UNIVERSIDADE FEDERAL DO CEARÁ
CENTRO DE CIÊNCIAS
DEPARTAMENTO DE MATEMÁTICA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

ESKO ANTERO HEINONEN

DIRICHLET PROBLEMS FOR MEAN CURVATURE AND p-HARMONIC EQUATIONS ON CARTAN-HADAMARD MANIFOLDS

FORTALEZA

2018

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Orientador: Prof. Dr. Jorge Lira
Coorientador: Prof. Dr. Ilkka Holopainen

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2018

Dados Internacionais de Catalogação na Publicação
Universidade Federal do Ceará
Biblioteca Universitária
Gerada automaticamente pelo módulo Catalog, mediante os dados fornecidos pelo(a) autor(a)

H38d Heinonen, Esko Antero.
Dirichet problems for mean curvature and p-harmonic equations on Carta-Hadamard manifolds / Esko Antero Heinonen. – 2018.
166 f.

Tese (doutorado) – Universidade Federal do Ceará, Centro de Ciências, Programa de Pós-Graduação em Matemática , Fortaleza, 2018.
Orientação: Prof. Dr. Jorge Herbert Sioares de Lira.
Coorientação: Prof. Dr. Ilkka Holopainen.

1. Variedades de Cartan-Hadamard. 2. Curvatura média. 3. p-Laplaciano. 4. Problema assintótico. 5. Equações diferenciais parciais não-lineares. I. Título.

CDD 510

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Aprovoda em: 6 / 3 / 2018.

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This thesis has been done under a Double Doctorate Degree agreement between the University of Helsinki and the Universidade Federal do Ceará.

Another version of the thesis (wihout Section 7) was published earlier in Helsinki by Unigrafia Oy (ISBN 978-951-51-3663-3 (paperback), ISBN 978-951-51-3664-0 (PDF)).

ACKNOWLEDGEMENTS

First and foremost I wish to thank my advisor Ilkka Holopainen for introducing me to the world of geometry and patiently guiding me through my PhD studies. I have always felt welcome when visiting his office with any kind of questions or concerns. I am also grateful to my second advisor Jorge Lira for his guidance and broadening of my mathematical adventure to the world of Killing graphs.

I extend my gratitude to Jean-Baptiste Castéras for collaboration and hosting my visits to Brussels. During the past years he has been almost like a third advisor for me.

I wish to thank my pre-examiners Professor Luis Alías and Professor Alexander Grigor'yan, and my opponent Professor Alberto Setti for their time spent reading my thesis and making it possible to defend in Helsinki. I'm also thankful to the defence committee (Jorge Lira, Ilkka Holopainen, Pacelli Bessa, Diego Moreira and Jaime Ripoll) in Fortaleza for all their efforts.

This work would not have been possible without the financial support of the Academy of Finland (project 252293), the Jenny and Antti Wihuri Foundation, the Centre of Excellence in Analysis and Dynamics Research, the Doctoral Programme in Mathematics and Statistics, and CNPq. I gratefully acknowledge this support.

During the spring 2017 I worked for almost six months at the Universidade Federal do Ceará in Fortaleza and I thank the faculty of the Department of Mathematics for their very warm welcome. I wish especially to thank Jorge Lira for hosting my visit and offering this possibility, and Marco Magliaro for helping me with all the practical things uncountably many times.

I have spent nine years at the Department of Mathematics and Statistics in Helsinki and I have been lucky to have so many friends who made the department such a good place to study and work. A special thanks to Olli for lunch company over the years and for all the conversations and support related to work, life and everything.

I wish to thank all my friends outside the math department, it has been important to have also other things to do and think.

None of this would have been possible without the support of my family. I am grateful for my parents, Päivi and Heimo, for supporting and encouraging me on the path I chose. I also thank my brothers Juho and Timo for all the fun along the years. Especially I wish to thank Erja for supporting and believing in me during this project, and I am grateful for all the love and happiness that she brings to my life.

RESUMO

O tema que dá unidade aos artigos [A,B,C,D,E] que compõem esta dissertação é a existência e não-existência de soluções contínuas, inteiras, de equações diferenciais não-lineares em uma variedade Riemanniana M. Os resultados de existência de tais soluções são demonstrados estudando-se o problema de Dirichlet assintótico sob diversas hipóteses relativas a geometria da variedade.

Funções que definem gráficos mínimos são estudadas nos artigos [A] e [D]. O artigo [A] lida com um resultado de existência, ao passo que, em [D], obtemos tanto resultados de existência quanto de não-existência com respeito a curvatura de M. Além disso, funções p-harmônicas são também estudadas em [D].

O artigo [B] lida com a existência de funções $\mathcal{A}$-harmônicas sob hipóteses de curvatura similares àquelas em [A]. No artigo [C], estudamos a existência de gráficos f-mínimos, os quais generalizam os gráficos mínimos usuais. Por fim, no artigo [E], tratamos de gráficos de Killing em produtos *warped*.

Antes de passar às ideias e resultados dos artigos de pesquisa. apresentamos alguns conceitos fundamentais da tese e um breve histórico das contribuições ao problema de Dirichlet assintótico. Dada a similaridade das técnicas em [A] e [B], tratamo-as conjuntamente na seção 3. O artigo [C] é, então, considerado na seção 4, o artigo [D] na seção 5 e, por fim, o artigo [E] na seção 6. No início das seções 3 – 6, descrevemos brevemente os métodos e técniicas usados nos artigos correspondentes.

Palavras-chave: Variedades de Cartan-Hadamard. Curvatura média. p-laplaciano. Problema assintótico. Equações diferenciais parciais não-lineares.

ABSTRACT

The unifying theme of the five articles, [A,B,C,D,E], forming this dissertation is the existence and non-existence of continuous entire non-constant solutions for nonlinear differential operators on a Riemannian manifold M. The existence results of such solutions are proved by studying the asymptotic Dirichlet problem under different assumptions on the geometry of the manifold.

