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JANIELLY GONÇALVES ARAÚJO

SINGULAR PERTURBATION METHODS AND OPTIMAL REGULARITY FOR DEGENERATE EQUATIONS.

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Thesis submitted to the Post-graduate Program of the Mathematical Departament of Universi dade Federal do Ceará in partial fulfillment of the necessary requirements for the degree of Ph.D in Mathematics. Area of expertise: Analysis

Advisor: Prof. Dr. Gleydson Chaves Ricarte

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I dedicate this work to my family and friends.

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"You can never quit. Winners never quit, and quitters never win."(Ted Turner)

RESUMO

Na primeira parte desse trabalho nós provamos regularidade Lipschitz interior e até a fronteira de soluções do problema de pertubação singular para uma equação reação/difusão governada pela equação p-Laplaciano normalizado

$$|\nabla u^\epsilon|^{2-p} \cdot \operatorname{div}\left(|\nabla u^\epsilon|^{p-2}\nabla u^\epsilon\right) = \beta_\epsilon(u^\epsilon),$$

onde o termo de reação é do tipo combustão. Nós obtemos o comportamento geométrico de soluções próximo as superfíceis ϵ-níveis, pela regularidade ótima e não-degenerecência geométrica sharp. Passamos o limite e investigamos propriedades da medida de Hausdorff da função limite.

Na segunda parte obtemos estimativas de regularidade ótima para soluções localmente limitada da equação duplamente não linear degenerada

$$u_t - \operatorname{div}(m|u|^{m-1}|\nabla u|^{p-2}\nabla u) = f,$$

onde $m > 1$, $p > 2$ e $f \in L^{q,r}$. Mais precisamente, mostramos que soluções são locamente de classe $C^{0,\beta}$, onde β depende explicitamente somente do expoente Hölder ótimo para soluções do caso homogêneo, da integrabilidade da f, das constantes p, m e da dimensão n.

Palavras-chave: Pertubação Singular. P-Laplaciano normalizado. Teoria de regularidade. Equação duplamente não linear . Degenerada. Regularidade ótima.

ABSTRACT

In the first part of this work we prove interior and up to boundary Lipschitz regularity of the viscosity solutions to a singular perturbation problem for a reaction-diffusion equation related to the normalized p-Laplacian equation

$$|\nabla u^\epsilon|^{2-p} \cdot \operatorname{div}\left(|\nabla u^\epsilon|^{p-2}\nabla u^\epsilon\right) = \beta_\epsilon(u^\epsilon),$$

where the reaction term is of combustion type. We obtain the precise geometric behavior of solutions near ϵ-level surfaces, by means of optimal regularity and sharp geometric nondegeneracy. We pass to the limit we investigate Hausdorff measure properties of the limit function.

In the second part the aim is to obtain sharp regularity estimates for locally bounded solutions of the degenerate doubly nonlinear equation

$$u_t - \operatorname{div}(m|u|^{m-1}|\nabla u|^{p-2}\nabla u) = f,$$

where $m > 1$, $p > 2$ and $f \in L^{q,r}$. More precisely, we show that solutions are locally of class $C^{0,\beta}$, where β depends explicitly only on the optimal Hölder exponent for solutions of the homogeneous case, the integrability of f, the constants p, m and the space dimension n.

Keywords: Singularly perturbed . Normalized p-Laplacian . Regularity theory. Doubly nonlinear . Degenerate. Sharp regularity.

SUMÁRIO

1 INTRODUCTION

This thesis is divided into two parts. In the first part of this work we are concerned about studying the limit as $\varepsilon \to 0$ of the solutions u^ε for an free boundary problem involving a class of degenerate/singular elliptic boundary-reaction-diffusion problem

$$\begin{cases} |\nabla u^\epsilon|^{2-p} \cdot \mathrm{div}\left(|\nabla u^\epsilon|^{p-2} \nabla u^\epsilon\right) &= \beta_\epsilon(u^\epsilon) \quad \text{in} \quad \Omega \\ u &= \varphi \qquad \text{on} \quad \partial\Omega \end{cases} \tag{E_ϵ}$$

Here, the nonlinear reaction term $\beta_\epsilon \colon \mathbb{R} \to \mathbb{R}_+$ is of combustion type satisfying

$$0 \leq \beta_\epsilon(t) \leq \frac{\mathcal{B}}{\epsilon} \chi_{(0,\epsilon)}(t), \quad \forall \;\; t \in \mathbb{R}_+, \tag{1.1}$$

for nonnegative constant $\mathcal{B} \geq 0$. For example,

$$\beta_\epsilon(t) := \frac{1}{\epsilon} \beta\left(\frac{t}{\epsilon}\right), \tag{1.2}$$

with $\beta \in C_0^{0,1}(\mathbb{R})$ satisfying

$$\beta \geq 0, \;\; \mathrm{supp}(\beta) = [0,1] \quad \text{and} \quad \int_0^1 \beta(t)dt = M$$

is a particular (simpler) case covered by the analysis to be developed here. This problem is a interesting model in combustion and flame propagation theory. It appears in the description of laminar flames as an asymptotic limit for high energy activation.

The idea is that Eq. (E_ϵ) approximates the free boundary problem description as follows: given a smooth bounded domain Ω, a smooth non-negative function $\varphi : \mathbb{R}^n \to \mathbb{R}$, compactly supported in Ω, such that it is possible to solve the free boundary problem

$$\begin{cases} \Delta_p u &= 0 \quad \text{in} \quad \Omega^+ \colon = \{u > 0\} \\ u &= \varphi \quad \text{on} \quad \partial\Omega \end{cases}, \tag{1.3}$$

in a certain sense that will be discussed later on. Here, $1 \leq p < \infty$ and $\Delta_p v \colon = \mathrm{div}(|\nabla v|^{p-2} \nabla v)$ denotes the p-Laplace operator. We do not impose a free boundary condition and thus the limiting problem is not understood as overdetermined. One of our main objectives in this paper is to show that the solutions u^ε of the singular perturbation problem (E_ϵ), converge to a solution to the free boundary problem (1.3), in a certain sense.

The modern mathematical treatment of free boundary problems arising from passing the limit in regularizing ones have been of large interest through the last years.

Its simplest form is (when $p=2$)

$$\Delta u^{\epsilon} = \beta_{\epsilon}(u^{\epsilon}). \tag{1.4}$$

The limiting free boundary problem obtained by letting ϵ go to zero in (1.4) was fully studied in the late 70's and early 80's. by Lewy-Stampacchia, Caffarelli, Kinderlehrer and Nirenberg, Alt and Phillips, among others. Back in 1938, Zeldovich and Frank-Kamenetski proposed the passage to the limit in this singular perturbation problem in ZELDOWITSCH and FRANK-KAMENETZKI (1992). The passage to the limit was not studied in a mathematically rigorous way until 1990 when Berestycki, Caffarelli and Nuremberg studied the case of d dimensional traveling waves (cf. BERESTYCKI (1990)). Later, in CAFFARELLI and VÁZQUEZ (1995), the general evolution problem in the one phase case (i.e., $u^{\epsilon} \geq 0$) was considered. Much research has been done on this matter ever since.(cf. CAFFARELLI, LEDERMAN, and WOLANSKI (1997b,a); RICARTE, TEYMURAZYAN, and URBANO (2016)).

The p-Laplacian version of approximating problems has been considered as well. For example, in DANIELLI, PETROSYAN, and SHAHGHOLIAN (2003), the authors study the limit u^{ϵ} as $\epsilon \to 0$ of the solutions u^{ϵ} of the one-phase equation $\Delta_p u^{\epsilon} = \beta_{\epsilon}(u^{\epsilon})$ in $\Omega \subset \mathbb{R}^d$. We also refer to MARTINEZ and WOLANSKI (2009) for more general quasilinear operators satisfying the natural growth condition of Liberman. The majority of the previous work on elliptic p-Laplace equation rely heavily on the variational structure of the equation. The equation (E_{ϵ}) does not have that structure. Therefore, we must take a completely different point of view using tools for equations in non-divergence form.

Another important line of research would be the study of fully nonlinear singular perturbation problem, that is,

$$\mathcal{F}(x, \nabla u^{\epsilon}, D^2 u^{\varepsilon}) = \beta_{\varepsilon}(u^{\varepsilon}), \tag{1.5}$$

where $\mathcal{F}(x, \vec{q}, M) \sim |\vec{q}|^{\gamma} \cdot F(M)$. Recently, this singular perturbation problem in the case $\gamma = 0$ has been studied by a least supersolution method in RICARTE and TEIXEIRA (2011), obtaining a nice geometric description of the limiting free boundary. The complete study of singular perturbation problem for more general fully nonlinear equations has been explored in RICARTE and DA SILVA (2015) for the case of $\gamma \geq 0$ (these techniques are in some sense stronger than variational methods). With this new machinery and for a didactical reason, we have chosen to present our theory for the normalized p-Laplacian operator. Basically because the main difficulty of dealing with complete elliptic operators lies, as we will point out, in the fact that there is no Euler-Lagrange functional associated to the equation (in contrast to previous work for linear operators and p-Laplace operators (cf. DANIELLI, PETROSYAN, and SHAHGHOLIAN (2003)), which depend heavily on the variational structure of the equation). Therefore, we must take a completely different point of view using tools for equations in non-divergence form. Our notion of solution will be viscosity solution instead of solutions in the sense of distributions.

We will focus on the uniform estimate of the solutions to (E_ϵ), we pass to the limit $(\varepsilon \to 0)$ and we show that, under suitable assumptions, limit functions are viscosity solutions to free boundary problem (1.3). We answer this question and prove interior Lipschitz estimates for the gradient of viscosity solutions to (E_ϵ). Afterwards using the ideas contained in KARAKHANYAN (2006); RICARTE and DA SILVA (2015), we prove an up to boundary uniform gradient estimate for solutions that generalizes this result up to $\partial\Omega$ for smooth enough $\partial\Omega$ and data.

We prove various results concerning the limits of u^ε, or, more precisely, we will study geometric properties of the limit function and its free boundary by establishing the same properties (estimates) for the approximating functions u^ε and its suitable level sets that will approach the free boundary of the limit in the Hausdorff distance.

In the second part we study sharp regularity issues for bounded weak solutions of the inhomogeneous degenerate doubly nonlinear equation (DNLE)

$$u_t - \operatorname{div}(m\,|u|^{m-1}|\nabla u|^{p-2}\nabla u) = f \in L^{q,r}(U_T) \tag{1.6}$$

for $m > 1$ and $p > 2$. The family of equations (1.6) generalizes two well-known cases: the porous media equation (PME), case $p = 2$, and the p-Laplacian equation (PLE), case $m = 1$. For the very particular case $m = 1$ and $p = 2$ we recover the standard heat equation $u_t = \Delta u$.

The main motivation for the study of this class of nonlinear evolution equations is their physical relevance, for example, in the study of non-Newtonian fluids, see LADYZHENSKAYA (1969), plasma physics, ground water problems, image-analysis, motion of viscous fluids and in the modeling of an ideal gas flowing isoentropically in a inhomogeneous porous medium LEIBENSON (1983).

The equation (1.6) exhibits a double nonlinear dependence, on both the solution u and its gradient ∇u that makes diffusion properties degenerate at points where the solution and its gradient vanish. Existence of weak solutions has been proven in STURM (2017a,b). Local boundedness of the gradient for locally bounded, strictly positive weak solutions has been investigated in SILJANDER (2010) and Harnack type inequalites for bounded weak solutions are proved in KINNUNEN and KUUSI (2007); VESPRI (1994). Besides, in IVANOV (1995, 1997); PORZIO and VESPRI (1993); VESPRI (1992), the Hölder regularity for bounded weak solutions is established. Here, we denote $0 < \alpha_* \leq 1$ the optimal Hölder exponent for solutions of the homogeneous case.

Hereafter in this paper we shall denote $U_T \equiv U \times (0, T)$, for a open and bounded set $U \subset \mathbb{R}^n$ and $T > 0$. In (1.6), we shall consider functions $f : U_T \to \mathbb{R}$ such that $f \in L^{q,r}(U_T) := L^r(0,T,L^q(U))$ satisfying conditions

$$\frac{1}{r} + \frac{n}{pq} < 1 \quad \text{and} \quad \frac{3}{r} + \frac{n}{q} > 2. \tag{1.7}$$

The first assumption is due to the standard minimal integrability condition that guarantees the existence of bounded weak solutions. The second one defines the borderline setting for the optimal Hölder regularity regime.

The greatest difficulty in the study of this equation is its doubly degeneracy. To work around this problem we adapt the techniques found in ARAÚJO, MAIA, and URBANO (2017),ARAÚJO, TEIXEIRA, and URBANO (2017a), ARAÚJO, TEIXEIRA, and URBANO (2017b), ARAUJO and ZHANG (2015) to our situation, and show the following result.

Theorem 1.1. *Let u be a locally bounded weak solution of (1.6) in G_1, with $f \in L^{q,r}(U_T)$, satisfying* (1.7)*. Then u is locally of class $C^{0,\beta}$ in space with*

$$\beta = \frac{\alpha(p-1)}{m+p-2}, \quad \textit{for} \quad \alpha = \min\left\{\alpha_\star^-, \frac{(m+p-2)[(pq-n)r-pq]}{q(p-1)[(r-1)(m+p-2)+1]}\right\}. \tag{1.8}$$

Moreover, u is locally $C^{0,\frac{\beta}{\theta}}$ in time for θ given by

$$\theta := p - \alpha(p-1)\left(1 - \frac{1}{m+p-2}\right). \tag{1.9}$$

Theorem 1.1 generalizes the cases studied in ARAÚJO, MAIA, and URBANO (2017); TEIXEIRA and URBANO (2014) where the authors determined the optimal Hölder exponents for weak solutions for the p-laplacian equation and the porous media equation. Such exponents coincide with (1.8) for the cases $p = 2$ and $m = 1$ respectively.

The number β in (1.8) is obtained as follows: in the case

$$\frac{(m+p-2)[(pq-n)r-pq]}{q(p-1)(r-1)[(m+p-2)+1]} < \alpha_*, \tag{1.10}$$

we have the exponent

$$\beta = \frac{(pq-n)r-pq}{q(r-1)[(m+p-2)+1]}.$$

In the case (1.10) is not satisfied, the exponent β is any number less than

$$\frac{\alpha_*(p-1)}{m+p-2} \leq \alpha_\star.$$

Some special borderline scenarios

By making a precise analysis on the exponent in (1.8) it is possible to observe how Hölder regularity for solutions of (1.6) behaves by approaching some integrability borderline cases.

Case $q = r = \infty$

By letting $q, r \to \infty$ we observe that

$$\frac{(m+p-2)[(pq-n)r-pq]}{q(p-1)[(r-1)(m+p-2)+1]} \longrightarrow \frac{p}{p-1} > 1.$$

Therefore after a certain integrability threshold, the optimal regularity exponent of the homogeneous case prevails in (1.8). It implies that solutions of (1.6) are locally $C^{0,\beta}$ for any

$$\beta < \frac{\alpha_*(p-1)}{m+p-2} < \alpha_\star.$$

Case $r = \infty$ **and** $q \searrow n/p$

Here we shall observe for the next two cases, how the Hölder regularity for solutions of (1.6) deteriorates explicitly by approaching the borderline integrability conditions in (1.7). Indeed, by assuming $f \in L^{\infty, \frac{n}{p}+\varepsilon}(U_T)$, Theorem 1.1 provides that for each $\varepsilon > 0$ universally small, solutions for the problem (1.6) are locally $C^{0,\beta(\varepsilon)}$ in space where

$$\beta(\varepsilon) = \frac{\varepsilon}{\frac{n}{p}+\varepsilon} \cdot \frac{p}{m+p-2}.$$

Case $r \searrow 1$ **and** $q = \infty$

By considering $f \in L^{1+\varepsilon, \infty}(U_T)$, Theorem 1.1 guarantees that for each number $\varepsilon > 0$ universally small, solutions are locally $C^{0,\delta(\varepsilon)}$ in space with exponent

$$\delta(\varepsilon) = \frac{\varepsilon(m+p-2)}{\varepsilon(m+p-2)+1} \cdot \frac{p}{m+p-2}.$$

Note that in both cases, $\beta(\varepsilon)$ and $\delta(\varepsilon)$ go to 0 as $\varepsilon \to 0$. In time, solutions are $C^{0,\gamma(\varepsilon)}$ for $\gamma(\varepsilon) = \beta(\varepsilon)/\theta(\varepsilon)$ where $\theta(\varepsilon) \to p$ as $\varepsilon \to 0$ so the exponent $\gamma(\varepsilon)$ also deteriorates as $\varepsilon \to 0$.

According to the second condition in (1.7), we observe that for the last two cases such regularity is optimal.

2 PRELIMINARES

2.1 Inhomogeneous Equations of p-Laplacian Type

The normalized p-Laplacian can be seen as the one-homogeneous version of the standard p-Laplacian and also as a combination of the Laplacian and the normalized infinity Laplacian,

$$\begin{aligned}\Delta_p^N v \quad &:= \quad |Du|^{2-p}\Delta_p u = \Delta u + (p-2)\Delta_\infty^N u \\ &= \quad \Delta u + (p-2)|Du|^{-2}\sum_{i,j} u_{ij}u_iu_j. \end{aligned} \tag{2.1}$$

Recently, a connection between the theory of stochastic tug-of-war games and non-linear equations of p-Laplacian type has been investigated. This connection started with the seminal work PERES *et al.* (2008) . Equations of type (2.1) have been suggested in connection to economics by NYSTROM and PARVIAINEN (2014).

To begin, note that the normalized p-Laplacian can be seen as a uniformly elliptic operator on the set $\mathcal{S}(u) = \{Du(X) \neq 0\}$. Moreover, for $p > 1$, it is easy to see that $\Lambda = \max(p-1, 1)$ and $\lambda = \min(p-1, 1)$. The normalized p-Laplacian enjoys the good properties of being uniformly parabolic and 1-homogeneous, the main difficulty in proving regularity results comes from the discontinuity at $\{Du = 0\}$. This difficulty can be resolved by adapting the notion of viscosity solution using the upper and lower semicontinuous envelopes (relaxations) of the operator, see CRANDALL, ISHII, and LIONS (1992).

Definition 2.1. Let Ω be a bounded domain, $1 < p < \infty$ and $f \in C(\Omega)$. An upper semicontinuous function u is a viscosity subsolution (supersolution) of

$$\Delta_p^N v = f(x) \quad \text{in} \quad \Omega, \tag{2.2}$$

provided that if for all $X_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that $u - \phi$ attains a local maximum (minimum) at X_0, then

$$\begin{cases} \Delta_p^N \phi(x_0) \leq f(x_0) \quad (\text{resp.} \quad \geq 0), \quad \text{if} \quad \nabla\phi(x_0) \neq 0 \\ -\Delta\phi(x_0) + (p-2)\lambda_{\max}(D^2\phi(x_0)) \leq f(x_0) \quad (\text{resp.} \quad \geq 0), \quad \text{if} \quad \nabla\phi(x_0) = 0 \quad \text{and} \quad p \geq 2 \\ -\Delta\phi(x_0) + (p-2)\lambda_{\min}(D^2\phi(x_0)) \leq f(x_0) \quad (\text{resp.} \quad \geq 0), \quad \text{if} \quad \nabla\phi(x_0) = 0 \quad \text{and} \quad 1 < p \leq 2 \end{cases}$$

A function $u \in C(\Omega) \cap L^\infty(\Omega)$ is called a viscosity provided it is both a viscosity subsolution and supersolution. We use up the following notation: given a symmetric matrix $M \in$

$\mathscr{S}(d)$, we shall denote by $\lambda_{\max}$ and $\lambda_{\min}$ its greatest and smallest eigenvalues, that is,

$$\lambda_{\max}(M) = \max_{|\eta|=1}\langle M\xi, \xi\rangle, \quad \lambda_{\min}(M) = \min_{|\eta|=1}\langle M\xi, \xi\rangle.$$

Existence of viscosity solutions of (2.2) has been proved using different techniques, including game-theoretic arguments by MANFREDI, PARVIAINEN, and ROSSI (2010). Let us also mention that the extremal cases $p = 1$ and $p = \infty$ have also received attention. The case $p \to 1$ is known as the mean curvature flow equation

$$\Delta_1^N u : = \Delta u + \frac{\langle D^2 u \cdot Du, Du\rangle}{|Du|^2} = f(x),$$

we refer the reader to the works of Evans and Spruck EVANS and SPRUCK (1991) who state analytical results and point out its connection to evolving hypersurfaces in $\mathbb{R}^n$. For $p \to \infty$ we obtain the normalized ∞-Laplacian

$$\Delta_\infty^N u : = \frac{\langle D^2 u \cdot Du, Du\rangle}{|Du|^2} = f(x).$$

This equation was firs studied by JUUTINEN and KAWOHL (2006a). Most of our discussion will focus on the case $p > 1$ of equation (2.2).

