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ADAILTON AZEVÊDO ARAÚJO FILHO

THE KALB RAMOND FIELD WITH SPONTANEOUS LORENTZ SYMMETRY BREAKING

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Dissertação de Mestrado apresentada ao Programa de Pós-Graduação em Física da Universidade Federal do Ceará, como requisito parcial para a obtenção do Título de Mestre em Física. Área de Concentração: Física da Matéria Condensada.

Orientador: Prof. Dr. Roberto Vinhaes Maluf Cavalcante.

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To God and to my family.

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For starting off, I would like to thank God for His infinite kindness and mercifulness which has been given to me. Even in the moments that I felt lost, lonely and small, He was right there relaxing my soul and make me figure a way out.

Every step that I gave through my life was under their soft wings. I grew up cocooned in their huge love heart, comforted by the clever words and motivated by their way of being. If I had a chance to live one more time, for sure, I would choose you two as being my parents again! My parents, I love you two SO MUCH. And thank you for everything.

When I was in the 5th semester, and then an undergraduate student, I knew someone who made me learn Electromagnetism I and II. After passing through some difficulties during my academic journey, I had felt quite vulnerable and confused. It turned out to be harder when I had just got my undergraduate degree and been approved to doing the master's level. Nevertheless, that same person took me in and picked me up when I needed someone most. I always remember what he had said: Adailton, don't be afraid! I took it on! Let's get started! And after that, everything for me added up. He made my academic life so soft that I couldn't have imagined. This person is someone who I'd better call him as my scientific father, Roberto Maluf.

Whenever we stumble upon difficulties around the journey of life, at least in few moments, there exist bad feelings which entirely fulfill us. We might fall down, weak, dumb... However, if you have someone who looks after you, you are able to make the guarantee that everything turns out to be easier (especially when you have been doing the same course, Physics). Despite have passed some months, nowadays, I remember that place which love was released at the first time. It was 26/08/2016. In agreement with the Greek's thoughts, which they had established the definition of happiness, which it would become to be known as a moment in which one wishes never ends. And therefore, I may guarantee for sure which the moments which we have been passing through deserve never end. Thank you for everything, Naiara Cipriano.

I would like to thank Stephen Hawking for encouraging me to do what in the currents days I like most, making science. Unfortunately he passed way on the same day of my birthday.

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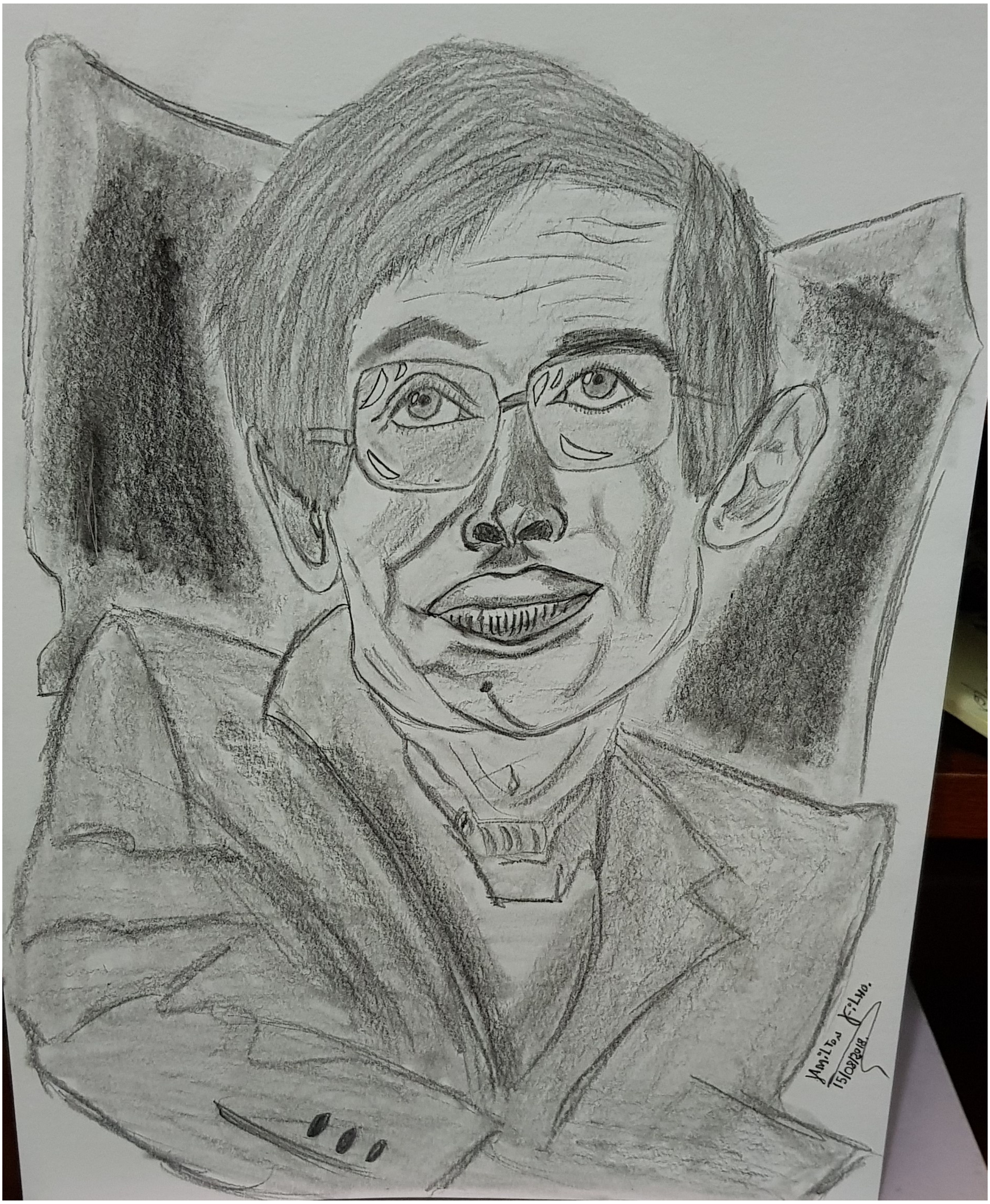
After passing through some experiences in the life, one is able to get used to either bad or good things. One of the most remarkable things which I learned is appreciating the love and brightness which are mean content in their smile. Whenever I realized it was happening, for sure, I wish I could stop the time for the sake of the occasion never ends. Walking through the life we stumble upon a huge kind of people who turn out to belong to our daily life. However, just a few make things happen. This is my sincere homage to the people who changed everything in my life. They took me in and raised me up with all kindness when I was five. With my whole heart, that is my eternal thankfulness to my grandparents for all attention and mercy which was given to me. Indeed, without them, anything could be possible, anything.



It is my honor to make a drawing doing a tribute to this humble person who was walking through the Earth’s crust by a wheelchair. He made his mind his own laboratory for taking the cosmos on. Whatever you are, please accept my sincere homage.

ABSTRACT

This work provides a study of some aspects of vector and tensor fields as well as their effects in the context of Lorentz spontaneous symmetry breaking. Notably, we have focused on bumblebee models in which there exist a vector field B_μ which acquires a nonzero vacuum expectation value. The study of the antisymmetric tensor, the Kalb-Ramond field, is provided as well showing a similar nonzero vacuum expectation value for the tensor field $B_{\mu\nu}$. The modified propagator of the Kalb-Ramond field was calculated. To accomplish this, we implemented a closed algebra with six projectors which were the requirement for inverting the wave operator associated with the Lagrangian density of the theory. The massive mode, in agreement with bumblebee models, do not propagate and, therefore, do not contribute to the calculation of the interparticle potential.

Keywords: Lorentz violation. Anti-symmetric tensor. Bumblebee. Weak field approximation. Propagator.

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Chapter 1

Introduction

1.1 *An overview*

"The standard model of particle physics describes forces and particles very well, but when you throw gravity into the equation, it all falls apart. You have to fudge the figures to make it work". - Lisa Randall [1]

Through the years, the physics has been built up by some concepts which afterwards turned out to be unified [2, 3]. In other words, it reflects some events with different phenomena which were recognized to be related to each other and some theories which were adjusted to fit in such an approach. One of the most remarkable unification happened in the early nineteenth century, the electromagnetism[1]. From 1771 through 1773, Henry Cavendish [4] tried to make an experiment (based on electrostatic theory) which would be known after his name, however, was Charles Augustin de Coulomb [5] who first built it up and added it up in 1785.

At the beginning of the nineteenth century, Hans Christian Oersted [6] realized that when an electric current on a wire was put nearby to the compass, its nail was deflected. Short after, André-Marie Ampère (1820–1825) [7] and Jean-Baptiste Biot Felix Savart (1820) [8] had established that the magnetic field could be produced by an alternating electric current. Following the odds of science in that epoch, a decisive step was given by Michael Faraday (1831) [9], who showed that instead of having an alternating electric current for producing a magnetic field, this field could produce an electric field as well. Equations which govern these situations were consistently added up by James Clerk Maxwell (1865) [10] which would

[1]Both physical phenomena, electricity and magnetism, were thought to be disconnected from each other. However, was with magnetism which was shown indeed a connection between them.

become to be known as Maxwell's equations.

In the late 1960s, another fundamental unification occurred which was about one hundred years after Maxwell's works. This uncovered a deep relationship between the electromagnetism and the weak interactions[2] [11].

Following scientific flux, a remarkable idea was introduced by Albert Einstein [12] who made the theory of relativity[3]. In this theory, one finds out a straight conceptual unification of space and time [13]. In nature, the merging of space and time into a continuous media turned out to represent a new insight where all physical phenomena could take place. Newtonian mechanics [14] was rather substituted by relativistic mechanics [13], and the previous ideas of absolute time were put away. Moreover, mass and energy were shown to be correlated with each other [15]!

Another remarkable theory was brought about mainly by Erwin Schrödinger [16], Werner Heisenberg [17], and Paul Dirac [18–21]. They discovered what would be known afterwards as Quantum Mechanics. It was verified to be a framework in which could describe correctly small bodies[4], the microscopic world [22, 23]. In this theory, instead of having classical variables, the new variables turn out to become operators (observables) [24–26]. Moreover, if two operators commute, their corresponding observables are able to be measured simultaneously.

In the context of knowledge of the fundamental forces, we had better look at the gravitational one. Although was known since long ago, it was first mathematically described by Sr. Isaac Newton [14]. Afterwards, gravity was reformulated within Einstein's theory of relativity [13]. In such an approach, the media where the events are able to take place is the so-called spacetime which borns on its own, and the gravitational force[5] emerges due to its dynamical curvature [27–29]. As anyone is used to knowing, Einstein's theory of relativity was released purely as a classical theory of gravitation rather than a quantum one. Passing through the second fundamental force, we stumble upon the electromagnetic one. In addition, it is worth remarking that Maxwell's theory is integrally consistent with Einstein's theory of relativity mentioned previously. Looking at the third fundamental one, which in this case, was considered as being the weak force. For instance, this is answerable for nuclear beta decay [30] in which a neutron decays into a proton[6]. Despite nuclear beta decay be known since the

[2]For well understanding, it is highly recommended overview some key facts which happened after its developments.

[3]It is worth mentioning that Einstein made two remarkable postulates: all inertial frame must be in agreement with each other and nothing could travel faster than light (causality).

[4]The fundamental particles and so forth.

[5]In the text, this force is considered the first fundamental one.

[6]By the way, in general, the processes which neutrinos are involved are mediated by the weak force [31–35].

late nineteenth century[7], it was not ascribed to a new force, rather it would take hold about the middle of the twentieth century. On the other hand, the weak interaction is much weaker than the electromagnetic one, that is why we are not able to see it working (its effects) in our daily life. For ending the discussion of these fundamental forces started above, the strong interaction [36–40] is the last one. In nowadays, it is also called the color force [41] and plays a role in holding the subatomic particles all together which are added up to forming the nuclei structure as we can see in nature. Their smallest fundamental constituents, which can not be seen in an isolated form, are called quarks [42] which are maintained coupled with each other by a tight interaction due to the color force.

It is time to come back to the subject of unification which was introduced previously. Within a consistent unified framework, in the late 1960s, the Glashow-Weinberg–Salam model of electroweak interactions [11, 43, 44] put both the electromagnetism and the weak interaction together. The theory was initially performed by regarding only massless particles whose carried the force. In nature, there exists a process called symmetry breaking [45–52] and afterwards Peter Higgs [53, 54] who gave a better interpretation which would give him the Nobel prize. Summarily, the particles W^+, W^-, and the Z_0[8], which were formulated for being massless, for the sake to maintaining the gauge invariance[9], [41, 55–58], could acquire mass due to the symmetry breaking process. However, in agreement with the predictions of the Standard Model, there is a remaining massless particle, the photon[10].

Due to the insufficiency of describing particles (small bodies) accurately, the classical electromagnetism turned out to be substituted by the quantum electrodynamics (QED), its quantum version [35]. In this theory, the photon appears to be a quantized package [41, 55, 56, 58] (a quantum) of the electromagnetic field. In accordance with it, the theory of the weak interactions is regarded to be a quantum version as well and named quantum electroweak.

On the other hand, in the case of the strong color interaction, the quantization procedure is provided as well, and the theory which governs it is considered to be quantum chromodynamics (QCD) [59–62]. Analogously to the others forces, the carriers of the color force are eight massless particles [41, 55]. They are colored gluons [63, 64], and in accordance with quarks, they are not able to be visualized in an isolated form in nature. The quarks appear in three different colors and can be felt by gluons because they carry color.

The quantum chromodynamics together with the electroweak theory encapsulates the so-called Standard Model of particle physics [65–67]. In this model, there are interactions

[7]How strength it is given by the Fermi constant.

[8]These three particles are the carriers of the weak interaction.

[9]The theory which is based on the gauge symmetry reflects a "well behavior theory". In other words, the theory which lies in the gauge group often turns out to be renormalizable!

[10]The photon is the carrier of the electromagnetic interaction.

between QCD and electroweak sector because some particles can feel both kinds of interactions. Until now, the Standard Model summarizes well the current knowledge of particle physics and some physicists believe that this model is only a step forward in the formulation of a complete theory [2]. As a non-perfect theory, the Standard Model has some problems such that: it does not suffice to explain the neutrino mass [68] and gravity. In addition, there is, however, a problem when one tries to incorporate the gravitational sector in the Standard Model. Due to the success of quantum theory, it is highly believed that gravity might have a quantum-like version. As a consequence of trying to make a quantum version of gravity, the resulting theory of quantum gravity has appeared to be an ill-defined [3] because it cannot be renormalized. For a better comprehension of the Big-Bang as well as certain properties of black holes, a quantum formulation of the quantum gravity seems to be reasonable. For the sake of formulating a consistent and complete theory which includes gravity, it might as well be required to build up a unified theory [3].

1.2 *String theory*

"String theory has the potential to show that all of the wondrous happenings in the universe - from the frantic dance of subatomic quarks to the stately waltz of orbiting binary stars; from the primordial fireball of the big bang to the majestic swirl of heavenly galaxies - are reflections of one, grand physical principle, one master equation" . - Brian Greene [69]

For a unification of all interactions present in nature, string theory is a remarkable candidate. Actually, in this theory, all kind of particles are considered to be unified since they emerge from string vibrations. In essence, string theory is a quantum theory and, since there exists gravitation within the theory, it is a quantum formulation of gravity. In this viewpoint, and bringing up the failure of Einstein's procedure to yield a quantum version, one might make out that all others interactions are needed to get a concise description of the quantum gravitational sector.

One interesting question which we could ask ourselves would be: how the string theory can give a unified approach? The answer lies in the deep of the theory. In this theory, particles are required to be the specific vibration modes of fundamental "microscopic" strings. Analogously to the violin strings[11], the vibrational string modes can be pointed out to correspond to the different particles which nature holds.

[11] It is capable of vibrating in different kinds of modes and each mode is ascribed to a different sound.

What about the particle decay in the context of string theory? How would be the interpretation? Whenever we stumble upon some physical decay process for instance $\gamma \longrightarrow \beta + \alpha$ (where an elementary particle γ decays into β and α), we might as well imagine a single vibrating string which can be considered as a particle γ which has broken into two strings which their vibrations are associated with particles β and α.

The treatment of uniqueness seems to be reasonable when one considers string theory[12]. It would probably be demotivating to have a lot of eligible candidates for a theory of all forces. The indicative which string theory is unique is because it does not have adjustable dimensionless parameters[13] which are in accordance with the dimensionality of spacetime. On the other hand, in agreement with we have mentioned above, the Standard Model of particle physics has an average of twenty parameters which should be reorganized conveniently and then, cannot be considered unique.

As we are used to knowing, the physical spacetime is composed by one time and three-dimensional space coordinates. In the SM, this information, which is used to added up the theory, is not derived. On the other hand, within string theory, the number of dimensions emerges from the calculation and instead of four, as one is used to knowing, there exist twenty-six dimensions!

For having more details, let us start off with some subdivisions of the theory. Firstly, there exist open and closed strings. Basically, while open strings have two endpoints, closed strings have no endpoints. Secondly, there exist bosonic strings (see Figure 1) which hold in 26 dimensions, and then, bosons can be represented by their vibrations.

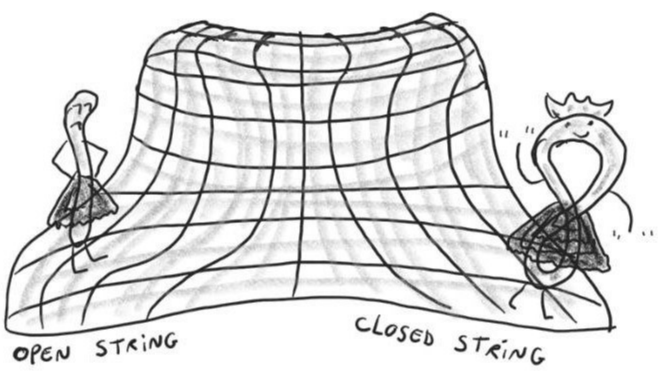

Figure 1: This picture represents the closed string keep dancing on its own. It produces particles associated with its vibration. By the way, the vibrations coming from closed strings may be associated with the massless Kalb–Ramond field which has a remarkable role in my work (see Chapter 3).

[12]Because all particles come from their vibrations.

[13]String theory has one dimensionful parameter, which is called the string length l_s. In a roughly speaking, it can be imagined as the "size" of string.

Another remarkable scenario, which may be most easily explained using covariant string theory, is when the Lorentz symmetry breakdown. It becomes natural when there exists an unstable perturbative string vacuum. The basic idea would be the Lorentz symmetry could be spontaneously broken by the generation of negative square masses due to the Lorentz tensor [70]. It was first investigated by Kostelecký and Samuel in 1989 [70] which had examined the covariant string field theory of open bosonic strings to figure out an attempt to explain the violation of the Poincaré group in the compactification of strings[14]. Moreover, that work provides the analysis whether right couplings were carried out for Lorentz symmetry breaking.

Within particle field theory, spontaneous symmetry breaking can occur when the symmetry of the ground state of the theory no longer exists. This situation arises when there exists a naïve perturbed vacuum which leaves the vacuum unstable. Some fields acquire nonzero vacuum expectation values and therefore, the symmetry is spontaneously broken.

In addition, there exists another approach which borns within string theory, the Kalb-Ramond field[15]. In essence, it is a quantum field which transforms as a 2-form, an anti-symmetric tensor. It was firstly required when Kalb and Ramond considered the direct coupling of the area elements of the world sheets, for the sake to generalize the electromagnetic potential[16] [71]. The Kalb–Ramond field [71] could show up with the dilaton and the metric tensor as being massless excitations coming from closed string [3, 71, 72].

For finishing, this work is divided as follows: In Chapter 2, we make a review of the consequences due to the Lorentz violation in the gravitational scenario. It is focused on bumblebee models that the graviton couples with a vector field B_μ. In Chapter 3, we study the spontaneous Lorentz symmetry breaking due to an anti-symmetric 2-tensor field in Minkowski spacetime. Besides, we calculate a new complete set of spin-projection operators, which sufficed to evaluate the propagator of the Kalb-Ramond field in the context of Lorentz violation. In Chapter 4, we make the conclusion. In the Appendices, we explain some concepts such as: how to obtain Einstein's equation from the variational method, the Higgs mechanism and the propagator of the Kalb-Ramond field.

[14] Because the compactification was regarded to be just an effective theory.

[15] It was named after Michael Kalb and Pierre Ramond [71].

[16] "The basic difference is considering in the fact that the electromagnetic potential is integrated over one-dimensional worldlines of particles to obtain one of its contributions to the action while the Kalb–Ramond field must be integrated over the two-dimensional worldsheet of the string" [3].

Chapter 2

Matter-gravity scattering

This section provides a review of the consequences due to the Lorentz violation in the gravitational scenario considering quantum gravity as an effective field theory [73]. It is focused on bumblebee models that the graviton couples with a vector field B_μ. For accomplishing the nonrelativistic potential between two scalar particles interacting gravitationally, the calculation of the scattering matrix was performed as well.