Minimal graphic functions are studied in articles [A] and [D]. Article [A] deals with an existence result whereas in [D] we give both existence and non-existence results with respect to the curvature of M. Moreover p-harmonic functions are studied in [D].

Article [B] deals with the existence of $\mathcal{A}$-harmonic functions under similar curvature assumptions as in [A]. In article [C] we study the existence of f-minimal graphs, which are generalisations of usual minimal graphs, and in the article [E] the Killing graphs on warped product manifolds.

Before turning to the ideas and results of the research articles, we present some key concepts of the thesis and give a brief history of the development of the asymptotic Dirichlet problem. Due to the similarity of the techniques in [A] and [B], we treat them together in Section 3. Article [C] is treated in Section 4, article [D] in Section 5 and article [E] in Section 6. At the beginning of the Sections 3 – 6 we briefly give the background of the methods and techniques used in the articles.

Keywords: Cartan-Hadamard manifolds. Mean curvature. p-Laplacian. Asymptotic problem. Nonlinear partial differential equations.

SUMÁRIO

1 PRELIMINARIES

This section is devoted to defining the key concepts of this thesis. Throughout the thesis we assume that M is an n-dimensional, $n \geq 2$, connected, non-compact orientable Riemannian manifold equipped with a Riemannian metric $\langle \cdot, \cdot \rangle$. The tangent space at each point $x \in M$ will be denoted by T_xM and the norm with respect to the Riemannian metric by $|\cdot|$. Unless otherwise specified, the integration will be with respect to the Riemannian volume form dm.

In the case of smooth functions $u\colon M \to \mathbb{R}$, the covariant derivation will be denoted by D or semicolon. The first covariant derivative agrees with the usual partial derivative and for the second covariant derivative we have

$$D_iD_ju = u_{i;j} = u_{ij} - \Gamma^k_{ij}u_k = u_{j;i} = D_jD_iu,$$

with $u_k = \partial u/\partial x^k$. The third covariant derivative is no more symmetric with respect to the last indices. If the Riemannian metric is given by $ds^2 = \sigma_{ij}dx^idx^j$ in local coordinates with inverse matrix (σ^{ij}), we will use a short hand notation $u^i = \sigma^{ij}D_ju$.

A *Cartan-Hadamard* (also *Hadamard*) manifold M is a simply connected Riemannian manifold whose all sectional curvatures satisfy

$$K_M \leq 0.$$

Basic examples of such manifolds are the Euclidean space $\mathbb{R}^n$, with zero curvature, and the hyperbolic space $\mathbb{H}^n$, with constant negative curvature. The name of these manifolds has its origin in the Cartan-Hadamard theorem which states that the exponential map is a diffeomorphism in the whole tangent space at every point of M.

Given a smooth function $k\colon [0,\infty) \to [0,\infty)$ we denote by $f_k\colon [0,\infty) \to \mathbb{R}$ the smooth non-negative solution to the initial value problem (Jacobi equation)

$$\begin{cases} f_k(0) = 0, \\ f_k'(0) = 1, \\ f_k'' = k^2 f_k. \end{cases}$$

These functions play an important role in estimates involving curvature bounds since they result to rotationally symmetric manifolds that can be used in various comparison theorems, e.g. Hessian and Laplace comparison (see Greene and Wu (1979)).

Recall that a rotationally symmetric manifold, also a model manifold, M_f is $\mathbb{R}^n$ equipped with a metric of the form $g^2 = dr^2 + f(r)^2d\theta^2$, where r is the distance to a pole o and $d\theta$ is the standard metric on the unit sphere $\mathbb{S}^{n-1}$. The sectional curvatures of a model manifold can be obtained from the radial curvature function, namely we have

$$K_{M_f}(P_x) = -\frac{f''\big(r(x)\big)}{f\big(r(x)\big)}\cos^2\alpha + \frac{1 - f'\big(r(x)\big)^2}{f\big(r(x)\big)^2}\sin^2\alpha, \tag{1.1}$$

where α is the angle between $\nabla r(x)$ and the 2-plane $P_x \subset T_xM$, and hence these manifolds offer examples of Cartan-Hadamard manifolds when $f'' \geq 0$. In the case of the radial sectional curvature the formula simplifies to

$$K_{M_f} = -\frac{f''}{f}.$$

For the verification of these formulae one could see e.g. Vähäkangas (2006).

1.1 Mean curvature equation and minimal surfaces

In 2-dimensional case we have a nice and simple interpretation. Let $\Omega \subset \mathbb{R}^2$ be an open set and $u\colon \Omega \to \mathbb{R}$ a C^2 function with graph $\Sigma_u = \{(x, u(x)) \colon x \in \Omega\}$. Keeping the boundary $\partial\Sigma_u$ fixed and making a smooth variation of the graph, we get that the critical points of the area functional

$$\int_\Omega \sqrt{1 + |\nabla u|^2}$$

are solutions to the *minimal graph equation*

$$\operatorname{div}\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0. \tag{1.2}$$

The graphs of solutions of (1.2) have the minimal area among all graphs with fixed boundary $\partial\Sigma_u$.

