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HENRIQUE VIANA OLIVEIRA

PROPOSITIONAL BELIEF MERGING AND DISTRIBUTIVE JUSTICE

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Tese apresentada ao Programa de Pós-Graduação em Ciência da Computação do Centro de Ciências da Universidade Federal do Ceará, como requisito parcial à obtenção do título de doutor em Ciência da Computação. Área de Concentração: Ciência da Computação

Orientador: Prof. Dr. João Fernando Lima Alcântara

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You live you learn

You love you learn

You cry you learn

You lose you learn

You bleed you learn

You scream you learn

You grieve you learn

You choke you learn

You laugh you learn

You choose you learn

You pray you learn

You ask you learn

You live you learn

(Alanis Morissette)

RESUMO

A Fusão de Crenças é um campo da Inteligência Artificial que estuda a fusão de fontes de informação independentes e igualmente confiáveis e é uma ferramenta importante para a tomada de decisões em sistemas multiagentes. Por outro lado, a Justiça distributiva, que é apresentada nas áreas de Economia, Ética, Filosofia Política e Psicologia Moral, preocupa-se em especificar o que se entende por distribuição justa de bens entre os membros de um grupo. Propomos integrar ambas as áreas, introduzindo e estudando a racionalidade de diferentes operadores de fusão de crenças, que levam em conta algumas teorias de justiça distributiva. As teorias do igualitarismo e do suficientismo são nosso foco central nesta tese. A partir do igualitarismo, estudamos os operadores de T-conorm e OWA da literatura da Lógica Difusa e, do suficientismo, estudamos os operadores *headcount*, *shortfall* e FGT. O aspecto da racionalidade desses operadores é medido por propriedades lógicas. Além das propriedades lógicas originais da fusão de crenças, consideramos outras propriedades da teoria da escolha social para descrever o comportamento de operadores igualitários e suficientistas. Provamos ainda quais são as condições necessárias para que esses operadores satisfaçam algumas propriedades lógicas igualitárias ou suficientistas.

Palavras-chave: Fusão de Crenças. Justiça Distributiva. Conjuntos Difusos. Igualitarismo. Suficientismo.

ABSTRACT

Belief merging is a field from Artificial Intelligence which studies the fusion of independent and equally reliable sources of information and it is an important tool for decision making in multi-agent systems. On the other hand, Distributive Justice, which is presented in the areas of Economics, Ethics, Political Philosophy and Moral Psychology, is concerned in specifying what is meant by a just distribution of goods among members of a group. We propose to integrate both areas by introducing and studying the rationality of different operators from belief merging, that take into account some theories of distributive justice. The theories of egalitarianism and sufficientarianism are our central focus in this thesis. From egalitarianism, we study T-conorm and OWA operators from the Fuzzy Logic literature, and from sufficientarianism, we study *headcount*, *shortfall* and FGT operators. The aspect of rationality of these operators are measured by logical properties. Besides the original logical properties from belief merging, we consider other properties from social choice theory to describe the behavior of egalitarian and sufficientarian operators. Furthermore, we prove what are the conditions needed to be achieved for these operators satisfy some egalitarian or sufficientarian logical properties.

Keywords: Belief Merging. Distributive Justice. Fuzzy Sets. Egalitarianism. Sufficientarianism.

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1 INTRODUCTION

This thesis proposes to combine the area of belief merging with the theories of distributive justice. Belief merging is an important issue in Artificial Intelligence and Databases and is concerned with the process of combining information obtained from different sources to produce a single consistent piece of information. On the other hand, theories of distributive justice, presented in the areas of Economics, Ethics, Political Philosophy and Moral Psychology, seek to specify what is meant by a just distribution of goods among members of society (in our case, the information among the belief bases).

Let us describe a practical example in which a group of friends are in a restaurant and they wish to choose to share a dish from the restaurant's menu. A natural choice for the group would be to open a voting process and the top rated dish would be the group's choice. But suppose one of the friends has food restrictions, and he may not eat certain types of food. Then what would be a fair/reasonable decision? Would an ordinary voting process still be a natural choice to guide the group's decision? Assume now some members of the group are very hungry, having spent all day without eating properly. In this case, a fair collective decision could take into account more elaborate dishes to obtain sufficient nutrients for everyone. Moreover, the final choice would be the one that tries the most to satisfy each one of the group. However, this does not always happen in reality and if we imagined a voting method to decide the outcome of the group, those aspects would certainly be disregarded. These cases just portrayed motivate the kinds of collective decision making we will exploit along this work.

How to make a fair and just choice in collective decisions is a challenging task and has been studied intensively in Artificial Intelligence (PHILLIPS-WREN; JAIN, 2006). Belief merging arises as one of the approaches to tackle this problem (KONIECZNY; PINO-PÉREZ, 1998). It is primarily used for decision making in multi-agent systems. The area of belief merging is related to the area of belief revision (ALCHOURRON *et al.*, 1985; KATSUNO; MENDELZON, 1991b; GÄRDENFORS, 1992), but the latter is concerned with the decision issues of a single agent, in which it his/her belief base and wishes to modify it depending on certain events occurring in the world real. Belief merging is exclusively used when engaging a group of agents. Each agent has his/her own source of information and the goal is to generate a result coherent with the whole group.

But why should we use belief merging? First, it is a logical approach to be understood and calculated; second, it offers a great flexibility in the choice of operators, since it is based on a minimal and intuitive set of logical postulates. Furthermore, these operators have been conceived to define the behavior of the decision process for the group.

And how do we decide if a choice of an operator is the best option for a group? One solution is to use the area of distributive justice to assist us in elucidating this process. Let us go back to the example of the friends in a restaurant. In the situation where only one of them has a food restriction, a reasonable decision would be to give a higher priority to that person with restriction. In the second situation where some of them explain that they do not like a lot of food, it would be good to try to help him, but this is not mandatory. The way in which this decision will be done is related to various theories of the distributive justice. For example, if we are going to give priority to the most restricted person or to only a subgroup of people; or all have the same priority level; or if some specific people have more priorities and others do not. All these circumstances are contained into a theory of distributive justice.

The most correct or just decision will depend on characteristics that are going to be taken into consideration to the group, that is, what current of theory of justice they are following to make a collective decision. Therefore, we are grounded on belief merging to represent the beliefs of the agents and make the decision of the group, and on distributive justice to assist us in this decision making.

1.1 Belief Merging

Belief merging is inserted in the area of Belief change theory, which has produced a lot of operators to model the different ways the beliefs of one (or some) agent(s) evolve over time. Some examples of these operators in the belief change theory, besides belief merging, are the belief revision (ALCHOURRON *et al.*, 1985; KATSUNO; MENDELZON, 1991b; GÄRDENFORS, 1992), belief update (KATSUNO; MENDELZON, 1991a; HERZIG; RIFI, 1998), belief extrapolation (SAINT-CYR; LANG, 2011), etc.

Belief merging aims at combining several pieces of information when there are no strict preference between them. The problem formulation faces several conflicting pieces of information coming from several sources of equal reliability, and it has to build a coherent description of the world from them.

As in belief revision, rationality postulates have been proposed to characterize belief merging operators. Indeed, these postulates are closely related to those in belief revision. Nevertheless there is an important difference, namely the social aspect of merging: one needs some postulates to say how to solve the conflicts between the sources of information. So it is possible to distinguish different families of merging operators, depending on their behavior with respect to the sources, like a majority behavior for instance. Similarly to belief revision, it is possible to state representation theorems that provide a constructive way to define merging operators satisfying all the desired logical properties.

There are numerous ways to define merging operators: model-based operators (LIN; MENDELZON, 1999; KONIECZNY; PINO-PÉREZ, 1998; KONIECZNY; PINO-PÉREZ, 1999), that select the interpretations that are the closest to the set of sources; formula-based operators (KONIECZNY *et al.*, 2004; BARAL *et al.*, 1990; BARAL *et al.*, 1991; EVERAERE *et al.*, 2007), that use a selection function on sets of formulas; DA^2 operators (KONIECZNY *et al.*, 2004), that generalize model-based operators and allow to take into account inconsistent sources; disjunctive operators (EVERAERE *et al.*, 2010a), that select the result of the merging inside the disjunction of the bases; conflict-based operators (EVERAERE *et al.*, 2008a), that use a vector of conflict in order to represent the conflict instead of the numerical distance of model-based operators; default-based operators (DELGRANDE; SCHAUB, 2004), that use renaming of the propositional variables of the language.

In this thesis, we assume each agent's source of information is described in terms of propositional logic formulas, and all the information have the same importance/priority. Notwithstanding, merging has also been studied in other representation frameworks, where new problems and possibilities arise. For instance, in (DELGRANDE *et al.*, 2006) Delgrande, Dubois and Lang propose a discussion on prioritized merging operators. Furthermore, there are merging operators for weighted formulae (BENFERHAT *et al.*, 2000; BENFERHAT *et al.*, 2002; KACI, 2011), first order logic (LIN; MENDELZON, 1996), logic programs (DELGRANDE *et al.*, 2009; HUÉ *et al.*, 2009; CREIGNOU *et al.*, 2014), constraint networks (KACI, 2011; KACI; TORRE, 2006) and argumentation frameworks (KACI, 2011; AMGOUD; KACI, 2007).

There are also related works towards applications of these belief merging techniques in many areas as text processing (HUNTER, 2002a; HUNTER, 2002b; HUNTER; SUMMERTON, 2006) and XML documents (HUNTER; LIU, 2006; MU *et al.*, 2007), requirement engineering (GHOSE; LIN, 2006) and cancer diagnosis (KAREEM *et al.*, 2017).

Merging operators are closely attached to social choice theory (ARROW *et al.*, 2002) and in particular to voting methods (ARROW *et al.*, 1963; ARROW *et al.*, 2002). In such contexts, it is worthwhile to study what are the consequences of well known social choice concepts when applied to merging scenarios. Two examples of these concepts of social choice theory associated to belief merging are strategy-proofness (EVERAERE *et al.*, 2007) and truth-tracking (EVERAERE *et al.*, 2010b). Strategy-proofness is about the resistance of strategic manipulation from the sources/agents. The truth-tracking issue study if the merging/voting methods are capable to identify the true state of the world if the sources/agents are sufficiently reliable.

Merging operators are also associated with distributive justice (RESCHER, 1982; COOK; HEGTVEDT, 1983). Questions of distributive justice, arise, naturally enough, in contexts where some sort of outcome could be provided by two or more agents (see next section). The reason why this is of interest to researchers is that in many cases disagreements are possible about what justice requires in a particular situation. The connection between these two areas, belief merging and distributive justice is the main focus of this thesis.

1.2 Distributive Justice

The term distributive justice (RESCHER, 1982; COOK; HEGTVEDT, 1983) refers to fairness in the way things are distributed, caring more about how it is decided who gets what, rather than what is distributed. In modern society, this is an important principle, as it is generally expected that all goods will be distributed throughout society in some manner. In a society with a limited amount of resources, the question of fair allocation is often a source of debate and contention. This is called distributive justice.

Some modern philosophers express the notion of distributive justice is not very old, probably originating in the 18th century, based on the idea society did not have a structure sophisticated enough to address allocation of resources with the intent of meeting everyone's needs.

Distributive principles vary in numerous dimensions. They vary in what is considered relevant to distributive justice (income, wealth, opportunities, jobs, welfare, utility, etc.); in the nature of the recipients of the distribution (individual persons, groups of persons, reference classes, etc.); and on what basis the distribution should be made (equality, maximization, according to individual characteristics, according to free transactions, etc.). In this entry, the focus is primarily on principles designed to cover the distribution of benefits and burdens of economic activity among individuals in a society.

1.2.1 The Nature of Justice

In order to characterize theories of distributive justice, we need firstly to know what is justice, which is a difficult term to define. Typically, we think there are four concepts associated with the definition of justice (FORSYTH, 2006). These are

1. **Fairness:** We must treat similar cases in the same way. For instance, it would be unfair if we were to respond to one murderer by putting him in jail, and then respond to another murderer by giving him an ice cream. Similar situations must be treated in similar ways.
2. **Equality:** Our treatment of people ought to reflect the fact we are all morally equal. There are no morally relevant differences between human beings which make it permissible to treat them differently. For instance, there is not one race or gender that is "better" than the others. To act otherwise is to engage in immoral discrimination.
3. **Desert:** People ought to get what they deserve (i.e., good deeds should be rewarded, and bad deeds should be punished). For instance, when a criminal gets away with their deed and goes unpunished, we typically think that an "injustice" has occurred.
4. **Rights:** There are certain moral claims that everyone ought to be able to exercise against others. For instance, we commonly think that everyone has a right to life, a right to the freedom of speech, and freedom of religion. When we say that we ought to "be able to exercise these claims against others" we mean that, if someone tries to violate one of your rights (for instance by trying to kill you), you have a legitimate claim against them, since they have an obligation not to violate your right (i.e., they are doing something morally wrong by violating your right).

Given these four features of justice, we might now be able to answer a closely related question: "What is the just distribution?" Or, in other words, "How much resource/how many goods should each person have?" Here are some common answers to that question:

- Each person should receive an **equal** share.
- Each person should receive a share, according to how much they **need**.
- Each person should receive a share, according to how much they **contribute**.
- Each person should receive a share, according to how much they **merit** it.

In this thesis, we will focus mainly on questions about **fairness** and **equality**. For this, we will consider theories of justice that encompass them.

1.2.2 Theories of Distributive Justice

Below we will list some theories of distributive justice we will consider directly or indirectly along this thesis. Although the numerous distributive principles vary along different dimensions, for simplicity, they are presented here in broad categories. Even though these are common classifications in the literature, it is important to keep in mind they necessarily involve over-simplification, particularly with respect to the criticisms of each of the groups of principles.

1. **Utilitarianism**: In the present work, utilitarianism (MILL, 1869; MILL, 1871; MYERSON, 1981) is the view that the moral value of a distribution of income/wealth/opportunity/utility is the non-weighed sum of each individual's income/wealth/opportunity/utility. The basic utilitarian approach to justice is to maintain that when we act to maximize utility, we are also acting justly (and vice versa). Thus, while utility maximization and justice are distinct concepts, in practice, achieving one also achieve the other; justice and utility converge. Utilitarianism is also distinguished by impartiality and agent-neutrality. Everyone's happiness counts the same. When one maximizes the good, it is the good impartially considered. My good counts for no more than anyone else's good. Further, the reason I have to promote the overall good is the same reason anyone else has to so promote the good.
2. **Egalitarianism**: This theory comes in many different versions (SCHEFFLER, 2003). Basically, an egalitarian favors equality of some sort: People should get the same, or be treated the same, or be treated as equals, in some respect, or enjoy an equality of social status. Egalitarian doctrines tend to rest on a background idea that all human persons are equal in fundamental worth or moral status. Egalitarianism is a versatile doctrine, because there are several different types of equality, or ways in which people might be treated the same, or might relate as equals, that might be thought desirable. However, all of the egalitarians approaches have something in common, that is the objective of decreasing inequality.
3. **Sufficientarianism**: Rather than being concerned with inequalities as such or with making the situation of everyone as good as possible, sufficientarian justice (FRANKFURT, 1987) aims at making sure that each of us has enough. The sufficientarian holds that justice requires that everybody gets "enough", not that everybody has the same. To flesh out this idea one needs answers to two questions: (1) Enough what? And (2) How much is enough?

Some authors tend to identify justice with the idea that all people should equally be assured the basic capability (real or effective freedom) to function in important valuable ways. "Basic capability" is capability at a threshold "good enough" level. Sufficientarians object that egalitarians make a fetish of distributive equality. The real problem of social justice is never merely that some have more than others, but that some do not have enough. If, for example, some people face grim, horrible life conditions, that is bad, and it would not in any way be better if we all equally faced such conditions.

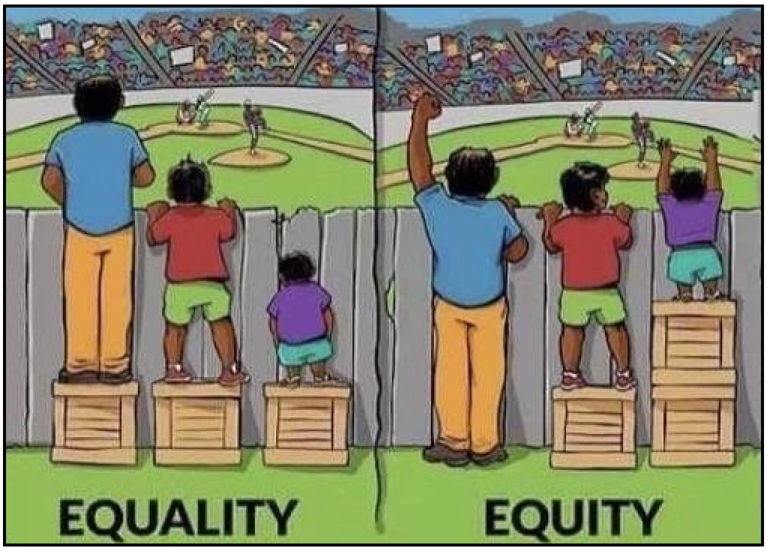
4. **Prioritarianism**: It is a view where an extra weight is given to worse-off individuals (ADLER, 2016). Prioritarianism resembles utilitarianism. The difference is under a prioritarian point of view, one does not weight the well-being of all individuals equally, but instead prioritizes those individuals that are worse-off. Suppose one can choose between gaining a benefit for a person who is very badly off or gaining an identical benefit for a person who is already very well off. Priority says the moral value of getting a same-sized benefit to a person is greater, the worse off in absolute terms she was, over the course of her life, without this benefit.

1.3 Motivation

There are two main subclasses of belief merging operators: majority operators which are related to utilitarianism, and arbitration operators which are related to egalitarianism (KONIECZNY; PINO-PÉREZ, 2002b). However, there is much less work on egalitarian operators than utilitarian operators and none related to other theories of distributive justice. In order to fill this gap, in (EVERAERE *et al.*, 2014) it was investigated possible translations into a belief merging framework of some egalitarian properties, as well as some egalitarian merging operators. It was studied how these properties interact with the standard rationality conditions considered in belief merging. It was a first approach to this topic and a lot of work about this issue is still needed.

Inspired by (EVERAERE *et al.*, 2014), this thesis has a main task to be performed: to study the rationality conditions that have been proposed to belief merging operators when we consider merging operators from different theories of distributive justice, especially egalitarianism and sufficientarianism. It has been stated that a "good" belief merging operator has to satisfy a series of rationality postulates. A question to be answered along this thesis is that if any theory of distributive justice produces a "good" belief merging operator. If not, our concern is in what conditions it is possible to turn a belief merging operator into a "good" one, without making the new operators lose the characteristics which insert them in their respective theories of distributive justice.

The starting point for this thesis is related to the maximum (*max*) operator. Regarding belief merging, *max* is responsible for making the decision based on the worst-case within a group (it gives absolute priority to the worst-case agent). This is a form of egalitarianism, because when we give priority to the worst agent within a group and helping it, we are certainly diminishing the group's inequality. The figure below illustrates somewhat the concept of egalitarianism brought by the *max*, which is based on equity. The general concept of equality states that everyone should have the same opportunity for a group to be fairer, while the concept of equity states that people with more needs should have more opportunities than the rest of the group.



From this idea we can think over other forms of priority for the agents by relaxing or restricting the *max* operator, or by combining it with other operators. For example, one can give priority not only to the worst agent, but to a subgroup of agents in a bad situation. Or else one can give a priority to the worst case within the group, but without forgetting the other agents in the group. These issues can be solved with several operators taken from different theories of justice.

We are going to focus on three approaches related to the worst case idea: the T-conorm operators, which are a generalization of the *max* and are a natural interpretation of the disjunction in the semantics of mathematical fuzzy logics (HÁJEK, 1998); the sufficientarian operators, which divide a group of worst cases and then treat each of them with the same priority; and the OWA operators, which generalize the *max* among other operators by allowing to consider not only the worst case, but also to combine it with other cases in the group.

In other words, we propose to answer if it is possible to represent different approaches of distributive justice in belief merging. We aim at achieving a similar behavior presented for majority or arbitration operators when considering other kinds of theory of distributive justice.

1.4 Objectives

In short, the main objectives of this thesis are

- To combine the area of belief merging with the theories of distributive justice;
- To explore new merging operators for belief merging;
- To propose new logical properties in belief merging to support the rationality analysis of the merging operators;
- To characterize the new merging operators in different classes of belief merging operators.

1.5 Main Contributions of this Thesis

The mains contributions of this thesis are

- We explore further new egalitarian operators. The first idea is to relax the maximum (*max*) operator and employ fuzzy connectives. When applied in belief merging, *max* operator proposes to minimize the worst cases in the group decision. T-conorms (ZADEH, 1983; KLEMENT *et al.*, 2000) are functions stronger than the *max* operator, which can be also used to capture the worst cases in some group decision problems. Thus, we expect to offer a new view about different merging operators with good logical properties and rationality.
- Besides the original logical properties (KONIECZNY; PINO-PÉREZ, 2011) for belief merging, we consider other egalitarian properties from social choice theory: the Hammond Equity Condition (EVERAERE *et al.*, 2014), Pigou-Dalton Principle (DALTON, 1920; EVERAERE *et al.*, 2014), the Harm Principle (ALCANTUD, 2011; CAPPELEN; TUNGODDEN, 2006; LOMBARDI *et al.*, 2013), Strong and Weak Pareto (TUNGODDEN, 2000). We prove that in some cases, restricted versions of these axioms may be satisfied by the new operators.
- We introduce the theory of Sufficientarianism in the propositional belief merging scene, based on two sufficientarian approaches: weak sufficientarianism (FRANKFURT, 1987) and strong sufficientarianism (SEGALL, 2014). This theory accommodates the concern for people who are badly off relative to a specific aspect (which we commonly call of

poverty or sufficiency). We show that, according to most versions, sufficientarian operators reject partially others theories of distributive justice, such as utilitarianism (concerned with the sum total of happiness of a group) and egalitarianism (which wants to promote equality for all people in a group). Overall, we deal with two sufficientarian operators: *headcount* and *shortfall*. Their objectives are minimize the number of people in a poverty situation and the amount of poverty in a group, respectively.

- We bring new logical properties from the sufficientarianism to the belief merging area. They are inspired and adapted from the work (TUNGODDEN; VALLENTYNE, 2005).
- We continue the investigation on egalitarian operators by introducing Ordered Weighted Averaging Operators (OWA) merging operators. They are a family of aggregation operators which assign weights to the values being aggregated. They are powerful enough to include many well-known operators such as the maximum, minimum and the simple average (YAGER; KACPRZYK, 1997), and many other operators depending on the values of the weights applied. As our main contributions, we define OWA merging operators and show their logical properties. As the operators defined in (EVERAERE *et al.*, 2014), OWA merging operators do not satisfy all the usual belief merging logical postulates. We show what conditions are required for an OWA merging operator to satisfy some missing logical postulates. We show that depending on the chosen weights, OWA merging operators can be included in a weak form of egalitarianism.

This thesis is structured as follows: In Chapter 2, we will make a survey in the area of belief merging, where we will compare results involving utilitarian and egalitarian merging operators. In Chapter 3, we will consider refinements of the maximum merging operator, which is a kind of egalitarian operator. We will propose some merging operators based on *discrimax* and T-conorms from the Fuzzy Logic literature. In Chapter 4, we will introduce the application of the sufficientarianism in the belief merging context. We will develop two classes of sufficientarian operators: weak and strong sufficientarian operators. In Chapter 5, we will apply OWA operators in propositional belief merging. Finally, in Chapter 6, we will conclude this thesis with our final considerations and future works.

1.6 Publications Related to this Thesis

We published some papers during this doctorate.

- VIANA, H.; ALCÂNTARA, J. Priority-Based Merging Operator Without Distance Measures. In: Multi-Agent Systems - 12th European Conference, EUMAS 2014, Prague, Czech Republic, December 18-19, 2014, Revised Selected Papers. Cham: Springer International Publishing, 2014. p. 398–413.

- VIANA, H.; ALCÂNTARA, J. Propositional Belief Merging with T-conorms. In: Multi-Agent Systems and Agreement Technologies - 14th European Conference, EUMAS 2016, and 4th International Conference, AT 2016, Valencia, Spain, December 15-16, 2016, Revised Selected Papers. Cham: Springer International Publishing, 2016. p. 405–420.
- VIANA, H.; ALCÂNTARA, J. Sufficientarian Propositional Belief Merging. In: Multi-Agent Systems and Agreement Technologies - 14th European Conference, EUMAS 2016, and 4th International Conference, AT 2016, Valencia, Spain, December 15-16, 2016, Revised Selected Papers. Cham: Springer International Publishing, 2016. p. 421–435.
- VIANA, H.; ALCÂNTARA, J. Aggregation with T-Norms and LexiT-Orderings and Their Connections with the Leximin Principle. In: BARRETO, G. A.; COELHO, R. (Ed.). Fuzzy Information Processing. Cham: Springer International Publishing, 2018. p. 179–191. ISBN 978-3-319-95312-0.
- “Propositional Belief Merging with OWA Operators” has been accepted in KR 2018, authors: Henrique Viana and João Alcântara.

2 PROPOSITIONAL BELIEF MERGING

2.1 Contributions of this Chapter

The main contributions of this chapter are listed below:

- We will make a survey about propositional belief merging, including the rationality aspects represented by logical postulates and conditions;
- We will show two forms of defining model-based belief merging: one based on distance measures and another one based on the notion of partial satisfiability. For the second one, we will also include our contribution by creating a different notion of merging, taking into account a priority for the agents based on their formulas;
- We will present a comparison between the utilitarianism and egalitarianism approaches from the distributive justice in the context of propositional belief merging;
- Moreover, we will show the proofs with respect to the rationality of some belief merging operators;
- Parts of this chapter have been published in EUMAS 2014, with the title "Priority-based Merging Operator without Distance Measures", whose authors are Henrique Viana and João Alcântara (VIANA; ALCÂNTARA, 2014).

2.2 Introduction

In many fields of Artificial Intelligence we are often confronted with multiple and conflicting sources of information. Systems organized around reasoning agents face the similar problem of resolving conflicts among contradictory knowledge or beliefs held by different agents. At the same time, one can employ these systems to extract additional knowledge that is not locally held by any agent, but collectively by all of them (LIN; MENDELZON, 1999).

Belief change theory has produced a lot of different operators to model the different ways the beliefs of one or some agents evolve over time. Among these operators, one can quote revision (ALCHOURRON *et al.*, 1985; KATSUNO; MENDELZON, 1991b; GÄRDENFORS, 1992), update (KATSUNO; MENDELZON, 1991a; HERZIG; RIFI, 1998), extrapolation (SAINT-CYR; LANG, 2011) and merging (LIN; MENDELZON, 1999; KONIECZNY; PINO-PÉREZ, 1998; KONIECZNY; PINO-PÉREZ, 1999; KONIECZNY; PINO-PÉREZ, 2002a; KONIECZNY *et al.*, 2004; KONIECZNY; PINO-PÉREZ, 2011).

Belief revision consists in incorporating some new information about a world, while belief update consists in incorporating into a belief base about an old state of the world a notification of some change in the world. We call belief extrapolation the process of completing initial belief sets stemming from observations by assuming minimal change.

Belief merging aims at combining several pieces of information coming from different sources through aggregation operators. The agent faces several conflicting pieces of information coming from several sources of (possibly) equal reliability, and he has to build a coherent description of the world from them.

Merging operators are useful in a lot of applications: to find a coherent information in a distributed database system, to solve conflict between several people or several agents, to find an answer in a decision-making committee, to take decision when information given by some captors is contradictory, etc (KONIECZNY; PINO-PÉREZ, 1998).

The aim of this chapter is to give an account for the main tools developed in last years in the area of belief merging. We will focus on the case where the pieces of information have logical representations, more specifically, using propositional logic. It is important to highlight that merging is a problem occurring in a lot of situations, some of them do not use propositional logic as representation language, but more structured languages. We can mention some extensions of propositional merging operators to some of these frameworks, namely weighted logics (KACI, 2011), first order logic (LIN; MENDELZON, 1996), logic programs (CREIGNOU *et al.*, 2014), constraint networks (KACI, 2011; KACI; TORRE, 2006) and argumentation frameworks (KACI, 2011; AMGOUD; KACI, 2007). Merging is also at work on numerical data (BLOCH *et al.*, 2001; KACI, 2011), but they are not specifically the focus of this thesis.

The chapter is structured as follows: in Section 2.3, we will do a survey about propositional belief merging, ranging from logical properties, examples of operators and comparison between utilitarian and egalitarian merging operators. In Section 2.4, we will consider a different framework for propositional belief merging, based on the notion of partial satisfiability. We will also propose a refinement for this framework as a contribution. In Section 2.5, we will conclude with some considerations about the results showed along the chapter.

2.3 Model-based Merging

There are numerous ways to define belief merging operators:

- Model-based operators (LIN; MENDELZON, 1999; KONIECZNY; PINO-PÉREZ, 1998;

KONIECZNY; PINO-PÉREZ, 1999) select the closest outcomes from the set of sources;

- Formula-based operators (KONIECZNY *et al.*, 2004; BARAL *et al.*, 1990; BARAL *et al.*, 1991; EVERAERE *et al.*, 2007) use a selection function on sets of formulas;
- DA^2 operators (KONIECZNY *et al.*, 2004) generalize model-based operators and allow to take into account inconsistent sources;
- Disjunctive operators (EVERAERE *et al.*, 2010a), that select the result of the merging inside the disjunction of the bases;
- Conflict-based operators (EVERAERE *et al.*, 2008a) use a vector of conflict to represent the conflict instead of the numerical distance of model-based operators;
- Default-based operators (DELGRANDE; SCHAUB, 2004) use renaming of the propositional variables of the language.

In the rest of this work we will be concerned only in the model-based operators, since we consider it the standard approach for belief merging merging. A complete survey in the subject can be found in (KONIECZNY; PINO-PÉREZ, 2011). In the sequel, we will consider some examples of model-based merging operators, their logical properties and their connections with distributive justice, more specifically with utilitarianism and egalitarianism.

2.3.1 Preliminaries

We will consider a propositional language $\mathscr{L}$ over a finite alphabet $\mathscr{P}$ of propositional letters. An outcome ω is a conjunction of propositional letters. The set of all outcomes is denoted by Ω. An outcome ω satisfies a formula φ (i.e., $\omega \models \varphi$) if and only if all interpretations that satisfy ω also satisfy φ (in the usual sense). Let φ be a formula, $mod(\varphi)$ denotes the set of models of φ, i.e., $mod(\varphi) = \{\omega \in \Omega \mid \omega \models \varphi\}$. A formula φ is consistent if and only if $\varphi \not\models \bot$.

A belief base K is simply a propositional formula, representing the beliefs of an agent. Let $K_1, \ldots, K_n$ be n belief bases (not necessarily different). We call belief set the multi-set $E = \{K_1, \ldots, K_n\}$ consisting of those n belief bases and representing n distinct agents. We denote $\bigwedge E$ the conjunction of the belief bases of E, i.e., $\bigwedge E = K_1 \wedge \cdots \wedge K_n$. The union of multi-sets will be denoted by $\sqcup$, e.g., $E = E_1 \sqcup \cdots \sqcup E_m$.

A pre-order $\leq$ over Ω is a reflexive and transitive relation on Ω. A pre-order is total if $\forall \omega_i, \omega_j \in \Omega$, $\omega_i \leq \omega_j$ or $\omega_j \leq \omega_i$. Let $\leq$ be a pre-order over Ω, we define $<$ as follows: $\omega_i < \omega_j$ iff $\omega_i \leq \omega_j$ and $\omega_j \not\leq \omega_i$, and $\approx$ as $\omega_i \approx \omega_j$ iff $\omega_i \leq \omega_j$ and $\omega_j \leq \omega_i$. We say $\omega_i \in min(mod(\varphi), \leq)$ iff $\omega_i \models \varphi$ and $\forall \omega_j \in mod(\varphi)$, $\omega_i \leq \omega_j$.

Let E_1, E_2 be two belief sets. E_1 and E_2 are equivalent, noted $E_1 \equiv E_2$, if and only if there is a bijection f from $E_1 = \{K_{11}, \ldots, K_{n1}\}$ to $E_2 = \{K_{12}, \ldots, K_{n2}\}$ such that $\models f(K_{i1}) \leftrightarrow K_{i2}$, for $1 \leq i \leq n$.

2.3.2 Logical Properties

We employ a logical definition for merging in the presence of Integrity Constraints (IC), i.e., the result of the merging has to obey a set of integrity constraints represented by a formula μ. We will consider merging operators Δ mapping a belief set E and an integrity constraint μ to a set of outcomes that represents the merging of E according to μ.

Definition 2.1 (IC Merging Operators) *(KONIECZNY; PINO-PÉREZ, 1999) Let E, E_1, E_2 be belief sets; K_1, K_2 be consistent belief bases; and μ, μ_1, μ_2 be propositional formulas. Δ_μ is an IC merging operator if and only if it satisfies the following logical postulates:*

(IC0) $\Delta_\mu(E) \models \mu$.

(IC1) *If μ is consistent, then $\Delta_\mu(E)$ is consistent.*

(IC2) *If $\bigwedge E$ is consistent with μ, then $\Delta_\mu(E) \equiv \bigwedge E \wedge \mu$.*

(IC3) *If $E_1 \equiv E_2$ and $\mu_1 \leftrightarrow \mu_2$, then $\Delta_{\mu_1}(E_1) \equiv \Delta_{\mu_2}(E_2)$.*

(IC4) *If $K_1 \models \mu$ and $K_2 \models \mu$, then $\Delta_\mu(\{K_1, K_2\}) \wedge K_1$ is consistent if and only if $\Delta_\mu(\{K_1, K_2\}) \wedge K_2$ is consistent.*

(IC5) $\Delta_\mu(E_1) \wedge \Delta_\mu(E_2) \models \Delta_\mu(E_1 \sqcup E_2)$.

(IC6) *If $\Delta_\mu(E_1) \wedge \Delta_\mu(E_2)$ is consistent, then $\Delta_\mu(E_1 \sqcup E_2) \models \Delta_\mu(E_1) \wedge \Delta_\mu(E_2)$.*

(IC7) $\Delta_{\mu_1}(E) \wedge \mu_2 \models \Delta_{\mu_1 \wedge \mu_2}(E)$.

(IC8) *If $\Delta_{\mu_1}(E) \wedge \mu_2$ is consistent, then $\Delta_{\mu_1 \wedge \mu_2}(E) \models \Delta_{\mu_1}(E)$.*

The meaning of these properties is the following: **(IC0)** ensures the result of merging satisfies the integrity constraint. **(IC1)** states that if the integrity constraint is consistent, then the result of merging will be consistent. **(IC2)** states that if there is no inconsistencies among the belief bases, the result of merging is simply the conjunction of the belief bases with the integrity constraint. **(IC3)** is the principle of irrelevance of syntax: the result of merging has to depend only on the expressed beliefs and not on their syntactical presentation. **(IC4)** is a fairness postulate meaning that the result of merging of two belief bases should not give preference to one of them. It is a condition aiming at ruling out operators that can give priority to one of the bases. **(IC5)** enunciates the following idea: if sets are viewed as expressing the beliefs of the members of a group, then if E_1 (corresponding to a first group) compromises on a set of alternatives which a formula A belongs to, and E_2 (corresponding to a second group) compromises on another set of outcomes which contains A too, then A has to be in the chosen outcomes if we join the two groups. **(IC5)** and **(IC6)** together state that if one could find two subgroups which agree on at least one outcome, then the result of the global merging will be exactly those outcomes the two groups agree on. **(IC7)** and **(IC8)** state that the notion of closeness is well-behaved, i.e., that an outcome that was chosen among all possible outcomes will remain the result of the merging if one restricts the possible choices.

Besides these nine postulates presented above, two main sub-classes of IC merging operators have been defined from two postulates: IC majority operators and IC arbitration operators. An IC majority operator aims at resolving conflicts by adhering to the majority wishes (related to utilitarianism), while IC arbitration operator has a more consensual behavior (related to egalitarianism).

Definition 2.2 (IC Majority Operator) *(KONIECZNY; PINO-PÉREZ, 1999) An IC merging operator is a majority operator if for any belief sets E_1 and E_2 it satisfies*

(Maj) $\exists n \Delta_\mu(E_1 \sqcup \underbrace{E_2 \sqcup \cdots \sqcup E_2}_{n}) \models \Delta_\mu(E_2)$.

This postulate states that if an information has a majority audience, then it will be the choice of the group.

Definition 2.3 (IC Arbitration Operator) *(KONIECZNY; PINO-PÉREZ, 1999) An IC merging operator is an arbitration operator if for any belief bases K_1 and K_2 it satisfies*

(Arb) *If* $\Delta_{\mu_1}(\{K_1\}) \equiv \Delta_{\mu_2}(\{K_2\}), \Delta_{\mu_1 \leftrightarrow \neg\mu_2}(\{K_1, K_2\}) \equiv (\mu_1 \leftrightarrow \neg\mu_2), \mu_1 \not\models \mu_2$, *and* $\mu_2 \not\models \mu_1$, *then* $\Delta_{\mu_1 \vee \mu_2}(\{K_1, K_2\}) \equiv \Delta_{\mu_1}(\{K_1\})$.

Unlike the majority operator, an arbitration operator is intended to satisfy each belief base as possible. According to (KONIECZNY; PINO-PÉREZ, 2002b), this postulate ensures this is the median of possible choices that are preferred. The idea of arbitration can be illustrated in the following scenario:

Example 2.1 *(KONIECZNY; PINO-PÉREZ, 2002a) Tom and David missed the soccer match yesterday between reds and yellows. So they do not know the result of the match. Tom listened in* *the morning that reds made a very good match. So he thinks that a win of reds is more plausible than a draw and that a draw is more reliable than a win of yellows. David was told that after that match yellows have now a lot of chances of winning the championship. From this information he infers that yellows win the match, or at least take a draw. Confronting their point of view, Tom and David agree on the fact that the two teams are of the same strength, and that they had the same chances of winning the match. What arbitration demand is that, with those information, Tom and David have to agree that a draw between the two teams is the more plausible result.*

Intuitively, we can analyze the arbitration as if the outcome ω_1 is preferred to the outcome ω_2 by the constraint c_1 and ω_1 is also more preferred to the outcome ω_3 using the constraint c_2 and ω_2, ω_3 are equally preferred using the constraints c_1 or c_2, then ω_1 is preferred to ω_2 and ω_3 using the constraints c_1 or c_2 (more details in Definition 2.14).

2.3.3 Example of Operators

In this section we give a model-theoretic characterization of merging operators in terms of functions on sets of outcomes. More exactly we show that each merging operator corresponds to a function from multi-sets of sets of formulas to sets of outcomes.

Example 2.2 *(REVESZ, 1993) Let us consider the academic example of a teacher who asks his three students among the languages SQL (denoted by s), O_2 (denoted by o) and Datalog (denoted by d) they would like to learn. The first student wants to learn only SQL or O_2, that is, $K_1 = \{(s \vee o) \wedge \neg d\}$. The second one wants to learn either Datalog or O_2 but not both, i.e., $K_2 = \{(\neg s \wedge d \wedge \neg o) \vee (\neg s \wedge \neg d \wedge o)\}$. For the last, the third one wants to learn the three languages: $K_3 = \{(s \wedge d \wedge o)\}$.*

With respect to Example 2.2, we have three propositional variables: s, d and o. The set of all possible outcomes is $\Omega = \{\omega_1, \ldots, \omega_8\}$, where $\omega_1 = \neg s \neg d \neg o$, $\omega_2 = \neg s \neg d o$, $\omega_3 = \neg s d \neg o$, $\omega_4 = \neg s d o$, $\omega_5 = s \neg d \neg o$, $\omega_6 = s \neg d o$, $\omega_7 = s d \neg o$ and $\omega_8 = s d o$. The outcome ω_1 may be viewed as $\omega_1 = \neg s \neg d \neg o$ or equivalently as $\omega_1 = \neg s \wedge \neg d \wedge \neg o$. The size of the set of outcomes is equal to $2^{|V|}$, in which V is the set of propositional variables.

We will recall now some famous families of operators: *sum*, *max*, *leximax* (or Generalized *max*), *min* and *leximin* (or Generalized *min*). First, it is needed to know the notion of a distance between outcomes.

Definition 2.4 (Distance Between Outcomes) *(KONIECZNY; PINO-PÉREZ, 1999) A distance measure between outcomes is a total function d from* $\Omega \times \Omega$ *to* $\mathbb{N}$ *such that for every* $\omega_i, \omega_j \in \Omega$*,*

- $d(\omega_i, \omega_j) = d(\omega_j, \omega_i)$*, and*
- $d(\omega_i, \omega_j) = 0$ *iff* $\omega_i = \omega_j$*.*

In the first works on model-based merging, the distance used was the Hamming distance between outcomes (DALAL, 1988), but any other distance may be used as well. The Hamming distance between outcomes characterizes the number of propositional variables that they differ. For example, the Hamming distance (denoted by d_H) between $\omega_1 = \neg s \neg d \neg o$ and $\omega_6 = s \neg d o$ is $d_H(\omega_1, \omega_6) = 2$ (i.e., they differ in two propositional variables). Other example of a well-known distance is the drastic distance (KONIECZNY *et al.*, 2004), denoted by d_D, which is defined as: $d_D(\omega_1, \omega_2) = 0$, if $\omega_1 = \omega_2$; $d_D(\omega_1, \omega_2) = 1$, otherwise.

We will also assume from now that our standard distance measure used in the merging will be the Hamming distance. Next, we define the distance between an outcome ω and a belief base K in the following way:

Definition 2.5 (Distance Between ω and K) *(KONIECZNY; PINO-PÉREZ, 1999) Let d be an distance measure. The distance between an outcome ω and a belief base K according to d is the minimum distance between this outcome and the models of the belief base K, i.e.,* $d(\omega, K) = \min_{\omega_i \models K} d(\omega, \omega_i)$*.*

Basically, the distance measure gives the closeness between an outcome and each formula of a belief base.

Example 2.3 *With respect to Example 2.2, the Hamming distance measures between each outcome w.r.t.* K_1, K_2 *and* K_3 *are showed in Table 1. For instance,* $d_H(\omega_1, K_1) = \min_{\omega_i \models K_1} d_H(\omega_1, \omega_i) = min(d_H(\omega_1, \omega_2), d_H(\omega_1, \omega_5), d_H(\omega_1, \omega_6)) = min(1, 1, 2) = 1$.

One simple way of defining the overall distance between an outcome and a belief set is to take the sum of the distances between all the outcomes of Ω and each knowledge base K_i. This will be our first merging operator, the *sum* operator:

Definition 2.6 (*sum* Operator) *(KONIECZNY; PINO-PÉREZ, 1999) Let E be a belief set, d a distance measure and ω an outcome. We define the distance between an outcome and a belief set based on sum as* $d_{sum}(\omega, E) = \sum_{K \in E} d(\omega, K)$*. Then we have the following pre-order:*

Ω	$d_H(\omega, K_1)$	$d_H(\omega, K_2)$	$d_H(\omega, K_3)$
$\omega_1 = \neg s \neg d \neg o$	1	1	3
$\omega_2 = \neg s \neg do$	0	0	2
$\omega_3 = \neg sd \neg o$	2	0	2
$\omega_4 = \neg sdo$	1	1	1
$\omega_5 = s \neg d \neg o$	0	2	2
$\omega_6 = s \neg do$	0	1	1
$\omega_7 = sd \neg o$	1	1	1
$\omega_8 = sdo$	1	2	0

Table 1 – The Hamming distances of K_1, K_2 and K_3.

$\omega_i \leq_E^{d,sum} \omega_j$ iff $d_{sum}(\omega_i, E) \leq d_{sum}(\omega_j, E)$. The operator $\Delta_\mu^{d,sum}$ is defined by $\Delta_\mu^{d,sum}(E) = min(mod(\mu), \leq_E^{d,sum})$.

The $\Delta_\mu^{d,sum}$ operator relies on the definition of the distance between an outcome ω and a belief set as the sum of the distances between ω and the belief bases of the belief set. The result of $\Delta_\mu^{d,sum}$ can be considered as the "election" of the most popular possible choices satisfying the integrity constraints (KONIECZNY; PINO-PÉREZ, 2002a). When two or more outcomes are the choices of the merging, we consider the result as the disjunction of these outcomes.

Example 2.4 *The results of sum merging operator w.r.t. Hamming distance for Example 2.3 are in Table 2. The resulting pre-order $\leq_E^{d_H,sum}$ is $\{\omega_2, \omega_6\} \leq_E^{d_H,sum} \{\omega_4, \omega_7, \omega_8\} \leq_E^{d_H,sum} \{\omega_3, \omega_5\} \leq_E^{d_H,sum} \omega_1$.*

Ω	$d_H(\omega, K_1)$	$d_H(\omega, K_2)$	$d_H(\omega, K_3)$	$d_{H\,sum}(\omega, E)$
$\omega_1 = \neg s \neg d \neg o$	1	1	3	5
$\omega_2 = \neg s \neg do$	0	0	2	**2**
$\omega_3 = \neg sd \neg o$	2	0	2	4
$\omega_4 = \neg sdo$	1	1	1	3
$\omega_5 = s \neg d \neg o$	0	2	2	4
$\omega_6 = s \neg do$	0	1	1	**2**
$\omega_7 = sd \neg o$	1	1	1	3
$\omega_8 = sdo$	1	2	0	3

Table 2 – Hamming distances between Ω and E w.r.t. *sum* operator.

In the example above we have that $\omega_2 \approx_E^{d_H,sum} \omega_6$ (both are the most preferred outcomes), $\omega_4 \approx_E^{d_H,sum} \omega_7 \approx_E^{d_H,sum} \omega_8$ and $\omega_3 \approx_E^{d_H,sum} \omega_5$. Consequently, when $\mu = \top$ (i.e., no constraint is applied), $\Delta_\mu^{d_H,sum}(E) = min(mod(\mu), \leq_E^{d_H,sum}) = \{\omega_2, \omega_6\} = \omega_2 \vee \omega_6 = (\neg s \wedge$

$\neg d \wedge o) \vee (s \wedge \neg d \wedge o)$. If we restrict only one programming language will be taught, i.e., $\mu_1 = (s \wedge \neg d \wedge \neg o) \vee (\neg s \wedge d \wedge \neg o) \vee (\neg s \wedge \neg d \wedge o)$, the result is $\Delta_{\mu_1}^{d_H,sum}(E) = min(mod(\mu_1), \leq_E^{d_H,sum}) = \omega_2 = (\neg s \wedge \neg d \wedge o)$.

Theorem 2.1 *(KONIECZNY; PINO-PÉREZ, 1999) $\Delta_\mu^{d,sum}$ is an IC majority merging operator.*

Theorem 2.1 ensures the merging operator $\Delta_\mu^{d,sum}$ satisfies the postulates **(IC0)-(IC8)** and **(Maj)**. In contrast, $\Delta_\mu^{d,sum}$ violates **(Arb)**.

Definition 2.7 (*max* Operator) *(LIN; MENDELZON, 1996) Let E be a belief set, d a distance measure and ω an outcome. We define the distance between an outcome and a belief set based on max as $d_{max}(\omega, E) = \max_{K \in E} d(\omega, K)$. Then we have the following pre-order: $\omega_i \leq_E^{d,max} \omega_j$ iff $d_{max}(\omega_i, E) \leq d_{max}(\omega_j, E)$. The operator $\Delta_\mu^{d,max}$ is defined by $\Delta_\mu^{d,max}(E) = min(mod(\mu), \leq_E^{d,max})$.*

This operator is very close to the minimax rule used in decision theory (SAVAGE, 1954). The minimax rule has been conceived to minimize the worst cases and similarly the operator $\Delta_\mu^{d,max}$ minimizes the most remote distance. The motivation behind $\Delta_\mu^{d,max}$ is to find the closest possible outcomes to the overall belief set (KONIECZNY; PINO-PÉREZ, 2002a).

Example 2.5 *The results of max merging operator w.r.t. Hamming distance for Example 2.3 are in Table 3. The resulting pre-order $\leq_E^{d_H,max}$ is $\{\omega_4, \omega_6, \omega_7\} \leq_E^{d_H,max} \{\omega_2, \omega_3, \omega_5, \omega_8\} \leq_E^{d_H,max} \omega_1$.*

Ω	$d_H(\omega, K_1)$	$d_H(\omega, K_2)$	$d_H(\omega, K_3)$	$d_{H max}(\omega, E)$
$\omega_1 = \neg s \neg d \neg o$	1	1	3	3
$\omega_2 = \neg s \neg d o$	0	0	2	2
$\omega_3 = \neg s d \neg o$	2	0	2	2
$\omega_4 = \neg s d o$	1	1	1	**1**
$\omega_5 = s \neg d \neg o$	0	2	2	2
$\omega_6 = s \neg d o$	0	1	1	**1**
$\omega_7 = s d \neg o$	1	1	1	**1**
$\omega_8 = s d o$	1	2	0	2

Table 3 – Hamming distances between Ω and E w.r.t. *max* operator.

We have $\Delta_\mu^{d_H,max}(E) = min(mod(\mu), \leq_E^{d_H,max}) = \{\omega_4, \omega_6, \omega_7\} = \omega_4 \vee \omega_6 \vee \omega_7 = (\neg s \wedge d \wedge o) \vee (s \wedge \neg d \wedge o) \vee (s \wedge d \wedge \neg o)$ when $\mu = \top$.

Note $\Delta_\mu^{d,max}$ is not an IC merging operator since it violates **(IC6)** (KONIECZNY; PINO-PÉREZ, 2002a). However, it satisfies a weakened version of it: **(IC6')** If $\Delta_\mu(E_1) \wedge \Delta_\mu(E_2)$

is consistent, then $\Delta_\mu(E_1 \sqcup E_2) \models \Delta_\mu(E_1) \vee \Delta_\mu(E_2)$. A merging operator is called IC quasi-merging operator iff it satisfies **(IC0)-(IC5)**, **(IC6')** and **(IC7)-(IC8)** (KONIECZNY; PINO-PÉREZ, 2002a). Furthermore, $\Delta_\mu^{d,max}$ satisfies **(Arb)** and violates **(Maj)**.

Theorem 2.2 *(KONIECZNY; PINO-PÉREZ, 2002a) $\Delta_\mu^{d,max}$ is an IC arbitration quasi-merging operator.*

In the sequel, we will present the generalization of the *max* operator.

Definition 2.8 (*leximax* Operator) *(KONIECZNY; PINO-PÉREZ, 1999) Let $E = \{K_1,\dots,K_n\}$ be a belief set. For each outcome ω, we build the list $(d_1^\omega,\dots,d_n^\omega)$ of distances between ω and the n belief bases in E, i.e., $d_i^\omega = d(\omega,K_i)$. Let $L_\omega^{d,E}$ be the list obtained from $(d_1^\omega,\dots,d_n^\omega)$ by sorting it in descending order. Let $\leq_{lex}$ be the lexicographical order between sequences of integers, i.e., $(x_1,\dots,x_n) \leq_{lex} (y_1,\dots,y_n)$ if (1) for all i, $x_i \leq y_i$ or (2) there exists i such that $x_i < y_i$ and for all $j < i$, $x_j \leq y_j$. We define the following pre-order: $\omega_i \leq_E^{d,leximax} \omega_j$ iff $L_{\omega_i}^{d,E} \leq_{lex} L_{\omega_j}^{d,E}$. The operator $\Delta_\mu^{d,leximax}$ is defined by $\Delta_\mu^{d,leximax}(E) = min(mod(\mu), \leq_E^{d,leximax})$.*

This operator has been tailored to capture the "*arbitration*" behavior of $\Delta_\mu^{d,max}$, but without losing the properties of an IC merging operator. This idea of using lexicographic operators comes from social choice theory (MOULIN, 1988) and its leximin functions (a generalization of *min* operator).