2.1 *Lorentz violation: the bumblebee models*

"The other energy is the Planck scale energy, which is sixteen orders of magnitude, or ten million billion times, greater than the weak scale energy: a whopping 10^{19} *GeV. The Planck scale energy determines the strength of gravitational interactions: Newton's law says that the strength is inversely proportional to the second power of that energy. And because the strength of gravity is small, the Planck scale mass (related to the Planck scale energy by* $E = mc^2$*) is big. A huge Planck scale mass is equivalent to extremely feeble gravity". - Lisa Randall [74]*

In theoretical physics, a renaming open problem is putting on an equal footing the Standard Model (SM), the theory which describes elementary particles, and General Relativity (GR), the theory which describes the gravitational force as a geometrical effect of spacetime deformation. If we were able to unify both theories, we could expect a fundamental theory of quantum gravity [75]. Although the Standard Model is well tested experimentally, it does not suffice to answer some questions that only a unified theory can do. Such a theory, if there exists, could provide some mechanisms for discovering and exploring new phenomena beyond those described by the Standard Model and General Relativity. Quantum gravity effects are relevantly regarded at high energy scale in order to the Planck mass $m_p \sim 1.22 \cdot 10^{19} GeV$, and then, up to now, as anyone could reach such scale, no evidence of any signatures of a supposed more fundamental physical phenomena have been found out. Although the Planck scale remains non-accessible experimentally, there exist some alternative ways of working on it. Some of them have been performed by exploring a different point of view which quantum gravity phenomena could be observed by highlighting its effects at attainable energies.

Figure 2: This Figure shows an illustration of the search for Lorentz violation as a signature of the Planck scale physics.

One of the most remarkable possibilities is Lorentz symmetry breaking [70] (see Figure 2). Lorentz violation (LV) can be seen in different contexts as, for instance, string theory [76], noncommutative field theories [77], warped brane worlds [78] and loop quantum gravity [79]. Kostelecký and Samuel proposed that due to interactions between strings could lead to spontaneous Lorentz symmetry breaking (see Figure 3).

Moreover, Kostelecký addressed the proposition of calling the Standard model plus Lorentz violation as the Standard Model Extension (SME). The SME provides a set of gauge invariant tensor operators that are in agreement with observer transformations [80] which could be used to forward Lorentz and CPT violation within the physical context.

Several works have been performed within different scenarios in the SME. For instance, CPT-even gauge was firstly examined by Kostelecký and Mewes [81]. This sector was also aimed for connecting with its classical solutions [82], consistency aspects [83] and pho-

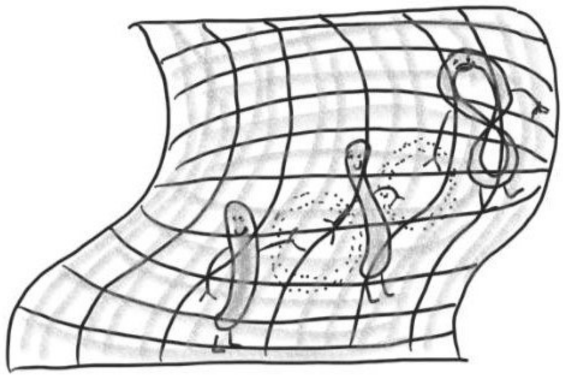

Figure 3: This picture shows some interactions (dotted circles) among open and closed strings on the worldsheet. This would lead to the so-called Lorentz spontaneous symmetry breaking which agrees with the Bianchi identity.

ton/fermion interactions [84, 85]. Some works have been pointed out Lorentz violation scenarios which have operators related to high dimensions giving remarkable results [86, 87]. Such high dimensional operators may also be considered within nonminimal interactions terms. On the other hand, CPT-odd nonminimal fermions coupling was first introduced in reference [88], which is trending some new developments [89, 90].

The gravitational approach has been well explored in the context of SME. It suffices to describe both explicit and spontaneous symmetry breaking. Nevertheless, whenever we are working on explicit symmetry breaking we stumble upon an incompatibility [91]. The continuity equation is no longer satisfied and hence, the Bianchi identity does not work. For the sake of maintaining the "previous blocks all together", we had better work on spontaneous symmetry breaking for addressing Lorentz violation in the gravitational scenario [92]. A general treatment of local Lorentz frame and diffeomorphism, within the gravitational sector of the SME, was accomplished by Bluhm and Kostelecký which says if one breaks Diffeomorphism automatically implies in breaking Lorentz symmetry and vice-versa. [93, 94]. For breaking symmetries spontaneously, is considered a vector field which acquires a nonzero vacuum expectation value (VEV) [94] (see Figures 4 and 5). It represents the so-called bumblebee models (see Figure 6), which was first introduced by Kostelecký and Bluhm [94].

Moreover, there are linearized equations which may be used for studying what happens to the post-Newtonian corrections [73, 95–97]. In reference [73], was investigated the low-energy effects considering Lorentz violation background in the gravitational sector of the Standard Model Extension.

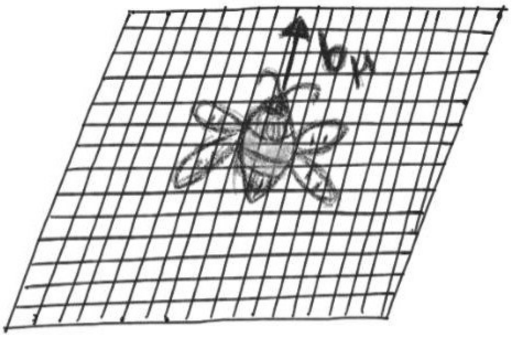

Figure 4: This picture is just an attempt of trying to visualize a preferred direction (a vector) in spacetime after occurring the Lorentz spontaneous symmetry breaking.

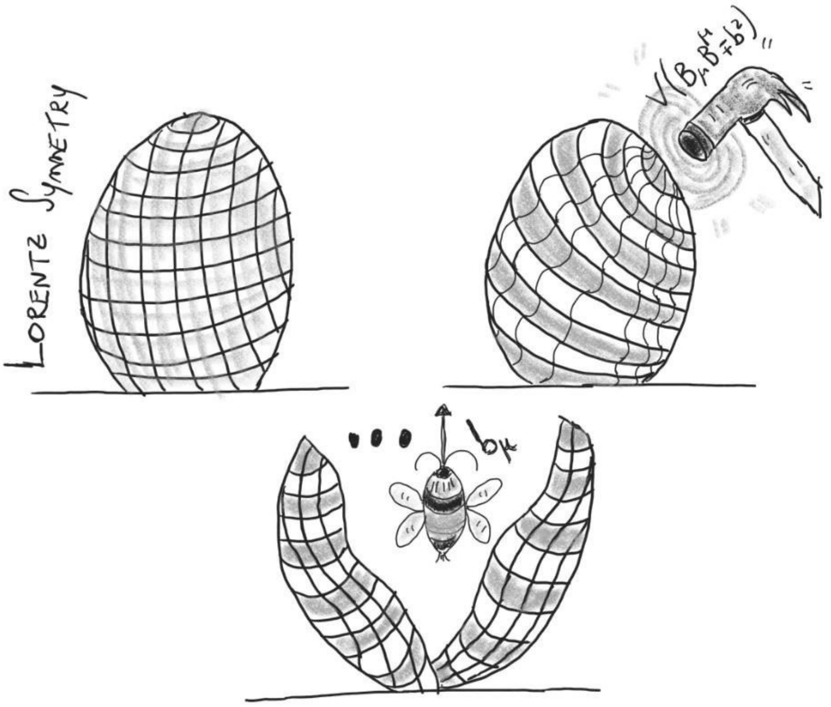

Figure 5: The illustration of the potential $V(B^{\mu}B_{\mu} \mp b^2)$ (the hammer) triggering the so-called Lorentz spontaneous symmetry breaking (crashed egg) as well as the following consequences which are the appearance of Nambu-Goldstone modes (dots) and the appearance of a preferred direction (a vector) in spacetime (bee with an arrow).

In this sense, using weak field limit, was determined the modified graviton propagator and checked its effects ascribed to Lorentz violation [73, 98]. It was shown that introducing a scalar field (coupled with the gravitational field) holds corrections to the so-called Newtonian potential [73]. Differently of what happens in the standard case (radial symmetry), rather it no longer has such radial symmetry showing a spatial anisotropy because of a term proportional to $b_i b_j \hat{x}^i \hat{x}^j$. Indeed, this result is in agreement with post-Newtonian calculations in the gravitational sector of the Standard Model Extension [94, 99, 100]. It is worth pointing out that a new

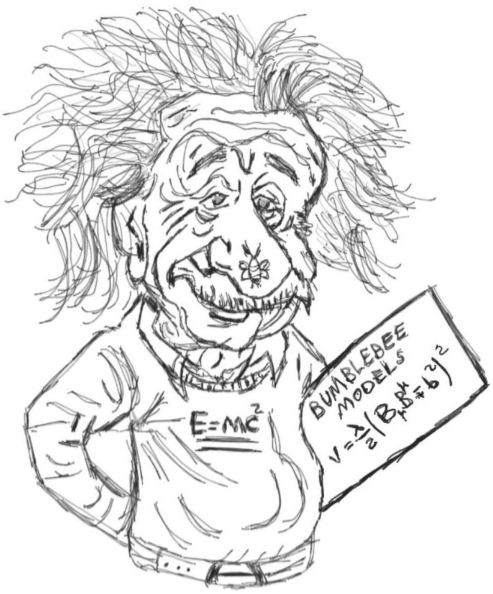

Figure 6: This Figure is a comic illustration of bumblebee models. On the sheet of paper, is written a smooth quadratic potential which triggers Lorentz spontaneous symmetry breaking.

term was found out in reference [73] which is proportional to $\nabla^2 \frac{1}{r} \sim \delta^{(3)}(\vec{x})$ and may be seen as a gravitational Darwin term [101].

2.2 *The mathematical model*

The simplest model regarding Lorentz-violating terms in the gravitational sector which combine fields that may break spontaneously local Lorentz frame is considered bellow as follows:

$$S = S_{EH} + S_{LV} + S_M. \tag{2.1}$$

For starting off, let us consider the first term in equation (2.1), which is represented by the Einstein-Hilbert action[1], given by

$$S_{EH} = \int \mathrm{d}^4x \sqrt{-g}\, \frac{2}{\kappa^2}(R - \Lambda), \tag{2.2}$$

where $\sqrt{-g}$ represents the determinant of the metric $g_{\mu\nu}$, R is the Ricci scalar defined as $R = g_{\mu\nu}R^{\mu\nu}$, Λ is the so-called cosmological constant and the gravitational coupling which is represented by $\kappa^2 = 32\pi G_n$. It was proposed to verify the effects due to the Lorentz violation in the context of nonrelativistic potential within gravitational sector[2].

The second term in equation (2.1) is the minimal model of the Standard Model Extension which encapsulates the coefficients (these will violate Lorentz symmetry) that are

[1] For more details how to develop Einstein's field equation from the Einstein Hilbert action as well as the conservation of the Einstein's tensor see Appendix A.

[2] We are able to neglect the consequences ascribed to Λ yielding $\Lambda = 0$ from now on.

coupled with the Riemann $R_{\mu\nu\alpha\beta}$, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R:

$$S_{LV} = \int \mathrm{d}^4x\sqrt{-g}\,\frac{2}{\kappa^2}\left(uR + s^{\mu\nu}R_{\mu\nu} + t^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\right), \tag{2.3}$$

where u, $s^{\mu\nu}$ and $t^{\mu\nu\alpha\beta}$ are dynamical fields with zero mass dimension. It is worth mentioning that since the $s^{\mu\nu}$ and $t^{\mu\nu\alpha\beta}$ couple with the Ricci tensor $R^{\mu\nu}$ and the Riemann tensor $R^{\mu\nu\alpha\beta}$ respectively, one might expect, for considering the calculations, the same symmetries of them. We consider the action (2.4) which is assumed to be a scalar (i.e. invariant under general coordinate transformations). Moreover, for the sake of obtaining the violation of the local Lorentz frame, a Higgs-like mechanism is presented[3].

Looking at the third and last term to the action (2.1) we stumble upon the matter-gravity couplings[4]. Nevertheless, it is focused on the effects due to the action (2.3), restricting the attention to the case whose the matter interacts exclusively with the gravitational field[5].

In agreement with the work which was made by Bailey and Kostelecký [94], let us consider a particular case where $t^{\mu\nu\alpha\beta} = 0$ for simplifying the calculations. One is able to note that u and $s^{\mu\nu}$ have summed 10 degrees of freedom, which may be regarded as an effective field theory with a vector field B_μ, whose dynamics is considered when one takes down the following action:

$$S_B = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \sigma B^\mu B^\nu R_{\mu\nu} - V\left(B^\mu B_\mu \mp b^2\right)\right], \tag{2.4}$$

where the field strength is given by $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\nu$, σ is the coupling constant (dimensionless), and b^2 plays the role of the vacuum expectation value due to B_μ field. The VEV is no longer zero and has the minimum value governed by $g_{\mu\nu}B^\mu B^\nu \pm b^2 = 0$ when the field gets a nonzero vacuum expectation value [70, 92, 94].

Reference [94] gives us the relation between the equation (2.3) and (2.4) which follows:

$$u = \frac{1}{4}\xi B^\alpha B_\alpha, \quad s^{\mu\nu} = \xi B^\mu B^\nu - \frac{1}{4}\xi g^{\mu\nu}B^\alpha B_\alpha, \quad t^{\mu\nu\alpha\beta} = 0, \tag{2.5}$$

where was considered $\sigma = 2\xi/\kappa^2$ for symplicity [73].

[3]If you are interested in more details about the Higgs Mechanism see Appendix B.

[4]They include all fields of the standard model as well as the possibility of interacting with u, $s^{\mu\nu}$ and $t^{\mu\nu\alpha\beta}$ which were mentioned previously.

[5]For further details involving the Lorentz-violation in the matter sector regarding the SME, one may check in reference [99] out.

2.3 *Perturbation in spacetime*

For verifying the effects due to the gravity-bumblebee coupling with the graviton propagator, one separates the dynamical fields as a vacuum solution plus fluctuations[6]. Then, the fields $g_{\mu\nu}, g^{\mu\nu}, B_\mu$ and B^μ can be rewritten as follows:

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu}, \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \qquad B_\mu = b_\mu + \tilde{B}_\mu, \qquad B^\mu = b^\mu + \tilde{B}^\mu - \kappa b_\nu h^{\mu\nu}, \tag{2.6}$$

where $\tilde{B}_\mu$ and $h_{\mu\nu}$[7] are the previous fluctuations which we have mentioned above and b_μ is the

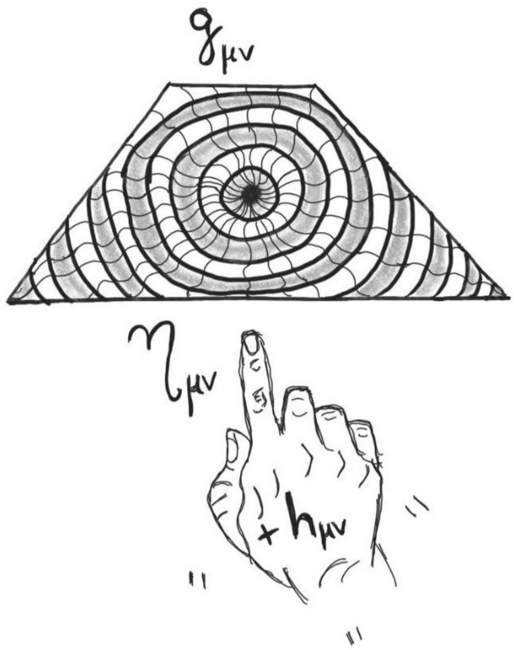

Figure 7: The metric $g_{\mu\nu}$ turns out to be regarded as Minkowski spacetime ($\eta_{\mu\nu}$) plus a small perturbation ($h_{\mu\nu}$).

vacuum expectation value of the bumblebee field. Before going on, let us demonstrate the last expression which appears in equation (2.6). Starting from $B_\mu = b_\mu + \tilde{B}_\mu$, one writes

$$\begin{aligned}
g_{\mu\nu} B^\nu &= \eta_{\mu\nu} b^v + \eta_{\mu\nu} \tilde{B}^\nu \\
B^\alpha &= g^{\mu\alpha} \left(\eta_{\mu\nu} b^\nu + \eta_{\mu\nu} \tilde{B}^\nu \right) \\
B^\alpha &= \left(\eta^{\mu\alpha} - \kappa h^{\mu\alpha} \right) \left(\eta_{\mu\nu} b^\nu + \eta_{\mu\nu} \tilde{B}^\nu \right) \\
B^\alpha &= \eta_{\mu\nu} \eta^{\mu\alpha} b^\nu + \eta_{\mu\alpha} \eta^{\mu\alpha} \tilde{B}^\nu - \kappa\, \eta_{\mu\nu} b^\nu h^{\mu\alpha} - \underbrace{\kappa\, \eta_{\mu\nu} \tilde{B}^\nu h^{\mu\alpha}}_{Second\ order} \\
B^\alpha &= b^\alpha + \tilde{B}^\alpha - b_\mu h^{\mu\alpha}.
\end{aligned} \tag{2.7}$$

[6] This is the standard procedure when one wants to solve an equation up to a perturbation.
[7] Another way of seeing it is thinking of a small fluctuation around the Minkowski space (see Figure 7).

And one might try to do the opposite. Starting with B^μ we could get B_μ as well. By the way, let us try to do this for the sake of verifying the veracity our previous result. It follows that

$$\begin{aligned}
B^\mu &= b^\mu + \tilde{B}^\mu - b_\nu h^{\mu\nu} \\
g^{\mu\sigma} B_\sigma &= \eta^{\mu\sigma} b_\sigma + \eta^{\mu\sigma} \tilde{B}^\sigma - b_\nu h^{\mu\nu} \\
B_\alpha &= g_{\mu\alpha}\eta^{\mu\sigma} b_\sigma + g_{\mu\alpha}\eta^{\mu\sigma}\tilde{B}_\sigma - g_{\mu\alpha} b_\nu h^{\mu\nu} \\
&= (\eta_{\mu\alpha} + h_{\mu\alpha})\,\eta^{\mu\sigma} b_\sigma + (\eta_{\mu\alpha} + h_{\mu\alpha})\eta^{\mu\sigma}\tilde{B}_\sigma - (\eta_{\mu\alpha} + h_{\mu\alpha})\, b_\nu h^{\mu\nu} \\
&= \eta_{\mu\alpha}\eta^{\mu\sigma} b_\sigma + \eta^{\mu\sigma} h_{\mu\alpha} b_\sigma + \eta_{\mu\alpha}\eta^{\mu\sigma}\tilde{B}_\sigma + \eta^{\mu\sigma} h_{\mu\alpha}\tilde{B}_\sigma - \eta_{\mu\alpha} h^{\mu\nu} b_\nu - h_{\mu\alpha} h^{\mu\nu} b_\nu \\
&= b_\alpha + \tilde{B}_\alpha + \eta^{\mu\sigma} h_{\mu\alpha} b_\sigma - \eta_{\mu\alpha} h^{\mu\nu} b_\nu \\
&= b_\alpha + \tilde{B}_\alpha \\
\longrightarrow\ B_\mu &= b_\mu + \tilde{B}_\mu,
\end{aligned} \tag{2.8}$$

as we should expect.