The normalized infinity Laplacian is related to certain geometric problems and was studied by [JUUTINEN and KAWOHL (2006b); LIU and YANG (2015)]. We refer to [EVANS (2007); KOHN and SERFATY (2006)] for game theoretic interpretations of these equations for the elliptic case. Recently, regularity issues for this problem were analyzed in ATTOUCHI, PARVIAINEN, and RUOSTEENOJA (2017) where the authors proved $C^{1,\alpha}$ estimates of viscosity solution to (2.2). More precisely: Assume that $p > 1$ and $f \in L^\infty(\Omega) \cap C(\Omega)$. There exists $\alpha = \alpha(p, n) > 0$ such that any viscosity solution u of (2.2) is in $C^{1,\alpha}_{\text{loc}}(\Omega)$. Moreover, for any $\Omega' \Subset \Omega$, we have

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}\right)$$

where $C = C(p, n, \Omega', \Omega) > 0$.

2.2 Doubly Nonlinear Equation: Energy estimates

We start with the definition of weak solution to (1.6) and we get energy estimates for Doubly Nonlinear equation based on the suitably choice of a function test and then performing various basic techniques from analysis to obtain an estimate of the desired form.

Definition 2.2. A locally bounded function

$$u \in C_{loc}(0,T;L^2_{loc}(U)),\ |u|^{\frac{(m+p-1)}{p}} \in L^p_{loc}(0,T;W^{1,p}_{loc}(U))$$

is a local, weak solution to (1.6), if for every compact set $K \subset U$ and every subinterval $[t_1,t_2] \subset (0,T]$, we have

$$\int_K u\varphi dx\,|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \{-u\varphi_t + m|u|^{m-1}|\nabla u|^{p-2}\nabla u.\nabla\varphi\}dxdt = \int_{t_1}^{t_2}\int_K f\varphi dxdt$$

for all test functions

$$\varphi \in W^{1,2}_{loc}(0,T;L^2(K)) \cap L^p_{loc}(0,T;W^{1,p}_0(K)).$$

All integrals in the above definition are convergent, since the gradient

$$\nabla|u|^{\frac{(m+p-1)}{p}} := \left(\frac{m+p-1}{p}\right)(\text{sgn } u)|u|^{\frac{m-1}{p}}\nabla u.$$

A alternative definition makes use of the Steklov average of a function $v \in L^1(U_T)$, defined for $0 < h < T$ by

$$v_h := \begin{cases} \frac{1}{h}\int_t^{t+h} v(\cdot,\tau)d\tau, & if \quad t \in (0,T-h], \\ 0 & if \quad t \in (T-h,T]. \end{cases} \tag{2.3}$$

Definition 2.3. A locally bounded function

$$u \in C_{loc}(0,T;L^2_{loc}(U)),\ |u|^{\frac{(m+p-1)}{p}} \in L^p_{loc}(0,T;W^{1,p}_{loc}(U))$$

is a local, weak solution to (1.6), if for every compact set $K \subset U$ and every $0 < t < T-h$, we have

$$\int_{K\times\{t\}} \{(u_h)_t\varphi + (m|u|^{m-1}|\nabla u|^{p-2}\nabla u)_h.\nabla\varphi\}dx = \int_{K\times\{t\}} f_h\varphi dx, \tag{2.4}$$

from all nonnegative $\varphi \in W^{1,p}_0(K)$.

One of the main tools we will use is the following Cacciopoli estimate.

Proposition 2.4. *Let u be a weak solution to* (1.6) *and* $K \times [t_1,t_2] \subset U \times [0,T]$. *There* *exists a constant C, depending only on $n, m, p, K \times [t_1, t_2]$, such that*

$$\sup_{t_1<t<t_2} \int_K u^2 \xi^p dx + \int_{t_1}^{t_2} \int_K |u|^{m-1} |\nabla u|^p \xi^p dxdt \leq C \int_{t_1}^{t_2} \int_K u^2 \xi^{p-1} \xi_t dxdt + \int_{t_1}^{t_2} \int_K |u|^{m+p-1} |\nabla \xi|^p dxdt + C\|f\|^2_{L^{q,r}},$$

for all $\xi \in C_0^\infty(K \times (t_1, t_2))$ such that $\xi \in [0, 1]$.

Demonstração. Taking $\varphi = u_h \xi^p$ as a test function in (2.4) and $t \in (t_1, t_2]$ arbitrary, we have

$$\begin{aligned} \int_{t_1}^{t} \int_K (u_h)_t u_h \xi^p dxd\tau &+ \int_{t_1}^{t} \int_K (m|u|^{m-1} |\nabla u|^{p-2} \nabla u)_h . \nabla u_h \xi^p dxd\tau \\ &+ p \int_{t_1}^{t} \int_K (m|u|^{m-1} |\nabla u|^{p-2} \nabla u)_h . \nabla \xi \xi^{p-1} dxd\tau \\ &= \int_{t_1}^{t} \int_K f_h u_h \xi^p dxd\tau. \end{aligned}$$

Integrating by parts and passing to the limit in $h \to 0$, we get

$$\begin{aligned} \int_{t_1}^{t} \int_K (u_h)_t u_h \xi^p dxd\tau &= \frac{1}{2} \int_{t_1}^{t} \int_K (u_h^2)_t \xi^p dxd\tau \\ &\longrightarrow \frac{1}{2} \int_K u^2 \xi^p (x, t) dx - \frac{1}{2} \int_K u^2 \xi^p (x, t_1) dx \\ &- \int_{t_1}^{t} \int_K u^2 \xi^{p-1} \xi_t dxd\tau. \end{aligned}$$

For $h \to 0$, we have

$$\int_{t_1}^{t} \int_K (m|u|^{m-1} |\nabla u|^{p-2} \nabla u)_h \nabla u_h \xi^p dxd\tau \longrightarrow m \int_{t_1}^{t} \int_K |u|^{m-1} |\nabla u|^p \xi^p dxd\tau.$$

Using Young's inequality and $h \to 0$,

$$p \int_{t_1}^{t} \int_K (m|u|^{m-1} |\nabla u|^{p-2} \nabla u)_h u_h \nabla \xi \xi^{p-1} dxd\tau$$

$$\longrightarrow mp \int_{t_1}^{t} \int_K |u|^{m-1} |\nabla u|^{p-2} \nabla u (u \nabla \xi) \xi^{p-1} dxd\tau$$

$$
\begin{aligned}
&\le \quad mp\int_{t_1}^{t}\int_K |u|^{m-1}|\xi\nabla u|^{p-1}|u\nabla\xi|dxd\tau \\
&\le \quad \gamma(m,p)\int_{t_1}^{t}\int_K |u|^{m-1}\xi^p|\nabla u|^p dxd\tau \\
&+ \quad \gamma(m,p)\int_{t_1}^{t}\int_K |u|^{m+p-1}|\nabla\xi|^p dxd\tau.
\end{aligned}
$$

Finally by Hölder inequality, we have

$$
\begin{aligned}
\int_K f_h u_h \xi^p dx &\le ||u_h\xi^p||_{\frac{q}{q-1},K}||f_h||_{q,K} \\
&\le C(K,q)||u_h\xi^p||_{2,K}||f_h||_{q,K} \\
&\le C(K,q)\left(\int_K u_h^2\xi^p dx\right)^{\frac{1}{2}}||f_h||_{q,K},
\end{aligned}
$$

where in the last inequality we use the fact that $\xi^p \ge \xi^{2p}$. Therefore, passing to the limit in $h \to 0$ and using Young's inequality,

$$
\begin{aligned}
\int_{t_1}^{t}\int_K fu\xi^p dxd\tau &\le C(K,q)|t-t_1|^{\frac{r-1}{r}}\left(\int_K u^2\xi^p dx\right)^{\frac{1}{2}}||f||_{L^{q,r}} \\
&\le \frac{1}{2}\int_K u^2\xi^p dx + C(t_1,t,K,q,r)||f||^2_{L^{q,r}}.
\end{aligned}
$$

Taking the supremum over $t \in (t_1, t_2]$ the result follows. $\square$

3 SINGULAR PERTUBATION PROBLEM

In this section we study nonnegative viscosity solution for the boundary-reaction-diffusion problem

$$\begin{cases} |\nabla u^\epsilon|^{2-p} \cdot div(|\nabla u^\epsilon|^{p-2}\nabla u^\epsilon) &= \beta_\epsilon(u^\epsilon) \quad in \quad \Omega \\ u &= \varphi \qquad on \quad \partial\Omega \end{cases} \tag{E_ϵ}$$

The nonlinear reaction term $\beta_\epsilon(t)$ *is of combustion type and is given by* (1.2). *One of our main objectives in this section is to show that the solutions* u^ϵ *of the singular perturbation problem* (E_ϵ), *are Lipschitz continuous up-to the boundary. The main originality of this section is to combine the* $C^{1,\alpha}$ *regularity to* (2.2) *introduced by ATTOUCHI, PARVIAINEN, and RUOSTEENOJA (2017) and the singular perturbation methods in RICARTE and TEIXEIRA (2011); RICARTE and DA SILVA (2015), to get the following results.*

Theorem 3.1 (Interior Uniform Lipschitz Estimate)**.** *Let* $\{u^\epsilon\}_{\epsilon>0}$ *be a viscosity solution of* (E_ϵ). *Given* $\Omega' \Subset \Omega$, *there exists a constant* C_0 *depending on dimension, ellipticity constants and on* Ω', *but independent of* $\epsilon > 0$, *such that*

$$\|\nabla u^\epsilon\|_{L^\infty(\Omega')} \le C_0.$$

Theorem 3.2 (Global uniform Lipschitz estimate)**.** *Let* $\{u^\epsilon\}$ *be a viscosity solution to the singular perturbation problem* (E_ϵ). *Then, if* $\|\varphi\|_{C^{1,\gamma}(\overline{\Omega})} \le \mathcal{A}$, *there exists a constant* $C = C(d, p, \mathcal{A}, \mathcal{B}, \Omega) > 0$, *independent of* ϵ, *such that*

$$\|\nabla u^\epsilon\|_{L^\infty(\overline{\Omega})} \le C_0.$$

3.1 Existence and properties of solutions

In Section we will need the following lemma which proves existence and properties for equations of the type (E_ϵ). *The idea is to obtain a solution of Perron's type, the least supersolution, stated in RICARTE and TEIXEIRA (2011), provides the existence of solutions to* (E_ϵ) *with the initial boundary data* $\varphi \in C^0(\partial\Omega)$. *We state our result independently of the* (E_ϵ) *context since it may be of independent interest.*

Theorem 3.3 (Least supersolution)**.** *Let* $g : [0, \infty) \to \mathbb{R}$ *be a bounded function, Lipschitz. Suppose* $\mathcal{F} : \mathbb{R}^d \times Sym(d) \to \mathbb{R}$ *a operator satisfying the following monotonicity condition*

$$\mathcal{F}(\vec{p}, N) \le \mathcal{F}(\vec{p}, M) \quad whenever \quad N \le M, \tag{3.1}$$

for any $\vec{p} \in \mathbb{R}^d$ *and* $N, M \in Sym(d)$*. If the equation*

$$\mathcal{F}(\nabla u, D^2 u) = g(u) \tag{3.2}$$

admits subsolution and supersolution $\underline{u}, \overline{u} \in C^0(\overline{\Omega})$ *respectively, and* $\underline{u} = \overline{u} = \varphi \in W^{2,\infty}(\partial\Omega)$*, then given the set of functions*

$$\mathscr{S} := \left\{ w \in C(\overline{\Omega}) \;\middle|\; w \text{ is a supersolution to (3.2), and } \underline{u} \leq w \leq \overline{u} \right\},$$

the function

$$v(x) := \inf_{w \in \mathscr{S}} w(x) \tag{3.3}$$

is a continuous viscosity solution to (3.2)*, safisfying* $u = \varphi$ *in* $\partial\Omega$*.*

Demonstração. By looking at the equation (3.2) as

$$\left[\mathcal{F}(\nabla u, D^2 u) - \lambda u\right] + (\lambda u - g(u)) = 0$$

let us denote the following operator

$$\mathcal{G}_f[u] = \mathcal{G}_f(X, u, \nabla u, D^2 u) := \mathcal{G}(\nabla u, D^2 u) - \lambda u + f(X).$$

Observe that $\mathcal{G}_f$ enjoys comparison principle, see for instance BIRINDELLI and DEMENGEL (2007). Also, we define

$$h(z) := \lambda z - g(z) \tag{3.4}$$

for some number $\lambda > 0$ sufficiently large such that $h'(z) \geq \lambda - g'(z) \geq \lambda/2$.

Now, we argue by finite induction. Let us consider $u_0 := \underline{u}$ and for each integer $k \geq 0$, u_{k+1} the solution of

$$\begin{cases} \mathcal{G}_{f_k}(X, u, \nabla u, D^2 u) = 0 & \text{in} \quad \Omega \\ u = \varphi & \text{on} \quad \partial\Omega. \end{cases} \tag{3.5}$$

where $f_k(X) := h(u_k(X))$.

In view of this, we claim for each $k > 0$, $u_k \leq u_{k+1}$ holds in Ω. Indeed, by (3.5) we notice that $\mathcal{G}_{f_0}[u_1] = 0 \leq \mathcal{G}_{f_0}[u_0]$ in the viscosity sense and so, comparison principle implies $u_0 \leq u_1$ in Ω. Now, we suppose $u_{k-1} \leq u_k$ in Ω. By taking $\lambda > 0$ sufficiently large in (3.4), h becomes increasing in the variable z which guarantees $\mathcal{G}_{f_k}[u_{k+1}] = 0 \leq \mathcal{G}_{f_k}[u_k]$ in the viscosity sense. Thence, using comparison principle again we have $u_k \leq u_{k+1}$ in Ω.

Also, we verify $u_k \leq \overline{u}$ holds for each $k > 0$. In fact, for $\overline{f}(X) := h(\overline{u}(X))$ we have $\mathcal{G}_{\overline{f}}[u_1] \geq 0 \geq \mathcal{G}_{\overline{f}}[\overline{u}]$ in the viscosity sense, so $u_1 \leq \overline{u}$ in Ω. By assuming $u_k \leq \overline{u}$ in Ω and taking account that $\mathcal{G}_{\overline{f}}[u_{k+1}] \geq 0 \geq \mathcal{G}_{\overline{f}}[\overline{u}]$ in the viscosity sense, we obtain $u_{k+1} \leq \overline{u}$

in Ω. Therefore, we derive the following increasing sequence

$$\underline{u} = u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_k \leq u_{k+1} \leq \cdots \leq \overline{u} \quad \text{in} \quad \Omega.$$

Besides this fact, by Harnack inequality CHARRO (2013), such sequence is locally bounded in $C^{0,\alpha}$. Then, up to a subsequence $\{u_k\}$ converges locally uniformly to a function u_∞ defined pointwise in Ω. In addition, with no loss of generality, we can assume $\mathcal{G}_{f_k}$ converges locally uniformly to

$$\mathcal{G}_\infty[\,u\,] = \mathcal{G}(\nabla u, D^2 u) - \lambda u + h(u_\infty)$$

and so, u_∞ is a viscosity solution of

$$\mathcal{F}(\nabla u, D^2 u) = g(u) \quad \text{in} \quad \Omega.$$

In order to finish the proof of Theorem 3.3, we check that u_∞ satisfies (3.35). For each $v \in \mathscr{S}$ and $k > 0$, we obtain

$$\mathcal{G}_{f_k}[\,v\,] \;\; = \;\; \mathcal{F}(\nabla v, D^2 v) - (h(v) - h(u_k)) - g(v). \tag{3.6}$$

Inductively, let us analyze the case $k = 0$ in (3.6). Since $u_0 = \underline{u} \leq v$ in Ω, we obtain

$$\mathcal{G}_{f_0}[u_1] = 0 \geq \mathcal{F}(\nabla v, D^2 v) - g(v) = \mathcal{G}_{f_0}[\,v\,]$$

in the viscosity sense. Thus comparison principle implies $u_1 \leq v$ in Ω. Analogously, for $u_k \leq v$ we obtain

$$\mathcal{G}_{f_k}[u_{k+1}] = 0 \geq \mathcal{G}_{f_k}[\,v\,]$$

and so $u_{k+1} \leq v$ in Ω. Therefore for any positive integer k there holds $u_k \leq v$ in Ω and by passing the limit as $k \to \infty$ we achieve

$$u_\infty(x) = \inf_{v \in \mathscr{S}} v(x).$$

□

To finish, the existence of a Perron's solution to

$$\begin{cases} \mathcal{F}(\nabla v, D^2 v) = \beta_\epsilon(v) & \text{in} \quad \Omega \\ \qquad v = \varphi & \text{on} \quad \partial\Omega, \end{cases} \tag{E_ϵ}$$

for $\varphi \in W^{2,\infty}(\partial\Omega)$ *and*

$$\mathcal{F}(Dv, D^2v) \colon = tr\left[\left(I + (p-2)\frac{\nabla\phi \otimes \nabla\phi}{|\nabla\phi|^2}\right) \cdot D^2\phi\right], \tag{3.7}$$

Is ensured as follows: for each $\varepsilon > 0$ *fixed, we choose* $\underline{u}^\varepsilon$ *and* $\overline{u}^\varepsilon$ *respectively as the solutions to the following boundary value problems:*

$$\mathcal{F}(\nabla\underline{u}^\varepsilon, D^2\underline{u}^\varepsilon) = \sup_{[0,\infty)} \beta_\varepsilon \quad \textit{and} \quad \mathcal{F}(\nabla\overline{u}^\varepsilon, D^2\overline{u}^\varepsilon) = \inf_{[0,\infty)} \beta_\varepsilon \quad \textit{in } \Omega, \tag{3.8}$$

satisfying $\underline{u}^\varepsilon = \overline{u}^\varepsilon = \varphi$ *on* $\partial\Omega$. *Existence of solutions to* (3.8) *follows by (BIRINDELLI and DEMENGEL, 2007, propositions 2 and 3). Moreover, by comparison principle, see also (CRANDALL, ISHII, and LIONS, 1992, theorem 3.3), solutions satisfy* $\underline{u}^\varepsilon \leq \overline{u}^\varepsilon$ *in* Ω. *Thus, for each* $\epsilon > 0$ *and* $\varphi \in W^{2,\infty}(\partial\Omega)$, *we then set*

$$u^\epsilon(X) \colon = \inf_{w \in \mathscr{S}} w(X).$$

Then u^ϵ *is a viscosity solution to* (E_ϵ) *and we will refer to* $\{u^\epsilon\}$ *as the family of the least supersolutions of problems* (E_ϵ).

For future reference, we record the properties of u^ϵ *in a Theorem.*

Theorem 3.4. *The least supersolution* u^ϵ *defined as above satisfies the following properties:*

a) $u^\epsilon \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$;
b) $\Delta_p^N u^\epsilon = \beta_\epsilon(u^\epsilon)$ *in* Ω, *in the viscosity sense;*
c) If $\varphi \geq 0$ *in* $\partial\Omega$ *then* $u^\epsilon \geq 0$;
d) there exists a universal constant $\Upsilon > 0$ *such that* $\|u^\varepsilon\|_{L^\infty(\Omega)} \leq \Upsilon$.