Example 2.6 *The results of leximax merging operator w.r.t. Hamming distance for Example 2.3 are in Table 4. The resulting pre-order $\leq_E^{d_H,leximax}$ is $\omega_6 \leq_E^{d_H,leximax} \{\omega_4,\omega_7\} \leq_E^{d_H,leximax} \omega_2 \leq_E^{d_H,leximax} \omega_8 \leq_E^{d_H,leximax} \{\omega_3,\omega_5\} \leq_E^{d_H,leximax} \omega_1$.*

Ω	$d_H(\omega,K_1)$	$d_H(\omega,K_2)$	$d_H(\omega,K_3)$	$L_\omega^{d_H,E}$
$\omega_1 = \neg s \neg d \neg o$	1	1	3	$(3,1,1)$
$\omega_2 = \neg s \neg do$	0	0	2	$(2,0,0)$
$\omega_3 = \neg sd \neg o$	2	0	2	$(2,2,0)$
$\omega_4 = \neg sdo$	1	1	1	$(1,1,1)$
$\omega_5 = s \neg d \neg o$	0	2	2	$(2,2,0)$
$\omega_6 = s \neg do$	0	1	1	**(1,1,0)**
$\omega_7 = sd \neg o$	1	1	1	$(1,1,1)$
$\omega_8 = sdo$	1	2	0	$(2,1,0)$

Table 4 – Hamming distances between Ω and E w.r.t. *leximax* operator.

When $\mu = \top$, we obtain $\Delta_\mu^{d_H,leximax}(E) = min(mod(\mu), \leq_E^{d_H,leximax}) = \omega_6 = (s \wedge \neg d \wedge o)$.

Theorem 2.3 *(KONIECZNY; PINO-PÉREZ, 1999) $\Delta_\mu^{d,leximax}$ is an IC arbitration merging operator.*

Note $\Delta_\mu^{d,leximax}$ satisfies **(IC0)-(IC8)** and **(Arb)**. It violates **(Maj)**. We will now introduce the *min* operator:

Definition 2.9 (*min* Operator) *Let E be a belief set, d a distance measure and ω an outcome. We define the distance between an outcome and a belief set based on min as $d_{min}(\omega, E) = \min_{K \in E} d(\omega, K)$. Then we have the following pre-order: $\omega_i \leq_E^{d,min} \omega_j$ iff $d_{min}(\omega_i, E) \leq d_{min}(\omega_j, E)$. The operator $\Delta_\mu^{d,min}$ is defined by $\Delta_\mu^{d,min}(E) = min(mod(\mu), \leq_E^{d,min})$.*

The idea of this operator is very simple: it produces outcomes satisfying at least one belief base of the belief set. It does not take into account how many belief bases are satisfied or the information of all belief bases.

Example 2.7 *The results of min merging operator w.r.t. Hamming distance for Example 2.3 are in the Table 5. The resulting pre-order $\leq_E^{d_H,min}$ is $\{\omega_2, \omega_3, \omega_5, \omega_6, \omega_8\} \leq_E^{d_H,min} \{\omega_1, \omega_4, \omega_7\}$.*

Ω	$d_H(\omega, K_1)$	$d_H(\omega, K_2)$	$d_H(\omega, K_3)$	$d_{H_{min}}(\omega, E)$
$\omega_1 = \neg s \neg d \neg o$	1	1	3	1
$\omega_2 = \neg s \neg d o$	0	0	2	**0**
$\omega_3 = \neg s d \neg o$	2	0	2	**0**
$\omega_4 = \neg s d o$	1	1	1	1
$\omega_5 = s \neg d \neg o$	0	2	2	**0**
$\omega_6 = s \neg d o$	0	1	1	**0**
$\omega_7 = s d \neg o$	1	1	1	1
$\omega_8 = s d o$	1	2	0	**0**

Table 5 – Hamming distances between Ω and E w.r.t. *min* operator.

When $\mu = \top$, we obtain $\Delta_\mu^{d_H,min}(E) = min(mod(\mu), \leq_E^{d_H,min}) = \omega_2 \vee \omega_3 \vee \omega_5 \vee \omega_6 \vee \omega_8 = (\neg s \wedge \neg d \wedge o) \vee (\neg s \wedge d \wedge \neg o) \vee (s \wedge \neg d \wedge \neg o) \vee (s \wedge \neg d \wedge o) \vee (s \wedge d \wedge o)$. We can see that the *min* operator is not a good option to distinguish the outcomes, as it is also reflected in its logical postulates.

Theorem 2.4 $\Delta_{\mu}^{d,min}$ *does not satisfy* ***(IC2)****,* ***(IC6)****,* ***(Maj)*** *and* ***(Arb)****. The other logical postulates are satisfied.*

Proof. See Appendix A.

$\Delta_{\mu}^{d,min}$ is not an IC merging operator since it violates **(IC2)** and **(IC6)**. Roughly speaking, $\Delta_{\mu}^{d,min}$ may not be as good as other operators, since it violates postulates **(IC2)**, **(IC6)**, (**Maj**) and (**Arb**), as also it does not discern well the outcomes. In defense of this operator, it satisfies an additional logical property:

Definition 2.10 (Temperance) *(EVERAERE et al., 2008b) A merging operator $\Delta_{\top}$ satisfies the temperance property iff for every belief set $E = \{K_1, \ldots, K_n\}$*

(Temp) $\Delta_{\top}(\{K_1, \ldots, K_n\}) \wedge K_i \not\models \bot$ *for each K_i.*

This definition shows that the merged base obtained using $\Delta_{\top}$ is consistent with every belief base of the belief set (when there is no integrity constraint involved). This postulate is a particular case of the consensus property **(CSS)** (BENFERHAT *et al.*, 2009), which is a stronger version of (**IC4**):

Definition 2.11 (Consensus) *(BENFERHAT et al., 2009) A merging operator Δ_{μ} satisfies the consensus property iff for every belief set $E = \{K_1, \ldots, K_n\}$*

(CSS) *$\forall K_i \in E$, if $K_i \models \mu$, then $\Delta_{\mu}(E) \wedge K_i \not\models \bot$.*

(CSS) is a stronger fairness postulate meaning the merging of a belief set should not give preference to any of the belief bases.

Theorem 2.5 $\Delta_{\mu}^{d,min}$ *satisfies* ***(Temp)*** *and* ***(CSS)****.*

Proof. See Appendix A.

It worths noting that the temperance property is not satisfied by many merging operators. In particular, none of the IC merging operators satisfies temperance (for example, *max* operator also does not satisfy it).

Proposition 2.1 *(EVERAERE et al., 2008b) There is no merging operator satisfying all of* ***(IC2****),* ***(IC6)****, and* ***(Temp)****.*

The *min* operator can also be generalized as showed previously with the *max* operator:

Definition 2.12 (*leximin* **Operator)** *(EVERAERE et al., 2005) Let $E = \{K_1, \ldots, K_n\}$ be a belief set. For each outcome ω, we build the list $(d_1^{\omega}, \ldots, d_n^{\omega})$ of distances between ω and the n belief bases in E, i.e., $d_i^{\omega} = d(\omega, K_i)$. Let $L_{\omega}^{d,E}$ be the list obtained from $(d_1^{\omega}, \ldots, d_n^{\omega})$ by sorting it in increasing order. Let $\leq_{lex}$ be the lexicographical order between sequences of integers. We define the following pre-order: $\omega_i \leq_E^{d,leximin} \omega_j$ iff $L_{\omega_i}^{d,E} \leq_{lex} L_{\omega_j}^{d,E}$. The operator $\Delta_{\mu}^{d,leximin}$ is defined by $\Delta_{\mu}^{d,leximin}(E) = min(mod(\mu), \leq_E^{d,leximin})$.*

As *leximax* operator, *leximin* is a refinement of *min* operator, since it captures the outcomes which satisfy more belief bases. Originally, it was introduced in (EVERAERE *et al.*, 2005) intended to refine the *k*-quota operators. Briefly, a *k*-quota merging of a belief set E results in models which satisfy at least k bases in E.

Example 2.8 *The results of leximin merging operator w.r.t. Hamming distance for Example 2.3 are in Table 6. The resulting pre-order $\leq_E^{d_H,leximin}$ is $\omega_2 \leq_E^{d_H,leximin} \omega_6 \leq_E^{d_H,leximin} \omega_8 \leq_E^{d_H,leximin} \{\omega_3, \omega_5\} \leq_E^{d_H,leximin} \{\omega_4, \omega_7\} \leq_E^{d_H,leximin} \omega_1$.*

Ω	$d_H(\omega, K_1)$	$d_H(\omega, K_2)$	$d_H(\omega, K_3)$	$L_{\omega}^{d_H,E}$
$\omega_1 = \neg s \neg d \neg o$	1	1	3	$(1,1,3)$
$\omega_2 = \neg s \neg d o$	0	0	2	**(0,0,2)**
$\omega_3 = \neg s d \neg o$	2	0	2	$(0,2,2)$
$\omega_4 = \neg s d o$	1	1	1	$(1,1,1)$
$\omega_5 = s \neg d \neg o$	0	2	2	$(0,2,2)$
$\omega_6 = s \neg d o$	0	1	1	$(0,1,1)$
$\omega_7 = s d \neg o$	1	1	1	$(1,1,1)$
$\omega_8 = s d o$	1	2	0	$(0,1,2)$

Table 6 – Hamming distances between Ω and E w.r.t. *leximin* operator.

When $\mu = \top$, we obtain $\Delta_{\mu}^{d_H,leximin}(E) = min(mod(\mu), \leq_E^{d_H,leximin}) = \omega_2 = (\neg s \wedge \neg d \wedge o)$. The *leximin* operator captures the idea of the priority for the outcomes which satisfy more belief bases. $\Delta_{\mu}^{d_H,leximin}(E) = \omega_2$ since ω_2 satisfies two belief bases. The other outcomes satisfy only one ($\omega_3, \omega_5, \omega_6$ and ω_8) or zero belief base (ω_1, ω_4 and ω_7). When two outcomes satisfy the same number of belief bases, we compare the distance measures of the belief bases (in ascending order) that are not satisfied as a tiebreaker condition.

Theorem 2.6 *(EVERAERE et al., 2005) $\Delta_{\mu}^{d,leximin}$ is an IC majority merging operator.*

Note $\Delta_{\mu}^{d,leximin}$ satisfies **(IC0)-(IC8)** and **(Maj)**. Furthermore, it violates **(Arb)**.

Table 7 summarizes the results involving the postulates satisfied by the operators presented in this section. Recall that all these operators satisfy **(IC0)-(IC1)**, **(IC3)-(IC5)**, **(IC6')** and **(IC7)-(IC8)**.

	(IC2)	**(IC6)**	**(Maj)**	**(Arb)**	**(Temp)/ (CSS)**
Δ_{μ}^{sum}	✓	✓	✓		
Δ_{μ}^{max}	✓			✓	
$\Delta_{\mu}^{leximax}$	✓	✓		✓	
Δ_{μ}^{min}					✓
$\Delta_{\mu}^{leximin}$	✓	✓	✓		

Table 7 – Summary of Logical Properties.

Now that we have a logical definition of IC merging operators, we will exhibit an alternative way to define IC merging operators. More precisely we will show that each IC merging operator corresponds to a family of pre-orders on possible worlds (outcomes). First, we have to introduce the notion of *syncretic assignment* among pre-orders:

Definition 2.13 (Syncretic Assignment) *(KONIECZNY; PINO-PÉREZ, 1999) A syncretic assignment is a function mapping each belief set E to a total pre-order $\leq_E$ over outcomes ($\omega_1, \omega_2 \in \Omega$) such that for any belief sets E, E_1, E_2 and for any belief bases K_1, K_2 the following conditions hold:*

1. If $\omega_1 \models E$ and $\omega_2 \models E$, then $\omega_1 \approx_E \omega_2$.

2. If $\omega_1 \models E$ and $\omega_2 \not\models E$, then $\omega_1 <_E \omega_2$.

3. If $E_1 \equiv E_2$, then $\leq_{E_1} = \leq_{E_2}$.

4. $\forall \omega_1 \models K_1\ \exists \omega_2 \models K_2$ such that $\omega_1 \leq_{K_1 \sqcup K_2} \omega_2$.

5. If $\omega_1 \leq_{E_1} \omega_2$ and $\omega_1 \leq_{E_2} \omega_2$, then $\omega_1 \leq_{E_1 \sqcup E_2} \omega_2$.

6. If $\omega_1 \leq_{E_1} \omega_2$ and $\omega_1 <_{E_2} \omega_2$, then $\omega_1 <_{E_1 \sqcup E_2} \omega_2$.

6'. If $\omega_1 <_{E_1} \omega_2$ and $\omega_1 <_{E_2} \omega_2$, then $\omega_1 <_{E_1 \sqcup E_2} \omega_2$.

The two first conditions ensure that the models of the belief set (if any) are the more plausible outcomes for the pre-order associated to the knowledge set. The third condition states that two equivalent knowledge sets have the same associated pre-orders. The fourth condition states that when merging two belief bases, for each model of the first one, there is a model of the second one that is at least as good as the first one. It ensures that the two knowledge bases are given the same consideration. The fifth condition says that if an outcome ω_1 is at least as plausible as an outcome ω_2 for a belief set E_1 and if ω_1 is at least as plausible as ω_2 for a belief set E_2, then if one joins the two belief sets, then ω_1 will still be at least as plausible as ω_2. The sixth condition strengthens the previous condition by saying that an outcome ω_1 is at least as plausible as an outcome ω_2 for a belief set E_1 and if ω_1 is strictly more plausible than ω_2 for a belief set E_2, then if one joins the two belief sets, then ω_1 will be strictly more plausible than ω_2. The condition 6' is a weakened version of condition 6. We can also define two particular syncretic assignments with additional conditions:

Definition 2.14 (Majority/Fair Syncretic Assignment) *(KONIECZNY; PINO-PÉREZ, 1999) A majority syncretic assignment is an assignment which satisfies the following:*

7. If $\omega_1 \leq_{E_2} \omega_2$, then $\exists n$ $\omega_1 <_{E_1 \sqcup \underbrace{E_2 \sqcup \cdots \sqcup E_2}_{n}} \omega_2$.

A fair syncretic assignment satisfies

8. If $\omega_1 <_{K_1} \omega_2$, $\omega_1 <_{K_2} \omega_3$ and $\omega_2 \approx_{K_1 \sqcup K_2} \omega_3$, then $\omega_1 <_{K_1 \sqcup K_2} \omega_2$.

Condition 7 guarantees that if an outcome ω_1 is stricly more preferred than an outcome ω_2 for a belief set E_2, then there is a quorum n of repetitions of the belief set from which ω_1 will be more preferred than ω_2 for the larger knowledge set $E_1 \sqcup \underbrace{E_2 \sqcup \cdots \sqcup E_2}_{n}$.

Condition 8 states that if an outcome ω_1 is more preferred than an outcome ω_2 for a belief base K_1, if ω_1 is more preferred than ω_3 for an other base K_2, and if ω_2 and ω_3 are equally preferred for the belief set $K_1 \sqcup K_2$, then ω_1 has to be more preferred than ω_2 and ω_3 for $K_1 \sqcup K_2$. Now we can state the following representation theorem for merging operators:

Theorem 2.7 *(KONIECZNY; PINO-PÉREZ, 1999) An operator Δ_μ is an IC merging operator (respectively IC majority merging operator or IC arbitration merging operator) if and only if there exists a syncretic assignment (respectively majority syncretic assignment or fair syncretic assignment) that maps each belief set E to a total pre-order $\leq_E$ such that $\Delta_\mu(E) = min(mod(\mu), \leq_E)$.*

When this equation holds we will say that the assignment represents the operator. This theorem states that the set of postulates **(IC0)-(IC8)**, **(Maj)** and **(Arb)** are equivalent to a syncretic assignment, i.e., the conditions correspond to the logical postulates. Keep in mind that when we talk about logical postulates we are considering a merging operator and when we talk about conditions, we are referring to pre-orders. Not always that it will have a translation between a logical postulate and a condition, but both can be considered when we are working with belief merging operators. This will be common in the next chapters when we will be using both logical postulates and conditions to present the properties of merging operators.

2.3.4 On Egalitarian Propositional Belief Merging

On Distributive Justice, the utilitarian principle asserts that the best social policy is that which provides the greatest total welfare to members of a group, where "total welfare" is measured by summing utility numbers for all individuals (MYERSON, 1981). Utilitarianism is the theory that individuals are best able to define their needs, desires and goals, and where they have freedom to make choices, the result will be the greatest possible satisfaction for the greatest number. Utilitarianism is the doctrine that actions are right if they are useful or for the benefit of a majority. The main postulate of belief merging related to the utilitarianism is **(Maj)** and some examples of operators satisfying it are *sum* and $leximin$.

In contrast to the utilitarianism, the egalitarian principle asserts that the best social policy is that which provides the greatest welfare subject to the constraint that all members should enjoy equal benefits from society. This principle leads to the same social choices as the "*maximin*" principle, which always maximizes the utility of the most unfortunate individuals in the group. So far, the main postulate of belief merging related to the egalitarianism is **(Arb)**, which is intended to find a consensual result between belief bases; some examples of operators satisfying it are *max* and $leximax$.

It is natural to consider egalitarian merging operators when we try to achieve a "fair" result. This could be important to ensure adhesion of agents towards the obtained group goals (SUZUMURA, 2009). As to the belief merging issue, when the aim is to find the correct state of the world, majority methods may seem more appealing.

In (EVERAERE *et al.*, 2014) the study of the egalitarianism was deepened and new egalitarian merging operators were introduced together with the analysis of other fairness conditions than arbitration. The methodology followed consists of investigating two equity conditions from social choice theory: the Hammond equity (HAMMOND, 1976) and Pigou-Dalton property (DALTON, 1920). The merging operators proposed were med^k (k-median), $leximed^k$ (or Generalized k-median) and cumulative *sum* (*csum* for short).

Definition 2.15 (*med^k* **Operator)** *(EVERAERE et al., 2014) Let $E = \{K_1, \ldots, K_n\}$ be a belief set. For each outcome ω we build the list $(d_1^\omega, \ldots, d_n^\omega)$ (sorted in increasing order) of distances between this outcome and the n belief bases in E, i.e., $d_i^\omega = d(\omega, K_i)$. Let $k \in]0,1]$ be a real number; the k-median $med^k(\{d_1^\omega, \ldots, d_n^\omega\}) = d^\omega_{\lceil n*k \rceil}$. Let $d_{med^k}(\omega, E) = med^k(\{d_1^\omega, \ldots, d_n^\omega\})$. Then we have the following pre-order: $\omega_i \leq_E^{d,med^k} \omega_j$ iff $d_{med^k}(\omega_i, E) \leq d_{med^k}(\omega_j, E)$. The operator Δ_μ^{d,med^k} is defined by $\Delta_\mu^{d,med^k}(E) = min(mod(\mu), \leq_E^{d,med^k})$.*

The idea of using the median value was very motivated by trying to be as fair as possible. Instead of focusing on a unique aggregation function, it was defined a full family of k-median aggregation functions. For $k = 0.5$, the usual notion of median is retrieved. For instance, $med^k(d_1, d_2, d_3, d_4, d_5) = med^{0.5}(0,1,3,4,7) = d_{\lceil 5*0.5 \rceil} = d_3 = 3$.

Example 2.9 *The results of $med^{0.5}$ merging operator w.r.t. Hamming distance for Example 2.3 are in Table 8. The resulting pre-order $\leq_E^{d_H,med^{0.5}}$ is $\omega_2 \leq_E^{d_H,med^{0.5}} \{\omega_1, \omega_4, \omega_6, \omega_7, \omega_8\} \leq_E^{d_H,med^{0.5}} \{\omega_3, \omega_5\}$.*

Ω	$d_H(\omega, K_1)$	$d_H(\omega, K_2)$	$d_H(\omega, K_3)$	$d_{H_{med^{0.5}}}(\omega, E)$
$\omega_1 = \neg s \neg d \neg o$	1	1	3	1
$\omega_2 = \neg s \neg d o$	0	0	2	**0**
$\omega_3 = \neg s d \neg o$	2	0	2	2
$\omega_4 = \neg s d o$	1	1	1	1
$\omega_5 = s \neg d \neg o$	0	2	2	2
$\omega_6 = s \neg d o$	0	1	1	1
$\omega_7 = s d \neg o$	1	1	1	1
$\omega_8 = s d o$	1	2	0	1

Table 8 – Hamming distances between Ω and E w.r.t. *$med^{0.5}$* operator.

For instance, when $\mu = \top$, we obtain $\Delta_\mu^{d_H,med^{0.5}}(E) = min(mod(\mu), \leq_E^{d_H,med^{0.5}}) = \omega_2 = (\neg s \wedge \neg d \wedge o)$. For the logical postulates, we have

Theorem 2.8 *(EVERAERE et al., 2014) For any real number $k \in]0,1]$, Δ_μ^{d,med^k} satisfies **(IC0)**, **(IC1)**, **(IC3)**, **(IC4)**, **(IC7)** and **(IC8)**. If $k \geq 0.5$, then Δ_μ^{d,med^k} satisfies **(Arb)**.*

The postulates **(IC2)**, **(IC5)**, **(IC6)** and **(Maj)** are not satisfied in general. Additionally, we have the following result:

Theorem 2.9 *Δ_μ^{d,med^k} does not satisfy **(IC6')**, **(Temp)** and **(CSS)** in general.*

Proof. See Appendix A.

A generalization of k-median operator is defined below.

Definition 2.16 (*$leximed^k$* **Operator)** *(EVERAERE et al., 2014) Let E be a belief set. Suppose $E = \{K_1, \ldots, K_n\}$. For each outcome ω we build the list $L_{\omega}^{d,E} = (d_1^{\omega}, \ldots, d_n^{\omega})$ (sorted in increasing order) of distances between this outcome and the n belief bases in E, i.e., $d_i^{\omega} = d(\omega, K_i)$. Let $k \in]0,1]$ be a real number, the k-median $med^k(\{d_1^{\omega}, \ldots, d_n^{\omega}\}) = d_{\lceil n*k \rceil}^{\omega}$. Let $\leq_{lexmed^k}$ be defined as $L_{\omega_1}^{d,E} \leq_{lexmed^k} L_{\omega_2}^{d,E}$ iff*

- *$med^k(\{d_1^{\omega_1}, \ldots, d_n^{\omega_1}\}) < med^k(\{d_1^{\omega_2}, \ldots, d_n^{\omega_2}\})$ or*
- *$med^k(\{d_1^{\omega_1}, \ldots, d_n^{\omega_1}\}) = med^k(\{d_1^{\omega_2}, \ldots, d_n^{\omega_2}\})$ and $L_{\omega_1}^{d,E} \backslash \{med^k(\{d_1^{\omega_1}, \ldots, d_n^{\omega_1}\})\} \leq_{lexmed^k} L_{\omega_2}^{d,E} \backslash \{med^k(\{d_1^{\omega_2}, \ldots, d_n^{\omega_2}\})\}$.*

Then we have the following pre-order: $\omega_i \leq_E^{d,leximed^k} \omega_j$ iff $L_{\omega_1}^{d,E} \leq_{lexmed^k} L_{\omega_2}^{d,E}$. The operator $\Delta_{\mu}^{d,leximed^k}$ is defined by $\Delta_{\mu}^{d,leximed^k}(E) = min(mod(\mu), \leq_E^{d,leximed^k})$.

Some standard operators are recovered by considering specific values of k: $\Delta_{\mu}^{d,leximed^k}$ with $k \in]0, \frac{1}{n}[$ corresponds to the *leximin* operator $\Delta_{\mu}^{d,leximin}$ (where n is the number of agents), and $\Delta_{\mu}^{d,leximed^1}$ to the *leximax* operator $\Delta_{\mu}^{d,leximax}$.

Example 2.10 *The results of $leximed^{0.5}$ merging operator w.r.t. Hamming distance for Example 2.3 are in Table 9. The resulting pre-order $\leq_E^{d_H,leximed^{0.5}}$ is $\omega_2 \leq_E^{d_H,leximed^{0.5}} \{\omega_6, \omega_8\} \leq_E^{d_H,leximed^{0.5}} \{\omega_4, \omega_7\} \leq_E^{d_H,leximed^{0.5}} \omega_1 \leq_E^{d_H,leximed^{0.5}} \{\omega_3, \omega_5\}$.*

Ω	$d_H(\omega, K_1)$	$d_H(\omega, K_2)$	$d_H(\omega, K_3)$	$L_{\omega}^{d_H,E}$
$\omega_1 = \neg s \neg d \neg o$	1	1	3	$(1,1,3)$
$\omega_2 = \neg s \neg d o$	0	0	2	**(0,0,2)**
$\omega_3 = \neg s d \neg o$	2	0	2	$(0,2,2)$
$\omega_4 = \neg s d o$	1	1	1	$(1,1,1)$
$\omega_5 = s \neg d \neg o$	0	2	2	$(0,2,2)$
$\omega_6 = s \neg d o$	0	1	1	$(0,1,1)$
$\omega_7 = s d \neg o$	1	1	1	$(1,1,1)$
$\omega_8 = s d o$	1	2	0	$(0,1,2)$

Table 9 – Hamming distances between Ω and E w.r.t. $leximed^{0.5}$ operator.

The result of the merging is $\Delta_{\mu}^{d_H,leximed^{0.5}}(E) = min(mod(\mu), \leq_E^{d_H,leximed^{0.5}}) = \omega_2 = (\neg s \wedge \neg d \wedge o)$ when $\mu = \top$.

Theorem 2.10 *(EVERAERE et al., 2014) $\Delta_{\mu}^{d,leximed^k}$ does not satisfy any of **(IC5)**, **(IC6)**, **(Maj)** and **(Arb)** in general. If $k \geq 0.5$, then $\Delta_{\mu}^{d,leximed^k}$ satisfies **(Arb)**.*

Additionally, we have the following result:

Theorem 2.11 *$\Delta_{\mu}^{d,leximed^k}$ does not satisfy **(IC6')**, **(Temp)** and **(CSS)**.*

Proof. See Appendix A.

The next merging operator comes from the notion of Lorenz curve (LORENZ, 1905), a famous representation of the inequalities of a distribution of income. The principle is to focus on the poorest (least satisfied) agents, by looking first as the utility of the poorest one, then at the sum of the utilities of the two poorest ones, etc. Following this, (EVERAERE *et al.*, 2014) was adhered to the notion of cumulative *sum* to define a new family of merging operators. First, the distance measure between a belief base and an outcome has to be translated into a satisfaction value. This process is formalized below:

Definition 2.17 (Cumulative *sum* Operator) *(EVERAERE et al., 2014) Let d be a distance between outcomes and E a belief set. Let $M = max\{d(\omega, \omega') \mid \omega, \omega' \in \Omega\}$. For an outcome ω, we consider the vector $L = (l^{\omega}_{\sigma(1)}, \ldots, l^{\omega}_{\sigma(n)})$ where $l^{\omega}_i = M - d(\omega, K_i)$ is the satisfaction value of belief base i for the outcome ω, and σ is the permutation of $\{1, \ldots, n\}$ sorting the l^{ω}_i in ascending order. Then we define the vector of accumulated satisfaction $W_d(\omega, E) = (W_1, \ldots, W_n)$, where $W_i = \sum_{k=1}^{i} l^{\omega}_{\sigma(k)}$. Let $d_{csum}(\omega, E) = \sum W_d(\omega, E)$ (the sum of its elements). Then we have the following pre-order: $\omega_i \leq_E^{d,csum} \omega_j$ iff $d_{csum}(\omega_i, E) \leq d_{csum}(\omega_j, E)$. The operator $\Delta_{\mu}^{d,csum}$ is defined by $\Delta_{\mu}^{d,csum}(E) = max(mod(\mu), \leq_E^{d,csum})$.*

Example 2.11 *The results of csum merging operator w.r.t. Hamming distance for Example 2.3 are in Table 10. The resulting pre-order $\leq_E^{d_H,csum}$ is $\omega_1 \leq_E^{d_H,csum} \{\omega_3, \omega_5\} \leq_E^{d_H,csum} \{\omega_8\} \leq_E^{d_H,csum} \{\omega_2, \omega_4, \omega_7\} \leq_E^{d_H,csum} \omega_6$.*

The result of the merging is $\Delta_{\mu}^{d_H,csum}(E) = max(mod(\mu), \leq_E^{d_H,csum}) = \omega_6 = (s \wedge \neg d \wedge o)$ when $\mu = \top$. Note that the operator selects as best outcomes those in which the distribution of the satisfaction is closer to the maximum value and that the inequality between agents is not high. In other words, it seeks to choose outcomes where the inequalities of the distribution of utilities are more stable.

Ω	l_1^ω	l_2^ω	l_3^ω	$W_{d_H}(\omega,E)$	$d_{H\,csum}(\omega,E)$
$\omega_1 = \neg s \neg d \neg o$	2	2	0	$(0,2,4)$	6
$\omega_2 = \neg s \neg d o$	3	3	1	$(1,4,7)$	12
$\omega_3 = \neg s d \neg o$	1	3	1	$(1,2,5)$	8
$\omega_4 = \neg s d o$	2	2	2	$(2,4,6)$	12
$\omega_5 = s \neg d \neg o$	3	1	1	$(1,2,5)$	8
$\omega_6 = s \neg d o$	3	2	2	$(2,4,7)$	**13**
$\omega_7 = s d \neg o$	2	2	2	$(2,4,6)$	12
$\omega_8 = s d o$	2	1	3	$(1,3,6)$	10

Table 10 – Hamming distances between Ω and E w.r.t. *csum* operator.

Theorem 2.12 *(EVERAERE et al., 2014) $\Delta_\mu^{d,csum}$ satisfies **(IC0)-(IC4)**, **(IC7)-(IC8)** and **(Maj)**.*

Again, postulates **(IC5)** and **(IC6)** are not satisfied. But differently from *med* and *leximed*, the cumulative *sum* belongs to a general family of belief merging operators, called pre-IC merging operators, obtained by relaxing the two postulates **(IC5)** and **(IC6)** into two natural conditions used in other aggregation theories contexts:

Definition 2.18 (Pre-IC Merging Operator) *(EVERAERE et al., 2014) A merging operator Δ is pre-IC merging operator iff it satisfies **(IC0)-(IC4)**, **(IC7)-(IC8)** and the following properties:*

(IC5b) $\Delta_\mu(K_1) \wedge \cdots \wedge \Delta_\mu(K_n) \models \Delta_\mu(\{K_, \ldots, K_n\})$

(IC6b) *If $\Delta_\mu(K_1) \wedge \cdots \wedge \Delta_\mu(K_n)$ is consistent, then $\Delta_\mu(\{K_, \ldots, K_n\}) \models \Delta_\mu(K_1) \wedge \cdots \wedge \Delta_\mu(K_n)$*

Thus, switching from IC operators to pre-IC ones simply consists in replacing the postulates **(IC5)** and **(IC6)** by the weaker **(IC5b)** and **(IC6b)**. Indeed, it is easy to see that **(IC5b)** (resp. **(IC6b)**) is implied by **(IC5)** (resp. **(IC6)**). As a consequence, we have that every IC merging operator is a pre-IC merging operator. Furthermore, a representation theorem suited to the pre-IC family is given by

Definition 2.19 (Pre-Syncretic Assignment) *(EVERAERE et al., 2014) A pre-syncretic assignment is a function mapping each belief set E to a total pre-order $\leq_E$ over outcomes ($\omega_1, \omega_2 \in \Omega$) such that for any belief sets E, E_1, E_2 and for any belief bases K_1, K_2 the following conditions hold:*

1. If $\omega_1 \models E$ and $\omega_2 \models E$, then $\omega_1 \approx_E \omega_2$.

2. If $\omega_1 \models E$ and $\omega_2 \not\models E$, then $\omega_1 <_E \omega_2$.

3. If $E_1 \equiv E_2$, then $\leq_{E_1} = \leq_{E_2}$.

4. $\forall \omega_1 \models K_1\ \exists \omega_2 \models K_2$ *such that* $\omega_1 \leq_{K_1 \sqcup K_2} \omega_2$.

5b. If $\forall i\ \omega_1 \leq_{K_i} \omega_2$, *then* $\omega_1 \leq_{K_1,\ldots,K_n} \omega_2$.

6b. If $\forall i\ \omega_1 \leq_{K_i} \omega_2$ *and* $\exists k\ \omega_1 <_{K_k} \omega_2$, *then* $\omega_1 <_{K_1,\ldots,K_2} \omega_2$.

Conditions 5b and 6b are taken from Pareto conditions, which are usual in social choice and multicriteria decision making. So they should be considered as minimal aggregation conditions to be satisfied. Besides, conditions 5 and 6 are much more demanding, since they constraint all unions of two belief sets.

Proposition 2.2 *(EVERAERE et al., 2014)* $\Delta_{\mu}^{d,csum}$ *is a pre-IC merging operator.*

Table 11 summarizes the differences of rationality between the three new merging operators showed above.

	(IC0-1)	**(IC2)**	**(IC3-4)**	**(IC5b)**	**(IC6b)**	**(IC7-8)**	**(Maj)**	**(Arb)**
Δ_{μ}^{d,med^k}	✓		✓			✓		$k \geq 0.5$
$\Delta_{\mu}^{d,lexmed^k}$	✓	✓	✓			✓		$k \geq 0.5$
$\Delta_{\mu}^{d,csum}$	✓	✓	✓	✓	✓	✓	✓	

Table 11 – Summary of Logical Properties (2).

The only egalitarian property that has been proposed so far for belief merging is arbitration, represented by **(Arb)** postulate. So a key issue we would like to address is to determine whether other egalitarian properties are possible in the belief merging framework, and, if so, how they relate with arbitration.

We present in the following lines a first alternative coming from social choice theory to characterize egalitarian behavior in belief merging. This condition, proposed by Hammond (HAMMOND, 1976) is known as the Hammond Equity condition (SEN, 1982), and can be expressed as follows:

Definition 2.20 *(SEN, 1982) If agent i is worse off than agent j both in outcomes u and in v, and if i is better off himself in u than in v, while j is better off in v than in u, and if furthermore all others are just as well off in u as in v, then u is socially better than v.*

In the utility theory (BARBERÀ *et al.*, 1998), the Hammond Equity can be defined in the following way: For all distinct agents i, j and let u, v be utility functions defined on $\mathbb{R}$, if

$u(k) = v(k)$ for every $k \neq i, j$ and $v(i) < u(i) \leq u(j) < v(j)$, then $u \prec v$ (u is more preferred than v). It is translated in the belief merging setting as constraints on the total pre-orders associated with the input profiles. These constraints concern profiles of arbitrary size. When distance-based merging operators are considered, this condition is equivalent to

Definition 2.21 (Hammond Equity) *(EVERAERE et al., 2014) Let d be a distance measure. A merging operator op satisfies the Hammond Equity property with respect to d iff for any belief set $E = \{K_1, \ldots, K_n\}$*

(HE) *If $\exists i, j \in \{1, \ldots, n\}$ such that $d(\omega, K_i) < d(\omega', K_i) \leq d(\omega', K_j) < d(\omega, K_j)$ and $\forall l \neq i, j$ $d(\omega, K_l) = d(\omega', K_l)$, then $\omega' <_E^{d,op} \omega$.*

Note that the **(HE)** property is defined as a condition of the syncretic assignment and not as a logical postulate of propositional belief merging.

Theorem 2.13 *(EVERAERE et al., 2014) Let d be any distance and op a merging operator satisfying strict non-decreasingness, i.e., if $x_i > x_i'$, then $op(x_1, \ldots, x_i, \ldots, x_m) > op(x_1, \ldots, x_i', \ldots, x_m)$. The IC merging operator $\Delta_\mu^{d,op}$ satisfies condition **(HE)** if and only if $op = leximax$.*

The condition of strict non-decreasingness is quite natural and not very demanding. Actually all the aggregation functions giving rise to IC merging operators we are aware of satisfy non-decreasingness. Given this theorem, defining other egalitarian distance-based merging operators requires to focus on other equity principles, or to weaken some IC postulates. We explore both ways in the sequel, but first we present a counterpoint of **(HE)**, which is called Hammond Inequity (BARBERÀ *et al.*, 1998) and it is satisfied by *leximin* function. When distance-based merging operators are considered, this condition is equivalent to

Definition 2.22 (Hammond Inequity) *Let d be a distance measure. A merging operator op satisfies the Hammond Inequity property with respect to d iff for any belief set $E = \{K_1, \ldots, K_n\}$*

(HI) *If $\exists i, j \in \{1, \ldots, n\}$ such that $d(\omega, K_i) < d(\omega', K_i) \leq d(\omega', K_j) < d(\omega, K_j)$ and $\forall l \neq i, j$ $d(\omega, K_l) = d(\omega', K_l)$, then $\omega <_E^{d,op} \omega'$.*

Theorem 2.14 $\Delta_\mu^{d,leximin}$ *satisfies **(HI)**.*

Proof. See Appendix A.

Now, we focus on another prioritarian condition from social choice, namely Pigou-Dalton transfer principle (DALTON, 1920). The idea underlying it is that every transfer from the most satisfied agent to the least satisfied one decreases the inequalities:

Definition 2.23 (Pigou-Dalton Transfer Principle) *(EVERAERE et al., 2014) Let d be a distance measure. An operator op satisfies the Pigou-Dalton transfer principle with respect to d iff for any belief set $E = \{K_1, \ldots, K_n\}$*

(PD) *If $\exists i, j \in \{1, \ldots, n\}$ such that $d(\omega, K_i) < d(\omega', K_i) \leq d(\omega', K_j) < d(\omega, K_j)$, $d(\omega', K_i) - d(\omega, K_i) = d(\omega, K_j) - d(\omega', K_j)$ and $\forall l \neq i, j$ $d(\omega, K_l) = d(\omega', K_l)$, then $\omega' <_E^{d,op} \omega$ (ω' is more preferred than ω).*

The axiom **(PD)** is a restriction of **(HE)**, where the difference between the distances has the same value. Consequently, we have that **(HE)** implies **(PD)**. The converse is not generally true. Pigou-Dalton introduces a very weak kind of distribution sensitivity (VALLENTYNE, 2010; ARROW *et al.*, 2002). It says that a transfer from a relatively better-off agent to a relatively worse-off, without reversing their ranking, is weakly improving. For example, it says that transfer $d(\omega', K_1) = d(\omega', K_2) = 2$ is better than transfer $d(\omega, K_1) = 0$ and $d(\omega, K_2) = 4$.

Theorem 2.15 *(EVERAERE et al., 2014) $\Delta_\mu^{d,leximax}$ and $\Delta_\mu^{d,csum}$ satisfy **(PD)**.*

Similar to the Pigou-Dalton principle, we can name another property, called Incremental Equity (ARROW *et al.*, 2002). Considering the distance-based merging operators, this property is defined as

Definition 2.24 (Incremental Equity) *Let d be a distance measure. An operator op satisfies the Incremental Equity condition iff for any belief set $E = \{K_1, \ldots, K_n\}$*

(IE) *If $\exists i, j \in \{1, \ldots, n\}$ such that $d(\omega, K_i) < d(\omega', K_i) \leq d(\omega', K_j) < d(\omega, K_j)$, $d(\omega', K_i) - d(\omega, K_i) = d(\omega, K_j) - d(\omega', K_j)$ and $\forall l \neq i, j$ $d(\omega, K_l) = d(\omega', K_l)$, then $\omega' \approx_E^{d,op} \omega$.*

Here, we observe a widening in the result of the implication, in comparison with the Pigou-Dalton principle; indeed this axiom is not interested in the values of transfer of the utilities between agents. As a result, we have that

Theorem 2.16 $\Delta_{\mu}^{d,sum}$ *satisfies **(IE)**.*

Proof. See Appendix A.

For instance, if we consider $d(\omega', K_1) = d(\omega', K_2) = 2$, $d(\omega, K_1) = 0$ and $d(\omega, K_2) = 4$, then $d_{sum}(\omega', \{K_1, K_2\}) = d_{sum}(\omega, \{K_1, K_2\}) = 4$. That is, only the result of the sums of utilities are considered, instead of the transfer of utilities between agents. Table 12 summarizes the satisfaction of these four logical properties with respect to all the merging operators defined until now.

	$\Delta_{\mu}^{d,sum}$	$\Delta_{\mu}^{d,max}$	$\Delta_{\mu}^{d,leximax}$	$\Delta_{\mu}^{d,min}$	$\Delta_{\mu}^{d,leximin}$	Δ_{μ}^{d,med^k}	$\Delta_{\mu}^{d,leximed^k}$	$\Delta_{\mu}^{d,csum}$
(HE)			✓					
(HI)					✓			
(PD)			✓					✓
(IE)	✓							

Table 12 – Summary of Logical Properties (3).

So far, we have seen that the number of merging operators satisfying these egalitarian conditions is quite low. In this thesis, we will be concerned with other different classes of operators that bring this egalitarian "flavor" to the propositional belief merging. We believe that the first walk to be made in this direction is to work with the variations of the *max* and *leximax* operators, which will be done in the next chapter.

To finish our overview over the propositional belief merging, we will show an alternative to distance-based merging, where the distance notion is dropped and a satisfaction measure is used instead. This approach is important since it brings a different vision from merging by applying a notion of priority among agents. It also provides some of the initial results obtained during this thesis.

2.4 Model-Based Merging without Distance Measures

2.4.1 PS-Merge

As showed in the previous section, several merging operators have been defined and characterized. The model-based merging operators obtain a belief base from a set of outcomes with the support of a distance measure on outcomes and an aggregation function (merging operator).

The major problem with distance-based merging operators is that evaluating the closeness between outcomes as a number may lead to lose too much information (EVERAERE *et al.*, 2008b). For example, the widely used Hamming distance (DALAL, 1988) assumes not only that propositional symbols are equally relevant to determine a distance between outcomes, but also that they are independent from each other and that nothing else is relevant. These assumptions are restrictive and give the Hamming distance very little flexibility (LAFAGE; LANG, 2001).

To overcome this issue, some characterizations of model-based merging operators were achieved by modifying the distance measure (EVERAERE *et al.*, 2008a; EVERAERE *et al.*, 2008b; KONIECZNY *et al.*, 2004; LAFAGE; LANG, 2001). In addition, merging operators without distance measures were also conceived. An alternative method of merging was proposed in (POZOS-PARRA; MACÍAS, 2007; MACÍAS; POZOS-PARRA, 2009; POZOS-PARRA *et al.*, 2011), which uses the notion of Partial Satisfiability instead of a distance measure, to define PS-Merge, a model-based merging operator which depends on the syntax of the belief bases (MACÍAS; POZOS-PARRA, 2007). It is not based on distance measures of models and we can say that it can provide more refined results than the usual distance measures since they are too loose to distinguish the closeness between outcomes. In this section we will review some notions and definitions about PS-Merge, changing a little the notation in order to fit it to this thesis.

Recall that a belief set $E = \{K_1, \ldots, K_n\}$ represents sets of belief bases K_i, and a belief base K_i is a finite and consistent set of propositional formulas. In this approach, each belief base K_i is restricted to a DNF (Disjunctive Normal Form) formula, i.e., it can be viewed as $K_i = (c_1 \vee \cdots \vee c_m)$ and $c_l = (x_1 \wedge \cdots \wedge x_k)$, where $x_1, \ldots, x_k$ are literals, i.e., a propositional variable or its negation.

Example 2.12 (Revisiting Example 2.2) *Let us consider the academic example of a teacher who asks his three students which among the languages SQL (denoted by s), O_2 (denoted by o) and Datalog (denoted by d) they would like to learn. The first student wants to learn only SQL or O_2, that is, $K_1 = (s \vee o) \wedge \neg d$. The second one wants to learn either Datalog or O_2 but not both, i.e., $K_2 = (\neg s \wedge d \wedge \neg o) \vee (\neg s \wedge \neg d \wedge o)$. For the last, the third one wants to learn the three languages: $K_3 = (s \wedge d \wedge o)$. First of all, we need to convert these preferences to the DNF format. We shall have $K_1 = (s \wedge \neg d) \vee (o \wedge \neg d)$, $K_2 = (\neg s \wedge d \wedge \neg o) \vee (\neg s \wedge \neg d \wedge o)$ and $K_3 = (s \wedge d \wedge o)$.*

The definition of outcome is as usual a function $\omega : \mathscr{P} \to \{0, 1\}$. For instance, when $\omega(s) = 1$, we say that the interpretation of the propositional variable s is true, whereas when

$\omega(s) = 0$, we say that its interpretation is false. We have that $\omega(s) = 1 \Leftrightarrow \omega(\neg s) = 0$. Remember that the outcome $\omega_1 = \neg s \neg d \neg o$ may be viewed as $\omega_1(\neg s) = 1, \omega_1(\neg d) = 1$ and $\omega_1(\neg o) = 1$.

Definition 2.25 (Partial Satisfiability) *(POZOS-PARRA et al., 2011) Let $K = \{c_1 \vee \cdots \vee c_m\}$ be a belief base in DNF. The partial satisfiability of the outcome ω w.r.t. K is given by $\omega(K) = max\{\omega(c_1), \ldots, \omega(c_m)\}$, where for each $c_i = (x_1 \wedge \cdots \wedge x_k)$, $1 \leq i \leq k$: $\omega(c_i) = \sum_{l=1}^{k} \left\{ \frac{\omega(x_l)}{k} \right\}$.*

Note that we are abusing the notation w.r.t. partial satisfiability function ω; it can be used to express the partial satisfiability of a belief base (K), of a clause (c_i) and of literals (x_l). The partial satisfiability of an outcome in a clause indicates the rate of the occurrences of its literals in the DNF formula. It is a little different from a distance measure, which measures the closeness of the outcome to its satisfaction: the lower this value, closer it is to its satisfaction.

Example 2.13 *From Example 2.12, we have $K_1 = \{(s \wedge \neg d) \vee (o \wedge \neg d)\}$, $K_2 = \{(\neg s \wedge d \wedge \neg o) \vee (\neg s \wedge \neg d \wedge o)\}$ and $K_3 = \{(s \wedge d \wedge o)\}$. The partial satisfiability of each outcome w.r.t. K_1, K_2 and K_3 is computed as*

Ω	$\omega(K_1)$	$\omega(K_2)$	$\omega(K_3)$
$\omega_1 = \neg s \neg d \neg o$	1/2	2/3	0
$\omega_2 = \neg s \neg d o$	1	1	1/3
$\omega_3 = \neg s d \neg o$	0	1	1/3
$\omega_4 = \neg s d o$	1/2	2/3	2/3
$\omega_5 = s \neg d \neg o$	1	1/3	1/3
$\omega_6 = s \neg d o$	1	2/3	2/3
$\omega_7 = s d \neg o$	1/2	2/3	2/3
$\omega_8 = s d o$	1/2	1/3	1

Table 13 – The partial satisfiability of K_1, K_2 and K_3.

For example, $\omega_1(K_1) = max\{\omega_1(c_1), \omega_1(c_2)\} = max\left\{\frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2}$.

Definition 2.26 (*sum* Operator) *(MACÍAS; POZOS-PARRA, 2009) Let $E = \{K_1, \ldots, K_n\}$ be a belief set and ω be an outcome. The partial satisfiability of ω w.r.t. E and sum is given by $\omega_{sum}(E) = \sum_{i=1}^{n} \{\omega(K_i)\}$. The binary relation $\leq_E^{ps,sum}$ is defined as $\omega \leq_E^{ps,sum} \omega'$ if and only if $\omega_{sum}(E) \leq \omega'_{sum}(E)$.*

Here, an outcome ω' is preferred to ω if the partial satisfiability of ω' is greater or equal to the partial satisfiability of ω.

Example 2.14 *After computing the partial satisfiability of the group of agents in Example 2.13, we can rank the outcome as* $\omega_1 \leq_E^{ps,sum} \omega_3 \leq_E^{ps,sum} \omega_5 \leq_E^{ps,sum} \{\omega_4, \omega_7, \omega_8\} \leq_E^{ps,sum} \{\omega_2, \omega_6\}$.

The results of each outcome w.r.t *sum* can be seen in the second column of Table 14. We can define this process as a merging operator in the following model-theoretical way:

Ω	$\omega_{sum}(E)$	$d_{H_{sum}}(\omega, E)$	$\omega_{min}(E)$	$d_{H_{max}}(\omega, E)$
$\omega_1 = \neg s \neg d \neg o$	1.166	5	0	3
$\omega_2 = \neg s \neg do$	**2.333**	**2**	0.333	2
$\omega_3 = \neg sd \neg o$	1.333	4	0	2
$\omega_4 = \neg sdo$	1.833	3	0.5	**1**
$\omega_5 = s \neg d \neg o$	1.666	4	0.333	2
$\omega_6 = s \neg do$	**2.333**	**2**	**0.666**	**1**
$\omega_7 = sd \neg o$	1.833	3	0.5	**1**
$\omega_8 = sdo$	1.833	3	0.333	2

Table 14 – The partial satisfiability w.r.t. *sum* and *min* operators.

Definition 2.27 (PS-Merge (*sum*)) *(POZOS-PARRA et al., 2011) Let* $E = \{K_1, \ldots, K_n\}$ *be a belief set and* μ *be an integrity constraint. The merging operator* $\Delta_\mu^{ps,sum}(E)$ *is defined as* $\Delta_\mu^{ps,sum}(E) = max(mod(\mu), \leq_E^{ps,sum})$.

Example 2.15 *The merging for the previous example when* $\mu = \top$ *results in* $mod(\Delta_\mu^{ps,sum}(E)) = \omega_2 \vee \omega_6 = (\neg s \wedge \neg d \wedge o) \vee (s \wedge \neg d \wedge o)$.

Another example of merging operator is the *min* function, which is a counterpart of the *max* function presented in the distance-based merging.

Definition 2.28 (*min* Operator) *(MACÍAS; POZOS-PARRA, 2009) Let* $E = \{K_1, \ldots, K_n\}$ *be a belief set and* ω *be an outcome. The partial satisfiability of* ω *w.r.t.* E *and min is given by* $\omega_{min}(E) = min(\omega(K_1), \ldots, \omega(K_n))$. *The binary relation* $\leq_E^{ps,min}$ *is defined as* $\omega \leq_E^{ps,min} \omega'$ *if and only if* $\omega_{min}(E) \leq \omega'_{min}(E)$.

Example 2.16 *After computing the partial satisfiability of the group of agents in Example 2.13, we can rank the outcomes as* $\{\omega_1, \omega_3\} \leq_E^{ps,min} \{\omega_2, \omega_5, \omega_8\} \leq_E^{ps,min} \{\omega_4, \omega_7\} \leq_E^{ps,min} \omega_6$.

The definition of the merging process in a model-theoretical way is

Definition 2.29 (PS-Merge (*min*)) *(MACÍAS; POZOS-PARRA, 2009) Let* $E = \{K_1, \ldots, K_n\}$ *be a belief set and* μ *be an integrity constraint. The merging operator* $\Delta_{\mu}^{ps,min}(E)$ *is defined as* $\Delta_{\mu}^{ps,min}(E) = max(mod(\mu), \leq_E^{ps,min})$.

PS-Merge differs from the distance-based merging (see Table 14) owing to the sensibility of the syntax of this approach. We can see a different result in the comparison with the *sum* operator in the outcomes ω_3 and ω_5. When the min/max operators are considered, this difference between the approaches is even wider.

The main question around this approach is what we gain when resorting to partial satisfiability instead of distance measures in the process of belief merging. In terms of expressivity, we can see that the results can be more disparate, since the range of possible values may be larger than the possible values of the distance measures seen before. The partial satisfiability is sensible to syntax, and the variation of the syntax is the feature that provides a larger range of values. Unfortunately, this will reflect in how much rational will be the merging operators. The way that the partial satisfiability is defined will imply in the loss of some logical properties.

Theorem 2.17 *(POZOS-PARRA; MACÍAS, 2007) Let* $op \in \{sum, min\}$ *be a merging operator.* $\Delta_{\mu}^{ps,op}$ *does not satisfy* ***(IC3)*** *and* ***(IC4)****. Furthermore,* $\Delta_{\mu}^{ps,min}$ *does not satisfy* ***(IC6)****.*

The postulate **(IC3)** is the principle of irrelevance of syntax: it says that the result of merging has to depend only on the expressed opinions (their semantics) and not on their syntactical presentation. The partial satisfiability is designed in a way that equivalent formulas can receive different evaluations. For instance, for the formula $(a \wedge b)$, both propositional variables a and b have the same value, which is $\frac{1}{2}$. The formula $(a \wedge b \wedge c) \vee (a \wedge b \wedge \neg c)$ is equivalent to the former, but the evaluation for each propositional variable will be different: their partial satisfiability value will be $\frac{1}{3}$.