Now, didactically let us open equation (2.4):

$$\begin{aligned}
\mathscr{L} &= \sqrt{-g}\,[-\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \sigma B^\mu B^\nu R_{\mu\nu} - V(B_\mu B^\mu \mp b^2)] \\
&= \sqrt{-g}\,[-\frac{1}{4}\,(\partial_\mu B_\nu - \partial_\nu B_\mu)\,(\partial^\mu B^\nu - \partial^\nu B^\mu) + \sigma B^\mu B^\nu R_{\mu\nu} - V(B_\mu B^\mu \mp b^2)] \\
&= \sqrt{-g}\,[-\frac{1}{4}\,(\partial_\mu B_\nu \partial^\mu B^\nu - \partial_\mu B_\nu \partial^\nu B^\mu - \partial_\nu B_\mu \partial^\mu B^\nu + \partial_\nu B_\mu \partial^\nu B^\mu) + \sigma B^\mu B^\nu R_{\mu\nu} - V(B_\mu B^\mu \mp b^2)] \\
&= \sqrt{-g}\,[-\frac{1}{2}\,(\partial_\mu B_\nu \partial^\mu B^\nu - \partial_\mu B_\nu \partial^\nu B^\mu) + \sigma B_\mu B_\nu R^{\mu\nu} - V(B_\mu B^\mu \mp b^2)].
\end{aligned} \tag{2.9}$$

Using the Euler-Lagrange for fields

$$\frac{\delta \mathscr{L}}{\delta B_\nu} = \partial_\mu \left(\frac{\delta \mathscr{L}}{\delta\,(\partial_\mu B_\nu)} \right), \tag{2.10}$$

and varying with respect to B_μ we have attained:

$$\begin{aligned}
&\blacksquare\ \ \frac{\delta \mathscr{L}}{\delta B_\nu} = \sqrt{-g}\,[\,2\,\sigma B_\mu R^{\mu\nu} - 2\,V' B^\nu]\,, \\
&\blacksquare\ \ \frac{\delta \mathscr{L}}{\delta\,(\partial_\mu B_\nu)} = \sqrt{-g}\left[-\frac{1}{2}\,(2\,\partial^\mu B^\nu - 2\,\partial^\nu B^\mu)\right] = -\sqrt{-g} B^{\mu\nu} \ \rightarrow\ -\partial_\mu \left(\sqrt{-g} B^{\mu\nu}\right),
\end{aligned} \tag{2.11}$$

note that

$$V' = \frac{\mathrm{d}V}{\mathrm{d}B_\mu},$$

and plugging them all together:

$$-\frac{1}{\sqrt{-g}}\,\partial^\mu \left(\sqrt{-g} B_{\mu\nu}\right) = -2V' B_\nu + 2\sigma B^\mu R_{\mu\nu}, \tag{2.12}$$

or,

$$\boxed{\frac{1}{\sqrt{-g}}\,\partial^{\mu}\left(\sqrt{-g}B_{\mu\nu}\right)-2V'B_{\nu}+2\sigma B^{\mu}R_{\mu\nu}=0.}$$

This equation of motion is totally in agreement with what Bailey and Kostelecký had done [94]. We choose the quadratic potential for triggering Lorentz violation:

$$V=\frac{\lambda}{2}\left(B^{\mu}B_{\mu}\mp b^{2}\right)^{2}. \tag{2.13}$$

For calculating the modification of the graviton due to Lorentz violation, we have to previously linearize the equation of motion (2.12). For accomplishing this, we have didactically separated equation (2.12) in three parts, I,II, and III as follows:

$$\underbrace{\frac{1}{\sqrt{-g}}\,\partial^{\mu}\left(\sqrt{-g}B_{\mu\nu}\right)}_{I}\underbrace{-2V'B_{\nu}}_{II}+\underbrace{2\sigma B^{\mu}R_{\mu\nu}}_{III}=0,$$

$$\begin{aligned}
\underline{I}:\quad &=\partial^{\mu}\left(\partial_{\mu}B_{\nu}-\partial_{\nu}B_{\mu}\right)=\partial^{\mu}\partial_{\mu}B_{\nu}-\partial^{\mu}\partial_{\nu}B_{\mu}=\Box B_{\nu}-\partial^{\mu}\partial_{\nu}B_{\mu}\\
&=\Box\left(b_{\nu}+\tilde{B}_{\nu}\right)-\partial^{\mu}\partial_{\nu}B_{\mu}=\Box\, b_{\nu}+\Box\,\tilde{B}_{\nu}-\partial_{\mu}\partial_{\nu}B^{\mu}\\
&=\Box\, b_{\nu}+\Box\,\tilde{B}_{\nu}-\partial_{\mu}\partial_{\nu}\left(b^{\mu}+\tilde{B}^{\mu}-\kappa\, b_{\nu}h^{\mu\nu}\right)\\
&=-\partial_{\mu}\partial_{\nu}\tilde{B}^{\mu}+\left(\Box\, b_{\nu}+\Box\,\tilde{B}_{\nu}\right)+\kappa\,\partial_{\mu}\partial_{\nu}b_{\nu}h^{\mu\nu}-\partial_{\mu}\partial_{\nu}b^{\mu}\\
&=-\partial_{\mu}\partial_{\nu}\tilde{B}^{\mu}+\Box\tilde{B}_{\nu}\\
&=\Box\,\eta_{\mu\nu}\tilde{B}^{\mu}-\partial_{\mu}\partial_{\nu}\tilde{B}^{\mu},\\[2ex]
\underline{II}:\quad &=-2\lambda\left[\left(b^{\mu}+\tilde{B}^{\mu}-\kappa b_{\nu}h^{\mu\nu}\right)\left(b_{\mu}+\tilde{B}_{\mu}\right)\mp b^{2}\right]\left[b_{\nu}+\tilde{B}_{\nu}\right]\\
&=-2\lambda\left[b^{2}+b^{\mu}\tilde{B}_{\mu}+b_{\mu}\tilde{B}^{\mu}-\kappa\, b_{\beta}b_{\alpha}h^{\mu\nu}\mp b^{2}\right]\left[b_{\nu}+\tilde{B}_{\nu}\right]\\
&=-2\lambda\left[b^{2}b_{\nu}+b^{2}\tilde{B}_{\nu}+2b_{\nu}b_{\mu}\tilde{B}^{\mu}-\kappa\, b_{\beta}b_{\nu}b_{\alpha}h^{\alpha\beta}\mp b^{2}b_{\nu}\mp b^{2}\tilde{B}_{\nu}\right]\\
&=-4\lambda b_{\nu}b_{\mu}\tilde{B}^{\mu}+2\lambda\kappa\, b_{\nu}b_{\alpha}b_{\beta}h^{\alpha\beta},\\[2ex]
\underline{III}:\quad &=2\sigma\left(b^{\mu}+\tilde{B}^{\mu}-\kappa\, b_{\nu}h^{\mu\nu}\right)R_{\mu\nu}\\
&=2\sigma\left(b^{\mu}R_{\mu\nu}+\tilde{B}^{\mu}R_{\mu\nu}-\kappa\, b_{\nu}h^{\mu\nu}R_{\mu\nu}\right)\\
&=2\sigma b^{\mu}R_{\mu\nu}.
\end{aligned} \tag{2.14}$$

Adding all these terms together:

$$\left(\Box\,\eta_{\mu\nu}-\partial_{\mu}\partial_{\nu}-4\lambda b_{\mu}b_{\nu}\right)\tilde{B}^{\mu}=-2\lambda\kappa\, b_{\nu}b_{\alpha}b_{\beta}h^{\alpha\beta}-2\sigma b^{\alpha}R_{\alpha\nu}, \tag{2.15}$$

where $R_{\alpha\nu} = R_{\alpha\nu}(h)$.

This above equation is regarded to be the linearized version of equation (2.12). After applying the Green's function method, and putting in momentum space, one is able to check the following ansatz:

$$\tilde{B}^{\mu} = \frac{\kappa\, p^{\mu} b_{\alpha} b_{\beta} h^{\alpha\beta}}{2(b\cdot p)} + \frac{2\sigma b_{\alpha} R^{\alpha\mu}}{p^2} - \frac{2\sigma p^{\mu} b_{\alpha} b_{\beta} R^{\alpha\beta}}{p^2(b\cdot p)} + \frac{\sigma p^{\mu} R}{4\lambda(b\cdot p)} - \frac{\sigma b^{\mu} R}{p^2} + \frac{\sigma\, b^2 p^{\mu} R}{p^2(b\cdot p)}\ , \quad (2.16)$$

note that, $p^2 = p^{\mu}p_{\mu} = p\cdot p$, and $b\cdot p = b_{\mu}p^{\mu}$.

Proof: Using the Green's function method,

$$\hat{\mathcal{O}}_{\mu\nu}\, G^{\nu\alpha}(x-y) = \delta^{\alpha}_{\mu}\, \delta^{(4)}(x-y), \quad (2.17)$$

it can be defined as a Fourier transform which follows

$$G^{\nu\alpha}(x-y) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4p\, e^{-ip\cdot(x-y)}\, G^{\nu\alpha}_{(p)}, \quad \delta^{(4)}(x-y) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4p\, e^{-ip\cdot(x-y)}. \quad (2.18)$$

Taking into account the left terms of the linearized version of the equation of the motion represented by (2.15):

$$\underbrace{(\eta_{\mu\nu}\Box - \partial_{\mu}\partial_{\nu} - 4\lambda\, b_{\mu}b_{\nu})}_{\hat{\mathcal{O}}_{\mu\nu}}\tilde{B}^{\mu}, \quad (2.19)$$

follows that

$$(\eta_{\mu\nu}\Box - \partial_{\mu}\partial_{\nu} - 4\lambda\, b_{\mu}b_{\nu})\frac{1}{(2\pi)^4}\int \mathrm{d}^4p\, e^{-ip\cdot(x-y)}\, G^{\nu\alpha}_{(p)} = \delta^{\alpha}_{\mu}\frac{1}{(2\pi)^4}\int \mathrm{d}^4p\, e^{-ip\cdot(x-y)}, \quad (2.20)$$

and,

$$(-\eta_{\mu\nu}\, p^2 + p_{\mu}\, p_{\nu} - 4\lambda\, b_{\mu}b_{\nu})\frac{1}{(2\pi)^4}\int \mathrm{d}^4p\, e^{-ip\cdot(x-y)}\, G^{\nu\alpha}_{(p)} = \delta^{\alpha}_{\mu}\frac{1}{(2\pi)^4}\int \mathrm{d}^4p\, e^{-ip\cdot(x-y)}. \quad (2.21)$$

This entails that

$$G^{\nu\alpha}_{(p)}(-\eta_{\mu\nu}\, p^2 + p_{\mu}\, p_{\nu} - 4\lambda\, b_{\mu}b_{\nu}) = \delta^{\alpha}_{\mu}. \quad (2.22)$$

Rewriting the Green's function as a linear combination with the basis vectors,

$$G^{\nu\alpha}_{(p)} = \tilde{a}\,\eta_{\nu\alpha} + \tilde{b}\, p^{\nu}p^{\alpha} + \tilde{c}\, b^{\nu}b^{\alpha} + \tilde{d}\,(p^{\nu}b^{\alpha} + p^{\alpha}b^{\nu}), \quad (2.23)$$

where $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$ are coefficients to be determined. The next step is just finding out what are these coefficients. To accomplish this, let us work on the following expression:

$$[\tilde{a}\,\eta_{\nu\alpha} + \tilde{b}\, p^{\nu}p^{\alpha} + \tilde{c}\, b^{\nu}b^{\alpha} + \tilde{d}\,(p^{\nu}b^{\alpha} + p^{\alpha}b^{\nu})]\,[-\eta_{\mu\nu}\, p^2 + p_{\mu}\, p^{\nu} - 4\lambda\, b_{\mu}b_{\nu}] = \delta^{\alpha}_{\mu}. \quad (2.24)$$

This above equation generates 16 terms which we must analyze carefully for the sake of clarity.

Then,

$$\begin{aligned}
&- \tilde{a}\, p^2 \delta^\alpha_\mu + \tilde{a}\, p_\mu p^\alpha - 4\, \tilde{a} \lambda\, b_\mu b^\alpha \\
&- \tilde{b}\, p^2 p_\mu p^\alpha + \tilde{b}\, p^2 p_\mu p^\alpha - 4\, \lambda\, \tilde{b}\, (b \cdot p) b_\mu p^\alpha \\
&- \tilde{c}\, p^2 b_\mu b^\alpha + \tilde{c}\, (b \cdot p) p_\mu b^\alpha - 4\, \lambda\, \tilde{c}\, b^2 b_\mu b^\alpha \\
&- \tilde{d}\, p^2 (p_\mu b^\alpha + b_\mu p^\alpha) + \tilde{d}\, (p^2 p_\mu b^\alpha + (b \cdot p)) p_\mu p^\alpha - \tilde{d}\, (4\, \lambda (b \cdot p) b_\mu b^\alpha + 4\, \lambda b^2 b_\mu p^\alpha) = \delta^\alpha_\mu .
\end{aligned} \tag{2.25}$$

Considering the following basis $\delta^\alpha_\mu, p_\mu p^\alpha, p_\mu b^\alpha, b_\mu p^\alpha, b_\mu b^\alpha$, we are able to solve the system of equations involving coefficients $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$.

For the basis δ^α_μ, we may separate the term involving only $\tilde{a}$ as follows:

$$-\tilde{a} p^2 = 1, \tag{2.26}$$

and therefore,

$$\boxed{\tilde{a} = -\frac{1}{p^2}.}$$

For the basis $p_\mu p^\alpha$, we may separate the terms involving $\tilde{a}$ and $\tilde{b}$ as follows:

$$\tilde{a} - \tilde{b}\, p^2 + \tilde{b}\, p^2 + \tilde{d}\, (b \cdot p) = 0, \tag{2.27}$$

and therefore,

$$\boxed{\tilde{d} = \frac{1}{p^2\, (b \cdot p)}.}$$

For the basis $b_\mu b^\alpha$, we may separate the terms involving $\tilde{a}$, $\tilde{c}$ and $\tilde{d}$ as follows:

$$-4\, \lambda\, \tilde{a} - \tilde{c}\, p^2 - 4\, \lambda\, \tilde{c}\, b^2 - 4\, \lambda\, \tilde{d}\, (b \cdot p) = 0, \tag{2.28}$$

and therefore,

$$\boxed{\tilde{c} = 0.}$$

For the basis $b_\mu p^\alpha$, we may separate the terms involving $\tilde{b}$ and $\tilde{d}$ as follows:

$$-4\, \lambda\, (b \cdot p) - \tilde{d}\, p^2 - 4\, \lambda\, \tilde{d}\, b^2 = 0, \tag{2.29}$$

and therefore,

$$\boxed{\tilde{b} = -\frac{(p^2 + 4\, \lambda\, b^2)}{4\, \lambda\, p^2\, (b \cdot p)^2}.}$$

Hence, the full Green's function is given by:

$$G^{\nu\alpha}_{(p)} = -\frac{1}{p^2}\eta^{\nu\alpha} - \frac{(p^2 + 4\,\lambda\, b^2)}{4\,\lambda\, p^2\,(b\cdot p)^2}\, p^\nu p^\alpha + \frac{1}{p^2\,(b\cdot p)}(p^\nu b^\alpha + p^\alpha b^\nu), \tag{2.30}$$

or,

$$\boxed{G^{\mu\nu}_{(p)} = -\frac{1}{p^2}\eta^{\mu\nu} - \frac{(p^2 + 4\,\lambda\, b^2)}{4\,\lambda\, p^2\,(b\cdot p)^2}\, p^\mu p^\nu + \frac{1}{p^2\,(b\cdot p)}(p^\mu b^\nu + p^\nu b^\mu).}$$

Lastly, to obtain equation (2.16) is needed to couple the Green's function with the renaming current, i.e. $\tilde{B}^\mu = G^{\mu\nu}_{(p)}(-2\lambda\kappa\, b_\nu b_\alpha b_\beta h^{\alpha\beta} - 2\sigma b^\alpha R_{\alpha\nu})$ as follows:

$$\begin{aligned}
\tilde{B}^\mu = &+\frac{2\lambda\kappa\,\eta^{\mu\nu} b_\nu b_\alpha b_\beta h^{\alpha\beta}}{p^2} + \frac{2\lambda\kappa\, b_\nu b_\alpha b_\beta h^{\alpha\beta}}{4\lambda p^2 (b\cdot p)^2}(4\lambda b^2 + p^2) p^\mu p^\nu - \frac{2\lambda\kappa\, b_\nu b_\alpha b_\beta h^{\alpha\beta}}{p^2 (b\cdot p)}(p^\mu b^\nu + p^\nu b^\mu) \\
&+ \frac{2\sigma\eta^{\mu\nu} b^\alpha R_{\alpha\nu}}{p^2} + \frac{2\sigma b^\alpha R_{\alpha\nu}}{4\lambda p^2 (b\cdot p)^2}(4\lambda b^2 + p^2) p^\mu p^\nu - \frac{2\sigma b^\alpha R_{\alpha\nu}}{p^2 (b\cdot p)}(p^\mu b^\nu + p^\nu b^\mu) \\
&= +\cancel{\frac{2\lambda\kappa\, b^\mu b_\alpha b_\beta h^{\alpha\beta}}{p^2}} \\
&+ \cancel{\frac{2\lambda\kappa\, b^2 p^\mu (b\cdot p) b_\alpha b_\beta h^{\alpha\beta}}{p^2 (b\cdot p)^2}} + \frac{\kappa\,(b\cdot p) p^\mu b_\alpha b_\beta h^{\alpha\beta} p^2}{2p^2 (b\cdot p)^2} \\
&- \cancel{\frac{2\lambda\kappa\, b^2 p^\mu b_\alpha b_\beta h^{\alpha\beta}}{p^2 (b\cdot p)}} - \cancel{\frac{2\lambda\kappa\, p^2 (b\cdot p) b^\mu b_\alpha b_\beta h^{\alpha\beta}}{p^2 (b\cdot p)}} \\
&+ \frac{2\sigma b_\alpha R^{\alpha\mu}}{p^2} \\
&+ \frac{2\sigma b^2 b^\alpha R_{\alpha\nu} p^\mu p^\nu}{p^2 (b\cdot p)^2} + \frac{\sigma b^\alpha R_{\alpha\nu} p^2 p^\mu p^\nu}{2\lambda p^2 (b\cdot p)^2} \\
&- \frac{2\sigma b^\alpha R_{\alpha\nu} p^\mu b^\nu}{p^2 (b\cdot p)} - \frac{2\sigma b^\alpha R_{\alpha\nu} p^\nu p^\mu}{p^2 (b\cdot p)},
\end{aligned} \tag{2.31}$$

where $(b\cdot p) = b_\mu\, p^\mu, b^2 = b_\mu\, b^\mu, p^2 = p_\mu\, p^\mu, p^\mu b^\nu R_{\mu\nu} = \frac{1}{2}(b\cdot p)R$. Hence we obtain,

$$\boxed{\tilde{B}^\mu = \frac{\kappa\, p^\mu b_\alpha b_\beta h^{\alpha\beta}}{2(b\cdot p)} + \frac{2\sigma b_\alpha R^{\alpha\mu}}{p^2} - \frac{2\sigma p^\mu b_\alpha b_\beta R^{\alpha\beta}}{p^2 (b\cdot p)} + \frac{\sigma p^\mu R}{4\lambda (b\cdot p)} - \frac{\sigma b^\mu R}{p^2} + \frac{\sigma b^2 p^\mu R}{p^2 (b\cdot p)}.}$$

□

If this above equation is a solution, hence must satisfy the previous relation which comes from the linearized version of the equation of motion $(\Box\,\eta_{\mu\nu} - \partial_\mu\partial_\nu - 4\lambda b_\mu b_\nu)\,\tilde{B}^\mu = -2\lambda\kappa\, b_\nu b_\alpha b_\beta h^{\alpha\beta} - 2\sigma b^\alpha R_{\alpha\nu}$. It is worth verifying whether $\tilde{B}^\mu$ indeed satisfies the linearized equation of motion. In this sense,

$$
\begin{aligned}
(\Box\,\eta_{\mu\nu}-\partial_\mu\partial_\nu-4\lambda b_\mu b_\nu)\tilde{B}^\mu =&+\cancel{\frac{\kappa\,\partial^\mu\partial^\mu p_\nu b_\alpha b_\beta h^{\alpha\beta}}{2(b\cdot p)}}+\cancel{\frac{2\sigma\partial_\mu\partial^\mu\eta_{\mu\nu}b_\alpha R^{\alpha\mu}}{p^2}}-\cancel{\frac{2\sigma\partial_\mu\partial^\mu p_\nu b_\alpha b_\beta R^{\alpha\beta}}{p^2(b\cdot p)}}\\
&+\cancel{\frac{\sigma\partial_\mu\partial^\mu p_\nu R}{4\lambda(b\cdot p)}}-\cancel{\frac{\sigma\partial_\mu\partial^\mu b_\nu R}{p^2}}+\cancel{\frac{\sigma\partial_\mu\partial^\mu p_\nu b^2 R}{p^2(b\cdot p)}}\\
&-\cancel{\frac{\kappa\,\partial_\mu\partial_\nu p^\mu b_\alpha b_\beta h^{\alpha\beta}}{2(b\cdot p)}}-\cancel{\frac{2\sigma\partial_\mu\partial_\nu b_\alpha R^{\alpha\mu}}{p^2}}+\cancel{\frac{2\sigma\partial_\mu\partial_\nu p^\mu b_\alpha b_\beta R^{\alpha\beta}}{p^2(b\cdot p)}}\\
&-\cancel{\frac{\sigma\partial_\mu\partial_\nu p^\mu R}{4\lambda(b\cdot p)}}+\cancel{\frac{\sigma\partial_\mu\partial_\nu b^\mu R}{p^2}}-\cancel{\frac{\sigma\partial_\mu\partial_\nu p^\mu b^2 R}{p^2(b\cdot p)}}\\
&-\frac{4\kappa\,p^\mu b_\alpha b_\beta h^{\alpha\beta}\lambda b_\mu b_\nu}{2(b\cdot p)}-\cancel{\frac{8\sigma b_\alpha R^{\alpha\mu}\lambda b_\mu b_\nu}{p^2}}+\cancel{\frac{8\sigma p^\mu b_\alpha b_\beta R^{\alpha\beta}\lambda b_\mu b_\nu}{p^2(b\cdot p)}}\\
&-\frac{4\sigma p^\mu R\lambda b_\mu b_\nu}{4\lambda(b\cdot p)}+\cancel{\frac{4\sigma b^\mu R\lambda b_\mu b_\nu}{p^2}}-\cancel{\frac{4\sigma p^\mu b^2 R\lambda b_\mu b_\nu}{p^2(b\cdot p)}}\\
=&-2\lambda\kappa\, b_\nu b_\alpha b_\beta h^{\alpha\beta}-2\sigma b^\alpha R_{\nu\alpha},
\end{aligned}
\tag{2.32}
$$

as we would like to verify.

Substituting (2.16) in (2.3), we are about to see the modification of the graviton due to the nonzero vacuum expectation value to b_μ field. After noting this, let us expand the graviton-bumblebee interaction[8]. However, there are some remarks which are needed to be fulfilled before stepping forward. First, it is quite recommended which one has the background concerned to the linearization of the $\sqrt{-g}$ which is unusual in textbooks on general relativity. Second, it is how to apply the previous knowledge in (2.4). As we have mentioned above, we might as well go inside to the process of linearizing $\sqrt{-g}$ which comes from some mathematical definitions as follows:

$$
g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu},\qquad \log(\det)=\mathrm{tr}(\log),\qquad \det(\eta+h)=\det(\eta)\det(1+\eta^{-1}h).
$$

[8]Remembering that we have considered only terms involving up to the second order.

