is a hypersurface in $\mathcal{R}^{n+1}(K)$ and g is a metric on M the covariant differentiation on M is defined as above but with respect to connection of the metric g. In this case for a smooth function v on M we denote by $\nabla_i v$ and $\nabla_{ij} v$ its first and second covariant derivatives and similarly for vector-valued functions and smooth symmetric tensors on M.

We now define the metric and the second fundamental form of M in the case when M is a graph of a smooth and positive function z on S^n, that is, $M = (u, z(u)),\ u \in S^n$. In spherical coordinates (u, ρ) in $\mathcal{R}^{n+1}(K)$ we let $R = \partial/\partial\rho$. The frame $\partial_1, ..., \partial_n, R$ is a local frame along M and a basis of tangent vectors on M is given by $r_i = \partial_i + z_i R, i = 1, ..., n$. The metric $g = g_{ij} du^i du^j$ on M induced from $\mathcal{R}^{n+1}(K)$ has coefficients

$$g_{ij} = f e_{ij} + z_i z_j \quad \text{and} \quad \det(g_{ij}) = f^{n-1}(f + |\nabla' z|^2) \det(e_{ij}). \tag{3}$$

Obviously, M is an embedded hypersurface. The inverse matrix $(g_{ij})^{-1}$ is given by

$$g^{ij} = \frac{1}{f}\left[e^{ij} - \frac{z^i z^j}{f + |\nabla' z|^2}\right] \quad \left(z^i = e^{ij} z_j\right). \tag{4}$$

The unit normal vector field on M is given by

$$N = \frac{\nabla' z - fR}{\sqrt{f^2 + f|\nabla' z|^2}}. \tag{5}$$

The second fundamental form b of M is the normal component of the covariant derivative in $\mathcal{R}^{n+1}(K)$ with respect to connection defined by the metric (1). In local coordinates its coefficients are given by ([12])

$$b_{ij} = \frac{f}{\sqrt{f^2 + f|\nabla' z|^2}}\left[-\nabla'_{ij} z + \frac{\partial \ln f}{\partial \rho} z_i z_j + \frac{1}{2}\frac{\partial f}{\partial \rho} e_{ij}\right], \tag{6}$$

Note that with our choice of the normal the second fundamental form of a sphere $z = const > 0$ is positive definite, since for $\mathcal{R}^{n+1}(K)$ $\partial f/\partial\rho > 0$.

The principal curvatures of M are the eigenvalues of the second fundamental form relative to the metric g and are the real roots, $\lambda_1, ..., \lambda_n$, of the equation

$$\det(b_{ij} - \lambda g_{ij}) = 0,$$

or, of the equivalent equation,

$$\det(a^i_j - \lambda \delta^i_j) = 0,$$

where

$$a^i_j = g^{ik} b_{kj}. \tag{7}$$

The elementary symmetric function of order $m,\ 1 \le m \le n$, of $\lambda = (\lambda_1, ..., \lambda_n)$ is

$$S_m(\lambda) = \sum_{i_1 < ... < i_n} \lambda_{i_1} \cdots \lambda_{i_m};$$

that is, $S_m(\lambda) = F(a^i_j)$, where F is the sum of the principal minors of (a^i_j) of order m. It follows from the preceding discussion that

$$F(a^i_j) \equiv F(u, z, \nabla'_1, ..., \nabla'_n z, \nabla'_{11} z, ..., \nabla'_{nn} z).$$

The equation (2) assumes now the form

$$F(a^i_j) = \bar{\psi}(u, z(u)), \tag{8}$$

where here and for the rest of the paper we put for convenience $\bar{\psi} \equiv \binom{n}{m}\psi$.

Let Γ be the connected component of $\{\lambda \in R^n \mid S_m(\lambda) > 0\}$ containing the positive cone $\{\lambda \in R^n \mid \lambda_1, ..., \lambda_n > 0\}$.

Definition 2.1 *A positive function $z \in C^2(S^n)$ is admissible for the operator F if the corresponding hypersurface $M = (u, z(u)), \quad u \in S^n$, is such that at every point of M with the choice of the normal as in (5), the principal curvatures $(\lambda_1, ..., \lambda_n) \in \Gamma$.*

It is known, [5], that

$$S_{m\lambda_i} \equiv \frac{\partial S_m}{\partial \lambda_i} > 0, \quad S_{m\lambda_i\lambda_j} \equiv \frac{\partial^2 S_m}{\partial \lambda_i \partial \lambda_j} > 0 \tag{9}$$

for all $\lambda \in \Gamma, \quad i \neq j$. For the first of the inequalities see also [4]. It is also known that the function $(S_m(\lambda))^{1/m}$ is concave on Γ [5].

The function

$$q(\rho) \equiv \frac{1}{2f(\rho)} \frac{df(\rho)}{d\rho} \tag{10}$$

will play an important role in our constructions. Note that for a sphere of radius c

$$F(a^i_j) = \binom{n}{m} q^m(\rho)_{|\rho=c}. \tag{11}$$

In $R^{n+1} \quad q(\rho) = \rho^{-1}$.