Demonstração. For the above discussions, the items a) and b) are proved. To prove c), suppose for the sake of contradiction, that u^ϵ solves (E_ϵ) in the viscosity sense, $u^\epsilon \geq 0$ on $\partial\Omega$ and $\mathcal{N} := \{X \in \Omega \mid u^\epsilon(X) < 0\}$ is nonempty. Clearly $u^\epsilon = 0$ on $\partial\mathcal{N} \cap \Omega$ and, since $u^\epsilon \geq 0$ on $\partial\Omega$, we conclude $u^\epsilon \geq 0$ on $\partial\mathcal{N}$. Now, in view that $\operatorname{supp}(\beta) = [0,1]$, we conclude u^ϵ satisfies $\Delta_p^N u^\epsilon = 0$ in $\mathcal{N}$. Then u^ϵ is also a viscosity solution to the homogeneous equation $\Delta_p u^\epsilon = 0$ in $\mathcal{N}$. Then u^ϵ is also a weak solution to homogeneous p-Laplacian equation (see JUUTINEN, LINDQVIST, and MANFREDI (2001)), which gives a contradiction to the maximum principle and the definition of $\mathcal{N}$. The item d) follows from the Alexandrov-Bakelman-Pucci (ABP) estimate to normalized p-Laplacian operators (see CHARRO (2013)). In fact, let u^ε be any viscosity solution of (E_ϵ) and $v^\varepsilon := u_\varepsilon - \|\varphi\|_\infty$. Note that $v_\varepsilon \leq 0$ on $\partial\Omega$ and

$$\Delta_p^N u^\epsilon = \beta_\epsilon(v^\epsilon) \geq 0.$$

Thus, the ABP estimate ((CHARRO, 2013, Theorem 3)) then implies

$$\sup_{\Omega_T}(v^{\varepsilon})^{+} \le C(p,d).$$

Thus, $u^{\varepsilon} \le \|\varphi\|_{\infty} + C(d,p) =: \Upsilon$. □

3.2 Interior Lipschitz Regularity

We derive interior uniform gradient estimates, which in particular provides compactness in the local uniform convergence topology. The strategy is the following: The proof concerns to analyze the gradient of u^{ϵ} *in two regions. Initially, we analyze the nice region* $\Omega_{\epsilon} :\ = \{y \in \Omega'; 0 \le u^{\epsilon} \le \epsilon\}$. *In this set,* u^{ϵ} *satisfies a inhomogeneous PDE. Thus, this regularity depends upon the priori estimates and Harnack inequality available for the equation* $\Delta_p^N u = f \in L^{\infty}$. *It is shown in ATTOUCHI, PARVIAINEN, and RUOSTEENOJA (2017), that solutions for normalized p-Laplacian type equations have at most* $C^{1,\alpha}$ *regularity. Afterwards we shall control the gradient of* u^{ϵ} *in the transition area* $\Gamma_{\epsilon} = \{u^{\epsilon} = \epsilon\}$. *The universal bound of the gradient of* u^{ϵ} *for points* X_0 *that are close to* Γ_{ϵ}, *in principle blows up when* x_0 *approaches* Γ_{ϵ}. *This step requires a more delicate analysis. The idea is will be to obtain an estimate of* $u^{\epsilon}(X_0)$ *in terms of the distance of dis*(X_0, Γ_{ϵ}). *In order to prove Theorem 3.1, we need to prove first some auxiliary results.*

Lemma 3.5. *Let* v *be a bounded nonnegative solution of*

$$0 \le \Delta_p^N v \le A\chi_{0<v<1}$$

in the ball B_1 *of* $\mathbb{R}^n$, *with* $v(0) \le 1$. *Then there is a constant* $C = C(n,p,A) > 0$ *such that*

$$\|v\|_{L^{\infty}(B_{1/4})} \le C.$$

Demonstração. Indeed, assume the contrary. Then there exists a sequence of functions $\{v_k\}$, $k = 1, 2, \ldots$, satisfying the assumptions of the lemma and such that

$$\max_{\overline{B}_{1/4}} v_k(X) > \frac{4}{3}k.$$

Consider the sets

$$\Omega_k :\ = \{X \in B_1 : v_k(X) > 1\} \quad \text{and} \quad \Gamma_k :\ = \partial\Omega_k \cap B_1.$$

Note thar $\mathcal{L}_p v_k = 0$ in Ω_k and thus $\Delta_p v_k = 0$ in Ω_k. Let now $\delta_k(X) :\ = \operatorname{dist}(X, B_1 \setminus \Omega_k)$ and define

$$\mathscr{P}_k :\ = \left\{X \in B_1 :\ \delta_k(X) \le \frac{1}{3}(1 - |X|)\right\} \supset B_1 \setminus \Omega_k.$$

Observe that $\overline{B}_{1/4} \subset \mathscr{P}_k$. In particular

$$m_k \colon = \sup_{\mathscr{P}_k}(1-|X|)v_k(X) \geq \frac{3}{4}\max_{\overline{B}_{1/4}} v_k(X) > k.$$

Since $v_k(X)$ is bounded (for fixed k), we will have $(1-|X|)v_k(X) \to 0$ as $|X| \to 1$, and therefore m_k will be attained at some point $X_k \in \mathscr{P}_k$:

$$(1-|X_k|)v_k(X_k) = \max_{\mathscr{P}_k}(1-|X|)v_k(X). \tag{3.9}$$

Clearly,

$$v_k(X_k) = \frac{m_k}{1-|X_k|} \geq m_k > k.$$

Since $X_k \in \mathscr{P}_k$, by the definition we will have

$$\delta_k \colon = \delta_k(X_k) \leq \frac{1}{3}(1-|X_k|). \tag{3.10}$$

Let now $Y_k \in \Gamma_k$ be a point where $\delta_k = \operatorname{dist}(X_k, \Gamma_k)$ is realized, so that

$$|X_k - Y_k| = \delta_k. \tag{3.11}$$

Then we will have two inclusions, $B_{2\delta_k}(Y_k) \subset B_1$ and $B_{\delta_k/2}(Y_k) \subset \mathscr{P}_k$, both consequences of (3.10)-(3.11). In particular, for $Z \in B_{\delta_k/2}(Y_k)$ the following inequality holds

$$\begin{aligned}(1-|Z|) &\geq (1-|X_k|) - |X_k - Z| \geq (1-|X_k|) - \frac{3}{2}\delta_k \\ &\geq \frac{1}{2}(1-|X_k|).\end{aligned}$$

This, in conjunction with (3.9), implies that

$$\max_{\overline{B}_{\delta_k/2}} v_k \leq 2v_k(X_k).$$

Next, since $B_{\delta_k}(X_k) \subset \Omega_k$, v_k satisfies $\Delta_p v_k = 0$ in $B_{\delta_k}(X_k)$. By the Harnack inequality for p-harmonic functions there is a constant $c = c(d,p) > 0$ such that

$$\min_{\overline{B}_{3\delta_k/4}(X_k)} v_k \geq c v_k(X_k).$$

In particular,

$$\max_{\overline{B}_{\delta_k/4}(Y_k)} v_k \geq c v_k(X_k).$$

Further, define

$$w_k(X) \colon = \frac{v_k(Y_k + \delta_k X)}{v_k(X_k)} \quad \text{for} \quad X \in B_2.$$

Summarizing the properties of v_k above, we see that w_k satisifes the following system

$$\begin{cases} 0 \le \Delta_p^N w_k \le \frac{\delta_k^p}{k^{p-1}} & \text{in} \quad B_2 \\ \max_{\overline{B}_{1/2}} w_k \le 2, \, \max_{\overline{B}_{1/4}} w_k \ge c > 0 \\ w_k \ge 0, \, w_k(0) \le \frac{1}{k} \end{cases}$$

Therefore, from a priori estimates (see (ATTOUCHI, PARVIAINEN, and RUOSTEENOJA, 2017, Theorem 1.1)), we can conclude that a subsequence of $\{w_k\}$ will converge in $C^{1,\alpha}$ norm on every compact subset of $B_{1/2}$ to a function w_0 that satisfies

$$\begin{cases} \Delta_p w_0 = 0 & \text{in} \quad B_{1/2} \\ \max_{\overline{B}_{1/4}} w_0 \ge c > 0, \\ w_0 \ge 0, \, w_0(0) = 0 \end{cases}$$

This, however, contradicts the strong maximum principle for p-harmonic functions. The lemma is proved. □

Lemma 3.6. *Let u_ε be a viscosity solution of (E_ϵ) in $B_{r_0}(X_0)$ such that $u^\varepsilon(X_0) \le 2\epsilon$. Then, there exists $C = C(d, r_0, p, \|\beta\|_\infty)$ such that, if $\epsilon \le 1$,*

$$|\nabla u^\varepsilon(X_0)| \le C.$$

Demonstração. Define the auxiliary function

$$v(Y) := \frac{1}{\epsilon} u^\epsilon(X_0 + \epsilon Y) \quad \text{in} \quad B_1.$$

Then if $\epsilon \le 1$, direct computations show that v satisfies

$$\Delta_p^N v = \beta(v) \quad \text{in} \quad B_1,$$

in the viscosity sense. Indeed, let $P(Y)$ be a paraboloid touching v, at some point Z_0, by below. So,

$$v(Z_0) = P(Z_0) \quad and \quad P(Y) < v(Y) \quad \forall \ Y \ne Z_0.$$

Then, $\tilde{P}(X) = \varepsilon P\left(\frac{X - X_0}{\varepsilon}\right)$ touching u^ε by below at $X_1 = X_0 + \varepsilon Z_0$, because

$$\tilde{P}(X_1) = \varepsilon P\left(\frac{X_1 - X_0}{\varepsilon}\right) = \varepsilon v(Z_0) = u^\varepsilon(X_1),$$

and for all $X \neq X_1$,

$$\tilde{P}(X) = \varepsilon P\left(\frac{X - X_0}{\varepsilon}\right) < \varepsilon v\left(\frac{X - X_0}{\varepsilon}\right) = u^{\varepsilon}(X).$$

As u^{ε} is the sense solution viscosity of (E_{ϵ}) we have

1. $\Delta_p^N \tilde{P}(X_1) \leq \beta_{\varepsilon}(u^{\varepsilon}(X_1))$, if $D\tilde{P}(X_1) \neq 0$. That is,

$$\Delta \tilde{P}(X_1) + \frac{(p-2)}{|D\tilde{P}(X_1)|^2} \sum_{i,j=1}^{n} D_{ij}\tilde{P} \cdot D_i\tilde{P} \cdot D_j\tilde{P} \leq \beta_{\varepsilon}(u^{\varepsilon}(X_1)) \tag{3.12}$$

Futhermore direct computation revels that

$$\begin{aligned} D_i\tilde{P}(X_1) &= D_iP(Z_0) \\ D_{ij}\tilde{P}(X_1) &= \frac{1}{\varepsilon} D_{ij}P(Z_0) \end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), we end up with

$$\Delta P(Z_0) + \frac{(p-2)}{|DP(Z_0)|^2} \sum_{i,j=1}^{n} D_{ij}P \cdot D_iP \cdot D_jP \leq \varepsilon\beta_{\varepsilon}(\varepsilon v(Z_0))$$

2. $\Delta \tilde{P}(X_1) - (p-2)\lambda_{\max}(D^2\tilde{P}(X_1)) \leq \beta_{\varepsilon}(u^{\varepsilon}(X_1))$, if $D\tilde{P}(X_1) = 0$ and $p \geq 2$. That is,

$$\begin{aligned} \Delta P(Z_0) - (p-2)\lambda_{\max}(D^2P(Z_0)) &= \varepsilon\Delta\tilde{P}(X_1) - \varepsilon(p-2)\lambda_{\max}(D^2\tilde{P}(X_1)) \\ &\leq \varepsilon\beta_{\varepsilon}(\varepsilon v(Z_0)). \end{aligned}$$

Taking a paraboloid touching v by above and arguing similarly, the result follows.
Thus, from the $C^{1,\alpha}$ regularity estimates (cf. ATTOUCHI, PARVIAINEN, and RUOSTEENOJA (2017), Theorem 1.1), we have

$$|\nabla v(0)| \leq C\{\|v\|_{L^{\infty}(B_{1/2})} + \|\beta\|_{\infty}\}, \tag{3.14}$$

for some universal constant $C > 0$. Since,

$$v(0) = \frac{1}{\epsilon} u^{\epsilon}(X_0) \leq 2,$$

it follows by Lemma 3.5 that

$$\|v\|_{L^{\infty}(B_{1/2})} \leq C, \tag{3.15}$$

for a universal constant $C > 0$. Combining (3.14) and (3.15) we get

$$|\nabla u^\epsilon(X_0)| = |\nabla v(0)| \leq C_0, \tag{3.16}$$

for some $C_0 > 0$ independent of ϵ. □

Lemma 3.7. *Let u^ϵ be a viscosity solution of (E_ϵ) in B_1 and $0 \in \partial\{u^\epsilon > \epsilon\}$. Then, for $X \in B_{1/4} \cap \{u^\epsilon > \epsilon\}$,*

$$u^\epsilon(X) \leq \epsilon + C\,dist(X, \{u^\epsilon \leq \epsilon\} \cap B_1),$$

with $C = C(d, p, \|\beta\|_\infty) > 0$.

Demonstração. For $X_0 \in B_{1/4} \cap \{u^\epsilon > \epsilon\}$ take, $m_0 = u^\epsilon(X_0) - \epsilon$ and

$$r\colon = \operatorname{dist}(X_0, \{u^\epsilon \leq \epsilon\} \cap B_1).$$

Since $0 \in \partial\{u^\epsilon > \epsilon\} \cap B_1$, we have that $r \leq 1/4$. We want to prove that,

$$m_0 \leq C(p, d, \|\beta\|_\infty) \cdot r.$$

Let us label

$$\mathcal{I} := \inf_{B_{r/2}(X_0)} (u^\epsilon - \epsilon).$$

Denote by $h^\epsilon(X) = u^\epsilon(X) - \epsilon$. Since, $B_r(X_0) \subset \{u^\epsilon > \epsilon\} \cap B_1$ then $h^\epsilon > 0$ in $B_r(X_0)$. Thus, we have that

$$\Delta_p^N h^\epsilon = 0 \quad \text{in} \quad B_r(X_0).$$

Thus, h^ϵ is also a viscosity solution (is also a weak solution) to the homogeneous p-Laplacian equation $\Delta_p h^\epsilon = 0$ in $B_r(X_0)$. Therefore, by Harnack's inequality there exists $c_1 = c_1(d, p) > 0$ such that,

$$\mathcal{I} = \inf_{B_{r/2}(X_0)} (u^\epsilon - \epsilon) \geq c_1 \sup_{B_{r/2}(X_0)} (u^\epsilon - \epsilon) \geq c_1 m_0.$$

For $\mu \gg 1$, define the auxiliary function in $B_r \setminus B_{r/2}$ by

$$\Psi(X) := e^{-\mu|X|^2} - e^{-\mu r^2}. \tag{3.17}$$

Then, by Lemma 7.2,

$$\Delta_p^N \Psi\colon = F_p\left(D^2\Psi, D\Psi\right) > 0 \quad \text{in} \quad B_r \setminus B_{r/2}$$

for $\mu = \frac{(d+p-2)}{2(p-1)r^2}$, where F_p is as in (7.6). Let now

$$\varrho(X) = c_2 m_0 \Psi(X - X_0) \quad \text{for} \quad X \in \overline{B_r(X_0)} \setminus B_{r/2}(X_0).$$

Then, again by Lemma 7.2, we have that, if we choose c_2 conveniently depending on d, p,

$$\begin{cases} F_p\left(D^2\rho, D\Psi\right) > 0, & \text{in} \quad \overline{B_r(X_0)} \setminus B_{r/2}(X_0) \\ \varrho(X) = 0, & \text{on} \quad \partial B_r(X_0) \\ \varrho(X) = c_1 m_0, & \text{on} \quad \partial B_{r/2}(X_0) \end{cases}$$

then

$$\varrho(X) = 0 \leq h_\epsilon \quad \text{on} \quad \partial B_r(X_0) \quad \text{and} \quad \varrho(X) = c_1 m_0 \leq h_\epsilon \quad \text{on} \quad \partial B_{r/2}(X_0),$$

by the comparison principle (see CRANDALL, ISHII, and LIONS (1992)) we have,

$$\varrho(X) \leq u^\epsilon(X) - \epsilon \quad \text{in} \quad \overline{B_r(X_0)} \setminus B_{r/2}(X_0). \tag{3.18}$$

Take $Z_0 \in \partial B_r(X_0) \cap \partial\{u^\epsilon > \epsilon\}$, then $Z_0 \in \overline{B_{1/2}}$ and

$$\varrho(Z_0) = u^\epsilon(Z_0) - \epsilon = 0. \tag{3.19}$$

Finally, by (3.16), (3.18) and (3.19) we have that,

$$|\nabla \varrho(Z_0)| \leq |\nabla u^\epsilon(Z_0)| \leq c_3.$$

On the other hand $|\nabla \varrho(Z_0)| = c_2 m_0 e^{-\mu r^2} 2\mu r \leq c_3$. Therefore,

$$m_0 \leq \frac{c_3 e^{\frac{(d+p-2)r^2}{2(p-1)r^2}}}{2\left(\frac{d+p-2}{2(p-1)r^2}\right) c_2 r} = \frac{(p-1)c_3 e^{\frac{d+p-2}{2(p-1)}}}{c_2(d+p-2)} \cdot r$$

and the result follows. □

Now, we can prove the main result of this section, Theorem 3.1.

Demonstração. Assume without loss of generality that $0 \in \partial\{u^\epsilon > \epsilon\}$. By Lemma 3.6 we know that if $X_0 \in \{u^\epsilon \leq 2\epsilon\} \cap B_{3/4}$ then,

$$|\nabla u^\epsilon(X_0)| \leq C_0$$

with $C_0 = C_0(d, p, \|\zeta\|_\infty)$. We now proceed our analysis to cover the open region $\{u^\epsilon > \epsilon\} \cap B_{1/8}$. For that, let us label

$$\Gamma_\epsilon := \{X \in \Omega' / u^\epsilon(X) = \epsilon\},$$

and fix a generic point X_1 inside $\{\epsilon < u^\epsilon\} \cap B_{1/8}$. In then, we compute the distance from X_1 to Γ_ϵ and call such a number r, i.e.,

$$r := \operatorname{dist}(X_1, \Gamma_\epsilon).$$

As $0 \in \partial\{u^\epsilon > \epsilon\}$ we have that $r \le 1/8$. Therefore $B_r(X_1) \subset \{u^\epsilon > \epsilon\} \cap B_{1/4}$ and then $\Delta_p^N u^\epsilon = 0$ in $B_r(X_1)$ and, by Lemma 3.7,

$$u^\epsilon(X) \le \epsilon + C_1 \cdot \operatorname{dist}(X, \{u^\epsilon \le \epsilon\}) \quad \text{in} \quad B_r(X_1). \tag{3.20}$$

Suppose that $\epsilon < \bar{c}r$ with $\bar{c}$ to be determined. Define the renormalized function $v_r \colon B_1 \to \mathbb{R}$ as

$$v_r(Y) := \frac{u^\epsilon(X_1 + rY) - \epsilon}{r}.$$

One easily verifies that v_r solves

$$\Delta_p^N v_r = r\beta_\epsilon(u^\epsilon(X_1 + rY)) =: \mathfrak{g}(Y),$$

in the viscosity sense. From geometric consideration, $u^\epsilon(X_1 + rY) > \epsilon$, for all $Y \in B_1$, thus, it follows from (1.1) that $\mathfrak{g}(Y) \equiv 0$. Thus,

$$\Delta_p v_r = 0 \quad \text{in} \quad B_1,$$

in the weak sense. Applying $C^{1,\alpha}$ regularity estimates for degenerate homogeneous equations (see LADYZHENSKAYA and URAL'TSEVA (1968)), we conclude

$$\begin{aligned} |\nabla u^\epsilon(X_1)| &= |\nabla v_r(0)| \le \frac{C}{r}\|u^\epsilon - \epsilon\|_{L^\infty(B_{r/2}(X_1))} \\ &\le \frac{C}{r}(\epsilon + \tilde{C}r) \le C(\bar{c} + \tilde{C}). \end{aligned} \tag{3.21}$$

Now suppose that $\epsilon \ge \bar{c}r$. By (3.33) we have

$$u^\epsilon(X_1) \le \epsilon + C_1 \cdot r \le \left(1 + \frac{C_1}{\bar{c}}\right)\epsilon < 2\epsilon,$$

if we choose $\bar{c}$ lager enough. By Lemma 3.6, we have

$$|\nabla u^\epsilon(X_1)| \le C(d, p, \|\beta\|_\infty).$$

□

3.3 Regularity result up to the boundary

In this subsection we shall prove that the solutions of the equations considered in the previous sections are $C^{0,1}$ up to the boundary if the data are sufficiently regular. The idea of the proof is to consider $C^{1,\alpha}$ Dirichlet data. It is shown in LADYZHENSKAYA and URAL'TSEVA (1968), that solutions to Dirichlet problem for p-Laplacian type equations have at most $C^{1,\alpha}$ regularity, for some $\alpha \in (0,1)$. Therefore our assumptions on the boundary data are optimal. More precisely, we shall prove a uniform gradient estimate up to the boundary for viscosity solutions of the singular perturbation problem (E_ϵ), where $0 \leq \varphi \in C^{1,\gamma}(\overline{\Omega})$, with $0 < \gamma < 1$, and, a bounded $C^{1,1}$ domain Ω (or $\partial\Omega$ for short). Throughout this paper we will assume the following bounds: $\|\varphi\|_{C^{1,\gamma}(\overline{\Omega})} \leq \mathcal{A}$ and $\|\beta\|_{L^\infty([0,1])} \leq \mathcal{B}$.

We make a pause to discuss some remarks which will be important throughout this work. Firstly, it is important to highlight that is always possible to perform a change of variables to flatten the boundary. Indeed, if $\partial\Omega$ is a $C^{1,1}$ set, the part of Ω near $\partial\Omega$ can be covered with a finite collection of regions that can be mapped onto half-balls by diffeomorphisms (with portions of $\partial\Omega$ being mapped onto the "flat"parts of the boundaries of the half-balls). Hence, we can use a smooth mapping, reducing this way the general case to that one on B_1^+, and, the boundary data would be given on $B_1 \cap \{X_d = 0\}$. We shall introduce some notations which will use throughout subsection.

- $\Gamma_X\colon = \{Y \in H_+ : |Y - \hat{Y}| \geq \frac{1}{2}|Y - X|\}$ *for* $X \in T\}$, *where* $H_+ = \{X_d > 0\}$ *and* $T = \{X_d = 0\}$.
- $B_r^+(X)\colon = B_r(X) \cap H_+$.
- $B'_r(X)$ *is the ball with center at* X *and radius* r *in* T.

We will now establish a universal bound for the Lipschitz norm of u^ε up to the boundary, Theorem 3.2. The proof will be divided into two cases.