It follows,

$$\begin{aligned}
\sqrt{-g} &= \sqrt{-\det(\eta+h)} = e^{\log\left(\sqrt{-\det(\eta+h)}\right)} = e^{\frac{1}{2}\log(-\det(\eta+h))} = e^{\left[\frac{1}{2}\log\left(-\det(\eta)\det(1+\eta^{-1}h)\right)\right]} \\
&= e^{\left[\log(-\det\eta)+\frac{1}{2}\log\left(\det\left(1+\eta^{-1}h\right)\right)\right]} = \sqrt{-\det\eta}e^{\left[\frac{1}{2}\log\left(\det\left(1+\eta^{-1}h\right)\right)\right]} \\
&= \sqrt{-\det\eta}e^{\left[\frac{1}{2}\mathrm{tr}\left(\log\left(\left(1+\eta^{-1}h\right)\right)\right)\right]} = \sqrt{-\det\eta}e^{\left[\frac{1}{2}\mathrm{tr}\left(\eta^{-1}h-\frac{1}{2}\left(\eta^{-1}h\right)^2+\ldots\right)\right]} \\
&= \sqrt{-\det\eta}e^{\left[\frac{1}{2}\mathrm{tr}\left(\eta^{-1}h\right)-\frac{1}{4}\mathrm{tr}\left(\eta^{-1}h\right)^2+\ldots\right]} \\
&= \sqrt{-\det\eta}\left[1+\frac{1}{2}\mathrm{tr}\left(\eta^{-1}h\right)-\frac{1}{4}\mathrm{tr}\left(\eta^{-1}h\right)^2+\frac{1}{2}\left(\frac{1}{2}\mathrm{tr}\left(\eta^{-1}h\right)-\frac{1}{4}\mathrm{tr}\left(\eta^{-1}h\right)^2\right)\right]+\mathscr{O}(h^3) \\
&= \sqrt{-\det\eta}\left[1+\frac{1}{2}\mathrm{tr}\left(\eta^{-1}h\right)-\frac{1}{4}\mathrm{tr}\left(\eta^{-1}h\right)^2+\frac{1}{8}\mathrm{tr}^2\left(\eta^{-1}h\right)^2\right]+\mathscr{O}(h^3) \\
&= \sqrt{-\det\eta}\left[1+\frac{1}{2}h^{\mu}{}_{\mu}-\frac{1}{4}h^{\mu\nu}h_{\mu\nu}+\frac{1}{8}\left(h^{\mu}{}_{\mu}\right)^2\right]+\mathscr{O}(h^3).
\end{aligned} \tag{2.33}$$

With the previous background established, we are now able to find out what would be the form of equation (2.4) after linearizing it up to the second order:

$$\begin{aligned}
\mathscr{L}_{LV} &= \sigma\sqrt{-g}B^{\mu}B^{\nu}R_{\mu\nu} = \sigma\left(1+\kappa\frac{1}{2}h^{\alpha}{}_{\alpha}\right)B_{\mu}B_{\nu}R^{\mu\nu} \\
&= \sigma\left(1+\frac{\kappa}{2}h^{\alpha}{}_{\alpha}\right)\left(b_{\mu}+\tilde{B}_{\nu}\right)\left(b_{\nu}+\tilde{B}_{\nu}\right)\left[R^{\mu\nu}(h)+R^{\mu\nu}(h^2)\right] \\
&= \sigma\left(1+\frac{\kappa}{2}h^{\alpha}{}_{\alpha}\right)\left[b_{\mu}b_{\nu}R^{\mu\nu}(h)+b_{\mu}\tilde{B}_{\nu}R^{\mu\nu}(h)+b_{\nu}\tilde{B}_{\mu}R^{\mu\nu}(h)+b_{\nu}b_{\nu}R^{\mu\nu}(h^2)\right]+\mathscr{O}(h^3) \\
&= \sigma\left(1+\frac{\kappa}{2}h^{\alpha}{}_{\alpha}\right)\left[b_{\mu}b_{\nu}R^{\mu\nu}(h)+2b_{\mu}\tilde{B}_{\nu}R^{\mu\nu}(h)+b_{\nu}b_{\nu}R^{\mu\nu}(h^2)\right]+\mathscr{O}(h^3) \\
&= \sigma\left[2\,b_{\mu}\tilde{B}_{\nu}R^{\mu\nu}(h)+b_{\nu}b_{\nu}R^{\mu\nu}(h^2)+\frac{\kappa}{2}h^{\alpha}{}_{\alpha}b_{\mu}b_{\nu}R^{\mu\nu}(h)\right]+\mathscr{O}(h^3),
\end{aligned} \tag{2.34}$$

$$\rightarrow \quad \boxed{\sigma\left[b_{\nu}b_{\nu}R^{\mu\nu}(h^2)+2\,b_{\mu}\tilde{B}_{\nu}R^{\mu\nu}(h)+\frac{\kappa}{2}h^{\alpha}{}_{\alpha}b_{\mu}b_{\nu}R^{\mu\nu}(h)\right]+\mathscr{O}(h^3).}$$

After passing through these calculation, let us plug (2.16) in (2.34), we obtain

$$
\begin{aligned}
\mathscr{L}_{LV} =&\xi \left[p^2 b_\mu b_\nu h^{\mu\nu} h^\alpha{}_\alpha + \frac{1}{2}(b\cdot p)^2 (h^\alpha{}_\alpha)^2 - \frac{1}{2}(b\cdot p)^2 h^{\mu\nu} h_{\mu\nu} + p^2 b_\mu b^\nu h^{\mu\alpha} h^\nu{}_\alpha\right] \\
&- \xi\left[\left(b_\mu b_\nu p_\alpha p_\beta + b_{(\mu} p_{\nu)}\, b_{(\alpha} p_{\beta)} h^{\mu\nu} h^{\alpha\beta}\right)\right] \\
&+ \frac{4\,\xi}{\kappa^2}\left(-2\,p^2 b_\mu b_\nu - 2\,b^2 p_\mu p_\nu + 4\,b\cdot p\, b_{(\mu} p_{\nu)} - \frac{p^2 p_\mu p_\nu}{4\,\lambda}\right) h^{\mu\nu} h^\alpha{}_\alpha \\
&+ \frac{4\,\xi^2}{\kappa^2}\left(2\,b_\mu b_\nu p_\alpha p_\beta - \, b_{(\mu} p_{\nu)}\, b_{(\alpha} p_{\beta)} + \frac{b^2 p_\mu p_\nu p_\alpha p_\beta}{p^2} - \frac{2\,b\cdot p\, p_\mu p_\nu \; b_{(\alpha} p_{\beta)}}{p^2} + \frac{p_\mu p_\nu p_\alpha p_\beta}{4\lambda}\right) h^{\mu\nu} h^{\alpha\beta} \\
&+ \frac{4\,\xi^2}{\kappa^2}\left(b^2 p^2 - (b\cdot p)^2 + \frac{p^4}{4\,\lambda}\right)(h^\alpha{}_\alpha)^2 \\
&+ \frac{4\,\xi^2}{\kappa^2}\left(p^2 b_\mu b_\nu - 2\,b\cdot p\; b_{(\mu} p_{\nu)} + \frac{(b\cdot p)\, p_\mu p_\nu}{p^2}\right) h^{\mu\nu} h^\nu{}_\alpha + \mathscr{O}(h^3).
\end{aligned}
\tag{2.35}
$$

It is worth mentioning that the first-order terms which appear in the gravity-bumblebee coupling constant ξ are quadratic order with b_μ.

The Lagrangian (2.35) may be written with the expanded Einstein-Hilbert Lagrangian in the position space:

$$
\mathscr{L}_{EH} = \partial h^{\mu\nu} \partial_\alpha h^\alpha{}_\nu - \partial_\mu h^{\mu\nu} \partial_\nu h + \frac{1}{2}\partial_\mu h\, \partial^\mu h - \frac{1}{2}\partial_\alpha h^{\mu\nu} \partial^\alpha h_{\mu\nu}. \tag{2.36}
$$

For the sake of convenience, we have added the convenient gauge fixing term for getting the effective Lagrangian, which is needed to obtain the modified graviton propagator as we have previously pointed out in this section:

$$
\mathscr{L}_{GF} = -\left(\partial_\mu h^{\mu\nu} - \frac{1}{2}\partial^\nu h\right)^2, \tag{2.37}
$$

where $\mathscr{L}_{GF}$ is the Lagrangian whose accounts the gauge fixing term. In this way, the kinetic term of the graviton, as mentioned above, can be seen as

$$
\mathscr{L}_K = -\frac{1}{2} h^{\mu\nu} \hat{\mathcal{O}}_{\mu\nu,\alpha\beta} h^{\alpha\beta}, \tag{2.38}
$$

where $\hat{\mathcal{O}}_{\mu\nu,\alpha\beta}$ is an operator which may be separated in two different parts

$$
\hat{\mathcal{O}}_{\mu\nu,\alpha\beta} = \hat{\mathscr{K}}_{\mu\nu,\alpha\beta} + \hat{\mathscr{V}}_{\mu\nu,\alpha\beta}, \tag{2.39}
$$

where $\hat{\mathscr{V}}_{\mu\nu,\alpha\beta}$ have terms concerned to $\mathscr{L}_{LV}$ and the $\hat{\mathscr{K}}_{\mu\nu,\alpha\beta}$ has a quadratic form given by:

$$
\hat{\mathscr{K}}_{\mu\nu,\alpha\beta} = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right)\left(-\partial^2\right). \tag{2.40}
$$

Analogously with what we have been encountering in quantum field theory textbooks [55, 56] for the scalar field, one can write the graviton propagator as

$$\langle 0|T\{h_{\mu\nu}(x)h_{\alpha\beta}(y)\}|0\rangle = i\, D_{\mu\nu,\alpha\beta}(x-y), \tag{2.41}$$

note that the $D_{\mu\nu,\alpha\beta}(x-y)$ operator satisfies the so-called Green's function equation:

$$\hat{\mathcal{O}}^{\mu\nu,}{}_{\lambda\sigma} D^{\lambda\sigma,\alpha\beta}(x-y) = \hat{\mathfrak{J}}^{\mu\nu,\alpha\beta}\delta^{(4)}(x-y), \tag{2.42}$$

where $\hat{\mathfrak{J}}^{\mu\nu,\alpha\beta}$ is given by:

$$\hat{\mathfrak{J}}^{\mu\nu,\alpha\beta} = \frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}\right). \tag{2.43}$$

After having seen the above considerations, for getting the graviton propagator is needed to invert the relation given by equation (2.39). The bumblebee models which we have been studying have Nambu-Goldstone (see Figure 5) and massive modes as well [96]. The following remarks involving causality and unitarity of these modes on the graviton propagator are important issues in which were well-investigated [80, 102]. Moreover, the full calculation of the graviton propagator on the presence of Lorentz violation is considered in reference [98]. Regarding the fact that the magnitude of b_μ is considered small as well as the coupling constant ξ, we have used for convenience the usual graviton propagator considered previously in equation (2.37) and we have calculated the Lorentz-violating terms in equation (2.39) as a perturbative method presented in [103, 104]. For accomplishing the calculations, we have considered the matricial identity

$$\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A}B\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A}B\frac{1}{A} + \frac{1}{A}B\frac{1}{A}B\frac{1}{A+B} = ... \tag{2.44}$$

The operator $\hat{\mathscr{K}}$ may be inverted and then, the graviton propagator may be written regarding momentum space

$$D_0^{\mu\nu,\alpha\beta}(q) = \frac{i}{2}\frac{\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta}}{q^2 + i\eta}. \tag{2.45}$$

Now, we can show the explicit form for $D^{\mu\nu,\alpha\beta}$:

$$D^{\mu\nu,\alpha\beta} = D_0^{\mu\nu,\alpha\beta} + D_{LV}^{\mu\nu,\alpha\beta}. \tag{2.46}$$

Let us take a look at the $D^{\mu\nu,\alpha\beta}_{LV}$, up to the second order:

$$
\begin{aligned}
(D^{\mu\nu,\alpha\beta}_{LV})_{\xi} =& i\xi b^2\left(\frac{g^{\alpha\beta}g^{\mu\nu}}{q^2}+\frac{q^{\alpha}q^{\beta}g^{\mu\nu}}{q^4}\right)+i\xi\frac{(b\cdot q)^2\left(g^{\mu\nu}g^{\alpha\beta}-g^{\beta\mu}g^{\alpha\nu}-g^{\beta\nu}g^{\alpha\mu}\right)}{2q^4}\\
&+i\xi\frac{(b\cdot q)\left(b^{\beta}b^{\alpha}g^{\mu\nu}+b^{\alpha}b^{\beta}g^{\mu\nu}+b^{\nu}b^{\mu}g^{\alpha\beta}+b^{\mu}b^{\nu}g^{\alpha\beta}\right)}{2q^4}\\
&+i\xi\frac{\left(b^{\alpha}b^{\mu}g^{\beta\nu}+b^{\beta}b^{\mu}g^{\alpha\nu}+b^{\alpha}b^{\nu}g^{\beta\mu}+b^{\beta}b^{\nu}g^{\alpha\mu}-2b^{\alpha}b^{\beta}g^{\mu\nu}-4b^{\mu}b^{\nu}g^{\alpha\beta}\right)}{2q^2}\\
&-\frac{4b^{\mu}b^{\nu}q^{\alpha}q^{\beta}+b^{\beta}b^{\nu}q^{\alpha}q^{\mu}+b^{\alpha}b^{\nu}q^{\beta}q^{\mu}+b^{\beta}b^{\mu}q^{\alpha}q^{\nu}+b^{\alpha}b^{\mu}q^{\beta}q^{\nu}}{2q^4},
\end{aligned}
\tag{2.47}
$$

and,

$$
\begin{aligned}
(D^{\mu\nu,\alpha\beta}_{LV})_{\xi^2} =& \frac{i\xi^2}{\kappa^2}b^2\left(\frac{12q^{\mu}q^{\nu}-12q^{\alpha}q^{\beta}g^{\mu\nu}}{q^4}+\frac{8q^{\alpha}q^{\beta}q^{\mu}q^{\nu}}{q^6}\right)\\
&+\frac{i\xi^2 g^{\alpha\beta}g^{\mu\nu}}{2\lambda\kappa^2}-\frac{i\xi^2}{\kappa^2}\frac{q^{\alpha}q^{\beta}g^{\mu\nu}}{q^{2\lambda}}+\frac{i\xi^2}{\kappa^2}\frac{3q^{\mu}q^{\nu}g^{\alpha\beta}}{q^2\lambda}\\
&+\frac{i\xi^2}{\kappa^2}\frac{2(b\cdot q)^2\left(q^{\alpha}q^{\mu}g^{\beta\nu}+q^{\beta}q^{\mu}g^{\alpha\nu}+q^{\alpha}q^{\nu}g^{\beta\mu}+q^{\beta}q^{\nu}g^{\alpha\mu}+2q^{\mu}q^{\nu}g^{\alpha\beta}-2q^{\alpha}q^{\beta}g^{\mu\nu}\right)}{q^6}\\
&+\frac{i\xi^2 b\cdot q}{\kappa^2}\frac{10\left(b^{\beta}q^{\alpha}g^{\mu\nu}+b^{\alpha}q^{\beta}g^{\mu\nu}-b^{\nu}q^{\mu}g^{\alpha\beta}-b^{\mu}q^{\nu}g^{\alpha\beta}\right)}{q^4}\\
&+\frac{i\xi^2 b\cdot q}{\kappa^2}\frac{8\left(b^{\beta}q^{\alpha}q^{\mu}q^{\nu}+b^{\alpha}q^{\beta}q^{\mu}q^{\nu}\right)}{q^6}\\
&-\frac{i\xi^2 b\cdot q}{\kappa^2}\frac{4\left(b^{\mu}q^{\alpha}g^{\beta\nu}-b^{\mu}q^{\beta}g^{\alpha\nu}-b^{\nu}q^{\alpha}g^{\beta\mu}-b^{\nu}q^{\beta}g^{\alpha\mu}\right)}{q^4}\\
&+\frac{i\xi^2}{\kappa^2}\frac{2\left(b^{\alpha}b^{\mu}g^{\beta\nu}+b^{\beta}b^{\mu}g^{\alpha\nu}+b^{\alpha}b^{\nu}g^{\beta\mu}+b^{\beta}b^{\nu}g^{\alpha\mu}-2b^{\alpha}b^{\beta}g^{\mu\nu}+2b^{\mu}b^{\nu}g^{\alpha\beta}\right)}{q^2}\\
&+\frac{i\xi^2}{\kappa^2}\frac{2\left(8b^{\mu}b^{\nu}q^{\alpha}q^{\beta}-b^{\beta}b^{\nu}q^{\alpha}q^{\mu}-b^{\alpha}b^{\nu}q^{\beta}q^{\mu}-b^{\beta}b^{\mu}q^{\alpha}q^{\nu}-b^{\alpha}b^{\mu}q^{\beta}q^{\nu}+\frac{2q^{\alpha}q^{\beta}q^{\mu}q^{\nu}}{\lambda}\right)}{q^4}.
\end{aligned}
\tag{2.48}
$$

Again it is worth mentioning that $(D^{\mu\nu,\alpha\beta}_{LV})_{\xi}$ and $(D^{\mu\nu,\alpha\beta}_{LV})_{\xi^2}$ are contributions due to $D^{\mu\nu,\alpha\beta}_{LV}$ which are proportional to ξ and ξ^2. Note that the product involving b^2, $(b\cdot q)^2$ and $(b\cdot q)b^{\mu}$ are first-order terms in the Lorentz violating coefficients u and $s^{\mu\nu}$ (see equation (2.3)) [73]. Therefore, one realizes that the $(D^{\mu\nu,\alpha\beta}_{LV})_{\xi}$ enhance only first-order terms regarding u and $s^{\mu\nu}$ and, moreover, they don't depend on the form of the bumblebee potential V [73]. Considering the second order term ξ, there exist expressions that are not associated with vector b_{μ} and depending only on the coupling potential terms.

2.4 *Modification of Newton's law*

In this current section we have shown the effects of Lorentz violation when one considers the modified tree-level propagator [73]. Among the variety of examples, perhaps the simplest one is the gravitational interaction of two distinguishable heavy particles which is governed by the nonrelativistic limit in the Newtonian potential. The aim is determining the scattering amplitude between two massive bosons particles mediated by the one-graviton exchange [73]. After calculating the scattering matrix amplitude, is taken the nonrelativistic limit and compared to the Born approximation for getting the modified potential taking into account the nonzero vacuum expectation value, b_μ. Considering the action for a real scalar field in curved spacetime

$$S_M = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right]. \tag{2.49}$$

Let us consider the following linearized relations which were expressed previously:

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa\, h^{\mu\nu}, \tag{2.50}$$

and,

$$\sqrt{-g} = \frac{\kappa}{2}h^\mu{}_\mu = 1 + \frac{\kappa}{2}\eta_{\mu\nu}h^{\mu\nu}, \tag{2.51}$$

it follows that

$$\begin{aligned}\mathscr{L}_M &= \left(1+\frac{\kappa}{2}\eta_{\mu\nu}h^{\mu\nu}\right)\left[\frac{1}{2}\left(\eta^{\mu\nu}-\kappa h^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right] \\ &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\kappa}{2}h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{\kappa}{4}h^{\mu\nu}\left(\eta_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi - \frac{1}{4}\eta_{\mu\nu}m^2\phi^2\right),\end{aligned} \tag{2.52}$$

$$\rightarrow \boxed{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\kappa}{2}h^{\mu\nu}\left[\partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\left(\partial_\alpha\phi\partial^\alpha\phi - m^2\phi^2\right)\right].}$$

If one considers the scattering process governed by two scalar particles labeled after their mass m_1 and m_2, the Feynman diagram which contributes to this process is

$$i\,\mathcal{M} = (-i\kappa)^2\, V^{\mu\nu}\left(p_1, -k_1, m_1\right) D_{\mu\nu,\alpha\beta}(q) V^{\alpha\beta}\left(p_2, -k_2, m_2\right), \tag{2.53}$$

where $q = p_2 - k_2 = -(p_1 - k_1)$ is considered the momentum transfer and the vertex $V^{\mu\nu}(p,k,m)$ is

$$V^{\mu\nu}(p,k,m) = -\frac{1}{2}\left[p^\mu k^\nu + p^\nu k^\mu - \eta^{\mu\nu}\left(p\cdot k + m^2\right)\right]. \tag{2.54}$$

Let us do a substitution of expressions (2.41) and (2.54) inside the scattering amplitude (2.53).