For ease of reference we state here two basic properties of the function $q(\rho)$. First note that it is strictly positive on the interval I (where f is defined). Further, since

$$\frac{\partial q}{\partial \rho} = -\frac{1}{f}, \tag{12}$$

it is strictly decreasing on I. Also, it follows directly from the definition of function f for each of the spaces $\mathcal{R}^{n+1}(K)$ that

$$f = \frac{1}{q^2 + K}. \tag{13}$$

3 C^0- estimates

Lemma 3.1 *Let $1 \leq m \leq n$ and let $\psi(X)$ be a positive continuous function defined on $\mathcal{R}^{n+1}(K) \setminus \{0\}$. Suppose there exist two numbers R_1 and R_2, $0 < R_1 < R_2 < a$, such that*

$$\psi(u, \rho) > q^m(\rho) \text{ for } u \in S^n, \ \rho < R_1, \tag{14}$$

$$\psi(u, \rho) < q^m(\rho) \text{ for } u \in S^n, \ \rho > R_2. \tag{15}$$

Let $z \in C^2(S^n)$ be a solution of equation (8). Then

$$R_1 \le z(u) \le R_2, \quad u \in S^n. \tag{16}$$

In applications a slightly different form of this estimate is sufficient.

Lemma 3.2 *Let $1 \le m \le n$ and let $\psi(X)$ be a positive continuous function in the annulus $\bar{\Omega}: \ u \in S^n, \quad \rho \in [R_1, R_2], \ 0 < R_1 < R_2 < a$. Suppose ψ satisfies the conditions:*

$$\psi(u, R_1) \ge q^m(R_1) \text{ for } u \in S^n, \tag{17}$$

$$\psi(u, R_2) \le q^m(R_2) \text{ for } \ u \in S^n. \tag{18}$$

Let $z \in C^2(S^n)$ be a solution of equation (8) and $R_1 \le z(u) \le R_2, \quad u \in S^n$. Then either $z \equiv R_1$, or $z \equiv R_2$, or

$$R_1 < z(u) < R_2, \quad u \in S^n. \tag{19}$$

Proof of Lemma 3.1. Suppose there exists a point $\bar{u} \in S^n$ such that $max_{S^n} z(u) = z(\bar{u}) > R_2$. At $\bar{u}$ $grad\ z = 0$ and $Hess(z) \le 0$. Then at $\bar{u}$

$$g^{ij} = \frac{1}{f} e^{ij}, \ b_{ij} = -Hess(z) + fqe_{ij} \ge fqe_{ij},$$

and $a^i_j \ge q\delta^i_j$. Consequently,

$$F(a^i_j) = \bar{\psi}(\bar{u}, R_2) \ge \tbinom{n}{m} q^m(\rho)_{|\rho = R_2}$$

which contradicts the inequality (15). Similarly it is shown that $R_1 \ge z(u)$.

The **Lemma 3.2** is a consequence of a strong maximum principle as in [1], Theorem 1.

4 C^1- estimate

Theorem 4.1 *Let $1 \le m \le n$ and let $\psi(X)$ be a positive C^1 function in the annulus $\bar{\Omega}: \ u \in S^n, \quad \rho \in [R_1, R_2], \ 0 < R_1 < R_2 < a$. Let $z \in C^3(S^n)$ be an admissible solution of equation (8) satisfying the inequalities*

$$R_1 \le z(u) \le R_2, \quad u \in S^n. \tag{20}$$

Suppose, in addition, that for all $u \in S^n$ and $\rho \in [R_1, R_2]$ ψ satisfies one of the following conditions:
if the sectional curvature $K = 0$ or 1 then

$$\frac{\partial}{\partial \rho}\left[\psi(u, \rho) q^{-m}(\rho)\right] \le 0; \tag{21}$$

if the sectional curvature $K = -1$ then

$$\frac{\partial}{\partial \rho}\left[\psi(u, \rho) f^{m/2}(\rho)\right] \le 0. \tag{22}$$

Then

$$|grad\ z| \le C \tag{23}$$

where C is a constant depending only on $m, n, R_1, R_2, \psi, grad\psi$.

Proof. It will be convenient to make the substitution

$$v(u) = q(\rho)|_{\rho = z(u)}.$$

Using (12) we get

$$v_i = -\frac{z_i}{f}, \quad \nabla'_{ij} v = \frac{1}{f}\left[-\nabla'_{ij} z + \frac{f_\rho}{f} z_i z_j\right],$$

where f_ρ denotes the derivative of f with respect to ρ. Then

$$\nabla'_{ij} v + v e_{ij} = \frac{1}{f}\left[-\nabla'_{ij} z + \frac{f_\rho}{f} z_i z_j + \frac{1}{2} f_\rho e_{ij}\right].$$

Using (4), (6) and (7) we obtain

$$g^{ij} = \frac{1}{f}\left[e^{ij} - \frac{f v^i v^j}{1 + f|\nabla' v|^2}\right] \quad (v^i = e^{ij} v_j),$$

$$b_{ij} = \frac{f}{\sqrt{1 + f|\nabla' v|^2}} (\nabla'_{ij} v + v e_{ij}),$$

$$a^i_j = \left[\frac{(1 + f|\nabla' v|^2) e^{is} - f v^i v^s}{(1 + f|\nabla' v|^2)^{3/2}}\right] (\nabla'_{sj} v + v e_{sj}). \tag{24}$$

Put

$$p^2 = v^2 + |\nabla' v|^2 + K, \quad P^{is} = p^2 e^{is} - v^i v^s, \quad W_{sj} = \nabla'_{sj} v + v e_{sj}.$$

Note that by (13) and (20) $v \geq c > 1$ for $K = -1$ and $v \geq c' > 0$ for $K = 0, 1$, where the constsnts c and c' depend only on R_1 and R_2. Using (13) with $q = v$ we rewrite (24) in the form

$$a^i_j = \frac{\sqrt{v^2 + K}}{p^3} P^{is} W_{sj}. \tag{25}$$

The C^0 bounds of v imply that $p \geq c = const >$ on S^n, where c depends only on R_1 and R_2. In order to estimate $|\nabla' v|$ we estimate the maximum of the function

$$\phi = p^2 \eta(v),$$

where η is a positive function to be specified later. This will give us an estimate of $|\nabla' v|$ and therefore of $|\nabla' z|$.