Case 1: *Lipschitz regularity up to the boundary in the region $\{0 \leq u^\varepsilon \leq \varepsilon\}$.*

Proposition 3.8. *Let u^ϵ be a viscosity solution to (E_ϵ). For $X \in \{0 \leq u^\varepsilon \leq \varepsilon\} \cap B_{1/2}^+$ there exists a universal constant $C_1 > 0$ independent of ϵ such that*

$$|\nabla u^\epsilon(X)| \leq C_1.$$

Demonstração. We denote by

$$\delta(X)\colon = \mathrm{dist}(X, \{X_d = 0\})$$

the vertical distance and

$$\hat{X} = \mathrm{Proj}_T X$$

is the vertical projection of X on T. If $\delta(X) \geq \epsilon$, then $B_\varepsilon(X) \subset B_1^+$ for $\varepsilon \ll 1$. Therefore, by Theorem 3.1, there is a universal constant $C_0 > 0$ independent of ϵ, such that

$$|\nabla u^\epsilon(X)| \leq C_0.$$

On the other hand, if $\delta(X) < \epsilon$, then it is sufficient to prove that there exists a universal constant $C_0 > 0$ independent of ε, such that

$$u^\epsilon(\hat{X}) \leq C_0 \epsilon. \tag{3.22}$$

Indeed, suppose that (3.22) holds. Consider $v\colon \overline{B}_1^+ \to \mathbb{R}$ to be the viscosity solution to the Dirichlet problem

$$\begin{cases} \Delta_p^N v = 0, & \text{in} \quad B_1^+ \\ \quad v = u^\epsilon, & \text{on} \quad \partial B_1^+. \end{cases}$$

Then v is also a weak solution to

$$\begin{cases} \operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0, & \text{in} \quad B_1^+ \\ \qquad\qquad v = u^\epsilon, & \text{on} \quad \partial B_1^+. \end{cases} \tag{3.23}$$

Thus, by results in LADYZHENSKAYA and URAL'TSEVA (1968), $v \in C^{1,\gamma}(\overline{B}^+_{3/4})$, for some $\gamma \in (0,1)$, with the following estimate

$$|\nabla v| \leq c(\|v\|_{L^\infty(B_1^+)} + \|\varphi\|_{C^{1,\gamma}}) \leq C \quad \text{in} \quad B^+_{3/4}$$

and by comparison principle we have $u_\epsilon \leq v$. Hence, it follows from assumption (3.22) that

$$u^\epsilon(Y) \leq v(Y) \leq v(\hat{X}) + C|Y - \hat{X}| \leq C\epsilon, \quad \text{if } Y \in B^+_{2\epsilon}(\hat{X})$$

Then, again applying $C^{1,\alpha}$ estimates, we obtain

$$|\nabla u^\epsilon(X)| \leq C_0(d, p, \mathcal{B}).$$

In order to prove (3.22) suppose there exists $\epsilon > 0$ such that

$$u^\epsilon(\hat{X}) \geq k\epsilon \quad \text{for} \quad k \gg 1.$$

We shall denote $r_0 := \operatorname{dist}(\hat{X}, \{0 \leq u^\varepsilon \leq \varepsilon\})$. Consider $X_0 \in \{0 \leq u^\varepsilon \leq \varepsilon\} \cap \partial B_{r_0}(\hat{X})$ a point to which the distance is realized $r_0 = |X_0 - \hat{X}|$. Thereafter, let $\Gamma_{\hat{X}}$ be the cone with vertex at $\hat{X} \in T$.

1. $X_0 \in \Gamma_{\hat{X}}$.

We have to $B_{r_0/2}(X_0) \subset B_1^+$. Now, let us define, $v^\epsilon : B_1 \to \mathbb{R}$ by

$$v^\epsilon(Y) := \frac{u^\epsilon(X_0 + (r_0/2)Y)}{\epsilon}.$$

Therefore, v^ϵ satisfies

$$|\nabla v|^{2-p} \cdot \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) = \frac{1}{\epsilon^2}\left(\frac{r_0}{2}\right)^2 \beta(v^\epsilon) \colon = \mathfrak{g}(Y),$$

where $\mathfrak{g}(Y) \colon = \epsilon^{-2}(r_0/2)^2\beta(v^\epsilon) \in L^\infty(B_1) \cap C(B_1)$, since $r_0 < \epsilon$.
Moreover, since $v^\epsilon(0) \leq 1$ it follows from Harnack inequality that $v^\epsilon(Y) \leq c$ for $Y \in B_{1/2}$, i.e.,

$$u^\epsilon(X) \leq c\epsilon, \quad X \in B_{r_0/4}(X_0).$$

Consider now $Z \in B'_{r_0}(\hat{X})$. It follows that

$$\varphi(Z) \geq \varphi(\hat{X}) - \mathcal{A} \cdot |Z - \hat{X}| \geq k\epsilon - r_0 \cdot \mathcal{A} \geq (k - \mathcal{A})\epsilon$$

since $r_0 < \epsilon$. Define the scaled function $w^\epsilon : B_1^+ \to \mathbb{R}$,

$$w^\epsilon(Y) := \frac{u^\epsilon(\hat{X} + r_0 Y)}{\epsilon}.$$

It readily follows that $\Delta_p^N w^\epsilon = 0$ in B_1^+ in the viscosity sense. Then w^ϵ is a weak solution to

$$\operatorname{div}\left(|\nabla w^\epsilon|^{p-2}\nabla w^\epsilon\right) = 0 \quad \text{in } B_1^+ \quad \text{and} \quad w^\epsilon \geq k - \mathcal{A} \quad \text{on } B'_1.$$

Therefore according to Lemma 2.1 in KARAKHANYAN (2006),

$$w^\epsilon \geq c(k - \mathcal{A}) \quad \text{in} \quad B^+_{3/4}.$$

In other words, we have reached that

$$u^\epsilon(X) \geq c\epsilon(k - \mathcal{A}) \quad \text{in} \quad B^+_{3r_0/4}(\hat{X}).$$

Hence

$$c\epsilon(k - \mathcal{A}) \leq u^\epsilon(\xi) \leq c\epsilon, \ \forall \, \xi \in \partial B_{3r_0/4}(\hat{X}) \cap \partial B_{r_0/4}(X_0)$$

which leads to a contradiction for $k \gg 1$.

2. $X_0 \notin \Gamma_{\hat{X}}$.

Choose $X_1 \in \{u^\varepsilon \le \varepsilon\}$ such that

$$r_1 = \operatorname{dist}(\hat{X}_0, \{u^\varepsilon \le \varepsilon\}) = |\hat{X}_0 - X_1|.$$

We have

$$r_1 \le |\hat{X}_0 - X_0| \le \frac{|X_0 - \hat{X}|}{2} = \frac{r_0}{2}. \tag{3.24}$$

From triangular inequality and (3.24) we have

$$|X_1 - \hat{X}| \le |X_1 - \hat{X}_0| + |\hat{X}_0 - \hat{X}| \le r_1 + r_0 \le \frac{r_0}{2} + r_0.$$

If $X_1 \in \Gamma_{\hat{X}_0}$ the result follows from previous analysis. Otherwise, let X_2 be such that

$$r_2 = \operatorname{dist}(\hat{X}_1, \{u^\varepsilon \le \varepsilon\}) = |\hat{X}_1 - X_2|.$$

As before we have

$$r_2 \le |\hat{X}_1 - X_1| \le \frac{|X_1 - \hat{X}_0|}{2} = \frac{r_1}{2} \le \frac{r_0}{4},$$

and so

$$|X_2 - \hat{X}| \le |\hat{X}_1 - X_2| + |\hat{X}_1 - \hat{X}| \le \frac{r_0}{4} + \frac{r_0}{2} < r_0.$$

Observe that this process must finish other a finite number of steps. Indeed, suppose that we have a sequence of points $X_j \in \partial\{u^\varepsilon \le \varepsilon\}$, $X_{j+1} \notin \Gamma_{\hat{X}_j}$ $(j = 1, 2, \dots)$ satisfying, $r_{j+1} \colon = \operatorname{dist}(\hat{X}_j, \{u^\varepsilon \le \varepsilon\}) = |X_{j+1} - \hat{X}_j|$ and

$$r_{j+1} \le \frac{r_j}{2} \le \frac{r_0}{2^{j+1}}. \tag{3.25}$$

Thus, it follows from (3.25) that

$$|X_j - \hat{X}| \le r_0 + r_0 \sum_{i=1}^{j} \frac{1}{2^i} \le 2r_0.$$

Therefore, up to a subsequence, $X_j \to \xi \in B'_{2r_0}(\hat{X})$ with $\varphi(\xi) = \varepsilon$. However,

$$\varphi(\xi) \ge \varphi(\hat{X}) - \mathcal{A} \cdot |\hat{X} - \xi| \ge \varepsilon(k - 2\mathcal{A}) \gg \varepsilon$$

for $k \gg 1$ which leads to a contradiction, and, hence the assertion (3.22) is proved. □

Case 2: *Lipschitz regularity in the region* $B^+_{1/8} \setminus \{u^\varepsilon \le \varepsilon\}$.

Lemma 3.9. *For* $X \in B'_{1/4}$ *with* $u^\epsilon(X) > \epsilon$, *there exists a constant* $c_0 = c_0(d, p) > 0$ *such* *that*

$$\varphi(X) \leq \epsilon + c_0 \cdot \delta_\varepsilon(X),$$

Demonstração. Let us suppose, for the sake of contradiction, that there exists an $\epsilon > 0$ and $X_0 \in B'_{1/4} \setminus \{u^\varepsilon \leq \varepsilon\}$ such that

$$\varphi(X_0) \geq \epsilon + k \cdot \delta_\varepsilon(X_0)$$

holds for $k \gg 1$, large enough. Let $Z = Z_\epsilon \in \partial\{u^\varepsilon \leq \varepsilon\}$ be a point that realizes the distance i.e. $\delta_\varepsilon \colon = \delta_\epsilon(X_0) = |X_0 - Z|$. We have two cases to analyze: If $Z \in \Gamma_{X_0}$, then the normalized function $v^\epsilon \colon B_1^+ \to \mathbb{R}$ given by

$$v^\epsilon(Y) := \frac{u^\epsilon(X_0 + \delta_\epsilon Y) - \varepsilon}{\delta_\epsilon},$$

satisfies $\operatorname{div}(|\nabla v^\epsilon|^{p-2}\nabla v^\epsilon) = 0$ in B_1^+ in the weak sense. Moreover, $v^\epsilon(Y) \geq 0$ in B_1^+. Now, for any $X \in B'_{\delta_\epsilon}(X_0)$ we should have for $k \gg 1$,

$$\begin{aligned} \varphi(X) &\geq \varphi(X_0) - \mathcal{A}\delta_\varepsilon \geq \varepsilon + k\delta_\varepsilon - \mathcal{A}\delta_\varepsilon \\ &\geq \varepsilon + \frac{k}{2}\delta_\varepsilon, \end{aligned}$$

i.e,

$$\frac{\varphi(X_0 + \delta_\varepsilon Y) - \varepsilon}{\delta_\varepsilon} \geq \frac{k}{2} \quad \text{in } B'_1$$

In other words, $v^\epsilon(Y) \geq ck$ for all $Y \in B'_1$. Hence, from Lemma 2.1 in KARAKHANYAN (2006), we have that $v^\epsilon \geq ck$ in $B^+_{3/4}$ in a more precise manner,

$$u^\epsilon(X) \geq \epsilon + Ck\delta_\epsilon, \quad X \in B^+_{3\delta_\epsilon/4}(X_0). \tag{3.26}$$

From now on, let us consider $\tilde{B} \colon = B_{\frac{\delta_\epsilon}{4}}(P)$, where $P = P_\epsilon := Z + \frac{X_0 - Z}{4}$. If we define $\omega^\varepsilon \colon = u^\epsilon - \epsilon$, then since $Z \in \partial\tilde{B}$, it follows that

$$\operatorname{div}\left(|\nabla\omega^\epsilon|^{p-2}\nabla\omega^\epsilon\right) = 0 \text{ in } \tilde{B}, \tag{3.27}$$

$$\omega^\epsilon(Z) = u^\varepsilon(Z) - \varepsilon = 0, \tag{3.28}$$

$$\frac{\partial\omega^\varepsilon}{\partial\nu}(Z) \leq |\nabla\omega^\epsilon(Z)| \leq C. \tag{3.29}$$

Therefore, from (3.27)-(3.29) we can apply Lemma 7.3, which gives $\omega^\epsilon(P) \leq C_0 \cdot \delta_\varepsilon$, i.e.,

$$u^\epsilon(P) \leq \epsilon + C\delta_\epsilon. \tag{3.30}$$

At a point P on $\partial B_{3\delta_\epsilon/4}(X_0)$ we have (according to (3.26) and (3.30))

$$\epsilon + kc\delta_\epsilon \leq u^\epsilon(P) \leq \varepsilon + C_0\delta_\epsilon$$

which gives a contradiction if k is chosen large enough. The second case, namely $Z \notin \Gamma_{X_0}$, it is treated similarly as in Theorem 3.8 and we omit the details here. □

Proposition 3.10. *Let u^ε be a viscosity solution to (E_ϵ). Suppose that $X \in B^+_{1/8}$ satisfies $u^\varepsilon(X) > \varepsilon$, then there exists a constant $C_0 = C_0(d, p, \mathcal{A}) > 0$ such that*

$$|\nabla u^\varepsilon(X)| \leq C_0.$$

Demonstração. The proof of the proposition consists of analysing three possible cases. We use the following notation

$$\delta_\varepsilon(X) \colon = \mathrm{dist}(X, \{u^\varepsilon \leq \varepsilon\}) \quad \text{and} \quad \delta(X) \colon = \mathrm{dist}(X, \{X_d = 0\}).$$

a) If $\delta_\varepsilon(X) \leq \delta(X)$, then there is a universal constant $C_0 > 0$, such that

$$|\nabla u^\epsilon(X)| \leq C_0.$$

In fact, we may assume with no loss of generality that $\delta_\epsilon(X) \leq \frac{1}{8}$. Otherwise, if we suppose that $\delta_\epsilon(X) > \frac{1}{8}$, then the result would follow from Theorem 3.1. From now on, we select $X_\epsilon \in \partial\{u^\varepsilon \leq \varepsilon\}$ a point which realizes distance, i.e., $\delta_\epsilon \colon = \delta_\varepsilon(X) = |X - X_\epsilon|$. Since

$$|X_\epsilon| \leq |X| + \delta_\epsilon \leq \frac{1}{4},$$

we must have that $X_\varepsilon \in B^+_{1/4} \cap \{u^\varepsilon \leq \varepsilon\}$. This way, applying Theorem 3.8, there exists a constant $C_1 = C(d, p, \mathcal{B}, \mathcal{A}) > 0$ such that

$$|\nabla u^\epsilon(X_\epsilon)| \leq C_1.$$

Defining the re-normalized function $v^\epsilon : B_1 \to \mathbb{R}$ as

$$v^\epsilon(Y) := \frac{u^\epsilon(X + \delta_\epsilon Y) - \epsilon}{\delta_\epsilon}.$$

As before v^ϵ satisfies

$$\operatorname{div}(|\nabla v^\epsilon|^{p-2}\nabla v^\epsilon) = 0 \text{ in } B_1, \tag{3.31}$$

$$v^\epsilon(Y_\epsilon) = 0, \tag{3.32}$$

$$|\nabla v^\epsilon(Y_\epsilon)| \leq C_1, \tag{3.33}$$

$$v^\epsilon(Y) \geq 0 \text{ for } Y \in B_1, \tag{3.34}$$

where $Y_\epsilon := \frac{X_\epsilon - X}{\delta_\epsilon} \in \partial B_1$. From (3.31)-(3.34) we are able to apply Lemma 2.2 of KARAKHANYAN (2006) and conclude that there exists a universal constant $c > 0$ such that $v^\varepsilon(0) \leq c$. Moreover, from Harnack inequality $v^\varepsilon \leq C_0$ in $B_{1/2}$. Therefore, by $C^{1,\alpha}$ regularity estimates (see ATTOUCHI, PARVIAINEN, and RUOSTEENOJA (2017)) we must have that

$$|\nabla u^\varepsilon(X)| = |\nabla v^\varepsilon(0)| \leq \frac{1}{\delta_\varepsilon}\|u^\varepsilon - \varepsilon\| \leq C_0,$$

and the Lemma is proved.

b) If $\delta(X) < \delta_\varepsilon(X) \leq 4\delta(X)$, then

$$|\nabla u^\epsilon(X)| \leq C_0$$

for some constant $C_0 = C_0(d, p, \mathcal{B}, \mathcal{A}) > 0$. In fact, similar to (a)), we may assume that $\delta_\epsilon \leq \frac{1}{8}$, otherwise, as in Theorem 3.1, the gradient bounded follows from local estimates. Define the scaled function $v^\epsilon \colon B_1 \to \mathbb{R}$ by

$$v^\epsilon(Y) := \frac{u^\epsilon(X + \delta Y) - \epsilon}{\delta}.$$

From Harnack inequality

$$v^\varepsilon \leq C v^\varepsilon(0) \sim \frac{1}{\delta} \quad \text{in } B_{1/2}.$$

Applying once more $C^{1,\alpha}$ regularity estimates, we obtain

$$|\nabla u^\epsilon(X)| = |\nabla v^\epsilon(0)| \leq \frac{C}{\delta}. \tag{3.35}$$

Therefore, the idea is to find an estimate for $u^\epsilon - \epsilon$ in terms of the vertical distance $\delta(X)$. To this end, consider h the viscosity solution to the Dirichlet problem

$$\begin{cases} \operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0, & \text{in } B_1^+ \\ v = u^\epsilon, & \text{in } \partial B_1^+. \end{cases} \tag{3.36}$$

Since v is a p-harmonic function type and $0 \leq u^\epsilon \leq 1$, it follows from up to boundary

$C^{1,\alpha}$ estimate (see LADYZHENSKAYA and URAL'TSEVA (1968), Lemma 2) that $v \in C^{1,\alpha}(\overline{B}^+_{3/4})$, moreover

$$|\nabla v(X)| \leq C(\|v\|_{L^\infty} + \|\varphi\|_{C^{1,\alpha}}) \leq C(1+\mathcal{A}).$$

From comparison principle, we have that $u_\epsilon \leq h$ in B^+_1. Hence,

$$u^\epsilon(X) \leq v(X) \leq v(\hat{X}) + C(2+\mathcal{A})|X-\hat{X}| \leq \varphi(\hat{X}) + C(2+\mathcal{A})\delta. \tag{3.37}$$

Now, we have that $|\hat{X}| \leq |X| + \delta \leq \frac{1}{4}$, and, consequently we are able to apply Lemma 3.9 which gives

$$\varphi(\hat{X}) \leq \epsilon + c_0 \cdot \mathrm{dist}(\hat{X}, \{u^\varepsilon \leq \varepsilon\}) \leq \epsilon + c_0(\delta_\epsilon + \delta) \leq \epsilon + 5c_0\delta. \tag{3.38}$$

Thus, it follow from (3.37) and (3.38) that $u^\epsilon(X) - \varepsilon \leq C_0\delta$, where $C_0\colon = C(5c_0 + C(1+\mathcal{A}))$. Finally, if we apply $C^{1,\alpha}$ estimate, Harnack inequality and estimate (3.35), respectively, we end up with

$$|\nabla u^\varepsilon(X)| = |\nabla v^\varepsilon(0)| \leq \frac{1}{\delta}\|u^\varepsilon - \varepsilon\|_{L^\infty(B_{1/2})} \leq C_0$$

which concludes the proof.

c) If $4\delta(X) < \delta_\varepsilon(X)$, then there exists a constant $C_0 = C_0(d, p, \mathcal{B}, \mathcal{A}) > 0$ such that

$$|\nabla u^\epsilon(X)| \leq C_0.$$

In fact, initially we will consider the case when $\delta_\epsilon \leq 1/8$. The following inclusion holds: $B^+_{\delta_\epsilon/2}(\hat{X}) \subset B^+_{1/4} \setminus \{u^\varepsilon \leq \varepsilon\}$. In fact, if $Y \in B^+_{\delta_\varepsilon/2}(\hat{X})$ then

$$|Y| \leq |Y - X| + |X| \leq 2\frac{\delta_\epsilon}{2} + |X| \leq \frac{1}{4}.$$

Now, using the same argument as in (3.36) we are able to estimate u^ϵ in $B^+_{\delta_\epsilon/2}(\hat{X})$ as follows

$$\begin{aligned} u^\epsilon(Y) &\leq u^\epsilon(\hat{Y}) + C(2+\mathcal{A})\frac{\delta_\epsilon}{2} \\ &\leq \epsilon + c_0 \cdot \mathrm{dist}(\hat{Y}, \{u^\epsilon \leq \epsilon\}) + C(1+\mathcal{A})\frac{\delta_\epsilon}{2}. \end{aligned}$$

Since the distance function is Lipschitz continuous with Lipschitz constant 1, we have

$$\mathrm{dist}(\hat{Y}, \{u^\varepsilon \leq \varepsilon\}) \leq \delta_\epsilon + |\hat{Y} - X| \leq 2\delta_\epsilon.$$

Therefore,

$$u^\epsilon(Y) \le \epsilon + \left(2c_0 + \frac{C(2+\mathcal{A})}{2}\right)\delta_\epsilon = \epsilon + c\delta_\epsilon.$$

Considering the function $v^\epsilon = u^\epsilon - \epsilon$ in $B^+_{\delta_\epsilon/2}(\hat{X})$, we have that

$$\operatorname{div}\left(|\nabla v^\epsilon|^{p-2}\nabla v^\epsilon\right) = 0 \quad \text{in } B^+_{\delta_\epsilon/2}(\hat{X}).$$

Thus, from up to boundary $C^{1,\alpha}$ estimate, we have

$$|\nabla u^\epsilon(X)| = |\nabla v^\epsilon(X)| \le C(c+\mathcal{A}).$$

On the other hand, for the case $\delta_\epsilon \ge 1/8$ we have the following inclusion $B^+_{1/16}(\hat{X}) \subset B_1 \setminus \{u^\varepsilon \le \varepsilon\}$. In this situation, since $\operatorname{supp}(\zeta_\epsilon) = [0, \epsilon]$,

$$\begin{cases} |\nabla u^\epsilon|^{2-p} \cdot \operatorname{div}\left(|\nabla u^\epsilon|^{p-2}\nabla u^\epsilon\right) = 0, & \text{in } B^+_{1/16}(\hat{X}) \\ 0 \le u^\epsilon \le C, & \text{in } \partial B^+_{1/16}(\hat{X}) \end{cases}$$

and, consequently, the estimate will follow from up to boundary $C^{1,\alpha}$ estimates.