We stumbled upon the sum of the two following terms:

$$i\,\mathcal{M} = i\,\mathcal{M}_0 + i\,\mathcal{M}_{LV}, \tag{2.55}$$

note that the first term is the usual amplitude which appears in reference [105]. It follows that

$$\begin{aligned} i\,\mathcal{M}_0 = &-\frac{i\kappa^2}{8q^2}\left[4\{k_1\cdot p_1\left(m_2^2 - k^2\cdot p^2\right) + k_1\cdot p_2 k_2\cdot p_1 + k_1\cdot k_2 p_1\cdot p_2\} - 2m_1^2\right] \\ &-\frac{i\kappa^2}{8q^2}\left[-2m_1^2\{4\left(m_2^2 - k_2\cdot p_2\right) + 2k_2\cdot p_2\}\right], \end{aligned} \tag{2.56}$$

which it is changed due to $i\,\mathcal{M}_{LV}$[9]. After some manipulations, one is able to see that the nonrelativistic limit is

$$i\,\mathcal{M}_{NR} = \frac{i\kappa^2 m_1^2 m_2^2}{2\vec{q}^2} - \frac{i\xi\vec{b}^2\kappa^2 m_1^2 m_2^2}{\vec{q}^2} + \frac{i\xi(\vec{b}\cdot\vec{q})^2\kappa^2 m_1^2 m_2^2}{2\vec{q}^4} + \frac{8i\xi^2 b_0^2 m_1^2 m_2^2}{\vec{q}^2} - \frac{i\xi^2 m_1^2 m_2^2}{2\lambda}. \tag{2.57}$$

Note that the first term gives us the tree-level result. Taking the Fourier transform, one yields the gravitational Newtonian potential [73]. Nevertheless, there are others matrix elements terms in which are concerned due to the spontaneous Lorentz violation. One is able to notice that the second and fourth terms may be absorbed by the coupling constant. The third and last terms must contribute to the matrix element and is discussed in the following text. For the sake of doing the connection with the Newtonian potential, is recommended follows the idea established in reference [106], and define the potential Fourier transformed considering the context of the nonrelativistic case

$$\begin{aligned} \langle f|iT|i\rangle &\equiv (2\pi)^4\,\delta^{(4)}(p-k)i\mathcal{M}(p_1,p_2\longrightarrow k_1,k_2) \\ &\approx -(2\pi)\delta(E_p - E_k)i\tilde{V}(\vec{q}), \end{aligned} \tag{2.58}$$

and the potential may be written in the canonical position space:

$$V(\vec{x}) = \frac{1}{2m_1}\frac{1}{2m_2}\int\frac{\mathrm{d}^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{x}}\tilde{V}(\vec{q}). \tag{2.59}$$

For solving the above equation, it considered two masses points m_1 and m_2 which lie on the coordinate vectors $\vec{x}_1$ and $\vec{x}_2$, where $\vec{x} = \vec{x}_1 - \vec{x}_2$[10] regarding $\vec{x}, \vec{q}$ and $\vec{b}$, one finds out the angular relations:

$$\begin{aligned} &\cos\theta = \vec{q}\cdot\vec{x}/qr, \quad \cos\theta_b = \vec{b}\cdot\vec{x}/br, \quad \cos\Psi = \vec{b}\cdot\vec{q}/bq, \\ &\cos\Psi = \sin\theta\sin\theta_b\cos(\phi-\phi_b) + \cos\theta\cos\theta\cos\theta_b \\ &q = \mid\vec{q}\mid, \quad r = \mid\vec{x}\mid, \quad b = \mid\vec{b}\mid. \end{aligned}$$

[9] It consists of a huge expression considering the contractions of the b_μ with the four-momenta (of the outgoing and incoming scalar field) and with the virtual momentum ascribed to the graviton as well.

[10] Off course, these are considered in an inertial cartesian coordinates background.

It follows that the background vector $\vec{b}$ is fixed in a certain direction in space of coordinates. θ_b and ϕ_b are also fixed angles which point out the dependence of the $V(\vec{x})$ [73]. Knowing all of these preliminaries, one is ready to evaluate the angular integration

$$\int_0^\infty \mathrm{d}q \int_0^\pi \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi e^{iqr\cos\theta} \cos^2\Psi = \frac{\pi^2 \sin^2\theta_b}{r}. \tag{2.60}$$

We must calculate the momentum integral on the q-variable,

$$\tilde{V}(\vec{x}) = -\frac{G_N m_1 m_2}{r}\left[1 - \frac{3}{2}\xi\vec{b}^2 - \frac{1}{2}\xi(\vec{b}\cdot\hat{x})^2\right] - G_N m_1 m_2 \left[\frac{\xi^2 b_0^2}{2\pi G_{Ns}}\frac{1}{r} - \frac{\xi^2}{8\lambda G_N}\delta^3(\vec{x})\right], \tag{2.61}$$

with $\hat{x} = \vec{x}/\mid\vec{x}\mid$. By the way, this above equation is in agreement with Refs. [94,99]. It is worth mentioning that the first-order corrections in ξ, the Newtonian potential keeps showing the usual behavior, being inversely proportional to the distance between two point masses. Moreover, this result has two remarkable features. Depending on the sign of the coupling constant ξ, we can have both attractive and repulsive behavior (i.e. $\xi > 0$ is repulsive and $\xi < 0$ is attractive) [73]. After some manipulations of equation (2.61) we have

$$V(\vec{x}) = -\frac{G_N m_1 m_2}{r}\left[1 + \frac{3}{2}\bar{s}^{00} + \frac{1}{2}\bar{s}^{ij}\hat{x}^i\hat{x}^j\right] + (...). \tag{2.62}$$

In accordance with reference [107], there exist a lot of discussions testing the range of gravity, which could be established experimentally the Lorentz-violating coefficients. In the following reference, is tested the accuracy of measuring how warped is the angle when a light beam deflects due to a massive body [108]. Further perspectives regarding an experimental approach are pointed out in Ref. [100] as well.

For finishing, the last term in (2.61) gives a nontrivial parcel which involves a Dirac delta function. The correction takes after a gravitational Darwin term and is ascribed to the remaining derivative ∂^4 which is presented in Lagrangian (2.35) at $\mathscr{O}(\xi^2)$ [73]. A similar correction is also realized when we had added the higher-order terms in the curvature for the pure-gravity Lagrangian (2.60).

Chapter 3

The Kalb-Ramond field

This section provides a study of spontaneous Lorentz symmetry breaking due to an anti-symmetric 2-tensor field in Minkowski spacetime. Considering a smooth quadratic potential, the spectrum of the theory exhibits both massless and massive modes. Furthermore, we show that the massive mode is non-propagating at leading order. Besides, the massless modes in the theory can be identified with the usual Kalb-Ramond field, carrying only one on-shell degree of freedom. We provide a new complete set of spin-projection operators, which sufficed to evaluate the propagator of the Kalb-Ramond field in the context of Lorentz violation.

3.1 Lorentz violation triggered by an antisymmetric 2-tensor

Some field theories in Minkowski spacetime may be built up from p-forms, including for instance the electrodynamics (1-form), and the antisymmetric p-tensors. There exist models which have a gauge-invariant kinetic term considering the appearance of an antisymmetric 2-tensor which was first considered within string theory, the Kalb-Ramond field [71]. Another reference, which involves a straightforward approach showing remarkable properties regarding dualities to different p-form theories, was considered as well [109].

Furthermore, models coupled with gravity in the context of the Lorentz violation involving an antisymmetric 2-tensor is considered in Ref. [97]. For adding the LV up, in a general Lagrange density, the terms might be constructed either explicitly or spontaneously [91]. Nevertheless, when one breaks explicitly the Lorentz symmetry, there exists an incompatibility with the so-called Bianchi identities which means a problem when one considers gravity [91, 92]. For the sake of overtaking this situation, there exists an alternative way, which is breaking the symmetry spontaneously. Basically, it is a consequence of the introduction of a potential term which triggers a nonzero vacuum expectation value for fields [97]. Moreover, considering the current work, the tensor field [71] namely, an antisymmetric 2-tensor, acquires a nonzero vacuum expectation value $b_{\mu\nu}$ triggering the Lorentz spontaneously symmetry breaking.

Whenever we have LV, it immediately accompanies the raising up of the so-called massless Nambu-Goldstone (NG) modes[1] [110]. If we take into account a smooth quadratic potential, the so-called massive modes may appear as well [111]. The importance of the NG and massive modes is remarkable when one tries to analyze the physical consequences of a theory with spontaneous Lorentz violation [102]. It is worth pointing out that their properties may be regarded to any field theory [110] (Appendix B).

Besides, when nonminimal curvature couplings are present, the Kalb-Ramond field is well described by all coefficients in the gravitational SME sector which are denoted as u, $s^{\mu\nu}$ and $t^{\mu\nu\alpha\beta}$ [92]. Nonzero $t^{\mu\nu\alpha\beta}$ coefficients coupled to gravitational field theory with dynamics governed by an antisymmetric 2- tensor was first considered in reference [97]. Moreover, it was shown that the post-Newtonian metric was affected by only u and $s^{\mu\nu}$, at least for the first order approximation [97].

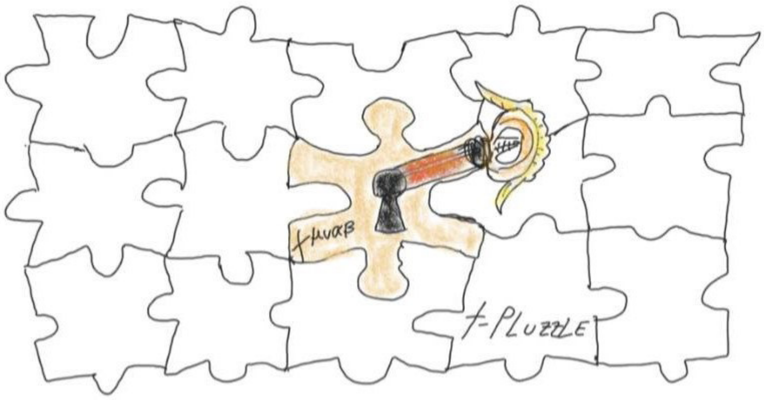

Figure 8: The *t-puzzle* was not undertaken up to date. Then, the phenomenological consequences of the contribution ascribed to the field $t^{\mu\nu\alpha\beta}$ remains an open question.

For the sake of stepping toward a fundamental explanation to $t^{\mu\nu\alpha\beta}$ (see Figure 8) and overcoming this deadlock which is the so-called *t-puzzle*[2] [112], we have provided some calculations involving a new closed algebra with six spin-projection operators to get the propagator of the Kalb-Ramond field theory in the context of Lorentz violation.

[1]They are associated with the field fluctuations ascribed to LV [93].

[2]So far there is no fundamental explanation well established in the literature for the $t^{\mu\nu\alpha\beta}$.

3.2 The model

For starting off, let us define the Lagrangian density which provides the dynamics for an antisymmetric 2-tensor ($B_{\mu\nu}$) in Minkowski spacetime:

$$\mathscr{L} = \frac{1}{6}H_{\mu\nu\alpha}H^{\mu\nu\alpha} - V + B_{\mu\nu}J^{\mu\nu}, \tag{3.1}$$

where $H_{\mu\nu\alpha}$ is the field strength defined as follows:

$$H_{\mu\nu\alpha} = \partial_\mu B_{\nu\alpha} + \partial_\alpha B_{\mu\nu} + \partial_\nu B_{\alpha\mu}, \tag{3.2}$$

V is the potential that triggers the spontaneous Lorentz symmetry breaking, and $J^{\mu\nu}$ is an antisymmetric current which comes from the coupling to the matter [97]. Besides, the field strength $H_{\mu\nu\alpha}$ satisfies the identity

$$\partial_\kappa H_{\lambda\mu\nu} - \partial_\lambda H_{\mu\nu\kappa} + \partial_\mu H_{\nu\kappa\lambda} - \partial_\nu H_{\kappa\lambda\mu} = 0, \tag{3.3}$$

showing that a 3-form is closed and there exists a gauge transformation of $B_{\mu\nu}$ which leaves equation (3.3) unchanged,

$$B_{\mu\nu}(x) \to B'_{\mu\nu}(x) = B_{\mu\nu}(x) + \partial_\mu\Lambda_\nu(x) - \partial_\nu\Lambda_\mu(x), \tag{3.4}$$

where Λ_μ is an arbitrary vector field which exhibits a gauge invariance as well:

$$\Lambda_\mu(x) \to \Lambda'_\mu(x) = \Lambda_\mu(x) + \partial_\mu\Sigma(x), \tag{3.5}$$

where Σ is an arbitrary scalar field.

Generically, we could ascribe to the potential V the dependence of $B_{\mu\nu}$, derivatives of $B_{\mu\nu}$, the metric $\eta_{\mu\nu}$, and the Levi-Civita tensor $\epsilon_{\mu\nu\alpha\beta}$. However, for the sake of simplicity, we provide a specific analysis for the choice of V in equation (3.1). From now on, we consider no matter coupling terms, then $J^{\mu\nu}$ simply vanishes.

In the context of Lorentz spontaneous symmetry breaking, the simplest case is when one considers the potential taking the form $V = V(X)$, where we may define $X \equiv B_{\mu\nu}B^{\mu\nu} - b_{\mu\nu}b^{\mu\nu}$ such that the potential acquires a nonzero vacuum expectation value for the field $B_{\mu\nu}$,

$$\langle B_{\mu\nu}\rangle \equiv b_{\mu\nu}. \tag{3.6}$$

Specifically, we choose a smooth quadratic potential which equation (3.1) turns out to be:

$$\mathscr{L}_{B,V} = \frac{1}{6}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{1}{2}\lambda(B_{\mu\nu}B^{\mu\nu} - b^2)^2, \tag{3.7}$$

where λ is a dimensionless positive constant and $b^2 \equiv b^{\mu\nu} b_{\mu\nu}$.

Since we are interested in the behavior of $B_{\mu\nu}$ around the vacuum expectation value $b_{\mu\nu}$, let us consider the decomposition:

$$B_{\mu\nu} = b_{\mu\nu} + \tilde{B}_{\mu\nu}, \tag{3.8}$$

where $\tilde{B}_{\mu\nu}$ is the vacuum fluctuation, and $b_{\mu\nu}$ satisfies the requirement $\partial_\mu b^{\mu\nu} = 0$. This assumption guarantees the translational invariance of the vacuum state and consequently the conservation of energy-momentum for the fluctuation $\tilde{B}_{\mu\nu}$.

Using equation (3.8) we are able to rewrite $\mathscr{L}_{B,V}$ in the linearized form

$$\begin{aligned} \mathscr{L}_{B,V} &= \frac{1}{6}\tilde{H}_{\mu\nu\alpha}\tilde{H}^{\mu\nu\alpha} - 2\lambda b_{\mu\nu} b_{\alpha\beta}\tilde{B}^{\mu\nu}\tilde{B}^{\alpha\beta} \\ &- 2\lambda b_{\alpha\beta}\tilde{B}^{\alpha\beta}\tilde{B}_{\mu\nu}\tilde{B}^{\mu\nu} - \frac{1}{2}\lambda(\tilde{B}_{\mu\nu}\tilde{B}^{\mu\nu})^2, \end{aligned} \tag{3.9}$$

where $\tilde{H}_{\mu\nu\alpha}$ is the field strength for the fluctuation $\tilde{B}_{\mu\nu}$. Let us note that in the first line of the above expression, there exists the presence of a mass term for $\tilde{B}_{\mu\nu}$ which is described by the mass matrix $m_{\mu\nu,\alpha\beta} = 4\lambda b_{\mu\nu} b_{\alpha\beta}$.

The goal is studying the propagation of the fluctuation $B_{\mu\nu}$ and for accomplishing this, we are focusing on the bilinear terms which yield the equation of motion

$$\partial_\mu \tilde{H}^{\mu\nu\alpha} + 4\lambda \tilde{B}_{\rho\sigma} b^{\rho\sigma} b^{\nu\alpha} = 0. \tag{3.10}$$

The solutions of Eq. (3.10) contain both massless NG and massive modes. They appear mixed by the mass matrix $m_{\mu\nu,\alpha\beta}$. To separate these modes and show the physical content of the theory, we introduce transverse and longitudinal projectors ascribed to the preferred direction induced by $b_{\mu\nu}$:

$$\Pi^{\parallel}_{\mu\nu,\alpha\beta} \equiv \frac{b_{\mu\nu} b_{\alpha\beta}}{b^2} \quad \text{and} \quad \Pi^{\perp}_{\mu\nu,\alpha\beta} \equiv \hat{\mathfrak{I}}_{\mu\nu,\alpha\beta} - \Pi^{\parallel}_{\mu\nu,\alpha\beta}, \tag{3.11}$$

where $\hat{\mathfrak{I}}_{\mu\nu,\alpha\beta}$ is the identity operator for an antisymmetric 2-tensor, and it is defined as follows

$$\hat{\mathfrak{I}}_{\mu\nu,\alpha\beta} = \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha}). \tag{3.12}$$

Therefore, the excitation $\tilde{B}_{\mu\nu}$ can be written in terms of the transverse and longitudinal components:

$$\tilde{B}_{\mu\nu} = A_{\mu\nu} + \beta\hat{b}_{\mu\nu}, \tag{3.13}$$

$$A_{\mu\nu} \equiv \Pi^{\perp}_{\mu\nu,\alpha\beta}\tilde{B}^{\alpha\beta} \quad \text{(transverse mode)}, \tag{3.14}$$

$$\beta\hat{b}_{\mu\nu} \equiv \Pi^{\parallel}_{\mu\nu,\alpha\beta}\tilde{B}^{\alpha\beta} \quad \text{(longitudinal mode)}, \tag{3.15}$$

with $A_{\mu\nu}b^{\mu\nu} = 0$, $\beta = \hat{b}_{\mu\nu}\tilde{B}^{\mu\nu}$ and $\hat{b}_{\mu\nu} = b_{\mu\nu}/\sqrt{b^2}$.

The equation of motion (3.10) may be rewritten as

$$\partial_\mu \tilde{G}^{\mu\nu\lambda} + \Box\beta\hat{b}^{\nu\lambda} + \partial_\mu\partial^\lambda\beta\hat{b}^{\mu\nu} + \partial_\mu\partial^\nu\beta\hat{b}^{\lambda\mu} + 4\lambda b^2\beta\hat{b}^{\nu\lambda} = 0, \tag{3.16}$$

where $\tilde{G}_{\mu\nu\lambda} \equiv \partial_\mu A_{\nu\lambda} + \partial_\lambda A_{\mu\nu} + \partial_v A_{\lambda\mu}$. Note that equation 3.16 satisfies the constraint:

$$b_{\lambda\nu}\partial^\nu\beta = 0. \tag{3.17}$$

Since the fields $A_{\mu\nu}$ and β are independent, we are able to extract their respective equations of motion assuming the constraint (3.17) and considering the projections (3.11). Hence,

$$\Box A_{\mu\nu} + \partial_\mu\partial^\alpha A_{\nu\alpha} + \partial_\nu\partial^\alpha A_{\alpha\mu} - \frac{2}{b^2}b_{\mu\nu}b_{\alpha\beta}\partial^\beta\partial_\lambda A^{\lambda\alpha} = 0, \tag{3.18}$$

$$\Box\beta + 4\lambda b^2\beta + 2\hat{b}_{\mu\nu}\partial^\nu\partial_\alpha A^{\alpha\mu} = 0. \tag{3.19}$$

These two equations show that the modes remain coupled and the dispersion relation, including the mass value, may not be correctly identified. On the other hand, the transverse components of $A_{\mu\nu}$, namely, those satisfying the condition $\partial_\mu A^{\mu\nu} = 0$, remain unaffected even when the massive mode β is nonzero. We may decouple equations (3.18) and (3.19) by noting that the constraint equation (3.17) yields to the massive mode β an additional requirement

$$b_{\mu\nu}p^\nu = 0, \tag{3.20}$$

which means that the vector associated with the massive mode is orthogonal to the vacuum expectation value $b_{\mu\nu}$. This condition entails the following dispersion relation

$$p^2 - 4\lambda b^2 = 0, \tag{3.21}$$

with the associated mass value given by $m_\beta^2 \equiv 4\lambda b^2$.

At first glance, this relation might suggest that the massive mode has a physical mass when $4\lambda b^2$ is positive. However, it is possible to show that there is a special observer frame in which $b_{\mu\nu}$ assumes a simplified block-diagonal form, which can be placed as

$$b_{\mu\nu} = \begin{pmatrix} 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & -d & 0 \end{pmatrix}, \tag{3.22}$$

such that $b^2 = -2(a^2 - d^2)$. Notice that the six parameters initially required to define $b_{\mu\nu}$ in an arbitrary referential is reduced to only two nonzero real numbers in this particular frame [97, 113]. We see this specific form of $b_{\mu\nu}$ combined with equation (3.17) entails that $\partial_\mu\beta = 0$, and then β is a constant at linear order. To satisfy the asymptotic boundary conditions, this amplitude must be zero. Therefore, there is no physical propagating massive mode at leading order. The remaining transverse mode $A_{\mu\nu}$ propagates as the usual Kalb-Ramond field, containing only one degree of freedom being like a real scalar field [114] .

The above results are in agreement with those obtained in Ref. [102], in which the authors exhibit the equivalence between the theory described in (3.7) and its dual defined as follows:

$$\mathscr{L}_{A,B,V} = \frac{1}{2}B_{\mu\nu}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} + \frac{1}{2}A_\mu A^\mu - V, \tag{3.23}$$

where $F_{\mu\nu}$ is the field strength for a vector field A_μ. Besides, a similar analysis was accomplished in the context of the bumblebee electrodynamics in Ref. [102].

3.3 The Kalb-Ramond propagator with Lorentz violation

After using equation (3.8), the Lagrange density (3.7) takes the form,

$$\mathscr{L}_{kin} = \frac{1}{6}\tilde{H}_{\mu\nu\alpha}\tilde{H}^{\mu\nu\alpha} - 2\lambda b_{\mu\nu}b_{\alpha\beta}\tilde{B}^{\mu\nu}\tilde{B}^{\alpha\beta}, \tag{3.24}$$

where we have separated only terms up to second order in $\tilde{B}_{\mu\nu}$. Naturally, the gauge symmetry no longer exists in (3.24) due to the term involving $b_{\mu\nu}$.