Let $\bar{u} \in S^n$ be the point where the $\max_{S^n} \phi(u)$ is attained, that is, $\max_{S^n} \phi(u) = \phi(\bar{u})$. Assume that $\bar{u}$ is the origin of a local coordinate system on S^n chosen so that at $\bar{u}$ the corresponding local frame of tangent vectors to S^n is orthonormal. Then at $\bar{u}$ the covariant derivatives coincide with the usual derivatives. At $\bar{u}$ we have

$$\phi_i = 2p \nabla'_i p \eta + p^2 \eta' v_i = 0 \quad \text{and} \quad \phi_{ii} (\equiv \nabla'_{ii} \phi) \leq 0, i = 1, 2, ..., n \quad (\eta' \equiv \frac{d\eta}{dv}). \tag{26}$$

The first of these conditions implies

$$p \nabla'_i p = v^s (\nabla'_{si} v + v e_{si}) = v^s W_{si} = -\frac{p^2}{2} \frac{\eta'}{\eta} v_i. \tag{27}$$

It follows from (25) that at $\bar{u}$

$$a_i^j = \frac{\sqrt{v^2+K}}{p}\left(e^{ik}W_{kj} + \frac{\eta'}{2\eta}v^i v_j\right). \tag{28}$$

The second condition in (26) together with (27) give

$$2(\nabla_i' v^s W_{si} + v^s \nabla_i' W_{si}) + \left[\frac{\eta''}{\eta} - 2\left(\frac{\eta'}{\eta}\right)^2\right] p^2 v_i^2 + p^2\frac{\eta'}{\eta}\nabla_{ii}' v \le 0, \tag{29}$$

where $\eta'' = \frac{d^2\eta}{dv^2}$.

By the Ricci identity on S^n

$$\nabla_i' W_{si} = \nabla_s' W_{ii}.$$

Applying it in (29) we get

$$2(\nabla_i' v^s W_{si} + v^s \nabla_s' W_{ii}) + \left[\frac{\eta''}{\eta} - 2\left(\frac{\eta'}{\eta}\right)^2\right] p^2 v_i^2 + p^2\frac{\eta'}{\eta}\nabla_{ii}' v \le 0. \tag{30}$$

Next, we differentiate covariantly on S^n the equation (8),

$$F_i^j \nabla_s' a_j^i = \bar{\psi}_s + \bar{\psi}_v v_s, \tag{31}$$

where the subscript v at $\bar{\psi}_v$ denotes differentiation with respect to v. Then, we multiply (31) by v^s and sum over s. This is a lengthy calculation and we break it down into several steps. Using (25) and (27) we obtain at $\bar{u}$

$$v^s\nabla_s'\frac{\sqrt{v^2+K}}{p^3} = \frac{\sqrt{v^2+K}}{p^3}\left[\frac{v}{v^2+K} + \frac{3}{2}\frac{\eta'}{\eta}\right]|\nabla' v|^2$$

and

$$v^s F_i^j \nabla_s'\frac{\sqrt{v^2+K}}{p^3}P^{ik}W_{kj} = \left[\frac{v}{v^2+K} + \frac{3}{2}\frac{\eta'}{\eta}\right] m\bar{\psi}|\nabla' v|^2. \tag{32}$$

Next, we have with repeated use of (27)

$$\begin{aligned} v^s\nabla_s' P^{ik}W_{kj} &= 2pp_s v^s e^{ik}W_{kj} - v^s\nabla_s' v^i v^k W_{kj} - v^i v^s \nabla_s' v^k W_{kj} \\ &= -\frac{p^3}{\sqrt{v^2+K}}\frac{\eta'}{\eta}|\nabla' v|^2 a_j^i + \frac{p^2}{2}\left(\frac{\eta'}{\eta}\right)^2 |\nabla' v|^2 v^i v_j - p^2\frac{\eta'}{\eta}\left(\frac{p^2}{2}\frac{\eta'}{\eta} + v\right) v^i v_j. \end{aligned}$$

Then, taking into account that $p^2 = v^2 + |\nabla' v|^2 + K$, we obtain

$$v^s\frac{\sqrt{v^2+K}}{p^3}F_i^j\nabla_s' P^{ik}W_{kj} = -\frac{\eta'}{\eta}|\nabla' v|^2 m\bar{\psi} - \frac{\sqrt{v^2+K}}{p}\left[\frac{v^2+K}{2}\left(\frac{\eta'}{\eta}\right)^2 + \frac{\eta'}{\eta}v\right]|Dv|^2, \tag{33}$$

where we put

$$|Dv|^2 \equiv F_i^j v^i v_j.$$

Multiply now (31) by v^s and sum over s. Then, using (32), (33), we get

$$\frac{\sqrt{v^2+K}}{p^3}v^sF_i^jP^{ik}\nabla'_sW_{kj}=\bar\psi_sv^s+\left[\bar\psi_v-\frac{mv\bar\psi}{v^2+K}\right]|\nabla'v|^2 \tag{34}$$
$$-\frac{\eta'}{2\eta}m\bar\psi|\nabla'v|^2+\frac{\sqrt{v^2+K}}{p}\left[\frac{v^2+K}{2}\left(\frac{\eta'}{\eta}\right)^2+\frac{\eta'}{\eta}v\right]|Dv|^2.$$

We transform (34) as follows. Assuming that $|\nabla'v|\neq 0$ (otherwise the needed estimate is obvious), we can rotate the local frame so that $\nabla'v=v^1\partial_1$. At $\bar u$ $e_{ij}=\delta_{ij}$ and it follows from (27) that at $\bar u$

$$\nabla'_{1i}v=v_{1i}=W_{1i}=0 \quad \text{when} \quad i>1.$$

By rotating the frame $\partial_2,...,\partial_n$ at $\bar u$ we can diagonalize the matrix (v_{ij}) at $\bar u$. Then by (28)

$$a_1^1=\frac{\sqrt{v^2+K}}{p}(v_{11}+v\delta_{11}+\frac{\eta'}{2\eta}v_1^2),\quad a_i^i=\frac{\sqrt{v^2+K}}{p}(v_{ii}+v\delta_{ii}) \ \text{ for } \ i>1,\quad a_j^i=0 \ \text{ if } \ i\neq j.$$

Consequently, the matrix (F_i^j) is diagonal at $\bar u$.