The postulate **(IC4)** is the fairness condition, which means that the result of merging of two belief bases should not give preference to one of them. By the definition of partial satisfiability, the formulas with only disjunctions tends to have a higher partial satisfiability value when compared to formulas with only conjunctions. For instance, if a belief base K_1 has the formula $(a \vee b)$ and a belief base K_2 has the formula $(\neg a \wedge \neg b)$, the result of the merging tends to give more preference to K_1. But it is possible to find a condition where the postulates **(IC3)** and **(IC4)** are satisfied. We introduce in the sequel the notion of normalization of a belief set:

Definition 2.30 (Normalization of a Belief Set) *Let $E = \{K_1, \ldots, K_n\}$ be a belief set, where each K_i is a formula in DNF format and V is the number of different propositional variables in E. We define an equivalent belief set $E' = \{K'_1, \ldots, K'_n\}$, where for each clause in K_i with less than V propositional variables, we can construct an equivalent K'_i, where each clause of K'_i has exactly V literals.*

Example 2.17 *The normalization $E' = \{K'_1, K'_2, K'_3\}$ of the belief set $E = \{K_1, K_2, K_3\}$ in Example 2.12, where $K_1 = (s \wedge \neg d) \vee (o \wedge \neg d)$, $K_2 = (\neg s \wedge d \wedge \neg o) \vee (\neg s \wedge \neg d \wedge o)$ and $K_3 = (s \wedge d \wedge o)$, is equal to $K'_1 = (s \wedge \neg d \wedge o) \vee (s \wedge \neg d \wedge \neg o) \vee (\neg s \wedge \neg d \wedge o)$, $K'_2 = K_2$ and $K'_3 = K_3$.*

Hence, we have the following result:

Theorem 2.18 *Let E be a belief set and E' the normalized version of E. We have that $\Delta_\mu^{ps,sum}(E') \equiv \Delta_\mu^{d_H,sum}(E)$ and $\Delta_\mu^{ps,min}(E') \equiv \Delta_\mu^{d_H,max}(E)$.*

Proof. See Appendix A.

This result can be extended to others merging operators; for instance, $\Delta_\mu^{ps,max}(E') \equiv \Delta_\mu^{d_H,min}(E)$, $\Delta_\mu^{ps,leximax}(E') \equiv \Delta_\mu^{d_H,leximin}(E)$, etc. Based on Definition 2.17 of cumulative *sum* operator, we can define other notion of merging based on satisfaction values, which has a close similarity with PS-Merge.

Definition 2.31 (Satisfaction Merging) *Let d be a distance between outcomes, op a merging operator and E be a belief set. Let $M = max(\{d(\omega, \omega') \mid \omega, \omega' \in \Omega\})$. For an outcome ω, we consider $l_i^{d,\omega} = M - d(\omega, K_i)$ as the satisfaction value of belief base i for the outcome ω. Then we have the following pre-order: $\omega_i \leq_E^{l,d,op} \omega_j$ iff $op(l_1^{d,\omega_i}, \ldots, l_n^{d,\omega_i}) \leq op(l_1^{d,\omega_j}, \ldots, l_n^{d,\omega_j})$. The operator $\Delta_\mu^{l,d,op}$ is defined by $\Delta_\mu^{l,d,op}(E) = max(mod(\mu), \leq_E^{l,d,op})$.*

Theorem 2.19 *Let E be a belief set and E' the normalized version of E. We have that $\Delta_\mu^{ps,op}(E') \equiv \Delta_\mu^{l,d_H,op}(E)$.*

Proof. See Appendix A.

We can see with these results PS-Merge has a strong connection with the Hamming distance. However, PS-Merge definition can be modified to match other distances measures. For instance, we can consider a drastic version of PS-Merge.

Definition 2.32 (Drastic Partial Satisfiability) *Let $K = \{c_1 \vee \cdots \vee c_m\}$ be a belief base. The drastic partial satisfiability of the outcome ω w.r.t. K is given by:*

$$\omega^D(K) = \begin{cases} 1, & \text{if } max\{\omega(c_1), \ldots, \omega(c_m)\} = 1 \\ 0, & \text{otherwise} \end{cases}$$

where for each $c_i = (x_1 \wedge \cdots \wedge x_k)$, $1 \leq i \leq k$: $\omega(c_i) = \sum_{l=1}^{k} \left\{ \frac{\omega(x_l)}{k} \right\}$.

Definition 2.33 (DPS-Merge) *Let $E = \{K_1, \ldots, K_n\}$ be a belief set and μ be an integrity constraint. The merging operator $\Delta_{\mu}^{dps,op}(E)$ is defined as $\Delta_{\mu}^{dps,op}(E) = max(mod(\mu), \leq_E^{dps,op})$, where the binary relation $\leq_E^{dps,op}$ is defined as $\omega \leq_E^{dps,op} \omega'$ if and only if $\omega_{op}^D(E) \leq \omega_{op}'^D(E)$ and $\omega_{op}^D(E) = op(\omega^D(K_1), \ldots, \omega^D(K_n))$..*

Theorem 2.20 *Let E be a belief set and E' the normalized version of E. We have that $\Delta_{\mu}^{dps,sum}(E') \equiv \Delta_{\mu}^{d_D,sum}(E) \equiv \Delta_{\mu}^{l,d_D,sum}(E)$ and $\Delta_{\mu}^{dps,min}(E') \equiv \Delta_{\mu}^{d_D,max}(E) \equiv \Delta_{\mu}^{l,d_D,min}(E)$.*

Proof. See Appendix A.

Indeed, Theorem 2.18 can be extended to others merging operators. In short, these theorems above show that in some conditions the partial satisfiability is equivalent to the distance-based merging.

2.4.2 Pr-Merge

In this subsection, we will consider mainly the problem of belief merging without distance measures, by refining the definition of PS-Merge through the weighting of the information in the belief bases. The resulting priority-based merging operator is dubbed Pr-Merge. Basically, the idea of priority consists in ranking the importance of each outcome in terms of the belief of each agent. In our work, we will measure the importance of an outcome by considering the number of propositions' appearance in the agents' belief bases.

Example 2.18 *The application of this merging is relevant in the following scenario: suppose that three friends are going to share a meal in a restaurant, which is constituted of a main dish and a drink. One person is very restrictive with relation to his/her beliefs, e.g., he/she prefers vegetarian food, while the others two have more choices to make than the first one, since they are non-vegetarian and there is a greater diversity of choices to make for both, and these possible options are considered equally satisfactory for them. Since the choices are more restricted and* *objective for the first person, it is natural that we need to give more priority to his/her desires, but without forgetting completely the desires of the other two people.*

The merging operator introduced in what follows will comprise this aspect: it will give more priority to agents expressing their beliefs in a simplified, objective or restricted way. On the other hand, it is extremely plausible to think of contexts where we should give more priority to agents expressing more beliefs (this view can be achieved later by changing a definition in the merging operator). The details about this approach will be explained during this section. Before, we will present some preliminary notions and the definition of the Pr-Merge. As well as with PS-Merge, here a belief base K_i is a finite and consistent set of propositional formulas in DNF.

Example 2.19 *Let us recall Example 2.2, in which a teacher asks his three students which among the languages SQL (denoted by s), O_2 (denoted by o) and Datalog (denoted by d) they would like to learn. The first student wants to learn only SQL or O_2, that is, $K_1 = (s \vee o) \wedge \neg d$. The second wants to learn either Datalog or O_2 but not both, i.e., $K_2 = (\neg s \wedge d \wedge \neg o) \vee (\neg s \wedge \neg d \wedge o)$. For the last, the third one wants to learn the three languages: $K_3 = (s \wedge d \wedge o)$.*

First of all, we need to convert these belief bases to the DNF format. We shall have $K_1 = (s \wedge \neg d) \vee (o \wedge \neg d)$, and consequently, $K_1 = \{c_1, c_2\}$, where $c_1 = (s \wedge \neg d)$ and $c_2 = (o \wedge \neg d)$. For the belief bases K_2 and K_3, we shall have $K_2 = \{c_3, c_4\}$ and $K_3 = \{c_5\}$, where $c_3 = (\neg s \wedge d \wedge \neg o)$, $c_4 = (\neg s \wedge \neg d \wedge o)$ and $c_5 = (s \wedge d \wedge o)$. Note the third agent has only one preferable choice $(s \wedge d \wedge o)$, while the first and second ones have both two preferable choices (for K_1, it is $(s \wedge \neg d)$ or $(o \wedge \neg d)$, and for K_2, it is $(\neg s \wedge d \wedge \neg o)$ or $(\neg s \wedge \neg d \wedge o)$). We can say that K_3 is more certain/restricted about his/her beliefs.

We can now begin with the notion of preference priority. In order to do this, we will work in two levels: the partial satisfiability of a specific agent ($K_i \in E$) and the preference priorities of a group of agents E (based on the partial satisfiability of each agent). These definitions are inspired by the works on the PS-Merge operator (MACÍAS; POZOS-PARRA, 2009; POZOS-PARRA; MACÍAS, 2007; POZOS-PARRA *et al.*, 2011).

To define the preference priority in our framework, we will assume that each clause of a belief base shares the same weight in the belief evaluation. For example, the formula $(s \wedge d \wedge o)$ of the belief base K_3 will have a priority weight 1 (because there is only one clause in the belief base), while the clauses $(s \wedge \neg d)$ and $(o \wedge \neg d)$ of the belief base K_1 will have both the priority weight $\frac{1}{2}$ (the sum of weights needs to be equal to 1). Formally, we will define this idea in two different ways:

Definition 2.34 (Preference Priority) *Let $E = \{K_1, \ldots, K_n\}$ be a belief set and ω an outcome. The preference priority of ω w.r.t. E is given by $\omega^+(E) = \sum_{i=1}^{n} \delta_i \times \omega(K_i)$, where $\delta_i = \frac{1}{a_i}$ and a_i is the number of clauses in the belief base K_i.*

This step reflects the preference priority of the group of agents, which will be a prioritized sum of the partial satisfiability of each individual belief base of the group. Intuitively, the higher is the number of choices made by an agent, the lower will be his/her preference priority among the group of agents.

For the sake of information, if we consider to prioritize agents that are expressing more choices, we must make a little change in the definitions above. In this case, we shall have $\omega^+(E) = \sum_{i=1}^{n} a_i \times \omega(K_i)$. We want to highlight that, although these two approaches express different ideas, they share similar properties.

Example 2.20 *The preference priority of the belief set E of Example 2.2 is shown in Table 15.*

Ω	$\omega^+(E)$
$\omega_1 = \neg s \neg d \neg o$	$1/4 + 1/3 + 0 = 7/12 \simeq$ **0.583**
$\omega_2 = \neg s \neg d o$	$1/2 + 1/2 + 1/3 = 4/3 \simeq$ **1.333**
$\omega_3 = \neg s d \neg o$	$0 + 1/2 + 1/3 = 5/6 \simeq$ **0.833**
$\omega_4 = \neg s d o$	$1/4 + 1/3 + 2/3 = 5/4 =$ **1.25**
$\omega_5 = s \neg d \neg o$	$1/2 + 1/6 + 1/3 = 6/6 =$ **1**
$\omega_6 = s \neg d o$	$1/2 + 1/3 + 2/3 = 3/2 =$ **1.5**
$\omega_7 = s d \neg o$	$1/4 + 1/3 + 2/3 = 5/4 =$ **1.25**
$\omega_8 = s d o$	$1/4 + 1/6 + 1 = 17/12 \simeq$ **1.416**

Table 15 – Preference Priority of E.

After computing the preference priorities, we can rank the outcomes and decide which one is the best option for the group:

Definition 2.35 *The binary relation $\leq_E^{pr}$ is defined as $\omega \leq_E^{pr} \omega'$ if and only if $\omega^+(E) \leq \omega'^+(E)$.*

Here, an outcome ω' is preferred to ω if the preference priority of ω' is greater or equal to the preference priority of ω.

Example 2.21 *After computing the preference priority in Example 2.20, we can rank the outcomes as* $\omega_1 \leq_E^{pr} \omega_3 \leq_E^{pr} \omega_5 \leq_E^{pr} \{\omega_4, \omega_7\} \leq_E^{pr} \omega_2 \leq_E^{pr} \omega_8 \leq_E^{pr} \omega_6$.

The best outcome is ω_6. Comparing our approach to that presented by the PS-Merge we will have

Ω	Pr-Merge $\omega(E)$	PS-Merge $\omega(E)$
$\omega_1 = \neg s \neg d \neg o$	0.583	1.16
$\omega_2 = \neg s \neg do$	1.333	**2.33**
$\omega_3 = \neg sd \neg o$	0.833	1.5
$\omega_4 = \neg sdo$	1.25	1.83
$\omega_5 = s \neg d \neg o$	1	1.67
$\omega_6 = s \neg do$	**1.5**	**2.33**
$\omega_7 = sd \neg o$	1.25	1.83
$\omega_8 = sdo$	1.416	1.83

Table 16 – Comparison between PS-Merge and Pr-Merge.

Note that, in general, the preferences between the outcomes are very similar. The difference appears with ω_2 and ω_8. The belief base $K_3 = (s \wedge d \wedge o)$ has a priority greater than the other bases, which will influence the result of ω_8 (only outcome satisfying K_3), increasing its final result, whereas it will decrease the result of the outcome ω_2, because it is not a good outcome to K_3 (ω_2 satisfies only one propositional variable of K_3). We can define this process as a merging operator in the following model-theoretical way:

Definition 2.36 (Pr-Merge) *Let* $E = \{K_1, \ldots, K_n\}$ *be a belief set and* μ *an integrity constraint. The merging operator* $\Delta_\mu^{pr}(E)$ *is defined as* $\Delta_\mu^{pr}(E) = max(mod(\mu), \leq_E^{pr})$, *where* $max(mod(\mu), \leq_E^{pr})$ *is the set of outcomes that satisfy* μ *and are maximal with respect to the relation* $\leq_E^{pr}$.

Example 2.22 *The merging operator* $\Delta_\mu^{pr}(E)$ *for the previous example when* $\mu = \top$ *shall result in* $\Delta_\mu^{pr}(E) = \omega_6 = (s \wedge \neg d \wedge o)$. *If we impose only one programming language will be taught, i.e.,* $\mu_1 = (s \wedge \neg d \wedge \neg o) \vee (\neg s \wedge d \wedge \neg o) \vee (\neg s \wedge \neg d \wedge o)$, *the result is* $\Delta_{\mu_1}^{pr}(E) = \omega_2 = (\neg s \wedge \neg d \wedge o)$.

To conclude this section, we want to emphasize our choice with respect to partial satisfiability. The approach introduced here is not restricted only to PS-Merge, i.e., it can be used with distance-based merging operators too. Indeed, the distance-based merging with priorities may be viewed as a particular case of the weighted sum aggregation function (KONIECZNY; PINO-PÉREZ, 2002a).

Formally, it can be defined in the following way: as said previously, the distance measure between an outcome and a belief base is defined as $d(\omega,K) = \min_{\omega' \models K} d(\omega,\omega')$, where $d(\omega,\omega')$ is the distance between outcomes. Using the *sum* as an aggregation function, we define the distance measure between an outcome and a belief set $E = \{K_1,\ldots,K_n\}$ as $d(\omega,E) = \sum_{i=1}^{n} \{d(\omega,K_i)\}$. When the weighted *sum* is considered as the aggregation function, we have $d(\omega,E) = \sum_{i=1}^{n} a_i \times d(\omega,K_i)$, where a_i is the number of clauses in the belief base K_i in our work. Consequently, the merging operator $\Delta_{\mu}^{d,op}(E)$, where $op \in \{sum, wsum\}$, is defined as $\Delta_{\mu}^{d,op}(E)) = min(mod(\mu), \leq_{E}^{d,op})$. The comparison between distance-based and partial satisfiability merging is showed below (when $d = d_H$):

Ω	$\Delta_{\mu}^{d_H,sum}$	Δ_{μ}^{ps}	$\Delta_{\mu}^{d_H,wsum}$	Δ_{μ}^{pr}
$\omega_1 = \neg s \neg d \neg o$	5	1.16	4	0.583
$\omega_2 = \neg s \neg d o$	**2**	**2.33**	2	1.333
$\omega_3 = \neg s d \neg o$	4	1.33	3	0.833
$\omega_4 = \neg s d o$	3	1.83	2	1.25
$\omega_5 = s \neg d \neg o$	4	1.66	3	1
$\omega_6 = s \neg d o$	**2**	**2.33**	**1.5**	**1.5**
$\omega_7 = s d \neg o$	3	1.83	2	1.25
$\omega_8 = s d o$	3	1.83	**1.5**	1.416

Table 17 – Comparison between PS-Merge, Pr-Merge and distance-based merging.

In short, we can see that a partial satisfiability-based merging is richer than a distance-based merging, since it gives us a more detailed evaluation of the outcomes. In terms of logical properties we have the following results.

Theorem 2.21 Δ_{μ}^{pr} *satisfies* ***(IC0)-(IC2)****,* ***(IC5)-(IC8)*** *and* ***(Maj)****.*

Proof. See Appendix A.

Δ_{μ}^{pr} does not satisfy **(IC3)** and **(IC4)**. The postulate **(IC3)** is the principle of irrelevance of syntax: it says that the result of merging has to depend only on the expressed opinions (their semantics) and not on their syntactical presentation. The preference priority is designed in a way that equivalent formulas can receive different evaluations. For instance, for the formula $(a \wedge b)$, both propositional variables a and b have the same value, which is $\frac{1}{2}$. The formula

$(a \wedge b \wedge c) \vee (a \wedge b \wedge \neg c)$ is equivalent to the former, but the evaluation for each propositional variable will be different: their preference priority value will be $\frac{1}{6}$.

Since **(IC4)** is not satisfied, it means that this merging operator tends to give preference to some specific belief bases. The postulate **(IC4)** is the fairness condition, which means that the result of merging of two belief bases should not give preference to one of them. By the definition of partial satisfiability, the formulas with only disjunctions tends to have a higher preference priority value when compared to formulas with only conjunctions. For instance, if a belief base K_1 has the formula $(a \vee b)$ and a belief base K_2 has the formula $(\neg a \wedge \neg b)$, the result of the merging tends to give more preference to K_1. This is not a bad result, since we intended from the beginning to give more priority to some agents.

2.5 Conclusions

This chapter presented a broad vision of merging operators included in the propositional belief merging literature. We considered the aspects of utilitarianism and egalitarianism contained in these operators. Besides, we showed an idea of priority among agents was also approached with an alternative definition for propositional belief merging.

As its central idea, the propositional belief merging owns a method to measure the utility of the agents, denoted by a distance measure, and an aggregation function between distances, which can have a more utilitarian or egalitarian flavour. In order to characterize how good are the distance measures and aggregation operators, series of logical postulates were defined to belief merging operators. Table 18 illustrate the main results showed in this chapter.

A merging operator is said to be an IC merging operator when it satisfies the logical postulates **(IC0)** to **(IC8)**. If it satisfies additionally **(Maj)** or **(Arb)**, it is called a majority IC merging operator or arbitration IC merging operator, respectively. Some example of majority IC merging operators are *sum* and *leximin* merging operators, whilst for arbitration IC merging operators, we have only *leximax* merging operator. Although satisfying **(Arb)**, *max* and *min* do not satisfy all of the logical postulates, since they violate **(IC6)**. In this case, they are called IC quasi-merging operators.

We showed some of the first results of other egalitarian merging operators, from (EVERAERE *et al.*, 2014). Three operators were *med* (median), *leximed* (generalized median) and *csum* (cumulative *sum*). In general, they are not IC merging operators, since they do not satisfy **(IC5)** and **(IC6)**.

Lastly, we described a different formalization of the model-based belief merging, where the distance measure is dropped and a partial satisfiability function is applied. Unfortunately, this approach does not result in an IC merging operator: this time **(IC3)** and **(IC4)** are not satisfied. These logical postulates are related to the sensibility of the syntax of the belief bases and the fairness of the merging. In fact, we can say the partial satisfiability is influenced by changes in the syntax of the belief bases, even if these changes produce an equivalent belief base. Furthermore, this syntax sensibility is also attached to a certain priority among agents. It happens that some agents have more priority than others, and the result of the merging may benefit them. In some situations, it can not be expected to give more advantage to some agents.

	(IC0)	**(IC1)**	**(IC2)**	**(IC3)**	**(IC4)**	**(IC5)**	**(IC6)**	**(IC7)**	**(IC8)**	**(Maj)**	**(Arb)**
$\Delta_\mu^{d,sum}$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	
$\Delta_\mu^{d,max}$	✓	✓	✓	✓	✓	✓		✓	✓		✓
$\Delta_\mu^{d,leximax}$	✓	✓	✓	✓	✓	✓	✓	✓	✓		✓
$\Delta_\mu^{d,min}$	✓	✓	✓	✓	✓	✓		✓	✓		
$\Delta_\mu^{d,leximin}$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	
Δ_μ^{d,med^k}	✓	✓		✓	✓			✓	✓		$k \geq 0.5$
$\Delta_\mu^{d,leximed^k}$	✓	✓	✓	✓	✓			✓	✓		$k \geq 0.5$
$\Delta_\mu^{d,csum}$	✓	✓	✓	✓	✓			✓	✓	✓	
$\Delta_\mu^{ps,sum}$	✓	✓	✓			✓	✓	✓	✓	✓	
$\Delta_\mu^{ps,min}$	✓	✓	✓			✓		✓	✓		✓
Δ_μ^{pr}	✓	✓	✓			✓	✓	✓	✓	✓	

Table 18 – Summary of Logical Properties (3).

3 PROPOSITIONAL BELIEF MERGING WITH REFINEMENTS OF MAXIMUM OPERATOR

3.1 Contributions of this Chapter

The main contributions of this chapter are listed below:

- We analyze in this chapter the impact of introducing *discrimax* and fuzzy T-conorm operators in belief merging. We prove *discrimax* and T-conorm merging operators can be included as a subtype of egalitarian operators based on equity;
- We also study how the different T-conorm operators behave with respect to their logical properties and how this affects their rationality;
- We introduce LexiT-Conorms as merging operators to avoid the problems brought by T-conorms and prove some results related to them;
- Some results of this chapter have been published in EUMAS 2016, with the title "Propositional Belief Merging with T-conorms", authors: Henrique Viana and João Alcântara (VIANA; ALCÂNTARA, 2016a); and in NAFIPS 2018, with the title "Aggregation with T-Norms and LexiT-Orderings and Their Connections with the Leximin Principle", authors: Henrique Viana and João Alcântara (VIANA; ALCÂNTARA, 2018).

3.2 Refinements of Maximum Operator

Utilitarianism sustains the idea that the best choice for a group is that which maximizes the utility of the group. Utility can be measured in several ways, but is usually related to the well-being of the agents. In the framework of model-based belief merging work, it represents the distance measure between outcomes. The *sum* merging operator (KONIECZNY; PINO-PÉREZ, 1999) is an example of utilitarian operator. Egalitarianism, on the other hand, is concerned with reaching a kind of equality for all agents. The *max* (LIN; MENDELZON, 1996) and *leximax* (KONIECZNY; PINO-PÉREZ, 1999) merging operators are examples of egalitarian operators. Intuitively, they promote equality of the group by favoring the agents with the worst well-being.

This chapter aims at exploring further these egalitarian operators. The first idea is to relax the *max* operator and employ fuzzy connectives. As it is known, T-conorms (ZADEH, 1983; KLEMENT *et al.*, 2000) are functions stronger than the *max* operator, which can be commonly used to capture the worst cases in some group decision problems. A motivation for this chapter is to offer a new view about different merging operators with good logical properties and rationality.

Then we will prove that extensions of *max* operator can still preserve some logical properties and additionally earn new specific properties. Besides the original logical properties (KONIECZNY; PINO-PÉREZ, 2011), we will consider three other egalitarian conditions: the Hammond Equity Condition (EVERAERE *et al.*, 2014), Pigou-Dalton Principle (DALTON, 1920; EVERAERE *et al.*, 2014) and the Harm Principle (ALCANTUD, 2011; CAPPELEN; TUNGODDEN, 2006; LOMBARDI *et al.*, 2013). We will prove that in some cases, restricted versions of these axioms may be satisfied by some T-conorms. Besides, we will make a connection between T-conorm operators and the *leximax* operator.

We also formalize a *discrimax* merging operator, an extension of the *max* operator. The idea for refining *max* using an operator between *max* and *leximax* is that even if we consider the worst case of the group and their resulting tiebreakers, a question that remains open is that for a chosen outcome, in what conditions the rest of the group is really better in an overall manner. For instance, *leximax* operator is not influenced by individual variations of the agents. Consider two vectors of distance measures representing outcomes, $a = (2,2,0)$ and $b = (2,2,1)$. For *max* and *leximax* operators, a is better than b. We can say that the agent 3 (the third position of the vector) is in a better condition in the outcome a than in b, while the other two agents stand in the same situation for both outcomes. Consider now the vectors $c = (0,2,2)$ and $d = (2,1,2)$. We can check that the first element is better in c than d, but the second element is worse in c than d (third element remains unchanged in both outcomes). When considering *max* and *leximax* operators, c is reckoned as better than d, but these operators are not always sensitive to individual changes. The question that comes up is in what conditions we can say that outcome c is better than d.

The chapter is structured as follows. We will start our contributions with the discrimax extension in Section 3.3. In Section 3.4, we will present some basic notions of T-conorms. In Subsection 3.4.1, we will find the main contributions of this chapter, where we will introduce different T-conorm operators and will explore their respective logical properties. In Subsection 3.4.2, we will investigate the connection between T-conorm operators and the egalitarian reasoning of the Leximax Principle. Finally, in Section 3.5 we will present our conclusions. In the sequel, inspired by (FORTEMPS; PIRLOT, 2004) we will consider the *discrimax* operator and its refinements as a form of justifying other tiebreaker conditions than those found in the definitions of *max* and *leximax* operators.

3.3 Belief Merging with Discrimax

Discrimax ordering is the dual notion of Discrimin ordering introduced in (DUBOIS *et al.*, 1995), and it will be considered here to fit the framework of propositional belief merging based on distance measures. Formallly, we can define a Discrimax ordering as

Definition 3.1 (Discrimax Ordering) *(DUBOIS et al., 1995) Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be vectors of values. The difference set, i.e., the set of points of view $D(a,b)$ on which the evaluations of alternatives a and b differ is defined as $D(a,b) = \{i \in \{1, \ldots, n\} \mid a_i \neq b_i\}$. The Discrimax ordering is defined as follows: $a \leq_{disc} b \Leftrightarrow a = b$ or $\max_{i \in D(a,b)} a_i \leq \max_{i \in D(a,b)} b_i$.*

Discrimax is based on the elimination of identical singleton elements at the same position (difference set) and the comparison of the maximum with the remaining elements. It refines *max* ordering and when the difference set is empty, it is equivalent to Maximum.

We will refer to each position of a vector as the value of an agent. Discrimax differs from Maximum since it does not take into account the cases where the agents remain with the same distance value. For instance, $a = (0,2,1)$ is worse than $b = (0,2,0)$ when Discrimax is considered (for maximum, they are equivalent). We can see that the first and second positions of vectors a and b are the same. Discrimax ignores these cases and consider only the remaining position.

The difference between Discrimax and Leximax is that Leximax can be seen as an application of Discrimax, but employing (decreasing) ordered vectors. Let again $a = (0,2,1)$ and $b = (0,2,0)$, we call $\bar{a} = (2,1,0)$ and $\bar{b} = (2,0,0)$ the decreasing ordered vectors of a and b, respectively. We can use the Discrimax operator and state that $\bar{a}$ is worse than $\bar{b}$. Note that Discrimax orderings are total orders, but not necessarily transitive. For example, let $a = (1,1,0.3)$, $b = (0.6,0.3,1)$ and $c = (0.3,1,0.3)$. We have $a \leq_{disc} b$, $b \leq_{disc} c$, but $a \not\leq_{disc} c$. The reason for this is because $\approx_{disc}$ is not necessarily transitive.

Proposition 3.1 *(FORTEMPS; PIRLOT, 2004) $a \leq_{lex} b \Rightarrow a \leq_{disc} b \Rightarrow a \leq_{max} b$.*

This proposition states that every Leximax ordering is also a Discrimax ordering, and every Discrimax ordering is also a Maximum ordering. That is, Leximax is a refinement of Discrimax and Discrimax is a refinement of Maximum.

Definition 3.2 (*discrimax* **Operator)** *Let $E = \{K_1, \dots, K_n\}$ be a belief set, d a distance measure and ω an outcome. We have the following order: $\omega_i \leq_E^{d,discrimax} \omega_j$ iff $(d(\omega_i, K_1), \dots, d(\omega_i, K_n)) \leq_{disc} (d(\omega_j, K_1), \dots, d(\omega_j, K_n))$. The operator $\Delta_\mu^{d,discrimax}$ is defined by $\Delta_\mu^{d,discrimax}(E) = min(mod(\mu), \leq_E^{d,discrimax})$.*

We have to pay attention now for a little problem of *discrimax* operator. Because of the lack of transitivity, we need to check its corresponding ordering for each outcome.

Example 3.1 *Regarding Example 2.2, the Discrimax orderings w.r.t. Hamming distance and E are $\omega_6 \leq_{disc} \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_7, \omega_8\}$, $\{\omega_4, \omega_7\} \leq_{disc} \{\omega_1, \omega_2, \omega_3, \omega_5, \omega_8\}$, $\omega_2 \leq_{disc} \{\omega_1, \omega_3, \omega_5, \omega_8\}$, $\omega_3 \leq_{disc} \{\omega_1, \omega_5, \omega_8\}$, $\omega_8 \leq_{disc} \{\omega_1, \omega_3, \omega_5\}$ and $\omega_5 \leq_{disc} \{\omega_1, \omega_3\}$. We obtain $\Delta_\mu^{d,discrimax} = \omega_6$ when $\mu = \top$.*

Let us turn now to the analysis of logical postulates and conditions of the *discrimax* operator. Besides the basic logical postulates, we will check some additional properties to make a result for the *discrimax* operator: they are the Strong Pareto and Anonymity.

Definition 3.3 (Strong Pareto) *(TUNGODDEN, 2000) Let $E = \{K_1, \dots, K_n\}$ be a belief set and d a distance measure. For all $\omega, \omega' \in \Omega$, if $\exists i \in \{1, \dots, n\}$ $d(\omega, K_i) < d(\omega', K_i)$ and $\forall j \neq i$, $d(\omega, K_j) \leq d(\omega', K_j)$, then $\omega <_E \omega'$.*

Strong Pareto might be interpreted as the principle of personal good, where the utility values refers to the good of the agents. An outcome where all utility values are higher or equal than other outcome (with at least one utility value higher), it might be considered more just. **(SP)** is slightly different from **(IC6)**. **(SP)** compares directly each distance value in the belief bases from E and **(IC6)** compares outcomes from belief sets.

Definition 3.4 (Anonymity) *(TUNGODDEN, 2000) Let $E = \{K_1, \dots, K_n\}$ be a belief set and $l_\omega^{d,E} = (d_1^\omega, \dots, d_n^\omega)$ the list of distances between the outcome ω and the n belief bases in E. For all $\omega, \omega' \in \Omega$, if $l_\omega^{d,E}$ is a permutation of $l_{\omega'}^{d,E}$, then $\omega \approx_E \omega'$.*

Anonymity is a condition of impartiality, which states that the identity of the agents should not matter in a justice relation. Thus, we have the following result:

Theorem 3.1 *$\Delta_\mu^{d,discrimax}$ satisfies **(IC0)**, **(IC2)**-**(IC5)**, **(IC7)**-**(IC8)**, **(Arb)**, **(HE)**, **(SP)** and **(A)**, but it violates **(IC1)**, **(IC6)** and **(Maj)**.*

Proof. See Appendix B.

The *discrimax* operator shows a weaker behavior of *leximax* operator when considering the IC logical postulates above. It does not satisfy **(IC1)** since it lacks transitivity and **(IC6)** since it can behave as the *max* operator. When *discrimax* is different from *max*, **(IC6)** is satisfied. In addition, we can relate *discrimax* with the **(HE)**, **(SP)** and **(A)** conditions:

Definition 3.5 (Discrimax Principle) *Let d be a distance measure and $E = \{K_1, \ldots, K_n\}$ a belief set. An order relation $\leq_E^{d,op}$ satisfies Discrimax Principle **(DM)** when $\omega \leq_E^{d,op} \omega'$ iff $(d(\omega,K_1),\ldots,d(\omega,K_n)) = (d(\omega',K_1),\ldots, d(\omega',K_n))$ or $\exists j \in \{1,\ldots,n\} : d(\omega,K_j) < d(\omega',K_j)$ and $\forall i\{1,\ldots,n\}$ $[d(\omega,K_i) \leq max(d(\omega',K_i), d(\omega',K_j))]$.*

It is easy to see that *discrimax* operator satisfies **(DM)** (this principle is the same stated in (FORTEMPS; PIRLOT, 2004) which is based on the relation named “no reason to regret”, and it was shown to be equivalent to the discrimin relation). Now, we have the following result about the Discrimax Principle.

Theorem 3.2 *If an order relation satisfies **(DM)**, then it satisfies **(HE)**, **(SP)** and **(A)**.*

Proof: See Appendix B.

(DM) is not an “if and only if” condition. That is, **(HE)**, **(SP)** and **(A)** does not imply **(DM)** because as in $\leq_{disc}$, the relation defined in **(DM)** is not transitive.

3.4 T-conorms

As we have already mentioned, *discrimax* extends *max* operator. Now we will show a quite distinct path to extend max operator by resorting to T-conorms. With regard to the *max* operation, the merging compares only the highest value of each outcome to take a decision. The *max* operator can also be viewed as the disjunction logic operator, i.e., $(a \vee b) = max\{a,b\}$, corresponding to a T-conorm in Fuzzy Logic (ZADEH *et al.*, 1996).

Definition 3.6 (T-conorm) *(KLEMENT et al., 2000) A binary function $\oplus : [0,1] \times [0,1] \rightarrow [0,1]$ is a T-conorm if it satisfies the following conditions:*

1. *$\oplus\{a,b\} = \oplus\{b,a\}$ (Commutativity);*
2. *$\oplus\{a,\oplus\{b,c\}\} = \oplus\{\oplus\{a,b\},c\}$ (Associativity);*
3. *$a \leq c$ and $b \leq d \Rightarrow \oplus\{a,b\} \leq \oplus\{c,d\}$ (Monotonicity);*

4. $\oplus\{a,0\} = a$ *(Neutral Element).*

Every T-conorm has an absorbent element, also called annihilator, which is the natural number 1, i.e., $\oplus\{a,1\} = 1$ (in this case, 1 can also be associated as an implicit veto). A T-conorm is called strict if it is continuous and strictly monotone (i.e., $\forall x,y,z \oplus \{x,y\} < \oplus\{x,z\}$ whenever $x < 1$ and $y < z$). Continuity, which is often required from fuzzy operators, expresses the idea that, roughly speaking, very small changes in truth values of elements should not macroscopically affect the truth value of their fuzzy operators. A T-conorm is called nilpotent if it is continuous and if each $a \in]0,1[$ is a nilpotent element. An element $a \in]0,1[$ is a nilpotent element of $\oplus$ if there exists some $n \in \mathbb{N}$ such that $\oplus \underbrace{\{a,\ldots,a\}}_{n} = 1$. Besides, for all T-conorm $\oplus$, $\oplus\{a,b\} \geq max\{a,b\}$ (DETYNIECKI *et al.*, 2002; KLEMENT *et al.*, 2002). Recall that for any $a,b \in \mathbb{R}$, $]a,b[= \{x \in \mathbb{R} : a < x < b\}$.

Definition 3.7 (Basic T-conorms) *(KLEMENT et al., 2000) The following are the four basic T-conorms:*

- *Maximum T-conorm:* $\oplus_{\boldsymbol{M}}\{x,y\} = max(x,y)$.
- *Probabilistic sum T-conorm:* $\oplus_{\boldsymbol{P}}\{x,y\} = x + y - x \cdot y$.
- *Łukasiewicz T-conorm:* $\oplus_{\boldsymbol{L}}\{x,y\} = min(x+y,1)$.
- *Drastic sum T-conorm:* $\oplus_{\boldsymbol{D}}\{x,y\} = \begin{cases} 1, & \textit{if } (x,y) \in]0,1] \times]0,1]; \\ max(x,y), & \textit{otherwise}. \end{cases}$

These four basic T-conorms are remarkable for several reasons. The drastic sum $\oplus_{\mathbf{D}}$ and the maximum $\oplus_{\mathbf{M}}$ are the largest and the smallest T-conorms, respectively (with respect to the pointwise order). The maximum $\oplus_{\mathbf{M}}$ is the only T-conorm where each $x \in [0,1]$ is an idempotent element (recall $x \in [0,1]$ is called an idempotent element of $\oplus$ if $\oplus\{x,x\} = x$). The probabilistic sum $\oplus_{\mathbf{P}}$ and the Łukasiewicz T-conorm $\oplus_{\mathbf{L}}$ are examples of two important subclasses of T-conorms, namely, the classes of strict and nilpotent T-conorms, respectively (more details in (KLEMENT; MESIAR, 2005)).

Definition 3.8 (Strength Between T-conorms) *(KLEMENT; MESIAR, 2005) Consider two T-conorms* $\oplus_1$ *and* $\oplus_2$*. If we have* $\oplus_1\{x,y\} \leq \oplus_2\{x,y\}$ *for all* $x,y \in [0,1]$*, then we say that* $\oplus_1$ *is weaker than* $\oplus_2$ *or, equivalently, that* $\oplus_2$ *is stronger than* $\oplus_1$*, and we write in this case* $\oplus_1 \leq \oplus_2$.

We shall write $\oplus_1 < \oplus_2$ if $\oplus_1 \leq \oplus_2$ and $\oplus_1 \neq \oplus_2$. The drastic sum $\oplus_{\mathbf{D}}$ is the strongest, and the Maximum $\oplus_{\mathbf{M}}$ is the weakest T-conorm, i.e., for each T-conorm $\oplus$ we have $\oplus_{\mathbf{M}} \leq \oplus \leq \oplus_{\mathbf{D}}$. Between the four basic T-conorms we have these strict inequalities: $\oplus_{\mathbf{M}} < \oplus_{\mathbf{P}} < \oplus_{\mathbf{L}} < \oplus_{\mathbf{D}}$. Many families of T-conorms can be defined by an explicit formula depending on a parameter λ. We now give a quick overview of them.

Definition 3.9 (Schweizer-Sklar T-conorms) *(SCHWEIZER; SKLAR, 1961) The family of Schweizer-Sklar T-conorms $(\oplus_{\lambda}^{\boldsymbol{SS}})_{\lambda \in [-\infty,\infty]}$ is given by*

$$\oplus_{\lambda}^{\boldsymbol{SS}}\{x,y\} = \begin{cases} \oplus_{\boldsymbol{M}}\{x,y\}, & \textit{if } \lambda = -\infty; \\ \oplus_{\boldsymbol{P}}\{x,y\}, & \textit{if } \lambda = 0; \\ \oplus_{\boldsymbol{D}}\{x,y\}, & \textit{if } \lambda = \infty; \\ 1-(max(((1-x)^{\lambda}+(1-y)^{\lambda}-1),0))^{\frac{1}{\lambda}}, & \textit{otherwise}. \end{cases}$$

This family of T-conorms is remarkable as it contains all four basic T-conorms. When $\lambda = 1$, $\oplus_{1}^{\mathbf{SS}} = \oplus_{\mathbf{L}}$. For the rest of the parameters we have the following strict inequalities: $\oplus_{\infty}^{\mathbf{SS}} > \ldots > \oplus_{1}^{\mathbf{SS}} > \oplus_{0}^{\mathbf{SS}} > \oplus_{-\infty}^{\mathbf{SS}}$.

Definition 3.10 (Frank T-conorms) *(BUTNARIU; KLEMENT, 1993) The family of Frank T-conorms $(\oplus_{\lambda}^{\boldsymbol{F}})_{\lambda \in [0,\infty]}$ is given by*

$$\oplus_{\lambda}^{\boldsymbol{F}}\{x,y\} = \begin{cases} \oplus_{\boldsymbol{M}}\{x,y\}, & \textit{if } \lambda = 0; \\ \oplus_{\boldsymbol{P}}\{x,y\}, & \textit{if } \lambda = 1; \\ \oplus_{\boldsymbol{L}}\{x,y\}, & \textit{if } \lambda = \infty; \\ 1-log_{\lambda}\left(1+\frac{(\lambda^{1-x}-1)(\lambda^{1-y}-1)}{\lambda-1}\right), & \textit{otherwise}. \end{cases}$$

The Frank family comprehends a series of T-conorms between the Łukasiewicz and the probabilistic sum T-conorms (for $\lambda \in [2,\infty[$). The Frank family has the following strict inequalities: $\oplus_{\infty}^{\mathbf{F}} > \ldots > \oplus_{2}^{\mathbf{F}} > \oplus_{1}^{\mathbf{F}} > \oplus_{0}^{\mathbf{F}}$.

Definition 3.11 (Yager T-conorms) *(YAGER, 1980) The family of Yager T-conorms $(\oplus_{\lambda}^{\boldsymbol{Y}})_{\lambda \in [0,\infty]}$ is given by*

$$\oplus_{\lambda}^{\boldsymbol{Y}}\{x,y\} = \begin{cases} \oplus_{\boldsymbol{D}}\{x,y\}, & \textit{if } \lambda = 0; \\ \oplus_{\boldsymbol{M}}\{x,y\}, & \textit{if } \lambda = \infty; \\ min((x^{\lambda}+y^{\lambda})^{\frac{1}{\lambda}},1), & \textit{otherwise}. \end{cases}$$

It is one of the most popular families for modeling the union of fuzzy sets. The idea is to use the parameter λ as a reciprocal measure for the strength of the logical operator "*or*". In this context, $\lambda = 0$ expresses the least demanding (i.e., largest) "*or*", and $\lambda = \infty$ the most demanding (i.e., smallest) "*or*". The Yager T-conorms comprehend a series of T-conorms between the drastic and the maximum T-conorms. When $\lambda = 1$, $\oplus_1^{\mathbf{Y}} = \oplus_{\mathbf{L}}$. The Yager family has the following strict inequalities: $\oplus_0^{\mathbf{Y}} > \oplus_1^{\mathbf{Y}} > \oplus_2^{\mathbf{Y}} > \ldots > \oplus_\infty^{\mathbf{Y}}$.

Definition 3.12 (Sugeno-Weber T-conorms) *(WEBER, 1983) The family of Sugeno-Weber T-conorms* $(\oplus_\lambda^{\boldsymbol{SW}})_{\lambda \in [-1,\infty]}$ *is given by*

$$\oplus_\lambda^{\boldsymbol{SW}}\{x,y\} = \begin{cases} \oplus_{\boldsymbol{P}}\{x,y\}, & \text{if } \lambda = -1; \\ \oplus_{\boldsymbol{D}}\{x,y\}, & \text{if } \lambda = \infty; \\ min(x+y+\lambda xy, 1), & otherwise. \end{cases}$$

Note that $(\oplus_\lambda^{\mathbf{SW}})_{\lambda > -1}$ are increasing functions of the parameter λ. The Sugeno-Weber family has the following strict inequalities: $\oplus_{-1}^{\mathbf{SW}} < \oplus_0^{\mathbf{SW}} < \oplus_1^{\mathbf{SW}} < \ldots < \oplus_\infty^{\mathbf{SW}}$.

3.4.1 Belief Merging with T-conorms

In this section we will present the main contributions of this chapter by analyzing the rationality of T-conorms merging operators through their logical postulates. We will also consider some additional logical postulates during this process.

Definition 3.13 ($\oplus$ Operator) *Let* $E = \{K_1, \ldots, K_n\}$ *be a belief set,* $\oplus$ *a T-conorm,* d *a distance measure and* ω *an outcome. Let* $M = max(\{d(\omega, \omega') \mid \omega, \omega' \in \Omega\})$. *We define the distance between an outcome and a belief set based on* $\oplus$ *as* $d_\oplus(\omega, E) = \bigoplus_{K \in E} \left\{ \frac{d(\omega, K)}{M} \right\}$. *Then we have the following pre-order:* $\omega_i \leq_E^{d,\oplus} \omega_j$ *iff* $d_\oplus(\omega_i, E) \leq d_\oplus(\omega_j, E)$. *The operator* $\Delta_\mu^{d,\oplus}$ *is defined by* $\Delta_\mu^{d,\oplus}(E) = min(mod(\mu), \leq_E^{d,\oplus})$.

The definition is very close to the merging operators defined previously. The difference comes from the fact that we need to adapt the distance measures to the interval $[0, 1]$.

Example 3.2 *The results for the probabilistic sum T-conorm operator w.r.t. Hamming distance for Example 2.2 are in the last column of Table 19. The resulting pre-order* $\leq_E^{d_H, \oplus_{\boldsymbol{P}}}$ *is* $\omega_6 \leq_E^{d_H, \oplus_{\boldsymbol{P}}} \omega_2 \leq_E^{d_H, \oplus_{\boldsymbol{P}}} \{\omega_4, \omega_7\} \leq_E^{d_H, \oplus_{\boldsymbol{P}}} \omega_8 \leq_E^{d_H, \oplus_{\boldsymbol{P}}} \{\omega_3, \omega_5\} \leq_E^{d_H, \oplus_{\boldsymbol{P}}} \omega_1$.

Observe that for any T-conorm the presence of the annihilator 1 on the evaluation of ω_1 works as an implicit veto for that outcome; if an outcome has the highest distance value for

Ω	$d_H(\omega,K_1)$	$d_H(\omega,K_2)$	$d_H(\omega,K_3)$	$d_{Hmax}(\omega,E)$	$L_{\omega}^{d_H,E}$	$d_{H\oplus_{\mathbf{P}}}(\omega,E)$
$\omega_1 = \neg s \neg d \neg o$	1	1	3	3	$(3,1,1)$	1
$\omega_2 = \neg s \neg do$	0	0	2	2	$(2,0,0)$	0.666
$\omega_3 = \neg sd \neg o$	2	0	2	2	$(2,2,0)$	0.8884
$\omega_4 = \neg sdo$	1	1	1	**1**	$(1,1,1)$	0.7032
$\omega_5 = s \neg d \neg o$	0	2	2	2	$(2,2,0)$	0.8884
$\omega_6 = s \neg do$	0	1	1	**1**	**(1, 1, 0)**	**0.5551**
$\omega_7 = sd \neg o$	1	1	1	**1**	$(1,1,1)$	0.7032
$\omega_8 = sdo$	1	2	0	2	$(2,1,0)$	0.7772

Table 19 – The Hamming distances of K_1, K_2, K_3 and E.

an agent, that outcome has to be rejected by the group. It brings a principle of justice where the worst scenarios inside a group need to be avoided. In other words, the use of T-conorms as a merging operator presupposes that there exists a consensus among the agents stating that if a choice is the worst for an agent, then this choice has to be the worst for the group. Now, we will show what logical properties the merging operators with T-conorms satisfy in the general case:

Theorem 3.3 *Let $\oplus$ be a T-conorm. $\Delta_{\mu}^{d,\oplus}$ satisfies **(IC0)-(IC5)**, **(IC7)** and **(IC8)**. $\Delta_{\mu}^{d,\oplus}$ does not satisfy **(Maj)**. The postulates **(IC6)** and **(Arb)** are not satisfied in general.*

Proof. See Appendix B.

This result is very similar to that for the *max* and *discrimax* operator (Theorem 2.2). The difference comes from the fact **(Arb)** is not satisfied in general for all T-conorms. The first important concern when dealing with T-conorms is the presence of the annihilator 1. The first logical postulate that we need to revisit is **(IC6)**. This postulate corresponds to the following syncretic assignment: 6. if $\omega <_{E_1} \omega'$ and $\omega \leq_{E_2} \omega'$, then $\omega <_{E_1 \sqcup E_2} \omega'$. It states if an outcome ω is strictly more preferable than an outcome ω' for a belief set E_1 and if ω is at least as preferable as ω' for a belief set E_2, then if one joins the two belief sets, we have ω will be strictly more preferable than ω'. Note that the presence of an annihilator is sufficient to falsify this condition. Consider that ω is equivalently preferable to ω' for a belief set E_2 ($\omega \approx_{E_2} \omega'$) and that $d_{\oplus}(\omega, E_2) = d_{\oplus}(\omega', E_2) = 1$. For any E_1, we will have $d_{\oplus}(\omega, E_1 \sqcup E_2) = d_{\oplus}(\omega', E_1 \sqcup E_2) = 1$, that is, $\omega \approx_{E_1 \sqcup E_2} \omega'$, which falsifies the condition 6 (and **(IC6)**). To overcome this issue, we will consider a weaker version of the logical postulate **(IC6)** with the presence of the annihilator 1:

(IC6-1) Let $d_{op}(\omega_i, E_2) \neq 1$, for $i = 1, 2$. If $\omega_1 <_{E_1} \omega_2$ and $\omega_1 \leq_{E_2} \omega_2$, then $\omega_1 <_{E_1 \sqcup E_2} \omega_2$.

This weaker version of **(IC6)** considers the principle when the annihilator 1 is safe to be used without falsifying it. The second important condition we will reconsider comes from the social choice theory, and it is related to egalitarianism between agents. It was proposed by Peter

J. Hammond (HAMMOND, 1976) and it is known as the Hammond Equity condition (SEN, 1982): If agent i is worse off than agent j both in ω and in ω', and if i is better off himself in ω than in ω', while j is better off in ω' than in ω, and if furthermore all others are just as well off in ω as in ω', then ω' is socially better than ω.

Intuitively, Hammond Equity **(HE)** (see Definition 2.21) is an egalitarian condition between two agents stating that an outcome is more preferred than another if the inequalities between agents is lower. For instance, according to **(HE)**, the tuple $\omega = \left(\frac{1}{5}, 1, \frac{2}{5}\right)$ representing the satisfaction of three agents, is less preferred than the tuple $\omega' = \left(\frac{2}{5}, \frac{3}{5}, \frac{2}{5}\right)$. The reason is because the inequality between $\frac{1}{5}$ and 1 is greater than the inequality between $\frac{2}{5}$ and $\frac{3}{5}$. We can say that in this case ω' is more stable than ω.

The next egalitarian operator reconsidered from the social choice literature is the Pigou-Dalton condition **(PD)** (DALTON, 1920; EVERAERE *et al.*, 2014). Pigou-Dalton is a special case of Hammond Equity when the difference between outcomes has the same value. This condition cannot be applied in the previous example of $\omega = \left(\frac{1}{5}, 1, \frac{2}{5}\right)$ and $\omega' = \left(\frac{2}{5}, \frac{3}{5}, \frac{2}{5}\right)$, since $\frac{2}{5} - \frac{1}{5} \neq 1 - \frac{3}{5}$. If we consider an outcome $\omega'' = \left(\frac{2}{5}, \frac{4}{5}, \frac{2}{5}\right)$, we can compare ω with ω'', and according with **(PD)** the outcome ω'' should be preferred. Intuitively, **(PD)** (see Definition 2.23) is an egalitarian principle which favors a better distribution of satisfaction between outcomes when the sum of the total amount is equal for both outcomes. Weaker versions of Hammond Equity and Pigou-Dalton conditions excluding the annihilator 1 are defined respectively as

(HE-1) If $\exists i, j \in \{1, \ldots, n\}$ such that $d(\omega, K_i) < d(\omega', K_i) < d(\omega', K_j) < d(\omega, K_j)$ and $\forall l \neq i, j$ $d(\omega, K_l) = d(\omega', K_l) \neq 1$, then $d_{op}(\omega', E) < d_{op}(\omega, E)$; and **(PD-1)** if $\exists i, j \in \{1, \ldots, n\}$ such that $d(\omega, K_i) < d(\omega', K_i) \leq d(\omega', K_j) < d(\omega, K_j)$, $d(\omega', K_i) - d(\omega, K_i) = d(\omega, K_j) - d(\omega', K_j)$ and $\forall l \neq i, j$ $d(\omega, K_l) = d(\omega', K_l) \neq 1$, then $d_{op}(\omega', E) < d_{op}(\omega, E)$.