More generally we define minimal graphic functions as follows. Let $\Omega \subset M$ be an open set. Then a function $u \in W^{1,1}_{\text{loc}}(\Omega)$ is a *(weak) solution of the minimal graph equation* if

$$\int_\Omega \frac{\langle \nabla u, \nabla \varphi \rangle}{\sqrt{1 + |\nabla u|^2}} = 0$$

for every $\varphi \in C_0^\infty(\Omega)$. Note that the integral is well-defined since

$$\sqrt{1 + |\nabla u|^2} \geq |\nabla u| \quad \text{a.e.},$$

and thus

$$\int_\Omega \frac{|\langle \nabla u, \nabla \varphi \rangle|}{\sqrt{1 + |\nabla u|^2}} \leq \int_\Omega \frac{|\nabla u||\nabla \varphi|}{\sqrt{1 + |\nabla u|^2}} \leq \int_\Omega |\nabla \varphi| < \infty.$$

The operator in (1.2) gives also the mean curvature of the graph Σ_u. Namely, if $\bar{N}$ is the unit normal vector field of Σ_u, then the mean curvature vector at point x is given by

$$\left(\operatorname{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\bar{N}(x)=\bar{H}(x)$$

and the (scalar) mean curvature is

$$\operatorname{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}=H(x). \tag{1.3}$$

Therefore it is also called the mean curvature operator. Recall that the mean curvature of a submanifold is the trace of the second fundamental form and general minimal (hyper) surfaces (not necessarily graphs of functions) are the surfaces having zero mean curvature.

Instead of minimal surfaces, one can also consider surfaces of constant mean curvature (CMC surfaces) or surfaces of prescribed mean curvature. In the latter case one considers solutions of (1.3) and H is a function defined on M or in more general situation in $M\times\mathbb{R}$, see Section 4 and [C].

It is well known that under certain conditions there exists a (strong) solution of (1.2) with given boundary values. Namely, let $\Omega \subset\subset M$ be a smooth relatively compact open set whose boundary has positive mean curvature with respect to inwards pointing unit normal. Then for each $\theta\in C^{2,\alpha}(\bar\Omega)$ there exists a unique $u\in C^\infty(\Omega)\cap C^{2,\alpha}(\bar\Omega)$ that solves the minimal graph equation (1.2) in Ω and has the boundary values $u|\partial\Omega=\theta|\partial\Omega$. Similar existence result holds also for the case of prescribed mean curvature equation (1.3) but with an assumption that the lower bound for the mean curvature of the boundary $\partial\Omega$ depends on the function H.

The standard strategy to prove these type of results is to obtain a priori height and gradient estimates for the solutions and then apply the continuity or Leray-Schauder method. For the proofs in the Euclidean case one should see the original papers Jenkins and Serrin (1968) and Serrin (1969) or the book Gilbarg and Trudinger (2001) where also more general equations are considered. For the Riemannian case see e.g. Spruck (2007) and Dajczer, Hinojosa, and de Lira (2008). In [C] we treat the more general case where the prescribed mean curvature depends also on the $\mathbb{R}$-variable of the product space $M\times\mathbb{R}$. Good references for the general theory of minimal surfaces are e.g. Colding and Minicozzi (2011) and Lawson (1977).

It is also useful to write the minimal graph equation in a non-divergence form

$$\frac{1}{W}g^{ij}D_iD_ju=0,$$

where $W=\sqrt{1+|\nabla u|^2}$,

$$g^{ij}=\sigma^{ij}-\frac{u^iu^j}{W^2} \tag{1.4}$$

and $u^i = \sigma^{ij} D_j u$. The induced metric on the graph of u is given by

$$g_{ij} = \sigma_{ij} + u_i u_j$$

with inverse (1.4). Similarly the mean curvature of the graph is given by

$$\frac{1}{W} g^{ij} D_i D_j u = nH.$$

For the derivation of these formulae, see e.g. Rosenberg, Schulze, and Spruck (2013).

1.2 $\mathcal{A}$-harmonic functions

The weak solutions of the quasilinear elliptic equation

$$\mathcal{Q}[u] = -\operatorname{div} \mathcal{A}_x(\nabla u) = 0 \tag{1.5}$$

are called $\mathcal{A}$-*harmonic* functions. Here the $\mathcal{A}$-harmonic operator (of type p), $\mathcal{A}\colon TM \to TM$, is subject to certain conditions; for instance $\langle \mathcal{A}(V), V\rangle \approx |V|^p$, $1 < p < \infty$, and $\mathcal{A}(\lambda V) = \lambda|\lambda|^{p-2}\mathcal{A}(V)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ (see [B] for the precise definition). The set of all such operators is denoted by $\mathcal{A}^p(M)$

To be more precise what we mean by a weak solution, let $\Omega \subset M$ be an open set and $\mathcal{A} \in \mathcal{A}^p(M)$. A function $u \in C(\Omega) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$ is $\mathcal{A}$-harmonic in Ω if it satisfies

$$\int_\Omega \langle \mathcal{A}(\nabla u), \nabla \varphi \rangle = 0 \tag{1.6}$$

for every test function $\varphi \in C_0^\infty(\Omega)$. If $|\nabla u| \in L^p(\Omega)$, then it is equivalent to require (1.6) for all $\varphi \in W_0^{1,p}(\Omega)$ by approximation. In the special case $\mathcal{A}(v) = |v|^{p-2}v$, yielding an equation

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0, \tag{1.7}$$

$\mathcal{A}$-harmonic functions are called *p-harmonic* and, in particular, if $p = 2$, we obtain the usual harmonic functions. Therefore we see that $\mathcal{A}$-harmonic functions are really a generalisation of harmonic functions.

As the properties of the harmonic functions can be studied with superharmonic functions, the $\mathcal{A}$-superharmonic functions play a similar role for the $\mathcal{A}$-harmonic functions. A lower semicontinuous function $u\colon \Omega \to (-\infty, \infty]$ is called $\mathcal{A}$-*superharmonic* if $u \not\equiv \infty$ in each component of Ω, and for each open $D \subset\subset \Omega$ and for every $h \in C(\bar{D})$, $\mathcal{A}$-harmonic in D, $h \le u$ on ∂D implies $h \le u$ in D. In the case of equation (1.7) these functions are called *p-superharmonic*. A very good standard reference for the study of nonlinear potential theory in the Euclidean case is the book Heinonen, Kilpeläinen, and Martio (1993). For the Riemannian setting see Holopainen (1990).