The matrix (P^{ik}) is also diagonal at $\bar u$ with $P^{11}=p^2-(v_1)^2=v^2+K$ and $P^{ii}=p^2$ for $i>1$. Thus, (34) becomes

$$\frac{\sqrt{v^2+K}}{p^3}v_1F_j^iP^{ji}\nabla'_1W_{ii}=\bar\psi_1v_1+\left[\bar\psi_v-\frac{mv\bar\psi}{v^2+K}\right]v_1^2 \tag{35}$$
$$-\frac{\eta'}{2\eta}m\bar\psi v_1^2+\frac{\sqrt{v^2+K}}{p}\left[\frac{v^2+K}{2}\left(\frac{\eta'}{\eta}\right)^2+\frac{\eta'}{\eta}v\right]F_1^1v_1^2.$$

In the chosen coordinates the inequality (30) assumes the form

$$2(\sum_s v_{is}W_{si}+v_1\nabla'_1W_{ii})+\left[\frac{\eta''}{\eta}-2\left(\frac{\eta'}{\eta}\right)^2\right]p^2v_i^2+p^2\frac{\eta'}{\eta}v_{ii}\leq 0. \tag{36}$$

By the first of the inequalities in (9) $F_i^i>0$ and $P^{ii}>0$ by the C^0 bounds, as it was explained in the beginning of this section. For each $i=1,2,...,n$ we multiply the inequality (36) by $F_j^iP^{ji}$ and sum over i. Then we obtain

$$v_1F_j^iP^{ji}\nabla'_1W_{ii}\leq -F_j^iP^{ji}\sum_s v_{is}W_{si}-\left[\frac{\eta''}{2\eta}-\left(\frac{\eta'}{\eta}\right)^2\right]p^2F_1^1P^{11}v_1^2-\frac{p^2}{2}\frac{\eta'}{\eta}F_j^iP^{ji}v_{ii}$$

$$=-F_j^iP^{ji}W_{ii}^2+\frac{vp^3}{\sqrt{v^2+K}}F_j^ia_i^j-\left[\frac{\eta''}{2\eta}-\left(\frac{\eta'}{\eta}\right)^2\right]p^2F_1^1P^{11}v_1^2-\frac{p^5}{2}\frac{\eta'}{\eta}\frac{F_j^ia_i^j}{\sqrt{v^2+K}}+\frac{p^2v}{2}\frac{\eta'}{\eta}\sum_i F_j^iP^{ji}$$

$$\leq\frac{vm\bar\psi p^3}{\sqrt{v^2+K}}-\frac{p^5}{2}\frac{\eta'}{\eta}\frac{m\bar\psi}{\sqrt{v^2+K}}-\left[\frac{\eta''}{2\eta}-\left(\frac{\eta'}{\eta}\right)^2\right]p^2F_1^1P^{11}v_1^2+\frac{p^2v}{2}\frac{\eta'}{\eta}\sum_i F_j^iP^{ji}. \tag{37}$$

Combining this inequality with (35) we obtain

$$vm\bar\psi - \frac{p^2}{2}\frac{\eta'}{\eta}m\bar\psi - \frac{\sqrt{v^2+K}}{p}\left[\frac{\eta''}{2\eta} - \left(\frac{\eta'}{\eta}\right)^2\right]F_1^1P^{11}v_1^2 + \frac{\sqrt{v^2+K}}{p}\frac{v}{2}\frac{\eta'}{\eta}\sum_i F_j^iP^{ji}$$

$$\geq \bar\psi_1 v_1 + \left[\bar\psi_v - \frac{mv\bar\psi}{v^2+K}\right]v_1^2 - \frac{\eta'}{2\eta}m\bar\psi v_1^2 + \frac{\sqrt{v^2+K}}{p}\left[\frac{v^2+K}{2}\left(\frac{\eta'}{\eta}\right)^2 + \frac{\eta'}{\eta}v\right]F_1^1v_1^2. \quad (38)$$

It can be shown that each of the conditions (21), (22) imply in each of the respective cases that

$$\bar\psi_v - \frac{mv\bar\psi}{v^2+K} \geq 0. \quad (39)$$

In order not to interrupt the present arguments we will postpone the proof of (39) till the end of this section.