The justification for this restriction is the same applied to **(IC6)**: if for any $l \neq i, j$, we have $s_d(\omega, K_l) = s_d(\omega', K_l) = 1$, then $d_{\oplus}(\omega, E) = d_{\oplus}(\omega', E) = 1$ for any T-conorm $\oplus$, and consequently falsifying both postulates. The last egalitarian property we want to consider comes from liberal egalitarianism (ALCANTUD, 2011; CAPPELEN; TUNGODDEN, 2006; LOMBARDI *et al.*, 2013); a theory of justice which combines the values of equality, personal freedom and personal responsibility. It is called Harm Principle (or Principle of Non-Interference).

Definition 3.14 (Harm Principle Condition) *(ALCANTUD, 2011)* ***(HP)*** *Let d be a distance measure, $E = \{K_1, \ldots, K_n\}$ be a belief set and $\omega_1 <_E^{d,op} \omega_2$, for any merging operator op. An operator op satisfies the Harm Principle Condition iff for all $\omega_1, \omega_2, \omega_1', \omega_2' \in \Omega$, consider ω_1', ω_2'*

such that $\exists i \in \{1,\ldots,n\}$, $d(\omega_1,K_i) < d(\omega_1',K_i)$, $d(\omega_2,K_i) < d(\omega_2',K_i)$ *and* $\forall j \neq i$ $d(\omega_1,K_j) = d(\omega_1',K_j)$, $d(\omega_2,K_j) = d(\omega_2',K_j)$. *If* $d(\omega_1',K_i) < d(\omega_2',K_i)$, *then* $\omega_1' <_E^{d,op} \omega_2'$.

In distributive justice theory, this condition embodies the idea that "an individual has the right to prevent society from acting against him in all circumstances of change in his welfare, provided that the welfare of no other individual is affected". In the distance-based merging framework, it can be seen as considering that an outcome ω_1 is more preferred than ω_2, and if occasionally an agent i has an increase of the distance value in ω_1 and ω_2, resulting in ω_1' and ω_2' respectively, ω_1' will be preferred to ω_2' if the distance measure in ω_1' is still lower than the distance in ω_2'. In other words, a single agent does not have the power of interference in the choice of the group when occurring an increase of distance measure (we can also see the non-satisfaction of the condition as a kind of veto power of agents). The equality emerges from the fact that no specific agent has the power to interfere in the decision of the group. Finally, we can continue with the analysis of logical postulates for each specific T-conorm.

Theorem 3.4 $\Delta_\mu^{d,\oplus_M}$ *satisfies* ***(Arb)*** *and* ***(HP)***, *but it does not satisfy* ***(IC6-***1***)***, ***(HE-***1***)***, ***(PD-***1***)*** *in the general case.* $\Delta_\mu^{d,\oplus_P}$ *satisfies* ***(IC6-***1***)*** *and* ***(PD-***1***)***, *but it does not satisfy* ***(Arb)***, ***(HE-***1***)*** *and* ***(HP)*** *in the general case.*

Proof. See Appendix B.

One interesting point to highlight is any strict T-conorm satisfies **(IC6-**1**)** (e.g., $\oplus_{\mathbf{P}}$ is a strict T-conorm).

Theorem 3.5 *Let* $\oplus$ *be a strict T-conorm, then* $\Delta_\mu^{d,\oplus}$ *satisfies* ***(IC6-***1***)***.

Proof. See Appendix B.

Below, we have results for the other basic T-conorms operators.

Proposition 3.2 $\Delta_\mu^{d,\oplus_L}$ *and* $\Delta_\mu^{d,\oplus_D}$ *do not satisfy* ***(IC6-***1***)***, ***(Arb)***, ***(HE-***1***)***, ***(PD-***1***)*** *and* ***(HP)***.

Proof. See Appendix B.

The drastic sum T-conorm is not continuous, which implies that little changes in the variables can change drastically the result and this reflects the loss of some important logical properties. The Łukasiewicz T-conorm is a nilpotent T-conorm; in this case, the presence of a nilpotent element reflects the loss of some properties.

Proposition 3.3 *Let $\oplus$ be a nilpotent T-conorm, then $\Delta_{\mu}^{d,\oplus}$ does not satisfy **(IC6-**1**)**, **(HE-**1**)**, **(PD-**1**)** and **(HP)** in the general case.*

Proof. See Appendix B.

The nilpotent element works as a sort of annihilator and then inherits all the problems discussed above. In the sequel, we will investigate deeper the behavior of T-conorms through some parameterized T-conorms to see in what conditions we can achieve egalitarian properties for the propositional belief merging. First, we will consider the Schweizer-Sklar T-conorms.

Theorem 3.6 *$\Delta_{\mu}^{d,\oplus_{\lambda}^{SS}}$ satisfies **(IC6-**1**)** and **(PD-**1**)** when $\lambda \in]-\infty, 0]$. Let $n \geq 3$ be the number of different propositional variables in the belief set E. $\Delta_{\mu}^{d,\oplus_{\lambda}^{SS}}$ satisfies **(Arb)**, **(HE-**1**)** and **(HP)** when $-\infty < \lambda \leq -\left\lfloor \frac{2n}{3} \right\rfloor$.*

Proof. See Appendix B.

Regarding Theorem 3.6, the interval $[1, \infty]$ comprises strictly increasing T-conorms from the Łukasiewicz T-conorm ($\oplus_{1}^{\mathbf{SS}}$) to the drastic sum T-conorm ($\oplus_{\infty}^{\mathbf{SS}}$). It is clear that all these conditions are falsified in this interval (since all T-conorms in this interval are weaker than Łukasiewicz T-conorm). Schweizer-Sklar T-conorm is strict for the interval $]-\infty, 0]$, therefore it satisfies **(IC6-**1**)** in this case. Considering this interval yet, we have that any Schweizer-Sklar T-conorm satisfies **(PD-**1**)**; and additionally satisfies **(Arb)** and **(HE-**1**)** when $-\infty < \lambda \leq -\left\lfloor \frac{2n}{3} \right\rfloor$, where *n* is the number of propositional variables in the belief set. Intuitively, as we decrease the parameter λ, we strengthen these conditions of egalitarian properties in the Schweizer-Sklar T-conorm.

Note that when the parameterized T-conorm gets closer to maximum T-conorm, it has stronger egalitarian properties (e.g., Harm Principle). When it gets closer to drastic sum and Łukasiewicz T-conorms, it tends to lose its logical properties. Now consider the Frank T-conorms:

Theorem 3.7 *$\Delta_{\mu}^{d,\oplus_{\lambda}^{F}}$ satisfies **(IC6-**1**)** and **(PD-**1**)** for $\lambda \in]0, \infty[$. $\Delta_{\mu}^{d,\oplus_{\lambda}^{F}}$ does not satisfy **(Arb)**, **(HE-**1**)** and **(HP)** in the general case.*

Proof. See Appendix B.

We observed previously that T-conorms converging to the maximum T-conorm tend to satisfy properties as **(Arb)**, **(HE-**1**)** and **(PD-**1**)**, while T-conorms converging to probabilistic sum T-conorm satisfy only **(PD-**1**)**. When the Frank T-conorm is considered, the convergence to

Łukasiewicz T-conorm ($\oplus_{\infty}^{\mathbf{F}} = \oplus_{\mathbf{L}}$) from probabilistic sum ($\oplus_{1}^{\mathbf{F}} = \oplus_{\mathbf{P}}$) still implies the satisfaction of **(PD-**1**)**. Besides, considering this convergence of Frank T-conorm to the maximum T-conorm, we can have an additional result for the interval $[0,1]$ (from $\oplus_{0}^{\mathbf{F}} = \oplus_{\mathbf{M}}$ to $\oplus_{1}^{\mathbf{F}} = \oplus_{\mathbf{P}}$).

Theorem 3.8 *Let $n \geq 3$ be the number of different propositional variables in the belief set E. $\Delta_{\mu}^{d,\oplus_{\lambda}^{F}}$ satisfies **(Arb)**, **(HE-**1**)** and **(HP)** when $0 < \lambda \leq 10^{-n}$.*

Proof. See Appendix B.

The limit of $0 < \lambda \leq 10^{-n}$ is rather loose, but it is a statement that there is an interval between maximum and probabilistic sum in the Frank T-conorm where **(Arb)**, **(HE-**1**)** and **(HP)** are satisfied. In the sequel, we will see the Yager family of T-conorms.

Theorem 3.9 *Let $n \geq 3$ be the number of different propositional variables in the belief set E. For $\lambda \in [2,\infty[$, $\Delta_{\mu}^{d,\oplus_{\lambda}^{Y}}$ satisfies **(Arb)** when $\lambda \geq \left\lfloor \frac{2n}{3} \right\rfloor$.*

Proof. See Appendix B.

Yager T-conorms comprise from drastic sum ($\oplus_{0}^{\mathbf{Y}}$), passing through Łukasiewicz T-conorm ($\oplus_{1}^{\mathbf{Y}}$), to maximum T-conorm ($\oplus_{\infty}^{\mathbf{Y}}$). Unlike the previous parameterized T-conorms, Yager T-conorms are nilpotent for $\lambda \in]0,\infty[$, which does not result in satisfying **(IC6-**1**)**, **(HE-**1**)**, **(PD-**1**)** and **(HP)** in the general case, but **(Arb)** can be still satisfied.

For the last, we analyze Sugeno-Weber family of T-conorms.

Theorem 3.10 *For $\lambda \in]-1,\infty]$, $\Delta_{\mu}^{d,\oplus_{\lambda}^{SW}}$ does not satisfy **(IC6-**1**)**, **(HE-**1**)**, **(PD-**1**)**, **(HP)** and **(Arb)** in the general case.*

Proof. See Appendix B.

Sugeno-Weber T-conorms are another class of nilpotent T-conorms. They range from drastic sum ($\oplus_{\infty}^{\mathbf{SW}}$) to Łukasiewicz ($\oplus_{0}^{\mathbf{SW}}$) and probabilistic sum T-conorms ($\oplus_{-1}^{\mathbf{SW}}$). As it is nilpotent, the conditions **(IC6-**1**)**, **(HE-**1**)**, **(PD-**1**)** and **(HP)** do not hold in the general case for any λ. The absence of convergence for maximum implies the falsification of **(Arb)**.

3.4.2 T-conorms and the Leximax Principle

In this section, we will use the results of (TUNGODDEN, 2000) to characterize an egalitarian property of some parameterized T-conorms. The Leximax principle will be the key to this analysis.

Definition 3.15 (Leximax Principle) *(TUNGODDEN, 2000) Let $E = \{K_1, \dots, K_n\}$ be a belief set. For each outcome ω, we build the list $(d_1^\omega, \dots, d_n^\omega)$ of distances between this outcome and the n belief bases in E, i.e., $d_i^\omega = d(\omega, K_i)$. Let $L_\omega^{d,E}$ be the list obtained from $(d_1^\omega, \dots, d_n^\omega)$ by sorting it in descending order. **(LM)** For all $\omega, \omega' \in \Omega$, (1) if there exists a position $k \leq n$ such that $d_k^\omega < d_k^{\omega'}$; and (2) for every $j < k$, $d_j^{\omega'} = d_j^\omega$, then $\omega <_E \omega'$ (ω is more preferred than ω'). Otherwise, $\omega \approx_E \omega'$.*

Basically, it is the same idea behind *leximax* operator (see Definition 2.8). The first important characterization we need to consider is

Theorem 3.11 *(TUNGODDEN, 2000) A Syncretic Assignment $\leq_E$ satisfies **(HE)**, **(SP)** and **(A)** if and only if it satisfies **(LM)**.*

This theorem can be used in belief merging to assert that any merging operator satisfying **(HE)** and **(SP)** and **(A)** is equivalent to the *leximax* operator. We turn now to the link between parameterized T-conorm merging operators and the Leximax principle. It is known that for every T-conorm $\oplus$, in which $\oplus \geq \oplus_{\mathbf{M}}$, and despite *max* operator does not satisfy properties as **(HE)** and **(PD)**, some T-conorms can satisfy weakened versions of them.

This analysis shows that some T-conorms present a similar (weaker) behavior to the *leximax* operator. What we want to achieve is that those T-conorms can also follow some weaker versions of the leximax principle. We introduce a restriction to the Leximax principle, named Leximax principle free from annihilator 1.

Definition 3.16 (Leximax Free From 1) *Let $E = \{K_1, \dots, K_n\}$ be a belief set. For each outcome ω we build the list $(d_1^\omega, \dots, d_n^\omega)$ of distances between this outcome and the n belief bases in E, i.e., $d_i^\omega = d(\omega, K_i)$. Let $L_\omega^{d,E}$ be the list obtained from $(d_1^\omega, \dots, d_n^\omega)$ by sorting it in descending order. **(LM-**1**)** For all $\omega, \omega' \in \Omega$, (1) if there exists a position $k \leq n$ such that $d_k^\omega < d_k^{\omega'}$; and (2) for every $j < k$, $d_j^{\omega'} = d_j^\omega \neq 1$, then $\omega <_E \omega'$ (ω is more preferred than ω'). Otherwise, $\omega \approx_E \omega'$.*

It is possible then to make a restricted characterization of the Leximax principle for a belief merging operator:

Definition 3.17 (Strong Pareto Free From 1) *Let $E = \{K_1, \dots, K_n\}$ be a belief set and d be a distance measure. **(SP-**1**)** For all $\omega, \omega' \in \Omega$, if $\exists i \in \{1, \dots, n\}$ such that $d(\omega, K_i) < d(\omega', K_i)$ and $\forall j \neq i$, $d(\omega, K_j) \leq d(\omega', K_j)$ and $d(\omega, K_j), d(\omega', K_j) \neq 1$, then $\omega <_E \omega'$.*

Corollary 3.1 *A belief merging operator Δ_μ satisfies **(HE-**1**)**, **(SP-**1**)** and **(A)** if and only if it satisfies **(LM-**1**)**.*

It comes directly from Theorem 3.11. This characterization restricts the Leximax principle when the annihilator is excluded from the possible distance values of the agents. As a consequence, we have

Corollary 3.2 *Let $n \geq 3$ be the number of propositional variables in the belief set E. $\Delta_\mu^{d,\oplus_\lambda^{SS}}$ and $\Delta_\mu^{d,\oplus_\lambda^{F}}$ satisfy **(LM-**1**)** when $\lambda \leq -\left\lfloor \frac{2n}{3} \right\rfloor$ and $0 < \lambda \leq 10^{-n}$, respectively.*

This result comes from Corollary 3.1 and Theorems 3.6 and 3.7. These operators satisfy **(HE-**1**)**, **(SP-**1**)** and **(A)** in those specific intervals. In other words, when the annihilator is not present in the merging, we can say these T-conorms have a behavior similar to the *leximax* operator. The last consideration of this subsection is about the Harm Principle. Although Hammond Equity and the Harm Principle are conceptually distinct and logically independent, it was proved the following result:

Theorem 3.12 *(MARIOTTI; VENEZIANI, 2008) A Syncretic Assignment $\leq_E$ satisfies **(HP)**, **(SP)** and **(A)** if and only if it satisfies **(LM)**.*

With the above theorem, it is possible to assert a different version of Corollary 3.1.

Corollary 3.3 *A belief merging operator Δ_μ satisfies **(HP)**, **(SP-**1**)** and **(A)** if and only if it satisfies **(LM-**1**)**.*

We just make clear that **(HP)** and **(HE)** are not logically equivalent. It is known that under **(A)**, Harm Principle implies Hammond Equity but the converse is not true (ALCANTUD, 2013).

3.4.3 Belief Merging with LexiT-conorms

In order to present the last contributions of this chapter, we will consider a refinement of the T-conorm operators. It is the same idea behind the *leximax* refinement of *maximum* operator. We will call it lexiT-conorm and it is the dual notion of lexiT-norms introduced in (YAGER *et al.*, 2005).

Definition 3.18 (LexiT-conorm) *(YAGER et al., 2005) Let $a = (a_1, a_2, \ldots, a_n) \in [0,1]^n$ and let $\oplus$ be a T-conorm. Let P_a be the power set of $\{a_1, a_2, \ldots, a_n\}$ excluding the empty set, that is, the set of all subsets of the indexed set $\{a_1, a_2, \ldots, a_n\}$ minus $\emptyset$. For any $A \in P_a$, we let $\oplus(A)$ indicate the T-conorm of the elements of A. Let $\bar{a} = (\bar{a}_1, \bar{a}_2, \ldots, a_{2^{\bar{n}}-1})$ be the $(2^n - 1)$-tuple of the family $\{\oplus(A) : A \in P_a\}$ put into descending order. On $[0,1]^{2^n-1}$ we have the lexicographic ordering $\leq_{lex}$ which is a linear ordering. The corresponding Lexi$\oplus$ procedure $a \leq_{Lexi\oplus} b$ is defined as follows:*

- *For $a, b \in [0,1]^n$, use $\oplus$ to construct $\bar{a}$ and $\bar{b} \in [0,1]^{2^n-1}$. Then $a \leq_{Lexi\oplus} b$ if and only if $\bar{a} \leq_{lex} \bar{b}$.*

In other words,

- *$a <_{Lexi\oplus} b$ if and only if there exists $k \geq 1$ such that $\bar{a}_k < \bar{b}_k$ and for $1 \leq i < k$, $\bar{a}_i = \bar{b}_i$;*
- *$a \approx_{Lexi\oplus} b$ if and only if $\bar{a}_i = \bar{b}_i$ for all $i = 1, 2, \ldots, 2^n - 1$.*

Here is a simple example: Take the probabilistic sum T-conorm $\oplus_{\mathbf{P}}\{x,y\} = x + y - x \cdot y$. Let $a = (0.2, 0.6)$ and $b = (0.3, 0.5)$. In this case, both P_a and P_b have 3 elements: $P_a = (\{0.2\}, \{0.6\}, \{0.2, 0.6\})$ and $P_b = (\{0.3\}, \{0.5\}, \{0.3, 0.5\})$. After calculating $\oplus\{A\}$ and $\oplus\{B\}$ for each $A \in P_a$ and $B \in P_b$ we get $\bar{a} = (0.68, 0.6, 0.2)$ and $\bar{b} = (0.65, 0.5, 0.3)$. Now, comparing $\bar{a}$ and $\bar{b}$ lexicographically, we see that $\bar{b} \leq_{lex} \bar{a}$, and consequently $b \leq_{Lexi\oplus_{\mathbf{P}}} a$. Before proceeding with the application of LexiT-conorms as a merging operator, note that we have the following property about LexiT-conorms:

Theorem 3.13 *(YAGER et al., 2005) Let $\oplus_{\mathbf{M}}$ be the Maximum T-conorm. Then for $a, b \in [0,1]^n$, $a \leq_{Leximax} b$ if and only if $a \leq_{Lexi\oplus_{\mathbf{M}}} b$.*

That is, if $\oplus$ is the Maximum T-conorm, Leximax and Lexi$\oplus$ are the same ordering.

Definition 3.19 (Lexi$\oplus$ Operator) *Let $E = \{K_1, \ldots, K_n\}$ be a belief set, $\oplus$ a T-conorm and d a distance measure. For each outcome, ω we build the list $l_\omega^{d,E} = (d_1^\omega, \ldots, d_n^\omega)$ of distances between this outcome and the n belief bases in E, i.e., $d_i^\omega = d(\omega, K_i)$. Let $M = max(\{d(\omega, \omega') \mid \omega, \omega' \in \Omega\})$ and $l_\omega^{d,\bar{E},\oplus} = (d_1^{\bar{\omega}}, \ldots, d_{2^n-1}^{\bar{\omega}})$ be the $(2^n - 1)$-tuple of the family $\left\{\oplus\left\{\frac{A}{M}\right\} : A \in P_{l_\omega^{d,E}}\right\}$ put in descending order. Let $\leq_{lex}$ be the lexicographical order between sequences of integers. We define the following pre-order: $\omega_i \leq_E^{d,lexi\oplus} \omega_j$ iff $l_{\omega_i}^{d,\bar{E},\oplus} \leq_{lex} l_{\omega_j}^{d,\bar{E},\oplus}$. The operator $\Delta_\mu^{d,lexi\oplus}$ is defined by $\Delta_\mu^{d,lexi\oplus}(E) = min(mod(\mu), \leq_E^{d,lexi\oplus})$.*

For strict T-conorms, computing the LexiT-conorm can be done in a simpler way.

Ω	$d_H(\omega,K_1)$	$d_H(\omega,K_2)$	$d_H(\omega,K_3)$	$d_{H\oplus_\mathbf{P}}(\omega,E)$	$l_\omega^{d_H,\bar{E},\oplus_\mathbf{P}}$
$\omega_1 = \neg s \neg d \neg o$	1	1	3	1	(1,1,1,1,0.777,0.333,0.333)
$\omega_2 = \neg s \neg do$	0	0	2	0.666	(0.666,0.666,0.666,0.666,0,0,0)
$\omega_3 = \neg sd \neg o$	2	0	2	0.888	(0.888,0.888,0.666,0.666,0.666,0.666,0)
$\omega_4 = \neg sdo$	1	1	1	0.703	(0.703,0.555,0.555,0.555,0.333,0.333,0.333)
$\omega_5 = s \neg d \neg o$	0	2	2	0.888	(0.888,0.888,0.666,0.666,0.666,0.666,0)
$\omega_6 = s \neg do$	0	1	1	**0.555**	**(0.555,0.555,0.333,0.333,0.333,0.333,0)**
$\omega_7 = sd \neg o$	1	1	1	0.703	(0.703,0.555,0.555,0.555,0.333,0.333,0.333)
$\omega_8 = sdo$	1	2	0	0.777	(0.777,0.777,0.666,0.666,0.333,0.333,0)

Table 20 – The Hamming distances of K_1, K_2, K_3 and E (2).

Theorem 3.14 *(WALKER et al., 2005) If an T-conorm $\oplus$ is strict, it takes at most n steps to determine whether or not $(a_1, a_2, \ldots, a_n) <_{Lexi\oplus} (b_1, b_2, \ldots, b_n)$.*

This Theorem points exactly what one needs to compute Lexi$\oplus$ with a strict T-conorm $\oplus$. From it, we can calculate $(a_1, a_2, \ldots, a_n) <_{Lexi\oplus} (b_1, b_2, \ldots, b_n)$ as follows: let $\bar{a} = (\bar{a_1}, \bar{a_2}, \ldots, \bar{a_n})$ be the n-tuple of $(a_1, a_2, \ldots, a_n)$ put into descending order, $(a_1, a_2, \ldots, a_n) <_{Lexi\oplus} (b_1, b_2, \ldots, b_n)$ if and only if $(\oplus\{\bar{a_1}, \bar{a_2}, \ldots, \bar{a_n}\}, \oplus\{\bar{a_1}, \bar{a_2}, \ldots, \bar{a_{n-1}}\}, \ldots, \oplus\{\bar{a_1}, \bar{a_2}\}, \bar{a_1}) <_{lex} (\oplus\{\bar{b_1}, \bar{b_2}, \ldots, \bar{b_n}\}, \oplus\{\bar{b_1}, \bar{b_2}, \ldots, \bar{b_{n-1}}\}, \ldots, \oplus\{\bar{b_1}, \bar{b_2}\}, \bar{b_1})$. We can now then simplify the definition of a strict Lexi$\oplus$ operator.

Definition 3.20 (Strict Lexi$\oplus$ Operator) *Let $E = \{K_1, \ldots, K_n\}$ be a belief set, $\oplus$ a strict T-conorm and d a distance measure. Let $M = max(\{d(\omega, \omega') \mid \omega, \omega' \in \Omega\})$. For each outcome, ω we build the list $l_\omega^{d,E} = (d_1^\omega, \ldots, d_n^\omega)$ of distances between this outcome and the n belief bases in E divided by M, i.e., $d_i^\omega = \frac{d(\omega, K_i)}{M}$. Let $l_\omega^{d,\bar{E},\oplus} = (d_1^{\bar{\omega}}, \ldots, d_n^{\bar{\omega}})$ be the n-tuple of $l_\omega^{d,E}$ put in descending order. Let $\leq_{lex}$ be the lexicographical order between sequences of integers. We define the following pre-order: $\omega_i \leq_E^{d,lexi\oplus} \omega_j$ iff $(\oplus\{d_1^{\bar{\omega}_i}, \ldots, d_n^{\bar{\omega}_i}\}, \oplus\{d_1^{\bar{\omega}_i}, \ldots, d_{n-1}^{\bar{\omega}_i}\}, \ldots, d_1^{\bar{\omega}_i})$ $\leq_{lex} (\oplus\{d_1^{\bar{\omega}_j}, \ldots, d_n^{\bar{\omega}_j}\}, \oplus\{d_1^{\bar{\omega}_j}, \ldots, d_{n-1}^{\bar{\omega}_j}\}, \ldots, d_1^{\bar{\omega}_j})$. The operator $\Delta_\mu^{d,lexi\oplus}$ is defined by $\Delta_\mu^{d,lexi\oplus}(E) = min(mod(\mu), \leq_E^{d,lexi\oplus})$.*

Using Table 20 as example, $l_{\omega_6}^{d_H,\bar{E},\oplus_\mathbf{P}} = (0.555, 0.555, 0.333, 0.333, 0.333, 0.333, 0)$ can now be computed as $(\oplus_\mathbf{P}\{0.333, 0.333, 0\}, \oplus_\mathbf{P}\{0.333, 0.333\}, 0.333) = (0.555, 0.555, 0.333)$. In terms of results, both forms of computation are equals. This is only a question of complexity, and since the class of strict T-conorms presented may have similar behavior to the *leximax* operator, we will see in the sequel that some LexiT-conorms may be equivalent to *leximax*, in terms of complexity and rationality. Let us turn now to the properties of LexiT-conorms operators. We will separate them in two parts.

Corollary 3.4 *Let $\oplus$ be a T-conorm. $\Delta_{\mu}^{d,lexi\oplus}$ satisfies **(IC0)**-**(IC5)**, **(IC7)** and **(IC8)**. $\Delta_{\mu}^{d,lexi\oplus}$ does not satisfy **(Maj)**. The postulate **(Arb)** is not satisfied in general.*

These results come directly from Theorem 3.3. As pointed in the previous sections, **(IC6)** is not satisfied in general and its weaker version (**IC6-**1) is only satisfied by strict T-conorms. This becomes different for LexiT-conorms:

Theorem 3.15 *Let $\oplus$ be a T-conorm, then $\Delta_{\mu}^{d,lexi\oplus}$ satisfies **(IC6)**.*

Proof. See Appendix B.

It occurs that **(IC6)** is satisfied by any LexiT-conorm, strict or nilpotent one.

Theorem 3.16 *Let $\oplus$ be a T-conorm. We have the following results:*

- *If $\Delta_{\mu}^{d,\oplus}$ satisfies (**HE-**1). then $\Delta_{\mu}^{d,lexi\oplus}$ satisfies **(HE)**;*
- *If $\Delta_{\mu}^{d,\oplus}$ satisfies (**PD-**1). then $\Delta_{\mu}^{d,lexi\oplus}$ satisfies **(PD)**;*
- *If $\Delta_{\mu}^{d,\oplus}$ satisfies (**SP-**1). then $\Delta_{\mu}^{d,lexi\oplus}$ satisfies **(SP)**.*

Proof. See Appendix B.

From Theorems 3.6, 3.8, 3.11 and 3.17 we have

Corollary 3.5 *Let $n \geq 3$ be the number of propositional variables in the belief set E. $\Delta_{\mu}^{d,lexi\oplus_{\lambda}^{SS}}$ and $\Delta_{\mu}^{d,lexi\oplus_{\lambda}^{F}}$ satisfy **(LM)** when $\lambda \leq -\left\lfloor \frac{2n}{3} \right\rfloor$ and $0 < \lambda \leq 10^{-n}$, respectively.*

Theses results illustrate that some LexiT-conorms have the similar behavior of the *leximax* operator, when compared with the logical properties, and produce a kind of egalitarian reasoning.

3.5 Conclusions

In this chapter, we proposed to use *discrimax* and T-conorm operators in the propositional belief merging. As we know, *discrimax* is a refinement of *max*, which is situated between *max* and *leximax* operators. T-conorms are generalization of the two-valued logical disjunction, i.e., the *max* operator. In belief merging, the *max* operator is equivalent to the minimax rule in decision theory: it tries to minimize the worst cases among the agents. Indeed, T-conorms and *discrimax* allow us to diversify the method of the minimax rule by applying generalized versions of the *max* operator.

The purpose of this chapter is to offer more diversity of egalitarian merging operators and explore their logical properties. In order to deepen this analysis, we considered other logical properties related to egalitarianism, more specifically, the arbitration, Hammond Equity, Pigou-Dalton principle, Strong Pareto, Harm Principle and their variations. These conditions are intended to express preference for a more just distribution among the agents, based on principles of equity or libertarianism. They are well-known conditions in Economics and they state a society is more stable when the distribution of income is somehow more balanced among all the individuals.

We make clear that we analyzed, in a great part of this chapter, weaker versions of the logical properties cited above; we restricted them since T-conorms have an absorbent element, also called annihilator (which is the value 1). T-conorm operators can be seen as merging operators with an implicit veto power: any agent having an outcome with distance value equal to 1 is capable to interpose the decision of the group. This restriction is responsible to weaken the conditions **(HE)**, **(PD)** and **(SP)** and the logical postulate **(IC6)**. When *discrimax* is considered, this restriction is discarded and we evaluate the original versions of logical postulates and conditions.

For the *discrimax* operator, we proved that it satisfies the so-called *discrimax* principle. The *discrimax* principle implies **(HE)**, **(SP)** and **(A)**, but the converse is not true. Therefore, *discrimax* principle is characterized as a weaker principle than *leximax* principle.

We chose in this chapter some of the most representative classes of T-conorms. First, we analyzed the four basic T-conorms: drastic sum, Łukasiewicz, probabilistic sum and maximum T-conorms. The lowest T-conorm *max* satisfies only the egalitarian properties **(Arb)** and **(HP)** (falsifies **(HE)**, **(PD)**, **(SP)** and their variants). The probabilistic sum falsifies **(Arb)** and **(HE-**1), but satisfies **(IC6-**1**)** and **(PD-**1). Łukasiewicz and drastic sum falsify all of them.

When analyzing the parameterized T-conorms, which are basically generalizations of some of the four basic T-conorms, we observed strict T-conorms converging to the maximum tend to satisfy **(HE-**1), **(HP)** and **(Arb)**, as found in the Schweizer-Sklar and Frank T-conorms. In fact, in these cases, we have a close connection between **(HE-**1), **(HP)** and **(Arb)**. The same idea does not follow from nilpotent T-conorms as they do not satisfy **(HE-**1) and**(HP)**. In general, every parameterized T-conorms exposed in this chapter satisfy **(PD-**1**)** in a specific interval (varying for each T-conorm), except the nilpotent T-conorms. For **(IC6-**1), we proved it is satisfied by the class of strict T-conorms, while it is not the case for nilpotent ones. With respect

	(IC6)	(IC6-1)	(Arb)	(PD-1)	(HE-1)	(PD)	(HE)	(HP)
$\Delta_{\mu}^{d,discrimax}$			✓	✓	✓	✓	✓	✓
$\Delta_{\mu}^{d,\oplus_{\mathbf{M}}}$			✓					✓
$\Delta_{\mu}^{d,\oplus_{\mathbf{P}}}$		✓		✓				
$\Delta_{\mu}^{d,\oplus_{\mathbf{L}}}$								
$\Delta_{\mu}^{d,\oplus_{\mathbf{D}}}$								
$\Delta_{\mu}^{d,\oplus_{\lambda}^{\mathbf{SS}}}$		$\lambda \in]-\infty,0]$	$-\infty < \lambda \leq -\lfloor \frac{2n}{3} \rfloor$	$\lambda \in]-\infty,0]$	$-\infty < \lambda \leq -\lfloor \frac{2n}{3} \rfloor$			$-\infty < \lambda \leq -\lfloor \frac{2n}{3} \rfloor$
$\Delta_{\mu}^{d,\oplus_{\lambda}^{\mathbf{F}}}$		$\lambda \in]0,\infty[$	$0 < \lambda \leq 10^{-n}$	$\lambda \in]0,\infty[$	$0 < \lambda \leq 10^{-n}$			$0 < \lambda \leq 10^{-n}$
$\Delta_{\mu}^{d,\oplus_{\lambda}^{\mathbf{Y}}}$			$\lambda \geq \lfloor \frac{2n}{3} \rfloor$					
$\Delta_{\mu}^{d,\oplus_{\lambda}^{\mathbf{SW}}}$								
$\Delta_{\mu}^{d,lexi\oplus_{\mathbf{M}}}$	✓	✓	✓					✓
$\Delta_{\mu}^{d,lexi\oplus_{\mathbf{P}}}$	✓	✓		✓				
$\Delta_{\mu}^{d,lexi\oplus_{\mathbf{L}}}$	✓	✓						
$\Delta_{\mu}^{d,lexi\oplus_{\mathbf{D}}}$	✓	✓						
$\Delta_{\mu}^{d,lexi\oplus_{\lambda}^{\mathbf{SS}}}$	✓	✓	$-\infty < \lambda \leq -\lfloor \frac{2n}{3} \rfloor$	$\lambda \in]-\infty,0]$	$-\infty < \lambda \leq -\lfloor \frac{2n}{3} \rfloor$	$\lambda \in]-\infty,0]$	$-\infty < \lambda \leq -\lfloor \frac{2n}{3} \rfloor$	$-\infty < \lambda \leq -\lfloor \frac{2n}{3} \rfloor$
$\Delta_{\mu}^{d,lexi\oplus_{\lambda}^{\mathbf{F}}}$	✓	✓	$0 < \lambda \leq 10^{-n}$	$\lambda \in]0,\infty[$	$0 < \lambda \leq 10^{-n}$	$\lambda \in]0,\infty[$	$0 < \lambda \leq 10^{-n}$	$0 < \lambda \leq 10^{-n}$
$\Delta_{\mu}^{d,lexi\oplus_{\lambda}^{\mathbf{Y}}}$	✓	✓	$\lambda \geq \lfloor \frac{2n}{3} \rfloor$					
$\Delta_{\mu}^{d,lexi\oplus_{\lambda}^{\mathbf{SW}}}$	✓	✓						

Table 21 – Summary of Logical Properties (4).

to **(Arb)**, Schweizer-Sklar, Frank and Yager T-conorms satisfy it in some specific intervals. Thus, it is possible to have a nilpotent T-conorm as an arbitration quasi-merging operator.

Since every merging operator we proposed is weaker than the *leximax* (it satisfies all of these properties), we proposed to demonstrate what kind of principle these operators satisfy. It is known *leximax* satisfies the Leximax principle, which is equivalent to satisfy the properties **(HE)**, **(SP)** and **(A)**. We weakened the Leximax principle and introduced the Leximax free from 1, which is equivalent to satisfy **(HE-**1**)**, **(SP-**1**)** and **(A)**. We showed that the T-conorms Schweizer-Sklar and Frank satisfy this principle.

In our quest to discover if there exists an operator similar to *leximax*, we extended the T-conorms to LexiT-conorms (its lexicographic version). LexiT-conorm avoids all the problems caused by the annihilator, as the immediate loss of **(IC6)**, **(HE)**, **(PD)** and **(SP)**, but it does not imply that all the LexiT-conorm operators satisfy them all. We showed that all LexiT-conorms satisfy **(IC6)** and for the other properties it follows from the previous achieved result: if a T-conorm operator satisfies **(HE-**1**)**, for example, its lexicographic version satisfies **(HE)**. The same idea is applied to the other properties. Finally, we can gather the results and characterize a hierarchy of the merging operators proposed in this chapter. We have the following results:

Theorem 3.17 *Let E be a belief set, d a distance measure, μ an integrity constraint, $\oplus$ a T-conorm and $\oplus^*$ be a parameterized T-conorm such that $\Delta_{\mu}^{d,\oplus^*}$ satisfies **(LM-**1**)**. We have*

1. $\Delta_{\mu}^{d,leximax}(E) \models \Delta_{\mu}^{d,\oplus^*}(E)$;

2. $\Delta_{\mu}^{d,leximax}(E) \models \Delta_{\mu}^{d,discrimax}(E)$;
3. $\Delta_{\mu}^{d,discrimax}(E) \models \Delta_{\mu}^{d,max}(E)$;
4. $\Delta_{\mu}^{d,leximax}(E) \equiv \Delta_{\mu}^{d,lexi\oplus^*}(E)$;
5. $\Delta_{\mu}^{d,lex\oplus}(E) \models \Delta_{\mu}^{d,\oplus}(E)$.

Proof. See Appendix B.

We can conclude this chapter by showing *leximax* operator is stronger than *max*, *discrimax*, and $\oplus^*$ operators. In addition it is equivalent to *lexi*$\oplus^*$ operator. When we consider an arbitrary T-conorm $\oplus$, it is not always the case that $\Delta_{\mu}^{d,leximax}(E) \models \Delta_{\mu}^{d,\oplus}(E)$. This is explained by the fact that many T-conorms are not subcases of the *maximum*, and consequently, they are not subcases of *leximax*. Table 21 summarizes all the results obtained in this chapter.

4 SUFFICIENTARIAN PROPOSITIONAL BELIEF MERGING

4.1 Contributions of this Chapter

The main contributions of this chapter are listed below:

- We introduce two notions of the theory of Sufficientarianism in the propositional belief merging: weak sufficientarianism and strong sufficientarianism;
- Three operators are considered: *headcount*, *shortfall* and Foster-Greer-Thorbecke indices. We analyze their logical properties and we define a different distributive principle, called Humanitarian Principle, which differs from the original idea of utilitarianism and egalitarianism (with its Leximax Principle);
- We prove that it is possible to define IC merging operators based on sufficientarianism;
- Part of this work has been published in "Sufficientarian Propositional Belief Merging", authors: Henrique Viana and João Alcântara, which was submitted on EUMAS 2016 (VIANA; ALCÂNTARA, 2016b).

4.2 Introduction

Sufficientarianism (FRANKFURT, 1987) is a theory of distributive justice which aims at ensuring each person has an adequate amount of benefits. For instance, we recognize the instrumental importance of having enough sleep, enough money, enough happiness and setting aside enough time. Obviously, this requires a criterion for how much is adequate. Typically, the criterion of adequacy is something like enough to meet basic needs, avoid poverty, or have a minimally decent life, which we refer commonly as the poverty line.

This principle accommodates the concern we normally have for people who are badly off in absolute terms. According to most versions, Sufficiency rejects partially others theories of distributive justice, such as utilitarianism (concerned with the sum total of happiness of a group) and egalitarianism (which promotes equality for all people in a group).

In the area of propositional belief merging, which studies the fusion of independent and equally reliable sources of information expressed in propositional logic, we need to consider some aspects of rationality and distributive justice. Indeed, there are already some belief merging operators based on utilitarianism and egalitarianism (EVERAERE *et al.*, 2014; KONIECZNY; PINO-PÉREZ, 1999; KONIECZNY; PINO-PÉREZ, 2011), but a study of sufficientarian operators in the context of belief merging is still missing.

There are two central views of sufficientarianism (SEGALL, 2014):

Weak Sufficientarianism: Any benefit below poverty line, no matter how small, and no matter to how few individuals, outweighs any benefit above poverty line, no matter how large, and no matter to how many individuals. Below poverty line equally large benefits matter more the worse off the recipient is.

Strong Sufficientarianism: Benefits that lift individuals above some poverty line level matter more than equally large benefits that don't (whether they occur above the poverty line or below it).

In this chapter, we will consider three operators of the theory of weak sufficientarianism in belief merging settings: the *headcount* and the *shortfall* operators and the Foster-Greer-Thorbecke indices. *Headcount* operator simply counts the number of people below the poverty line and aims at minimizing the number of people below this line. On the other hand, *shortfall* operator adds up each person's *shortfall* from the poverty line (or the amount that they need to reach the poverty line). The objective is also to minimize the amount of *shortfall* in a group. The Foster-Greer-Thorbecke indices (FOSTER *et al.*, 1984; FOSTER *et al.*, 2010) are a family of poverty metrics which generalize *headcount* and *shortfall* methods. We will prove these operators have a different rationality from others previously defined for belief merging by showing they satisfy different logical postulates. Consequently, we will extend these operators to their corresponding strong sufficientarianism version.

The chapter is structured as follows. In Section 4.3, we will introduce the *headcount* and *shortfall* operators of sufficiency for belief merging and will explore their respective logical properties. In Section 4.4, we will compare the differences between the weak sufficientarian and the egalitarian reasoning. In Section 4.5, we will introduce the Foster-Greer-Thorbecke index operators and will prove their additional properties. In Section 4.6, we extend these merging operators to turn them IC merging operators. In Section 4.7, we consider the strong sufficientarianism in the propositional belief merging. Finally, in Section 4.8 we will conclude the chapter.

4.3 Weak Sufficientarian Belief Merging

In this section, we propose a characterization of a sufficientarian merging operator, based on the IC merging operators postulates and the syncretic assignment. Besides, we present two different sufficientarian merging operators, as well as additional logical postulates and their relation with each operator.

The idea of weak sufficientarianism is commonly traced back to Harry Frankfurt's doctrine of sufficiency (FRANKFURT, 1987), which inspired and motivated a number of versions of sufficientarianism in recent works (HIROSE, 2014). Frankfurt claims that the doctrine of sufficiency aims at maximizing the number of individuals at or above sufficiency (sometimes denoted as poverty line). We can translate Frankfurt's claim into our framework's point of view as

Definition 4.1 (Frankfurt Sufficientarianism) *(FRANKFURT, 1987)* ***(FS)*** *An outcome ω is at least as good as another ω' if and only if the number of agents at or above sufficiency in ω is at least as large as that in ω'.*

We want to show in the following subsections that the sufficientarian principle can be a plausible tool in belief merging. Although it differs from utilitarian and egalitarian operators (EVERAERE *et al.*, 2014), it still can exhibit some interesting properties. First, we will focus on two different operators: *headcount* and *shortfall* (HIROSE, 2014; VALLENTYNE, 2010).

4.3.1 The headcount Operator

One of the simplest measures is the *headcount* measure, which originally simply counts the number of agents below a poverty line:

The Headcount Claim: we should maximize the number of agents who secure enough.

This principle assesses outcomes solely in terms of the number of agents who have secured enough in each outcome. Benefits to those who do not reach the sufficiency do not improve the assessment of the outcome. As for the framework of belief merging with a distance measure, we will consider the distance between an outcome and a belief base as the measure of sufficiency.

Ω	$d_H(\omega,K_1)$	$d_H(\omega,K_2)$	$d_H(\omega,K_3)$	$hc(\omega,d_H,E,1)$
ω_1	1	1	3	1
ω_2	0	0	2	1
ω_3	2	0	2	2
ω_4	1	1	1	**0**
ω_5	0	2	2	2
ω_6	0	1	1	**0**
ω_7	1	1	1	**0**
ω_8	1	2	0	1

Table 22 – *Headcount* of Hamming distances between Ω and E for $s = 1$.

Definition 4.2 (*headcount* Operator) *Let $E = \{K_1, \ldots, K_n\}$ be a belief set, d a distance measure, ω an outcome and $s \geq 0$ a threshold. We define the number of belief bases in E above s as $hc(\omega,d,E,s) = \#(\{K_i \in E \mid d(\omega,K_i) > s\})$, where $\#(A)$ is the cardinal of the set A. Then we have the following pre-order: $\omega_i \leq_E^{d,hc_s} \omega_j$ iff $hc(\omega_i,d,E,s) \leq hc(\omega_j,d,E,s)$. The merging operator Δ_μ^{d,hc_s} is defined by $\Delta_\mu^{d,hc_s}(E) = min(mod(\mu), \leq_E^{d,hc_s})$.*

Keep in mind we are counting the agents above the threshold s (our poverty line), since we are working with distance measures and the welfare of an agent is calculated as how close its distance measure is from 0. Note that when $s = 0$, we have $\Delta_\mu^{d,hc_0} \equiv \Delta_\mu^{d_D,sum}$, that is, the *headcount* merging operator is equivalent to the distance-based merging with the drastic distance and the *sum* operator.

Example 4.1 *The results of headcount merging operator w.r.t. Hamming distance and $s = 1$ for Example 2.3 are found in Table 22. The resulting pre-order $\leq_E^{d_H,hc_1}$ is $\{\omega_4, \omega_6, \omega_7\} \leq_E^{d_H,hc_1}$ $\{\omega_1, \omega_2, \omega_8\} \leq_E^{d_H,hc_1} \{\omega_3, \omega_5\}$.*

In this example, for $s = 1$, the merging operator is counting the number of agents in which the Hamming distance value is greater than 1 in a belief base. The outcomes ω_4, ω_6 and ω_7 are the result of the merging (when $\mu = \top$). Note that the sufficientarian principle is only worried if the agents are below or equal the threshold s and not about their specific values (ω_4, ω_6 and ω_7 are equivalent, independently of their values). To begin with our analysis involving logical postulates of this weak sufficientarian operator, first we will discuss about the basic IC postulates.

Theorem 4.1 Δ_μ^{d,hc_s} *satisfies **(IC0)**-**(IC1)**, **(IC3)**-**(IC8)**. The postulate **(IC2)** is not satisfied in the general case. Additionally, Δ_μ^{d,hc_s} satisfies both **(Arb)** and **(Maj)**.*

Proof. See Appendix C.

When $s = 0$, the postulate **(IC2)** is satisfied (i.e., it is equivalent to $\Delta_{\mu}^{d_D,sum}$). The reason why **(IC2)** is not always true comes from the fact that even if an outcome is not a consensus between agents, it can be a choice of the merging (e.g., outcome ω_4 in Example 4.1).

Let us take a closer look on postulate **(IC2)**: If $\bigwedge E$ is consistent with μ, then $\Delta_{\mu}(E) \equiv \bigwedge E \wedge \mu$. It states the result of belief merging needs to be complete and sufficient with the consensus among agents (if it exists). This postulate corresponds to syncretic assignments 1 and 2 (KONIECZNY; PINO-PÉREZ, 2002a): 1. If $\omega \models \bigwedge E$ and $\omega' \models \bigwedge E$, then $\omega \approx_E \omega'$; 2. If $\omega \models \bigwedge E$ and $\omega' \not\models \bigwedge E$, then $\omega <_E \omega'$. We argue the sufficientarian principle is weaker than **(IC2)**, since the result of belief merging needs to be only sufficient w.r.t. the consensus among agents (if it exists). Formally, we have

(IC2'): If $\bigwedge E$ is consistent with μ, then $\bigwedge E \wedge \mu \models \Delta_{\mu}(E)$.

In other words, there are some choices of the merging that are not necessarily the consensus of the group. The corresponding syncretic assignments for **(IC2')** are

1. If $\omega \models \bigwedge E$ and $\omega' \models \bigwedge E$, then $\omega \approx_E \omega'$; and

2'. If $\omega \models \bigwedge E$ and $\omega' \not\models \bigwedge E$, then $\omega \leq_E \omega'$.

Proposition 4.1 Δ_{μ}^{d,hc_s} *satisfies **(IC2')**.*

Interestingly, the *headcount* operator satisfies both **(Maj)** and **(Arb)**. It is not a new result, since it was already proved in (KONIECZNY; PINO-PÉREZ, 2002b) that the operator $\Delta_{\mu}^{d_D,sum}$ and the family of full sense operators Δ_{μ}^{d,sum^n} satisfy also both **(Maj)** and **(Arb)**. Based on these results, we can say that each agent is relevant for the merging and the opinion of the majority is the priority. The arbitration property guarantees two agents will have a more consensual behavior in their decisions.

To finish this first part of the chapter, we will bring new logical postulates for this sufficientarian operator, which come from the literature of liberal egalitarianism (ALCANTUD, 2011; CAPPELEN; TUNGODDEN, 2006; LOMBARDI *et al.*, 2013), a theory of justice which seeks to combine values of equality, personal freedom and personal responsibility. The first postulate we will discuss is a weaker version of the Harm Principle (LOMBARDI *et al.*, 2013):

Definition 4.3 (Weak Harm Principle Condition) *(LOMBARDI et al., 2013) **(WHP)** Let $E = \{K_1, \ldots, K_n\}$ be a belief set, a distance measure d, a merging operator op and $\omega_1 <_E^{d,op} \omega_2$.*

For all $\omega_1, \omega_2, \omega_1', \omega_2' \in \Omega$, *consider* ω_1', ω_2' *such that* $\exists i \in \{1, \dots, n\}$, $d(\omega_1, K_i) < d(\omega_1', K_i)$, $d(\omega_2, K_i) < d(\omega_2', K_i)$ *and* $\forall j \neq i$ $d(\omega_1, K_j) = d(\omega_1', K_j)$, $d(\omega_2, K_j) = d(\omega_2', K_j)$. *If* $d(\omega_1', K_i) < d(\omega_2', K_i)$ *then* $\omega_1' \leq_E^{d,op} \omega_2'$.

The Weak Harm Principle assigns a veto power to agents in situations in which they suffer a harm and no other agent is affected. This veto power is weak as it only applies to certain welfare configuration (individual preferences after the satisfaction loss must coincide with group's initial preferences) and, crucially, the agent cannot force group's preferences to coincide with her own. The counterpart of the Harm Principle, where a gain in the agent *i*'s distance value is considered, it is called Individual Benefit Principle (ALCANTUD, 2011) and defined as

Definition 4.4 (Individual Benefit Principle Condition) *(LOMBARDI et al., 2013)* ***(IBP)*** *Let* $E = \{K_1, \dots, K_n\}$ *be a belief set, a distance measure d, a merging operator op and* $\omega_1 <_E^{d,op} \omega_2$. *For all* $\omega_1, \omega_2, \omega_1', \omega_2' \in \Omega$, *consider* ω_1', ω_2' *such that* $\exists i \in \{1, \dots, n\}$, $d(\omega_1', K_i) < d(\omega_1, K_i)$, $d(\omega_2', K_i) < d(\omega_2, K_i)$ *and* $\forall j \neq i$ $d(\omega_1, K_j) = d(\omega_1', K_j)$, $d(\omega_2, K_j) = d(\omega_2', K_j)$. *If* $d(\omega_1', K_i) < d(\omega_2', K_i)$ *then* $\omega_1' <_E^{d,op} \omega_2'$.

The intuition is the same of Harm Principle, but now there is a decrease in agent *i*'s distance value in ω_1' and ω_2'. This condition can be weakened too:

Definition 4.5 (Weak Individual Benefit Principle Condition) ***(WIBP)*** *Let* $E = \{K_1, \dots, K_n\}$ *be a belief set, a distance measure d, a merging operator op and* $\omega_1 <_E^{d,op} \omega_2$. *For all* ω_1, ω_2, $\omega_1', \omega_2' \in \Omega$, *consider* ω_1', ω_2' *such that* $\exists i \in \{1, \dots, n\}$, $d(\omega_1', K_i) < d(\omega_1, K_i)$, $d(\omega_2', K_i) < d(\omega_2, K_i)$ *and* $\forall j \neq i$ $d(\omega_1, K_j) = d(\omega_1', K_j)$, $d(\omega_2, K_j) = d(\omega_2', K_j)$. *If* $d(\omega_1', K_i) < d(\omega_2', K_i)$ *then* $\omega_1' \leq_E^{d,op} \omega_2'$.

Now we can relate *headcount* merging operator with the conditions presented above:

Theorem 4.2 Δ_μ^{d,hc_s} *satisfies* ***(WHP)*** *and* ***(WIBP)***. *The conditions* ***(HP)*** *and* ***(IBP)*** *are not satisfied in the general case.*

Proof. See Appendix C.

We highlight that the *sum* merging operator does not satisfy any of these properties. The *max* and *leximax* operators satisfy all four postulates. In this sense, the *headcount* operator has an intermediate behavior for these postulates when compared with the basic merging operators *sum* and *max*/*leximax*.