$\square$

4 GEOMETRIC CONSEQUENCES OF THE LEVEL SETS

In this section we discuss some geometric consequences of the sharp control of solutions, established in the previous two sections.

4.1 Nondegeneracy

In this section, we shall derive the proof of growth of minimal solutions u^ϵ *away from* ϵ*-level surfaces. This property implies that solutions cannot degenerate and imposes very restrictive behavior of the free boundary in terms of its geometric measure theoretical properties. The proof will be based on appropriate barrier functions. In the next proposition, we construct a radially symmetric supersolution to* E_ϵ*, where its value in a inner disk is much smaller than the value on the boundary. To this end, we shall look at degenerate elliptic equations of the form*

$$\Delta_p^N u = \beta(u), \qquad in\ \mathbb{R}^d,$$

where the reaction term satisfies the mild non-degeneracy assumption:

$$\inf_{t\in[a,b]} \zeta(t) > 0, \tag{4.1}$$

Hereafter, we will denote the distance of a point in the non-coincidence set $X \in \Omega \cap \{u^\epsilon > 0\}$ *to the approximating transition boundary,* Γ_ϵ*, by*

$$d_\epsilon(X_0) := dist(X_0, \{u^\epsilon \leq \epsilon\}).$$

Proposition 4.1 (Barrier). *Let* $0 < a < b < 1$ *be fixed. Assume* $0 \in \Omega$*. Given* $0 < \eta < dist(0, \partial\Omega)$*, there exists a radially symmetric function* $\Theta_\epsilon \in C^{1,1}_{loc}(\mathbb{R}^d)$ *and universal constants* $\epsilon_0 > 0$ *and* $\kappa_0 > 0$ *such that, if* $\epsilon < \epsilon_0$ *then*

a) $\Theta_\epsilon \equiv a\epsilon$ *in* $B_{\frac{1}{4}\eta}$*;*

b) $\Theta_\epsilon \geq \kappa_0\eta$ *in* $\mathbb{R}^d \setminus B_\eta$*;*

c) Θ_ϵ *satisfies* $\Delta_p^N\Theta_\epsilon \leq \beta_\epsilon(\Theta_\epsilon)$ *in* $\mathbb{R}^d$*.*

Demonstração. We will initially study the case $\epsilon = 1$. The radially symmetric viscosity supersolution Θ_ϵ will be constructed out from Θ_1, based on a scaling argument. For α and A_0 positive numbers (to be chosen *a posteriori*), consider the radially symmetric function $\Theta\colon \mathbb{R}^d \to \mathbb{R}$ defined as follows,

$$\Theta(X) = \Theta_L(X) := \begin{cases} a & \text{for } \ 0 \leq |X| < L; \\ A_0(|X|-L)^2 + a & \text{for } \ L \leq |X| < L + \sqrt{\frac{b-a}{A_0}}; \\ \psi(L) - \phi(L)|X|^{-\alpha} & \text{for } \ |X| \geq L + \sqrt{\frac{b-a}{A_0}}. \end{cases} \tag{4.2}$$

where

$$\phi(L) = \frac{2}{\alpha}\sqrt{(b-a)A_0}\left(L+\sqrt{\frac{b-a}{A_0}}\right)^{1+\alpha} \text{ and } \psi(L) = b+\phi(L)\left(L+\sqrt{\frac{b-a}{A_0}}\right)^{-\alpha}. \tag{4.3}$$

Indeed, clearly $\Theta_L \in C^{1,1}_{\text{loc}}(\mathbb{R}^d)$. So, we can compute the second order derivatives of Θ_L almost everywhere. Our first aim is to show, provided the appropriate parameters, that Θ_L satisfies pointwise

$$\Delta_p^N \Theta_L(X) \le \beta(\Theta_L(X)) \quad \text{in} \quad \mathbb{R}^d. \tag{4.4}$$

For $0 \le |X| < L$ the inequality (4.4) is true, since

$$\Delta\Theta_L(X) + (p-2)\lambda_{\max}(D^2\Theta_L(X)) = 0 \le \beta(\Theta_L(X)).$$

In the region $L \le |X| < L + \sqrt{\frac{b-a}{A_0}}$, we have

$$D_i\Theta_L(X) = 2A_0\frac{(|X|-L)}{|X|}X_i$$

and

$$D_{ij}\Theta_L(X) = 2A_0\left[\left(\frac{1}{|X|^2} - \frac{(|X|-L)}{|X|^3}\right)X_i\cdot X_j + \frac{(|X|-L)}{|X|}\delta_{ij}\right].$$

Moreover,

$$|\nabla\Theta_L(X)| = 2A_0(|X|-L)$$

and

$$D^2\Theta_L(X) = 2A_0\left[\left(\frac{1}{|X|^2} - \frac{(|X|-L)}{|X|^3}\right)X\otimes X + \frac{(|X|-L)}{|X|}Id\right] \le 4A_0\cdot Id.$$

Note that for $|X| = L$, we have $\nabla\Theta_L(X) = 0$, therefore

$$\Delta\Theta_L(X) + (p-2)\lambda_{\max}(D^2\Theta_L(X)) = 0 \le \beta(\Theta_L(X)).$$

In the region $L < |X| < L + \sqrt{\frac{b-a}{A_0}}$, we obtain

$$\begin{aligned}
\Delta_\infty^N\Theta_L(X) &:= \frac{1}{|\nabla\Theta_L(X)|^2}\sum_{i,j=1}^n D_i\Theta_L\cdot D_j\Theta_L\cdot D_{ij}\Theta_L \\
&= \frac{8{A_0}^3(|X|-L)^2}{4{A_0}^2(|X|-L)^2|X|^2}\sum_{i,j=1}^n\left[\left(\frac{1}{|X|^2} - \frac{(|X|-L)}{|X|^3}\right){X_i}^2{X_j}^2 + \frac{(|X|-L)}{|X|}X_iX_j\delta_{ij}\right] \\
&= \frac{2A_0}{|X|^2}\left[\left(\frac{1}{|X|^2} - \frac{(|X|-L)}{|X|^3}\right)|X|^4 + \frac{(|X|-L)}{|X|}|X|^2\right] \\
&= 2A_0.
\end{aligned}$$

Using the estimates above, we get

$$\begin{aligned}\Delta_p^N \Theta_L(X) &= \Delta\Theta_L(X) + (p-2)\Delta_\infty^N \Theta_L(X) \\ &\leq 4A_0 d + (p-2)2A_0\end{aligned}$$

By construction

$$a \leq \Theta_L(X) < A_0 \frac{(b-a)}{A_0} + a = b.$$

and so, for A_0 sufficiently small, we get

$$\Delta_p^N \Theta_L(X) \leq \inf_{t\in[a,b]} \beta(t) \leq \beta(\Theta_L(X)).$$

Finally, let us turn our attention to the set $|X| \geq L + \sqrt{\frac{b-a}{A_0}}$. Direct computation shows that

$$D_i\Theta_L(X) = \alpha\phi(L)\frac{X_i}{|X|^{\alpha+2}}$$

and

$$D_{ij}\Theta_L(X) = \alpha\phi(L)|X|^{-(\alpha+2)}\left(-\frac{(\alpha+2)}{|X|^2}X_iX_j + \delta_{ij}\right),$$

Therefore,

$$|\nabla\Theta_L(X)| = \alpha\frac{\phi(L)}{|X|^\alpha}$$

and

$$D^2\Theta_L(X) = \alpha\phi(L)|X|^{-(\alpha+2)}\left(-\frac{(\alpha+2)}{|X|^2}X\otimes X + Id\right)$$

hence

$$\begin{aligned}\Delta_\infty^N\Theta_L(X) &:= \frac{1}{|\nabla\Theta_L(X)|^2}\sum_{i,j=1}^n D_i\Theta_L\cdot D_j\Theta_L\cdot D_{ij}\Theta_L \\ &= \frac{|X|^{2\alpha}}{(\phi(L))^2\alpha^2}\left[\left(\frac{\alpha\phi(L)}{|X|^{\alpha+2}}\right)^3\sum_{i,j=1}^n\left(\frac{-(\alpha+2)}{|X|^2}X_i{}^2X_j{}^2 + X_iX_j\delta_{ij}\right)\right] \\ &= \frac{|X|^{2\alpha}}{(\phi(L))^2\alpha^2}\left[\left(\frac{\alpha\phi(L)}{|X|^{\alpha+2}}\right)^3\right](-(\alpha+2)|X|^2 + |X|^2) \\ &= \frac{|X|^{2\alpha}}{(\phi(L))^2\alpha^2}\left(-\alpha^3\phi(L)^3(\alpha+1)\frac{1}{|X|^{3\alpha+4}}\right) \\ &= \frac{-\alpha(\alpha+1)\phi(L)}{|X|^{\alpha+4}}.\end{aligned}$$

Therefore,

$$\Delta_p^N\Theta_L(X) = \frac{\alpha\phi(L)}{|X|^{(\alpha+2)}}(-(\alpha+2)+d) + (p-2)\left[\frac{-\alpha(\alpha+1)\phi(L)}{|X|^{\alpha+4}}\right].$$

Since $p \geq 2$, taking

$$\alpha \geq d-2,$$

we get

$$\Delta_p^N \Theta_L(X) \leq 0 \leq \beta(\Theta_L(X)).$$

Therefore, Θ_L satisfies (4.4). To finish the proof, we show that there exists a universal constant $\kappa_0 > 0$ such that

$$\Theta_L(X) \geq 4\kappa_0 L \quad \text{for} \quad |X| \geq 4L. \tag{4.5}$$

In fact, by (4.3)

$$|X| \geq 4L \geq 2(L+L_0) = 2\left(\frac{\phi(L)}{\psi(L)-b}\right)^{\frac{1}{\alpha}}$$

and hence

$$\Theta_L(X) = \psi(L) - \phi(L)|X|^{-\alpha} \geq \psi(L) - 2^{-\alpha}(\psi(L)-b) \geq C_\alpha \psi(L),$$

for $\alpha > 1$. Therefore,

$$\Theta_L(X) \geq 4\kappa_0 L,$$

where $\kappa_0 > 0$ depends on d and $(b-a)$. For the general case, set

$$\eta \colon = \frac{1}{3} d_\epsilon(0) \quad \text{and} \quad \epsilon_0 \colon = \frac{\eta}{4L_0}.$$

Define

$$\Theta_\epsilon(X) \colon = \epsilon \cdot \Theta_{\frac{\eta}{4\epsilon}}\left(\frac{X}{\epsilon}\right).$$

The following equation holds in the viscosity sense in

$$\Delta_p^N \Theta_\epsilon(X) \leq \beta_\epsilon(\Theta_\epsilon(X)) \quad \text{in} \quad \mathbb{R}^d.$$

Using (4.5) we verify that for $\varepsilon \leq \varepsilon_0$,

$$\begin{aligned} \Theta_\varepsilon &= a\varepsilon \quad \text{in} \quad B_{\frac{1}{4}\eta}; \\ \Theta_\varepsilon &\geq \kappa_0 \eta \quad \text{in} \quad \mathbb{R}^d \setminus B_\eta. \end{aligned}$$

□

Theorem 4.2. *Let $\{u_\epsilon\}_{\epsilon>0}$ be Perron's solution of (E_ϵ). There exists a universal constant $c > 0$ such that for any $X_0 \in \{u^\varepsilon > \varepsilon\}$ and $0 < \varepsilon \leq d_\varepsilon(X_0) \ll 1$ one has*

$$u^\varepsilon(X_0) \geq c d_\varepsilon(X_0).$$

Demonstração. Without loss of generality we assume that $0 \in \{u^\varepsilon > \varepsilon\}$. Let Θ_ε the radially symmetric function given by Proposition 4.1. We claim that there exists a $Z_0 \in \partial B_\eta$ such that

$$\Theta_\varepsilon(Z_0) \le u^\varepsilon(Z_0). \tag{4.6}$$

Indeed, if

$$\Theta_\varepsilon(X) > u^\varepsilon(X) \quad \text{in} \quad \partial B_\eta,$$

then the auxiliary function

$$v^\varepsilon := \min\{\Theta_\varepsilon, u^\varepsilon\}$$

would be a super-solution to (E_ϵ). By Proposition 4.1-(a), we have

$$\Theta_\varepsilon(0) = a\varepsilon < u^\varepsilon(0)$$

and so v^ε is strictly below u^ε, which contradicts the minimality of u^ε. Therefore, by Proposition 4.1-(b), we obtain

$$\kappa_0\eta \le \Theta_\varepsilon(Z_0) \le u^\varepsilon(Z_0) \le \sup_{B_\eta} u^\varepsilon. \tag{4.7}$$

Furthermore, u^ε solves (in the viscosity sense)

$$c_0 \le \Delta_p^N u^\varepsilon \le c_1 \quad \text{in} \quad B_{3\eta}.$$

Hence, by Harnack inequality, see CHARRO *et al.* (2013), we get

$$\sup_{B_\eta} u^\varepsilon \le C u^\varepsilon(0).$$

for C universal. Thus, by (4.7)

$$u^\varepsilon(0) \ge \frac{\kappa_0\eta}{C}.$$

Finally, by taking $\eta > 0$ small enough we conclude

$$u^\varepsilon(0) \ge c\,\eta = c d_\varepsilon(0).$$

for some $0 < c < 1$ (independent of ε).

□

4.2 Strong nondegeneracy

As a consequence of the Lipschitz regularity, Theorem 3.1 and Theorem 4.2, we are able to completely control u^ε in terms of $d_\varepsilon(X_0)$.

Corollary 4.3. *For a sub-domain $\Omega' \Subset \Omega$, there exists $C > 0$, depending on universal* *parameters and Ω', such that for $X_0 \in \Omega' \cap \{u_\varepsilon > \varepsilon\}$ and $\varepsilon \leq d_\varepsilon(X)$, there holds*

$$C^{-1} d_\varepsilon(X_0) \leq u^\varepsilon(X_0) \leq C\, d_\varepsilon(X_0).$$

Demonstração. The inequality from below is exactly the Theorem 4.2. Now take $Y_0 \in \partial\{u^\varepsilon > \varepsilon\} \cap \Omega'$, such that $|Y_0 - X_0| = d_\varepsilon(X_0)$. From Theorem 3.1,

$$u^\varepsilon(X_0) \leq C\, d_\varepsilon(X_0) + u^\varepsilon(Y_0) = C\ d_\varepsilon(X_0) + \varepsilon \leq C_1\ d_\varepsilon(X_0),$$

and the corollary is proved. □

Now we prove a strong non-degeneracy result for Lipschitz solutions of non-linear singular equation that have a linear rate of growth away from the level surfaces. Strong non-degeneracy means that $\max u^\varepsilon$ *on the boundary of a ball $B_r \subset \{u^\varepsilon > \varepsilon\}$ is of the order of r. It is a more accurate control on the rate growth of u^ε away from ε-level surfaces. This result will be of fundamental importance to estimate the Hausdorff measure of free boundaries later on.*

Theorem 4.4. *Given $\Omega' \Subset \Omega$, there exists a universal constant $c > 0$ such that, for $X_0 \in \{u^\epsilon > \epsilon\}$, $\epsilon \ll \rho \ll 1$, there holds*

$$c\,\rho \leq \sup_{B_\rho(X_0)} u^\epsilon \leq c^{-1}(\rho + u^\epsilon(X_0)).$$

Demonstração. As in the proof of Theorem 4.2, taking $\Theta_\varepsilon(X) = \varepsilon\Theta_{\frac{\rho}{4\varepsilon}}(X)$, we have

$$u^\epsilon(Z) > \Theta_\epsilon(Z),$$

for some point $Z \in \partial B_\rho(X_0)$. To conclude it, we note that

$$\kappa \cdot \rho \leq \Theta_\epsilon(Z) < u^\epsilon(Z) \leq \sup_{B_\rho(X_0)} u^\epsilon.$$

The upper estimate follows directly from Lipschitz regularity. □

Remark 4.5. Given $X_0 \in \{u^\epsilon > \epsilon\}$, $\epsilon \ll \rho$ and $\rho \ll 1$ universally small, we have from the strong non-degeneracy that there exists $Y_0 \in B_\rho(X_0)$ such that

$$u^\epsilon(Y_0) \geq c_0\rho.$$

By Lipschitz continuity, for $Z \in B_{\kappa\rho}(Y_0)$, we get

$$u^\epsilon(Z) - C\kappa\rho \geq u^\epsilon(Y_0).$$

Then, by estimates above, it is possible to choose $0 < \kappa \ll 1$ universally small such that

$$Z \in B_{\kappa\rho}(Y_0) \cap B_\rho(X_0) \quad \text{and} \quad u^\epsilon(Z) > \epsilon.$$

Finally, we conclude that there exists a portion of $B_\rho(X_0)$ with volume of order $\sim \rho^d$ within $\{u^\epsilon > \epsilon\}$. By this fact, we are ready to obtain uniform positive density along level sets of minimal solutions to Eq. (E_ϵ).

Corollary 4.6. *Given $X_0 \in \{u^\epsilon > \epsilon\}$, $\epsilon \ll \rho$ and $\rho \ll 1$ universally small, there exists a universal constant $0 < c_0 < 1$ such that*

$$\frac{\mathscr{L}^d(B_\rho(X_0) \cap \{u^\epsilon > \epsilon\})}{\mathscr{L}^d(B_\rho(X_0))} \geq c_0\,,$$

where $\mathscr{L}^d(A)$ is the Lebesgue measure of the set A.

Demonstração. Following the lines of Remark 4.5, we check

$$\mathscr{L}^d(B_\rho(X_0) \cap \{u^\epsilon > \epsilon\}) \geq \mathscr{L}^d(B_\rho(X_0) \cap B_{\kappa\rho}(Y_0)) = c_0\, \mathscr{L}^d(B_\rho(X_0)),$$

for some universal constant $0 < c_0 \ll 1$.

□

For solutions of (E_ϵ) the Harnack inequality is valid for balls that touch the free boundary along the ε-layers, i.e., $\partial\{u^\varepsilon > \varepsilon\}$.

Theorem 4.7. *Let u^ε be a solution of (E_ϵ). Let also $X_0 \in \{u^\varepsilon > \varepsilon\}$ and $\varepsilon \ll c := d_\varepsilon(X_0)$. Then,*

$$\sup_{B_{\frac{c}{2}}(X_0)} u^\varepsilon(X) \leq C \inf_{B_{\frac{c}{2}}(X_0)} u^\varepsilon(X)$$

for a universal constant $C > 0$ independent of ε.

Demonstração. Let Z_1, Z_2 be extremal points for u^ε in $\overline{B_{\frac{c}{2}}(X_0)}$, i.e.,

$$\inf_{B_{\frac{c}{2}}(X_0)} u^\varepsilon(X) = u^\varepsilon(Z_1) \quad \text{and} \quad \sup_{B_{\frac{c}{2}}(X_0)} u^\varepsilon(X) = u^\varepsilon(Z_2).$$

Since $d_\varepsilon(Z_1) \geq \frac{c}{2}$, by Corollary 4.3

$$u^\varepsilon(Z_1) \geq C_1 c. \tag{4.8}$$

Moreover, by Theorem 4.4

$$u^\varepsilon(Z_2) \leq C_2 \left(\frac{c}{2} + u^\varepsilon(X_0)\right). \tag{4.9}$$

Taking $Y \in \partial\{u^\varepsilon > \varepsilon\}$ such that $c = |X_0 - Y|$, we get from Corollary 4.3 and Theorem

4.4

$$u^\varepsilon(X_0) \le \sup_{B_c(Y)} u^\varepsilon \le C_2(c + u^\varepsilon(Y)) \le C_3 c. \tag{4.10}$$

Combining (4.8), (4.9) and (4.10), we have

$$\begin{aligned} u^\varepsilon(Z_2) &\le C_2\left(\frac{c}{2} + C_3 c\right) \\ &= c\left(\frac{C_2}{2} + C_2 C_3\right) \\ &\le \left(\frac{C_2 + 2C_2C_3}{2C_1}\right) u^\varepsilon(Z_1) \\ &= C u^\varepsilon(Z_1), \end{aligned}$$

and we conclude

$$\sup_{B_{\frac{c}{2}}(X_0)} u^\varepsilon(X) \le C \inf_{B_{\frac{c}{2}}(X_0)} u^\varepsilon(X).$$

□

4.3 Porosity of the ϵ-level surfaces

As a consequence of the growth rate and the non-degeneracy property, we get porosity of level sets.

Definition 4.8. A set $E \subset \mathbb{R}^d$ is said to be porous with porosity $\delta > 0$, if $\exists\, R > 0$ such that

$$\forall X \in E,\ \ \forall r \in (0, R),\ \ \exists Y \in \mathbb{R}^d \text{ such that } B_{\delta r}(Y) \subset B_r(X) \setminus E.$$

A porous set of porosity δ has Hausdorff dimension not exceeding $d - c\delta^d$, where $c = c(d) > 0$ is a constant depending only on d, see ZAJÍČEK (1976). In particular, a porous set has Lebesgue measure zero (see, for example, ZAJÍČEK (1976)).