Using (39) we strengthen the inequality (38) by deleting the term with $\bar\psi_v - \frac{mv\bar\psi}{v^2+K}$. In addition, we simplify it by using the fact that $P^{11} = v^2 + K$ and also regroup the terms. Then (38) becomes

$$\bar\psi_1 v_1 \leq vm\bar\psi - \frac{(v^2+K)m\bar\psi\eta'}{2\eta} + \frac{pv\sqrt{v^2+K}}{2}\frac{\eta'}{\eta}\sum_i F_i^i + J\sqrt{v^2+K}v_1F_1^1, \quad (40)$$

where

$$J \equiv \frac{\eta'}{\eta}\left[\frac{v(v^2+K)}{2pv_1} - \frac{pv}{2v_1} - \frac{v_1v}{p}\right] - \frac{v_1(v^2+K)}{2p}\left[\frac{\eta''}{\eta} - \left(\frac{\eta'}{\eta}\right)^2\right].$$

We claim that the function η can be chosen so that $J \leq 0$. First note that without loss of generality it may be assumed that $\max_{S^n}|\nabla'v| \geq \max_{S^n}\sqrt{v^2+K}$. Otherwise, the required estimate is trivial. With this assumption we have the following estimates

$$\frac{pv}{2|\nabla'v|} + \frac{|\nabla'v|v}{p} \leq (1+\frac{\sqrt{2}}{2})\max_{S^n} v \ (\equiv A),$$

$$\frac{|\nabla'v|(v^2+K)}{2p} \geq \frac{\min_{S^n}(v^2+K)}{2\sqrt{2}} \ (\equiv B).$$

By the C^0-estimates $A > 0,\ B > 0$. We choose now the function η by setting

$$\eta(v) = \exp\{Q\frac{B}{A}\exp\{-\frac{Av}{B}\}\},$$

where Q is a positive constant to be specified later. Then at $\bar u$ we have

$$\frac{\eta'}{\eta} = -Q\exp\{-\frac{Av}{B}\}, \quad \frac{\eta''}{\eta} - \left(\frac{\eta'}{\eta}\right)^2 = Q\frac{A}{B}\exp\{-\frac{Av}{B}\}$$

and

$$J = Q\exp\{-\frac{Av}{B}\}\left\{\left[-\frac{v(v^2+K)}{2pv_1} + \frac{pv}{2v_1} + \frac{v_1v}{p}\right] - \frac{A}{B}\frac{v_1(v^2+K)}{2p}\right\} < 0.$$

Consequently, the last term on the right side of (40) can be deleted.

We consider now the remaining terms in (40). We have

$$\frac{pv\sqrt{v^2+K}}{2}\frac{\eta'}{\eta}\sum_i F_i^i = -Q\exp\{-\frac{Av}{B}\}\frac{pv\sqrt{v^2+K}}{2}\sum_i F_i^i.$$

Since $\bar{\psi} > 0$, the $\sum_i F_i^i$ admits a positive lower bound depending only on $\bar{\psi}$; see [9]. Therefore,

$$\exp\{-\frac{Av}{B}\}\frac{v\sqrt{v^2+K}}{2}\sum_i F_i^i \geq c > 0$$

where c is a constant depending only on $\bar{\psi}, R_1, R_2, m, n$. Thus, we can choose Q so that

$$Q\exp\{-\frac{Av}{B}\}\frac{v\sqrt{v^2+K}}{2}\sum_i F_i^i - \max_{\bar{\Omega}}|grad\bar{\psi}| \geq c_1 > 0$$

with the choice of Q dependent only on $\bar{\psi}, R_1, R_2, m, n, |grad\psi|$. Then the inequality (40) assumes the form

$$c_1 p \leq vm\bar{\psi} + Q\frac{(v^2+K)m\bar{\psi}}{2}\exp\{-\frac{Av}{B}\}$$

which implies a bound on p at $\bar{u}$. Then

$$\max_{S^n}\phi \leq c_2 < \infty,$$

where c_2 depends only on $\bar{\psi}, R_1, R_2, m, n, |grad\psi|$. This implies the required bound (23).

In order to complete the proof it remains to establish (39). Consider first the case $K = 1$. We transform the condition (21) in Theorem 4.1 as follows. Using (12), (13) we get

$$\frac{\partial}{\partial\rho}\left[\psi q^{-m}\right] = q^{-m}[\psi_\rho + mq^{-1}(q^2+K)\psi] \leq 0, \tag{41}$$

where $\psi_\rho = \frac{\partial\psi}{\partial\rho}$. Using the relation $v = q(\rho)$, we obtain with the use of (12) and (13)

$$\frac{\partial\rho}{\partial v} = -f = -\frac{1}{q^2+K}, \quad \psi_v = -\frac{\psi_\rho}{v^2+K}$$

and it follows from (41) that

$$-\psi_v + mv^{-1}\psi \leq 0. \tag{42}$$

Since $K > 0$, (42) implies that

$$\bar{\psi}_v - \frac{mv\bar{\psi}}{v^2+K} > \bar{\psi}_v - \frac{m\bar{\psi}}{v} \geq 0 \tag{43}$$

and (39) is established.

Suppose now that $K = 0$. Then, arguing as in the case $K = 1$, we conclude that the left hand side of condition (21) is transformed into

$$\psi_v - mv^{-1}\psi,$$

which together with (21) implies (39).

Finally, consider the case when $K = -1$. It follows from condition (22) and definition of q that

$$-\psi_v(v^2 + K) + mv\psi \leq 0.$$

The proof of Theorem 4.1 is now complete.

Remark 1. In the case $K > 0$ the proof of the gradient estimate can be completed by setting $\eta \equiv 1$ in (38). Then it follows from (38) that

$$vm\bar{\psi} \geq \bar{\psi}_1 v_1 + \left[\bar{\psi}_v - \frac{v}{v^2 + K} m\bar{\psi}\right] v_1^2. \tag{44}$$

Together with (43) this establishes the required estimate (23).

Remark 2. If in Theorem 4.1 $m = n$ then the the estimate (23) is true without the conditions (21) and (22); see [11, 12]. In this case it can be shown that $\nabla' v = 0$ at the point where the $\max_{S^n} p^2$ is attained. This obviously implies an estimate of $|\nabla' v|$ by the $\max_{S^n}(v^2 + K)$.