Ω	$d_H(\omega, K_1)$	$d_H(\omega, K_2)$	$d_H(\omega, K_3)$	$sh(\omega, d_H, E, 1)$
ω_1	1	1	3	2
ω_2	0	0	2	1
ω_3	2	0	2	2
ω_4	1	1	1	**0**
ω_5	0	2	2	2
ω_6	0	1	1	**0**
ω_7	1	1	1	**0**
ω_8	1	2	0	1

Table 23 – *Shortfall* of Hamming distances between Ω and E for $s = 1$.

4.3.2 The shortfall Operator

Let us consider another measure of aggregation. The *shortfall* measure simply adds up each agent's total gap from the distance measure (where an agent's *shortfall* is zero if her distance value is at or below s). The total *shortfall* operator simply adds up the *shortfall* from s across agents above s, and takes the unweighted sum to be the measure of the disvalue of the group (HIROSE, 2014). Differently from *headcount* operator, which tries to minimize the number of agents above s, the *shortfall* is concerned with the total amount of deficit of the agents above s, and aims at minimizing it.

Definition 4.6 (*shortfall* Operator) *Let $E = \{K_1, \ldots, K_n\}$ be a belief set, d a distance measure, ω an outcome and $s \geq 0$ a threshold. We define the shortfall of belief bases in E above s in ω as $sh(\omega, d, E, s) = \sum_{d(\omega, K_i) > s} d(\omega, K_i) - s$. Then we have the following pre-order: $\omega_i \leq_E^{d, sh_s} \omega_j$ iff $sh(\omega_i, d, E, s) \leq sh(\omega_j, d, E, s)$. The merging operator Δ_μ^{d, sh_s} is defined by $\Delta_\mu^{d, sh_s}(E) = min(mod(\mu), \leq_E^{d, sh_s})$.*

We can see this approach is prioritarian for those satisfaction values above s. The relative overall goodness of an outcome is judged on the basis of a sum of different agent's well-being where it is determined by the disvalue of an agent's *shortfall* from s.

Example 4.2 *The results of shortfall merging operator w.r.t. Hamming distance and $s = 1$ for Example 2.3 are in Table 23. The resulting pre-order $\leq_E^{d_H, sh_1}$ is $\{\omega_4, \omega_6, \omega_7\} \leq_E^{d_H, sh_1} \{\omega_2, \omega_8\} \leq_E^{d_H, sh_1} \{\omega_1, \omega_3, \omega_5\}$.*

Shortfall operator is influenced by variations of the distance values. In the above example, we can see this change with respect to outcome ω_1. The total *shortfall* of ω_1 is equal to 2 and it is equivalent to ω_3 and ω_5. Regarding the *headcount* operator in Example 4.1, the outcome ω_1 is more preferred than ω_3 and ω_5, because only one agent has the distance value above $s = 1$, against two agents for ω_3 and ω_5. With respect to logical postulates some alterations also occur:

Theorem 4.3 Δ_{μ}^{d,sh_s} *satisfies* ***(IC0), (IC1), (IC2'), (IC3)-(IC8)*** *and* ***(Maj)****. The postulates* ***(IC2), (Arb), (HP), (WHP), (IBP)*** *and* ***(WIBP)*** *are not satisfied in the general case.*

Proof. See Appendix C.

The *shortfall* merging operator is a majority sufficientarian operator. The difference between *headcount* and *shortfall* operators appears when some egalitarian and libertarian conditions are considered, as in the loss of logical postulates **(Arb)**, **(WHP)** and **(WIBP)**.

4.4 A Humanitarian Principle

In this section, we will present some logical postulates which characterize the behavior of sufficientarian merging operators. We consider them as representative of a humanitarian principle and, in light of this principle, we show *shortfall* operator is more just than *headcount* operator. We will include some positions in the general category of egalitarian perspectives of distributive justice presented in (TUNGODDEN, 2000). Some families of egalitarian properties were defined in this work and we will use a particular one, modified to fit into our framework of belief merging.

Definition 4.7 (Weak Povertymax for s**)** *(WPM-s) Let $E = \{K_1, \ldots, K_n\}$ be a belief set, d be a distance measure and $s \geq 0$. For all ω, ω', if (1) there exists a $k \leq n$ such that $d(\omega, K_k) < d(\omega', K_k)$ and $s < d(\omega', K_k)$; (2) every position i that $s < d(\omega, K_i)$ implies $d(\omega, K_i) \leq d(\omega', K_i)$, then $\omega <_E^{d,op_s} \omega'$.*

Weak Povertymax differs from the *leximax* principle by giving priority to those agents above the threshold s, while the leximax gives absolute priority to the worst off agent (also referred as equity promotion (TUNGODDEN, 2000)). We argue that **(WPM-s)** can be seen as a humanitarian condition, since it tries to favor a group of agents instead of prioritizing a unique agent. The agents below the threshold s are not considered essential for the group's choice. By way of illustration, **(WPM-s)** implies the loss of a single agent satisfaction value s outweighs any gain of any number of agents above s. Now, consider the following new condition:

Definition 4.8 (Weak Absolute Priority of those Above s**)** *(**WAPA-**s) Let $E = \{K_1, \ldots, K_n\}$ be a belief set, d be a distance measure and $s \geq 0$. For all ω, ω', if there exist j, k such that (1) $d(\omega', K_j) < d(\omega, K_j) \leq s$; (2) $s \leq d(\omega, K_k) < d(\omega', K_k)$; (3) for $i \neq j, k$, $d(\omega, K_i) = d(\omega', K_i)$, then $\omega \leq_E^{d,op_s} \omega'$.*

With the addition of Strong Pareto **(SP)** and Anonymity **(A)** (TUNGODDEN, 2000), we can achieve an important result:

Theorem 4.4 *(TUNGODDEN, 2000) If a Syncretic Assignment satisfies **(WAPA-**s**)**, **(SP)** and **(A)**, then it satisfies **(WPM-**s**)**.*

By taking into account the humanitarian concern, Povertymax shows there are alternatives to the Leximax Principle of justice. Such an egalitarian position deals with both the claim of equality promotion and the humanitarian perspective. Hence, we have the following results for the *headcount* and *shortfall* operators.

Theorem 4.5 *Δ_μ^{d,hc_s} satisfies **(WAPA-**s**)** and **(A)**, but **(SP)** and **(WPM-**s**)** are not satisfied in general. Δ_μ^{d,sh_s} satisfies **(WAPA-**s**)**, **(A)** and **(WPM-**s**)**, but **(SP)** is not satisfied in general.*

Proof. See Appendix C.

Note that **(SP)** is not satisfied because s is not considered in its definition. If we imagine a situation where all the distance values of the outcomes are below s, we have these outcomes are equivalent, independently of their values. Moreover, an operator may satisfy **(WPM-**s**)**, even if it does not satisfy **(WAPA-**s**)**, **(SP)** and **(A)** (Theorem 4.4 is not an "if and only if" condition). **(WPM-**s**)** is not satisfied by Δ_μ^{d,hc_s} since the conclusion $\omega <_E^{d,op_s} \omega'$ is too strong for this operator. However, we may consider a weaker version of **(WPM-**s**)**.

Definition 4.9 (Weaker Povertymax for s**)** *(**W_1PM-**s) Let $E = \{K_1, \ldots, K_n\}$ be a belief set, d be a distance measure and $s \geq 0$. For all $\omega, \omega' \in \Omega$, if (1) there exists a $k \leq n$ such that $d(\omega, K_k) < d(\omega', K_k)$ and $s < d(\omega', K_k)$; (2) every position i that $s < d(\omega, K_i)$ implies $d(\omega, K_i) \leq d(\omega', K_i)$, then $\omega \leq_E^{d,op_s} \omega'$.*

Besides, we can also consider the condition of Weak Pareto, instead of Strong Pareto:

Definition 4.10 (Weak Pareto) *(WP) Let $E = \{K_1, \ldots, K_n\}$ be a belief set and d a distance measure. For all $\omega, \omega' \in \Omega$, if $\forall i \in \{1, \ldots, n\}$ $d(\omega, K_i) \leq d(\omega', K_i)$, then $\omega \leq_E^{d,op_s} \omega'$.*

Then the following result is obtained:

Corollary 4.1 *If a pre-order satisfies* **(WAPA-***s***)**, **(WP)** *and* **(A)**, *it satisfies* **(W$_1$PM-***s***)**.

It follows from Theorem 4.4. Therefore, along with Theorem 4.5 and Corollary 4.1, Δ_{μ}^{d,hc_s} satisfies these properties:

Corollary 4.2 Δ_{μ}^{d,hc_s} *satisfies* **(WAPA-***s***)**, **(WP)**, **(A)** *and* **(W$_1$PM-***s***)**.

Headcount satisfies **(W$_1$PM-***s***)** because when $d(\omega, K_k) > s$, the conclusion is that $\omega \approx_E^{d,op_s} \omega'$; otherwise, it concludes that $\omega <_E^{d,op_s} \omega'$. Weak Pareto is achieved because the distance measures below s are equivalent w.r.t to the pre-order $\leq$ and do not bring any impact to the conclusion of the property (note that *shortfall* also satisfies Weak Pareto). For now, we established a weak Humanitarian Principle for sufficientarian merging operators. In the next sections we will strengthen this principle.

4.5 Generalizing *headcount* and *shortfall* Operators

The *headcount* and *shortfall* operators present a basic kind of humanitarian principle. In this section, we will extend the Weak Povertymax principle by introducing a more egalitarian feature to it. We will achieve this by employing a Foster-Greer-Thorbecke index operator.

The Foster-Greer-Thorbecke indices (FOSTER *et al.*, 1984; FOSTER *et al.*, 2010) are a family of poverty metrics generalizing *headcount* and *shortfall* operators. It is a class of poverty measures having the formula $FGT^{\alpha} = \frac{1}{n}\sum_{i=1}^{m}\left(\frac{s-x_i}{s}\right)^{\alpha}$, where s is the poverty line, x_i is the i-th lowest income (or other standard of living indicator), n is the total population, m is the number of agents who are below the poverty line, and $\alpha \geq 0$ is a "poverty aversion" parameter.

The FGT class is based on the normalized gap $g_i = \frac{(s-x_i)}{s}$ of a poor agent i, which is the income shortfall expressed as a share of the poverty line. Viewing $g_i^{\alpha} = \frac{(s-x_i)^{\alpha}}{s}$ as the measure of individual poverty for a poor agent (that one below the poverty line), and 0 as the respective measure for non-poor agents, FGT^{α} is the average poverty in the given population.

The case $\alpha = 0$ yields a distribution of individual poverty levels in which each poor agent has poverty level 1; the average across the entire population is simply the *headcount* ratio, i.e., $FGT^0 = \frac{hc}{n}$ (observe it differs from *headcount* by applying the ratio of the group of agents, although that does not influence their logical properties in the belief merging). The case $\alpha = 1$ employs the normalized gap g_i of a poor person i as a poor agent's poverty level, thereby differentiating among the poor; the average becomes the poverty gap measure $FGT^1 = \frac{sh}{n}$ (as in *headcount*, this representation is equivalent to a *shortfall* operator in terms of logical properties). The case $\alpha = 2$ squares the normalized gap and thus weights the gaps by the gaps; this yields to the squared gap measure $FGT^2 = \frac{1}{n}\sum_{i=1}^{m}\left(\frac{s-x_i}{s}\right)^2$. As α tends to infinity, the condition of the poorest is all that matters (i.e., they have absolute priority).

When we are considering the distance based merging, we need to make some adjusts in the definition of FGT (remind that we consider those agents above some threshold s when we deal with distance measures instead of those agents below a poverty line).

Definition 4.11 (FGT^α Operator) *Let $E = \{K_1, \ldots, K_n\}$ be a belief set, d a distance measure, ω an outcome and $s \geq 0$ a threshold. We define the FGT^α index of belief bases in E below s in an outcome ω as $FGT(\alpha, \omega, d, E, s) = \frac{1}{n}\sum_{d(\omega,K_i)>s}\left(\frac{d(\omega,K_i)-s}{s}\right)^\alpha$. Consequently, we have the following pre-order: $\omega_i \leq_E^{d,FGT_s^\alpha} \omega_j$ iff $FGT(\alpha, \omega_i, d, E, s) \leq FGT(\alpha, \omega_j, d, E, s)$. The operator $\Delta_\mu^{d,FGT_s^\alpha}$ is defined by $\Delta_\mu^{d,FGT_s^\alpha}(E) = min(mod(\mu), \leq_E^{d,FGT_s^\alpha})$.*

In terms of the logical properties already presented in this chapter, we can easily prove FGT^α operator, for $\alpha \geq 2$, is equivalent to the logical properties of *shortfall*. Although FGT^α and *shortfall* differ a little in their definition (*shortfall* has a tie condition in case of equal shortfalls), this does not change their logical properties.

We want to emphasize FGT^α can satisfy a stronger condition than the two previous described operators. Let us define now a stronger restriction to the **(wAPA-*s*)** property:

Definition 4.12 (weak Absolute Priority of those Above s) *(**wAPA-**s) Let $E = \{K_1, \ldots, K_n\}$ be a belief set, d be a distance measure and $s \geq 0$. For all ω, ω', if there exist j,k such that: (1) $d(\omega', K_j) < d(\omega, K_j)$, and $s < d(\omega, K_j)$; (2) $s \leq d(\omega, K_k) < d(\omega', K_k)$, and $d(\omega, K_j) \leq d(\omega, K_k)$; (3) for $i \neq j,k$, $d(\omega, K_i) = d(\omega', K_i)$, then $\omega <_E^{d,op_s} \omega'$.*

Now we have that the distance measure of an outcome ω is above s in both positions j and k. Even in this situation, ω is still more preferred than ω'. This property suggests that we will give absolute preference to the worst off agents. It is similar to the Hammond Equity condition, but only considering the case where the agents are above s.

Theorem 4.6 *$\Delta_\mu^{d,FGT_s^\alpha}$ satisfies (**wAPA-**s), when $\alpha \geq \frac{n}{2}$, such that n is the number of propositional variables in the belief set.*

Proof. See Appendix C.

As (TUNGODDEN, 2000), we can make an analogy between **(wAPA-*s*)** and Leximax Principle, considering the cases where the agents are above s:

Definition 4.13 (Leximax Above s) *(**LMA-**s) Let $E = \{K_1, \ldots, K_n\}$ be a belief set. For each outcome ω, we build the list $(d_1^\omega, \ldots, d_n^\omega)$ of distances between this outcome and the n belief bases in E, i.e., $d_i^\omega = d(\omega, K_i)$. Let $L_\omega^{d,E}$ be the list obtained from $(d_1^\omega, \ldots, d_n^\omega)$ by sorting it in descending order. For all $\omega, \omega' \in \Omega$, (1) if there exists a position $k \leq n$ such that $d_k^\omega < d_k^{\omega'}$; (2) $s < d_k^\omega$; and (3) for every $j < k$, $d_j^{\omega'} = d_j^\omega$, then $\omega <_E \omega'$ (ω is more preferred than ω'). Otherwise, $\omega \approx_E \omega'$.*

(LMA-*s*) is silent in conflicts below s, but gives absolute priority to the worse off in any other conflict.

Theorem 4.7 *Given some $s > 0$, if a Syncretic Assignment satisfies **(WAPA-**s**)**, **(wAPA-**s**)**, **(SP)** and **(A)**, then it satisfies **(LMA-**s**)**.*

Proof. See Appendix C.

This result guarantees the Syncretic Assignment is equivalent to the Leximax Principle when only utility level above s is considered. With this theorem we can observe that can exist sufficientarian operators with a behavior similar to egalitarian operators.

Proposition 4.2 $\Delta_\mu^{d,FGT_s^\alpha}$ *does not satisfy **(LMA-**s**)** and **(SP)** in general.*

Unfortunately, **(LMA-*s*)** is not satisfied owing to the falsification of **(SP)**. For this operator, **(SP)** is a required condition to satisfy **(LMA-*s*)**. In fact, we saw in this section all sufficientarian operators do not satisfy **(IC2)** and **(SP)**. A question that arises is in what conditions we can verify these properties. In other words, if there is a sufficientarian operator that is also a merging operator (satisfy all the basic logical postulates). That questions will be tackled in the next section.

4.6 Sufficientarian IC Merging Operators

In the previous sections, it was showed a first access to sufficientarian operators with propositional belief merging. However, none of them is, in fact, an IC merging operator, since **(IC2)** is not implied by them. In order to overcome such weakness, we give now a further step by presenting a class of belief merging operators: axiological sufficientarian merging operators. The axiological sufficientarian belief merging operators have been conceived to satisfy all the eight IC postulates by combining sufficientarianism with utilitarianism.

The doctrine of sufficiency has established itself as a distinctive position among many theories of distributive justice. It has attracted many proponents and they have proposed several variants of the doctrine of sufficiency that retains its general spirit (CHUNG, 2017). For instance, in (CASAL, 2007), it was pointed out that the doctrine of sufficiency (or sufficientarianism) is committed to both the "positive thesis" and the "negative thesis":

- **The Positive Thesis:** It is morally important for people to have enough resources.
- **The Negative Thesis:** Once everybody has enough resources, whether somebody has more or less than others has absolutely no moral significance.

In general terms, proponents and opponents of sufficientarianism find the positive thesis plausible. So, the usual target of criticisms is focused on sufficientarianism's commitment to the negative thesis. According to the negative thesis, the moral insignificance of some agents having more than others is limited to situations in which everybody has enough resources. This means that there is room for sufficientarians to give priority to those who are below such threshold if we discard the negative thesis condition.

There are different ways to approach the violation of the negative thesis. One of them is to depict the sufficientarianism as an axiological principle, i.e., a criterion for ranking the outcomes in terms of goodness. In (HIROSE, 2014), it was proposed the Axiological Sufficientarianism, which can be translated as a hybrid doctrine of sufficiency + utilitarianism:

- **The Positive Thesis:** It is morally important for people to have enough resources.
- **The Utilitarian Negative Thesis:** Once everybody has enough resources, whether somebody has more resources than others has moral significance.

It is possible to have a merging operator and to be sufficientarian at the same time. We will change *headcount*, *shortfall* and FGT index in the following way:

Definition 4.14 (Axiological Sufficientarian Operators) *Let $E = \{K_1, \ldots, K_n\}$ be a belief set, d a distance measure, ω an outcome and $s \geq 0$ a threshold.*

1. *We define the number of belief bases in E above s as $hc(\omega, d, E, s) = \#(\{K_i \in E \mid d(\omega, K_i) > s\})$, where $\#(A)$ is the cardinal of the set A. Then we have the following pre-order: $\omega_i \leq_E^{d,ahc_s} \omega_j$ iff (if $hc(\omega_i, d, E, s) = hc(\omega_j, d, E, s)$ then $d_{sum}(\omega_i, E) \leq d_{sum}(\omega_j, E)$; else*

$hc(\omega_i,d,E,s) \leq hc(\omega_j,d,E,s))$. The merging operator Δ_μ^{d,ahc_s} is defined by $\Delta_\mu^{d,ahc_s}(E) = min(mod(\mu), \leq_E^{d,ahc_s})$.

2. *We define the shortfall of belief bases in E above s in ω as $sh(\omega,d,E,s) = \sum_{d(\omega,K_i)>s} d(\omega,K_i) - s$. Then we have the following pre-order: $\omega_i \leq_E^{d,ash_s} \omega_j$ iff (if $sh(\omega_i,d,E,s) = sh(\omega_j,d,E,s)$ then $d_{sum}(\omega_i,E) \leq d_{sum}(\omega_j,E)$; else $sh(\omega_i,d,E,s) \leq sh(\omega_j,d,E,s))$. The merging operator Δ_μ^{d,ash_s} is defined by $\Delta_\mu^{d,ash_s}(E) = min(mod(\mu), \leq_E^{d,ash_s})$.*
3. *We define the FGT^α index of belief bases in E below s in an outcome ω as: $FGT(\alpha,\omega,d,E,s) = \frac{1}{n}\sum_{d(\omega,K_i)>s}\left(\frac{d(\omega,K_i)-s}{s}\right)^\alpha$. Consequently, we have the following pre-order: $\omega_i \leq_E^{d,aFGT_s^\alpha} \omega_j$ iff (if $FGT(\alpha,\omega_i,d,E,s) = FGT(\alpha,\omega_j,d,E,s)$ then $d_{sum}(\omega_i,E) \leq d_{sum}(\omega_j,E)$; else $FGT(\alpha,\omega_i,d,E,s) \leq FGT(\alpha,\omega_j,d,E,s))$. The operator $\Delta_\mu^{d,aFGT_s^\alpha}$ is defined by $\Delta_\mu^{d,aFGT_s^\alpha}(E) = min(mod(\mu), \leq_E^{d,aFGT_s^\alpha})$.*

When the outcomes are not equally sufficient, these operators behave as the original *headcount*, *shortfall* and FGT operators. Otherwise, they behave as the *sum* operator among the distance measures. These changes are enough to achieve the satisfaction of the logical postulate **(IC2)** and still preserve the remaining basic logical postulates:

Theorem 4.8 *Δ_μ^{d,ahc_s}, Δ_μ^{d,ash_s} and $\Delta_\mu^{d,aFGT_s^\alpha}$ satisfy **(IC0)-(IC8)** and **(Maj)**. Additionally, Δ_μ^{d,ahc_s} satisfies **(Arb)**.*

Proof. See Appendix C.

Now, these operators can be called merging operators, since they satisfy **(ICO)-(IC8)**. Δ_μ^{d,ahc_s} is a majority/arbitration merging operator; Δ_μ^{d,ash_s} and $\Delta_\mu^{d,aFGT_s^\alpha}$ are majority merging operators. However, for Δ_μ^{d,ahc_s}, the properties **(WHP)** and **(WIBP)** are not satisfied anymore (*headcount* operator satisfies them). An axiological operator is also robust enough to satisfy Strong Pareto for both *headcount* and *shortfall*.

Theorem 4.9 *Δ_μ^{d,ahc_s}, Δ_μ^{d,ash_s} and $\Delta_\mu^{d,aFGT_s^\alpha}$ satisfy **(SP)**. Additionally, Δ_μ^{d,ahc_s} satisfies **(WAPA-s)**.*

Proof. See Appendix C.

Δ_μ^{d,ahc_s} and Δ_μ^{d,ash_s} preserve the results from Δ_μ^{d,hc_s} and Δ_μ^{d,sh_s}, respectively, while they additionally satisfy **(SP)**. For $\Delta_\mu^{d,aFGT_s^\alpha}$, the situation is even better: it satisfies now **(SP)** and **(LMA-*s*)** when α is restricted to $\alpha \geq \frac{n}{2}$, such that *n* is the number of propositional variables in the belief set.

4.7 Strong Sufficientarian Belief Merging

In this section, we shed light on the role strong sufficientarianism can play on aggregation. Recalling (SEGALL, 2014), there are two central views of sufficientarianism:

- **Weak Sufficientarianism:** Any benefit above s, no matter how small, and no matter to how few agents, outweighs any benefit below s, no matter how large, and no matter to how many agents. Above s, equally large benefits matter more the worse off the agent is;
- **Strong Sufficientarianism:** Benefits that lift agents below some threshold level s matter more than equally large benefits that do not, whether they occur above or below s.

In previous sections, weak sufficientarian merging operators were described. Now we will show a first approach to strong sufficientarian merging operators. For this, we will define strong versions of the *headcount* and *shortfall* operators:

Definition 4.15 (*Strong headcount* Operator) *Let $E = \{K_1, \dots, K_n\}$ be a belief set, d a distance measure, ω an outcome and $s \geq 0$ a threshold. We define the number of belief bases in E across s w.r.t. ω and ω' as $shc(\omega, \omega', d, E, s) = \#(\{K_i \in E \mid d(\omega, K_i) \geq s > d(\omega', K_i)\})$, where $\#(A)$ is the cardinal of the set A. Then we have the following pre-order: $\omega_i \leq_E^{d,shc_s} \omega_j$ iff $shc(\omega_i, \omega_j, d, E, s) \leq shc(\omega_j, \omega_i, d, E, s)$. The merging operator Δ_μ^{d,shc_s} is defined by $\Delta_\mu^{d,shc_s}(E) = min(mod(\mu), \leq_E^{d,shc_s})$.*

This definition comes directly from strong sufficientarianism statement: an outcome ω is better than ω' if the number of agents above s in ω is less than in ω'. Furthermore, it is silent for the case where the number of agents above s are equal in the outcomes. Note that, although different definitions, *headcount* and strong *headcount* operators produce exactly the same result in terms of merging operator and logical properties (they are in fact the same operator).

Proposition 4.3 $\Delta_\mu^{d,shc_s} \equiv \Delta_\mu^{d,hc_s}$.

Now, let us analyze the definition of the strong *shortfall* operator:

Definition 4.16 (Strong *shortfall* Operator) *Let $E = \{K_1, \dots, K_n\}$ be a belief set, d a distance measure, ω an outcome and $s \geq 0$ a threshold. We define the shortfall of belief bases in E across s w.r.t. ω and ω' in ω as $ssh(\omega, \omega', d, E, s) = \sum_{d(\omega,K_i) \geq s > d(\omega',K_i)} d(\omega, K_i) - s$. Then we have the following pre-order: $\omega_i \leq_E^{d,ssh_s} \omega_j$ iff $ssh(\omega_i, \omega_j, d, E, s) \leq ssh(\omega_j, \omega_i, d, E, s)$. The merging operator Δ_μ^{d,ssh_s} is defined by $\Delta_\mu^{d,ssh_s}(E) = min(mod(\mu), \leq_E^{d,ssh_s})$.*

The same scenario that occurred for the strong *headcount* will not occur with the Strong *shortfall* operator: the strong *shortfall* justice relation is not equivalent to the original *shortfall* justice relation (although they satisfy tha same IC logical postulates). Now, considering the humanitarian principle, we can state the strong version of **(WPM-s)** below:

Definition 4.17 (Strong Povertymax for s) *(SPM-s) Let $E = \{K_1, \ldots, K_n\}$ be a belief set, d be a distance measure and $s \geq 0$. For all $\omega, \omega' \in \Omega$, if (1) there exists a $k \leq n$ such that $d(\omega, K_k) < d(\omega', K_k)$ and $s < d(\omega', K_k)$; (2) every position i that $d(\omega', K_i) < d(\omega, K_i)$ implies $s < d(\omega', K_i)$ or $d(\omega, K_i) \leq s$, then $\omega \leq_E^{d,op_s} \omega'$.*

As in **(WPM-s)**, this principle is followed by some other properties. We will describe them below:

Definition 4.18 (Indifference for those Above or Below s) *(IAB-s) Let $E = \{K_1, \ldots, K_n\}$ be a belief set, d be a distance measure and $s \geq 0$. For all $\omega, \omega' \in \Omega$, if there exist j,k such that (1) $d(\omega', K_j) < d(\omega, K_j) < s$; (2) $s \leq d(\omega, K_k) < d(\omega', K_k)$; (3) for $i \neq j,k$, $d(\omega, K_i) = d(\omega', K_i)$, then $\omega \approx_E^{d,op_s} \omega'$.*

The addition of **(IAB-s)** comes from the definition of strong sufficientarianism: benefits that lift agents below some threshold level s matter more than equally large benefits that do not, whether they occur above or below s. For the strong operators we introduced, we considered null any benefit occurring above or below s.

Proposition 4.4 *If a pre-order satisfies **(WAPA-s)**, **(IAB-s)**, **(WP)** and **(A)**, then it satisfies **(SPM-s)**.*

We have the following results for the strong *headcount*.

Theorem 4.10 *Δ_μ^{d,shc_s} satisfies **(WAPA-s)**, **(IAB-s)**, **(WP)**, **(A)** and **(SPM-s)**.*

Proof. See Appendix C.

This proposition shows *headcount* operator is a weaker and strong sufficientarian operator. The same can not be said about strong *shortfall* operators:

Theorem 4.11 *Δ_μ^{d,ssh_s} satisfies **(WAPA-s)**, **(IAB-s)**, **(WP)** and **(SPM-s)**, but **(A)** is not satisfied in general.*

Proof. See Appendix C.

It is possible to strengthen the **(SPM-**s**)** property as follows:

Definition 4.19 (Stronger Povertymax for s**)** *(*S_1***PM-****s) Let* $E = \{K_1, \ldots, K_n\}$ *be a belief set, d be a distance measure and* $s \geq 0$*. For all* $\omega, \omega' \in \Omega$*, if (1) there exists a* $k \leq n$ *such that* $d(\omega, K_k) < d(\omega', K_k)$ *and* $s < d(\omega', K_k)$*; (2) every position i that* $d(\omega', K_i) < d(\omega, K_i)$ *implies* $s < d(\omega', K_i)$ *or* $d(\omega, K_i) \leq s$*, then* $\omega <_E^{d,op_s} \omega'$*.*

This condition can be achieved similarly as shown in Proposition 4.4, but replacing the Weak Pareto with the Strong Pareto property.

Proposition 4.5 *If a pre-order* $\leq_f$ *satisfies* ***(WAPA-****s),* ***(IAB-****s),* ***(SP)*** *and* ***(A)****, then it satisfies (*S_1***PM-****s).*

For the two strong operators described in this section, we have this result.

Theorem 4.12 Δ_μ^{d,shc_s} *and* Δ_μ^{d,ssh_s} *satisfy neither (*S_1***PM-****s) nor* ***(SP)****.*

Proof. See Appendix C.

In order to satisfy **(SP)**, we can also define an axiological version of the strong sufficientarian merging operators.

Definition 4.20 (Strong Axiological Sufficientarian Operators) *Let* $E = \{K_1, \ldots, K_n\}$ *be a belief set, d a distance measure,* ω *an outcome and* $s \geq 0$ *a threshold.*

1. *We define the number of belief bases in E across s w.r.t.* ω *and* ω' *as* $shc(\omega, \omega', d, E, s) = \#(\{K_i \in E \mid d(\omega, K_i) \geq s > d(\omega', K_i)\})$*, where* $\#(A)$ *is the cardinal of the set A. Then we have the following pre-order:* $\omega_i \leq_E^{d,sahc_s} \omega_j$ *iff (if* $shc(\omega_i, \omega_j, d, E, s) = shc(\omega_j, \omega_i, d, E, s)$ *then* $d_{sum}(\omega_i, E) \leq d_{sum}(\omega_j, E)$*; else* $shc(\omega_i, \omega_j, d, E, s) \leq shc(\omega_j, \omega_i, d, E, s)$*). The merging operator* $\Delta_\mu^{d,sahc_s}$ *is defined by* $\Delta_\mu^{d,sahc_s}(E) = min(mod(\mu), \leq_E^{d,sahc_s})$*.*
2. *We define the shortfall of belief bases in E across s w.r.t.* ω *and* ω' *in* ω *as* $ssh(\omega, \omega', d, E, s) = \sum_{d(\omega,K_i) \geq s > d(\omega',K_i)} d(\omega, K_i) - s$*. Then we have the following pre-order:* $\omega_i \leq_E^{d,sash_s} \omega_j$ *iff (if* $ssh(\omega_i, \omega_j, d, E, s) = ssh(\omega_j, \omega_i, d, E, s)$ *then* $d_{sum}(\omega_i, E) \leq d_{sum}(\omega_j, E)$*; else* $ssh(\omega_i, \omega_j, d, E, s) \leq ssh(\omega_j, \omega_i, d, E, s)$*). The merging operator* $\Delta_\mu^{d,sash_s}$ *is defined by* $\Delta_\mu^{d,sash_s}(E) = min(mod(\mu), \leq_E^{d,sash_s})$*.*

Finally, for the strong axiological operators, using all the results obtained until now we have:

Proposition 4.6 $\Delta_{\mu}^{d,sahc_s}$ *satisfies* **(WAPA-***s***)**, **(IAB-***s***)**, **(SP)**, **(A)** *and* **(S$_1$PM-***s***)**. $\Delta_{\mu}^{d,sash_s}$ *satisfies* **(WAPA-***s***)**, **(IAB-***s***)**, **(SP)** *and* **(S$_1$PM-***s***)**, *but does not satisfy* **(A)** *in general.*

4.8 Conclusions

In this chapter, we investigated another theory of distributive justice called sufficientarianism. It is a prioritarian approach concerned with alleviating the inequalities among groups of agents who have not reached a sufficient condition. It is an alternative to the egalitarianism, where the inequalities are remedied by giving absolute preference to the worst off agents in a group. The sufficientarian claim considers not only one, but everyone in the group of the less favored agents. It is important since it brings a more humanitarian approach to the distributive justice. In the literature, these inequalities are calculated mainly using Poverty index measurement. We applied three of them: the *headcount* operator, the *shortfall* operator and the Foster-Greer-Thorbecke (FGT) index.

We showed a principle of sufficientarianism is presented in a logical postulate that we named **(IC2')**. Furthermore, we showed sufficientarian operators may satisfy additional logical postulates, e.g., the *headcount* operator satisfies properties like **(WHP)** and **(WIBP)**.

A point of discussion in this work was about conditions for a humanitarian distribution of justice. We found *headcount* operator is weaker than *shortfall* with respect to a property called Povertymax (named Weaker Povertymax for *s* or **(W$_1$PM-***s***)**), which is a humanitarian alternative to the Leximax Principle. The *shortfall* operator establishes a strong version of Povertymax (named Weak Povertymax for *s* or **(WPM-***s***)**).

We also extended *headcount* and *shortfall* operators by resorting to the FGT index, a poverty measure index. This index is defined through a parameter $\alpha \geq 0$, called poverty aversion parameter. It puts higher weight on the poverty of the poorest agents as long as α increases, giving preference for those in necessity (below the poverty line). We showed FGT index operator does not satisfy a new principle called Leximax Above s or **(LMA-s)**. It is a corresponding version of Leximax Principle for the sufficientarian reasoning. In other words, it is the Leximax Principle applied only for those agents who have not reached a sufficient condition. In fact, a strong property was needed to satisfy **(LMA-s)**, which is the satisfaction of Strong Pareto **(SP)**.

	(IC2')	(IC2)	(Maj)	(Arb)	(W$_1$PM-s)	(WPM-s)	(LMA-s)	(SPM-s)	(SP)	(A)
Δ_{μ}^{d,hc_s}	✓		✓	✓	✓			✓		✓
Δ_{μ}^{d,sh_s}	✓		✓		✓	✓				✓
$\Delta_{\mu}^{d,FGT_s^{\alpha}}$	✓		✓		✓	$\alpha > 1$				✓
Δ_{μ}^{d,ahc_s}	✓	✓	✓	✓	✓	✓		✓	✓	✓
Δ_{μ}^{d,ash_s}	✓	✓	✓		✓	✓			✓	✓
$\Delta_{\mu}^{d,aFGT_s^{\alpha}}$	✓	✓	✓		✓	$\alpha > 1$	$\alpha \geq \frac{n}{2}$		✓	✓
Δ_{μ}^{d,shc_s}	✓		✓	✓	✓			✓		✓
Δ_{μ}^{d,ssh_s}	✓		✓		✓	✓		✓		
$\Delta_{\mu}^{d,sahc_s}$	✓	✓	✓	✓	✓	✓		✓	✓	✓
$\Delta_{\mu}^{d,sash_s}$	✓	✓	✓		✓	✓		✓	✓	

Table 24 – Summary of Logical Properties (5).

We verified none of these three operators satisfies **(SP)**. In order to bring Strong Pareto to them, we made these operators a hybrid between sufficientarianism and utilitarianism: the operator behaves as a utilitarian operator when a condition is satisfied, otherwise, it is sufficientarian. This change benefits these operators not only satisfying **(SP)**, but all the basic logical postulates **(IC0)** to **(IC8)**, making them merging operators. Finally, with this change, the merging operator FGT index is capable to fulfill **(LMA-s)**, when $\alpha \geq \frac{n}{2}$ (n is the number of propositional variables in the belief set).

Lastly, we began to analyze another form of sufficientarianism called strong sufficientarianism (we assume the classical one is treated as the weak sufficientarianism). We glimpsed two different kinds of operators: the strong *headcount* and strong *shortfall*. As a result, we showed that the strong version of *headcount* does not bring anything new to the results of the aggregation, since they are equivalent to its classical version. However, the strong *shortfall* loses the Anonymity **(A)** logical property. Consequently, we proved these strong sufficientarian operators satisfy the Strong Povertymax property. Finally, we considered an axiological version of the strong sufficientarian operators to overcome again the loss of **(SP)**.

Some work still needs to be done. For instance, what other characterizations of sufficientarianism can be proposed, along with the axiological and strong sufficientarianism? For the Strong sufficientarianism, a deep research in its logical properties is still missing. Besides, a future work about other possible strong operators that satisfy all logical properties (in special a version of strong *shortfall* which satisfies **(A)**) is envisaged.

Besides, an integration of sufficientarianism with other distributive justice theories are an interesting research field. The original definition of the doctrine of sufficiency may be sometimes vague and imprecise. Some works in this line are being glimpsed (CHUNG, 2017), and their relation with the belief merging is a work that deserves to be considered.

Another relevant point about sufficientarianism is that some authors argue this theory has the necessity of a threshold (e.g., the poverty line). We conjecture it is possible to avoid this fixed value and work with other parameters, as for example the mean of the utilities (each outcome would have its own mean value). The consequences of this representation as well as its rationalization and intuition deserve more attention and a deeper analysis.

Table 24 summarizes the main contributions of this chapter. Recall that all of these operators satisfy **(IC0)**, **(IC1)**, **(IC3)**-**(IC8)**.

5 PROPOSITIONAL BELIEF MERGING WITH OWA OPERATORS

5.1 Contributions of this Chapter

The main contributions of this chapter are listed below:

- We introduce Ordered Weighted Averaging (OWA) operators as belief merging operators. They are a family of aggregation functions which include many well-known operators such as the maximum, minimum and the simple average. We show that OWA merging operators can also be included as a subtype of egalitarian operators. More specifically, as pre-IC merging operators;
- We study how the different OWA operators behave with respect to their logical properties and how this affects their rationality;
- Part of this chapter has been accepted in KR 2018, with the title "Propositional Belief Merging with OWA Operators", authors: Henrique Viana and João Alcântara.

5.2 Introduction

Aggregation of information are basic concerns for all kinds of knowledge based systems, from image processing to decision making, from pattern recognition to machine learning. From a general point of view we can say that aggregation has as purpose the simultaneous use of different pieces of information (provided by several sources/agents) in order to come to a conclusion or a decision (DETYNIECKI, 2001).

The Ordered Weighted Averaging (OWA) operators were originally introduced in (YAGER, 1988) to provide means for aggregating information associated with the satisfaction to multiple criteria. They have proved to be a useful family of aggregation operators for many different types of problems. A fundamental aspect of this operation is to assign weights to the values being aggregated.

When regarding egalitarian operators, it is natural to consider merging operators which tries to achieve a "fair" result. In (EVERAERE *et al.*, 2014), two egalitarian conditions coming from social choice theory were translated into the propositional belief merging framework: Hammond equity, and Pigou-Dalton condition (DALTON, 1920; HAMMOND, 1976). Besides, two new families of belief merging operators based on the median and on a cumulative sum were introduced. A general family of belief merging operators called pre-IC merging operators was defined, by weakening two IC logical postulates. The egalitarian operators defined in (EVERAERE *et al.*, 2014) do not satisfy all the usual IC postulates. However, the ones based on the cumulative sum were proved to be in the family of pre-IC merging operators.

One of the aims of this chapter is to continue this investigation on egalitarian operators, by introducing OWA merging operators. As our main contributions, we will define OWA merging operators and show their logical properties. As the operators defined in (EVERAERE *et al.*, 2014), OWA merging operators will not satisfy all the usual IC logical postulates. We will show what conditions need to be achieved for an OWA merging operator to satisfy some missing IC logical postulates. We will show that depending on the chosen weights, OWA merging operators can be in the family of IC or pre-IC merging operators.

We will consider some families of OWA operators during this chapter: S-OWA merging operators, which combines *max* and *min* with the *sum* operator and seeks to join the theories of utilitarianism and egalitarianism; Step-OWA merging operators, which extends the notion of giving the priority for the worst case by giving the priority to any case among the group; Window-OWA merging operators extends step-OWA merging operators considering a group of agents to giving the priority instead of only one; Buoyancy Measure merging Operators have the idea of giving more priority to the worst cases and less priority to the best cases; and $leximax$ Like OWA merging operators, which simulates the $leximax$ operator by giving some specific configuration of weights for the group.

The chapter is organized as follows. First, in Section 5.3 we will review some notions about OWA operators and define OWA merging operators. We will show the results involving OWA merging operators and in Section 5.4, we will provide some families of OWA merging operators, with their respective results. Finally, in Section 5.5 we will conclude the chapter with some discussions about all the results obtained.

5.3 Belief Merging with OWA Operators

5.3.1 Ordered Weighted Averaging Operators

In this section we review the basic concepts associated with OWA operators (YAGER, 1988). They are a parameterized family of aggregation operators which include many well-known operators such as the maximum, minimum and the simple average (YAGER; KACPRZYK, 1997).

Definition 5.1 (OWA Operator) *(YAGER, 1988) An OWA operator of dimension n is a mapping*

$f_W : \mathbb{R}^n \rightarrow \mathbb{R}$ *that has an associated n vector of weights* $W = [w_1, w_2, \ldots, w_n]$*, such that (1)* $w_i \in [0,1]$ *and (2)* $\sum_i w_i = 1$*. Furthermore* $f_W(a_1, \ldots, a_n) = \sum_j w_j b_j$*, where* b_j *is the jth largest of the* a_i *in* $(a_1, \ldots, a_n)$*.*

A fundamental aspect of this operation is the re-ordering step, in particular an aggregate a_i is not associated with a particular weight w_i but rather a weight is associated with a particular ordered position of aggregate.

Example 5.1 *Let* $W = [0.4, 0.1, 0.3, 0.2]$ *be a vector of weights, then* $f_W(0.7, 1, 0.3, 0.6) = (0.4)(1) + (0.1)(0.7) + (0.3)(0.6) + (0.2)(0.3) = 0.71$.

Note that different OWA operators are distinguished by their vector of weights. In (YAGER, 1988) it was pointed out three important cases of vectors:

1. $W^* = [1, 0, \ldots, 0]$;
2. $W_* = [0, \ldots, 0, 1]$;
3. $W_A = \left[\frac{1}{n}, \ldots, \frac{1}{n}\right]$.

W^* gives weight only to the highest value of a vector (whilst W_* gives it to the lowest value) and the rest of the values have no associated weight. W_A associates an equal weight to all values in a vector. It can easily be seen that

1. $f_{W^*}(a_1, \ldots, a_n) = \max_i(a_i)$;
2. $f_{W_*}(a_1, \ldots, a_n) = \min_i(a_i)$;
3. $f_{W_A}(a_1, \ldots, a_n) = \frac{1}{n}\sum_i a_i$.

Some important properties can be associated with the OWA operators. We shall now discuss some of these. For any OWA operator f,

$$f_{W_*}(a_1, \ldots, a_n) \leq f_W(a_1, \ldots, a_n) \leq f_{W^*}(a_1, \ldots, a_n).$$

Thus, the *minimum* and *maximum* operators are its boundaries. Besides, any OWA operator is commutative. Let $a = (a_1, a_2, \ldots, a_n)$ be a set of values and let $(d_1, d_2, \ldots, d_n)$ be any permutation of a. Then for any OWA operator

$$f_W(a_1, a_2, \ldots, a_n) = f_W(d_1, d_2, \ldots, d_n).$$

A third characteristic associated with these operators is monotonicity. Assume $a = (a_1, a_2, \ldots, a_n)$ and $c = (c_1, c_2, \ldots, c_n)$ such that for each i, $a_i \geq c_i$. Then

$f_W(a_1, a_2, \ldots, a_n) \geq f_W(c_1, c_2, \ldots, c_n)$.

Another characteristic associated with these operators is idempotency. If $a_i = x$ for all i, then for any OWA operator

$$f_W(a_1,\ldots,a_n) = x.$$

5.3.2 OWA Merging Operators

An OWA Merging Operator may be defined directly in the following way:

Definition 5.2 (OWA Merging Operators) *Let d be a distance measure and $E = \{K_1,\ldots,K_n\}$ a belief set. For each outcome ω, we consider the vector $L_E^{\omega} = (l_1^{\omega},\ldots,l_n^{\omega})$ where $l_i^{\omega} = d(\omega, K_{\sigma(i)})$ is the distance between $K_{\sigma(i)}$ and ω, and σ is the permutation of $\{1,\ldots,n\}$ such that $l_i^{\omega} \geq l_{i+1}^{\omega}$ for every $1 \leq i < n$. Then we define the vector $W = [w_1,\ldots,w_n]$, where $w_i \in [0,1]$ and $\sum_i w_i = 1$. Let $d(W, L_E^{\omega}) = \sum_{i=1}^{n} w_i l_i^{\omega}$. Then we have the following pre-order: $\omega_i \leq_E^{d,W} \omega_j$ iff $d(W, L_E^{\omega_i}) \leq d(W, L_E^{\omega_j})$. The OWA merging operator $\Delta_{\mu}^{d,W}$ is defined by $\Delta_{\mu}^{d,W}(E) = min(mod(\mu), \leq_E^{d,W})$.*

The idea for an OWA merging operator is to give the possibility of allowing different priorities for the information in a group. Different from *max* and *leximax*, which give priority to the worst case, an OWA is flexible enough to give more or less priority for any position in a group, and consequently dealing with different degrees of priorities. As a first result about the logical properties of OWA merging operators, we have

Theorem 5.1 *$\Delta_{\mu}^{d,W}$ satisfies **(IC0)**, **(IC1)**, **(IC3)**, **(IC4)**, **(IC5b)**, **(IC7)** and **(IC8)**.*

Proof. See Appendix D.

In general, an OWA merging operator is not an IC or pre-IC merging operator, since it does not satisfy **(IC2)**, **(IC5)**, **(IC6)** or **(IC6b)**. However, it is possible to state some conditions which validates some of these logical properties.

Theorem 5.2 *$\Delta_{\mu}^{d,W}$ satisfies **(IC6b)** if and only if $w_i \neq 0$, for all $w_i \in W$.*

Proof. See Appendix D.

(IC6b) is equivalent to Strong Pareto, which can be translated as if $\forall i\ d(\omega', K_i) \leq d(\omega, K_i)$ and $\exists j\ d(\omega', K_j) < d(\omega, K_j)$, then $\omega' < \omega$. Thus, the existence of $w_j = 0$ is sufficient to falsify this condition.

Theorem 5.3 $\Delta_\mu^{d,W}$ *satisfies **(IC2)** if and only if* $w_1 \neq 0$*, where* $w_1 \in W$.

Proof. See Appendix D.

The reason why **(IC2)** is not always true comes from the fact that even if an outcome does not have a consensus between agents, it can still be a choice of the merging operator, e.g., (0,0,0,0) can as preferred as (0,1,0,0) if we ignore the worst cases in both vectors, and that is equivalent to consider $w_1 = 0$. Consequently, any OWA merging operator $\Delta_\mu^{d,W}$ is a pre-IC merging operator when it satisfies Theorems 5.2 and 5.3.

Corollary 5.1 *Let* $W = [w_1, w_2, \ldots, w_n]$ *be a vector of weights.*

- *If* $w_i = \frac{1}{n}$ *for all* $w_i \in W$ *and* $W = |n|$*, then* $\Delta_\mu^{d,W}$ *satisfies **(Maj)**, **(IC5)** and **(IC6)**.*
- *If* $w_1 = 1$ *and* $w_i = 0$ *for all* $i \neq 1$*, then* $\Delta_\mu^{d,W}$ *satisfies **(IC5)**. If* $w_n = 1$ *and* $w_i = 0$ *for all* $i \neq n$*, then* $\Delta_\mu^{d,W}$ *satisfies **(IC5)**.*

These restrictions come directly from the *sum*, *max* and *min* operators. It is not known if there are two-sided conditions for **(IC5)**, **(IC6)** and **(Maj)** in relation with OWA merging operators. The same holds for **(Arb)**.

Theorem 5.4 *Let d be a distance measure,* ω *an interpretation and* $m = max(\{d(\omega, \omega') \mid \omega, \omega' \in \Omega\})$*. If* $w_1 > (m-1)w_2$*, then* $\Delta_\mu^{d,[w_1,w_2]}$ *satisfies **(Arb)** .*

Proof. See Appendix D.

Here, **(Arb)** is a two-agents property that states when the weight of the worst case is greater than the best case in these conditions, we are giving absolute preference to the worst case. When we refer to **(PD)**, we have the following result:

Theorem 5.5 $\Delta_\mu^{d,W}$ *satisfies **(PD)** if and only if* $w_1 > w_2 > w_3 > \ldots > w_n$*, for* $W = [w_1, \ldots, w_n]$.

Proof. See Appendix D.

In other words, if we are prioritizing the worst case, and the weights are successively decreasing for the next cases, we are guaranteeing a more balanced merging for the group.

5.4 Families of OWA Operators

The OWA operators have a great flexibility in the choice of the types of aggregation based on the choice of their weights. In this section, we shall look at some families of OWA operators (YAGER, 1993a). These families will be specified by a few parameters which can be used to generate the weights.

5.4.1 S-OWA Merging Operators

Two families of OWA weights were introduced by Yager and Filev (YAGER; FILEV, 1994) and are called S-OWA operators. More specifically, the first one is "*orlike*" (a family of operators closer to *maximum*) and the second is "*andlike*" (a family of operators closer to *minimum*).

Definition 5.3 (*orlike* **S-OWA Operators)** *(YAGER; FILEV, 1994) The "orlike" S-OWA operators, denoted by $f_{W_{SO,\alpha}}$, are defined as a family of weights $W_{SO,\alpha} = [w_1,\dots,w_n]$ such that for $\alpha \in [0,1]$,*

- $w_1 = \frac{1}{n}(1-\alpha)+\alpha$,
- $w_i = \frac{1}{n}(1-\alpha)$, *for* $i = 2,\dots,n$.

Using these weights, we get an equivalent form for this operator: $f_{W_{SO,\alpha}}(a_1,\dots,a_n) = \alpha \max_i(a_i) + \dfrac{1}{n}(1-\alpha)\sum_i a_i$. Thus we are getting a weighted average of the *max* and the average of the values in the vector. Note that when $\alpha = 0$, we get $\frac{1}{n}\sum_i a_i$ and when $\alpha = 1$, we get $\max_i(a_i)$.

Definition 5.4 (*andlike* **S-OWA Operators)** *(YAGER; FILEV, 1994) The "andlike" S-OWA operators, denoted by $f_{W_{SA,\alpha}}$, are defined as a family of weights $W_{SA,\alpha} = [w_1,\dots,w_n]$ such that for $\alpha \in [0,1]$,*

- $w_i = \frac{1}{n}(1-\alpha)$, *for* $i \neq n$,
- $w_n = \frac{1}{n}(1-\alpha)+\alpha$.

Using these weights we get $f_{W_{SA,\alpha}}(a_1,\dots,a_n) = \alpha \min_i(a_i) + \dfrac{1}{n}(1-\alpha)\sum_i a_i$. Analogous to $f_{W_{SO,\alpha}}$, we are getting now a weighted average of the *min* and the average of the values in the vector.