For acquiring the Feynman propagator we put the kinetic Lagrangian into the bilinear form

$$\mathscr{L}_{kin} = \frac{1}{2}\tilde{B}^{\mu\nu}\hat{\mathcal{O}}_{\mu\nu,\alpha\beta}\tilde{B}^{\alpha\beta}, \tag{3.25}$$

where the operator $\hat{\mathcal{O}}_{\mu\nu,\alpha\beta}$ is antisymmetric in the indices $(\mu\nu)$, $(\alpha\beta)$, and symmetric under the interchange of the pairs $(\mu\nu)$ and $(\alpha\beta)$. Therefore, the operator takes the form

$$\hat{\mathcal{O}}_{\mu\nu,\alpha\beta} = -\frac{\Box}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha}) + \frac{1}{2}(\partial_\mu\partial_\beta\eta_{\nu\alpha} - \partial_\nu\partial_\beta\eta_{\mu\alpha} - \partial_\mu\partial_\alpha\eta_{\nu\beta} + \partial_\nu\partial_\alpha\eta_{\mu\beta}) - 4\lambda b_{\mu\nu}b_{\alpha\beta}. \tag{3.26}$$

By definition, the Feynman propagator is

$$\langle 0|T\left[\tilde{B}_{\mu\nu}(x)\tilde{B}_{\alpha\beta}(y)\right]|0\rangle = i\,\hat{\mathcal{O}}^{-1}_{\mu\nu,\alpha\beta}\,\delta^{(4)}(x-y). \tag{3.27}$$

To invert operator $\hat{\mathcal{O}}_{\mu\nu,\alpha\beta}$, a closed algebra, which involves spin-projection operators, is invoked. The set of spin-projection operators for the Lorentz-invariant antisymmetric

	$\Pi^{(1)}$	$\Pi^{(2)}$	$\Pi^{(3)}$
$\Pi^{(1)}$	$\Pi^{(1)}$	0	$\Pi^{(3)} - P^{(2)} - 2P^{(4)}$
$\Pi^{(2)}$	0	$\Pi^{(2)}$	$\Pi^{(2)} + 2\Pi^{(4)}$
$\Pi^{(3)}$	$\Pi^{(3)} - \Pi^{(2)} - 2\Pi^{(5)}$	$\Pi^{(2)} + 2\Pi^{(5)}$	$\Pi^{(3)}$
$\Pi^{(4)}$	$\Pi^{(4)} + 2\Pi^{(6)}$	$-2\Pi^{(6)}$	0
$\Pi^{(5)}$	0	$\Pi^{(5)}$	$\Pi^{(5)} + \frac{(b_{\mu\nu}p^{\nu})^2}{p^2b^2}(\Pi^{(1)} + \Pi^{(2)} - \Pi^{(3)})$
$\Pi^{(6)}$	0	$\Pi^{(6)}$	$\Pi^{(6)} + \frac{(b_{\mu\nu}p^{\nu})^2}{p^2b^2}\Pi^{(4)}$

Table 1: The closed algebra for the spin-projection operators.

2-tensor are defined in Ref. [115]

$$\begin{aligned} \Pi^{(1)}_{\mu\nu,\alpha\beta} &= \frac{1}{2}(\theta_{\mu\alpha}\theta_{\nu\beta} - \theta_{\mu\beta}\theta_{\nu\alpha}), \\ \Pi^{(2)}_{\mu\nu,\alpha\beta} &= \frac{1}{4}(\theta_{\mu\alpha}\omega_{\nu\beta} - \theta_{\nu\alpha}\omega_{\mu\beta} - \theta_{\mu\beta}\omega_{\nu\alpha} + \theta_{\nu\beta}\omega_{\mu\alpha}), \end{aligned} \tag{3.28}$$

where

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu}, \quad \omega_{\mu\nu} = \frac{\partial_\mu\partial_\nu}{\Box}, \tag{3.29}$$

are transverse and longitudinal operators respectively.

Note that the spin-projection operators satisfy the orthogonality relation as follows:

$$\Pi^{(i)}_{\mu\nu,\rho\sigma}\Pi^{(j)}_{\rho\sigma,\alpha\beta} = \delta^{ij}\Pi^{(i)}_{\mu\nu,\alpha\beta}, \tag{3.30}$$

with $i, j = 1, 2$ and the tensorial completeness relation:

$$\left[\Pi^{(1)} + \Pi^{(2)}\right]_{\mu\nu,\alpha\beta} = \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha}) = \hat{\mathcal{I}}_{\mu\nu,\alpha\beta}. \tag{3.31}$$

It is needed to introduce some additional operators besides the usual ones defined in (3.28) to account for the mass matrix generated by the Lorentz violation as follows:

$$\begin{aligned} \Pi^{(3)}_{\mu\nu,\alpha\beta} &= \Pi^{\perp}_{\mu\nu,\alpha\beta}, \\ \Pi^{(4)}_{\mu\nu,\alpha\beta} &= \frac{1}{2}\left(\omega_{\mu\lambda}\,\Pi^{\parallel}_{\nu\lambda,\alpha\beta} - \omega_{\nu\lambda}\,\Pi^{\parallel}_{\mu\lambda,\alpha\beta}\right), \\ \Pi^{(5)}_{\mu\nu,\alpha\beta} &= \frac{1}{2}\left(\omega_{\alpha\lambda}\,\Pi^{\parallel}_{\mu\nu,\beta\lambda} - \omega_{\beta\lambda}\,\Pi^{\parallel}_{\mu\nu,\alpha\lambda}\right), \\ \Pi^{(6)}_{\mu\nu,\alpha\beta} &= \frac{1}{4}\left(\omega_{\mu\alpha}\,\Pi^{\parallel}_{\nu\rho,\beta\sigma}\,\omega^{\rho\sigma} - \omega_{\nu\alpha}\,\Pi^{\parallel}_{\mu\rho,\beta\sigma}\,\omega^{\rho\sigma} - \omega_{\mu\beta}\,\Pi^{\parallel}_{\nu\rho,\alpha\sigma}\,\omega^{\rho\sigma} + \omega_{\nu\beta}\,\Pi^{\parallel}_{\mu\rho,\alpha\sigma}\,\omega^{\rho\sigma}\right), \end{aligned}$$

where $\Pi^{\perp}$ and $\Pi^{\parallel}$ were defined in (3.11).

These new operators together with the spin-projection operators (3.28) satisfy a closed algebra explicitly shown in Table 1 and Table 2.

Now we are ready to calculate the propagator. Let us write both operators $\hat{\mathcal{O}}_{\mu\nu,\alpha\beta}$

	$\Pi^{(4)}$	$\Pi^{(5)}$	$\Pi^{(6)}$
$\Pi^{(1)}$	0	$\Pi^{(5)}+2\Pi^{(6)}$	0
$\Pi^{(2)}$	$\Pi^{(4)}$	$-2\Pi^{(6)}$	$\Pi^{(6)}$
$\Pi^{(3)}$	$\Pi^{(4)}+\frac{(b_{\mu\nu}p^{\nu})^2}{p^2b^2}(\Pi^{(1)}+\Pi^{(2)}-\Pi^{(3)})$	0	$\Pi^{(6)}+\frac{(b_{\mu\nu}p^{\nu})^2}{p^2b^2}\Pi^{(5)}$
$\Pi^{(4)}$	$-\frac{(b_{\mu\nu}p^{\nu})^2}{p^2b^2}\Pi^{(4)}$	$\Pi^{(6)}$	$-\frac{(b_{\mu\nu}p^{\nu})^2}{p^2b^2}\Pi^{(6)}$
$\Pi^{(5)}$	$\frac{(b_{\mu\nu}p^{\nu})^2}{2p^2b^2}(\Pi^{(1)}+\Pi^{(2)}-\Pi^{(3)})$	$-\frac{(b_{\mu\nu}p^{\nu})^2}{p^2b^2}\Pi^{(5)}$	$-\frac{(b_{\mu\nu}p^{\nu})^2}{2p^2b^2}\Pi^{(5)}$
$\Pi^{(6)}$	$\frac{(b_{\mu\nu}p^{\nu})^2}{2p^2b^2}\Pi^{(4)}$	$-\frac{(b_{\mu\nu}p^{\nu})^2}{p^2b^2}\Pi^{(6)}$	$\frac{(b_{\mu\nu}p^{\nu})^2}{2p^2b^2}\Pi^{(6)}$

Table 2: The closed algebra for the spin-projection operators.

and $\hat{\mathcal{O}}^{-1}_{\mu\nu,\alpha\beta}$ as a linear combination of the projectors $\{\Pi^{(1)},\Pi^{(2)},\Pi^{(3)},\Pi^{(4)},\Pi^{(5)},\Pi^{(6)}\}$, such that

$$\begin{aligned}\hat{\mathcal{O}}_{\mu\nu,\alpha\beta} &= a_1\Pi^{(1)}+a_2\Pi^{(2)}+a_3\Pi^{(3)}+a_4\Pi^{(4)}+a_5\Pi^{(5)}+a_6\Pi^{(6)},\\ \hat{\mathcal{O}}^{-1}_{\mu\nu,\alpha\beta} &= c_1\Pi^{(1)}+c_2\Pi^{(2)}+c_3\Pi^{(3)}+c_4\Pi^{(4)}+c_5\Pi^{(5)}+c_6\Pi^{(6)},\end{aligned} \tag{3.32}$$

with the coefficients a_i being known scalar functions from the momentum and the VEV $b_{\mu\nu}$, and c_i are coefficients to be determined.

For our specific case, the operator $\hat{\mathcal{O}}_{\mu\nu,\alpha\beta}$ can be expanded in the form

$$\hat{\mathcal{O}}_{\mu\nu,\alpha\beta}=(-\Box-4\lambda b^2)\Pi^{(1)}_{\mu\nu,\alpha\beta}-4\lambda b^2\Pi^{(2)}_{\mu\nu,\alpha\beta}+4\lambda b^2\Pi^{(3)}_{\mu\nu,\alpha\beta}. \tag{3.33}$$

Taking into account that $\hat{\mathcal{O}}_{\mu\nu,\alpha\beta}\,\hat{\mathcal{O}}^{-1}_{\mu\nu,\alpha\beta}=\hat{\mathcal{I}}_{\mu\nu,\alpha\beta}$, and after performing the necessary algebra with the help of Table 1 and Table 2, we find the following result in the momentum space:

$$\hat{\mathcal{O}}^{-1}_{\mu\nu,\alpha\beta}=\frac{1}{p^2}\Pi^{(1)}_{\mu\nu,\alpha\beta}+\frac{b^2}{(b_{\rho\sigma}p^{\sigma})^2}(\Pi^{(4)}_{\mu\nu,\alpha\beta}+\Pi^{(5)}_{\mu\nu,\alpha\beta}). \tag{3.34}$$

It is worth mentioning that only coefficients c_1 and c_4 turned out to be different than zero, c_3 was equal to zero, c_2 and c_6 were the divergent ones. Nevertheless, this divergence does not mean "a problem" since the terms which keep with c_2 and c_6 are dependent of momenta, and therefore, they do not contribute to the S-Matrix.

Looking at above expression (3.34), we notice the presence of the pole $p^2=0$. So, a massless excitation is present, and may be ascribed to the transverse mode $A_{\mu\nu}$. Moreover, it is worth mentioning that the dependence of the parameter λ simply vanished. The pole $(b_{\rho\sigma}p^{\sigma})^2=0$ indicates the presence of a non-physical mode, naturally identified with the longitudinal mode. We can conclude that no physical mass term was generated by the spontaneous Lorentz symmetry breaking. Finally, we can observe that the longitudinal mode does not modify the interparticle potential, since the Lorentz-symmetry violating term associated with the projection operators $\Pi^{(4)}_{\mu\nu,\alpha\beta}$ and $\Pi^{(5)}_{\mu\nu,\alpha\beta}$ do not contribute to any observable associated with the S-Matrix at tree-level approximation.

Chapter 4

Conclusion

In Chapter 1, we discussed literally some physical topics which were the basis for the modern physics based on the concepts of unification.

In Chapter 2, we studied some aspects of vector fields in the context of Lorentz violation. We made a review of an action which provides the simplest gravity model involving tensors, which breaks Lorentz symmetry, coupled with the gravitational field. We focused on the gravitational sector for the minimal SME, using a particular case, the so-called bumblebee models. There exist some features in these models which are the appearance of massless and massive modes. Nevertheless, the massive mode may not propagate. We made a review of the correction to the Newtonian potential as well.

In Chapter 3, we considered spontaneous Lorentz symmetry breaking due to an anti-symmetric 2-tensor field triggered by a smooth quadratic potential in Minkowski space-time. The theory provides the appearance of both massless and massive modes. We showed that the solutions of the equation of motion obey some constraints that lead to the massive mode being non-propagating at leading order. These results are in agreement with Ref. [97]. Besides, we have evaluated the modified Kalb-Ramond propagator in the presence of Lorentz-violating terms using a closed algebra involving six spin projection operators. The analysis of the propagator poles showed that no physical mass was generated by spontaneous Lorentz symmetry breaking. Moreover, the massive mode could not modify the interparticle potential. The determination of the exact form of the Kalb-Ramond propagator allows the application of tensor calculation techniques for some interesting problems. The issue whether the massive mode propagates at higher orders is an interesting open question [97], and the calculation of the radiative corrections may elucidate this subject. Moreover, we may use $\tilde{B}_{\mu\nu}$ propagator to access corrections at higher orders in the gravitational scenario [73, 98]. Some investigations in this direction are now under development.

APPENDIX A – THE EINSTEN-HILBERT ACTION

This section was based on the following references: *Variational Principle Approach to General Relativity* and *A General Relativity Workbook* [116], [117] respectively. In this way, if one is interested in further details, please see these references mentioned above.

From the variational principle, we are able to derive all possible equations of motion in classical field theory. This approach may be applied to the development of the Einstein's equation as well. In agreement with what we had studied in classical mechanics, here we are on the verge of using the principle of least action

$$\delta \int \mathrm{d}^4x \, \mathscr{L} = 0. \tag{A.1}$$

It is quite important to take into account two remarks: it must be built up regarding the metric tensor $g_{\mu\nu}$ which is the dynamical variable when one works on General Relativity and the action must be invariant under Lorentz transformations. Moreover, the Lagrangian density $\mathscr{L}$ is given by

$$S_{EH} = \int \sqrt{-g} R. \tag{A.2}$$

This is the so-called Einstein-Hilbert action. So, for the following calculations of physical equation, we have

$$\begin{aligned}
&\delta_{EH} \left(\int \mathrm{d}^4x \, \sqrt{-g} \, R \right), \\
&= \int \mathrm{d}^4x \, \delta \left(\sqrt{-g} g^{\mu\nu} R_{\mu\nu} \right) \\
&= \int \mathrm{d}^4x \, \left[\underbrace{\delta \left(\sqrt{-g} \right) g^{\mu\nu} R_{\mu\nu}}_{1^o} + \underbrace{\sqrt{-g} \delta \left(g^{\mu\nu} \right) R_{\mu\nu}}_{2^o} + \underbrace{\sqrt{-g} g^{\mu\nu} \left(R_{\mu\nu} \right)}_{3^o} \right].
\end{aligned}$$

For sake of organizing the following calculation, we have explicitly pointed out the first, second and third terms of the integral respectively. And noting some relation when we did General Relativity:

$$R^{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = \partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} + \Gamma^\rho{}_{\rho\sigma} \Gamma^\sigma{}_{\nu\mu} - \Gamma^\rho{}_{\nu\sigma} \Gamma^\sigma{}_{\mu\rho}, \tag{A.3}$$

and,

$$\nabla_\rho \, \delta \, \Gamma^\rho{}_{\mu\nu} = \partial_\rho \, \delta \, \Gamma^\rho{}_{\mu\nu} + \Gamma^\rho{}_{\rho\sigma} \delta \, \Gamma^\sigma{}_{\nu\mu} - \Gamma^\sigma{}_{\mu\rho} \delta \, \Gamma^\rho{}_{\nu\sigma} - \Gamma^\sigma{}_{\nu\rho} \delta \, \Gamma^\rho{}_{\mu\sigma}, \tag{A.4}$$

and

$$\nabla_\nu \, \delta \, \Gamma^\rho{}_{\mu\rho} = \partial_\nu \, \delta \, \Gamma^\rho{}_{\mu\rho} + \Gamma^\rho{}_{\nu\sigma} \delta \, \Gamma^\sigma{}_{\mu\rho} - \Gamma^\sigma{}_{\nu\mu} \delta \, \Gamma^\rho{}_{\rho\sigma} - \Gamma^\sigma{}_{\nu\rho} \delta \, \Gamma^\rho{}_{\mu\sigma}. \tag{A.5}$$

It follows that the 3^o term is

$$\begin{aligned} \underbrace{\sqrt{-g}\, g^{\mu\nu} \left(R_{\mu\nu}\right)}_{3^o} = + \sqrt{-g} g^{\mu\nu} \{ & \partial_\rho \sigma \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} + \delta \left(\Gamma^\rho{}_{\rho\sigma}\right) \Gamma^\rho{}_{\mu\nu} + \Gamma^\rho{}_{\rho\sigma} \delta \left(\Gamma^\rho{}_{\mu\nu}\right) - \\ & - \delta \left(\Gamma^\rho{}_{\nu\sigma}\right) \Gamma^\sigma{}_{\mu\rho} - \Gamma^\rho{}_{\nu\sigma} \delta \left(\Gamma^\sigma{}_{\mu\rho}\right) \}, \end{aligned} \tag{A.6}$$

however, note that

$$\begin{aligned} \nabla_\rho \left(\delta \, \Gamma^\rho{}_{\mu\nu}\right) - \nabla_\nu \, \delta \left(\Gamma^\rho{}_{\mu\rho}\right) = + \ & \partial_\rho \, \sigma \left(\Gamma^\rho{}_{\mu\nu}\right) + \Gamma^\rho{}_{\rho\sigma} \delta \left(\Gamma^\sigma{}_{\mu\nu}\right) - \Gamma^\sigma{}_{\mu\rho} \delta \left(\Gamma^\rho{}_{\rho\sigma}\right) - \partial_\nu \, \sigma \left(\Gamma^\rho{}_{\mu\rho}\right) - \\ & - \Gamma^\rho{}_{\nu\sigma} \delta \left(\Gamma^\sigma{}_{\mu\rho}\right) + \Gamma^\sigma{}_{\mu\nu} \delta \left(\Gamma^\rho{}_{\rho\sigma}\right). \end{aligned} \tag{A.7}$$

This looks exactly like the previous equation! Then, the variation of the Ricci tensor is in shorter notation

$$\underbrace{\sqrt{-g}\, g^{\mu\nu} \delta \left(R_{\mu\nu}\right)}_{3^o} = \sqrt{-g}\, g^{\mu\nu} \{ \nabla_\rho \, \delta \left(\Gamma^\rho{}_{\mu\nu}\right) - \nabla_\nu \, \delta \left(\Gamma^\rho{}_{\mu\rho}\right) \}, \tag{A.8}$$

and,

$$\int \mathrm{d}^4 x \, \underbrace{\sqrt{-g}\, g^{\mu\nu} \delta \left(R_{\mu\nu}\right)}_{3^o} = \int \mathrm{d}^4 x \, \sqrt{-g}\, g^{\mu\nu} \{ \nabla_\rho \left(\delta \, \Gamma^\rho{}_{\mu\nu}\right) - \nabla_\nu \, \delta \left(\Gamma^\rho{}_{\mu\rho}\right) \}. \tag{A.9}$$

For we continue this procedure, we had better know some following relations:

$$\nabla_\rho \left[g^{\mu\nu} \delta \left(\Gamma^\rho{}_{\mu\nu}\right)\right] = \nabla_\rho \left(g^{\mu\nu}\right) \delta \left(\Gamma^\rho{}_{\mu\nu}\right) + g^{\mu\nu} \nabla_\rho \left(\delta \left(\Gamma^\rho{}_{\mu\nu}\right)\right), \tag{A.10}$$

and,

$$\nabla_\nu \left[g^{\mu\nu} \delta \left(\Gamma^\rho{}_{\mu\rho}\right)\right] = \nabla_\nu \left(g^{\mu\nu}\right) \delta \left(\Gamma^\rho{}_{\mu\rho}\right) + g^{\mu\nu} \nabla_\nu \left(\delta \left(\Gamma^\rho{}_{\mu\rho}\right)\right), \tag{A.11}$$

and,

$$\int \mathrm{d}^4 x \, \sqrt{-g} \{ \nabla_\rho \left[g^{\mu\nu} \delta \left(\Gamma^\rho{}_{\mu\nu}\right)\right] - \nabla_\rho \, g^{\mu\nu} \delta \left(\Gamma^\rho{}_{\mu\nu}\right) - \nabla_\nu \, \delta \left(\Gamma^\sigma{}_{\mu\rho}\right) - \nabla_\nu \left[g^{\mu\nu} \delta \left(\Gamma^\rho{}_{\mu\rho}\right)\right] \}, \tag{A.12}$$

$$
\begin{aligned}
&= \int \mathrm{d}^4x \sqrt{-g} \left\{ \nabla_\rho \left[g^{\mu\nu} \delta \left(\Gamma^{\rho}{}_{\mu\nu} \right) \right] - \nabla_\nu \left[g^{\mu\nu} \delta \left(\Gamma^{\rho}{}_{\mu\rho} \right) \right] - \nabla_\rho \left(g^{\mu\nu} \right) \delta \left(\Gamma^{\rho}{}_{\mu\nu} \right) + \nabla_\nu \left(g^{\mu\nu} \right) \delta \left(\Gamma^{\rho}{}_{\mu\rho} \right) \right\} \\
&= \int \mathrm{d}^4x \sqrt{-g} \left\{ \nabla_\rho \left[g^{\mu\nu} \delta \left(\Gamma^{\rho}{}_{\mu\nu} \right) \right] - \nabla_\nu \left[g^{\mu\nu} \delta \left(\Gamma^{\rho}{}_{\mu\rho} \right) \right] \right\} \\
&= \int \mathrm{d}^4x \sqrt{-g} \, \nabla_\rho \{ \underbrace{\left[g^{\mu\nu} \delta \left(\Gamma^{\rho}{}_{\mu\nu} \right) - g^{\mu\rho} \delta \left(\Gamma^{\nu}{}_{\mu\nu} \right) \right]}_{J^\rho \text{ for convenience}} \} \\
&= \int \mathrm{d}^4x \sqrt{-g} \, \nabla_\rho J^\rho .
\end{aligned}
\tag{A.13}
$$