Theorem 4.9. *Let u^ϵ be a solution of (E_ϵ). Then the level sets $\partial\{u^\epsilon > \epsilon\}$ are porous with porosity constant independent of ϵ.*

Demonstração. Let $R > 0$ and $X_0 \in \Omega$ be such that $\overline{B_{4R}(X_0)} \subset \Omega$. Consider $X \in \partial\{u^\epsilon > \epsilon\} \cap B_R(X_0)$. For each $r \in (0, R)$ we have $\overline{B_r(X)} \subset B_{2R}(X_0) \subset \Omega$. Let $Y \in \partial B_r(X)$ such that $u^\epsilon(Y) = \sup\limits_{\partial B_r(X)} u^\epsilon$. By non-degeneracy

$$u^\epsilon(Y) \ge cr, \tag{4.11}$$

where $c > 0$ is a constant. On the other hand, we know that near the free boundary

$$u^\epsilon(Y) \le C d_\epsilon(Y), \tag{4.12}$$

where $C > 0$ is a constant, and $d_\epsilon(Y)$ is the distance of Y from the set $\overline{B_{2R}(X_0)} \cap \Gamma_\epsilon$.

Now, from (4.11) and (4.12) we get

$$d_\epsilon(Y) \geq \delta r \tag{4.13}$$

for a positive constant $\delta < 1$. Let now $Y^* \in [X,Y]$ be such that $|Y - Y^*| = \frac{\delta r}{2}$, then it is not hard to see that

$$B_{\frac{\delta}{2}r}(Y^*) \subset B_{\delta r}(Y) \cap B_r(X). \tag{4.14}$$

Indeed, for each $Z \in B_{\frac{\delta}{2}r}(Y^*)$

$$|Z - Y| \leq |Z - Y^*| + |Y - Y^*| < \frac{\delta r}{2} + \frac{\delta r}{2} = \delta r,$$

and

$$|Z - X| \leq |Z - Y^*| + \big(|X - Y| - |Y^* - Y|\big) < \frac{\delta r}{2} + \left(r - \frac{\delta r}{2}\right) = r,$$

and (4.14) follows. Since by (4.13) $B_{\delta r}(Y) \subset B_{d_\varepsilon(Y)}(Y) \subset \{u^\epsilon > \epsilon\}$, then

$$B_{\delta r}(Y) \cap B_r(X) \subset \{u^\varepsilon > \varepsilon\},$$

which together with (4.14) provides

$$B_{\frac{\delta}{2}r}(Y^*) \subset B_{\delta r}(Y) \cap B_r(X) \subset B_r(X) \setminus \partial\{u^\varepsilon > \varepsilon\} \subset B_r(X) \setminus \partial\{u^\epsilon > \epsilon\} \cap B_R(X_0).$$

□

5 PASSAGE TO THE LIMIT AS $\epsilon \to 0$

In this section, we study the properties of the limit

$$u := \lim_{k\to\infty} u^{\epsilon_k},$$

for a subsequence $\epsilon_k \to 0$ for the problems (E_ϵ). It is essentially a combination of all the results in the paper together with the results in RICARTE and TEIXEIRA (2011) since we have proved uniform Lipschitz regularity and linear growth properties for least supersolutions u^ε. From uniform Lipschitz regularity the family $\{u^\epsilon\}$ is pre-compact in $C^{0,1}_{loc}(\Omega)$. Hence, up to a subsequence, there exists a limiting function u, obtained as the uniform limit of u^ϵ, as $\epsilon \to 0$. One readily verifies that a limiting function u satisfies the same properties.

Theorem 5.1. *Let $\{u^\epsilon\}_{\epsilon>0}$ be a family of nonnegative viscosity solution of (E_ϵ). For any subsequence $\epsilon_k \to 0$ there exists a function $u \in C^{0,1}_{loc}(\Omega)$ such that*

i) $0 \le u \le K_0$ *in* Ω *and*

$$u \in C^{0,1}_{loc}$$

iii) $\Delta_p u = 0$ *in* $\Omega \setminus \partial\{u > 0\}$*, in the viscosity sense thus, weak sense ;*

iv) For every $K \Subset \Omega$*, there exists positive constant* C, ρ_K *under control such that:*

$$C^{-1} dist(X_0, \mathfrak{F}(u)) \le u(X) \le C \cdot dist(X_0, \mathfrak{F}(u)) \quad for \quad dist(X_0, \mathfrak{F}(u)) \le \rho_K,$$

where $\mathfrak{F}(u) = \partial\{u > 0\}$*;*

v) For $X_0 \in K \cap (\{u > 0\} \cup \mathfrak{F}(u))$*, there exists* C_K, D_K *and* ρ_K *under control such that:*

$$D_K^{-1}\rho \le \sup_{B_\rho(X_0)} u \le D_K(\rho + u(X_0)) \quad for \quad \rho \le \rho_K.$$

vi) (L^p-compactness) $\nabla u^{\epsilon_k} \to \nabla u$ *in* $L^p_{loc}(\Omega)$*.*

Demonstração. In fact, i) and ii) follow from Lipschitz regularity with Ascoli-Arzela Theorem. Now, let $B_r(X_0) \Subset \{u > 0\}$. There exists $\tau > 0$ where $u \ge \tau$ in $B_r(X_0)$. From the uniform convergence, for k large enough $u^{\epsilon_k} \ge \tau/2$ in $B_r(X_0)$. If we take $\epsilon^k < \tau/2$, since $\epsilon^k \to 0$, then $\beta_{\epsilon_k}(u^{\epsilon_k}) = 0$ in $B_r(X_0)$ and we have

$$|\nabla u^{\epsilon_k}|^{2-p} \cdot \mathrm{div}\left(|\nabla u^{\epsilon_k}|^{p-2}\nabla u^{\epsilon_k}\right) = 0 \quad \text{in} \quad B_r(X_0).$$

By the equivalence result proved in JUUTINEN, LINDQVIST, and MANFREDI (2001), u is a weak solution to the homogeneous p-Laplacian equation $\Delta_p u^{\epsilon_k} = 0$ in $B_r(X_0)$. This way, from $C^{1,\beta}$ interior estimates we may assume $\nabla u^{\epsilon_k} \to \nabla u$ locally uniformly in $B_r(X_0)$.

In particular, if $\eta \in C_0^\infty(B_r(X_0))$ we have that

$$\int_{B_r(X_0)} \left\langle |\nabla u^{\epsilon_k}|^{p-2}\nabla u^{\epsilon_k} - |\nabla u|^{p-2}\nabla u, \nabla \eta \right\rangle dx \to 0 \quad \text{as} \quad k \to \infty$$

and this proves iii). To prove iv), let $X_0 \in K$ such that for κ_0 defined as in Theorem **??**

$$0 < d_0 = \text{dist}(X_0, \mathfrak{F}(u)) \leq \kappa_0.$$

Let us consider $B_\delta(X_0)$ such that $0 < \delta < d_0$. Since $u > 0$ in $\overline{B_\delta(X_0)}$ it follows once more by the uniform convergence that $u^{\epsilon_k} > \epsilon_k$ in $B_\delta(X_0)$ for $k \gg 1$. In particular,

$$u^{\epsilon_k}(X_0) \geq c\delta.$$

Passing to the limit as $k \to \infty$ and then letting $\delta \to d_0$ we get the linear growth estimate. The upper estimate follows readily from local Lipschitz continuity of u. Finally, the strong nondegeneracy follows from Theorem 5.2 in RICARTE and TEIXEIRA (2011).

To prove vi), let $\psi \in C_0^\infty(\Omega)$ be a nonnegative function and $\delta > 0$. Take $v(X) = (u(X) - \delta)^+\psi(X)$ as a test function. Since $\Delta_p u = 0$ in $\{u > 0\}$, we obtain

$$0 = \int_{\{u>\delta\}} |\nabla u|^{p-2}\nabla u \cdot \nabla v dX =$$

$$\begin{aligned} = & \int_{\{u>\delta\}} |\nabla u|^{p-2}\nabla u \cdot \nabla u \psi(X) dX + \int_{\{u>\delta\}} |\nabla u|^{p-2}\nabla u \cdot \nabla \psi dX \\ - & \delta \int_{\{u>\delta\}} |\nabla u|^{p-2}\nabla u \cdot \nabla \psi dX. \end{aligned}$$

Letting $\delta \to 0$, we conclude

$$\int_{\{u>0\}} |\nabla u|^p \psi(X) dX = -\int_{\{u>0\}} |\nabla u|^{p-2}\nabla u \cdot u(X) \nabla \psi dX \tag{5.1}$$

On the other hand, as u^{ϵ_k} is a viscosity solution to $|\nabla u^{\epsilon_k}|^{2-p}\Delta_p u^{\epsilon_k} = \zeta_{\epsilon_k}(u^{\epsilon_k})$, and by the equivalence result proved in ATTOUCHI, PARVIAINEN, and RUOSTEENOJA (2017), u^{ϵ_k} is a weak solution to the equation $\Delta_p u^{\epsilon_k} = |\nabla u^{\epsilon_k}|^{p-2}\zeta_{\epsilon_k}(u^{\epsilon_k})$, i.e.,

$$\int_\Omega |\nabla u^{\epsilon_k}|^{p-2}\nabla u^{\epsilon_k} \cdot \nabla \eta dX = -\int_\Omega |\nabla u^{\epsilon_k}|^{p-2}\zeta_{\epsilon_k}(u^{\epsilon_k})\eta dX, \quad \forall\, \eta \in C_0^\infty(\Omega).$$

Taking $\eta = (u^{\epsilon_k} - \delta)^+\psi$ we have

$$\int_{\{u^{\epsilon_k}>\delta\}} |\nabla u^{\epsilon_k}|^p \psi dX + \int_{\{u^{\epsilon_k}>\delta\}} |\nabla u^{\epsilon_k}|^{p-2}\nabla u^{\epsilon_k} u^{\epsilon_k} \nabla \psi dX - \delta \int_{\{u^{\epsilon_k}>\delta\}} |\nabla u^{\epsilon_k}|^{p-2}\nabla u^{\epsilon_k} \nabla \psi dX$$

$$= -\int_{\{u^{\epsilon_k}>\delta\}} |\nabla u^{\epsilon_k}|^{p-2}\zeta_{\epsilon_k}(u^{\epsilon_k})u^{\epsilon_k}\psi dX + \delta\int_{\{u^{\epsilon_k}>\delta\}} |\nabla u^{\epsilon_k}|^{p-2}\zeta_{\epsilon_k}(u^{\epsilon_k})\psi dX.$$

Letting $\delta \to 0$ and the observation $|\nabla u^{\epsilon_k}|^{p-2}\zeta_{\epsilon_k}(u^{\epsilon_k})u^{\epsilon_k} \geq 0$ yields

$$\begin{aligned}\int_{\{u^{\epsilon_k}>0\}} |\nabla u^{\epsilon_k}|^p\psi dX &= -\int_{\{u^{\epsilon_k}>0\}} |\nabla u^{\epsilon_k}|^{p-2}\nabla u^{\epsilon_k}u^{\epsilon_k}\nabla\psi dX - \int_{\{u^{\epsilon_k}>0\}} |\nabla u^{\epsilon_k}|^{p-2}\zeta_{\epsilon_k}(u^{\epsilon_k})u^{\epsilon_k}\psi dX \\ &\leq -\int_{\{u^{\epsilon_k}>0\}} |\nabla u^{\epsilon_k}|^{p-2}\nabla u^{\epsilon_k}u^{\epsilon_k}\nabla\psi dX.\end{aligned}$$

Therefore,

$$\int_\Omega |\nabla u^{\epsilon_k}|^p\psi(X)dX \leq -\int_\Omega |\nabla u^{\epsilon_k}|^{p-2}\nabla u^{\epsilon_k}\cdot\nabla\psi u^{\epsilon_k}(X)dX. \tag{5.2}$$

Using the uniform convergence of u^{ϵ_k} to u and the weak convergence of $|\nabla u^{\epsilon_k}|^{p-2}\nabla u^{\epsilon_k}$ to $|\nabla u|^{p-2}\nabla u$ in $L^{p/(p-1)}_{\text{loc}}(\Omega)$, we infer from (5.1) and (5.2) that

$$\limsup_{k\to\infty}\int |\nabla u^{\epsilon_k}|^p\psi \leq \int |\nabla u|^p\psi. \tag{5.3}$$

Since $\nabla u^{\epsilon_k} \rightharpoonup \nabla u$ in $L^p_{\text{loc}}(\Omega)$, we have

$$\int |\nabla u|^p\psi \leq \liminf_{k\to\infty}\int |\nabla u^{\epsilon_k}|^p\psi. \tag{5.4}$$

It follows from (5.3), (5.4), and from a simple compactness argument that $\nabla u^{\epsilon_k} \to \nabla u$ in $L^p_{\text{loc}}(\Omega)$. □

For any set $A \subset \mathbb{R}^d$, and $\delta > 0$ fixed, let $\mathcal{N}_\delta(A)$ denote the δ-neighborhood of A, that is,

$$\mathcal{N}_\delta(A) := \{X \in \mathbb{R}^d : \text{dist}(X, A) < \delta\}.$$

Definition 5.2. We say that a set sequence $\{A_k\}_{k\geq 1}$ converge (locally) for a set A in the distance Hausdorff if given a compact K and $\delta > 0$ exist one $k_0 = k_0(K, \delta) \in \mathbb{N}$ such that, for all $k > k_0$ we have

$$A_k \cap K \subset \mathcal{N}_\delta(A) \cap K$$

$$A \cap K \subset \mathcal{N}_\delta(A_k) \cap K.$$

Let us continue our discussion on the limiting geometric properties obtained as $\epsilon \to 0$ in (E_ϵ). In the sequel, we show that the set $\{u > 0\}$ is the limit, in the Hausdorff distance, of $\{u^\epsilon > \epsilon\}$ as $\epsilon \to 0$.

Theorem 5.3. *Let u be a limiting function, $u := \lim_{k\to\infty} u^{\epsilon_k}$. Then*

$$\partial\{u^{\epsilon_k} > \epsilon\} \to \partial\{u > 0\}, \quad \text{as } k \to \infty$$

in the Hausdorff distance.

Demonstração. We have to show that given $\delta > 0$, for $k \gg 1$, there hold

$$\begin{aligned} \{u^{\epsilon_k} > C_1\epsilon_k\} &\subset \mathcal{N}_\delta(\{u > 0\}) \\ \{u > 0\} &\subset \mathcal{N}_\delta(\{u^{\epsilon_k} > C_1\epsilon_k\}), \end{aligned}$$

for some $C_1 \gg 1$. We will prove the first inclusion. Let us assume by contradiction that there exist a subsequence $\epsilon_k \to 0$ and points $X_k \in \{u^{\epsilon_k} > C_1\epsilon_k\} \cap \Omega'$ such that

a) $\text{dist}(X_k, \{u^{\epsilon_k} > C_1\epsilon_k\}) > \delta$.

b) $X_k \to X_0$ with $\text{dist}(X_0, \mathfrak{F}(u)) \geq \delta$

From b), we conclude that $u(X_0) = 0$. Indeed, we have that $u^{\epsilon_k}(X_k) > C_1\epsilon_k$, doing $\epsilon_k \to 0$, $u(X_0) \geq 0$. If $u(X_0) > 0$ then $X_0 \in \{\ u > 0\}$ and therefore $\text{dist}(X_0, \mathfrak{F}(u)) = 0$, contradicting b). For $k \gg 1$, $u^{\epsilon_k}(X_0) < \epsilon_k$ and $|X_k - X_0| \leq \frac{1}{10}\text{dist}(\Omega', \Omega)$, so we can find $Y_k \in (X_k, X_0)$ such that $u^{\epsilon_k}(Y_k) = \epsilon_k$ and so, $d_{\epsilon_k}(X_k) \ll 1$. By the strong nondegeneracy, we can find $Z_k \in \overline{B_\rho(X_k)}$ such that

$$u^{\epsilon_k}(Z_k) = \sup_{B_\rho(X_k)} u^{\epsilon_k} \geq c_0\rho \quad \text{for} \quad \rho \ll 1.$$

Take $\rho = \frac{1}{8}\delta$. But, for $|X_k - X_0| < \rho$, we have $B_\rho(X_k) \subset B_{\frac{\delta}{2}}(X_0)$. Finally, up to a subsequence, $Z_k \to Z_0 \in B_{\frac{\delta}{4}}(X_0)$, and, since $u^{\epsilon_k}(Z_k) \to u(Z_0)$, we would conclude

$$0 = \sup_{B_{\delta/2}(X_0)} u \geq u(Z_0) \geq C\rho$$

a contradiction. Similarly, we conclude the second inclusion. Suppose the inclusion does not hold. It means that there exists a sequence $X_k \in \{u > 0\} \cap \Omega'$ such that

$$\text{dist}(X_k, \{u^{\epsilon_k} > C_1\epsilon_k\}) > \delta. \tag{5.5}$$

By strong nondegeneracy, and taking $k \gg 1$, we get

$$u^{\epsilon_k}(Y_k) = \sup_{B_{\frac{\delta}{2}}(X_k)} u^{\epsilon_k} \geq \frac{1}{2} \cdot \sup_{B_{\frac{\delta}{2}}(X_k)} u \geq c\delta \geq C_1\epsilon_k$$

for some $Y_k \in \overline{B_{\frac{\delta}{2}}(X_k)} \cap \{u^{\epsilon_k} > C_1\epsilon_k\}$, which contradicts to (5.5). Similarly, we obtain the other inclusion. □

Definition 5.4. Let $u \in C(\Omega)$ and let $A \subset \Omega$. We say that u is locally uniformly nondegenerate in A if for every $\Omega' \Subset \Omega$ there exists constant $C = C(\Omega')$, $r = r(\Omega')$ such that for any $X_0 \in A \cap \Omega'$

$$C^{-1}\rho \leq \fint_{\partial B_\rho(X_0)} u\, dS \leq C\,\rho \quad \text{for} \quad \rho \leq r. \tag{5.6}$$

Theorem 5.5. *Under the assumptions of Theorem 6.17, there exists universal constants $C > 0$ and $\rho_0 > 0$ depending on Ω' and universal parameters such that, for any $X_0 \in \mathfrak{F}(u)$ and $\rho \leq \rho_0$, there holds* (5.6).

Demonstração. By Lipschitz continuity, it is easily to check that upper estimate is valid. To prove the other inequality, we initially observe that by non-degeneracy and Lipschitz continuity we immediately verify such an estimate for the volume average,i.e., there exists a constant $C_1 = C_1(\Omega') > 0$ such that,

$$\fint_{B_\rho(X_0)} udX \geq C_1^{-1}\rho. \tag{5.7}$$

From the fact that $u \geq 0$ is locally Lipschitz and satisfies $\Delta_p u = 0$ in $\{u > 0\}$ in the viscosity sense, the same conclusion holds for the area average as in the statement of (5.6). In fact, suppose by contradiction, that this is not the case. Then, we could find a sequence of point $\{X_m\}_{\{m\geq 1\}} \subset \mathfrak{F}(u) \cap \Omega'$ such that

$$\frac{1}{\rho_m}\fint_{\partial B_{\rho_m}(X_m)} udS = o(1) \quad \text{as} \quad m \to \infty.$$

We now consider the sequence of rescaled functions,

$$v_m(X) \colon = \frac{1}{\rho_m} u(X_m + \rho_m X),$$

which, up to a subsequence, converges uniformly in compact subsets of $\mathbb{R}^d$, to a Lipschitz function $V \geq 0$. Furthermore, $\Delta_p V = 0$ in $\{V > 0\}$. Note that for any $0 < r < 1$,

$$\frac{1}{\rho_m}\fint_{\partial B_{\rho_m}(X_m)} udX = o(1).$$

On the other hand

$$\frac{1}{r\rho_m}\fint_{\partial B_{\rho_m}(X_m)} udS = \fint_{\partial B_r(0)} v_m dS.$$

Therefore, when $m \to \infty$, we obtain for all $0 < r \leq 1$ fixed

$$\begin{aligned}\fint_{\partial B_r(0)} VdS &= \lim_{m\to\infty} \fint_{\partial B_r(0)} v_m dS \\ &= \lim_{m\to\infty} \frac{1}{\rho_m}\fint_{\partial B_{\rho_m}(X_m)} udS \\ &= 0.\end{aligned}$$

Thus,

$$\fint_{\partial B_r(0)} V(Y)dS(Y) = 0, \quad \forall\, r \in (0,1].$$

Integrating the above from $r = 0$ to $r = 1$ we obtain

$$\int_{B_1(0)} V(X)dX = 0.$$

However, from non-degeneracy,

$$\fint_{B_1(0)} V(X)dX = \fint_{B_1(0)} v_m(X)dX = o(1) \geq c > 0,$$

which gives us a contradiction. □

Now, we show that the positive set $\{u > 0\}$ has uniform density along the free transition boundary $\mathfrak{F}(u)$.