5 C^2-estimate

Let $z \in C^4(S^n)$ be an admissible solution of equation (8). Let M be a hypersurface in $\mathcal{R}^{n+1}(1)$ given as a graph of z over S^n. In this section an estimate of the maximal principal curvature of the hypersurface M is obtained. Such an estimate together with the C^0- and C^1- a priori estimates in sections 3 and 4 implies an a priori estimate of the C^2 norm of solutions to the equation (8).

Many of our considerations here are valid for $\mathcal{R}^{n+1}(K)$ with $K = -1, 0, 1$ and may be useful in other instances. For that reason we will state and prove some of the preliminary results for an arbitrary space form. Unfortunately, the arguments in the proof of Theorem 5.2 are valid only when sectional curvature K is equal to either 1 or 0.

5.1 More local formulas

It is convenient to use a common framework to model $\mathcal{R}^{n+1}(K)$ in which the hyperbolic space $\mathcal{R}^{n+1}(-1)$ is modeled as the upper sheet of the two-sheeted hyperboloid in the $(n+2)$-dimensional Minkowski space with Lorentz metric and the elliptic space $\mathcal{R}^{n+1}(1)$ as the upper hemisphere of S^{n+1} in Euclidean space R^{n+2}. We can combine all three cases (including Euclidean space) by introducing the space

$$L^{n+2} = \{p = (p_0, p_1, \ldots, p_{n+1}) |\; p_0, p_1, \ldots, p_{n+1} \in R\}$$

with the metric

$$\langle , \rangle = K dp_0^2 + dp_1^2 + \ldots dp_{n+1}^2.$$

In this setting $\mathcal{R}^{n+1}(K)$ is identified with the appropriate hypersurface

$$\{p \in L^{n+2}; \langle p, p \rangle = K, \}$$

where in case of $K = -1$ we take $p_0 \geq 1$, when $K = 1$ we take $p_0 > 0$ and when $K = 0$ we take $p_0 = 0$.

Let S^n be a unit sphere centered at the origin and lying in the hyperplane $p_0 = 0$ in L^{n+2}. Put $e_0 = (1, 0, \ldots, 0)$. When $K = \pm 1$ we represent the hypersurface M in $\mathcal{R}^{n+1}(K)$ defined by function $z(u),\ u \in S^n$, as

$$X(u) = c(z(u))\, e_0 + s(z(u))\, u, \tag{45}$$

where u is treated as a point on S^n and also as a unit vector and

$$s(\rho) = \sqrt{f(\rho)}, \quad c = \frac{ds}{d\rho}.$$

When $K = 0$

$$X(u) = z(u)u.$$

As in section 2.1 we let $u^1, ..., u^n$ be some local coordinates on M and $X_i = \partial_i X,\ i = 1, ..., n$, the corresponding local frame of tangent vectors. The unit normal N to M (as a submanifold of L^{n+2}) oriented in the inward direction is given by

$$N = \frac{1}{\sqrt{f(z) + |\nabla' z|^2}} (Kf(z)e_0 + \nabla' z - c(z)s(z)u). \tag{46}$$

We record here the Weingarten, Codazzi, Gauss, and Ricci equations on M.

$$\nabla_i N = -b_{is} g^{sk} X_k, \tag{47}$$

$$\nabla_{ij} N = -\sum_{s,k} \nabla_j b_{is} g^{sk} X_k - \sum_{s,k} b_{is} g^{sk} b_{kj} N + K b_{ij} X, \tag{48}$$

$$\nabla_i b_{jk} = \nabla_k b_{ji}, \tag{49}$$

$$\nabla_{ij} X = b_{ij} N - K g_{ij} X \tag{50}$$

$$R_{ijkl} = b_{ik} b_{jl} - b_{il} b_{jk} + K(g_{ik} g_{jl} - g_{il} g_{jk}), \tag{51}$$

$$\nabla_l \nabla_k b_{ij} - \nabla_k \nabla_l b_{ij} = \sum_l b_{il} R_{ljkl} + \sum_l b_{jl} R_{likl}, \tag{52}$$

where ∇_i and ∇_{ij} denote covariant differentiation in the metric g on M with respect to some local coordinates on M.

5.2 An estimate of the maximal normal curvature of M

Let $k_1 \geq ... \geq k_n$ be the principal curvatures of M. Since the function $\bar{\psi}$ in (8) is positive, it follows that $\sum_i k_i > 0$ on M and therefore $k_1 > 0$.

Lemma 5.1 *Let $1 \leq m \leq n$ and let $\psi(X)$ be a positive C^2 function in the annulus $\bar{\Omega}:\ u \in S^n,\ \ \rho \in [R_1, R_2],\ 0 < R_1 < R_2 < a$. Let $z \in C^4(S^n)$ be an admissible solution of equation (8) in $\mathcal{R}^{n+1}(1)$. Let $\bar{u}$ be a point on M and coordinates $u^1, ..., u^n$ with origin at $\bar{u}$ in some* *neighborhood U of $\bar{u}$ are such that the frame $X_1, ..., X_n$, is orthonormal in metric g on U and the second fundamental form b_{ij} is diagonal at $\bar{u}$. Then at $\bar{u}$*

$$\bar{\psi}_{II} - (1 - \frac{1}{m})\frac{\bar{\psi}_I^2}{\bar{\psi}} \le \sum_i F_i^i \nabla_{ii} b_{11} + b_{11} \sum_i F_i^i b_{ii}^2 - b_{11}^2 m\bar{\psi} + K(m\bar{\psi} - b_{11} \sum_i F_i^i), \tag{53}$$

where

$$\bar{\psi}_I \equiv \nabla_1 \bar{\psi} + \bar{\psi}_z \nabla_1 z \quad \left(\bar{\psi}_z = \frac{\partial \bar{\psi}}{\partial z}\right)$$

and

$$\bar{\psi}_{II} \equiv \nabla_{11} \bar{\psi} + 2\nabla_1 \bar{\psi}_z \nabla_1 z + \bar{\psi}_{zz} (\nabla_1 z)^2 + \bar{\psi}_z \nabla_{11} z \quad \left(\bar{\psi}_{zz} = \frac{\partial^2 \bar{\psi}}{\partial z^2}\right).$$