Using Gauss's law in 4-Dimension we have

$$
\int_\Upsilon \mathrm{d}^4x \sqrt{-g} \, \nabla_\rho J^\rho = \int_\Omega \mathrm{d}^3x \sqrt{|h|} \, \eta_\rho J^\rho , \tag{A.14}
$$

and then,

$$
\delta S_{EH(3)} = \int_\Omega \mathrm{d}^3x \sqrt{|h|} \, \eta_\rho J^\rho = 0. \tag{A.15}
$$

We can easily realize that the volume element is due to the covariant divergence of a vector, and using Gauss's law, it is equal to a boundary condition at infinity. For the sake of vanishing the variation at infinity, let us consider it, without loss of generality, be zero. Hence, this whole term contributes with nothing to the total divergence! Now, considering the 1^o term in the action $\delta S_{EH(1)}$ one is capable of seeing that

$$
\delta S_{EH(1)} = \delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} \, g_{\mu\nu} \delta \left(g^{\mu\nu} \right) . \tag{A.16}
$$

After doing these previous calculations, we can plug them all together and obtain the main action which is

$$
\begin{aligned}
\delta S_{EH} &= \delta S_{EH(1)} + \delta S_{EH(2)} + \delta S_{EH(3)} \\
&= \int \mathrm{d}^4x \left[-\frac{1}{2} \sqrt{-g} \, g_{\mu\nu} \delta \left(g^{\mu\nu} \right) R_{\mu\nu} + \sqrt{-g} \, \delta \left(g^{\mu\nu} \right) R_{\mu\nu} + 0 \right] \\
&= \int \mathrm{d}^4x \sqrt{-g} \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right] \delta \left(g^{\mu\nu} \right) ,
\end{aligned}
$$

where R is the Ricci scalar defined as $R = g^{\mu\nu} R_{\mu\nu}$. The functional derivative must satisfy:

$$
\delta S = \int \mathrm{d}^4x \sum_k \left(\frac{\delta S}{\delta \Psi^k} \delta \Psi^k \right) , \qquad \delta \left(\Psi^k \right) = \delta \left(g^{\mu\nu} \right) \rightarrow \Psi^k = g^{\mu\nu} , \tag{A.17}
$$

and,

$$
\rightarrow \frac{\delta S}{\delta \Psi^k} = \frac{\delta S}{g^{\mu\nu}} = \sqrt{-g} \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right] \rightarrow \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0, \tag{A.18}
$$

then,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \tag{A.19}$$

This is called the Einstein's equation in absence of matter. We can define

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = G^{\mu\nu},$$

where $G^{\mu\nu}$ is called the Einstein's tensor. We can prove that

$$\nabla_\mu G^{\mu\nu} = 0. \tag{A.20}$$

Proof: Starting with the Bianchi Identity

$$\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\sigma\mu} + \nabla_\mu R_{\alpha\beta\nu\sigma} = 0, \tag{A.21}$$

and multiplying $g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}$ in both sides,

$$\begin{aligned}
&\nabla_\sigma g^{\gamma\sigma}\underbrace{g^{\alpha\mu}g^{\beta\nu}R_{\alpha\beta\mu\nu}}_{g^{\alpha\mu}g^{\beta\nu}R_{\alpha\beta\mu\nu}=R} + \nabla_\nu g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}\underbrace{R_{\alpha\beta\sigma\mu}}_{R_{\alpha\beta\sigma\mu}=-R_{\beta\alpha\sigma\mu}} + \nabla_\mu g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}\underbrace{R_{\alpha\beta\nu\sigma}}_{R_{\alpha\beta\nu\sigma}=-R_{\beta\alpha\nu\sigma}} = 0\\
&\nabla_\sigma g^{\gamma\sigma}R - \nabla_\nu g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}R_{\beta\alpha\sigma\mu} - \nabla_\mu g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}R_{\beta\alpha\nu\sigma} = 0\\
&\nabla_\sigma g^{\gamma\sigma}R - \nabla_\nu g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}R_{\sigma\mu\beta\alpha} - \nabla_\mu g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}R_{\mu\sigma\beta\alpha} = 0\\
&\nabla_\sigma g^{\gamma\sigma}R - \nabla_\nu g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}R_{\mu\sigma\alpha\beta} - \nabla_\mu g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}R_{\mu\sigma\beta\alpha} = 0\\
&\nabla_\sigma g^{\gamma\sigma}R - \nabla_\nu g^{\gamma\sigma}g^{\beta\nu}R^{\alpha}{}_{\sigma\alpha\beta} - \nabla_\mu g^{\gamma\sigma}g^{\alpha\mu}R^{\alpha}{}_{\sigma\beta\alpha} = 0\\
&\nabla_\sigma g^{\gamma\sigma}R - \nabla_\nu g^{\gamma\sigma}g^{\beta\nu}R_{\sigma\beta} - \nabla_\mu g^{\gamma\sigma}g^{\alpha\mu}R_{\sigma\alpha} = 0\\
&\nabla_\sigma g^{\gamma\sigma}R - \nabla_\sigma g^{\gamma\sigma}g^{\beta\sigma}R_{\sigma\beta} - \nabla_\sigma g^{\gamma\sigma}g^{\alpha\sigma}R_{\sigma\alpha} = 0\\
&\nabla_\sigma g^{\gamma\sigma}R - 2\nabla_\sigma R^{\gamma\sigma} = 0\\
&\to \nabla_\sigma\left(\underbrace{R^{\gamma\sigma} - \frac{1}{2}g^{\mu\nu}R}_{G^{\gamma\sigma}}\right) = 0\\
&\to \nabla_\sigma G^{\gamma\sigma} = 0, \quad \nabla_\mu G^{\mu\nu} = 0.
\end{aligned} \tag{A.22}$$

□

For the sake of completeness to get the generalized field equation (plus matter), let us introduce one more field besides those three previous fields that we had already put,

$$\frac{1}{16\pi G}S_{EH} + S_M \to \frac{\delta\bar{S}}{\delta g^{\mu\nu}} = \frac{1}{16\pi G}\frac{\delta S_{EH}}{\delta g^{\mu\nu}} + \frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{A.23}$$

it follows that

$$\frac{\delta \bar{S}}{\delta g^{\mu\nu}} = \underbrace{\frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\sqrt{-g}}_{\frac{1}{16\pi G}\frac{\delta S_{EH}}{\delta g^{\mu\nu}}}, \tag{A.24}$$

$$\rightarrow \frac{1}{\sqrt{-g}}\frac{\delta \bar{S}}{\delta g^{\mu\nu}} = \frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 0.$$

Conveniently, we define the Energy-momentum tensor as

$$T^{\mu\nu} = 2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{A.25}$$

and therefore, we have the full Einstein's equation:

$$\frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = T^{\mu\nu}. \tag{A.26}$$

APPENDIX B – THE HIGGS MECHANISM

This section was based on *"Lectures on quantum field theory"* [58]. Therefore, for the sake of further details of calculations, please see this reference pointed out above.

It is worth enunciating the Goldstone theorem which establishes that *being a Lorentz invariant theory with a positive defined metric in Hilbert space, if a continuous symmetry is spontaneously broken, then there must appear massless excitations*[1].

Starting off with the Lagrangian:

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) + m^2(\phi^\dagger\phi) - \frac{\lambda}{4}(\phi^\dagger\phi)^2, \tag{B.1}$$

where we have defined the covariant derivative D_μ as

$$D_\mu\phi = \partial_\mu\phi + iqA_\mu\phi. \tag{B.2}$$

The Lagrangian introduced above (B.1) has some symmetries

$$\delta\phi = -ia(x)\phi(x), \tag{B.3}$$

and,

$$\delta\phi^\dagger = ia(x)\phi^\dagger(x), \tag{B.4}$$

and,

$$\delta A_\mu = \frac{1}{q}\partial_\mu a(x). \tag{B.5}$$

Let us decompose the field ϕ in terms of σ and χ,

$$\phi = \frac{1}{\sqrt{2}}(\sigma + i\chi), \tag{B.6}$$

and differently of scalar field and complex scalar field theory which have the vacuum expectation value $\langle\phi\rangle = \langle\phi^\dagger\rangle = 0$, the "new" vacuum is infinite degenerated! The "new" vacuum

[1] Within a pedagogical approach, it turns out to be just perturbations around the minimum.

expectation value is

$$\langle\sigma\rangle = \frac{2m}{\sqrt{\lambda}}, \quad \langle\chi\rangle = 0. \tag{B.7}$$

For the sake of further analyses, let us rewrite Lagrangian (B.1) in terms of (B.6) and noting that we are on the verge of using the definition $\phi^\dagger \overleftrightarrow{\partial_\mu} \phi = \partial_\mu\phi^\dagger\phi - \phi^\dagger\partial_\mu\phi$ as follows:

$$\begin{aligned}
\mathscr{L} =& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) + m^2\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2 \\
=& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial_\mu\phi^\dagger\partial^\mu\phi - iqA^\mu\phi^\dagger\overleftrightarrow{\partial_\mu}\phi + q^2A_\mu A^\mu\phi^\dagger\phi + m^2\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2 \\
=& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{m^2}{2}(\sigma^2+\chi^2) \\
& - qA^\mu\chi\overleftrightarrow{\partial_\mu}\sigma + \frac{q^2}{2}A_\mu A^\mu(\sigma^2+\chi^2) - \frac{\lambda}{16}(\sigma^2+\chi^2)^2.
\end{aligned} \tag{B.8}$$

Shifting the vacuum fields variables $\sigma \to \sigma + \langle\sigma\rangle = \sigma + \frac{2m}{\sqrt{\lambda}} = \sigma + v$ and $\chi \to \chi + \langle\chi\rangle = \chi$, Lagrangian (B.8) could be rewritten as

$$\begin{aligned}
\mathscr{L} =& -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - qA^\mu\chi\overleftrightarrow{\partial_\mu}(\sigma+v) + \frac{q^2}{2}A_\mu A^\mu((\sigma+v)^2+\chi^2) \\
& + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{m^2}{2}((\sigma+v)^2+\chi^2) - \frac{\lambda}{16}((\sigma+v)^2+\chi^2)^2 \\
=& -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - m^2\sigma^2 \\
& + \frac{m^2v^2}{4} + \frac{q^2v^2}{2}A_\mu A^\mu + qvA^\mu\partial_\mu\chi - qA^\mu\chi\overleftrightarrow{\partial_\mu}\sigma + \frac{q^2}{2}A^\mu A_\mu(\sigma^2+\chi^2+2v\sigma) \\
& - \frac{\lambda}{16}(\sigma^4 + 2\sigma^2\chi^2 + \chi^4 + 4v\sigma^3 + 4v\sigma\chi^2),
\end{aligned} \tag{B.9}$$

where the v was introduced conveniently to make the calculations easier. From the above equation, we can select only quadratic terms:

$$\begin{aligned}
\mathscr{L}_{Quadratic} =& -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{q^2v^2}{2}A_\mu A^\mu \\
& + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - m^2\sigma^2 + qvA^\mu\partial_\mu\chi.
\end{aligned} \tag{B.10}$$

Trying to leave equation (B.10) in terms of a unique gauge field, one might consider

$$B_\mu = A_\mu + \frac{1}{qv}\partial_\mu\chi, \tag{B.11}$$

and now the quadratic diagonalized Lagrangian turns out to be

$$\mathscr{L}_{QD} = -\frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B^\mu)(\partial^\mu B^\nu - \partial^\nu B^\mu) + \frac{q^2v^2}{2}B^\mu B_\mu + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - m^2\sigma^2, \tag{B.12}$$

where, $q^2v^2 = 4q^2m^2/\lambda$ and hence,

$$\mathscr{L}_{QD} = -\frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B^\mu)(\partial^\mu B^\nu - \partial^\nu B^\mu) + \underbrace{\frac{2q^2m^2}{\lambda}}_{Mass\ term!} B^\mu B_\mu + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - m^2\sigma^2. \quad \text{(B.13)}$$

We may explicitly see the mass term raising up! And the bosonic χ field has entirely disappeared from the spectrum of the theory as one may see in equation (B.13). One might ask: what about the Nambu-Goldstone theorem (in which ensure that if there exists a continuous spontaneous symmetry breaking the massless bosons should appear)? So where is it? The action of diagonalizing the theory turned out to make a wrong theory which may not describe what happens in nature? The intriguing point is that even though does not exist the NG bosons, the theory which we have been working on is in agreement with unitarity[2]!

What has happened is the Nambu-Goldstone mode χ, the massless mode, which comes from the complex scalar field, has combined with the massless gauge field to give it an additional degree of freedom. In other words, the most remarkable feature is that the gauge field B_μ has become massive as we can see directly from equation (B.13). Furthermore, there exists an additional remark which is worth being pointed out. It is the fact that B_μ has acquired three degrees of freedom without losing unitarity.

This phenomenon described above (.i.e. the acquiring mass of a gauge field by absorbing a Nambu-Goldstone mode) is known as the Higgs mechanism! And the scalar field which is responsible for this procedure is conventionally called the Higgs field [53, 54].

So, the Nambu-Goldstone theorem fails in this case? The answer lies in the same fact which happens in the Gupta-Bleuler method[3] [118, 119]. The point is whether a gauge theory is written in a Lorentz invariant way or not. Moreover, the crucial point is that the metric of the Hilbert space becomes indefinite when one asks for a gauge theory with Lorentz invariance. Then, there may exist states with negative norm! This can be seen in the electromagnetism approach, where the field A_μ is a four-vector and hence must have four independent polarization states. However, for being in agreement with experiments, the physical states have only transverse polarization. In this way, the others two states must cancel each other for their effects be neglected.

Finally, the crucial point in the Nambu-Goldstone theorem is the requirement of having Lorentz invariance as well as a positive defined metric in Hilbert space. However, when one deals with a gauge theory, the Goldstone theorem does not hold. We have got two remarkable benefits in losing the massless bosons within the theory. First, we have a well-established procedure for giving mass to gauge bosons (see Figure 9) which is totally in agreement with

[2] At least from the field degrees of freedom viewpoint.

[3] This method is widely known when one tries to do a quantization of the electromagnetic field.

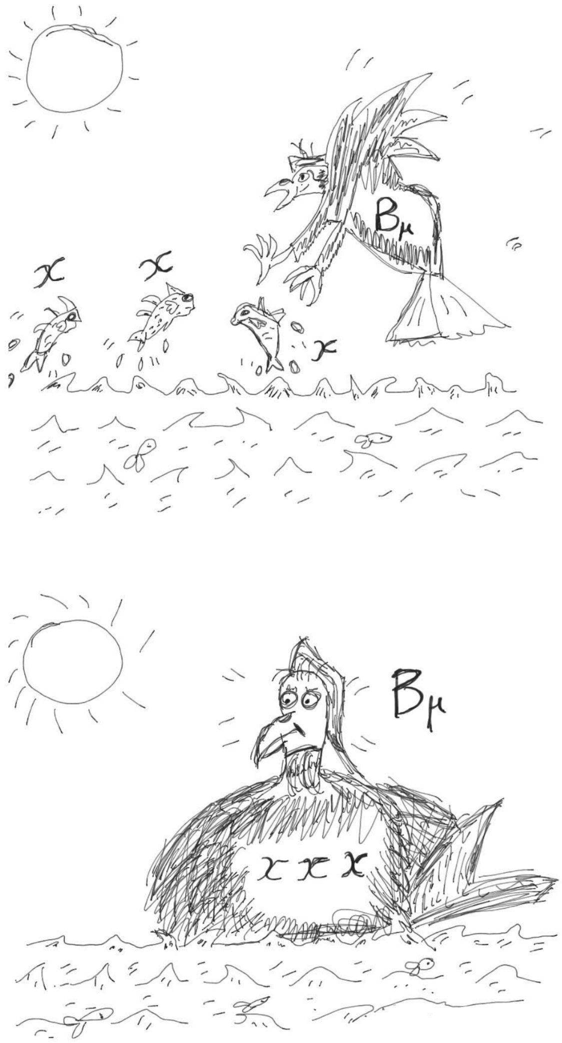

Figure 9: This Figure shows the boson B_μ eating some Nambu Goldstone bosons which allow the acquirement of an extra degree of freedom (mass).

renormalizability and unitarity of the theory! Second, it is quite important when one develops a gauge theory for the weak interactions.

APPENDIX C – THE KALB-RAMOND PROPAGATOR

This section just a proof of how to calculate the Kalb-Ramond propagator using exclusively the Green's function method rather than projectors.

Proof: For calculating the Kalb-Ramond propagator, we had better take clearly all the following definitions:

$$
\begin{aligned}
&\bullet\ H^{\lambda\mu\nu} = \partial^{\lambda}B^{\mu\nu} + \partial^{\mu}B^{\nu\lambda} + \partial^{\nu}B^{\lambda\mu},\\
&\bullet\ \mathscr{L}_{FULL} = \mathscr{L}_{KR} + \mathscr{L}_{GF} = \underbrace{\frac{1}{6}H^{\lambda\mu\nu}H_{\lambda\mu\nu} + B_{\mu\nu}J^{\mu\nu}}_{\mathscr{L}_{KR}} + \underbrace{\frac{1}{2\beta}\partial_{\mu}B^{\mu\nu}\partial_{\alpha}B^{\alpha}{}_{\nu}}_{\mathscr{L}_{GF}},\\
&\bullet\ \partial_{\lambda}H^{\lambda\mu\nu} = J^{\mu\nu},\\
&\bullet\ \hat{\mathfrak{O}}^{\mu\nu,\gamma\sigma}G_{\gamma\sigma,\alpha\beta}(x-y) = \hat{\mathfrak{J}}^{\mu\nu,}{}_{\alpha\beta}\,\delta^{(4)}(x-y),\\
&\bullet\ \hat{\mathfrak{J}}^{\mu\nu,\alpha\beta} = \frac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\beta}\eta^{\nu\alpha}),\\
&\bullet\ \hat{\mathfrak{J}}^{\mu\nu,}{}_{\alpha\beta} = \frac{1}{2}(\delta^{\mu}_{\alpha}\,\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\,\delta^{\nu}_{\alpha}),\\
&\bullet\ \delta^{(4)}(x-y) = \frac{1}{(4\pi)^4}\int \mathrm{d}^4p\, e^{-ip\cdot(x-y)},\\
&\bullet\ G_{\gamma\sigma,\alpha\beta}(x-y) = \frac{1}{(4\pi)^4}\int \mathrm{d}^4p\, e^{-ip\cdot(x-y)}\tilde{G}_{\gamma\sigma,\alpha\beta}(p),
\end{aligned}
\tag{C.1}
$$

where, $H^{\lambda\mu\nu}$ is the field strength, $\mathscr{L}_{FULL}$ is the full Lagrangian, $\mathscr{L}_{KR}$ is the Lagrangian of the free Kalb-Ramond field, $\mathscr{L}_{GF}$ is the Lagrangian of the gauge fixing term, $\hat{\mathfrak{O}}^{\mu\nu,\gamma\sigma}$ is the operator, $G_{\gamma\sigma,\alpha\beta}(x-y)$ is the Green's function, $\hat{\mathfrak{J}}^{\mu\nu,\alpha\beta}$ is the antisymmetric identity and $\delta^{(4)}(x-y)$ is the four-dimensional delta function.