The question about the solvability of the Dirichlet problem (also the asymptotic one, see Section 1.3) for $\mathcal{A}$-harmonic functions can be approached via the Perron's method which reduces the problem to the question about the regularity of the boundary points. Recall that a boundary point x_0 is regular if

$$\lim_{x\to x_0} \overline{H}_f(x) = f(x_0)$$

for every continuous boundary data f. Here $\overline{H}_f$ is the upper Perron solution. For precise definitions see [B] and for a complete treatment Heinonen, Kilpeläinen, and Martio (1993).

1.3 Asymptotic Dirichlet problem on Cartan-Hadamard manifolds

Cartan-Hadamard manifolds can be compactified by adding the *asymptotic boundary* (also *sphere at infinity*) $\partial_\infty M$ and equipping the resulting space $\bar{M} := M \cup \partial_\infty M$ with the *cone topology*, making $\bar{M}$ homeomorphic to the closed unit ball. The asymptotic boundary $\partial_\infty M$ consists of equivalence classes of geodesic rays under the equivalence relation

$$\gamma_1 \sim \gamma_2 \quad \text{if} \quad \sup_{t\geq 0} \operatorname{dist}\big(\gamma_1(t), \gamma_2(t)\big) < \infty.$$

Equivalently it can be considered as the set of geodesic rays emitting from a fixed point $o \in M$, which justifies the name sphere at infinity.

The basis for the cone topology in $\bar{M}$ is formed by cones

$$C(v,\alpha) := \{y \in M \setminus \{x\} \colon \sphericalangle(v, \dot\gamma_0^{x,y}) < \alpha\}, \quad v \in T_xM,\ \alpha > 0,$$

truncated cones

$$T(v,\alpha,R) := C(v,\alpha) \setminus \bar{B}(x,R), \quad R > 0,$$

and all open balls in M. Cone topology was first introduced in Eberlein and O'Neill (1973).

This construction allows us to define the main concept of this thesis, namely the *asymptotic Dirichlet problem* (also *Dirichlet problem at infinity*) for a quasilinear elliptic operator Q:

Problem. *Let $\theta\colon \partial_\infty M \to \mathbb{R}$ be a continuous function. Does there exist a continuous function $u\colon \bar{M} \to \mathbb{R}$ with*

$$\begin{cases} Q[u] = 0 & \text{in } M; \\ u|\partial_\infty M = \theta, \end{cases}$$

and if yes, is the function u unique?

In the case such function u exists for every $\theta \in C(\partial_\infty M)$, we say that the asymptotic Dirichlet problem in $\bar{M}$ is *solvable*. As we will see, the solvability of this problem depends on the geometry of the manifold M, but the uniqueness of the solutions depends also on the operator Q. For the usual Laplace, $\mathcal{A}$-harmonic and minimal graph operators we have the uniqueness but more complicated operators may not satisfy maximum principles and hence also the uniqueness of solutions will be lost (see Section 4).

2 BACKGROUND OF THE ASYMPTOTIC DIRICHLET PROBLEM

In this section we give a brief history of the asymptotic Dirichlet problem and developments before [A],[B],[C], [D] and [E]. We will denote by M a Cartan-Hadamard manifold with sectional curvature K_M. Point $o \in M$ will be a fixed point and $r = d(o, \cdot)$ is the distance to o. By P_x we denote a 2-dimensional subspace of T_xM.

2.1 Harmonic functions

The study of the harmonic functions on Cartan-Hadamard manifolds has its origin in Greene and Wu (1979) where they proposed the conjecture that if the sectional curvatures of the manifold M satisfy

$$K_M \le -\frac{C}{r^2}, \quad C > 0,$$

outside a compact set, then there exists a bounded non-constant harmonic function on M. One way to show the existence of such functions is to try to solve the asymptotic Dirichlet problem with continuous boundary data on $\partial_\infty M$.

The study of the asymptotic Dirichlet problem began in the beginning of 1980's when Choi (1984) gave a definition of the problem and showed that it can be solved on a general n-dimensional Cartan-Hadamard manifold by assuming that the sectional curvatures have an upper bound $K_M \le -a^2$, for some constant $a > 0$, and that any two points on the boundary $\partial_\infty M$ can be separated by convex neighbourhoods. In Anderson (1983) Anderson showed that such neighbourhoods can be constructed by assuming that the sectional curvatures are bounded between two negative constants, resulting to the following.

Theorem. *Assume that the sectional curvatures of M satisfy*

$$-b^2 \le K_M \le -a^2, \tag{2.1}$$

where $0 < a \le b$ are arbitrary constants. Then the asymptotic Dirichlet problem is uniquely solvable.

Sullivan (1983) solved the asymptotic Dirichlet problem independently at the same time by assuming (2.1) and using probabilistic methods. In Anderson and Schoen

(1985) Anderson and Schoen gave an identification of the Martin boundary of M under the assumption (2.1).

A slightly different setting was considered in Ballmann (1989), and Ballmann and Ledrappier (1994) when studying the Dirichlet problem on negatively curved rank 1 manifolds. Ancona considered Gromov hyperbolic graphs Ancona (1988) and Gromov hyperbolic manifolds Ancona (1990). In Ancona (1987) he solved the asymptotic Dirichlet problem by assuming an upper bound for the sectional curvatures and that balls up to a fixed radius are L-bi-Lipschitz equivalent to an open set in $\mathbb{R}^n$.

In Cheng (1993) Cheng introduced the pointwise pinching condition

$$|K_M(P_x)| \leq C_K |K_M(P'_x)| \tag{2.2}$$

for the sectional curvatures, and solved the problem assuming (2.2) and positive bottom spectrum for the Laplacian. Here $C_K > 0$ is a constant and $P_x, P'_x \subset T_xM$ are any 2-dimensional subspaces containing the radial vector field. It is worth noting that (2.2) allows the curvature to behave very freely along different geodesic rays.