Definition 5.5 (S-OWA Merging Operators) *Let d be a distance measure, ω an outcome and $E = \{K_1,\dots,K_n\}$ a belief set.*

- *Let $d_{SO}(\omega,E,\alpha) = \alpha \times \max_{K\in E} d(\omega,K) + (1-\alpha) \times \sum_{K\in E} \dfrac{d(\omega,K)}{|E|}$, where $\alpha \in [0,1]$. Then we have the following pre-order: $\omega_i \leq_E^{d,SO_\alpha} \omega_j$ iff $d_{SO}(\omega_i,E,\alpha) \leq d_{SO}(\omega_j,E,\alpha)$. The operator Δ_μ^{d,SO_α} is defined by $\Delta_\mu^{d,SO_\alpha}(E) = min(mod(\mu), \leq_E^{d,SO_\alpha})$.*

- *Let* $d_{SA}(\omega, E, \alpha) = \alpha \times \min_{K \in E} d(\omega, K) + (1-\alpha) \times \sum_{K \in E} \frac{d(\omega, K)}{|E|}$*, where* $\alpha \in [0,1]$*. Then we have the following pre-order:* $\omega_i \leq_E^{d,SA_\alpha} \omega_j$ *iff* $d_{SA}(\omega_i, E, \alpha) \leq d_{SA}(\omega_j, E, \alpha)$*. The operator* Δ_μ^{d,SA_α} *is defined by* $\Delta_\mu^{d,SA_\alpha}(E) = min(mod(\mu), \leq_E^{d,SA_\alpha})$*.*

Then we have the following results:

Corollary 5.2 *Considering the S-OWA merging operators:*

1. Δ_μ^{d,SO_1} *satisfies **(IC0)-(IC4), (IC5), (IC7), (IC8)** and **(Arb)**;*
2. Δ_μ^{d,SA_1} *satisfies **(IC0), (IC1), (IC3), (IC4), (IC5), (IC7)** and **(IC8)**;*
3. Δ_μ^{d,SO_0} *and* Δ_μ^{d,SA_n} *satisfy **(IC0)-(IC8)** and **(Maj)**;*
4. *For* $0 < \alpha < 1$, Δ_μ^{d,SO_α} *and* Δ_μ^{d,SA_α} *satisfy **(IC0)-(IC4), (IC5b), (IC6b), (IC7)** and **(IC8)**.*

Item 1. is equivalent to *max* operator; items 2. and 4. come from Theorems 5.1, 5.2 and 5.3; and item 3. is equivalent to *sum* operator. This corollary shows when $0 < \alpha < 1$, Δ_μ^{d,SO_α} and Δ_μ^{d,SA_α} are pre-IC merging operators.

5.4.2 Step-OWA Merging Operators

We shall now present another family of OWA operators, called step-OWA operators (YAGER, 1993a).

Definition 5.6 (Step-OWA Operators) *(YAGER, 1993a) The Step-OWA operator is denoted by* $f_{W_{step(k)}}$ *and their weights are defined as* $W_{step(k)} = [w_1, \ldots, w_n]$ *such that* $1 \leq k \leq n$ *and*

- $w_k = 1$,
- $w_i = 0$, *for* $i \neq k$.

Thus with a step-OWA operator we have just one non-zero weight and that is the kth weight. Note that when $k = 1$ we get f_{W^*} and when $k = n$ we get f_{W_*}. It is easily seen that $f_{W_{step(k)}}(a_1, \ldots, a_n) = b_k$ where b_k is the kth largest of the values.

Definition 5.7 (Step-OWA Merging Operators) *Let* d *be a distance measure and* $E = \{K_1, \ldots, K_n\}$ *a belief set. For each outcome* ω*, we consider the vector* $L_E^\omega = (l_1^\omega, \ldots, l_n^\omega)$ *where* $l_i^\omega = d(\omega, K_{\sigma(i)})$ *is the distance between* K_i *and* ω*, and* σ *is the permutation of* $\{1, \ldots, n\}$ *such that* $l_i^\omega \geq l_{i+1}^\omega$ *for every* $1 \leq i < n$*. Let* $k > 0$ *and* $d_{step}(L_E^\omega, k) = l_k^\omega$*, if* $k \leq n$*; 0, otherwise. Additionally,* $d_{step}(L_E^\omega, k) = d_{step}(L_E^\omega, n)$*, if* $k > n$*. Then we have the following pre-order:*

$\omega_i \leq_E^{d,step_k} \omega_j$ iff $d_{step}(L_E^{\omega_i},k) \leq d_{step}(L_E^{\omega_j},k)$. The operator $\Delta_\mu^{d,step_k}$ is defined by $\Delta_\mu^{d,step_k}(E) = min(mod(\mu), \leq_E^{d,step_k})$.

Corollary 5.3 *$\Delta_\mu^{d,step_k}$ satisfies **(IC0)**, **(IC1)**, **(IC3)**, **(IC4)**, **(IC5b)**, **(IC7)** and **(IC8)**. If $k = 1$, then $\Delta_\mu^{d,step_k}$ satisfies **(Arb)**.*

It follows from Theorems 5.1, 5.2, 5.3 and 5.4. $\Delta_\mu^{d,step_k}$ shows a similar behavior of the k-median aggregation functions, introduced in (EVERAERE *et al.*, 2014). In fact, they satisfy the same logical postulates.

5.4.3 Window-OWA Merging Operators

We now consider a family of OWA operators called Window-OWA operators. These operators are determined by two parameters k and m which are responsible to generate the vector of weights.

Definition 5.8 (Window-OWA Operators) *(YAGER, 1993a) The Window-OWA operator is denoted by $f_{W_{window(k,m)}}$ and their weights are defined as $W_{window(k,m)} = [w_1, \ldots, w_n]$ such that $1 \leq k \leq m \leq n$ and*

- *$w_i = 0$, for $i < k$ or $m < i$,*
- *$w_j = \frac{1}{m-k+1}$, for $k \leq j \leq m$.*

We see that these window-OWA operators have $m-k+1$ non-zero weights, all with the same weight $\frac{1}{m-k+1}$, and that k is the place where the non-zero weights start. Using these class of weights, we get $f_{W_{window(k,m)}}(a_1, \ldots, a_n) = \frac{1}{m-k+1} \sum_{j=k}^{m} b_j$, where b_j is the jth largest value of the vector. Thus we are taking a window of elements, starting at k and going until m and then averaging these elements. Note that when $m = k$ we have an equivalence with the step-OWA $f_{W_{step(k)}}$. When $k = m = 1$ (resp. $k = m = n$), we have the case where the OWA operator is equivalent to f_{W^*} (resp. f_{W_*}). Lastly, when $k = 1$ and $m = n$, then we have the case where the operator is equivalent to f_{W_A}.

Definition 5.9 (Window-OWA Merging Operators) *Let d be a distance measure and $E = \{K_1, \ldots, K_n\}$ a belief set. For each outcome ω, we consider the vector $L_E^\omega = (l_1^\omega, \ldots, l_n^\omega)$ where $l_i^\omega = d(\omega, K_{\sigma(i)})$ is the distance between K_i and ω, and σ is the permutation of $\{1, \ldots, n\}$ such that $l_i^\omega \geq l_{i+1}^\omega$ for every $1 \leq i < n$. Let $d_{window}(L_E^\omega, k, m) = l_k^\omega + l_{k+1}^\omega + \cdots + l_m^\omega$, where*

$k \leq m$. If $k < 1$, then $d_{window}(L_E^\omega, k, m) = d_{window}(L_E^\omega, 1, m)$. If $m > n$, then $d_{window}(L_E^\omega, k, m) = d_{window}(L_E^\omega, k, n)$. We have the following pre-order: $\omega_i \leq_E^{d,window_{k,m}} \omega_j$ iff $d_{window}(L_E^{\omega_i}, k, n) \leq d_{window}(L_E^{\omega_j}, k, m)$. The operator $\Delta_\mu^{d,window_{k,m}}$ is defined by $\Delta_\mu^{d,window_{k,m}}(E) = min(mod(\mu), \leq_E^{d,window_{k,m}})$.

Corollary 5.4 *Considering the Window-OWA operators, we obtain*

1. *$\Delta_\mu^{d,window_{k,m}}$ satisfies **(IC0)**, **(IC1)**, **(IC3)**, **(IC4)**, **(IC5b)**, **(IC7)** and **(IC8)**;*
2. *$\Delta_\mu^{d,window_{1,m}}$ satisfies additionally **(IC2)**;*
3. *$\Delta_\mu^{d,window_{1,n}}$ satisfies **(IC0)-(IC8)** and **(Maj)**.*

Item 1. follows from Theorem 5.1; item 2. follows from Theorem 5.3 and item 3. is equivalent to *sum* operator. $\Delta_\mu^{d,window_{k,m}}$ is an extension of $\Delta_\mu^{d,step_k}$ and can gain additional logical postulates, depending on the choice of the weights.

5.4.4 Buoyancy Measure Merging Operators

We now introduce a class of OWA operators called buoyancy measures (YAGER, 1993a; YAGER, 1993b; YAGER, 1993c):

Definition 5.10 (Buoyancy Measures) *(YAGER, 1993a) We say that f_W is a buoyancy measure if the weights $W = [w_1, \ldots, w_n]$ satisfy the condition $w_i \geq w_j$, for $i < j$. We call f_W a buoyancy measure extensive if the condition is made slightly stronger, that is, $w_i > w_j$, for $i < j$.*

Thus, a buoyancy measure gives a non increasing weight for the values in the vector. In other words, the largest values have more or equal weight than the smallest values.

Definition 5.11 (Bouyancy Merging Operators) *Let d be a distance measure and $E = \{K_1, \ldots, K_n\}$ a belief set. For each outcome ω, we consider the vector $L_E^\omega = (l_1^\omega, \ldots, l_n^\omega)$ where $l_i^\omega = d(\omega, K_{\sigma(i)})$ is the distance between K_i and ω, and σ is the permutation of $\{1, \ldots, n\}$ such that $l_i^\omega \geq l_{i+1}^\omega$ for every $1 \leq i < n$. Then we define the vectors*

- *$W_b = [w_1, \ldots, w_n]$, where $w_i \in [0,1]$, $\sum_i w_i = 1$ and $w_i \geq w_j$ for $i < j$, and*
- *$W_{eb} = [w_1, \ldots, w_n]$, where $w_i \in [0,1]$, $\sum_i w_i = 1$ and $w_i > w_j$ for $i < j$.*

Let $d(W, L_E^\omega) = \sum_{i=1}^{n} w_i l_i^\omega$, for $W \in \{W_b, W_{eb}\}$. We have the following pre-order: $\omega_i \leq_E^{d,W} \omega_j$ iff $d(W, L_E^{\omega_i}) \leq d(W, L_E^{\omega_j})$. The operators Δ_μ^{d,W_b} and $\Delta_\mu^{d,W_{eb}}$ are defined by $\Delta_\mu^{d,W_b}(E) = min(mod(\mu), \leq_E^{d,W_b})$ and $\Delta_\mu^{d,W_{eb}}(E) = min(mod(\mu), \leq_E^{d,W_{eb}})$, respectively.

Corollary 5.5 *Δ_μ^{d,W_b} and $\Delta_\mu^{d,W_{eb}}$ satisfy **(IC0)-(IC4)**, **(IC5b)**, **(IC7)** and **(IC8)**. Additionally,*

- *$\Delta_\mu^{d,W_{eb}}$ satisfies **(PD)**;*
- *If $w_n > 0$, then $\Delta_\mu^{d,W_{eb}}$ satisfies **(IC6b)**.*

It follows from Theorems 5.1, 5.2, 5.3 and 5.5. Bouyancy merging operators have a behavior similar to the cumulative *sum* $(csum)$ operator (EVERAERE *et al.*, 2014). The *csum* is a weighted sum of the distances between belief bases, but the vector of weights are fixed for the number of belief bases. If $E = (K_1, \ldots, K_n)$, then the we have the following set of weights for *csum*: $W = [w_1, w_2, \ldots, w_n] = [n, n-1, \ldots, 1]$. Therefore, a *csum* is not an OWA operator, since the sum of weights is higher than 1, but it preserves the idea of ordering among the weights and some results related to the logical postulates. For example, if a bouyancy merging operator is extensive and all the weights are greater than 0, then it is equivalent to *csum* merging operator in terms of logical properties.

5.4.5 $leximax$ Like OWA Merging Operators

We shall now consider an OWA operator that simulates the $leximax$ ordering. It is based on the operator defined in (YAGER, 1997), and it is assumed that we have $\delta \in]0,1]$, which can be seen as a degree of dispersion with $leximax$.

Definition 5.12 ($leximax$ like OWA Operators) *We say f_{W_δ} is a $leximax$ like OWA Operator if their weights are defined as $W_\delta = [w_1, \ldots, w_n]$, such that $\delta \in]0,1]$ and*

- *$w_i = \frac{\delta^{i-1}}{(1+\delta)^i}$, for $i \neq n$,*
- *$w_n = \frac{\delta^{n-1}}{(1+\delta)^{n-1}}$.*

The idea behind this operator is to give the highest weight to the highest value of a vector and this weight decreases to the consequent values. Depending of the value of δ, the difference of weights are so large that the operator gives an absolute priority to the highest value than the other values of the vector. That is the idea we find in the $leximax$ ordering.

Theorem 5.6 *Let d be a distance measure, ω an interpretation and $m = max(\{d(\omega,\omega') \mid \omega,\omega' \in \Omega\})$. Consider $W = [w_1, \ldots, w_n]$, where $w_i = \frac{\delta^{i-1}}{(1+\delta)^i}$, for $i \neq n$ and $w_n = \frac{\delta^{n-1}}{(1+\delta)^{n-1}}$. If $\delta \leq \frac{1}{m}$, then Δ_μ^{d,W_δ} satisfies **(HE)**.*

Proof. See Appendix D.

In other words, $\Delta_{\mu}^{d,W_{\delta}}$ is not equivalent to $\Delta_{\mu}^{d,leximax}(E)$, but as every belief set E has a finite number of belief bases, which are also finite, it is possible to find a δ' such that $\Delta_{\mu}^{d,W_{\delta'}}(E) \equiv \Delta_{\mu}^{d,leximax}(E)$.

5.5 Conclusions

In this chapter, we proposed Ordered Weighted Averaging (OWA) operators as belief merging operators. The class of OWA operators includes *maximum*, *minimum* and the simple average. Besides, it is powerful enough to encompass other operators such as the median, cumulative *sum* and $leximax$.

OWA merging operators exploit a set of weights to generate different kind of priorities among the information. For instance, *maximum* and $leximax$ operators are known to give absolute priority to the worst information in a group. With a set of weights, we are free to determine what information has more priority or not. It is important because, for example, we can explore different forms of egalitarianism.

	(IC2)	**(IC5)**	**(IC6)**	**(IC6b)**	**(Maj)**	**(Arb)**	**(PD)**	**(HE)**
$\Delta_{\mu}^{d,W}$	THM. 5.3	COR. 5.1	COR. 5.1	THM. 5.2	COR. 5.1	THM. 5.4	THM. 5.5	THM. 5.6
$\Delta_{\mu}^{d,SO_{\alpha}}$	$0<\alpha<1$	$\alpha=1$		$0<\alpha<1$	$\alpha=0$			
$\Delta_{\mu}^{d,SA_{\alpha}}$	$0<\alpha<1$	$\alpha=1$		$0<\alpha<1$	$\alpha=n$			
$\Delta_{\mu}^{d,step_k}$						$k=1$		
$\Delta_{\mu}^{d,window_{k,m}}$	$k=1$	$k=1, m=n$	$k=1, m=n$	$k=1, m=n$	$k=1, m=n$			
Δ_{μ}^{d,W_b}	COR. 5.5							
$\Delta_{\mu}^{d,W_{eb}}$	COR. 5.5			COR. 5.5			COR. 5.5	

Table 25 – Summary of Logical Properties (6).

Regarding egalitarian operators, generally they do not satisfy all the usual IC postulates in belief merging. Some are included in a family of belief merging operators called pre-IC merging operators. As a result of this chapter, we showed some OWA merging operators that are included in this family.

The choice of the weights plays a fundamental role in the relation of the satisfaction of some IC logical postulates. In general, logical postulates like **(IC2)**, **(IC5)**, **(IC6)**, **(Maj)** and **(Arb)** are not satisfied by OWA merging operators. We showed in this chapter that when some conditions are met, these properties can hold. Furthermore, we still explored two egalitarian conditions: Pigou-Dalton and Hammond Equity. We also proved that these conditions can be satisfied when some restrictions are applied to the weights. Therefore, OWA merging operators are powerful enough to represent IC and pre-IC merging operators.

As future work we intend to find necessary and sufficient conditions to satisfy **(IC5)**, **(IC6)**, **(Maj)**, **(Arb)** and **(HE)**. Another task involves to investigate new families of OWA operators and under which conditions they are IC or pre-IC merging operators.

Table 25 summarizes the main contributions of this chapter. Remember that all of these operators satisfy **(IC0)**, **(IC1)**, **(IC3)**, **(IC4)**, **(IC5b)**, **(IC7)** and **(IC8)**.

6 FINAL CONCLUSIONS

In this thesis, we proposed to combine the area of belief merging with the theories of distributive justice. As said in the beginning of this work, the distributive justice is associated with the four concepts of justice. They are **Equality**, **Fairness**, **Desert** and **Rights**. We focused mainly on these two first concepts of justice along the chapters.

Equality states that our treatment of agents ought to reflect they are all morally equal. There are no morally relevant differences between agents which make it permissible to treat them differently. As a consequence, it states that each agent should receive an equal share of amounts. In the context of belief merging, the operators *max* and *leximax* are related with this concept, as they promote a merging with less injustice by giving preference to the worst cases in a group. Using this as a starting point, in Chapter 3, we considered two *max* refinements of belief merging operators: they are Discrimax and T-conorms.

Discrimax is based on the elimination of identical singleton elements at the same position (difference set) and the comparison of the maximum with the remaining elements. It refines *max* as when the difference set is empty, it is equivalent to Maximum. On the other hand, T-conorms are functions stronger than the max operator, using the interval [0,1] instead of the classical values 0 or 1.

The big problem when it comes to *max* in belief merging is the loss of postulate **(IC6)**, which is bypassed with its extension to *leximax*. The same can be observed with Discrimax and T-conorms. With the Discrimax it is even worse as it also loses **(IC1)** (due to the loss of transitivity). Again, this can also be remedied by a *lexi* version of these operators. We extend T-conorms to lexiT-conorms with the aim of making them an IC merging operator (satisfying all the basic postulates). In the literature, there is also a *lexi* version of Discrimax, called lexidiscrimax (FORTEMPS; PIRLOT, 2004), but it has been proved to be equivalent to leximax and, therefore, it was not used in this work.

Still speaking of equality, another important point is the question of the contribution of each agent in a group. In chapter 4, we considered the concept of sufficientarianism in belief merging. In this approach, we give preference to agents who have not contributed enough (they have not reached an estimated level, called commonly as the sufficiency line). These agents are in a need situation and our goal is to help them to improve. We can also call this approach as equality from necessity.

We used as base in the study of sufficientarianism the operators of *headcount*, *shortfall* and a generalization of them called FGT index. *Headcount* counts the number of agents who are in need. On the other hand, *shortfall* counts the amount of value of these agents who are in need.

Sufficientarianism has an interesting feature that says that not always the best group of agents (or the best outcomes) will be exactly the result of merging. This is equivalent to the loss of the basic postulate **(IC2)**. To circumvent this and turn the sufficientarian operators into IC merging operators, we resorted to another doctrine of sufficiency, called axiological sufficientarianism. It considers important to help the agents in need, and in addition, the amount of the total contribution of the group is also taken into account. That is enough to bring the satisfaction of **(IC2)**. With this we show that the operators of axiological sufficientarianism are IC merging operators.

We can cite two main contributions for this thesis. The first one is to bring different forms of distributive justice to belief merging and check if they are compatible with IC belief merging operators. We have shown that equality and necessity are concepts of egalitarianism that are in some way compatible with belief merging, through lexiT-conorms and axiological sufficientarianism. It is an important result because the area of propositional belief merging was grounded on only two subclasses of operators: majority (based on utilitarianism) and arbitration (based on egalitarianism) operators, and we brought new options for operators besides them.

The second contribution is to analyze these theories of distributive justice, through the logical properties that characterize their behaviors. We analyzed the leximax principle from a starting point, which is one of the main characterizations when we deal with egalitarianism. In Chapter 3, we saw the relationship between the Discrimax, the T-conorms and lexiT-conorms with this principle. In Chapter 4, we have seen the Povertymax principle, which is deeply related to sufficientarianism. We showed how the *headcount* and *shortfall* operators relate to it and we also compared the leximax and Povertymax principles, that is, we compared the egalitarianism and sufficiency in terms of merging operators. All this work contributes to hierarchize these operators with respect to their rationality, i.e., the satisfaction of the logical properties.

Finally, in chapter 5, we looked at OWA operators in belief merging. The motivation comes from the notion of **Fairness**, in the sense that we must treat similar cases in the same way. OWA operators are powerful operators that ranges from the maximum to the minimum, and allow us to give different priorities for each level of information. They do not give preference to the agents themselves, but to the values presented within the group of agents. Similar cases within the group can be seen as the order of the information, such as the worst case of the group, the second worst case, etc... OWA operators allow us to treat them together or separately, among all possible combinations.

We showed that OWA operators offer a weaker form of egalitarianism when compared to *max*, *leximax*, Discrimax or T-conorms, since they do not satisfy, in general, three postulates: **(IC2)**, **(IC5)** and **(IC6)**. However, there are some classes of OWA operators that have interesting properties. We have proved there are some OWA classes that are characterized by being pre-IC merging operators, a weaker form of IC merging operators, and some classes of operators that are also IC merging operators, and it is even possible to represent *leximax* depending on the chosen parameters. Pre-IC merging operators are a general family of belief merging operators where the Pareto-related conditions **(IC5)** and **(IC6)** are replaced by unanimity conditions, denoted by **(IC5b)** and **(IC6b)**. They are relevant because it is known in the literature that egalitarian operators tend not to be compatible with Pareto conditions (TUNGODDEN; VALLENTYNE, 2005), that is, Pareto conditions are not a mandatory property to be satisfied when we are considering Distributive Justice.

6.1 Future Works

It is possible to extend the results of this thesis to work on belief merging with **Desert** and **Rights**. As for Desert, it uses the idea that agents ought to get what they deserve (i.e., good deeds should be rewarded, and bad deeds should be punished). For instance, we can define a belief merging model in which agents receive weights that represent rewards or punishments, something similar to what happens with the Pr-Merge model or OWA operators, but in this case, rather than prioritizing information, it is given to the agents themselves.

An initial research on this area was done, based on these works (FEINBERG, 1970; MILLER, 1976; MILLER, 1990; SADURSKI, 2010; DICK, 1975; LAMONT, 1997), where it were presented a catalog of types of desert claims: a student might deserve a high grade in virtue of having written a good paper; an athlete might deserve a prize in virtue of having excelled in a competition; a successful researcher might deserve an expression of gratitude in virtue of having perfected a disease-preventing serum; etc. In all these familiar cases, the *deserver* is an agent. Desert claims also typically involve a desert. A typical desert claim is that someone deserves something from someone on some basis. For example, consider the claim that a certain student deserves a high grade from her teacher because she did an excellent work in the course.

This is the thing that the *deserver* is said to deserve. When we consider our framework of belief merging this claim is a little simpler: an agent deserves a better well-being from the choice of the group in virtue of she has a high grade of deserving. The well-being measured here is the distance value between interpretations and belief bases, and the grade of deserving is an arbitrary numerical value.

At a first view, when we deal with Desert on belief merging, the main logical postulates that conflict with the theory of distributive justice are **(IC2)**, **(IC3)** and **(IC4)**. The problem with **(IC2)** comes from the same situation of the sufficientarianism. **(IC3)** and **(IC4)** also cannot be satisfied because these operators may have the same problem of the Pr-merge: they can be sensitive to syntax and are tended to give more preference to some agents than the others. The main question about this approach is if there exists a desertarian merging operator that is an IC merging operator.

With relation to the idea of inserting **Rights** in belief merging, one direction to explore is the connection between the belief merging and the Judgment Aggregation areas (PIGOZZI, 2016). The question judgment aggregation addresses is how we can define aggregation procedures that preserve individual rationality at the collective level. Thus, the formal approach to judgment aggregation can serve to cast light on the dependence between individual and collective beliefs. We can think in a model of belief merging where if a right of an agent is violated, he/she has a legitimate claim against them, since they have an obligation to respect the rights of each agent.

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APPENDIX A – PROOF THEOREMS - CHAPTER 2

Theorems 2.4 and 2.5. **Proof: (IC0)** By definition $\Delta_{\mu}^{d,min}(E) \subseteq mod(\mu)$.

(IC1) *min* is a function with values in $\mathbb{N}$, so if $mod(\mu) \neq \emptyset$, there is always a minimal model ω of μ such that for every model ω' of μ, $d_{min}(\omega,E) \leq d_{min}(\omega',E)$. So $\omega \models \Delta_{\mu}^{d,min}(E)$ and $\Delta_{\mu}^{d,min}(E) \models \perp$.

(IC2) As a counterexample, suppose that $(d(\omega,K_1,\omega,K_2)) = (0,0)$ and $(d(\omega',K_1,\omega',K_2)) = (0,1)$. We have that $\omega,\omega' \in \Delta_{\mu}^{d,min}(E)$, but $\omega' \not\models \bigwedge E \wedge \mu$.

(IC3) Assume that $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$. Hence we can find a permutation δ such that for every $i \in 1,\ldots,n, K_{\delta(i)} \equiv K_i'$. Now, since $d(\omega,K_{\delta(i)}) = d(\omega,K_i')$ one gets $d_{min}(\omega,E_1) = min(d(\omega,K_1),\ldots,d(\omega,K_n)) = d_{min}(\omega,E_2)$. Consequently $\Delta_{\mu}^{d,min}(E_1) \equiv \Delta_{\mu}^{d,min}(E_2)$.

(IC4) Suppose that $\Delta_{\mu}^{d,min}(\{K_1,K_2\}) \wedge K_1 \not\models \perp$ and that $\Delta_{\mu}^{d,min}(\{K_1,K_2\}) \wedge K_2 \not\models \perp$. As a consequence, we have $\min_{\omega \models K_1} min(d(\omega,K_1),d(\omega,K_2)) = \min_{\omega \models K_2} min(d(\omega,K_1),d(\omega,K_2))$, because $\min_{\omega \models K_1} min(0,d(\omega,K_2)) = \min_{\omega \models K_2} min(d(\omega,K_1),0) = 0$.

(IC5) It is enough to show that the following property holds: if $d_{min}(\omega,E_1) \leq d_{min}(\omega',E_1)$ and $d_{min}(\omega,E_2) \leq d_{min}(\omega',E_2)$, then $d_{min}(\omega,E_1 \sqcup E_2) \leq d_{min}(\omega',E_1 \sqcup E_2)$. It is easy to see that this property is satisfied.

(IC6) It is enough to show that the following property holds: if $d_{min}(\omega,E_1) < d_{min}(\omega',E_1)$ and $d_{min}(\omega,E_2) \leq d_{min}(\omega',E_2)$, then $d_{min}(\omega,E_1 \sqcup E_2) < d_{min}(\omega',E_1 \sqcup E_2)$. This property is not satisfied by *min*. A counter-example is $E_1 = \{K_1,K_2\}$, where $K_1 = \{(a \wedge \neg b)\}$, $K_2 = \{(a \wedge b)\}$ and $E_2 = \{K_3\}$, where $K_3 = \{(a \wedge b) \vee (\neg a \wedge \neg b)\}$. We have $d_{min}(\omega_4,E_2) = 0 < d_{min}(\omega_2,E_2) = 1$ and $d_{min}(\omega_4,E_1) = 0 \leq d_{min}(\omega_2,E_2) = 0$, but $d_{min}(\omega_4,E_1 \sqcup E_2) = 0 = d_{min}(\omega_2,E_1 \sqcup E_2) = 0$.

Ω	$d_H(\omega,K_1)$	$d_H(\omega,K_2)$	$d_{Hmin}(\omega,E_1)$	$d_{Hmin}(\omega,E_2)$	$d_{Hmin}(\omega,E_1 \sqcup E_2)$
$\omega_1 = \neg a \neg b$	1	2	1	0	0
$\omega_2 = a \neg b$	0	1	0	1	0
$\omega_3 = \neg ab$	2	1	1	1	1
$\omega_4 = ab$	1	0	0	0	0

(IC6') It is enough to show that the following property holds: if $d_{min}(\omega,E_1) < d_{min}(\omega',E_1)$ and $d_{min}(\omega,E_2) < d_{min}(\omega',E_2)$, then $d_{min}(\omega,E_1 \sqcup E_2) < d_{min}(\omega',E_1 \sqcup E_2)$. It is easy to see that this property is satisfied.

(IC7) Suppose $\omega \models \Delta_{\mu_1}^{d,min}(E) \wedge \mu_2$. For any $\omega' \models \mu_1$, we have $d_{min}(\omega,E) \leq d_{min}(\omega',E)$. Hence $\omega' \models \mu_1 \wedge \mu_2, d_{min}(\omega,E) \leq d_{min}(\omega',E)$. Subsequently $\omega \models \Delta_{\mu_1 \wedge \mu 2}^{d,min}(E)$.

(IC8) Suppose that $\Delta_{\mu_1}^{d,min}(E) \wedge \mu_2$ is consistent. Then there exists a model ω' of $\Delta_{\mu_1}^{d,min}(E) \wedge \mu_2$. Consider a model ω of $\Delta_{\mu_1 \wedge \mu_2}^{d,min}(E)$ and suppose that $\omega \not\models \Delta_{\mu_1}^{d,min}(E)$. We have $d_{min}(\omega', E) < d_{min}(\omega, E)$, and since $\omega' \models \mu_1 \wedge \mu_2$, we have $\omega \notin min(mod(\mu_1 \wedge \mu_2), \leq_E^{d,min})$, hence $\omega \not\models \Delta_{\mu_1 \wedge \mu_2}^{d,min}(E)$. Contradiction.

(Maj) $\Delta_{\mu}^{d,min}$ does not satisfy **(Maj)**. We can find a counter-example where the repetition of one base does not change the result. Consider the following counter-example: Let $\mu = \top$, $E_1 = \{K_1\} = \{a \wedge b\}$ and $E_2 = \{K_2\} = \{\neg a \wedge \neg b\}$. Clearly, we have $\Delta_{\mu}^{d,min}(E_1 \sqcup \underbrace{E_2 \sqcup \cdots \sqcup E_2}_{n}) \not\equiv \Delta_{\mu}^{d,min}(E_2)$ for any $n \in \mathbb{N}$.

(Arb) Δ_{μ}^{min} does not satisfy **(Arb)**. Consider the following counter-example: $K_1 = \{a \wedge b\}, K_2 = \{\neg a \wedge \neg b\}, \mu_1 = \neg(a \wedge b)$ and $\mu_2 = a \vee b$. We have that $\Delta_{\mu_1 \vee \mu_2}^{d,min}(\{K_1, K_2\}) = (\omega_1 \vee \omega_4) \not\equiv (\omega_2 \vee \omega_3) = \Delta_{\mu_1}^{d,min}(\{K_1\})$.

Ω	$d_H(\omega, K_1)$	$d_H(\omega, K_2)$	$d_{H\,min}(\omega, \{K_1, K_2\})$
$\omega_1 = \neg a \neg b$	2	0	**0**
$\omega_2 = a \neg b$	**1**	**1**	1
$\omega_3 = \neg a b$	**1**	**1**	1
$\omega_4 = ab$	0	2	**0**

(Temp) Let $E = \{K_1, \ldots, K_n\}$. For each K_i, there is ω that $\omega \models K_i$. By definition $d(\omega, K_i) = 0$ and then, $d_{min}(\omega, E) = 0$. Consequently, $\omega \in mod(\Delta_{\top}^{d,min}(E))$.

(CSS) Let $E = \{K_1, \ldots, K_n\}$. For each K_i, there is ω that $\omega \models K_i$. By definition $d(\omega, K_i) = 0$ and then, $d_{min}(\omega, E) = 0$. Consequently, if $\omega \models \mu$, then $\omega \in mod(\Delta_{\mu}^{d,min}(E))$.

□

Theorems 2.9 and 2.11. **Proof: (IC6')** (Counterexample taken from (EVERAERE *et al.*, 2014)) Suppose $k = 0.5$, and ω_1 such that $d_{med^{0.5}}(\omega_1, E_1) = med^{0.5}(0,0) = 0$, $d_{med^{0.5}}(\omega_1, E_2) = med^{0.5}(3,4,4) = 4$ and accordingly $d_{med^{0.5}}(\omega_1, E_1 \sqcup E_2) = med^{0.5}(0,0,3,4,4) = 3$. Suppose ω_2 such that $d_{med^{0.5}}(\omega_2, E_1) = med^{0.5}(1,1) = 1$, $d_{med^{0.5}}(\omega_2, E_2) = med^{0.5}(2,6,7) = 6$ and $d_{med^{0.5}}(\omega_2,\ E_1 \sqcup E_2) = med^{0.5}(1,1,2,6,\ 7) = 2$. Let $\mu = \omega_1 \vee \omega_2$, $mod(\Delta_{\mu}^{d,med^{0.5}}(E_1)) = \{\omega_1\}$ and
$mod(\Delta_{\mu}^{d,med^{0.5}}(E_2)) = \{\omega_1\}$, whereas $mod(\Delta_{\mu}^{d,med^{0.5}}(E_1 \sqcup E_2)) = \{\omega_2\}$. This example shows that **(IC6')** is not satisfied.

(Temp)/(CSS) It is easy to see that these properties are not satisfied (the example above can even be used to construct the counterexamples).

Theorem 2.11 is similar to Theorem 2.9.

□

Theorem 2.14. **Proof:** **(HI)** Let $E = \{K_1, \dots, K_n\}$. Suppose that for $\omega, \omega' \in \Omega$, $\exists i, j \in \{1, \dots, n\}$ such that $d(\omega, K_i) < d(\omega', K_i) \leq d(\omega', K_j) < d(\omega, K_j)$ and $\forall l \neq i, j\ d(\omega, K_l) = d(\omega', K_l)$ holds. Then we can build the lists $L_{\omega}^{d,E} = (d_1^{\omega}, \dots, d_n^{\omega})$ and $L_{\omega'}^{d,E} = (d_1^{\omega'}, \dots, d_n^{\omega'})$ by sorting the elements in increasing order, where $d_i^{\omega} = d(\omega, K_i)$.

- If $d_1^{\omega} = d(\omega, K_i)$, then $d(\omega, K_i) < d(\omega', K_l)$, for all $l \neq i$. Then $L_{\omega}^{d,E} <_{lex} L_{\omega'}^{d,E}$ and $\omega <_E^{d,leximin} \omega'$.
- Otherwise, $d_1^{\omega} = d(\omega, K_l)$, where $l \neq i, j$. In this case, $d(\omega, K_l) = d(\omega', K_l)$. Then, the previous step is repeated for d_2^{ω}. Consequently, there will be an i, where $d_i^{\omega} < d_i^{\omega'}$ and then $\omega <_E^{d,leximin} \omega'$.

□

Theorem 2.16. **Proof:** **(IE)** Let $E = \{K_1, \dots, K_n\}$. Suppose that for $\omega, \omega' \in \Omega$, $\exists i, j \in \{1, \dots, n\}$ such that $d(\omega, K_i) < d(\omega', K_i) \leq d(\omega', K_j) < d(\omega, K_j)$, $d(\omega', K_i) - d(\omega, K_i) = d(\omega, K_j) - d(\omega', K_j)$ and $\forall l \neq i, j\ d(\omega, K_l) = d(\omega', K_l)$ holds. Since $d(\omega', K_i) - d(\omega, K_i) = d(\omega, K_j) - d(\omega', K_j)$, we have that $d(\omega', K_i) + d(\omega', K_j) = d(\omega, K_j) + d(\omega, K_i)$. By definition, $d_{sum}(\omega, E) = \sum(d(\omega, K_1), \dots, d(\omega, K_i), \dots, d(\omega, K_j), \dots, d(\omega, K_n)) = \sum(d(\omega', K_1), \dots, d(\omega', K_i), \dots, d(\omega', K_j), \dots, d(\omega', K_n)) = d_{sum}(\omega', E)$. Then $\omega \approx_E^{d,sum} \omega'$.

□

Theorem 2.18. **Proof:** Let $E = \{K_1, \dots, K_n\}$ be a belief set and $E' = \{K_1', \dots, K_n'\}$ its corresponding normalized belief set. We want to show that $\Delta_{\mu}^{ps,sum}(E') \equiv \Delta_{\mu}^{d,sum}(E)$ and $\Delta_{\mu}^{ps,min}(E') \equiv \Delta_{\mu}^{d,max}(E)$. Let V be the number of propositional variables of E and $op \in \{sum, min\}$. Then the partial satisfiability of each $K_i \in E'$ (i.e., $\omega(K_i)$) ranges from $\{0, \frac{1}{V}, \dots, \frac{V}{V}\}$. It is easy to see that there exists a correspondence of the partial satisfiability with the Hamming distance, which ranges from $\{0, \dots, V\}$. If the partial satisfiability of a belief base K_i is equal to m, this means that its highest value clause satisfies m literals. Consequently, the Hamming distance of K_i (i.e., $d_H(\omega, K_i)$) is equal to $V - m$, that is, it is needed to change $V - m$ literals in order to satisfy K_i.

Let $\omega \in \Delta_{\mu}^{ps,sum}(E')$. By definition, the outcome ω is in $max(mod(\mu), \leq_{E'}^{ps,sum})$, that is, the sum of the partial satisfiability of ω is greater or equal than any other ω'. This implies that the Hamming distance between ω and E w.r.t. the *sum* operator is less or equal to any other ω'. Consequently, ω is in $min(mod(\mu), \leq_E^{d_H,sum})$ and $\omega \in \Delta_{\mu}^{d_H,sum}(E)$. The same ideia can be used to show that $\Delta_{\mu}^{ps,min}(E') \equiv \Delta_{\mu}^{d,max}(E)$.

□

Theorem 2.19. **Proof:** The proof is very similar to the Theorem 2.18. The only difference in the construction is that the partial satisfiability $\omega(K_i)$ of each $K_i \in E'$ ranges from $\{0, \frac{1}{V}, \ldots, \frac{V}{V}\}$ and the satisfiability $l_i^{d_H,\omega}$ ranges from $\{0, 1, \ldots, V\}$ and the mapping is direct: 0 maps to 0, and for every other m, it maps to $\frac{1}{m}$.

□

Theorem 2.20. **Proof:** Similar to the previous proofs, but with the restriction that the drastic distance values include only $\{0, 1\}$.

□

Theorem 2.21. **Proof:** **(IC0)** By definition, $\Delta_\mu^{pr,op}(E) \subseteq mod(\mu)$.

(IC1) The function $\omega(E)$ maps to values in $\mathbb{R}$, so if $mod(\mu) \neq \emptyset$, there is a model ω of μ such that for every model ω' of μ, $\omega(E) \geq \omega'(E)$. So $\omega \models \Delta_\mu^{pr}(E)$ and $\Delta_\mu^{pr}(E) \not\models \perp$.

(IC2) By assumption, $\bigwedge E$ is consistent and without loss of generality let $E = \{K_1, \ldots, K_n\}$. There exists ω such that $\omega \models (c_{11} \vee \cdots \vee c_{1k}) \wedge \cdots \wedge (c_{n1} \vee \cdots \vee c_{nm})$, where $K_1 = \{c_{11}, \ldots, c_{1k}\}, \ldots, K_n = \{c_{n1}, \ldots, c_{nm}\}$. By definition, $\omega(K_1) = max\{\omega(c_{11}), \ldots, \omega(c_{1n})\}$ and as $\omega \models (c_{11} \vee \cdots \vee c_{1n})$, there is a clause c_{1j} such that $\omega \models c_{1j}$. It is easy to see that this clause has the maximum value, i.e. $\omega(c_{ij}) = 1$ (see the Definition 4). Thus, $\omega(K_1)$ will also receive the maximum possible value. The same idea holds for every K_i, $1 \leq i \leq n$. Hence, as $\omega(E) = \sum_{i=1}^{n} \frac{1}{a_i} \times \omega(K_i)$, for every ω', $\omega(E) \geq \omega'(E)$ (the same holds for $\omega_\times(E)$). So $\omega \models \Delta_\mu^{pr}(E)$ if and only if $\omega \models \bigwedge E \wedge \mu$.

(IC5) In order to show that the operator satisfy **(IC5)**, it is enough to guarantee that the following property holds: if $\omega(E_1) \geq \omega'(E_1)$ and $\omega(E_2) \geq \omega'(E_2)$, then $\omega(E_1 \sqcup E_2) \geq \omega'(E_1 \sqcup E_2)$. We can see clearly that this is satisfied.

(IC6) In order to show that the operator satisfy **(IC6)**, it is enough to guarantee that the following property holds: if $\omega(E_1) > \omega'(E_1)$ and $\omega(E_2) \geq \omega'(E_2)$, then $\omega(E_1 \sqcup E_2) > \omega'(E_1 \sqcup E_2)$. We can see clearly that this is satisfied.

(IC7) Suppose that $\omega \models \Delta_{\mu_1}^{pr}(E) \wedge \mu_2$. For any $\omega' \models \mu_1$, we have $\omega(E) \geq \omega'(E)$. Hence, for any $\omega' \models \mu_1 \wedge \mu_2$, we have $\omega(E) \geq \omega'(E)$. Subsequently $\omega \models \Delta_{\mu_1 \wedge \mu_2}^{pr}(E)$.

(IC8) Suppose that $\Delta_{\mu_1}^{pr}(E) \wedge \mu_2$ is consistent. Then there exists a model ω' of $\Delta_{\mu_1}^{pr}(E) \wedge \mu_2$. Consider a model ω of $\Delta_{\mu_1 \wedge \mu_2}^{pr}(E)$ and suppose that $\omega \not\models \Delta_{\mu_1}^{pr}(E)$. In this case $\omega'(E) > \omega(E)$, and since $\omega' \models \mu_1 \wedge \mu_2$, we have $\omega \notin \Delta_{\mu_1 \wedge \mu_2}^{pr}(E) = max(mod(\mu_1 \wedge \mu_2), \leq_E^{pr})$, hence $\omega \not\models \Delta_{\mu_1 \wedge \mu_2}^{pr}(E)$. Contradiction.

(Maj) Showing that the operator satisfies **(Maj)** is easy from the properties of sum. Since $\omega(E) = \sum_{i=1}^{n} \frac{1}{a_i} \times \omega(K_i)$, without loss of generality we can assume two cases: (i) let ω be a model for $\Delta_\mu^{pr}(E_1 \sqcup E_2)$ and for all ω', $\omega(E_2) \geq \omega'(E_2)$. In this case, we also have that ω is a model for $\Delta_\mu^{pr}(E_2)$, and for every n, $\Delta_\mu^{pr}(E_1 \sqcup E_2^n) \models \Delta_\mu^{pr}(E_2)$; (ii) let ω be a model for $\Delta_\mu^{pr}(E_1 \sqcup E_2)$ and there is a ω' such that $\omega(E_2) < \omega'(E_2)$. In this case we can always find a number n of repetitions to E_2 such that ω' will be a model for $\Delta_\mu^{pr}(E_1 \sqcup E_2^n)$, i.e., $\omega'(E_2) \times n + \omega'(E_1) > \omega(E_2) \times n + \omega(E_1)$. Consequently, $\Delta_\mu^{pr}(E_1 \sqcup E_2^n) \models \Delta_\mu^{pr}(E_2)$.

$\square$

APPENDIX B – PROOF THEOREMS - CHAPTER 3

Theorem 3.1. **Proof: (IC0)** By definition, $mod(\Delta_{\mu}^{d,discrimax}(E)) \subseteq mod(\mu)$.

(IC1) As discrimax is not transitive we do not have a guarantee that always exists a ω such that $\omega \models \Delta_{\mu}^{d,discrimax}(E)$.

(IC2) By assumption, $\bigwedge E$ is consistent and without loss of generality let $E = \{K_1, \ldots, K_n\}$. There exists ω such that $\omega \models (c_{11} \vee \cdots \vee c_{1k}) \wedge \cdots \wedge (c_{n1} \vee \cdots \vee c_{nm})$, where $K_1 = \{c_{11} \vee \cdots \vee c_{1k}\}, \ldots, K_n = \{c_{n1} \vee \cdots \vee c_{nm}\}$. By definition, $d(\omega, K_i) = 0$ if $\omega \models K_i$, so we have $(d(\omega,K_1), \ldots, d(\omega,K_n)) = (0, 0, \ldots, 0)$. Therefore, for every ω', $(d(\omega,K_1), \ldots, d(\omega,K_n))$ $\leq_{disc}$ $(d(\omega',K_1), \ldots, d(\omega',K_n))$. So $\omega \models \Delta_{\mu}^{d,discrimax}(E)$ if and only if $\omega \models \bigwedge E \wedge \mu$.

(IC3) Let $E_1 = \{K_1, \ldots, K_n\}$ and $E_2 = \{K'_1, \ldots, K'_n\}$ be belief sets and suppose that $E_1 \equiv E_2$ and $\mu_1 \leftrightarrow \mu_2$. Hence we can find a function f such that for every $i, K_i \equiv f(K'_i)$, and if $(d(\omega,K_1), \ldots, d(\omega,K_n)) \leq_{disc} (d(\omega',K_1), \ldots, d(\omega',K_n))$, then $(d(\omega, f(K'_1)), \ldots, d(\omega, f(K'_n)))$ $\leq_{disc}$ $(d(\omega', f(K'_1)), \ldots, d(\omega', f(K'_n)))$. Consequently, $\Delta_{\mu_1}^{d,discrimax}(E_1) \equiv \Delta_{\mu_1}^{d,discrimax}(E_2)$.

(IC4) Let K_1 and K_2 be a belief bases and suppose that $\Delta_{\mu}^{d,discrimax}(\{K_1, K_2\}) \wedge K_1 \not\models \bot$. We have $\min\limits_{\omega \models K_1} (d(\omega,K_1),\, d(\omega,K_2)) = \min\limits_{\omega \models K_1} (0, d(\omega,K_2))$. By the definition of distance, it holds that $\min\limits_{\omega \models K_1} d(\omega,K_2) = \min\limits_{\omega \models K_2} d(\omega,K_1)$. Therefore, $\min\limits_{\omega \models K_1} (d(\omega,K_1),\, d(\omega,K_2)) = \min\limits_{\omega \models K_1} (0, d(\omega,K_2)) = \min\limits_{\omega \models K_2} (d(\omega,K_1), 0) = \min\limits_{\omega \models K_2} (d(\omega,K_1), d(\omega,K_2))$. Then,
$\Delta_{\mu}^{d,discrimax}(\{K_1, K_2\}) \wedge K_2 \not\models \bot$.

(IC5) It is enough to guarantee that if $(d(\omega,K_1), \ldots, d(\omega,K_n)) \leq_{disc} (d(\omega',K_1), \ldots, d(\omega',K_n))$ and $(d(\omega,K'_1), \ldots, d(\omega,K'_n)) \leq_{disc} (d(\omega',K'_1), \ldots, d(\omega',K'_n))$, then $(d(\omega,K_1), \ldots, d(\omega,K_n), d(\omega,K'_1), \ldots, d(\omega,K'_n)) \leq_{disc} (d(\omega',K_1), \ldots, d(\omega',K_n), d(\omega',K'_1), \ldots, d(\omega',K'_n))$. It is easy to see that this holds.

(IC6) *discrimax* falsifies **(IC6)** when it is the case that $discrimax = max$.

(IC7) Suppose that $\omega \models \Delta_{\mu_1}^{d,discrimax}(E) \wedge \mu_2$. For any $\omega' \models \mu_1$, we have $(d(\omega,K_1), \ldots, d(\omega,K_n)) \leq_{disc} (d(\omega',K_1), \ldots, d(\omega',K_n))$. Hence, for any $\omega' \models \mu_1 \wedge \mu_2$, we have $(d(\omega,K_1), \ldots, d(\omega,K_n)) \leq_{disc} (d(\omega',K_1), \ldots, d(\omega',K_n))$. As result, $\omega \models \Delta_{\mu_1 \wedge \mu_2}^{d,discrimax}(E)$.

(IC8) Suppose that $\Delta_{\mu_1}^{d,discrimax}(E) \wedge \mu_2$ is consistent. Then there exists a model ω' of $\Delta_{\mu_1}^{d,discrimax}(E) \wedge \mu_2$. Consider a model ω of $\Delta_{\mu_1 \wedge \mu_2}^{d,discrimax}(E)$ and suppose that $\omega \not\models \Delta_{\mu_1}^{d,discrimax}(E)$. In this case, $(d(\omega',K_1), \ldots, d(\omega',K_n)) <_{disc} (d(\omega,K_1), \ldots, d(\omega,K_n))$, and since $\omega' \models \mu_1 \wedge \mu_2$, we have $\omega \notin mod(\Delta_{\mu_1 \wedge \mu_2}^{d,discrimax}(E)) = min(mod\ (\mu_1 \wedge \mu_2), \leq_E^{d,discrimax})$. Hence $\omega \not\models \Delta_{\mu_1 \wedge \mu_2}^{d,discrimax}(E)$. Contradiction.

(Arb) We can show that the following property holds: If $\omega <_{K_1}^{d,discrimax} \omega'$, $\omega <_{K_2}^{d,discrimax} \omega''$ and $\omega' \approx_{\{K_1,K_2\}}^{d,discrimax} \omega''$, then $\omega <_{\{K_1,K_2\}}^{d,discrimax} \omega'$. Suppose that $\omega <_{K_1}^{d,discrimax} \omega', \omega <_{K_2}^{d,discrimax} \omega''$ and $\omega' \approx_{\{K_1,K_2\}}^{d,discrimax} \omega''$. Consequently, $d(\omega,K_1) < d(\omega',K_1)$, $d(\omega,K_2) < d(\omega'',K_2)$ and $(d(\omega',K_1), d(\omega',K_2)) =_{disc} (d(\omega'',K_1), d(\omega'',K_2))$. W.l.o.g. assume that $d(\omega,K_1) \leq d(\omega,K_2)$, then we have $(d(\omega,K_1), d(\omega, K_2)) <_{disc} (d(\omega',K_1), d(\omega',K_2))$. Therefore, $\omega <_{\{K_1,K_2\}}^{d,discrimax} \omega'$.

(HE) Suppose that $\exists i, j \in \{1,\dots,n\}$ such that $d(\omega,K_i) < d(\omega',K_i) < d(\omega',K_j) < d(\omega,K_j)$ and $\forall l \neq i, j \; d(\omega,K_l) = d(\omega',K_l)$, then $\max_{k \in D(\omega,\omega')} d(\omega',K_k) < \max_{k \in D(\omega,\omega')} d(\omega,K_k) \Rightarrow d(\omega',K_j) < d(\omega,K_j)$, where $D(\omega,\omega') = \{k \in \{1,\dots,n\} \mid d(\omega,K_i) \neq d(\omega',K_i)\}$. Therefore, $\omega' <_{E}^{d,discrimax} \omega$.

(SP) Similar to **(IC6)**.

(A) By definition, $a \leq_{disc} b \Leftrightarrow a = b$ or $\max_{i \in D(a,b)} a_i \leq \max_{i \in D(a,b)} b_i$. If b is a permutation of a then $a = b$ or $\max_{i \in D(a,b)} a_i = \max_{i \in D(a,b)} b_i$. Therefore, $a \approx_{disc} b$.

□

Theorem 3.2. **Proof:** Suppose that **(DM)** holds. Assume that $\exists i, j \in \{1,\dots,n\}$ such that $d(\omega,K_i) < d(\omega',K_i) \leq d(\omega',K_j) < d(\omega,K_j)$ and $\forall l \neq i, j \; d(\omega,K_l) = d(\omega',K_l)$. Then, it is true that $\omega' <_{E}^{d,op} \omega$, because by the definition of **(DM)**, $\exists j \in \{1,\dots,n\} : d(\omega,K_j) < d(\omega',K_j)$ and $\forall i\{1,\dots,n\}[d(\omega,K_i) \leq max(d(\omega',K_i), d(\omega',K_j))]$. **(SP)** also follows directly from **(DM)**.

For **(A)**, note that $l_{\omega}^{d,E} = (d_1^{\omega},\dots,d_n^{\omega})$, the list of distances, has a maximum value $max = d_k^{\omega}$ and for any permutation σ, $\exists i, d_{\sigma(i)}^{\omega} \leq d_k^{\omega}$. If $d_{\sigma(i)}^{\omega} < d_k^{\omega}$, then $\forall j$, $d_{\sigma(j)}^{\omega} \leq d_k^{\omega}$ and consequently, $\omega \leq_{E}^{d,op} \sigma(\omega)$ (with the same idea we can show the converse and that $\omega \approx_{E}^{d,op} \sigma(\omega)$). If $d_{\sigma(i)}^{\omega} = d_k^{\omega}$, we can repeat the same idea used before to find another maximum less than d_k^{ω} and consequently show that $\omega \leq_{E}^{d,op} \sigma(\omega)$. The case where σ is equal to the identity is trivial.

□

Theorem 3.3. **Proof:** **(IC0)** By definition, $\Delta_{\mu}^{d,\oplus}(E) \subseteq mod(\mu)$.