Proof. First we calculate the first covariant derivative (in the metric g) of the equation (8) with respect to u_1. This derivative is given by

$$\sum_{i,j} F_i^j \nabla_1 a_i^j = \bar{\psi}_I, \tag{54}$$

The second covariant derivative is given by

$$\sum_{\substack{i,j,k,s \\ i \ne s,\ j \ne k}} \frac{\partial F_i^j}{\partial a_s^k} \nabla_1 a_s^k \nabla_1 a_j^i + \sum_{i,j} F_i^j \nabla_{11} a_j^i = \bar{\psi}_{II} \tag{55}$$

Note that the metric g is constant with respect to the operator ∇ and therefore at $\bar{u}$

$$\nabla_1 a_i^j = \nabla_1 b_{ij}.$$

Taking into account that F_j^i and b_{ij} are both diagonal at $\bar{u}$, it follows from (54) that

$$\sum_i F_i^i \nabla_1 b_{ii} = \bar{\psi}_I \quad \text{at} \quad \bar{u}. \tag{56}$$

Similarly, at $\bar{u}$ we have

$$\sum_{\substack{i,j,k,s \\ i \ne s,\ j \ne k}} \frac{\partial F_i^j}{\partial a_s^k} \nabla_1 a_s^k \nabla_1 a_i^j = \sum_{i \ne j} \frac{\partial^2 F}{\partial b_{jj} \partial b_{ii}} [\nabla_1 b_{jj} \nabla_1 b_{ii} - (\nabla_1 b_{ij})^2].$$

Since $F^{1/m}(a_i^j) = S_m^{1/m}(k_1, ..., k_n)$, we can use the second inequality in (9) to delete the term with the negative sign on the right. Then

$$\sum_{\substack{i,j,k,s \\ i \ne s,\ j \ne k}} \frac{\partial F_i^j}{\partial a_s^k} \nabla_1 a_s^k \nabla_1 a_j^i \le \sum_{i \ne j} \frac{\partial^2 F}{\partial b_{jj} \partial b_{ii}} \nabla_1 b_{jj} \nabla_1 b_{ii}. \tag{57}$$

Using concavity of $F^{1/m}(a_i^j)$ we get

$$\sum_{i \ne j} \frac{\partial^2 F}{\partial b_{jj} \partial b_{ii}} \nabla_1 b_{jj} \nabla_1 b_{ii} \le (1 - \frac{1}{m}) \frac{1}{F} \Big(\sum_i F_i^i \nabla_1 b_{ii} \Big)^2 = (1 - \frac{1}{m}) \frac{\bar{\psi}_I^2}{\bar{\psi}}.$$

The equality on the right follows from (56). This inequality, (57) and (55) give

$$\sum_i F_i^i \nabla_{11} b_{ii} \geq \bar{\psi}_{II} - (1 - \frac{1}{m}) \frac{\bar{\psi}_I^2}{\bar{\psi}}. \tag{58}$$

We transform the left side of this inequality as follows. Using (49) and (52) we obtain

$$\nabla_{11} b_{ii} = \nabla_{1i} b_{1i} = \nabla_{i1} b_{1i} + \sum_k b_{1k} R_{kii1} + \sum_k b_{ik} R_{k1i1}.$$

By (49) $\nabla_{i1} b_{1i} = \nabla_{ii} b_{11}$ and applying the Gauss equations (51) we get at $\bar{u}$

$$\nabla_{11} b_{ii} = \nabla_{ii} b_{11} + \sum_k b_{1k} (b_{ki} b_{i1} - b_{k1} b_{ii} + K(\delta_{ki}\delta_{i1} - \delta_{k1}\delta_{ii}))$$

$$+ \sum_k b_{ik} (b_{ki} b_{11} - b_{k1} b_{1i} + K(\delta_{ki}\delta_{11} - \delta_{k1}\delta_{1i}))$$

$$= \nabla_{ii} b_{11} + b_{11} b_{ii}^2 - b_{11}^2 b_{ii} + K(b_{11}\delta_{1i}\delta_{1i} - b_{11}\delta_{ii} + b_{ii} - b_{i1}\delta_{1i}).$$

Noting that at $\bar{u}$ $\sum_i F_i^i b_{ii} = m\bar{\psi}$, we get

$$\sum_i F_i^i \nabla_{11} b_{ii} = \sum_i F_i^i \nabla_{ii} b_{11} + b_{11} \sum_i F_i^i b_{ii}^2 - b_{11}^2 m\bar{\psi} + K(m\bar{\psi} - b_{11} \sum_i F_i^i). \tag{59}$$

This expression and (58) give (53).

Theorem 5.2 *Let $1 \leq m \leq n$ and let $\psi(X)$ be a positive C^2 function in the annulus $\bar{\Omega}: \ u \in S^n, \ \ \rho \in [R_1, R_2], \ 0 < R_1 < R_2 < a$. Let $z \in C^4(S^n)$ be an admissible solution of equation (8) in $\mathcal{R}^{n+1}(1)$ satisfying the inequalities*

$$R_1 \leq z(u) \leq R_2, \quad u \in S^n \tag{60}$$

and

$$|\nabla' z| \leq C = const \quad on \quad S^n. \tag{61}$$

Then

$$\| z \|_{C^2(S^n)} \leq C_1, \tag{62}$$

where the constant C_1 depends only on $m, n, R_1, R_2, C, \| \psi \|_{C^2(\bar{\Omega})}$.