Trying to relax the assumption (2.1), the first result allowing the curvature to approach zero was due to Hsu and March (1985) with assumption

$$-b^2 \leq K_M \leq -C/r^2$$

for some constants $b > 0$ and $C > 2$. On the other hand, Borbély (1992) allowed the curvature to decay with assumption

$$-be^{\lambda r} \leq K_M \leq -a$$

for some constants $b \geq a > 0$ and $\lambda < 1/3$.

In 2003 Hsu (2003) solved the asymptotic Dirichlet problem already under very general curvature assumptions, namely his first result allowed the upper bound behave like $K_M \leq -\alpha(\alpha-1)/r^2$ for $\alpha > 0$ and instead of a lower bound for the sectional curvatures, he assumed a Ricci lower bound $-r^{2\beta} \leq \mathrm{Ric}$ with $\beta < \alpha - 2$. His second result assumed a constant sectional curvature upper bound $-a$ but allowed the Ricci lower bound to decay as

$$-h(r)^2 e^{2ar} \leq \mathrm{Ric},$$

where h is a function satisfying $\int_0^\infty rh(r)\,dr < \infty$.

2.2 $\mathcal{A}$- and p-harmonic functions

Investigation of the nonlinear setting was started by Pansu (1989) in 1988 when he showed the existence of non-constant bounded p-harmonic functions, with $p >$

$(n-1)b/a$ and gradients in L^p, under the curvature assumption (2.1). His proof was based on study of the L^p-cohomology and it also gave non-existence for $p \le (n-1)a/b$.

In Holopainen (2002) Holopainen showed that the direct approach of Anderson and Schoen (1985) can be generalised to work also in the case of p-harmonic functions under the assumption (2.1). Few years later Holopainen, Lang, and Vähäkangas (2007) proved the existence of non-constant bounded p-harmonic functions in Gromov hyperbolic metric measure spaces X equipped with a Borel regular locally doubling measure.

Vähäkangas (2007) replaced Cheng's (Cheng (1993)) assumption on the spectrum of the Laplacian by a curvature upper bound $K_M \le -\phi(\phi-1)/r^2$ and was able to generalise the techniques used by Cheng to show the existence of non-constant bounded $\mathcal{A}$-harmonic functions assuming also (2.2).

Holopainen and Vähäkangas (2007) (see also the unpublished licentiate thesis Vähäkangas (2006)) generalised the approach of Holopainen (2002) and Anderson and Schoen (1985) even further to allow very general curvature bounds

$$-(b\circ r)^2 \le K_M \le -(a\circ r)^2,$$

where a and b are functions satisfying assumptions (Holopainen and Vähäkangas, 2007, (A1)-(A7)) (see also [C, Section 4]). As a special case they obtain e.g. the following.

Theorem. *Let M be a Cartan-Hadamard manifold of dimension $n \ge 2$. Suppose that*

$$-r(x)^{2(\phi-2)-\varepsilon} \le K(P_x) \le -\frac{\phi(\phi-1)}{r(x)^2}, \tag{2.3}$$

$r(x) \ge R_0$, for some constants $\phi > 1$ and $\varepsilon, R_0 > 0$. Then the asymptotic Dirichlet problem for p-Laplacian is solvable for every $p \in (1, 1+(n-1)\phi)$.

And assuming a constant curvature upper bound $-k$, they can also allow the curvature to decay exponentially. Namely under the curvature bounds

$$-r(x)^{-2-\varepsilon}e^{2kr(x)} \le K_M(P_x) \le -k \tag{2.4}$$

they solve the Dirichlet problem for every $p \in (1,\infty)$.

In the unpublished preprint Vähäkangas (2009) Vähäkangas proved the existence of $\mathcal{A}$-harmonic functions under curvature assumptions similar to (2.3) and (2.4). His technique adapted the method of Cheng (1993) using Sobolev and Caccioppoli-type inequalities together with complementary Young functions. Recently Casteras, Holopainen, and Ripoll (To appeara) refined the methods of Vähäkangas (2009) and improved the curvature upper bound to (almost) optimal, assuming

$$-\frac{\big(\log r(x)\big)^{2\tilde{\varepsilon}}}{r(x)^2} \le K_M(P_x) \le -\frac{1+\varepsilon}{r(x)^2\log r(x)} \tag{2.5}$$

for some constants $\varepsilon > \tilde{\varepsilon} > 0$.

2.3 Minimal graphic functions

In this subsection we mention also some results that do not concern directly the asymptotic Dirichlet problem but are still related to the study of this thesis. Readers interested in the general theory of minimal surfaces could see e.g. the survey Meeks and Pérez (2012).

The theory of minimal surfaces is very classical and has its origin in the 18th century. One of the most interesting questions was the Plateau's problem raised originally by Lagrange (1760) in 1760, named after the Belgian physicist Joseph Plateau (1801-1883), and finally solved independently by Douglas (1931) and Radó (1930) in the beginning of 1930's. Another interesting aspect is the Bernstein-type problem that deals with minimal hypersurfaces in $\mathbb{R}^n$. The 3-dimensional case was proved by Bernstein (1927) in 1915-1917.

In 1968 Jenkins and Serrin (1968) proved the solvability of the Dirichlet problem on bounded domains $\Omega \subset \mathbb{R}^n$ whose boundary has non-negative mean curvature. Serrin (1969) gave a classical existence result for the prescribed mean curvature graphs in $\mathbb{R}^n$ and more recently Guio and Sa Earp (2005) considered similar Dirichlet problem in the hyperbolic space.

Nelli and Rosenberg (2002) constructed catenoids, helicoids and Scherk-type surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and they also proved the solvability of the asymptotic Dirichlet problem in $\mathbb{H}^2$.