Theorem 5.6. *Given $\Omega' \Subset \Omega$, there exists a universal constant $c_0 > 0$, such that for $X_0 \in \mathfrak{F}(u) \cap \Omega'$ there holds*

$$\frac{\mathcal{L}^d(B_\rho(X_0) \cap \{u > 0\})}{\mathcal{L}^d(B_\rho(X_0))} \geq c_0, \tag{5.8}$$

for $\rho \ll 1$. In particular, $\mathcal{L}^d(\mathfrak{F}(u)) = 0$.

Demonstração. By Theorem 5.5, for any $X_0 \in \mathfrak{F}(u) \cap \Omega'$ and $\rho \leq r$ we have

$$\frac{2}{\rho}\fint_{B_{\rho/2}(X_0)} u(X)dX \geq C.$$

So, there exist $Y_0 \in \overline{B}_{\rho/2}(X_0)$ such that $u(Y_0) \geq \frac{C\rho}{2}$. Let $Z_0 \in \mathfrak{F}(u)$ such that

$$|Y - Z_0| = \operatorname{dist}(Y_0, \Omega \setminus \{u > 0\}).$$

So, $Z_0 \in \tilde{\Omega}$ where $\Omega' \Subset \bigcup_{X \in \Omega'} B_r(X) \Subset \Omega$. By Lipschitz regularity

$$\frac{C\rho}{2} \leq u(Y_0) \leq L \cdot |Y_0 - Z_0| = L \cdot \operatorname{dist}(Y_0, \Omega \setminus \{u > 0\}).$$

Now, setting $\rho = \rho(\Omega') \ll 1$,

$$B_{r\rho}(Y_0) \subset B_{2r\rho}(Y_0) \subset \{u > 0\} \cap B_\rho(X_0).$$

The fact that $\mathcal{L}^d(\mathfrak{F}(u)) = 0$ follows from the Lebesgue Differentiation Theorem and simple covering argument. □

We obtain the Hausdorff measure estimates for the free boundaries of Lipschitz solutions of the normalized p-Laplacian equations.The ultimate goal is to prove that the limiting free transition boundary $\mathfrak{F}(u)$ has locally finite $\mathcal{H}^{d-1}$-Hausdorff measure.

Proposition 5.7. *Let u be a limiting function*

$$u = \lim_{\epsilon_k \to 0} u^{\epsilon_k}.$$

Then, for every $K \Subset \Omega$ there exists constants κ_1, κ_2 depending on p, $[u]_{C^{0,1}(K)}$ and ρ_K such that for any $X_0 \in \mathfrak{F}(u) \cap K$

$$\kappa_1 \leq \fint_{B_\rho(X_0)} |\nabla u|^p dX \leq \kappa_2 \quad \textit{for} \quad \rho \leq \rho_K.$$

Demonstração. Initially, note that one of the inequalities above is immediate from the Lipschitz regularity of u, since

$$\fint_{B_\rho(X_0)} |\nabla u|^p dX \leq ([u]_{C^{0,1}(K)})^p.$$

To prove the other inequality, we observe that there exist subballs $B^1, B^2 \subset B \colon = B_\rho(X_0)$ such that:

a) The radius of B^1 and B^2 are comparable to ρ;
b) $u \geq \frac{3}{8} D_K \rho$ in B^1 and $u \leq \frac{1}{4} D_K \rho$ in B^2, where D_K is the constant given by Theorem 6.17-v).

First, we may suppose without losing generality that $B_\rho(X_0) \Subset K$, since otherwise, we increase K to a set $\tilde{K}$ to contain it. From Theorem 5.5, there exists a X_1 in $\overline{B_{\rho/2}(X_0)}$ such that:

$$u(X_1) = \fint_{B_{\rho/2}(X_0)} u(X) dX \geq \frac{C_K}{2} \rho.$$

This way, setting $r_1 = \min\left\{\frac{C_K}{10[u]_{C^{0,1}(K)}}, \frac{1}{4}\right\}\rho$ and $r_2 = \min\left\{\frac{C_K}{5[u]_{C^{0,1}(K)}}, \frac{1}{4}\right\}\rho$, we have

$$\begin{aligned} u &\geq \frac{3C_K\rho}{10} \quad \text{in} \quad B^1 \colon = B_{r_1}(X_1) \subset B \\ u &\leq \frac{C_K\rho}{5} \quad \text{in} \quad B^2 \colon = B_{r_2}(X_0) \subset B. \end{aligned}$$

Now, if we set $m \colon = \fint_{B_\rho(X_0)} u(X) dx$, we have that $|u - m| > LC_K\rho$ in at least one of the subballs B^1, B^2 for a universal constant $L > 0$. Indeed, if this is not the case, we could find sequences $\{X_n\} \subset B^1$ and $\{Y_n\} \subset B^2$ such that

$$\frac{|u(X_n) - m|}{C_K\rho} < \frac{1}{n}, \quad \frac{|u(Y_n) - m|}{C_K\rho} < \frac{1}{n}, \ \forall n$$

yielding

$$\frac{|u(X_n) - u(Y_n)|}{C_k \rho} \to 0$$

contradicting b). Now, by Poincare's inequality, there exists $D = D(C_K, [u]_{C^{0,1}}) > 0$ and a dimensional constant $C > 0$ such that

$$L^p C_K^p \rho^p C \leq \fint_{B_\rho(X_0)} |u - m|^p dX \leq C\rho^p \fint_{B_\rho(X_0)} |\nabla u|^p dX.$$

This finishes the proof. □

Proposition 5.8 (Energy estimate on level set strip)**.** *Let $u \in C^{0,1}(\overline{B}_1)$ be a limiting function*

$$u = \lim_{\epsilon_k \to 0} u^{\epsilon_k}.$$

For $X_0 \in \mathfrak{F}(u) \cap B_{1/2}$, $\kappa > 0$ and $0 < \rho < 1/2$, we have

$$\int_{\{0<u<\kappa\} \cap B_\rho(X_0)} |\nabla u|^p dX \leq C\kappa\rho^{d-1}$$

where $C = C(p, d, [u]_{C^{0,1}}) > 0$.

Demonstração. In fact, by setting

$$u_{\kappa,\lambda} = (\min(u, \kappa) - \lambda)^+, \ \ 0 < \lambda < \kappa.$$

Now, by Gauss-Green Theorem we have

$$\begin{aligned} \int_{B_\rho(X_0)} u_{\kappa,\lambda} \Delta_p u \, dx \ &+ \int_{B_\rho(X_0)} \langle |Du|^{p-2} Du, Du_{\kappa,\lambda} \rangle dx \\ &= \int_{\partial B_\rho(X_0)} u_{\kappa,\lambda} \langle |\nabla u|^{p-2} Du, \nu \rangle d\mathcal{H}^{d-1} \end{aligned}$$

Thus,

$$\begin{aligned} \int_{B_\rho(X_0) \cap \{\lambda < u < \kappa\}} |\nabla u|^p dX &= \int_{\partial B_\rho(X_0)} u_{\kappa,\lambda} \langle |\nabla u|^{p-2} \nabla u, \nu \rangle d\mathcal{H}^{d-1} \\ &\leq \left([u]_{C^{0,1}}^{p-1} d\omega_d \rho^{d-1} \right) \kappa \leq C\kappa\rho^{d-1}. \end{aligned}$$

□

Now, we are ready to prove the Hausdorff measure estimate for the free boundary. It is known in Geometric Measure Theory that Hausdorff measure of compact sets is closely related to their Minkowski content.

Theorem 5.9. *Given $\Omega' \Subset \Omega$, there exists a constant $C > 0$, depending on Ω' and* *universal parameters such that, for* $X_0 \in \mathfrak{F}(u) \cap \Omega'$,

$$\mathcal{H}^{d-1}(\mathfrak{F}(u) \cap B_\rho(X_0)) \leq C\rho^{d-1}.$$

Demonstração. Let $\{B_j\}$ be a covering of $\mathfrak{F}(u) \cap B_\rho(X_0)$ by balls of radius δ such that they have finite overlapping of at most $K = K(d)$ balls, i.e.,

$$\mathfrak{F}(u) \cap B_\rho(X_0) \subset \bigcup B_j, \quad B_j = B_\delta(X_j), \quad X_j \in \mathfrak{F}(u) \cap B_\rho(X_0), \quad \sum \chi_{B_j} \leq K.$$

This way, for $\delta > 0$ small enough

$$\mathcal{N}_\delta(\mathfrak{F}(u)) \cap B_{\rho/2}(X_0) \subset \mathcal{N}_\delta(\mathfrak{F}(u)) \cap B_{\rho-\delta}(X_0) \subset \bigcup 2B_j$$

and

$$\bigcup B_j \subset B_{\rho+\delta}(X_0) \subset B_{2\rho}(X_0).$$

So,

$$\mathscr{L}^d[\mathcal{N}_\delta(\mathfrak{F}(u)) \cap B_{\rho/2}(X_0)] \leq \sum \mathscr{L}^d(2B_j) = 2^d \sum \mathscr{L}^d(B_j).$$

By Corollary 5.7 and finite overlapping there exists of $\kappa_1 = \kappa_1(p, [u]_{C^{0,1}}) > 0$ such that

$$2^d \sum \mathscr{L}^d(B_j) \leq 2^d \kappa_1^{-1} \sum \int_{B_j} |\nabla u|^p dX \leq 2^d K \kappa_1^{-1} \int_{\bigcup B_j} |\nabla u|^p dX.$$

By Lipschitz regularity, we have for any $X \in \bigcup B_j$,

$$u(X) < \delta [u]_{C^{0,1}}.$$

From Proposition 5.8

$$\int_{\bigcup B_j} |\nabla u|^p dX \leq \int_{B_{2\rho}(X_0) \cap \{0 < u < [u]_{C^{0,1}}\}} |\nabla u|^p dX \leq C\delta [u]_{C^{0,1}} (2\rho)^{d-1},$$

i.e.,

$$\mathscr{L}^d[\mathcal{N}_\delta(\mathfrak{F}(u)) \cap B_{\rho/2}(X_0)] \leq \left(2^{2d-1} K \kappa_1^{-1} C [u]_{C^{0,1}}\right) \delta \rho^{d-1},$$

where C is a constant under control whose dependence is given in Proposition 5.8. So, there exists a constant $\tilde{C} = \tilde{C}(d)$ such that

$$\mathcal{H}^{d-1}(\mathfrak{F}(u) \cap B_\rho(X_0)) \leq \tilde{C} \sum \mathcal{H}^{d-1}(\partial B_j) \leq \frac{\tilde{C}}{\delta} \sum \mathscr{L}^d(B_j).$$

Since $\{B_j\}$ has finite overlapping, then

$$\begin{aligned}\frac{\tilde{C}}{\delta}\sum \mathscr{L}^d(B_j) &\leq \frac{K\tilde{C}}{\delta}\mathscr{L}^d\left(\bigcup B_j\right) \leq \frac{K\tilde{C}}{\delta}\mathscr{L}^d[\mathcal{N}_\delta(\mathfrak{F}(u)) \cap B_{\rho+\delta}(X_0)] \\ &\leq KC\tilde{C}(\rho+\delta)^{d-1} = KC\tilde{C}\rho^{d-1} + o(\delta).\end{aligned}$$

Letting $\delta \to 0$, we finish the proof.

□

6 OPTIMAL HÖLDER REGULARITY OF DOUBLY NONLINEAR EQUATION

In this section we are interested in local regularity properties for weak solutions of the Doubly Nonlinear Equation

$$u_t - \operatorname{div}\left(m|u|^{m-1}|\nabla u|^{p-2}\nabla u\right) = f \quad \text{in} \quad \Omega_T, \tag{6.1}$$

where $m > 1$, $p > 2$ and $f \in L^{q,r}$. The problem with this equation is that we can not add constants to solutions like in the linear case and because of this we adapt the technique of ARAÚJO, MAIA, and URBANO (2017),ARAÚJO, TEIXEIRA, and URBANO (2017a), ARAÚJO, TEIXEIRA, and URBANO (2017b), ARAUJO and ZHANG (2015) to our situation, and we prove the Theorem 1.1.

6.1 Modus Operandi: Geometric tangential proceeding

We start our fine regularity analysis by fixing the intrinsic geometric setting for our problem. Given $0 < \alpha < 1$, let

$$\theta := p - \alpha\left((p-1) - \frac{(p-1)}{m+p-2}\right), \tag{6.2}$$

which clearly satisfies the bounds

$$1 + \frac{(p-1)}{m+p-2} < \theta < p.$$

For such θ, define the intrinsic θ-parabolic cylinder as

$$G_\rho := (-\rho^\theta, 0) \times B_\rho(0), \quad \rho > 0.$$

In the sequel we show that for a certain smallness regime require to the parameters of the equation (1.6) that u can be approximated by homogeneous functions.

Lemma 6.1. *Given $\delta > 0$, there exists $0 < \epsilon \ll 1$ such that if $\|f\|_{L^{q,r}(G_1)} \leq \epsilon$ and u a weak solution of* (1.6) *in G_1, with $\|u\|_{\infty,G_1} \leq 1$, then there exists a ϕ such that*

$$\phi_t - \operatorname{div}(m|\phi|^{m-1}|\nabla\phi|^{p-2}\nabla\phi) = 0, \quad in \quad G_{1/2} \tag{6.3}$$

and

$$\|u - \phi\|_{\infty,G_{1/2}} \leq \delta.$$

Demonstração. Suppose, for the sake of contradiction, that, for some $\delta_0 > 0$, there exists a sequence

$$(u^j)_j \in C_{loc}(0, T; L^2_{loc}(B_1)), \quad |u^j|^{\frac{m+p-1}{p}} \in L^p_{loc}(0, T; W^{1,p}_{loc}(B_1))$$

and a sequence $(f^j)_j \in L^{q,r}(G_1)$ such that

$$u^j_t - \operatorname{div}(m|u^j|^{m-1}|\nabla u^j|^{p-2}\nabla u^j) = f^j \ in \ G_1 \tag{6.4}$$

$$\|u^j\|_{\infty,G_1} \leq 1, \tag{6.5}$$

$$\|f^j\|_{L^{q,r}(G_1)} \leq 1/j, \tag{6.6}$$

but still, for any j and any solution ϕ of the homogeneous equation (6.3) in $G_{1/2}$,

$$\|u^j - \phi\|_{\infty,G_{1/2}} > \delta_0. \tag{6.7}$$

Consider a cutoff function $\xi \in C^\infty_0(G_1)$, such that $\xi \in [0,1], \xi \equiv 1$ in $G_{1/2}$ and $\xi \equiv 0$ near $\partial_p G_1$. Thus, since u^j is a solution of (1.6), we can apply the Caccioppoli estimate of Proposition 2.4 to get

$$\begin{aligned} \sup_{t_1<t<t_2} \int_K u^2 \xi^p dx + \int_{-1}^0 \int_{B_1} |u^j|^{m-1} |\nabla u^j|^p \xi^p dx dt &\leq C \int_{-1}^0 \int_{B_1} (u^j)^2 \xi^{p-1} |\xi_t| dx dt \\ &+ \int_{-1}^0 \int_{B_1} |u^j|^{m+p-1} |\nabla \xi|^p dx dt \\ &+ C\|f\|^2_{L^{q,r}} \\ &\leq \tilde{c}, \end{aligned} \tag{6.8}$$

where we use (6.5) and (6.6).
Define $v^j = |u^j|^{\frac{m+p-1}{p}}$; thus

$$|\nabla v^j|^p = \left(\frac{m+p-1}{p}\right)^p |u^j|^{m-1} |\nabla u^j|^p$$

and we get, by (6.8),

$$\|\nabla v^j\|^p_{p,G_{1/2}} \leq \int_{-1}^0 \int_{B_1} |\nabla v^j|^p \xi^p dx dt \leq \left(\frac{m+p-1}{p}\right)^p \tilde{c}. \tag{6.9}$$

Then, for a subsequence,

$$\nabla v^j \rightharpoonup \psi$$

weakly in $L^p(G_{1/2})$. Note that the equibounded sequence $(u^j)_j$ is also equicontinuous, by

PORZIO and VESPRI (1993) and, by Arzelà-Ascoli theorem, along a subsequence

$$u^j \longrightarrow \phi,$$

uniformly in $G_{1/2}$. We can identify $\psi = \nabla v$ once we have the pointwise convergence

$$v^j = |u^j|^{\frac{m+p-1}{p}} \longrightarrow |\phi|^{\frac{m+p-1}{p}} =: v.$$

Passing to the limit in (6.4), we find that ϕ solves (6.3) which contradicts (6.7) for $j \gg 1$.

Indeed, if $K \subset B_{1/2}$ is a compact set and $(t_1, t_2] \subset (-\left(\frac{1}{2}\right)^\theta, 0]$ we have

$$\left| \int_K u^j \varphi dx\Big|_{t_1}^{t_2} - \int_K \phi\varphi dx\Big|_{t_1}^{t_2} \right| = \left| \int_K (u^j - \phi)\varphi dx\Big|_{t_1}^{t_2} \right| \longrightarrow 0.$$

Note also that

$$\left| \int_{t_1}^{t_2} \int_K (-u^j \varphi_t + \phi\varphi_t) dxdt \right| \leq \int_{t_1}^{t_2} \int_K |u^j - \phi||\varphi_t| dxdt \longrightarrow 0.$$

Now consider

$$h^j = |u^j|^{\frac{m-1}{p}} \quad , \quad h = |\phi|^{\frac{m-1}{p}}$$

$$g^j = |\nabla v^j|^{p-2}\nabla v^j \quad , \quad g = |\nabla v|^{p-2}\nabla v.$$

So give $\varphi \in W^{1,2}_{loc}(0, T; L^2(K)) \cap L^p_{loc}(0, T; W^{1,p}_0(K))$, we have

$$\left| \int_{t_1}^{t_2} \int_K h^j g^j \nabla\varphi dxdt - \int_{t_1}^{t_2} \int_K hg\nabla\varphi dxdt \right| = \left| \int_{t_1}^{t_2} \int_K (h^j g^j - hg)\nabla\varphi dxdt \right| \longrightarrow 0$$

since $h^j g^j \rightharpoonup hg$ in $L^p(G_{1/2})$. To finish, consider the function

$$h^j(x,t) = \int_K f^j(x,t)\varphi(x,t) dx$$

and estimate

$$|h^j(x,t)| \leq \|\varphi\,(x,t)\,\|_{\frac{q}{q-1},K} \|f^j\|_{q,K}.$$

Then we obtain

$$|h^j(x,t)|^r \leq \|\varphi\|^r_{\frac{q}{q-1},K}\|f^j\|^r_{q,K}.$$

Thus,

$$\left(\int_{t_1}^{t_2} |h^j(x,t)dt|^r\right)^{1/r} \leq \|\varphi\|_{\frac{q}{q-1},r}\|f^j\|_{q,r}, \tag{6.10}$$

and

$$\begin{aligned}\left|\int_{t_1}^{t_2}\int_K f^j(x,t)\varphi(x,t)dxdt\right| &\leq \int_{t_1}^{t_2}\left|\int_K f^j(x,t)\varphi(x,t)dx\right|dt\\ &= \|h^j\|_{1,K}\\ &\leq C\|h^j\|_{r,K}\\ &\leq C\|\varphi\|_{\frac{q}{q-1},r}\|f^j\|_{q,r}\\ &\leq C\|\varphi\|_{\frac{q}{q-1},r}\frac{1}{j}.\end{aligned}$$

Therefore for $j \to \infty$,

$$\int_{t_1}^{t_2}\int_K f^j(x,t)\varphi(x,t)dxdt \longrightarrow 0.$$

□

We developed a geometric iteration in a certain intrinsic scaling. Here we consider

$$\beta = \frac{\alpha(p-1)}{m+p-2}$$

where α is defined as in (1.8). The following result provides the first step in the iteration process to be implemented.