(IC1) The function $d_{\oplus}(\omega,E)$ maps to values in the interval $[0,1]$, so if $mod(\mu) \neq \emptyset$, there is a model ω of μ such that for every model ω' of μ, $d_{\oplus}(\omega,E) \leq d_{\oplus}(\omega',E)$. So $\omega \models \Delta_{\mu}^{d,\oplus}(E)$ and $\Delta_{\mu}^{d,\oplus}(E) \not\models \perp$.

(IC2) By assumption, $\bigwedge E$ is consistent and without loss of generality let $E = \{K_1,\dots, K_n\}$. There exists ω such that $\omega \models (c_{11} \vee \cdots \vee c_{1k}) \wedge \cdots \wedge (c_{n1} \vee \cdots \vee c_{nm})$, where $K_1 = \{c_{11} \vee \cdots \vee c_{1k}\},\dots,K_n = \{c_{n1} \vee \cdots \vee c_{nm}\}$. By definition, $d(\omega,K_i) = 0$ if $\omega \models K_i$, so we have

$d_{\oplus}(\omega, E) = \oplus\left\{\frac{d(\omega,K_1)}{M}, \ldots, \frac{d(\omega,K_n)}{M}\right\} = \{0,0,\ldots,0\} = 0$. Therefore, for every ω', $d_{\oplus}(\omega, E) \leq d_{\oplus}(\omega', E)$. So $\omega \models \Delta_{\mu}^{d,\oplus}(E)$ if and only if $\omega \models \bigwedge E \wedge \mu$.

(IC3) Let $E_1 = \{K_1, \ldots, K_n\}$ and $E_2 = \{K'_1, \ldots, K'_n\}$ be belief sets and suppose that $E_1 \equiv E_2$ and $\mu_1 \leftrightarrow \mu_2$. Hence we can find a permutation σ such that for every $i \in \{1, \ldots, n\}, K_{\sigma(i)} \equiv K'_i$. Now, since $\oplus$ satisfies symmetry (i.e. for any permutation σ, $\oplus\{x_1, \ldots, x_m\} = \oplus\{\sigma(x_1, \ldots, x_m)\}$) one gets $d_{\oplus}(\omega, E_1) = \oplus\left\{\frac{d(\omega,K_1)}{M}, \ldots, \frac{d(\omega,K_n)}{M}\right\} = d_{\oplus}(\omega, E_2)$. Consequently, $\Delta_{\mu_1}^{d,\oplus}(E_1) \equiv \Delta_{\mu_1}^{d,\oplus}(E_2)$.

(IC4) Let K_1 and K_2 be a belief bases and suppose that $\Delta_{\mu}^{d,\oplus}(\{K_1, K_2\}) \wedge K_1 \not\models \bot$. We have $\min_{\omega \models K_1} \oplus\{d(\omega, K_1), d(\omega, K_2)\} = \min_{\omega \models K_1} \oplus\{0, d(\omega, K_2)\} = \min_{\omega \models K_1} d(\omega, K_2)$. By the definition of distance, it holds that $\min_{\omega \models K_1} d(\omega, K_2) = \min_{\omega \models K_2} d(\omega, K_1)$. Therefore, $\min_{\omega \models K_1} \oplus\{d(\omega, K_1), d(\omega, K_2)\} = \min_{\omega \models K_1} d(\omega, K_2) = \min_{\omega \models K_2} d(\omega, K_1) = \min_{\omega \models K_2} \oplus\{d(\omega, K_1), 0\} = \min_{\omega \models K_2} \oplus\{d(\omega, K_1), d(\omega, K_2)\}$. Then, $\Delta_{\mu}^{d,\oplus}(\{K_1, K_2\}) \wedge K_2 \not\models \bot$.

(IC5) In order to show that the operator satisfy **(IC5)**, it is enough to guarantee that the following property holds: if $d_{\oplus}(\omega, E_1) \leq d_{\oplus}(\omega', E_1)$ and $d_{\oplus}(\omega, E_2) \leq d_{\oplus}(\omega', E_2)$, then $d_{\oplus}(\omega, E_1 \sqcup E_2) \leq d_{\oplus}(\omega', E_1 \sqcup E_2)$. Let $E_1 = \{K_{11}, \ldots, K_{1n}\}$, $E_2 = \{K_{21}, \ldots, K_{2m}\}$ and suppose that $d_{\oplus}(\omega, E_1) \leq d_{\oplus}(\omega', E_1)$ and $d_{\oplus}(\omega, E_2) \leq d_{\oplus}(\omega', E_2)$ hold. By definition,

$$\oplus\left\{\frac{d(\omega,K_{11})}{M}, \ldots, \frac{d(\omega,K_{1n})}{M}\right\} \leq \oplus\left\{\frac{d(\omega',K_{11})}{M}, \ldots, \frac{d(\omega',K_{1n})}{M}\right\} \text{ and}$$
$$\oplus\left\{\frac{d(\omega,K_{21})}{M}, \ldots, \frac{d(\omega,K_{2m})}{M}\right\} \leq \oplus\left\{\frac{d(\omega',K_{21})}{M}, \ldots, \frac{d(\omega',K_{2m})}{M}\right\}.$$

Then,

$$\oplus\left\{\frac{d(\omega,K_{11})}{M}, \ldots, \frac{d(\omega,K_{1n})}{M}, \frac{d(\omega,K_{21})}{M}, \ldots, \frac{d(\omega,K_{2m})}{M}\right\} \leq$$
$$\oplus\left\{\frac{d(\omega',K_{11})}{M}, \ldots, \frac{d(\omega',K_{1n})}{M}, \frac{d(\omega,K_{21})}{M}, \ldots, \frac{d(\omega,K_{2m})}{M}\right\}$$
$$\text{and } \oplus\left\{\frac{d(\omega,K_{21})}{M}, \ldots, \frac{d(\omega,K_{2m})}{M}, \frac{d(\omega',K_{11})}{M}, \ldots, \frac{d(\omega',K_{1n})}{M}\right\} \leq$$
$$\oplus\left\{\frac{d(\omega',K_{21})}{M}, \ldots, \frac{d(\omega',K_{2m})}{M}, \frac{d(\omega',K_{11})}{M}, \ldots, \frac{d(\omega',K_{1n})}{M}\right\},$$

as if $a \geq b$ then $\oplus\{a, c\} \geq \oplus\{b, c\}$ (monotonicity). Thus, by commutativity,

$$\oplus\left\{\frac{d(\omega,K_{11})}{M}, \ldots, \frac{d(\omega,K_{1n})}{M}, \frac{d(\omega,K_{21})}{M}, \ldots, \frac{d(\omega,K_{2m})}{M}\right\} \leq$$
$$\oplus\left\{\frac{d(\omega,K_{21})}{M}, \ldots, \frac{d(\omega,K_{2m})}{M}, \frac{d(\omega',K_{11})}{M}, \ldots, \frac{d(\omega',K_{1n})}{M}\right\} \text{ and}$$
$$\oplus\left\{\frac{d(\omega,K_{21})}{M}, \ldots, \frac{d(\omega,K_{2m})}{M}, \frac{d(\omega',K_{11})}{M}, \ldots, \frac{d(\omega',K_{1n})}{M}\right\} \leq$$
$$\oplus\left\{\frac{d(\omega',K_{11})}{M}, \ldots, \frac{d(\omega',K_{1n})}{M}, \frac{d(\omega',K_{21})}{M}, \ldots, \frac{d(\omega',K_{2m})}{M}\right\}.$$

By transitivity, we have that

$$\oplus\left\{\frac{d(\omega,K_{11})}{M},\dots,\frac{d(\omega,K_{1n})}{M},\frac{d(\omega,K_{21})}{M},\dots,\frac{d(\omega,K_{2m})}{M}\right\}\leq$$
$$\oplus\left\{\frac{d(\omega',K_{11})}{M},\dots,\frac{d(\omega',K_{1n})}{M},\frac{d(\omega',K_{21})}{M},\dots,\frac{d(\omega',K_{2m})}{M}\right\}.$$

That is, $d_{\oplus}(\omega,E_1 \sqcup E_2) \leq d_{\oplus}(\omega',E_1 \sqcup E_2)$.

(IC7) Suppose that $\omega \models \Delta^{d,\oplus}_{\mu_1}(E) \wedge \mu_2$. For any $\omega' \models \mu_1$, we have $d_{\oplus}(\omega,E) \leq d_{\oplus}(\omega',E)$. Hence, for any $\omega' \models \mu_1 \wedge \mu_2$, we have $d_{\oplus}(\omega,E) \leq d_{\oplus}(\omega',E)$. As result, $\omega \models \Delta^{d,\oplus}_{\mu_1\wedge\mu_2}(E)$.

(IC8) Suppose that $\Delta^{d,\oplus}_{\mu_1}(E) \wedge \mu_2$ is consistent. Then there exists a model ω' of $\Delta^{d,\oplus}_{\mu_1}(E) \wedge \mu_2$. Consider a model ω of $\Delta^{d,\oplus}_{\mu_1\wedge\mu_2}(E)$ and suppose that $\omega \not\models \Delta^{d,\oplus}_{\mu_1}(E)$. In this case, $d_{\oplus}(\omega',E) < d_{\oplus}(\omega,E)$, and since $\omega' \models \mu_1 \wedge \mu_2$, we have $\omega \notin mod(\Delta^{d,\oplus}_{\mu_1\wedge\mu_2}(E)) = min(mod(\mu_1 \wedge \mu_2), \leq^{d,\oplus}_{E})$. Hence $\omega \not\models \Delta^{d,\oplus}_{\mu_1\wedge\mu_2}(E)$. Contradiction.

(Maj) We can find a counter-example where the repetition of one base does not change the result. Consider the following counter-example: Let $\mu = \top$, $E_1 = \{K_1\} = \{\{a \wedge b\}\}$ and $E_2 = \{K_2\} = \{\{\neg a \wedge \neg b\}\}$. Clearly, we have $\Delta^{d,\oplus}_{\mu}(E_1 \sqcup \underbrace{E_2 \sqcup \cdots \sqcup E_2}_{n}) \not\equiv \Delta^{d,\oplus}_{\mu}(E_2)$ for any T-conorm $\oplus$ and $n \in \mathbb{N}$.

□

Theorem 3.4. **Proof:** -$\Delta^{d,\oplus_{\mathbf{M}}}_{\mu}$:

(Arb) To show that the operator satisfies **(Arb)**, we can show that the following property holds: If $\omega <^{d,\oplus_{\mathbf{M}}}_{K_1} \omega', \omega <^{d,\oplus_{\mathbf{M}}}_{K_2} \omega''$ and $\omega' \approx^{d,\oplus_{\mathbf{M}}}_{\{K_1,K_2\}} \omega''$, then $\omega <^{d,\oplus_{\mathbf{M}}}_{\{K_1,K_2\}} \omega'$.

Suppose that $\omega <^{d,\oplus_{\mathbf{M}}}_{K_1} \omega', \omega <^{d,\oplus_{\mathbf{M}}}_{K_2} \omega''$ and $\omega' \approx^{d,\oplus_{\mathbf{M}}}_{\{K_1,K_2\}} \omega''$. Then we have $d(\omega,K_1) < d(\omega',K_1)$, $d(\omega,K_2) < d(\omega'',K_2)$ and $max\left\{\frac{d(\omega',K_1)}{M},\frac{d(\omega',K_2)}{M}\right\} = max\left\{\frac{d(\omega'',K_1)}{M},\frac{d(\omega'',K_2)}{M}\right\}$. W.l.o.g. assume that $d(\omega,K_1) < d(\omega,K_2)$ and that if

- $max\left\{\frac{d(\omega',K_1)}{M},\frac{d(\omega',K_2)}{M}\right\} = \frac{d(\omega',K_1)}{M}$. Then, $max\left\{\frac{d(\omega,K_1)}{M},\frac{d(\omega,K_2)}{M}\right\} = \frac{d(\omega,K_2)}{M}$ and $\frac{d(\omega,K_2)}{M} < \frac{d(\omega',K_1)}{M}$. Suppose that $\frac{d(\omega,K_2)}{M} \not< \frac{d(\omega',K_1)}{M}$. As $\frac{d(\omega,K_2)}{M} < \frac{d(\omega'',K_2)}{M}$, then $max\left\{\frac{d(\omega'',K_1)}{M},\frac{d(\omega'',K_2)}{M}\right\} = \frac{d(\omega'',K_2)}{M} \neq \frac{d(\omega',K_1)}{M}$. Contradiction.
  Therefore, $\omega <^{d,\oplus_{\mathbf{M}}}_{\{K_1,K_2\}} \omega'$.
- $max\left\{\frac{d(\omega',K_1)}{M},\frac{d(\omega',K_2)}{M}\right\} = \frac{d(\omega',K_2)}{M}$. We have that $\frac{d(\omega,K_1)}{M} < \frac{d(\omega',K_2)}{M}$ and $\frac{d(\omega,K_2)}{M} < \frac{d(\omega',K_2)}{M}$ (previous item). Therefore, $\omega <^{d,\oplus_{\mathbf{M}}}_{\{K_1,K_2\}} \omega'$.

(HP) Suppose that $\omega_1 <^{d,\oplus_{\mathbf{M}}}_{E} \omega_2$. Then, by definition $max\left\{\frac{d(\omega_1,K_1)}{M},\dots,\frac{d(\omega_1,K_n)}{M}\right\} < max\left\{\frac{d(\omega_2,K_1)}{M},\dots,\frac{d(\omega_2,K_n)}{M}\right\}$. Consider ω_1', ω_2' such that $\exists i \in \{1,\dots,n\}$, $d(\omega_1,K_i) < d(\omega_1',K_i)$, $d(\omega_2,K_i) < d(\omega_2',K_i)$ and $\forall j \neq i$ $d(\omega_1,K_j) = d(\omega_1',K_j)$, $d(\omega_2,K_j) = d(\omega_2',K_j)$.

If $d(\omega_2',K_i) > d(\omega_1',K_i)$, then we need to show that $max\left\{\frac{d(\omega_1,K_1)}{M},\ldots,\frac{d(\omega_1',K_i)}{M},\ldots,\frac{d(\omega_1,K_n)}{M}\right\} < max\left\{\frac{d(\omega_2,K_1)}{M},\ldots,\frac{d(\omega_2',K_i)}{M},\ldots,\frac{d(\omega_2,K_n)}{M}\right\}$:

- If $max\left\{\frac{d(\omega_1,K_1)}{M},\ldots,\frac{d(\omega_1,K_n)}{M}\right\} = \frac{d(\omega_1,K_1)}{M}$, then it is clear that $max\left\{\frac{d(\omega_1,K_1)}{M},\ \ldots,\frac{d(\omega_1',K_i)}{M},\ldots,\frac{d(\omega_1,K_n)}{M}\right\} < max\left\{\frac{d(\omega_2,K_1)}{M},\ldots,\frac{d(\omega_2',K_i)}{M},\ldots,\frac{d(\omega_2,K_n)}{M}\right\}$, because $\frac{d(\omega_2',K_i)}{M} > \frac{d(\omega_1',K_i)}{M}$.
- If $max\left\{\frac{d(\omega_1,K_1)}{M},\ldots,\frac{d(\omega_1,K_n)}{M}\right\} \neq \frac{d(\omega_1,K_1)}{M}$, but $max\left\{\frac{d(\omega_1,K_1)}{M},\ldots,\frac{d(\omega_1',K_i)}{M},\ \ldots,\frac{d(\omega_1,K_n)}{M}\right\} = \frac{d(\omega_1',K_1)}{M}$. Then $max\left\{\frac{d(\omega_1,K_1)}{M},\ldots,\frac{d(\omega_1',K_i)}{M},\ldots,\frac{d(\omega_1,K_n)}{M}\right\} <$ $max\left\{\frac{d(\omega_2,K_1)}{M},\ldots,\frac{d(\omega_2',K_i)}{M},\ldots,\frac{d(\omega_2,K_n)}{M}\right\}$ is true since $\frac{d(\omega_2',K_i)}{M} > \frac{d(\omega_1',K_i)}{M}$.
- If $max\left\{\frac{d(\omega_1,K_1)}{M},\ldots,\frac{d(\omega_1,K_n)}{M}\right\} \neq \frac{d(\omega_1,K_1)}{M}$, and $max\left\{\frac{d(\omega_1,K_1)}{M},\ldots,\frac{d(\omega_1',K_i)}{M},\ \ldots,\frac{d(\omega_1,K_n)}{M}\right\} \neq \frac{d(\omega_1',K_1)}{M}$. Then $max\left\{\frac{d(\omega_1,K_1)}{M},\ldots,\frac{d(\omega_1',K_i)}{M},\ldots,\frac{d(\omega_1,K_n)}{M}\right\} <$ $max\left\{\frac{d(\omega_2,K_1)}{M},\ldots,\frac{d(\omega_2',K_i)}{M},\ldots,\frac{d(\omega_2,K_n)}{M}\right\}$ since $max\left\{\frac{d(\omega_1,K_1)}{M},\ldots,\frac{d(\omega_1,K_n)}{M}\right\}$ $< max\left\{\frac{d(\omega_2,K_1)}{M},\ldots,\frac{d(\omega_2,K_n)}{M}\right\}$.

-$\Delta_\mu^{d,\oplus_\mathbf{P}}$:

(IC6-1**)** To show that **(IC6-**1**)** is satisfied for $\Delta_\mu^{d,\oplus_\mathbf{P}}$, it is enough to guarantee that the following property holds: if $d_{\oplus_\mathbf{P}}(\omega,E_1) < d_{\oplus_\mathbf{P}}(\omega',E_1)$ and $d_{\oplus_\mathbf{P}}(\omega,E_2) \leq d_{\oplus_\mathbf{P}}(\omega',E_2) \neq 1$, then $d_{\oplus_\mathbf{P}}(\omega,E_1 \sqcup E_2) < d_{\oplus_\mathbf{P}}(\omega',E_1 \sqcup E_2)$. We can see that this is satisfied.

(PD-1**)** The Pigou-Dalton (with annihilator 1) condition can be defined alternatively in the following way: Let d be a distance measure. An operator op satisfies the Pigou-Dalton principle iff for any belief set $E = \{K_1,\ldots,K_n\}$, if $\exists i,j \in \{1,\ldots,n\}$ such that $d(\omega,K_i) < d(\omega',K_i) \leq d(\omega',K_j) < d(\omega,K_j)$, $d(\omega',K_i) = d(\omega,K_i)+\delta$, $d(\omega',K_j) = d(\omega,K_j)-\delta$ and $\forall l \neq i,j$ $d(\omega,K_l) = d(\omega',K_l)$, then $d_{op}(\omega',E) < d_{op}(\omega,E)$.

So, assume that $d(\omega,K_i) = x$, $d(\omega,K_j) = y$, $d(\omega',K_i) = x+\delta$ and $d(\omega',K_j) = y-\delta$. We want to show that $\frac{x}{M}+\frac{y}{M}-\frac{x}{M}\cdot\frac{y}{M} > \frac{(x+\delta)}{M}+\frac{(y-\delta)}{M}-\frac{(x+\delta)}{M}\cdot\frac{(y-\delta)}{M}$, then

$$
\begin{aligned}
\frac{x}{M}+\frac{y}{M}-\frac{x}{M}\cdot\frac{y}{M} &> \frac{x}{M}+\frac{y}{M}-(\frac{x}{M}\cdot\frac{y}{M}-\frac{\delta}{M}\cdot\frac{x}{M}+\frac{\delta}{M}\cdot\frac{y}{M}-\frac{\delta^2}{M^2}),\\
&> \frac{x}{M}+\frac{y}{M}-(\frac{x}{M}\cdot\frac{y}{M}+\frac{\delta}{M^2}\cdot(\underbrace{y-x}_{>\delta}-\delta)).
\end{aligned}
$$

So, we have $\omega' <_E^{d,\oplus_\mathbf{P}} \omega$.

(Arb) To show that the operator does not satisfy **(Arb)**, we can show that the following property does not hold: If $\omega <_{K_1}^{d,\oplus_\mathbf{P}} \omega'$, $\omega <_{K_2}^{d,\oplus_\mathbf{P}} \omega''$ and $\omega' \approx_{\{K_1,K_2\}}^{d,\oplus_\mathbf{P}} \omega''$, then $\omega <_{\{K_1,K_2\}}^{d,\oplus_\mathbf{P}}$

ω'. Suppose that $M=4$, $d(\omega,K_1)=d(\omega,K_2)=2$, $d(\omega',K_1)=d(\omega'',K_2)=3$ and $d(\omega',K_2)=d(\omega'',K_1)=0$. Then,

- $\omega <^{d,\oplus_{\mathbf{P}}}_{K_1} \omega'$, since $d(\omega,K_1)<d(\omega',K_1)$.
- $\omega <^{d,\oplus_{\mathbf{P}}}_{K_2} \omega''$, since $d(\omega,K_2)<d(\omega'',K_2)$.
- $\omega' \approx^{d,\oplus_{\mathbf{P}}}_{\{K_1,K_2\}} \omega''$, since $\frac{3}{4}+0-\frac{3}{4}\cdot 0=0+\frac{3}{4}-0\cdot\frac{3}{4}=\frac{3}{4}$.

But, $\omega \approx^{d,\oplus_{\mathbf{P}}}_{\{K_1,K_2\}} \omega'$, since $\frac{1}{2}+\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2}=\frac{3}{4}=\frac{3}{4}+0-\frac{3}{4}\cdot 0$.

(HE-1**)** A counterexample for $\oplus_{\mathbf{P}}$ is $M=4$, $d(\omega,K_1)=3, d(\omega,K_2)=0, d(\omega',K_1)=d(\omega',K_2)=2$. We have $d(\omega,K_1)>d(\omega',K_1)\geq d(\omega',K_2)>d(\omega,K_2)$, but $\omega' <^{d,\oplus_{\mathbf{P}}}_{\{K_1,K_2\}} \omega$ is false.

(HP) A counterexample for $\oplus_{\mathbf{P}}$ is $M=6$, $d(\omega_1,K_1)=4, d(\omega_1,K_2)=0, d(\omega_2,K_1)=1, d(\omega_2,K_2)=4$, $d(\omega_1',K_1)=4, d(\omega_1',K_2)=4, d(\omega_2',K_1)=1, d(\omega_2',K_2)=5$.

$\omega_1 <^{d,\oplus_{\mathbf{P}}}_{K_1,K_2} \omega_2$ since $\frac{4}{6}+\frac{0}{6}-\frac{4}{6}\cdot\frac{0}{6}=0.666<0.722=\frac{1}{6}+\frac{4}{6}-\frac{1}{6}\cdot\frac{4}{6}$, but $\omega_1' <^{d,\oplus_{\mathbf{P}}}_{K_1,K_2} \omega_2'$ is false since $\frac{4}{6}+\frac{4}{6}-\frac{4}{6}\cdot\frac{4}{6}=0.888>0.861=\frac{1}{6}+\frac{5}{6}-\frac{1}{6}\cdot\frac{5}{6}$.

$\square$

Theorem 3.5. **Proof:** Let $\oplus$ be a strict T-conorm, i.e., $\oplus\{x,y\}<\oplus\{x,z\}$ whenever $x<1$ and $y<z$. Assume that $d_{\oplus}(\omega,E_1)<d_{\oplus}(\omega',E_1)$ $(y<z)$ and $d_{\oplus}(\omega,E_2)\leq d_{\oplus}(\omega',E_2)\neq 1$ $(x\leq x'<1)$. Assume in the worst case that $d_{\oplus}(\omega,E_2)=d_{\oplus}(\omega',E_2)\neq 1$. As $\oplus$ is strict, by its definition we have then $d_{\oplus}(\omega,E_1\sqcup E_2)<d_{\oplus}(\omega',E_1\sqcup E_2)$. Clearly, this is the definition of **(IC6-**1**)**.

$\square$

Proposition 3.2. **Proof: (IC6-**1**)** The property $d_{\oplus}(\omega,E_1)<d_{\oplus}(\omega',E_1)$ and $d_{\oplus}(\omega,E_2)\leq d_{\oplus}(\omega',E_2)\neq 1 \Rightarrow d_{\oplus}(\omega,E_1\sqcup E_2)<d_{\oplus}(\omega',E_1\sqcup E_2)$ does not hold for the T-conorms $\oplus_{\mathbf{L}}$ and $\oplus_{\mathbf{D}}$, a counter-example is $E_1=\{K_1,K_2\}$, where $K_1=\{(\neg a\wedge b)\}$, $K_2=\{(a\wedge b)\}$ and $E_2=\{K_3,K_4\}$, where $K_3=\{(a\wedge b)\vee(\neg a\wedge\neg b)\}$ and $K_4=\{(\neg b\wedge a)\}$.

(Arb) Consider the following counter-example: $K_1=\{a\wedge b\}, K_2=\{\neg a\wedge\neg b\}, \mu_1=\neg(a\wedge b)$ and $\mu_2=a\vee b$. We have that $\Delta^{d,\oplus}_{\mu_1\vee\mu_2}(\{K_1,K_2\})\not\equiv\Delta^{d,\oplus}_{\mu_1}(\{K_1\})$, when $\oplus\in\{\oplus_{\mathbf{L}},\oplus_{\mathbf{D}}\}$.

(HE-1**)**, and **(PD-**1**)**. It is easy to see some examples where these properties are not satisfied for $\oplus_{\mathbf{L}}$ and $\oplus_{\mathbf{D}}$. For instance, when $M=5$, $d(\omega,K_i)=1$, $d(\omega,K_j)=3$, $d(\omega',K_i)=2$ and $d(\omega',K_j)=2$, **(HE-**1**)** and **(PD-**1**)** are not satisfied.

(HP) A counterexample for $\oplus_{\mathbf{L}}$ is $M=4$, $d(\omega_1,K_1)=1, d(\omega_1,K_2)=2, d(\omega_2,K_1)=2, d(\omega_2,K_2)=3$, $d(\omega_1',K_1)=1, d(\omega_1',K_2)=3, d(\omega_2',K_1)=2, d(\omega_2',K_2)=4$. A counterexample for $\oplus_{\mathbf{D}}$ is $M=4$, $d(\omega_1,K_1)=1, d(\omega_1,K_2)=0, d(\omega_2,K_1)=1, d(\omega_2,K_2)=1$, $d(\omega_1',K_1)=$

$1, d(\omega_1', K_2) = 1, d(\omega_2', K_1) = 1, d(\omega_2', K_2) = 2.$

$\square$

Proposition 3.3. **Proof:** Let $\oplus$ be a nilpotent T-conorm, i.e., there is an element $a \in]0,1[$ called nilpotent element that there exists some $n \in \mathbb{N}$ such that $\oplus \underbrace{\{a,\ldots,a\}}_{n} = 1$. For **(IC6-1)**, assume that $d_\oplus(\omega, E_1) < d_\oplus(\omega', E_1)$ and $d_\oplus(\omega, E_2) \leq d_\oplus(\omega', E_2) \neq 1$. If $d_\oplus(\omega, E_1) \geq a$ and $d_\oplus(\omega, E_2) \geq a$, then it is possible that $d_\oplus(\omega, E_1 \sqcup E_2) = 1$ and consequently $d_\oplus(\omega', E_1 \sqcup E_2) = 1$, falsifying the postulate.

The idea is similar for **(HE-1)** and **(PD-1)**. The problem comes from the condition $\forall l \neq i,j\ d(\omega, K_l) = d(\omega', K_l) \neq 1$ contained in both postulates. If $\forall l \neq i,j\ d(\omega, K_l) = d(\omega', K_l) \geq a$, then possibly $d_\oplus(\omega, E) = d_\oplus(\omega', E) = 1$, falsifying the postulates.

For **(HP)**, it is similar: for $\forall j \neq i$ if $d(\omega, K_j) \geq a$, then possibly $d_\oplus(\omega, E) = 1$.

$\square$

Theorem 3.6. **Proof:** **(IC6-1)** It is enough to guarantee that the following property holds: if $d_{\oplus_\lambda^{\mathrm{SS}}}(\omega, E_1) < d_{\oplus_\lambda^{\mathrm{SS}}}(\omega', E_1)$ and $d_{\oplus_\lambda^{\mathrm{SS}}}(\omega, E_2) \leq d_{\oplus_\lambda^{\mathrm{SS}}}(\omega', E_2) \neq 1$, then $d_{\oplus_\lambda^{\mathrm{SS}}}(\omega, E_1 \sqcup E_2) < d_{\oplus_\lambda^{\mathrm{SS}}}(\omega', E_1 \sqcup E_2)$, for $\lambda \in]-\infty,-1]$ (for $\lambda = 0$, $\oplus_\lambda^{\mathbf{SS}} = \oplus_{\mathbf{P}}$).

Assume that $d_{\oplus_\lambda^{\mathrm{SS}}}(\omega, E_1) = x_1$, $d_{\oplus_\lambda^{\mathrm{SS}}}(\omega, E_2) = y_1$, $d_{\oplus_\lambda^{\mathrm{SS}}}(\omega', E_1) = x_2$, $d_{\oplus_\lambda^{\mathrm{SS}}}(\omega', E_2) = y_2$, $x_1 < x_2$ and $y_1 \leq y_2 \neq 1$. We want to show that $1 - (max(((1-x_1)^\lambda + (1-y_1)^\lambda - 1), 0))^{\frac{1}{\lambda}} < 1 - (max(((1-x_2)^\lambda + (1-y_2)^\lambda - 1), 0))^{\frac{1}{\lambda}}$. As $x_1 < x_2$ ($y_1 \leq y_2$), then $(1-x_1)^\lambda < (1-x_2)^\lambda$ ($(1-y_1)^\lambda \leq (1-y_2)^\lambda$), since $\lambda \in]-\infty,-1]$. Then,

$$
\begin{aligned}
1 - (max(((1-x_1)^\lambda + (1-y_1)^\lambda - 1), 0))^{\frac{1}{\lambda}} &< 1 - (max(((1-x_2)^\lambda + (1-y_2)^\lambda - 1), 0))^{\frac{1}{\lambda}}, \\
(max((\underbrace{(1-x_1)^\lambda + (1-y_1)^\lambda}_{>1} - 1), 0))^{\frac{1}{\lambda}} &> (max((\underbrace{(1-x_2)^\lambda + (1-y_2)^\lambda}_{>1} - 1), 0))^{\frac{1}{\lambda}}, \\
max((\underbrace{(1-x_1)^\lambda + (1-y_1)^\lambda}_{>1} - 1), 0) &< max((\underbrace{(1-x_2)^\lambda + (1-y_2)^\lambda}_{>1} - 1), 0), (\lambda \in]-\infty,-1]) \\
(1-x_1)^\lambda + (1-y_1)^\lambda - 1 &< (1-x_2)^\lambda + (1-y_2)^\lambda - 1, \\
(1-x_1)^\lambda + (1-y_1)^\lambda &< (1-x_2)^\lambda + (1-y_2)^\lambda.
\end{aligned}
$$

To prove **(HE-1)**, we have to show that if $\exists i,j \in \{1,\ldots,n\}$ such that $d(\omega, K_i) > d(\omega', K_i) \geq d(\omega', K_j) > d(\omega, K_j)$ and $\forall l \neq i,j\ d(\omega, K_l) = d(\omega', K_l) \neq 1$, then $\omega' <_E^{d,\oplus_\lambda^{\mathrm{SS}}} \omega$. Let n be the number of variables in E and $\lambda = -\left\lfloor \frac{2n}{3} \right\rfloor$. We will consider only the worst cases, where $d(\omega, K_i)$ and $d(\omega, K_j)$ have the lowest values and $d(\omega', K_i)$ and $d(\omega', K_j)$ have the highest values. Proving for these cases, will also guarantee the result for any arbitrary values. For instance, if $n = 4$, we can check the property for the case where $d(\omega, K_i) = 3$, $d(\omega, K_j) = 0$ and $d(\omega', K_i) = d(\omega', K_j) = 2$.

To show **(HE-**1**)**, we need to guarantee only that $\left(1 - \frac{d(\omega,K_i)}{M}\right)^{\lambda} + \left(1 - \frac{d(\omega,K_j)}{M}\right)^{\lambda} > \left(1 - \frac{d(\omega',K_i)}{M}\right)^{\lambda} + \left(1 - \frac{d(\omega',K_j)}{M}\right)^{\lambda}$.

For $n \geq 3$ and $\lambda = -\frac{2n}{3}$:

$$\left(1 - \frac{d(\omega, K_i)}{n}\right)^{\lambda} + \left(1 - \frac{d(\omega, K_j)}{n}\right)^{\lambda} > \left(1 - \frac{d(\omega', K_i)}{n}\right)^{\lambda} + \left(1 - \frac{d(\omega', K_j)}{n}\right)^{\lambda},$$

$$\left(1 - \frac{y}{n}\right)^{\lambda} + (1-0)^{\lambda} > 2.\left(1 - \frac{y-1}{n}\right)^{\lambda},$$

$$\left(\frac{n-y}{n}\right)^{-\frac{2n}{3}} > 2.\left(\frac{n-y+1}{n}\right)^{-\frac{2n}{3}}, (1 \leq y \leq n-2)$$

$$\left(\frac{n}{n-y}\right)^{\frac{2n}{3}} > 2.\left(\frac{n}{n-y+1}\right)^{\frac{2n}{3}},$$

$$\frac{n^{\frac{2n}{3}}}{(n-y)^{\frac{2n}{3}}} > 2.\frac{n^{\frac{2n}{3}}}{(n-y+1)^{\frac{2n}{3}}}, (x = n-y)$$

$$\frac{1}{x^{\frac{2n}{3}}} > 2.\frac{1}{(x+1)^{\frac{2n}{3}}},$$

$$2.x^{\frac{2n}{3}} < (x+1)^{\frac{2n}{3}},$$

$$8.x^{2n} < (x+1)^{2n}.$$

We have the following results:

- $\binom{2n}{1}.x^{2n-1} = 2n.x^{2n-1} > 2x.x^{2n-1} = 2.x^{2n}$. (since $n > x$)
- $\binom{2n}{2}.x^{2n-2} = 2n.(2n-1).x^{2n-2} = 4.x^{2n} - 2.x^{2n-1} = 2.x^{2n}.\underbrace{(2 - \frac{1}{x})}_{\geq 1}.$
- $\binom{2n}{3}.x^{2n-3} = 2n.(2n-1).(2n-2).x^{2n-3} = 8.x^{2n} - 12.x^{2n-1} + 4.x^{2n-2} = 4.x^{2n}.\underbrace{(2 - \frac{3}{x} + \frac{1}{x^2})}_{> \frac{1}{2}}.$
- $\binom{2n}{4}.x^{2n-4} = 2n.(2n-1).(2n-2).(2n-3).x^{2n-4} = 8.x^{2n} - 36.x^{2n-1} + 44.x^{2n-2} - 12.x^{2n-3} = 4.x^{2n}.\underbrace{(2 - \frac{9}{x} + \frac{11}{x^2} - \frac{3}{x^3})}_{>0}.$

Then, $8.x^{2n} < x^{2n} + \underbrace{\binom{2n}{1}.x^{2n-1}}_{>2.x^{2n}} + \underbrace{\binom{2n}{2}.x^{2n-2}}_{>2.x^{2n}} + \underbrace{\binom{2n}{3}.x^{2n-3}}_{>2.x^{2n}} +$

$\underbrace{\binom{2n}{4}.x^{2n-4}}_{\approx x^{2n} \text{ to } >2.x^{2n}} + \cdots + 1.$

(Arb) To prove that the operator satisfies **(Arb)**, we can show that the following property holds: If $\omega <_{K_1}^{d,\oplus_\lambda^{\mathbf{SS}}} \omega', \omega <_{K_2}^{d,\oplus_\lambda^{\mathbf{SS}}} \omega''$ and $\omega' \approx_{\{K_1,K_2\}}^{d,\oplus_\lambda^{\mathbf{SS}}} \omega''$, then $\omega <_{\{K_1,K_2\}}^{d,\oplus_\lambda^{\mathbf{SS}}} \omega'$. W.l.o.g. we can consider that $d(\omega',K_1) > d(\omega,K_1) \geq d(\omega,K_2) > d(\omega',K_2)$ and treat the problem as a restriction of the Hammond Equity condition. Consequently, $\omega <_{\{K_1,K_2\}}^{d,\oplus_\lambda^{\mathbf{SS}}} \omega'$ will hold when $\lambda \leq -\lfloor \frac{2n}{3} \rfloor$.

(HP) It comes as a consequence of the fact that **(HE-**1**)** + **(SP-**1**)** + **(A)** is equivalent to **(HP)** + **(SP-**1**)** + **(A)**.

(PD-1**)** This condition can be defined in the following way: If $\exists i,j \in \{1,\ldots,n\}$, $\delta > 0$ such that $d(\omega,K_i) > d(\omega',K_i) \geq d(\omega',K_j) > d(\omega,K_j)$, $d(\omega',K_i) = d(\omega,K_i) - \delta$ and $d(\omega',K_j) = d(\omega,K_j) + \delta$ and $\forall l \neq i,j \; d(\omega,K_l) = d(\omega',K_l) \neq 1$, then $\omega' <_E^{d,\oplus_\lambda^{\mathbf{SS}}} \omega$.

So, assume that $d(\omega,K_i) = x$, $d(\omega,K_j) = y$, $d(\omega',K_i) = x - \delta$ and $d(\omega',K_j) = y + \delta$. We want to show that $1 - (max(((1-x)^\lambda + (1-y)^\lambda - 1), 0))^{\frac{1}{\lambda}} > 1 - (max(((1-(x-\delta))^\lambda + (1-(y+\delta))^\lambda - 1), 0))^{\frac{1}{\lambda}}$. It is equivalent to show that (considering $\lambda \in]-\infty, -1]$):

$$\left(1 - \frac{x}{M}\right)^\lambda + \left(1 - \frac{y}{M}\right)^\lambda > \left(1 - \frac{x-\delta}{M}\right)^\lambda + \left(1 - \frac{y+\delta}{M}\right)^\lambda,$$

$$\left(\frac{M-x}{M}\right)^\lambda + \left(\frac{M-y}{M}\right)^\lambda > \left(\frac{M-x+\delta}{M}\right)^\lambda + \left(\frac{M-y-\delta}{M}\right)^\lambda.$$

Assume that $M = n$, $M - x = a$, $M - y = b$ and $\delta = c$. As $x > y$ and $x - \delta \geq y + \delta$, when $x + \delta = y - \delta$, then we have $y = x - 2.\delta - \frac{d}{n}$ and:

$$
\begin{aligned}
\left(\frac{a}{n}\right)^{\lambda}+\left(\frac{b}{n}\right)^{\lambda} &> \left(\frac{a}{n}-\frac{c}{n}\right)^{\lambda}+\left(\frac{b}{n}+\frac{c}{n}\right)^{\lambda}, (\text{ let } \lambda=-m)\\
\left(\frac{a}{n}\right)^{-m}+\left(\frac{a}{n}-\frac{2.c}{n}-\frac{d}{n}\right)^{-m} &> \left(\frac{a}{n}-\frac{c}{n}\right)^{-m}+\left(\frac{a}{n}-\frac{c}{n}-\frac{d}{n}\right)^{-m},\\
\left(\frac{a}{n}\right)^{-m}+\left(\frac{a-2.c-d}{n}\right)^{-m} &> \left(\frac{a-c}{n}\right)^{-m}+\left(\frac{a-c-d}{n}\right)^{-m},\\
\frac{n^m}{a^m}+\frac{n^m}{(a-2.c-d)^m} &> \frac{n^m}{(a-c)^m}+\frac{n^m}{(a-c-d)^m},\\
\frac{1}{a^m}+\frac{1}{(a-2.c-d)^m} &> \frac{1}{(a-c)^m}+\frac{1}{(a-c-d)^m},\\
1+\frac{a^m}{(a-2.c-d)^m} &> \frac{a^m}{(a-c)^m}+\frac{a^m}{(a-c-d)^m},\\
(a-2.c-d)^m+a^m &> \frac{a^m.(a-2.c-d)^m}{(a-c)^m}+\frac{a^m.(a-2.c-d)^m}{(a-c-d)^m},\\
(a-c)^m.(a-c-d)^m.((a-2.c-d)^m+a^m) &> a^m.(a-2.c-d)^m.((a-c-d)^m+(a-c)^m),\\
(a-c)^m.(a-2c-d)^m.((a-c-d)^m-a^m) &> a^m.(a-c-d)^m.((a-2c-d)^m-(a-c)^m),\\
(a-c-d)^m-a^m &> a^m.(a-c-d)^m.\underbrace{\frac{((a-2c-d)^m-(a-c)^m)}{(a-2c-d)^m.(a-c)^m}}_{<\frac{1}{a^m}}.
\end{aligned}
$$

So, we have $\omega' <_E^{ps,\oplus_\lambda^{\mathbf{SS}}} \omega$.

$\square$

Theorem 3.7. **Proof: (IC6-**1**)** It is enough to guarantee that the following property holds: if $d_{\oplus_\lambda^{\mathbf{F}}}(\omega,E_1) < d_{\oplus_\lambda^{\mathbf{F}}}(\omega',E_1)$ and $d_{\oplus_\lambda^{\mathbf{F}}}(\omega,E_2) \leq d_{\oplus_\lambda^{\mathbf{F}}}(\omega',E_2) \neq 1$, then $d_{\oplus_\lambda^{\mathbf{F}}}(\omega,E_1 \sqcup E_2) < d_{\oplus_\lambda^{\mathbf{F}}}(\omega',E_1 \sqcup E_2)$, for $\lambda \in [1,\infty[$. Assume that $d_{\oplus_\lambda^{\mathbf{F}}}(\omega,E_1) = x_1$, $d_{\oplus_\lambda^{\mathbf{F}}}(\omega,E_2) = y_1$, $d_{\oplus_\lambda^{\mathbf{F}}}(\omega',E_1) = x_2$, $d_{\oplus_\lambda^{\mathbf{F}}}(\omega',E_2) = y_2$, $x_1 < x_2$ and $y_1 \leq y_2$. So,

$$
1-log_\lambda\left(1+\underbrace{\frac{(\lambda^{1-\frac{x_2}{M}}-1).(\lambda^{1-\frac{y_2}{M}}-1)}{\lambda-1}}_{<}\right) > 1-log_\lambda\left(1+\underbrace{\frac{(\lambda^{1-\frac{x_1}{M}}-1).(\lambda^{1-\frac{y_1}{M}}-1)}{\lambda-1}}_{>}\right).
$$

Then, $d_{\oplus_\lambda^{\mathbf{F}}}(\omega,E_1 \sqcup E_2) < d_{\oplus_\lambda^{\mathbf{F}}}(\omega',E_1 \sqcup E_2)$.

(PD-1) We need to show that

$$1 - log_\lambda\left(1 + \frac{(\lambda^{1-x}-1)(\lambda^{1-y}-1)}{\lambda-1}\right) > 1 - log_\lambda\left(1 + \frac{(\lambda^{1-(x-\delta)}-1)(\lambda^{1-(y+\delta)}-1)}{\lambda-1}\right),$$

$$log_\lambda\left(1 + \frac{(\lambda^{1-x}-1)(\lambda^{1-y}-1)}{\lambda-1}\right) < log_\lambda\left(1 + \frac{(\lambda^{1-(x-\delta)}-1)(\lambda^{1-(y+\delta)}-1)}{\lambda-1}\right),$$

$$1 + \frac{(\lambda^{1-x}-1)(\lambda^{1-y}-1)}{\lambda-1} < 1 + \frac{(\lambda^{1-(x-\delta)}-1)(\lambda^{1-(y+\delta)}-1)}{\lambda-1},$$

$$\frac{(\lambda^{1-x}-1)(\lambda^{1-y}-1)}{\lambda-1} < \frac{(\lambda^{1-(x-\delta)}-1)(\lambda^{1-(y+\delta)}-1)}{\lambda-1},$$

$$(\lambda^{1-x}-1)(\lambda^{1-y}-1) < (\lambda^{1-(x-\delta)}-1)(\lambda^{1-(y+\delta)}-1),$$

$$\lambda^{2-x-y} - \lambda^{1-x} - \lambda^{1-y} + 1 < \lambda^{2-x-y} - \lambda^{1-x+\delta} - \lambda^{1-y-\delta} + 1,$$

$$-(\lambda^{1-x} + \lambda^{1-y}) < -(\lambda^{1-x+\delta} + \lambda^{1-y-\delta}),$$

$$\lambda^{1-x} + \lambda^{1-y} > \lambda^{1-x+\delta} + \lambda^{1-y-\delta},$$

$$\lambda^{1-\frac{x'}{M}} + \lambda^{1-\frac{y'}{M}} > \lambda^{1-\frac{x'+\delta}{M}} + \lambda^{1-\frac{y'-\delta}{M}},$$

$$\lambda^{\frac{M-x'}{M}} + \lambda^{\frac{M-y'}{M}} > \lambda^{\frac{M-x'-\delta}{M}} + \lambda^{\frac{M-y'+\delta}{M}},$$

$$\lambda^{\frac{a}{n}} + \lambda^{\frac{a-2c-d}{n}} > \lambda^{\frac{a-c}{n}} + \lambda^{\frac{a-c-d}{n}},$$

$$\lambda^{\frac{a}{n}}.(1 + \lambda^{\frac{-2c-d}{n}}) > \lambda^{\frac{a}{n}}.(\lambda^{-\frac{c}{n}} + \lambda^{\frac{-c-d}{n}}),$$

$$1 + \frac{1}{\lambda^{\frac{2c+d}{n}}} > \frac{1}{\lambda^{\frac{c}{n}}} + \frac{1}{\lambda^{\frac{c+d}{n}}},$$

$$\frac{\lambda^{\frac{2c+d}{n}} + 1}{\lambda^{\frac{2c+d}{n}}} > \frac{\lambda^{\frac{c+d}{n}} + \lambda^{\frac{c}{n}}}{\lambda^{\frac{2c+d}{n}}},$$

$$\lambda^{\frac{2c+d}{n}} + 1 > \lambda^{\frac{c+d}{n}} + \lambda^{\frac{c}{n}}.$$

(HE-1) A counterexample for $\oplus_\lambda^{\mathbf{F}}$ is $M = 4$, $d(\omega, K_1) = 3, d(\omega, K_2) = 0, d(\omega', K_1) = d(\omega', K_2) = 2$. We have $d(\omega, K_1) > d(\omega', K_1) \geq d(\omega', K_2) > d(\omega, K_2)$, but $\omega' <_{\{K_1,K_2\}}^{d,\oplus_\lambda^{\mathbf{F}}} \omega$ is false when $\lambda = 2$. The same example can be used to create a counterexample for **(Arb)**.

(HP) A counterexample for $\oplus_{\mathbf{F}}$ is $M = 6$, $d(\omega_1, K_1) = 4, d(\omega_1, K_2) = 0, d(\omega_2, K_1) = 1, d(\omega_2, K_2) = 4$, $d(\omega_1', K_1) = 4, d(\omega_1', K_2) = 4, d(\omega_2', K_1) = 1, d(\omega_2', K_2) = 5$, for $\lambda = 3$.

□

Theorem 3.8. **Proof:** To prove **(HE-1)**, we have to show that if $\exists i, j \in \{1, \ldots, n\}$ such that $d(\omega, K_i) > d(\omega', K_i) \geq d(\omega', K_j) > d(\omega, K_j)$ and $\forall l \neq i, j\ d(\omega, K_l) = d(\omega', K_l) \neq 1$, then $\omega' <_E^{d,\oplus_\lambda^{\mathbf{F}}} \omega$. Let n be the number of variables in E and $\lambda = 10^{-n}$. We will consider only the worst cases, where $d(\omega, K_i)$ and $d(\omega, K_j)$ have the lowest values and $d(\omega', K_i)$ and $d(\omega', K_j)$ have the highest values. Proving for these cases, will also guarantee the result for any arbitrary values. For instance, if $n=4$, we can check the property for the case where $d(\omega,K_i)=3$, $d(\omega,K_j)=0$ and $d(\omega',K_i)=d(\omega',K_j)=2$.

To show **(HE-**1**)**, we need to guarantee only that

$$1-log_\lambda\left(1+\frac{(\lambda^{1-\frac{d(\omega,K_i)}{n}}-1)(\lambda^{1-\frac{d(\omega,K_j)}{n}}-1)}{\lambda-1}\right) > 1-log_\lambda\left(1+\frac{(\lambda^{1-\frac{d(\omega',K_i)}{n}}-1)(\lambda^{1-\frac{d(\omega',K_j)}{n}}-1)}{\lambda-1}\right),$$

$$log_\lambda\left(1+\frac{(\lambda^{1-\frac{d(\omega,K_i)}{n}}-1)(\lambda^{1-\frac{d(\omega,K_j)}{n}}-1)}{\lambda-1}\right) < log_\lambda\left(1+\frac{(\lambda^{1-\frac{d(\omega',K_i)}{n}}-1)(\lambda^{1-\frac{d(\omega',K_j)}{n}}-1)}{\lambda-1}\right),$$

$$1+\frac{(\lambda^{1-\frac{d(\omega,K_i)}{n}}-1)(\lambda^{1-\frac{d(\omega,K_j)}{n}}-1)}{\lambda-1} < 1+\frac{(\lambda^{1-\frac{d(\omega',K_i)}{n}}-1)(\lambda^{1-\frac{d(\omega',K_j)}{n}}-1)}{\lambda-1},$$

$$(\lambda^{1-\frac{d(\omega,K_i)}{n}}-1)(\lambda^{1-\frac{d(\omega,K_j)}{n}}-1) < (\lambda^{1-\frac{d(\omega',K_i)}{n}}-1)(\lambda^{1-\frac{d(\omega',K_j)}{n}}-1),$$

$$(\lambda^{1-\frac{y}{n}}-1)(\lambda^{1-\frac{0}{n}}-1) < (\lambda^{1-\frac{y-1}{n}}-1)(\lambda^{1-\frac{y-1}{n}}-1),$$

$$(10^{-(n-y)}-1)(10^{-n}-1) < (10^{-(n-y+1)}-1)(10^{-(n-y+1)}-1),\ (\lambda=10^{-n})$$

$$10^{-(2n-y)}-10^{-(n-y)}-10^{-n}+1 < 10^{-2(n-y+1)}-2.10^{-(n-y+1)}+1,$$

$$10^{-(2n-y)}-10^{-(n-y)}-10^{-n} < 10^{-2(n-y+1)}-2.10^{-(n-y+1)},$$

$$10^{-(n+1)}-10^{-1}-10^{-n} < 10^{-4}-2.10^{-2},\ (y\leq n-1)$$

$$10^{-(n+1)}-10^{-n} < 10^{-4}-2.10^{-2}+10^{-1},$$

$$\frac{1}{1\underbrace{00\ldots0}_{n+1}}-\frac{1}{1\underbrace{00\ldots0}_{n}} < \frac{1}{10000}-\frac{2}{100}+\frac{1}{10},\ (n\geq 3)$$

$$\frac{1}{10000}-\frac{1}{1000} < \frac{1}{10000}-\frac{2}{100}+\frac{1}{10},$$

$$\frac{1}{1000} > \frac{2}{100}-\frac{1}{10},$$

$$\frac{1}{1000} > -\frac{8}{100}.$$

(Arb) To prove that the operator satisfies **(Arb)**, we can show that the following property holds: If $\omega <^{d,\oplus^{\mathbf{F}}_\lambda}_{K_1} \omega', \omega <^{d,\oplus^{\mathbf{F}}_\lambda}_{K_2} \omega''$ and $\omega' \approx^{d,\oplus^{\mathbf{F}}_\lambda}_{\{K_1,K_2\}} \omega''$, then $\omega <^{d,\oplus^{\mathbf{F}}_\lambda}_{\{K_1,K_2\}} \omega'$. W.l.o.g. we can consider that $d(\omega',K_1) > d(\omega,K_1) \geq d(\omega,K_2) > d(\omega',K_2)$ and treat the problem as a restriction of the Hammond Equity condition. Consequently, $\omega <^{d,\oplus^{\mathbf{F}}_\lambda}_{\{K_1,K_2\}} \omega'$ will hold when $0<\lambda\leq 10^{-n}$.