Theorem. *Let Γ be a continuous rectifiable Jordan curve in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$, that is a vertical graph. Then, there exists a minimal vertical graph on $\mathbb{H}^2$ having Γ as asymptotic boundary. The graph is unique.*

In 2005 Meeks and Rosenberg (2005) developed the theory of properly embedded minimal surfaces in $N \times \mathbb{R}$, where N is a closed orientable Riemannian surface but the existence of entire minimal surfaces in product spaces $M \times \mathbb{R}$ really draw attention after the papers Collin and Rosenberg (2010) and Gálvez and Rosenberg (2010). In Collin and Rosenberg (2010) Collin and Rosenberg constructed a harmonic diffeomorphism from $\mathbb{C}$ onto $\mathbb{H}$ and hence disproved the conjecture of Schoen and Yau (1997). Gálvez and Rosenberg generalised this result to Hadamard surfaces whose curvature is bounded from above by a negative constant. A key tool in their constructions was to solve the Dirichlet problem on unbounded ideal polygons with alternating boundary values $\pm\infty$ on the sides of the ideal polygons.

Spruck (2007) established a priori gradient estimates and existence results for graphs of constant positive mean curvature in product spaces $N \times \mathbb{R}$, where N is n-dimensional simply connected and complete Riemannian manifold. Many of these results apply also to the case of zero mean curvature and especially the gradient estimate has been used in later works considering the asymptotic Dirichlet problem.

Sa Earp and Toubiana (2008) constructed minimal vertical graphs over unbounded domains in $\mathbb{H}^2 \times \mathbb{R}$ taking prescribed boundary data. do Espírito-Santo and Ripoll (2011) considered the existence of solutions to the exterior Dirichlet problem on simply connected manifolds with negative sectional curvature. Here the idea is to find minimal hypersurfaces on unbounded domains with compact boundary assuming zero boundary values.

do Espí rito Santo, Fornari, and Ripoll (2010) proved the solvability of the asymptotic Dirichlet problem with negative constant upper bound for the sectional curvature and an assumption on the isometry group of the manifold.

Rosenberg, Schulze, and Spruck (2013) studied minimal hypersurfaces in $N \times \mathbb{R}_+$ with N complete Riemannian manifold having non-negative Ricci curvature and sectional curvatures bounded from below. They proved so-called half-space properties both for properly immersed minimal surfaces and for graphical minimal surfaces. In the latter, a key tool was a gradient estimate for solutions of the minimal graph equation.

Ripoll and Telichevesky (2015) showed the existence of entire bounded nonconstant solutions for slightly larger class of operators, including minimal graph operator, by studying the strict convexity (SC) condition of the manifold. Similar class of operators was studied also by Casteras, Holopainen, and Ripoll (To appearb) but instead of considering the SC condition, they solved the asymptotic Dirichlet problem by using similar barrier functions as in Holopainen and Vähäkangas (2007). Both of these gave the existence of minimal graphic functions under the assumption (2.4) and the latter also included (2.3).

The method of Cheng adapted also to the case of minimal graphs and in Casteras, Holopainen, and Ripoll (To appeara) Casteras, Holopainen and Ripoll proved the following.

Theorem. *Let M be a Cartan-Hadamard manifold of dimension $n \geq 3$ and suppose that*

$$-\frac{\big(\log r(x)\big)^{2\tilde{\varepsilon}}}{r(x)^2} \leq K_M(P_x) \leq -\frac{1+\varepsilon}{r(x)^2 \log r(x)} \tag{2.6}$$

holds for some constants $\varepsilon > \tilde{\varepsilon} > 0$ and r large enough. Then the asymptotic Dirichlet problem is uniquely solvable.

Telichevesky (2016) considered the Dirichlet problem on unbounded domains Ω proving the existence of solutions provided that $K_M \leq -1$, the ordinary boundary of Ω is mean convex and that Ω satisfies the SC condition at infinity. The SC condition was studied by Casteras, Holopainen and Ripoll also in Casteras, Holopainen, and Ripoll (2015) and they proved that the manifold M satisfies the SC condition under very general curvature assumption. As special cases they obtain the bound (2.6) and

$$-ce^{(2-\varepsilon)r(x)}e^{e^{r(x)/e^3}} \leq K_M \leq -\phi e^{2r(x)}$$

for some constants $\phi > 1/4$, $\varepsilon > 0$ and $c > 0$.

2.4 Rotationally symmetric manifolds

The situation on rotationally symmetric manifolds is slightly different from the general n-manifolds and hence we decided to treat them separately, although the problems on these manifolds has been studied at the same time as on the general manifolds. In Choi (1984) Choi gave also a definition of the asymptotic Dirichlet problem with respect to a pole on model manifolds and in the case of a Cartan-Hadamard model, it coincides with the previous definition.

As in the case of general manifolds, the study of the existence results begun with the harmonic functions. In 1977 Milnor (1977) proved that a 2-dimensional rotationally symmetric surface M_f possess non-constant harmonic functions if and only if $\int_1^\infty 1/f(s)\,ds < \infty$. In terms of curvature bounds this gives the existence when

$$K_{M_f} \leq -\frac{1+\varepsilon}{r^2 \log r}. \tag{2.7}$$

Choi (1984) extended this result and proved that if (2.7) holds outside a compact set, then the asymptotic Dirichlet problem with respect to a pole is solvable for all $n \geq 2$.

March (1986) studied the behaviour of the Brownian motion and used the invariant σ-field to characterise the existence of harmonic functions in terms of the curvature function, obtaining the following result.

Theorem. *Let M_f be a model manifold with negative radial curvature. Then there exist non-constant bounded harmonic functions if and only if*

$$\int_1^\infty \Big(f(s)^{n-3} \int_s^\infty f(t)^{1-n} dt \Big) ds < \infty. \tag{2.8}$$

In 2-dimensional case this corresponds to the curvature bound (2.7) and when $n \geq 3$, (2.8) is equivalent to

$$K_{M_f} \leq -\frac{1/2+\varepsilon}{r^2 \log r}.$$

Murata (1992) gave an analytic proof that (2.8) is equivalent to either (i) M_f does not have strong Liouville property or (ii) the asymptotic Dirichlet problem is solvable. A simple analytic proof for the existence part of March's result can be found from Vähäkangas' licentiate thesis Vähäkangas (2006).