Lemma 6.2. *Let u a weak solution of* (1.6) *in G_1. There exists $\epsilon > 0$, and $0 < \lambda \ll 1/2$, depending only on m, n, p and α such that if $\|f\|_{L^{q,r}(G_1)} \leq \epsilon$, $\|u\|_{\infty,G_1} \leq 1$ and*

$$|u(0,0)| \leq \frac{1}{4}\lambda^\beta,$$

then

$$\|u\|_{\infty,G_\lambda} \leq \lambda^\beta.$$

Demonstração. Let $0 < \delta < 1$, to be chosen later using the last lemma, we obtain $0 <$

$\epsilon \ll 1$ and a solution ϕ of (6.3) in $G_{1/2}$ such that

$$\|u-\phi\|_{\infty,G_{1/2}} \leq \delta.$$

From the available regularity theory (see KUUSI, SILJANDER, and URBANO (2012); PORZIO and VESPRI (1993)), ϕ is locally $C_x^{\alpha_*} \cap C_t^{\alpha_*/2}$, for some $0 < \alpha_* < 1$. Thus we obtain

$$\sup_{(x,t)\in G_\lambda} |\phi(x,t)-\phi(0,0)| \leq C\lambda^{\frac{\alpha_*(p-1)}{m+p-2}},$$

for $\lambda \ll 1$, to be chosen soon, and $C > 1$ universal. In fact, for $(x,t) \in G_\lambda$

$$\begin{aligned}
|\phi(x,t)-\phi(0,0)| &\leq |\phi(x,t)-\phi(0,t)| + |\phi(0,t)-\phi(0,0)| \\
&\leq c_1|x-0|^{\alpha_*} + c_2|t-0|^{\frac{\alpha_*}{2}} \\
&\leq c_1\lambda^{\alpha_*} + c_2\lambda^{\frac{\theta}{2}\alpha_*} \\
&\leq c_1\lambda^{\frac{\alpha_*(p-1)}{m+p-2}} + c_2\lambda^{\frac{\alpha_*(p-1)}{m+p-2}} \\
&\leq C\lambda^{\frac{\alpha_*(p-1)}{m+p-2}}
\end{aligned}$$

since $\theta > 1 + \dfrac{(p-1)}{m+p-2} > \dfrac{2(p-1)}{m+p-2}$. We will choose $\lambda \ll 1/2$ can therefore estimate

$$\begin{aligned}
\sup_{G_\lambda}|u| &\leq \sup_{G_{1/2}}|u-\phi| + \sup_{G_\lambda}|\phi-\phi(0,0)| \\
&+ |\phi(0,0)-u(0,0)| + |u(0,0)| \\
&\leq 2\delta + C\lambda^{\frac{\alpha_*(p-1)}{m+p-2}} + \frac{1}{4}\lambda^\beta.
\end{aligned} \tag{6.11}$$

Now finally fix the constants, choosing λ and δ as

$$\lambda := \left(\frac{1}{4C}\right)^{\frac{m+p-2}{(\alpha_*-\alpha)(p-1)}} \quad \text{and} \quad \delta := \frac{1}{4}\lambda^\beta$$

and fixing also $\varepsilon > 0$, through Lemma 6.1. The result follows from estimate (6.11). □

Theorem 6.3. *Let u a local weak solution of* (1.6) *in G_1. There exists $\epsilon > 0$, and $0 < \lambda \ll 1/2$, depending only on m, n, p and α, such that if $\|f\|_{L^{q,r}(G_1)} \leq \epsilon$, $\|u\|_{\infty,G_1} \leq 1$ and*

$$|u(0,0)| \leq \frac{1}{4}(\lambda^k)^\beta,$$

then

$$\|u\|_{\infty,G_\lambda} \leq (\lambda^k)^\beta. \tag{6.12}$$

Demonstração. The proof is by induction on $k \in \mathbb{N}$. If $k = 1$, (6.12) holds due to Lemma 6.2. Now suppose that the conclusion holds for k and let's show it also holds for $k + 1$. Consider the following function $v : G_1 \to \mathbb{R}$ defined by

$$v(x,t) = \frac{u(\lambda^k x, \lambda^{k\theta} t)}{\lambda^{\beta k}}. \tag{6.13}$$

We have that

$$v_t(x,t) = \lambda^{k\theta - \beta k} u_t(\lambda^k x, \lambda^{k\theta} t)$$

and

$$\nabla v(x,t) = \lambda^{k - \beta k} \nabla u(\lambda^k x, \lambda^{k\theta} t).$$

Thus,

$$\operatorname{div}(m(v(x,t))^{m-1} |\nabla v(x,t)|^{p-2} \nabla v(x,t))$$

$$= \lambda^{(p-\alpha(p-1))k} \operatorname{div}(m(u(\lambda^k x, \lambda^{k\theta} t))^{m-1} |\nabla u(\lambda^k x, \lambda^{k\theta} t)|^{p-2} \nabla u(\lambda^k x, \lambda^{k\theta} t)).$$

Recalling (6.2), we conclude, since u is a local weak solution of (1.6) in G_1, that

$$\begin{aligned} v_t - \operatorname{div}(m(v(x,t))^{m-1} |v(x,t)|^{p-2} v(x,t)) &= \lambda^{(p-\alpha(p-1))k} f(\lambda^k x, \lambda^{k\theta} t) \\ &= \tilde{f}(x,t). \end{aligned}$$

Now

$$\begin{aligned} \|\tilde{f}\|^r_{L^{q,r}(G_1)} &= \int_{-1}^0 \left(\int_{B_1} \lambda^{(p-\alpha(p-1))kq} |f(\lambda^k x, \lambda^{k\theta} t)|^q dx \right)^{\frac{r}{q}} dt \\ &= \int_{-1}^0 \left(\int_{B_{\lambda^k}} \lambda^{(p-\alpha(p-1))kq - kn} |f(x, \lambda^{k\theta} t)|^q dx \right)^{\frac{r}{q}} dt \\ &= \lambda^{[(p-\alpha(p-1))kq - kn]\frac{r}{q}} \int_{-1}^0 \left(\int_{B_{\lambda^k}} |f(x, \lambda^{k\theta} t)|^q dx \right)^{\frac{r}{q}} dt \\ &= \lambda^{[(p-\alpha(p-1))kq - kn]\frac{r}{q} - k\theta} \int_{-\lambda^{k\theta}}^0 \left(\int_{B_{\lambda^k}} |f(x,t)|^q dx \right)^{\frac{r}{q}} dt. \end{aligned}$$

To apply the Lemma 6.2 we need to have

$$[(p - \alpha(p-1))kq - kn]\frac{r}{q} - k\theta \geq 0,$$

that is,

$$k\left[[(p-\alpha(p-1))q-n]\frac{r}{q}-\left(p-\alpha\left((p-1)-\frac{(p-1)}{m+p-2}\right)\right)\right]\geq 0.$$

Since $k>0$, we have

$$\alpha\leq\frac{(m+p-2)[(pq-n)r-pq]}{q(p-1)[r(m+p-2)-(m+p-3)]}.$$

Choosing the optimal

$$\alpha=\frac{(m+p-2)[(pq-n)r-pq]}{q(p-1)[r(m+p-2)-(m+p-3)]}$$

we have

$$\|\tilde{f}\|_{L^{q,r}(G_1)}=\|f\|_{L^{q,r}((-\lambda^{\theta k},0)\times B_{\lambda^k})}\leq\|f\|_{L^{q,r}(G_1)}\leq\epsilon$$

which entitles v to Lemma 6.2. Note that $||v||_{\infty,G_1}\leq 1$, due to the induction hypothesis, and

$$|v(0,0)|=\left|\frac{u(0,0)}{(\lambda^k)^\beta}\right|\leq\left|\frac{\frac{1}{4}(\lambda^{k+1})^\beta}{(\lambda^k)^\beta}\right|\leq\frac{1}{4}\lambda^\beta.$$

It then follows that

$$||v||_{\infty,G_\lambda}\leq\lambda^\beta,$$

which is the same as

$$||u||_{\infty,G_{\lambda^{k+1}}}\leq\lambda^{\beta(k+1)}.$$

The induction is complete. □

We next show the smallness regime required in the previous theorem is not restrictive and generalize it to cover the case of any small radius.

Theorem 6.4. *Let u be a local weak solution of (1.6) in $G_{1/2}$ then, for every $0<r<\lambda$, if*

$$|u(0,0)|\leq\frac{1}{4}r^\beta$$

we have

$$||u||_{\infty,G_r}\leq Cr^\beta.$$

Demonstração. Take

$$v(x,t) = \rho u(\rho^a x, \rho^{(m-1)+(p-2)+pa} t)$$

with ρ, a to be fixed, which is a solution of (1.6) with

$$\tilde{f}(x,t) = \rho^{(m-1)+(p-1)+pa} f(\rho^a x, \rho^{(m-1)+(p-2)+pa} t).$$

In fact, let

$$v(x,t) = \rho u(\rho^a x, \rho^b t).$$

We have

$$v_t(x,t) = \rho^{1+b} u_t(\rho^a x, \rho^b t)$$

and

$$\nabla v(x,t) = \rho^{1+a} \nabla u(\rho^a x, \rho^b t).$$

So we obtain

$$\operatorname{div}(m(v(x,t))^{m-1} |\nabla v(x,t)|^{p-2} \nabla v(x,t))$$

$$= \rho^{(m-1)+(p-1)+pa} \operatorname{div}(m[u(\rho^a x, \rho^b t)]^{m-1} |\nabla u(\rho^a x, \rho^b t)|^{p-2} \nabla u(\rho^a x, \rho^b t)).$$

Now we choose b such that

$$1 + b = (m-1) + (p-1) + pa.$$

Therefore, we have

$$v_t - \operatorname{div}(m(v(x,t))^{m-1} |\nabla v(x,t)|^{p-2} \nabla v(x,t))$$

$$= \rho^{(m-1)+(p-1)+pa} f(\rho^a x, \rho^{(m-1)+(p-2)+pa} t) := \tilde{f}(x,t).$$

We have still

$$||v||_{\infty, G_1} \leq \rho ||u||_{\infty, G_1}$$

and

$$\|\tilde{f}\|^r_{L^{q,r}(G_1)} = \int_{-1}^{0} \left(\int_{B_1} \rho^{((m-1)+(p-1)+pa)q} |f(\rho^a x, \rho^{(m-1)+(p-2)+pa} t)|^q dx \right)^{\frac{r}{q}} dt$$

$$= \rho^{[((m-1)+(p-1)+pa)q-an]\frac{r}{q}-[(m-1)+(p-2)+pa]} \int_{-\rho^{(m-1)+(p-2)+pa}}^{0} \left(\int_{B_{\rho^a}} |f(x,t)|^q dx \right)^{\frac{r}{q}} dt$$

$$\leq \rho^{[(m-1)+(p-1)+pa]r-a(n\frac{r}{q}+p)-[(m-1)+(p-2)]} \|f\|^r_{L^{q,r}(G_1)}.$$

Now choosing $a > 0$ such that

$$[(m-1)+(p-1)+pa]r - a(n\frac{r}{q}+p) - [(m-1)+(p-2)] > 0,$$

which is always possible, and $0 < \rho << 1$, we enter the smallness regime required by Theorem 6.3 , i.e.,

$$\|v\|_{\infty,G_1} \leq 1$$

and

$$\|\tilde{f}\|_{L^{q,r}(G_1)} \leq \epsilon.$$

Given $0 < r < \lambda$, there exists $k \in \mathbb{N}$ such that

$$\lambda^{k+1} < r \leq \lambda^k.$$

Since

$$|u(0,0)| \leq \frac{1}{4} r^\beta \leq \frac{1}{4}(\lambda^k)^\beta,$$

it follows from Theorem 6.3 that

$$||u||_{\infty,G_{\lambda^k}} \leq (\lambda^k)^\beta.$$

Then,

$$||u||_{\infty,G_r} \leq ||u||_{\infty,G_{\lambda^k}} \leq (\lambda^k)^\beta \leq \left(\frac{r}{\lambda}\right)^\beta = Cr^\beta$$

where $C = \lambda^{-\beta}$. □

6.2 Proof of the Theorem 1.1

In this section we will prove the main result of our work, studying the Hölder continuity at the origin, proving there is a uniform constant B such that

$$||u - u(0,0)||_{\infty,G_r} \leq Br^\beta. \quad (6.14)$$

Demonstração. Since u is continuous we can define

$$\kappa := (4|u(0,0)|)^{\frac{1}{\beta}} \geq 0.$$

Now take any radius $0 < r < \lambda$. We will analyze the possible cases.

1. If $\kappa \leq r < \lambda$ then, by Theorem 6.3,

$$\sup_{G_r} |u(x,t) - u(0,0)| \leq Cr^{\lambda} + |u(0,0)| \leq \left(C + \frac{1}{4}\right) r^{\beta}. \tag{6.15}$$

2. If $0 < r < \kappa$ we consider the function

$$w(x,t) := \frac{u(\kappa x, \kappa^{\theta} t)}{\kappa^{\beta}}.$$

Note that $|w(0,0)| = \frac{1}{4}$ and w solves in G_1

$$w_t - \operatorname{div}(mw^{m-1}|\nabla w|^{p-2}\nabla w) = \kappa^{(p-\alpha(p-1))} f(\kappa x, \kappa^{\theta} t).$$

Since $|u(0,0)| = \frac{1}{4}\kappa^{\beta}$, using Theorem 6.4, we have

$$||w||_{\infty,G_1} \leq \kappa^{-\gamma} ||u||_{\infty,G_{\kappa}} \leq C.$$

With this uniform estimate in hand, and using local $C^{0,\alpha}$ regularity estimates, we find that there exists a radius ρ_* depending only on the data, such that

$$|w(x,t)| \geq \frac{1}{8}, \qquad \forall (x,t) \in G_{\rho_*}.$$

This implies that, in G_{ρ_*}, w solves a uniformly parabolic equation of the form

$$w_t - \operatorname{div}(a(x,t)|\nabla w|^{p-2}\nabla w) = f \in L^{q,r}$$

with continuous coefficients satisfying the bounds $0 < c \leq a(x,t) < d$. By Theorem 7.1, we have

$$w \in C^{0,\gamma}(G_{\rho_*}), \qquad \text{with } \gamma = \frac{(pq-n)r - pq}{q[(p-1)r - (p-2)]} > \beta.$$

Therefore,

$$\sup_{(x,t)\in G_r} |w(x,t) - w(0,0)| \leq Cr^{\gamma} \qquad \forall\ 0 < r < \frac{\rho_*}{2},$$

this is

$$\sup_{(x,t)\in G_r} \left| \frac{u(\kappa x, \kappa^{\theta} t)}{\kappa^{\beta}} - \frac{u(0,0)}{\kappa^{\beta}} \right| \leq Cr^{\gamma} \qquad \forall\ 0 < r < \frac{\rho_*}{2}.$$

Since $\beta < \gamma$, we conclude

$$\sup_{(x,t)\in G_{\kappa r}} |u(x,t) - u(0,0)| \leq C(\kappa r)^\beta \qquad \forall\ \ 0 < \kappa r < \kappa \frac{\rho_*}{2},$$

and, relabelling, we obtain

$$\sup_{(x,t)\in G_r} |u(x,t) - u(0,0)| \leq C r^\beta \qquad \forall\ \ 0 < r < \kappa \frac{\rho_*}{2}. \tag{6.16}$$

3. If $\kappa \frac{\rho_*}{2} \leq r < \kappa$, we have

$$\begin{aligned} \sup_{(x,t)\in G_r} |u(x,t) - u(0,0)| &\leq \sup_{(x,t)\in G_\kappa} |u(x,t) - u(0,0)| \\ &\leq C\kappa^\beta \leq \left(\frac{2r}{\rho_*}\right)^\beta = C' r^\beta. \end{aligned} \tag{6.17}$$

Taking $B = \max\{C + \frac{1}{4}, C'\}$, the result follows for every $0 < r < \lambda$.

□

7 APPENDIX

In section 6.2 we use optimal regularity estimates for solutions of equations

$$u_t - \text{div}(\gamma(x,t)|\nabla u|^{p-2}\nabla u) = f \quad \text{in} \quad U_T, \tag{7.1}$$

with continuous coefficients, i.e

$$|\gamma((x,t)) - \gamma(x_0,t)| \leq L\omega(|x - x_0|)). \tag{7.2}$$

The $\omega(.)$ denotes a modulus of continuity; that is, $\omega(.)$ is concave and non-decreasing such that $\lim_{s\downarrow 0}\omega(s) = 0$ and also satisfying the bounds

$$0 < \nu \leq \gamma(x,t) \leq L \tag{7.3}$$

for some structure constants $0 < \nu \leq 1 \leq L$. The function $f \in L^{q,r}(U_T)$ where

$$\frac{1}{r} + \frac{n}{pq} < 1 \quad \text{and} \quad \frac{2}{r} + \frac{n}{q} > 1. \tag{7.4}$$

Theorem 7.1. *A locally bounded weak solution of (7.1), where γ satisfies the conditions (7.2) and (7.3) and $f \in L^{q,r}$, satisfying (7.4) is Hölder continuous in the space variables, with exponent*

$$\alpha = \frac{(pq-n)r - pq}{q[(p-1)r - (p-2)]}$$

and locally Hölder continuous in time with exponent $\frac{\alpha}{\theta}$ for θ given by

$$\theta := \alpha + p - (p-1)\alpha = 2\alpha + (1-\alpha)p. \tag{7.5}$$

The proof of the above theorem is found in TEIXEIRA and URBANO (2014), where they prove the result to more general degenerate parabolic equations

$$u_t - \text{div}A(x,t,\nabla u) = f \in L^{q,r}(U_T).$$

In Section 3.3 we will needed an explicit family of subsolutions and supersolutions in an annulus. Next, we recall a Hopf's type lemma below for a future reference. We denote

$$F_p(D\phi, D^2\phi) = \text{tr}\left[\left(I + (p-2)\frac{\nabla\phi \otimes \nabla\phi}{|\nabla\phi|^2}\right)\cdot D^2\phi\right]. \tag{7.6}$$

Lemma 7.2. *Let $\phi_\mu = \lambda e^{-\mu|X|^2}$, for $\lambda > 0$, $r_1 > r_2 > 0$ then there exists $\mu = \mu(r_2, p, d) >$*

0 *such that*

$$F_p(D\phi, D^2\phi) > 0 \quad in \quad B_{r_1} \setminus B_{r_2},$$

where F_p *is as in* (7.6).

Demonstração. First, note that

$$\begin{aligned} D_i\phi &= -2\lambda\mu X_i e^{-\mu|X|^2} \quad \text{and} \quad |D\phi| = 2\lambda\mu|X|e^{-\mu|X|^2} & (7.7)\\ D^2\phi &= \lambda\left(4\mu^2 X\otimes X - 2\mu \mathrm{I}\right)e^{-\mu|X|^2}. & (7.8)\end{aligned}$$

Computing, we have for $\eta = \frac{X}{|X|}$

$$\begin{aligned} F_p(D\phi, D^2\phi) &= \mathrm{tr}\left[\left(I + (p-2)\frac{\nabla\phi\otimes\nabla\phi}{|\nabla\phi|^2}\right)\cdot D^2\phi\right] \\ &= \lambda(4\mu^2|X|^2 - 2\mu d)e^{-\mu|X|^2} \\ &+ (p-2)\left\langle \lambda(4\mu^2 X\otimes X - 2\mu\mathrm{I})e^{-\mu|X|^2}\cdot\eta, \eta\right\rangle \\ &= \lambda(4\mu^2|X|^2 - 2\mu d)e^{-\mu|X|^2} \\ &+ 4(p-2)\mu^2|X|^2\lambda e^{-\mu|X|^2}\left\langle(\eta\otimes\eta)\cdot\eta, \eta\right\rangle \\ &- 2(p-2)\lambda\mu e^{-\mu|X|^2}\left\langle\eta,\eta\right\rangle & (7.9)\\ &= \lambda\left[4(p-1)\mu^2|X|^2 - 2(d+p-2)\mu\right]\cdot e^{-\mu|X|^2} & (7.10)\\ &\geq \lambda\left[4(p-1)\mu^2 r_2^2 - 2(d+p-2)\mu\right]\cdot e^{-\mu|X|^2}. & (7.11)\end{aligned}$$

Therefore if μ is large enough, depending only on r_2, p and d, we have

$$F_p(D\phi, D^2\phi) > 0 \quad \text{in} \quad B_{r_1} \setminus B_{r_2}.$$

□

Lemma 7.3. *Let* u *be a viscosity solution to*

$$\begin{cases} F_p(Du, D^2u) = f & in \quad B_r(z) \\ u \geq 0 & in \quad B_r(z). \end{cases}$$

If for some $x_0 \in \partial B_r(z)$,

$$u(x_0) = 0 \quad and \quad \frac{\partial u}{\partial\nu}(x_0) \leq \theta,$$

where ν *is the inward normal vector at* x_0*, then there exists a universal constant* $C > 0$ *such that*

$$u(z) \leq C\theta r.$$

Demonstração. Observe, that considering the scale function

$$v(y) = \frac{u(x+ry)}{r}, \quad y \in B_1$$

we can reduce that $r = 1$. Introduce the function

$$\omega(y) = \gamma \cdot (e^{-\lambda|x|^2} - e^{-\lambda}),$$

where the positive constants γ and λ will be determined below. By (7.11),

$$F_p(D\omega, D^2\omega) \geq \gamma \left[(p-1)\lambda^2 - 2(d+p-2)\lambda\right] \cdot e^{-\lambda|x|^2}.$$

Therefore, $F_p(D\omega, D^2\omega) \geq 0$ in $B_1 \setminus B_{1/2}$ if $\lambda \geq \frac{2(p+d-2)}{p-1}$. By Harnack inequality

$$v(0) \leq \sup_{B_{1/2}} v \leq c \inf_{B_{1/2}} v.$$

Hence $v(y) \geq \frac{1}{c}v(0)$ in $B_{1/2}$. Choosing $\gamma = \frac{v(0)}{c(e^{-\lambda/4}-e^{-\lambda})}$ we have

$$\omega \leq v \quad \text{on} \quad \partial B_1 \cup \partial B_{1/2}$$

and comparison principle gives, that $\omega \leq v$ in $B_1 \setminus B_{1/2}$. Then

$$\frac{\partial \omega}{\partial \nu}(x_0) \leq \frac{\partial v}{\partial \nu}(x_0).$$

Explicity this means, that $\gamma\lambda e^{-\lambda} \leq \Theta$, i.e,

$$v(0) \leq \frac{\Theta c(e^{-\lambda/4} - e^{\lambda})}{\lambda e^{-\lambda}}.$$

Returning to the function u the assertion follows. □

8 CONCLUSION

In the development of this thesis it was possible to understand the importance of two problems presented in her.

The first problem presents the singular perturbation method that is a fundamental tool for the study of problems that present certain difficulties in preventing us from working with the initial problem itself.

The second problem gives us the explicit exponent of the Holder continuity of the solutions of the doubly nonilnear equation, explaining and improving the mathematical models governed by this equation.

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