Proof. We estimate the maximal principal curvature of M. Such an estimate together with the C^0- and C^1- estimates implies an estimate of $\| z \|_{C^2(S^n)}$. We preserve here the notation used in the proof of the preceding lemma. Put

$$\tau(u) = \langle N(u), e_0 \rangle, \quad \eta(u) = \langle X(u), e_0 \rangle$$

It follows from (60) and (61) that the function τ on M is uniformly bounded away from 0 and ∞. Let

$$\omega(u) = \log \frac{b_{11}(u)}{\tau(u)}. \tag{63}$$

Again, because of the estimates (60) and (61), in order to estimate the maximal curvature k_1 on M it suffices to estimate $\max_M \omega$.

The function ω is similar to the function g in [6], section 4, but here we work in elliptic space. Also, in our proof we do not use the special local coordinates used in [6] and this simplifies the computations.

Let $\bar{u} \in M$ be a point where the function ω attains its maximum and the coordinates $u^1, ..., u^n$ are as in Lemma 5.1. Then $F_i^j = \frac{\partial F}{\partial b_{ii}}$ when $i = j$ and $F_i^j = 0$ otherwise. Note that at $\bar{u}$ the covariant derivatives coincide with the usual derivatives.

At $\bar{u}$ we have $\nabla_i \omega = 0$, which implies

$$\frac{\nabla_i b_{11}}{b_{11}} = \frac{\nabla_i \tau}{\tau}, \quad i = 1, 2, ..., n, \tag{64}$$

and

$$\nabla_{ii}\omega = \frac{\nabla_{ii} b_{11}}{b_{11}} - \left(\frac{\nabla_i b_{11}}{b_{11}}\right)^2 - \frac{\nabla_{ii}\tau}{\tau} + \left(\frac{\nabla_i \tau}{\tau}\right)^2 \leq 0, \quad i = 1, 2, ..., n. \tag{65}$$

Squaring (64) and substituting in (65) we get

$$\frac{\nabla_{ii} b_{11}}{b_{11}} \leq \frac{\nabla_{ii}\tau}{\tau}. \tag{66}$$

Using definitions of functions τ and η and the Weingarten, Codazzi and Gauss equations (with $K = 1$), we obtain

$$\nabla_i \tau = -b_{ii}\nabla_i \eta, \quad \nabla_{ii}\tau = -\sum_s \nabla_s \eta \nabla_s b_{ii} - \tau b_{ii}^2 + \eta b_{ii}, \quad \nabla_{ii}\eta = \tau b_{ii} - \eta \delta_{ii}.$$

Substituting these expressions into (66) we get

$$\frac{\nabla_{ii} b_{11}}{b_{11}} \leq -\frac{1}{\tau}\sum_s \nabla_s \eta \nabla_s b_{ii} - b_{ii}^2 + \frac{\eta b_{ii}}{\tau}.$$

At $\bar{u}$ F_j^i is diagonal and $F_i^i > 0$ by (9). Multiplying the last inequality by F_i^i, summing over i and taking into account that $\sum_i F_i^i b_{ii} = m\bar{\psi}$, we obtain

$$\frac{1}{b_{11}}\sum_i F_i^i \nabla_{ii} b_{11} \leq \frac{1}{\tau}\sum_s \nabla_s \eta \nabla_s \bar{\psi} + \frac{m\bar{\psi}\eta}{\tau} - \sum_i F_i^i b_{ii}^2. \tag{67}$$

Consider now the inequality (53) in Lemma 5.1. We use the estimate (67) to bound the first term on the right of (53). Also, we strengthen the inequality (53) by deleting the term $-b_{11}\sum_i F_i^i$. Then (53) assumes the form

$$\bar{\psi}_{II} - (1 - \frac{1}{m})\frac{\bar{\psi}_I^2}{\bar{\psi}} \leq \frac{b_{11}}{\tau}\left(\sum_s \nabla_s \eta \nabla_s \bar{\psi} + m\bar{\psi}\eta\right) + m\bar{\psi} - b_{11}^2 m\bar{\psi}. \tag{68}$$

Next, we observe that the $\max_M |\bar{\psi}_I|$ is bounded by a constant depending only on $\bar{\psi}$, its first derivatives, and C^1- norm of z. Similarly, $\max_M |\bar{\psi}_{II}|$ is bounded by a constant depending on the same quantities, $\| \bar{\psi} \|_{C^2(\bar{\Omega})}$ and $\max_M |\nabla'_{11} z|$. On the other hand, it follows from

(6) that $|\nabla'_{11} z| < c_2 |b_{11}| + c_3$, where the constants c_2, c_3 depend only on R_1, R_2 and C in (61). Then, it follows from (68) that b_{11} is bounded by a constant depending only on $m, n, R_1, R_2, \psi, grad\psi, |gradz|$. It follows from (63) that $max_M b_{11}$ is bounded by a constant depending on the same quantities. Now, (6) implies the required estimate (62).

Remark. Essentially the same arguments give also the estimate (62) in the case when $K = 0$ treated earlier in [11] for $m = n$ and in [6] for general m, $1 \leq m \leq n$. The required modifications reduce to replacing the function $\tau(u)$ by $\tau(u) = \langle N(u), X(u) \rangle$, setting $\eta(u) \equiv 0$, and using Lemma 5.1 with $K = 0$. The calculations using the Weingarten, Codazzi, Gauss, and Ricci equations should also be adjusted accordingly.

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