In 2012 Ripoll and Telichevesky (2012) considered the asymptotic Dirichlet problem for the minimal graph equation. They proved the existence of entire non-constant bounded minimal graphic functions on 2-dimensional Hadamard surfaces assuming (2.8), i.e. the curvature upper bound (2.7). Idea in the proofs in Vähäkangas (2006) and Ripoll and Telichevesky (2012) is to use (2.8) to construct barriers at infinity.

2.5 Non-existence of solutions

By the non-existence results in $\mathbb{R}^n$, it is already clear that the curvature upper bound must be strictly negative but the discussion about the rotationally symmetric case and the theorems replacing the sectional curvature lower bound with the pinching condition (2.2) raise a question about the necessity of the lower bound. However, when M is a general n-dimensional Cartan-Hadamard manifold it is not enough to assume only the curvature upper bound.

Concerning results in this direction, Ancona (1994) proved in 1994 the following.

Theorem. *There exists a 3-dimensional Cartan-Hadamard manifold with $K_M \leq -1$ such that the asymptotic Dirichlet problem for the Laplacian is not solvable.*

His construction of such manifold was based on probabilistic methods. Namely, he proved the non-solvability of the asymptotic Dirichlet problem by showing that Brownian motion almost surely exits M at a single point on the asymptotic boundary.

Borbély (1998) constructed similar manifold using analytic arguments and later Ulsamer (2004) showed that Borbély's manifold can be constructed also with probabilistic methods, and generalised the Anconas result to higher dimensions. Arnaudon, Thalmaier, and Ulsamer (2009) continued the probabilistic study of these manifolds.

Holopainen (2016) generalised Borbély's example to cover also the p-harmonic functions and then Holopainen and Ripoll (2015) proved that the same example works also for the minimal graph equation. These results show that apart from the 2-dimensional or the rotationally symmetric setting, one really needs to have a control also on the lower bound.

It is also worth pointing out two closely related results by Greene and Wu (1982) that partly answer the question about the optimal curvature upper bound. Firstly, in (Greene and Wu, 1982, Theorem 2 and Theorem 4) they showed that an n-dimensional, $n \neq 2$, Cartan-Hadamard manifold with asymptotically non-negative sectional curvature is isometric to $\mathbb{R}^n$. Secondly, in (Greene and Wu, 1982, Theorem 2) they showed that an odd dimensional Riemannian manifold with a pole $o \in M$ and everywhere non-positive or everywhere non-negative sectional curvature is isometric to $\mathbb{R}^n$ if $\liminf_{s\to\infty} s^2 k(s) = 0$, where $k(s) = \sup\{|K(P_x)|\colon x \in M,\, d(o,x) = s, P_x \in T_xM \text{ two-plane}\}$.

3 POINTWISE PINCHING CONDITION FOR THE SECTIONAL CURVATURES

3.1 Background

To solve the asymptotic Dirichlet problem for the Laplacian, Anderson and Schoen, Anderson and Schoen (1985), solve the problem

$$\begin{cases} \Delta u_R = -\Delta f & \text{in } B(o,R), \\ u_R = 0 & \text{on } \partial B(o,R) \end{cases} \tag{3.1}$$

in geodesic balls and then construct a barrier function to be able to extract a converging subsequence from $(u_R + f)$ when $R \to \infty$. This process relays highly on the curvature assumption $-b^2 \leq K_M \leq -a^2$.

Assuming only a pointwise pinching condition

$$|K_M(P_x)| \leq C_K |K_M(P'_x)| \tag{3.2}$$

and positivity of the first eigenvalue of the Laplacian, Cheng (1993) was able to relax the curvature assumptions of Anderson and Schoen. To prove the claim, it is still necessary to extract the converging subsequence and show the correct boundary values at infinity but for this end Cheng's approach did not use barriers. His proof of convergence is based on an L^p-norm estimate, namely, he proves an upper bound for the L^p-norm of a solution in compact subsets in terms of the L^p-norm of $|\nabla f|$.

In order to show the correct boundary values of u at infinity, Cheng uses the assumption $|\nabla f| \in L^p$ and Moser iteration technique to prove that the supremum of $|u|^p$ on a ball $B(x,(1-\varepsilon)R)$, $\varepsilon \in (0,1)$, is bounded in terms of the integral of $|u|^p$ over $B(x,R)$. The last step is to show that the gradient of radially constant function is in L^p and this is the step requiring condition (3.2).

Vähäkangas (2007) replaced the assumption on the eigenvalue by a curvature upper bound

$$K_M(P_x) \leq -\frac{\phi(\phi-1)}{r(x)^2}, \quad \phi > 1,$$

and showed that the same result holds also for the p-Laplacian. The approach in his proof was essentially the same as Cheng's. In Vähäkangas (2009) Vähäkangas refined this argument with help of Young functions and was able to prove the solvability result for $\mathcal{A}$-harmonic functions under the curvature assumptions of Holopainen and Vähäkangas (2007).

These ideas involving Young functions was also used in Casteras, Holopainen, and Ripoll (To appeara) where Casteras, Holopainen and Ripoll solved the asymptotic

Dirichlet problem for the minimal graph equation and the $\mathcal{A}$-harmonic equation under the assumption (2.6).

3.2 Articles [A] and [B] revisited

In [A] we generalise the result of Vähäkangas (2007) and prove that under the same curvature assumptions the asymptotic Dirichlet problem is solvable also for the minimal graph equation. To be more precise, our main theorem is the following.