Firstly, we have to find out the full equation of motion. Let us do this separately working on $\mathscr{L}_{GF}$ and $\mathscr{L}_{KR}$. Taking into account the $\mathscr{L}_{GF}$:

$$\begin{aligned}
\mathscr{L}_{GF} &= \frac{1}{2\beta}\partial_\mu B^{\mu\nu}\partial_\alpha B^{\alpha}{}_{\nu} \\
&= \frac{1}{2\beta}\eta^{\nu\rho}\partial_\mu B^{\mu\nu}\partial_\alpha B^{\alpha\rho} = \frac{1}{8\beta}\eta^{\nu\rho}\partial_\mu(\eta^{\mu\sigma}\eta^{\nu\gamma} - \eta^{\nu\sigma}\eta^{\mu\gamma})B_{\sigma\gamma}\,\partial_\alpha(\eta^{\alpha\mu}\eta^{\rho\nu} - \eta^{\rho\mu}\eta^{\alpha\nu})B_{\mu\nu} \\
&= \frac{1}{8\beta}(\delta^{\rho}_{\gamma}\,\partial^\sigma - \delta^{\rho}_{\sigma}\,\partial^\gamma)B_{\sigma\gamma}(\partial^\mu\eta^{\rho\nu} - \partial^\nu\eta^{\rho\mu})B_{\mu\nu} \\
&= \frac{1}{8\beta}(\delta^{\rho}_{\gamma}\,\eta^{\rho\nu}\partial^\sigma\partial^\mu - \delta^{\rho}_{\gamma}\,\eta^{\rho\mu}\partial^\sigma\partial^\nu - \delta^{\rho}_{\sigma}\,\eta^{\rho\nu}\partial^\gamma\partial^\mu + \delta^{\rho}_{\sigma}\,\eta^{\rho\mu}\partial^\gamma\partial^\nu)B_{\sigma\gamma}B_{\mu\nu} \\
&= \frac{1}{8\beta}(\eta^{\gamma\nu}\partial^\sigma\partial^\mu - \eta^{\gamma\mu}\partial^\sigma\partial^\nu + \eta^{\sigma\mu}\partial^\gamma\partial^\nu - \eta^{\sigma\nu}\partial^\gamma\partial^\mu)B_{\sigma\gamma}B_{\mu\nu},
\end{aligned} \tag{C.2}$$

therefore,

$$\frac{\delta\mathscr{L}_{GF}}{\delta B_{\mu\nu}} = \frac{1}{4\beta}(\eta^{\gamma\nu}\partial^\sigma\partial^\mu - \eta^{\gamma\mu}\partial^\sigma\partial^\nu + \eta^{\sigma\mu}\partial^\gamma\partial^\nu - \eta^{\sigma\nu}\partial^\gamma\partial^\mu)B_{\sigma\gamma}. \tag{C.3}$$

Secondly, let us introduce the remaining part, $\mathscr{L}_{KR}$:

$$\begin{aligned}
\mathscr{L}_{kR} =& \frac{1}{6}\,(\,\partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu})(\partial^\lambda B^{\mu\nu} + \partial^\mu B^{\nu\lambda} + \partial^\nu B^{\lambda\mu}\,) + B_{\mu\nu}J^{\mu\nu} \\
=& \frac{1}{6}\,[\,\partial_\lambda B_{\mu\nu}\partial^\lambda B^{\mu\nu} + \partial_\lambda B_{\mu\nu}\partial^\mu B^{\nu\lambda} + \partial_\lambda B_{\mu\nu}\partial^\nu B^{\lambda\mu} \\
&+ \partial_\mu B_{\nu\lambda}\partial^\lambda B^{\mu\nu} + \partial_\mu B_{\nu\lambda}\partial^\mu B^{\nu\lambda} + \partial_\mu B_{\nu\lambda}\partial^\nu B^{\lambda\mu} \\
&+ \partial_\nu B_{\lambda\mu}\partial^\lambda B^{\mu\nu} + \partial_\nu B_{\lambda\mu}\partial^\mu B^{\nu\lambda} + \partial_\nu B_{\lambda\mu}\partial^\nu B^{\lambda\mu}\,] + B_{\mu\nu}J^{\mu\nu} \\
=& \frac{1}{6}\,[\,3\,\partial_\mu B_{\nu\lambda}\partial^\mu B^{\nu\lambda} + 3\,\partial_\mu B_{\nu\lambda}\partial^{\lambda\mu} + 3\,\partial_\mu B_{\nu\lambda}\partial^\lambda B^{\mu\nu}\,] + B_{\mu\nu}J^{\mu\nu} \\
=& \frac{1}{2}\,[\,\partial_\mu B_{\nu\lambda}\partial^\mu B^{\nu\lambda} + \partial_\mu B_{\nu\lambda}\partial^\nu B^{\lambda\mu} + \partial_\mu B_{\nu\lambda}\partial^\lambda B^{\mu\nu}\,] + B_{\mu\nu}J^{\mu\nu} \\
=& \frac{1}{2}\,\partial_\mu B_{\nu\lambda}\,(\,H^{\lambda\mu\nu}\,) + B_{\mu\nu}J^{\mu\nu},
\end{aligned} \tag{C.4}$$

and therefore,

$$\begin{aligned}
D_\lambda\left(\frac{\delta\mathscr{L}_{KR}}{\delta(\partial_\lambda B_{\mu\nu})}\right) &= \frac{1}{2}\,D_\lambda\,[\,\partial^\lambda B^{\mu\nu} + \partial^\lambda B^{\mu\nu} + \partial^\mu B^{\nu\lambda} + \partial^\mu B^{\nu\lambda} + \partial^\nu B^{\lambda\mu} + \partial^\nu B^{\lambda\mu}\,] \\
&= \frac{1}{2}\,D_\lambda\,[\,2\,\partial^\lambda B^{\mu\nu} + 2\,\partial^\mu B^{\nu\lambda} + 2\,\partial^\nu B^{\lambda\mu}\,] \\
&= D_\lambda\,[\,\partial^\lambda B^{\mu\nu} + \partial^\mu B^{\nu\lambda} + \partial^\nu B^{\lambda\mu}\,] \\
&= D_\lambda\,H^{\lambda\mu\nu} \\
&= \partial_\lambda\,H^{\lambda\mu\nu} = \Box B^{\mu\nu} + \partial_\lambda\partial^\mu B^{\nu\lambda} + \partial_\lambda\partial^\nu B^{\lambda\mu}.
\end{aligned} \tag{C.5}$$

Considering some "transformations":

$$\begin{aligned}
\therefore\ B^{\mu\nu} &= \frac{1}{2}(\eta^{\mu\sigma}\eta^{\nu\gamma} - \eta^{\nu\sigma}\eta^{\mu\gamma})B_{\sigma\gamma}, \\
\therefore\ B^{\nu\lambda} &= \frac{1}{2}(\eta^{\nu\sigma}\eta^{\lambda\gamma} - \eta^{\lambda\sigma}\eta^{\nu\gamma})B_{\sigma\gamma}, \\
\therefore\ B^{\lambda\mu} &= \frac{1}{2}(\eta^{\lambda\sigma}\eta^{\mu\gamma} - \eta^{\mu\sigma}\eta^{\lambda\gamma})B_{\sigma\gamma},
\end{aligned} \tag{C.6}$$

we have

$$\longrightarrow \left[\frac{\Box}{2}(\eta^{\mu\sigma}\eta^{\nu\gamma} - \eta^{\nu\sigma}\eta^{\mu\gamma}) + \frac{1}{2}\partial_\lambda\partial^\mu(\eta^{\nu\sigma}\eta^{\lambda\gamma} - \eta^{\lambda\sigma}\eta^{\nu\gamma}) + \frac{1}{2}\partial_\lambda\partial^\nu(\eta^{\lambda\sigma}\eta^{\mu\gamma} - \eta^{\mu\sigma}\eta^{\lambda\gamma})\right] B_{\sigma\gamma}, \tag{C.7}$$

or

$$\longrightarrow \left[-\frac{\Box}{2}(\eta^{\nu\sigma}\eta^{\mu\gamma} - \eta^{\mu\sigma}\eta^{\nu\gamma}) - \frac{1}{2}(\eta^{\nu\gamma}\partial^\sigma\partial^\mu - \eta^{\nu\sigma}\partial^\gamma\partial^\mu + \eta^{\mu\sigma}\partial^\gamma\partial^\nu - \eta^{\mu\gamma}\partial^\sigma\partial^\nu)\right] B_{\sigma\gamma}, \tag{C.8}$$

and,

$$\frac{\delta\mathscr{L}_{KR}}{\delta B_{\mu\nu}} = J^{\mu\nu}. \tag{C.9}$$

For the sake of obtaining the full equation of motion, we have summed all these parts:

$$\begin{aligned}
&\left[-\frac{\Box}{2}(\eta^{\nu\sigma}\eta^{\mu\gamma} - \eta^{\mu\sigma}\eta^{\nu\gamma}) - \frac{1}{2}(\eta^{\nu\gamma}\partial^\sigma\partial^\mu - \eta^{\nu\sigma}\partial^\gamma\partial^\mu + \eta^{\mu\sigma}\partial^\gamma\partial^\nu - \eta^{\mu\gamma}\partial^\sigma\partial^\nu)\right. \\
&\left.-\frac{1}{4\beta}(\eta^{\gamma\mu}\partial^\sigma\partial^\nu - \eta^{\gamma\nu}\partial^\sigma\partial^\mu + \eta^{\sigma\nu}\partial^\gamma\partial^\mu - \eta^{\sigma\mu}\partial^\gamma\partial^\nu\right] B_{\sigma\gamma} = J^{\mu\nu}.
\end{aligned} \tag{C.10}$$

In other words,

$$\hat{\mathcal{O}}^{\mu\nu,\gamma\sigma}B_{\sigma\gamma} = J^{\mu\nu}. \tag{C.11}$$

Take a look in the symmetry of the operator:

$$\begin{aligned}
\hat{\mathcal{O}}^{\mu\nu,\gamma\sigma} &= -\hat{\mathcal{O}}^{\nu\mu,\gamma\sigma}, \\
\hat{\mathcal{O}}^{\mu\nu,\gamma\sigma} &= -\hat{\mathcal{O}}^{\mu\nu,\sigma\gamma}, \\
\hat{\mathcal{O}}^{\mu\nu,\gamma\sigma} &= \hat{\mathcal{O}}^{\gamma\sigma,\mu\nu}.
\end{aligned} \tag{C.12}$$

Taking this above operator $\hat{\mathcal{O}}^{\mu\nu,\gamma\sigma}$ in the Green's equation,

$$\hat{\mathcal{O}}^{\mu\nu,\gamma\sigma}G_{\gamma\sigma,\alpha\beta}(x-y) = \hat{\mathcal{I}}^{\mu\nu,}{}_{\alpha\beta}\,\delta^{(4)}(x-y), \tag{C.13}$$

we have got:

$$\Big[-\frac{\Box}{2}(\eta^{\nu\sigma}\eta^{\mu\gamma}-\eta^{\mu\sigma}\eta^{\nu\gamma})-\frac{1}{2}(\eta^{\nu\gamma}\partial^{\sigma}\partial^{\mu}-\eta^{\nu\sigma}\partial^{\gamma}\partial^{\mu}+\eta^{\mu\sigma}\partial^{\gamma}\partial^{\nu}-\eta^{\mu\gamma}\partial^{\sigma}\partial^{\nu})$$
$$-\frac{1}{4\beta}(\eta^{\gamma\mu}\partial^{\sigma}\partial^{\nu}-\eta^{\gamma\nu}\partial^{\sigma}\partial^{\mu}+\eta^{\sigma\nu}\partial^{\gamma}\partial^{\mu}-\eta^{\sigma\mu}\partial^{\gamma}\partial^{\nu}\Big]\times \tag{C.14}$$
$$\frac{1}{(4\pi)^4}\int \mathrm{d}^4p\, e^{-ip\cdot(x-y)}\tilde{G}_{\gamma\sigma,\alpha\beta}(p)=\frac{1}{2}(\delta^{\mu}_{\alpha}\,\delta^{\nu}_{\beta}-\delta^{\mu}_{\beta}\,\delta^{\nu}_{\alpha})\,\frac{1}{(4\pi)^4}\int \mathrm{d}^4p\, e^{-ip\cdot(x-y)}\,,$$

it follows that

$$\Big[+\frac{p^2}{2}(\eta^{\nu\sigma}\eta^{\mu\gamma}-\eta^{\mu\sigma}\eta^{\nu\gamma})+\frac{1}{2}(\eta^{\nu\gamma}p^{\sigma}p^{\mu}-\eta^{\nu\sigma}p^{\gamma}p^{\mu}+\eta^{\mu\sigma}p^{\gamma}p^{\nu}-\eta^{\mu\gamma}p^{\sigma}p^{\nu})$$
$$+\frac{1}{4\beta}(\eta^{\gamma\mu}p^{\sigma}p^{\nu}-\eta^{\gamma\nu}p^{\sigma}p^{\mu}+\eta^{\sigma\nu}p^{\gamma}p^{\mu}-\eta^{\sigma\mu}p^{\gamma}p^{\nu})\Big]\,\tilde{G}_{\gamma\sigma,\alpha\beta}(p)=\frac{1}{2}(\delta^{\mu}_{\alpha}\,\delta^{\nu}_{\beta}-\delta^{\mu}_{\beta}\,\delta^{\nu}_{\alpha})\,. \tag{C.15}$$

The big deal is finding out what is $\tilde{G}_{\gamma\sigma,\alpha\beta}(p)$. For accomplishing this, let us suppose that it may be written as a linear combination of the basis vectors $(\eta_{\sigma\beta}\,\eta_{\gamma\alpha}-\eta_{\gamma\beta}\,\eta_{\sigma\alpha})$ and $(\eta_{\sigma\alpha}\,p_{\beta}p_{\gamma}-\eta_{\sigma\beta}\,p_{\alpha}p_{\gamma}+\eta_{\gamma\beta}\,p_{\alpha}p_{\sigma}-\eta_{\gamma\alpha}\,p_{\beta}p_{\sigma})$. Then,

$$\tilde{G}_{\gamma\sigma,\alpha\beta}(p)=a\,(\eta_{\sigma\beta}\,\eta_{\gamma\alpha}-\eta_{\gamma\beta}\,\eta_{\sigma\alpha})+b\,(\eta_{\sigma\alpha}\,p_{\beta}p_{\gamma}-\eta_{\sigma\beta}\,p_{\alpha}p_{\gamma}+\eta_{\gamma\beta}\,p_{\alpha}p_{\sigma}-\eta_{\gamma\alpha}\,p_{\beta}p_{\sigma}). \tag{C.16}$$

Being quite patient, let us solve the system of the equation and ensure the value of a and b. It follows that

$$\Big[+\frac{p^2}{2}(\eta^{\nu\sigma}\eta^{\mu\gamma}-\eta^{\mu\sigma}\eta^{\nu\gamma})+\frac{1}{2}(\eta^{\nu\gamma}p^{\sigma}p^{\mu}-\eta^{\nu\sigma}p^{\gamma}p^{\mu}+\eta^{\mu\sigma}p^{\gamma}p^{\nu}-\eta^{\mu\gamma}p^{\sigma}p^{\nu})$$
$$+\frac{1}{4\beta}(\eta^{\gamma\mu}p^{\sigma}p^{\nu}-\eta^{\gamma\nu}p^{\sigma}p^{\mu}+\eta^{\sigma\nu}p^{\gamma}p^{\mu}-\eta^{\sigma\mu}p^{\gamma}p^{\nu})\Big]\times \tag{C.17}$$
$$a\,(\eta_{\sigma\beta}\,\eta_{\gamma\alpha}-\eta_{\gamma\beta}\,\eta_{\sigma\alpha})+b\,(\eta_{\sigma\alpha}\,p_{\beta}p_{\gamma}-\eta_{\sigma\beta}\,p_{\alpha}p_{\gamma}+\eta_{\gamma\beta}\,p_{\alpha}p_{\sigma}-\eta_{\gamma\alpha}\,p_{\beta}p_{\sigma})=$$
$$=\frac{1}{2}(\delta^{\mu}_{\alpha}\,\delta^{\nu}_{\beta}-\delta^{\mu}_{\beta}\,\delta^{\nu}_{\alpha})\,.$$

and

$$\begin{aligned}
\longrightarrow\ & +\underbrace{a\,\frac{p^2}{2}(\eta^{\nu\sigma}\eta^{\mu\gamma}-\eta^{\mu\sigma}\eta^{\nu\gamma})(\eta_{\sigma\beta}\,\eta_{\gamma\alpha}-\eta_{\gamma\beta}\,\eta_{\sigma\alpha})}_{(1)}\\
& +\underbrace{b\,\frac{p^2}{2}(\eta^{\nu\sigma}\eta^{\mu\gamma}-\eta^{\mu\sigma}\eta^{\nu\gamma})(\eta_{\sigma\alpha}\,p_\beta p_\gamma-\eta_{\sigma\beta}\,p_\alpha p_\gamma+\eta_{\gamma\beta}\,p_\alpha p_\sigma-\eta_{\gamma\alpha}\,p_\beta p_\sigma)}_{(2)}\\
& +\underbrace{a\,\frac{1}{2}(\eta^{\nu\gamma}p^\sigma p^\mu-\eta^{\nu\sigma}p^\gamma p^\mu+\eta^{\mu\sigma}p^\gamma p^\nu-\eta^{\mu\gamma}p^\sigma p^\nu)(\eta_{\sigma\beta}\,\eta_{\gamma\alpha}-\eta_{\gamma\beta}\,\eta_{\sigma\alpha})}_{(3)}\\
& +\underbrace{b\,\frac{1}{2}(\eta^{\nu\gamma}p^\sigma p^\mu-\eta^{\nu\sigma}p^\gamma p^\mu+\eta^{\mu\sigma}p^\gamma p^\nu-\eta^{\mu\gamma}p^\sigma p^\nu)(\eta_{\sigma\alpha}\,p_\beta p_\gamma-\eta_{\sigma\beta}\,p_\alpha p_\gamma+\eta_{\gamma\beta}\,p_\alpha p_\sigma-\eta_{\gamma\alpha}\,p_\beta p_\sigma)}_{(4)}\\
& +\underbrace{a\,\frac{1}{4\beta}(\eta_{\sigma\beta}\,\eta_{\gamma\alpha}-\eta_{\gamma\beta}\,\eta_{\sigma\alpha})(\eta^{\gamma\mu}p^\sigma p^\nu-\eta^{\gamma\nu}p^\sigma p^\mu+\eta^{\sigma\nu}p^\gamma p^\mu-\eta^{\sigma\mu}p^\gamma p^\nu)}_{(5)}\\
& +\underbrace{b\,\frac{1}{4\beta}(\eta_{\sigma\alpha}\,p_\beta p_\gamma-\eta_{\sigma\beta}\,p_\alpha p_\gamma+\eta_{\gamma\beta}\,p_\alpha p_\sigma-\eta_{\gamma\alpha}\,p_\beta p_\sigma)(\eta^{\gamma\mu}p^\sigma p^\nu-\eta^{\gamma\nu}p^\sigma p^\mu+\eta^{\sigma\nu}p^\gamma p^\mu-\eta^{\sigma\mu}p^\gamma p^\nu)}_{(6)}\,.
\end{aligned} \tag{C.18}$$

Following carefully the six steps:

$$\begin{aligned}
&(1)\ \therefore\ \ a\,p^2\,\underbrace{[\delta^\mu_\alpha\delta^\nu_\beta-\delta^\nu_\alpha\delta^\mu_\beta]}_{basis},\\
&(2)\ \therefore\ \ b\,p^2\,\underbrace{[\delta^\nu_\alpha p^\beta p^\mu-\delta^\nu_\beta p^\alpha p^\mu+\delta^\mu_\beta p^\alpha p^\nu-\delta^\mu_\alpha p^\beta p^\nu]}_{basis},\\
&(3)\ \therefore\ \ a\,\underbrace{[\delta^\nu_\alpha p^\beta p^\mu-\delta^\nu_\beta p^\alpha p^\mu+\delta^\mu_\beta p^\alpha p^\nu-\delta^\mu_\alpha p^\beta p^\nu]}_{basis},\\
&(4)\ \therefore\ \ -b\,p^2\,\underbrace{[\delta^\nu_\alpha p^\beta p^\mu-\delta^\nu_\beta p^\alpha p^\mu+\delta^\mu_\beta p^\alpha p^\nu-\delta^\mu_\alpha p^\beta p^\nu]}_{basis},\\
&(5)\ \therefore\ \ -a\,\frac{1}{2\beta}\,\underbrace{[\delta^\nu_\alpha p^\beta p^\mu-\delta^\nu_\beta p^\alpha p^\mu+\delta^\mu_\beta p^\alpha p^\nu-\delta^\mu_\alpha p^\beta p^\nu]}_{basis},\\
&(6)\ \therefore\ \ b\,p^2\,\frac{1}{2\beta}\,\underbrace{[\delta^\nu_\alpha p^\beta p^\mu-\delta^\nu_\beta p^\alpha p^\mu+\delta^\mu_\beta p^\alpha p^\nu-\delta^\mu_\alpha p^\beta p^\nu]}_{basis}\,.
\end{aligned} \tag{C.19}$$

Finally, we need to solve the system

$$a\,p^2=\frac{1}{2}\ ,\ \longrightarrow\ a=\frac{1}{2p^2}, \tag{C.20}$$

and

$$-\frac{a}{2\beta} + a + \frac{bp^2}{2\beta}\ , \ \longrightarrow \ b = \frac{1}{2p^4}(1-2\beta). \tag{C.21}$$

Therefore, replacing a and b, the full Green's function is given by:

$$\tilde{G}_{\gamma\sigma,\alpha\beta}(p) = \frac{1}{2\,p^2}\,(\eta_{\sigma\beta}\,\eta_{\gamma\alpha} - \eta_{\gamma\beta}\,\eta_{\sigma\alpha}) + \frac{(1-2\beta)}{2p^4}\,(\eta_{\sigma\alpha}\,p_\beta p_\gamma - \eta_{\sigma\beta}\,p_\alpha p_\gamma + \eta_{\gamma\beta}\,p_\alpha p_\sigma - \eta_{\gamma\alpha}\,p_\beta p_\sigma). \tag{C.22}$$

$$\boxed{\tilde{G}_{\gamma\sigma,\alpha\beta}(p) = \frac{1}{2\,p^2}\,(\eta_{\sigma\beta}\,\eta_{\gamma\alpha} - \eta_{\gamma\beta}\,\eta_{\sigma\alpha}) + \frac{(1-2\beta)}{2p^4}\,(\eta_{\sigma\alpha}\,p_\beta p_\gamma - \eta_{\sigma\beta}\,p_\alpha p_\gamma + \eta_{\gamma\beta}\,p_\alpha p_\sigma - \eta_{\gamma\alpha}\,p_\beta p_\sigma).}$$

□

This is the Kalb-Ramond propagator in which agrees with the well-established result in the literature [120–122]. We can see that the propagator possesses a pole with zero mass evidencing that the Kalb-Ramond field is non-massive. Moreover, the additional parameter, which came from the gauge fixing term (β), do not contribute to the S-Matrix.

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