(HP) It comes as a consequence of the fact that **(HE-**1**)** + **(SP-**1**)** + **(A)** is equivalent to **(HP)** + **(SP-**1**)** + **(A)**.

□

Theorem 3.9. **Proof:** To prove **(Arb)**, we can show that the following property holds: If $\omega <_{K_1}^{d,\oplus_\lambda^Y} \omega', \omega <_{K_2}^{d,\oplus_\lambda^Y} \omega''$ and $\omega' \approx_{\{K_1,K_2\}}^{d,\oplus_\lambda^Y} \omega''$, then $\omega <_{\{K_1,K_2\}}^{d,\oplus_\lambda^Y} \omega'$. W.l.o.g. we can consider that $d(\omega',K_1) > d(\omega,K_1) \geq d(\omega,K_2) > d(\omega',K_2)$ and treat the problem as a restriction of the Hammond Equity condition.

We will consider only the worst cases, where $d(\omega,K_i)$ and $d(\omega,K_j)$ have the lowest values and $d(\omega',K_i)$ and $d(\omega',K_j)$ have the highest values. Proving for these cases, will also guarantee the result for any arbitrary values. For instance, if $n=4$, we can check the property for the case where $d(\omega,K_i)=3$, $d(\omega,K_j)=0$ and $d(\omega',K_i)=d(\omega',K_j)=2$. Let n be the number of variables in K_1,K_2 and $\lambda = \lfloor \frac{2n}{3} \rfloor$. We need to guarantee that $d_{\oplus_\lambda^Y}(\omega,\{K_1,K_2\}) > d_{\oplus_\lambda^Y}(\omega',\{K_1,K_2\})$, that is

$$min\left(\left(\left(\frac{d(\omega,K_i)}{n}\right)^\lambda + \left(\frac{d(\omega,K_j)}{n}\right)^\lambda\right)^{\frac{1}{\lambda}},1\right) > min\left(\left(\left(\frac{d(\omega',K_i)}{n}\right)^\lambda + \left(\frac{d(\omega',K_j)}{n}\right)^\lambda\right)^{\frac{1}{\lambda}},1\right),$$

$$\left(\left(\frac{d(\omega,K_i)}{n}\right)^\lambda + \left(\frac{d(\omega,K_j)}{n}\right)^\lambda\right)^{\frac{1}{\lambda}} > \left(\left(\frac{d(\omega',K_i)}{n}\right)^\lambda + \left(\frac{d(\omega',K_j)}{n}\right)^\lambda\right)^{\frac{1}{\lambda}},$$

$$\left(\frac{d(\omega,K_i)}{n}\right)^\lambda + \left(\frac{d(\omega,K_j)}{n}\right)^\lambda > \left(\frac{d(\omega',K_i)}{n}\right)^\lambda + \left(\frac{d(\omega',K_j)}{n}\right)^\lambda,$$

$$\left(\frac{y}{n}\right)^\lambda > \left(\frac{y-1}{n}\right)^\lambda + \left(\frac{y-1}{n}\right)^\lambda,$$

$$\left(\frac{y}{n}\right)^\lambda > 2.\left(\frac{y-1}{n}\right)^\lambda,$$

$$\left(\frac{y}{n}\right)^{\frac{2n}{3}} > 2.\left(\frac{y-1}{n}\right)^{\frac{2n}{3}},$$

$$\frac{y^{\frac{2n}{3}}}{n^{\frac{2n}{3}}} > 2.\frac{(y-1)^{\frac{2n}{3}}}{n^{\frac{2n}{3}}},$$

$$y^{\frac{2n}{3}} > 2.(y-1)^{\frac{2n}{3}},$$

$$y^{2n} > 8.(y-1)^{2n}. \text{ (Similar to Theorem 6).}$$

Yager T-conorm is nilpotent, i.e., it is continuous and if each $a \in]0,1[$ is a nilpotent element. An element $a \in]0,1[$ is called a nilpotent element of $\oplus$ if there exists some $n \in \mathbb{N}$ such that $\oplus \underbrace{\{a,\ldots,a\}}_{n} = 1$. In other terms, there are other elements besides the annihilator that results in 1 with the T-conorm application. We will show that for the possible highest value different from 1 (which is $\frac{n-1}{n}$), $\lambda > log_{\left(\frac{n}{n-1}\right)}2$ is sufficient to guarantee $d_{\oplus_\lambda^Y}(\omega,\{K_1,K_2\}) < 1$. We need to guarantee that $min(\left(\left(\frac{1}{n}\right)^\lambda + \left(\frac{1}{n}\right)^\lambda\right)^{\frac{1}{\lambda}},0) < 1$, that is,

$$
\begin{aligned}
\left(\left(\frac{n-1}{n}\right)^{\lambda}+\left(\frac{n-1}{n}\right)^{\lambda}\right)^{\frac{1}{\lambda}} &< 1,\\
2.\left(\frac{n-1}{n}\right)^{\lambda} &< 1,\\
\sqrt[\lambda]{2}.\left(\frac{n-1}{n}\right) &< 1,\\
\sqrt[\lambda]{2} &< \left(\frac{n}{n-1}\right),\\
log(\sqrt[\lambda]{2}) &< log\left(\frac{n}{n-1}\right),\\
\frac{1}{\lambda}.log2 &< log\left(\frac{n}{n-1}\right),\\
log2 &< \lambda.log\left(\frac{n}{n-1}\right),\\
\frac{log2}{log\left(\frac{n}{n-1}\right)} &< \lambda,\\
log_{\left(\frac{n}{n-1}\right)}2 &< \lambda.
\end{aligned}
$$

As $\lambda = \left\lfloor \frac{2n}{3} \right\rfloor > log_{\left(\frac{n}{n-1}\right)}2$, then we still guarantee $1 > d_{\oplus^{\mathrm{Y}}_{\lambda}}(\omega, \{K_1, K_2\}) > d_{\oplus^{\mathrm{Y}}_{\lambda}}(\omega', \{K_1, K_2\})$ and **(Arb)** is satisfied.

□

Theorem 3.10. **Proof: (Arb)** A counterexample for $\oplus^{\mathbf{SW}}_{\lambda}$ is $d(\omega, K_1) = 3, d(\omega, K_2) = 0, d(\omega', K_1) = d(\omega', K_2) = 2, d(\omega'', K_1) = 0, d(\omega'', K_2) = 3$. We have $\omega' <^{d,\oplus^{\mathbf{SW}}_{\lambda}}_{K_1} \omega, \omega' <^{d,\oplus^{\mathbf{SW}}_{\lambda}}_{K_2} \omega''$ and $\omega \approx^{d,\oplus^{\mathbf{SW}}_{\lambda}}_{\{K_1,K_2\}} \omega''$, but $\omega' <^{d,\oplus^{\mathbf{SW}}_{\lambda}}_{\{K_1,K_2\}} \omega$ is false when $\lambda \geq 0$.

□

Theorem 3.15. **Proof: (IC6)** We need to show that if $\omega_1 \leq^{d,lexi\oplus}_{E_1} \omega_2$ and $\omega_1 <^{d,lexi\oplus}_{E_2} \omega_2$, then $\omega_1 <^{d,lexi\oplus}_{E_1 \sqcup E_2} \omega_2$. For strict T-conorms:

- Case 1: $l^{d,\bar{E}_1,\oplus}_{\omega_1} = (d^{\omega_1}_1, \ldots, d^{\omega_1}_{2^n-1}) = (d^{\omega_2}_1, \ldots, d^{\omega_2}_{2^n-1}) = l^{d,\bar{E}_1,\oplus}_{\omega_2} = (1,1,1,\ldots,1)$. In this case, we have $l^{d,E_1\bar{\sqcup}E_2,\oplus}_{\omega_1} = (1,1,1,\ldots,1,d^{\omega_1}_k,\ldots,d^{\omega_1}_{2^m-1}) < (1,1,1,\ldots,1,d^{\omega_2}_k,\ldots,d^{\omega_2}_{2^m-1}) = l^{d,E_1\bar{\sqcup}E_2,\oplus}_{\omega_2}$. It is easy to see that $d^{\omega_1}_k < d^{\omega_2}_k$ because $\omega_1 <^{d,lexi\oplus}_{E_2} \omega_2$.
- Case 2: $l^{d,\bar{E}_1,\oplus}_{\omega_1} = (d^{\omega_1}_1, \ldots, d^{\omega_1}_{2^n-1}) = (d^{\omega_2}_1, \ldots, d^{\omega_2}_{2^n-1}) = l^{d,\bar{E}_1,\oplus}_{\omega_2}$. In this case,
  - If $\exists i, d^{\omega_1}_i = 1$, we can use an argument similar to the Case 1.
  - If $\forall i, d^{\omega_1}_i \neq 1$, then $l^{d,E_1\bar{\sqcup}E_2,\oplus}_{\omega_1} = (d^{\omega_1}_1, \ldots, d^{\omega_1}_{2^m-1}) < (,d^{\omega_2}_1, \ldots, d^{\omega_2}_{2^m-1}) = l^{d,E_1\bar{\sqcup}E_2,\oplus}_{\omega_2}$. It is easy to see that $d^{\omega_1}_1 < d^{\omega_2}_1$ since the values from $l^{d,\bar{E}_1,\oplus}_{\omega_1}$ do not interfere in the result.

- Case 3: $l_{\omega_1}^{d,\bar{E}_1,\oplus} = (d_1^{\omega_1},\ldots,d_{2^n-1}^{\omega_1}) <_L (d_1^{\omega_2},\ldots,d_{2^n-1}^{\omega_2}) = l_{\omega_2}^{d,\bar{E}_1,\oplus}$. Similar to the Case 2.

For nilpotent T-conorms: A T-conorm is called nilpotent if it is continuous and if each $a \in]0,1[$ is a nilpotent element. An element $a \in]0,1[$ is called a nilpotent element of $\oplus$ if there exists some $n \in \mathbb{N}$ such that $\oplus \underbrace{\{a,\ldots,a\}}_{n} = 1$.

For the special case where there is a nilpotent element, we can use similar argument of the strict T-conorms since we can ignore the 1's from the vector and compare only then remaining values. For the other cases, the proof is analagous to the strict case.

$\square$

Theorem 3.16. **Proof:** Similar to the Theorem 3.15.

$\square$

Theorem 3.17. **Proof:**

- If $\Delta_\mu^{d,\oplus}$ satisfies **(HE-**1**)**. then $\Delta_\mu^{d,lexi\oplus}$ satisfies **(HE)**;

Suppose that $\Delta_\mu^{d,\oplus}$ satisfies **(HE-**1**)**. By Definition, **(HE-**1**)** If $\exists i,j \in \{1,\ldots,n\}$ such that $d(\omega,K_i) < d(\omega',K_i) < d(\omega',K_j) < d(\omega,K_j)$ and $\forall l \neq i,j\ d(\omega,K_l) = d(\omega',K_l) \neq 1$, then $d_\oplus(\omega',E) < d_\oplus(\omega,E)$.

We need to show that $\Delta_\mu^{d,lexi\oplus}$ satisfies **(HE)**. By Definition, **(HE-**1**)** If $\exists i,j \in \{1,\ldots,n\}$ such that $d(\omega,K_i) < d(\omega',K_i) < d(\omega',K_j) < d(\omega,K_j)$ and $\forall l \neq i,j\ d(\omega,K_l) = d(\omega',K_l)$, then $\omega' <_E^{d,lexi\oplus} \omega$. We need to show only the case where $\exists l, d(\omega,K_l) = d(\omega',K_l) = 1$, since the other case are proved from the assumption. If theres exists such l, then $l_{\omega'}^{d,\bar{E},\oplus} = (d_1^{\omega'},\ldots,d_{2^n-1}^{\omega'}) = (1,1,\ldots,1,d_k^{\omega'},\ldots,d_{2^n-1}^{\omega'})$ and $l_{\omega}^{d,\bar{E},\oplus} = (d_1^{\omega},\ldots,d_{2^n-1}^{\omega}) = (1,1,\ldots,1,d_k^{\omega},\ldots,d_{2^n-1}^{\omega})$. Therefore, we have $l_{\omega'}^{d,\bar{E},\oplus} <_{lex} l_{\omega_j}^{d,\bar{E},\oplus}$ by the assumption done before.

- If $\Delta_\mu^{d,\oplus}$ satisfies **(PD-**1**)**. then $\Delta_\mu^{d,lexi\oplus}$ satisfies **(PD)**; and
- If $\Delta_\mu^{d,\oplus}$ satisfies **(SP-**1**)**. then $\Delta_\mu^{d,lexi\oplus}$ satisfies **(SP)**.

These cases are analogous to the first case. $\square$

APPENDIX C – PROOF THEOREMS - CHAPTER 4

Theorem 4.1. **Proof: (IC0)** By definition, $mod(\Delta_{\mu}^{d,hc_s}(E)) \subseteq mod(\mu)$.

(IC1) The function $hc(\omega,d,E,s)$ maps to values in $\mathbb{N}$, so if $mod(\mu) \neq \emptyset$, there is a model ω of μ such that for every model ω' of μ, $hc(\omega,d,E,s) \leq hc(\omega',d,E,s)$. So $\omega \models \Delta_{\mu}^{d,hc_s}(E)$ and $\Delta_{\mu}^{d,hc_s}(E) \not\models \bot$.

(IC3) Let $E_1 = \{K_1,\ldots,K_n\}$ and $E_2 = \{K'_1,\ldots,K'_n\}$ be belief sets and suppose that $E_1 \equiv E_2$ and $\mu_1 \leftrightarrow \mu_2$. Hence we can find a permutation σ such that for every $i \in \{1,\ldots,n\}, K_{\sigma(i)} \equiv K'_i$. Now, since hc satisfies symmetry one gets $hc(\omega,d,E_1,s) = hc(\omega,d,E_2,s)$. Consequently, $\Delta_{\mu_1}^{d,hc_s}(E_1) \equiv \Delta_{\mu_1}^{d,hc_s}(E_2)$.

(IC4) Let K_1 and K_2 be a belief bases and suppose that $\Delta_{\mu}^{d,hc_s}(\{K_1,K_2\}) \wedge K_1 \not\models \bot$. We have $\min\limits_{\omega \models K_1} hc(\omega,d,\{K_1,K_2\},s) = 0$, if $d(\omega,K_2) < s$; 1, otherwise ($d(\omega,K_1) = 0$). By the definition of distance, it holds that $\min\limits_{\omega \models K_1} d(\omega,K_2) = \min\limits_{\omega \models K_2} d(\omega,K_1)$. Therefore, $\min\limits_{\omega \models K_1} hc(\omega,d,\{K_1,K_2\},$ $s) = \min\limits_{\omega \models K_2} hc(\omega,d,\{K_1,K_2\},s)$. Then, $\Delta_{\mu}^{d,hc_s}(\{K_1,K_2\}) \wedge K_2 \not\models \bot$.

(IC5) In order to show that the operator satisfy **(IC5)**, it is enough to guarantee that the following property holds: if $hc(\omega,d,E_1,s) \leq hc(\omega',d,E_1,s)$ and $hc(\omega,d,E_2,s) \leq hc(\omega',d,E_2,s)$, then $hc(\omega,d,E_1 \sqcup E_2,s) \leq hc(\omega',d,E_1 \sqcup,s)$. Clearly, we can see that this property holds.

(IC6) In order to show that the operator satisfy **(IC6)**, it is enough to guarantee that the following property holds: if $hc(\omega,d,E_1,s) < hc(\omega',d,E_1,s)$ and $hc(\omega,d,E_2,s) \leq hc(\omega',d,E_2,s)$, then $hc(\omega,d,E_1 \sqcup E_2,s) < hc(\omega',d,E_1 \sqcup,s)$. Clearly, we can see that this property holds.

(IC7) Suppose that $\omega \models \Delta_{\mu_1}^{d,hc_s}(E) \wedge \mu_2$. For any $\omega' \models \mu_1$, we have $hc(\omega,d,E,s) \leq hc(\omega',d,E,s)$. Hence, for any $\omega' \models \mu_1 \wedge \mu_2$, we have $hc(\omega,d,E,s) \leq hc(\omega',d,E,s)$. As result, $\omega \models \Delta_{\mu_1 \wedge \mu_2}^{d,hc_s}(E)$.

(IC8) Suppose that $\Delta_{\mu_1}^{d,hc_s}(E) \wedge \mu_2$ is consistent. Then there exists a model ω' of $\Delta_{\mu_1}^{d,hc_s}(E) \wedge \mu_2$. Consider a model ω of $\Delta_{\mu_1 \wedge \mu_2}^{d,hc_s}(E)$ and suppose that $\omega \not\models \Delta_{\mu_1}^{d,hc_s}(E)$. In this case, $hc(\omega',d,E,s) < hc(\omega,d,E,s)$, and since $\omega' \models \mu_1 \wedge \mu_2$, we have $\omega \notin mod(\Delta_{\mu_1 \wedge \mu_2}^{d,hc_s}(E)) = min(mod(\mu_1 \wedge \mu_2), \leq_E^{d,hc_s})$. Hence $\omega \not\models \Delta_{\mu_1 \wedge \mu_2}^{d,hc_s}(E)$. Contradiction.

(Maj) Showing that the operator hc satisfies **(Maj)** is easy since since it is based on the sum operator.

(Arb) To show that the operator satisfies **(Arb)**, we can show that the following property holds: If $\omega <_{K_1}^{d,hc_s} \omega', \omega <_{K_2}^{d,hc_s} \omega''$ and $\omega' \approx_{\{K_1,K_2\}}^{d,hc_s} \omega''$, then $\omega <_{\{K_1,K_2\}}^{d,hc_s} \omega'$.

Suppose that $\omega <_{K_1}^{d,hc_s} \omega', \omega <_{K_2}^{d,hc_s} \omega''$ and $\omega' \approx_{\{K_1,K_2\}}^{d,hc_s} \omega''$:

- $\omega <_{K_1}^{d,hc_s} \omega'$ implies $d(\omega,K_1) < s < d(\omega',K_1)$;
- $\omega <_{K_2}^{d,hc_s} \omega''$ implies $d(\omega,K_2) < s < d(\omega'',K_2)$;
- $\omega' \approx_{\{K_1,K_2\}}^{d,hc_s} \omega''$ implies $hc(\omega',d,\{K_1,K_2\},s) = hc(\omega'',d,\{K_1,K_2\},s) \geq 1$.

So, $hc(\omega,d,\{K_1,K_2\},s) = 0 < hc(\omega',d,\{K_1,K_2\},s)$. Therefore, $\omega <_{\{K_1,K_2\}}^{d,hc_s} \omega'$ holds.

□

Theorem 4.2. **Proof:** **(WHP)** We need to show: for all $\omega_1, \omega_2, \omega_1', \omega_2' \in \Omega$, suppose $E = \{K_1, \ldots, K_n\}$, a distance measure d and $\omega_1 <_E^{d,hc_s} \omega_2$. Consider ω_1', ω_2' such that $\exists i \in \{1,\ldots,n\}$, $d(\omega_1,K_i) < d(\omega_1',K_i)$, $d(\omega_2,K_i) < d(\omega_2',K_i)$ and $\forall j \neq i$ $d(\omega_1,K_j) = d(\omega_1',K_j)$, $d(\omega_2,K_j) = d(\omega_2',K_j)$. If $d(\omega_2',K_i) > d(\omega_1',K_i)$ then $\omega_1' \leq_E^{d,hc_s} \omega_2'$.

Let $d(\omega_1,K_i) < d(\omega_1',K_i)$, $d(\omega_2,K_i) < d(\omega_2',K_i)$ and $\forall j \neq i$ $d(\omega_1,K_j) = d(\omega_1',K_j)$, $d(\omega_2,K_j) = d(\omega_2',K_j)$. Suppose that $d(\omega_2',K_i) > d(\omega_1',K_i)$, the we have the following cases:

- $d(\omega_2',K_i) > d(\omega_1',K_i) > s$. As $\omega_1 <_E^{d,hc_s} \omega_2$, then $hc(\omega_1,d,E,s) < hc(\omega_2,d,E,s)$.
  - if $d(\omega_1,K_i), d(\omega_2,K_i) < s$. In this case $hc(\omega_1',d,E,s) = hc(\omega_1,d,E,s) + 1 < hc(\omega_2,d,E,s) + 1 = hc(\omega_2',d,E,s)$.
  - if $d(\omega_1,K_i), d(\omega_2,K_i) > s$. In this case $hc(\omega_1',d,E,s) = hc(\omega_1,d,E,s) < hc(\omega_2,d,E,s) = hc(\omega_2',d,E,s)$.
  - if $d(\omega_1,K_i) < s < d(\omega_2,K_i)$. In this case $hc(\omega_1',d,E,s) = hc(\omega_1,d,E,s) + 1 \leq hc(\omega_2,d,E,s) = hc(\omega_2',d,E,s)$.
  - if $d(\omega_1,K_i) > s > d(\omega_2,K_i)$. In this case $hc(\omega_1',d,E,s) = hc(\omega_1,d,E,s) < hc(\omega_2,d,E,s) < hc(\omega_2,d,E,s) + 1 = hc(\omega_2',d,E,s)$.

Therefore, $\omega_1' \leq_E^{d,hc_s} \omega_2'$.

(WIBP) Analogous to **(WHP)**.

We can see clearly that **(HP)** and **(IBP)** are not always true. Consider the third case of the **(WHP)** proof: - if $d(\omega_1,K_i) < s < d(\omega_2,K_i)$. In this case $hc(\omega_1',d,E,s) = hc(\omega_1,d,E,s) + 1 \leq hc(\omega_2,d,E,s) = hc(\omega_2',d,E,s)$. If this case happens, then we have $\omega_1' \leq_E^{d,hc_s} \omega_2'$ and not necessarily $\omega_1' <_E^{d,hc_s} \omega_2'$.

□

Theorem 4.3. **Proof:** **(Arb)** To show that the operator does not satisfy **(Arb)**, we can show that the following property does not hold: If $\omega <_{K_1}^{d,sh_s} \omega', \omega <_{K_2}^{d,sh_s} \omega''$ and $\omega' \approx_{\{K_1,K_2\}}^{d,sh_s} \omega''$, then $\omega <^{d,sh_s}_{\{K_1,K_2\}} \omega'$. Suppose that $s = 1$, $d(\omega,K_1) = d(\omega,K_2) = 2$, $d(\omega',K_1) = d(\omega'',K_2) = 3$ and $d(\omega',K_2) = d(\omega'',K_1) = 0$. Then,

- $\omega <^{d,sh_1}_{K_1} \omega'$, since $sh(\omega,d,K_1,1) = 1 < 2 = sh(\omega',d,K_1,1)$.
- $\omega <^{d,sh_1}_{K_2} \omega''$, since $sh(\omega,d,K_2,1) = 1 < 2 = sh(\omega'',d,K_2,1)$.
- $\omega' \approx^{d,sh_1}_{\{K_1,K_2\}} \omega''$, since $sh(\omega',d,\{K_1,K_2\},1) = 2 = 2 = sh(\omega'',d,\{K_1,K_2\},1)$.

But, $\omega \approx^{d,sh_1}_{\{K_1,K_2\}} \omega'$, because $sh(\omega,d,\{K_1,K_2\},1) = 1+1 = 2+0 = sh(\omega',d,\{K_1, K_2\},1)$.

The proofs for the other logical postulates are similar to the previous theorems.

□

Theorem 4.5. **Proof:** -Δ^{d,hc_s}_{μ}: We need to show that **(WAPA-*s*)** Let $E = \{K_1,\dots,K_n\}$, d be a distance measure and $s \geq 0$. For all ω,ω', if there exist j,k such that (1) $s \geq d(\omega,K_j) > d(\omega',K_j)$; (2) $d(\omega',K_k) > d(\omega,K_k) \geq s$; (3) for $i \neq j,k$, $d(\omega,K_i) = d(\omega',K_i)$, then $\omega \leq^{d,hc_s}_{E} \omega'$.

- (1) $s \geq d(\omega,K_j) > d(\omega',K_j)$ implies $hc(\omega,d,K_j,s) = hc(\omega',d,K_j,s) = 0$;
- (2) $d(\omega',K_k) > d(\omega,K_k) \geq s$ implies $hc(\omega,d,K_k,s) = hc(\omega',d,K_k,s) = 1$;
- (3) for $i \neq j,k$, $d(\omega,K_i) = d(\omega',K_i)$ implies $hc(\omega,d,K_i,s) = hc(\omega',d,K_i,s)$;

Then $\omega \approx^{d,hc_s}_{E} \omega'$ and therefore, $\omega \leq^{d,hc_s}_{E} \omega'$.

- It is easy to check that Δ^{d,hc_s}_{μ} satisfies **(A)**.

-Δ^{d,sh_s}_{μ}: Similar to Δ^{d,hc_s}_{μ}.

- **(WPM-*s*)**: Let $E = \{K_1,\dots,K_n\}$, d be a distance measure and $s \geq 0$. For all ω,ω', if (1) there exists a $k \leq n$ such that $d(\omega',K_k) > d(\omega,K_k)$ and $d(\omega',K_k) > s$; (2) every position i that $d(\omega,K_i) > s$ implies $d(\omega',K_i) \geq d(\omega,K_i)$, then $\omega \leq^{d,op_s}_{E} \omega'$.

Suppose that (1) and (2) hold. If $d(\omega,K_k) \leq s$ then $sh(\omega,d,E,s) < sh(\omega',d,E,s)$ and $\omega <^{d,sh_s}_{E} \omega'$. If $d(\omega,K_k) > s$ then $sh(\omega,d,E,s) < sh(\omega',d,E,s)$ and $\omega <^{d,sh_s}_{E} \omega'$.

□

Theorem 4.6. **Proof:** We need to show **weak Absolute Priority of those Above s:** Let $E = \{K_1,\dots,K_n\}$, d be a distance measure and $s \geq 0$. For all ω,ω', if there exist j,k such that: (1) $d(\omega,K_j) > d(\omega',K_j)$, and $d(\omega,K_j) > s$; (2) $d(\omega',K_k) > d(\omega,K_k) \geq s$, and $d(\omega,K_k) \geq d(\omega,K_j)$; (3) for $i \neq j,k$, $d(\omega,K_i) = d(\omega',K_i)$, then $\omega <^{d,op_s}_{E} \omega'$.

Consider $\alpha = \frac{n}{2}$, where n is the number of propositional variables in the belief set.

- (1) $d(\omega,K_j) > d(\omega',K_j)$, and $d(\omega,K_j) > s$;
- (2) $d(\omega',K_k) > d(\omega,K_k) \geq s$, and $d(\omega,K_k) \geq d(\omega,K_j)$;
- (3) for $i \neq j,k$, $d(\omega,K_i) = d(\omega',K_i)$.

W.l.o.g. we can assume that (3) for $i \neq j,k$, $d(\omega,K_i) = d(\omega',K_i) < s$. With (1) and (2) we have

$$FGT(\frac{n}{2},\omega,d,E,s) < FGT(\frac{n}{2},\omega',d,E,s)$$

$$\frac{1}{n}\left(\left(\frac{d(\omega,K_j)-s}{s}\right)^{\frac{n}{2}} + \left(\frac{d(\omega,K_k)-s}{s}\right)^{\frac{n}{2}}\right) < \frac{1}{n}\left(\frac{d(\omega',K_k)-s}{s}\right)^{\frac{n}{2}}$$

$$\left(\frac{d(\omega,K_j)-s}{s}\right)^{\frac{n}{2}} + \left(\frac{d(\omega,K_k)-s}{s}\right)^{\frac{n}{2}} < \left(\frac{d(\omega',K_k)-s}{s}\right)^{\frac{n}{2}}$$

$$(d(\omega,K_j)-s)^{\frac{n}{2}} + (d(\omega,K_k)-s)^{\frac{n}{2}} < (d(\omega',K_k)-s)^{\frac{n}{2}}$$

Assume that in the worst case $d(\omega,K_k) = d(\omega,K_j)$, then

$2.(d(\omega,K_k)-s)^{\frac{n}{2}} < (d(\omega',K_k)-s)^{\frac{n}{2}}$

Since $d(\omega,K_k)$ and $d(\omega',K_k)$ ranges from 0 to n we can show the following property

$$2.n^{\frac{n}{2}} < (n+1)^{\frac{n}{2}}$$
$$\sqrt{2}.n^n < (n+1)^n$$
$$\sqrt{2}.n^n < n^n + n.n^{n-1} + \cdots + 1$$
$$\sqrt{2}.n^n < n^n + n^n + \cdots + 1.$$

Therefore, $\omega <_E^{d,FGT_s^{\frac{n}{2}}} \omega'$.

□

Theorem 4.7. **Proof:** The proof is similar to the one found in **THM.** (TUNGODDEN, 2000) Given some $s \geq 0$, if a pre-order relation satisfies **(WAPA-**s**)**, **(SP)** and **(A)**, then it satisfies **(WPM-**s**)**.

Sketch of the proof: We need to prove **Leximax Above s (LMA-**s**):** Let $E = \{K_1,\ldots,K_n\}$ be a belief set. For each outcome ω, we build the list $(d_1^{\omega},\ldots,d_n^{\omega})$ of distances between this outcome and the n belief bases in E, i.e., $d_i^{\omega} = d(\omega,K_i)$. Let $L_{\omega}^{d,E}$ be the list obtained from $(d_1^{\omega},\ldots,d_n^{\omega})$ by sorting it in descending order. For all $\omega,\omega' \in \Omega$, (1) if there exists a position $k \leq n$ such that $d_k^{\omega'} > d_k^{\omega}$; (2) $d_k^{\omega} > s$; and (3) for every $j < k$, $d_j^{\omega'} = d_j^{\omega}$, then $\omega <_E \omega'$ (ω is more preferred than ω'). Otherwise, $\omega \approx_E \omega'$.

For any $\omega,\omega' \in \Omega$ satisfying (1), (2) and (3) we can construct a list of intermediate outcomes $\omega_1,\ldots,\omega_{n-k}$ such that $\omega <_E \omega'_{k+1} <_E \omega'_{k+2} <_E \ldots <_E \omega'_{n-k} <_E \omega'$, taking the position k and comparing individually with the positions $k' \in \{k+1,k+2,\ldots,n\}$. For each comparison,

- If $d_{k'}^{\omega'} > s$, we can use **(wAPA-*s*)**, **(SP)** and **(A)** to show that $\omega <_E \omega'_{k'}$;
- If $d_{k'}^{\omega'} < s$, we can use **(WAPA-*s*)**, **(SP)** and **(A)** to show that $\omega <_E \omega'_{k'}$.

We repeat this process $n-k$ times to show that $\omega <_E \omega'$.

□

Theorem 4.8. **Proof:** We will concentrate only in the proof of **(IC2)** If $\bigwedge E$ is consistent with μ, then $\Delta_\mu(E) \equiv \bigwedge E \wedge \mu$. Suppose that $\bigwedge E$ is consistent with μ. Then there is $\omega \models \bigwedge E \wedge \mu$. Consequently, $hc(\omega,d,E,s) = 0$ or $sh(\omega,d,E,s) = 0$ and $\omega \in \Delta_\mu(E)$. For the converse, suppose that there is a ω' such that $hc(\omega',d,E,s) = 0$, $sh(\omega',d,E,s) = 0$ or $FGT(\alpha,\omega_i,d,E,s) = 0$ and $\omega \not\models \bigwedge E \wedge \mu$. We have that $\omega' \notin \Delta_\mu(E)$, since $\sum_{K\in E} d(\omega,K) < \sum_{K\in E} d(\omega',K)$.

□

Theorem 4.9. **Proof:** We will show that they satisfy **(SP)** (the proof of the other properties follows a similar reasoning). **(SP):** Let $E = \{K_1,\dots,K_n\}$ and d be a distance measure. For all $\omega,\omega' \in \Omega$, if $\exists i \in \{1,\dots,n\}$ $d(\omega,K_i) < d(\omega',K_i)$ and $\forall j \neq i$, $d(\omega,K_j) \le d(\omega',K_j)$, then $\omega <_E^{d,op_s} \omega'$. Suppose that For all $\omega,\omega' \in \Omega$, if $\exists i \in \{1,\dots,n\}$ $d(\omega,K_i) < d(\omega',K_i)$ and $\forall j \neq i$, $d(\omega,K_j) \le d(\omega',K_j)$. We have two cases to analyze:

1. there is a $k \in \{1,\dots,n\}$ such that $d(\omega,K_k) > s$. In this case it is easy to see that $hc(\omega,d,E,s) < hc(\omega',d,E,s)$ or $sh(\omega,d,E,s) < sh(\omega',d,E,s)$.
2. for all $k \in \{1,\dots,n\}$, $d(\omega,K_k) \le s$. We have that $\sum_{K\in E} d(\omega,K) < \sum_{K\in E} d(\omega',K)$.

Therefore, $\omega <_E^{d,ahc_s} \omega'$ or $\omega <_E^{d,ash_s} \omega'$.

□

Theorems 4.10 and 4.11. **Proof:** We will show the proofs for **(IAB-*s*)** and **(A)**. The rest is similar to the previous Theorems.

(IAB-*s*) Let $A = \{1,\dots,n\}$ be a set of agents, $\Omega = \{\omega_1,\dots,\omega_m\}$ be a set of outcomes, where each $\omega_i = (\omega_i^1,\dots,\omega_i^n)$ and $s \ge 0$. Suppose that for $\omega_i,\omega_j \in \Omega$, we have (i) there exist k,k' such that $s \le \omega_j^k < \omega_i^k$; (ii) $\omega_i^{k'} < \omega_j^{k'} < s$; and (iii) for $l \neq k,k'$, $\omega_i^l = \omega_j^l$. Then it follows that $shc(\omega_i,\omega_j,s) = shc(\omega_j,\omega_i,s)$ and $ssh(\omega_i,\omega_j,s) = shc(\omega_j,\omega_i,s)$, since (i) and (ii) do not alter in the strong *headcount* and *shortfall*, and (iii) adds the same value for strong *headcount* and *shortfall* in both outcomes. Therefore, $\omega_i \approx_{shc_s} \omega_j$ and $\omega_i \approx_{ssh_s} \omega_j$.

(A) Δ_μ^{d,ssh_s} does not satisfy **(A)**. As a counterexample, suppose that we have the outcome $\omega = (4,3,0,0)$, its permutation $\omega' = (0,4,3,0)$ and $s = 2$. We have that $ssh(\omega,\omega',2) = 2 > 1 = ssh(\omega',\omega,2)$. Therefore, $\omega >_{ssh_2} \omega'$.

□

Theorem 4.12. **Proof:** **(S_1PM-s)** Let $A = \{1,\ldots,n\}$ be a set of agents, $\Omega = \{\omega_1,\ldots,\omega_m\}$ be a set of outcomes, where each $\omega_i = (\omega_i^1,\ldots,\omega_i^n)$ and $s \geq 0$. Suppose we have $\omega_i, \omega_j \in \Omega$, such that (i) there exists a $k \leq n$ such that $\omega_i^k < \omega_j^k$ and $\omega_i^k < s$; and (ii) for every position l such that $\omega_j^l \leq \omega_i^l$, we obtain $\omega_j^l \geq s$ or $\omega_i^l < s$; By (i), suppose that $\omega_i^k < \omega_j^k < s$. In this case, we will have $shc(\omega_i, \omega_j, s) = ssh(\omega_j, \omega_i, s)$ and $ssh(\omega_i, \omega_j, s) = ssh(\omega_j, \omega_i, s)$. Therefore, $\omega_i \not<_{shc_s} \omega_j$ and $\omega_i \not<_{ssh_s} \omega_j$.

(SP) It is similar for $\leq_{hc_s}$ and $\leq_{sh_s}$.

□

APPENDIX D – PROOF THEOREMS - CHAPTER 5

Theorem 5.1. **Proof:** **(IC0)** By definition, $\Delta_{\mu}^{d,W}(E) \subseteq mod(\mu)$.

(IC1) $d(W, L_E^{\omega})$ is a function with values in $\mathbb{R}$, so if $mod(\mu) \neq \emptyset$, there is always a minimal model ω of μ such that for every model ω' of μ, $d(W, L_E^{\omega}) \leq d(W, L_E^{\omega'})$. Then, $\omega \models \Delta_{\mu}^{d,W}(E)$ and $\Delta_{\mu}^{d,W}(E) \not\models \perp$.

(IC3) Assume that $E_1 \equiv E_1'$ and $\mu_1 \equiv \mu_1'$. Hence we can find a permutation δ such that for every $i \in \{1, \ldots, n\}$, $K_{\delta(i)} \equiv K_i'$. Since $d(\omega, K_{\delta(i)}) = d(\omega, K_i')$ one gets $d(W, L_{E_1}^{\omega}) = w_1 l_1^{\omega} + \cdots + w_n l_n^{\omega} = d(W, L_{E_1'}^{\omega})$. Consequently, $\Delta_{\mu_1}^{d,W}(E_1) \equiv \Delta_{\mu_1'}^{d,W}(E_1')$.

(IC4) This postulate is equivalent to show that $\forall \omega \models K_1$, $\exists \omega' \models K_2$ and $\omega' \leq_{\{K_1,K_2\}}^{d,W} \omega$ (KONIECZNY; PINO-PÉREZ, 2002a). We have that $d(\omega, K_1) = 0$ and $d(\omega, K_2) = \min_{\omega' \models K_2} d(\omega, \omega')$, so choose $\omega' \models K_2$ and that $d(\omega, \omega') = d(\omega, K_2)$. Then $d(\omega', K_1) = \min_{\omega'' \models K_1} d(\omega', \omega'') \leq d(\omega', \omega)$, and $d(\omega', K_2) = 0$. So $d(W, L_{\{K_1,K_2\}}^{\omega'}) = w_1 l_1 = w_1 d(\omega', K_1) \leq w_1 d(\omega, \omega') = w_1 d(\omega, K_2) = d(W, L_{\{K_1,K_2\}}^{\omega})$, i.e., $\omega' \leq_{\{K_1,K_2\}}^{d,W} \omega$.

(IC5b) We can show that if for all i, $d(\omega, K_i) \leq d(\omega', K_i)$, then $d(W, L_E^{\omega}) \leq d(W, L_E^{\omega'})$. Suppose that for all i, $d(\omega, K_i) \leq d(\omega', K_i)$. Then, $w_1 l_1^{\omega} + \cdots + w_n l_n^{\omega} \leq w_1 l_1^{\omega'} + \cdots + w_n l_n^{\omega'}$. This is true because for all i, $l_i^{\omega} \leq l_i^{\omega'}$ (a consequence of the assumption).

(IC7) Suppose $\omega \models \Delta_{\mu_1}^{d,W} \wedge \mu_2$. For any $\omega' \models \mu_1$, we have $d(W, L_E^{\omega}) \leq d(W, L_E^{\omega'})$. Hence, for any $\omega' \models \mu_1 \wedge \mu_2, d(W, L_E^{\omega}) \leq d(W, L_E^{\omega'})$. This means $\omega \models \Delta_{\mu_1 \wedge \mu_2}^{d,W}(E)$.

(IC8) Suppose that $\Delta_{\mu_1}^{d,W}(E_1) \wedge \mu_2$ is consistent. Then there exists an outcome ω' such that $\omega' \models \Delta_{\mu_1}^{d,W}(E) \wedge \mu_2$. Consider a model ω of $\Delta_{\mu_1 \wedge \mu_2}^{d,W}(E)$ and suppose by absurd that $\omega \not\models \Delta_{\mu_1}^{d,W}(E)$. We have $d(W, L_E^{\omega'}) < d(W, L_E^{\omega})$ and since $\omega' \models \mu_1 \wedge \mu_2$, we know $\omega \notin min(mod(\mu_1 \wedge \mu_2), \leq_E^{d,W})$, hence $\omega \not\models \Delta_{\mu_1 \wedge \mu_2}^{d,W}(E)$. Contradiction.

$\square$

Theorem 5.2. **Proof:** ($\Rightarrow$) By contrapositive, suppose that $\exists i \in \{1, \ldots, n\}, w_i = 0$. We need to show that **(IC6b)** is not satisfied. Consider the following counterexample: $d(\omega, K_i) < d(\omega', K_i)$ such that $d(\omega, K_{\sigma(i)}) = l_i^{\omega}$ and $d(\omega', K_{\sigma(i)}) = l_i^{\omega'}$ (i.e., they are the i-th greatest element of L_E^{ω} and $L_E^{\omega'}$, respectively). For all $j \neq i$, let $d(\omega, K_j) = d(\omega', K_j)$. We have then $d(W, L_E^{\omega}) = w_1 l_1^{\omega} + \cdots + \underbrace{w_i l_i^{\omega}}_{=0} + \cdots + w_n l_n^{\omega} = w_1 l_1^{\omega'} + \cdots + \underbrace{w_i l_i^{\omega'}}_{=0} + \cdots + w_n l_n^{\omega'} = d(W, L_E^{\omega'})$, which falsifies **(IC6b)**.

($\Leftarrow$)Assume $\Delta_{\mu}^{d,[1]}(K_1) \wedge \cdots \wedge \Delta_{\mu}^{d,[1]}(K_n) \not\models \perp$ and $w_i \neq 0$ for any $w_i \in W$. We will show $\Delta_{\mu}^{d,W}(\{K_1, \ldots, K_n\}) \models \Delta_{\mu}^{d,[1]}(K_1) \wedge \cdots \wedge \Delta_{\mu}^{d,[1]}(K_n)$: If $\omega \models \Delta_{\mu}^{d,W}(\{K_1, \ldots, K_n\})$, then by definition $\omega \models \mu$ and for every $\omega' \models \mu$, $w_1 l_1^{\omega} + \cdots + w_n l_n^{\omega} \leq w_1 l_1^{\omega'} + \cdots + w_n l_n^{\omega'}$. By absurd, suppose $\omega \not\models \Delta_{\mu}^{d,[1]}(K_1) \wedge \cdots \wedge \Delta_{\mu}^{d,[1]}(K_n)$.

As $\Delta_{\mu}^{d,[1]}(K_1) \wedge \cdots \wedge \Delta_{\mu}^{d,[1]}(K_n) \not\models \perp$, there exists $\omega'' \in \Omega$ such that $\omega'' \models \Delta_{\mu}^{d,[1]}(K_1) \wedge \cdots \wedge \Delta_{\mu}^{d,[1]}(K_n)$. This means $w'' \in min(mod(\mu), \leq_{K_i}^{d,[1]})$ for every $i \in \{1,\ldots,n\}$, i.e., $w'' \models \mu$ and for every $w' \in \Omega$ such that $w' \models \mu$, we have $d([1], L_{(K_i)}^{\omega''}) = l_i^{\omega''} \leq d([1], L_{(K_i)}^{\omega'_i}) = l_i^{\omega'}$ for every $i \in \{1,\ldots,n\}$. In particular, $l_i^{\omega''} \leq l_i^{\omega}$ for every $i \in \{1,\ldots,n\}$. As $\omega \not\models \Delta_{\mu}^{d,[1]}(K_1) \wedge \cdots \wedge \Delta_{\mu}^{d,[1]}(K_n)$, we know $\omega'' \neq \omega$. Then there exists $j \in \{1,\ldots,n\}$ such that $l_j^{\omega''} < l_j^{\omega}$. Given the monotonicity of OWA operators and $w_i \neq 0$ for any $w_i \in W$, we obtain $d(W, L_{\{K_1,\ldots K_n\}}^{\omega''}) = \sum_{i=1}^{n} w_i l_i^{\omega''} < d(W, L_{\{K_1,\ldots K_n\}}^{\omega}) = \sum_{i=1}^{n} w_i l_i^{\omega}$. It is an absurd as $w'' \models \mu$ and $\sum_{i=1}^{n} w_i l_i^{\omega} \leq \sum_{i=1}^{n} w_i l_i^{\omega'}$ for every $\omega' \models \mu$.

□

Theorem 5.3. **Proof:** $(\Rightarrow)$ Assume that $\Delta_{\mu}^{d,W}$ satisfies **(IC2)**. Then there is a ω such that $d(\omega, K_i) = 0, \forall i \in \{1,\ldots,n\}$. So, $d(W, L_E^{\omega}) < d(W, L_E^{\omega'})$, for every ω' which $\exists j$ where $d(\omega, K_j) \neq 0$. Suppose that $w_1 = 0$. Then, consider the a ω'' such that $d(\omega'', K_1) = 1$ and $d(\omega'', K_i) = 0$, for $i \neq 1$. Then $d(W, L_E^{\omega}) = w_1 l_1^{\omega} + \cdots + w_n l_n^{\omega} = 0 = w_1 l_1^{\omega''} + \cdots + w_n l_n^{\omega''} = d(W, L_E^{\omega''})$. Contradiction.

$(\Leftarrow)$ Assume $w_1 \neq 0$ and $\bigwedge E$ is consistent with μ, i.e., there exists $\omega \in \Omega$ such that $\omega \models K_1 \wedge \ldots K_n \wedge \mu$. Then, there exists $\omega \in \Omega$ such that $d(\omega, K_1) = \cdots = d(\omega, K_n) = 0$ and as consequence, $d(W, L_{\{K_1,\ldots,K_n\}}^{\omega}) = 0$. According to the definition, this leads to $\omega' \models \Delta_{\mu}^{d,W}(\{K_1,\ldots,K_n\})$ iff $w \models \mu$ and $d(W, L_{\{K_1,\ldots,K_n\}}^{\omega'}) = 0$. We need to show $\omega \models \Delta_{\mu}^{d,W}(\{K_1,\ldots, K_n\})$ iff $\omega \models K_1 \wedge \ldots \wedge K_n \wedge \mu$: $\omega \models \Delta_{\mu}^{d,W}(\{K_1,\ldots,K_n\})$ iff $w \models \mu$ and $d(W, L_{\{K_1,\ldots,K_n\}}^{\omega}) = 0$ iff $w \models \mu$ and $\sum_{i=1}^{n} w_i l_i^{\omega} = 0$ iff (as $w_1 \neq 0$) $w \models \mu$ and $l_1^{\omega} = 0$ iff (as l_i^{ω} is in descending order) $w \models \mu$ and $l_1^{\omega} = \cdots = l_n^{\omega} = 0$ iff $w \models \mu$ and $d(w, K_j) = 0$ for every $j \in \{1,\ldots,n\}$ iff $\omega \models K_1 \wedge \ldots \wedge K_n \wedge \mu$.

□

Theorem 5.4. **Proof:** We can use the following property equivalent to **(Arb)** (KONIECZNY *et al.*, 2004): If $d(\omega, K_1) < d(\omega', K_1), d(\omega, K_2) < d(\omega'', K_2)$ and $\omega' \approx_{\{K_1,K_2\}}^{d,[w_1,w_2]} \omega''$, then $\omega <_{\{K_1,K_2\}}^{d,[w_1,w_2]} \omega'$. Suppose that $d(\omega, K_1) < d(\omega', K_1), d(\omega, K_2) < d(\omega'', K_2)$ and $\omega' \approx_{\{K_1,K_2\}}^{d,[w_1,w_2]} \omega''$. In the worst case we have that $d(\omega, K_1) = d(\omega, K_2) = m - 1$, $d(\omega', K_1) = d(\omega'', K_2) = m$, $d(\omega'', K_1) = d(\omega', K_2) = 0$, where m is the number of propositional variables in the belief set E. In this case, for $\omega <_{\{K_1,K_2\}}^{d,[w_1,w_2]} \omega'$ to be true, we need to have $(m-1)(w_1 + w_2) < m w_1 + 0 w_2 \Rightarrow (m-1)(w_1 + w_2) < m w_1 \Rightarrow m w_1 - w_1 + m w_2 - w_2 < m w_1 \Rightarrow w_1 > (m-1) w_2$.

$\square$

Theorem 5.5. **Proof:** ($\Rightarrow$) Suppose that $\Delta_{\mu}^{d,W}$ satisfies **(PD)** and there is $j > i$, such that $w_j \leq w_i$. By assumption, there is ω', where $l_i^{\omega} < l_i^{\omega'} \leq l_j^{\omega'} < l_j^{\omega}$, $l_i^{\omega'} - l_i^{\omega} = l_j^{\omega} - l_j^{\omega'}$ and $\forall k \neq i, j, l_k^{\omega} = l_k^{\omega'}$. Then, $w_j l_j^{\omega} + w_i l_i^{\omega} \leq w_j l_j^{\omega'} + w_i l_i^{\omega'} \Rightarrow w_j(l_j^{\omega} - l_j^{\omega'}) \leq w_i(l_i^{\omega'} - l_i^{\omega}) \Rightarrow w_j \leq w_i$. Consequently, $\omega \leq_E^{d,W} \omega'$. Contradiction.

($\Leftarrow$) Assume $w_1 > w_2 > w_3 > \ldots > w_n$ and $\exists i, j \in \{1, \ldots, n\}$ such that $d(\omega, K_i) < d(\omega', K_i) \leq d(\omega', K_j) < d(\omega, K_j)$, $d(\omega', K_i) - d(\omega, K_i) = d(\omega, K_j) - d(\omega', K_j)$ and $\forall l \neq i$, $j\ d(\omega, K_l) = d(\omega', K_l)$. So $d(W, L_E^{\omega}) > d(W, L_E^{\omega'}) \Rightarrow w_j l_j^{\omega} + w_i l_i^{\omega} + \sum_{k \neq i,j} w_k l_k^{\omega} > w_j l_j^{\omega'} + w_i l_i^{\omega'} + \sum_{k \neq i,j} w_k l_k^{\omega'} \Rightarrow w_j l_j^{\omega} + w_i l_i^{\omega} > w_j l_j^{\omega'} + w_i l_i^{\omega'} \Rightarrow w_j(l_j^{\omega} - l_j^{\omega'}) > w_i(l_i^{\omega'} - l_i^{\omega}) \Rightarrow w_j > w_i$. Therefore, $\omega' <_E^{d,W} \omega$ and $\Delta_{\mu}^{d,W}$ satisfies **(PD)**.

$\square$

Theorem 5.6. **Proof:** Let $\delta = \frac{1}{m}$, $w_i = \frac{\delta^{i-1}}{(1+\delta)^i}$, for $i \neq n$ and $w_n = \frac{\delta^{n-1}}{(1+\delta)^{n-1}}$. Now assume $\exists i, j \in \{1, \ldots, n\}$ such that $d(\omega, K_i) < d(\omega', K_i) \leq d(\omega', K_j) < d(\omega, K_j)$ and $\forall l \neq i, j\ d(\omega, K_l) = d(\omega', K_l)$. Then, $w_j l_j^{\omega} + w_i l_i^{\omega} + \sum_{k \neq i,j} w_k l_k^{\omega} > w_j l_j^{\omega'} + w_i l_i^{\omega'} + \sum_{k \neq i,j} w_k l_k^{\omega'} \Rightarrow w_j l_j^{\omega} + w_i l_i^{\omega} > w_j l_j^{\omega'} + w_i l_i^{\omega'}$.

Let us assume the worst case where $l_j^{\omega} = m, l_i^{\omega} = 0, l_j^{\omega'} = l_i^{\omega'} = m - 1$, $w_j = w_x$ and $w_i = w_{x+1}$. So, $m\frac{(\frac{1}{m})^{x-1}}{(\frac{m+1}{m})^x} > (m-1)\frac{(\frac{1}{m})^{x-1}}{(\frac{m+1}{m})^x} + (m-1)\frac{(\frac{1}{m})^x}{(\frac{m+1}{m})^{x+1}} \Rightarrow \frac{m^2}{(m+1)^x} > \frac{m^2-m}{(m+1)^x} + \frac{m^2-m}{(m+1)^{x+1}} \Rightarrow \frac{m^2}{(m+1)^x} > \frac{(m^2-m)(m+1)^x + (m^2-m)(m+1)^{x+1}}{(m+1)^x(m+1)^{x+1}} \Rightarrow m^2(m+1)^{x+1} > (m^2 - m)((m+1)^x + (m+1)^{x+1}) \Rightarrow m(m+1)^{x+1} > m(m+1)^x + m(m+1)^{x+1} - (m+1)^x - (m+1)^{x+1} \Rightarrow (m+1)^{x+1} > m(m+1)^x - (m+1)^x \Rightarrow (m+1)(m+1)^x > (m-1)(m+1)^x$. Then, $\omega' <_E^{d,W} \omega$ and $\Delta_{\mu}^{d,W}(E)$ satisfies **(HE)**.

$\square$