Theorem 3.3 ([A, Theorem 1.3]). *Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$ and let $\phi > 1$. Assume that*

$$K(P) \leq -\frac{\phi(\phi-1)}{r(x)^2}, \tag{3.4}$$

where $K(P)$ is the sectional curvature of any two-dimensional subspace $P \subset T_xM$ containing the radial vector $\nabla r(x)$, with $x \in M \setminus B(o, R_0)$. Suppose also that there exists a constant $C_K < \infty$ such that

$$|K(P)| \leq C_K |K(P')|$$

whenever $x \in M \setminus B(o, R_0)$ and $P, P' \subset T_xM$ are two-dimensional subspaces containing the radial vector $\nabla r(x)$. Moreover, suppose that the dimension n and the constant ϕ satisfy the relation

$$n > \frac{4}{\phi} + 1. \tag{3.5}$$

Then the asymptotic Dirichlet problem for the minimal graph equation is uniquely solvable for any boundary data $f \in C(\partial_\infty M)$.

We notice that if we choose the constant ϕ in the curvature assumption to be bigger than 4, then our theorem holds in every dimension $n \geq 2$. Similarly, if we let the dimension n to be at least 5, we can take the constant ϕ to be as close to 1 as we wish.

In [B] we improve the results of Vähäkangas (2007, 2009) and Casteras, Holopainen, and Ripoll (To appeara) and show that in the case of $\mathcal{A}$-harmonic functions it is possible to solve the asymptotic Dirichlet problem assuming only the pinching condition (3.2) and a weaker curvature upper bound. A localised argument proving the $\mathcal{A}$-regularity of points $x_0 \in \partial_\infty M$ leads to the main theorem of [B].

Theorem 3.6 ([B, Theorem 1.3]). *Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$. Assume that*

$$K(P) \leq -\frac{1+\varepsilon}{r(x)^2 \log r(x)}, \tag{3.7}$$

for some constant $\varepsilon > 0$, where $K(P)$ is the sectional curvature of any two-dimensional subspace $P \subset T_xM$ containing the radial vector $\nabla r(x)$, with $x \in M \setminus B(o, R_0)$. Suppose *also that there exists a constant* $C_K < \infty$ *such that*

$$|K(P)| \leq C_K |K(P')| \tag{3.8}$$

whenever $x \in M \setminus B(o, R_0)$ *and* $P, P' \subset T_x M$ *are two-dimensional subspaces containing the radial vector* $\nabla r(x)$*. Then the asymptotic Dirichlet problem for the* $\mathcal{A}$*-harmonic equation is uniquely solvable for any boundary data* $f \in C(\partial_\infty M)$ *provided that* $1 < p < n\alpha/\beta$*.*

In the case of usual Laplacian we have $\alpha = \beta = 1$ and $p = 2$. Hence we obtain the following special case.

Corollary 3.9 ([B, Corollary 1.6])**.** *Let* M *be a Cartan-Hadamard manifold of dimension* $n \geq 3$ *and assume that the assumptions* (3.7) *and* (3.8) *are satisfied. Then the asymptotic Dirichlet problem for the Laplace operator is uniquely solvable for any boundary data* $f \in C(\partial_\infty M)$*.*

It is also worth pointing out that in dimension $n = 2$ the condition (3.2) is trivially satisfied since at any point $x \in M$ there exists only one tangent plane P_x. Therefore it is enough to assume only a curvature upper bound and we obtain the following corollaries.

Corollary 3.10. *Let* M *be a 2-dimensional Cartan-Hadamard manifold and let* $\phi > 4$*. Assume that*

$$K(P) \leq -\frac{\phi(\phi - 1)}{r(x)^2},$$

where $K(P)$ *is the sectional curvature of a two-dimensional subspace* $P \subset T_x M$ *containing the radial vector* $\nabla r(x)$*, with* $x \in M \setminus B(o, R_0)$*. Then the asymptotic Dirichlet problem for the minimal graph equation is uniquely solvable for any boundary data* $f \in C(\partial_\infty M)$*.*

Corollary 3.11. *Let* M *be a 2-dimensional Cartan-Hadamard manifold or* n*-dimensional rotationally symmetric Cartan-Hadamard manifold satisfying the curvature upper bound* (3.7)*. Then the asymptotic Dirichlet problem for the* $\mathcal{A}$*-harmonic equation is uniquely solvable for any boundary data* $f \in C(\partial_\infty M)$ *provided that* $1 < p < n\alpha/\beta$*.*

As it was pointed out in Casteras, Holopainen, and Ripoll (To appeara) (see also [D, Theorem 5.1]), the curvature upper bound and the range of p, $1 < p < n\alpha/\beta$, in Theorem 3.6 are in a sense optimal. Namely, if we assume that

$$K(P) \geq -\frac{1}{r(x)^2 \log r(x)}$$

and consider $\mathcal{A}$-harmonic operator of type $p \geq n$, it follows that M is p-parabolic, i.e. every bounded $\mathcal{A}$-harmonic function (of type p) is constant.

In Cheng's proof one of the key points was to show the L^p-bound for a solution u and in the proofs of Theorems 3.3 and 3.6 we need a similar estimate. However, instead of just considering the norm of u, we take an auxiliary smooth function $\varphi\colon [0, \infty) \to [0, \infty)$, related to Young functions, and show the bound for $\varphi(|u-\theta|/c)$. In [A] θ is a radial extension of the boundary data function and in [B] it is a certain continuous function that can also be thought as a boundary data. Once we have the integral estimate, it remains to show that we can bound the supremum of $\varphi(|u-\theta|)$ in $B(x,s/2)$ in terms of the integral of $\varphi(|u-\theta|)$ over $B(x,s)$. Together these estimates guarantee that $u(x)\to\theta(x_0)$ as $x\to x_0\in\partial_\infty M$.

3.2.1 *Integral bounds for solutions*

Vähäkangas (Vähäkangas, 2009, Lemma 2.17) proved an integral estimate for $\mathcal{A}$-harmonic functions under the curvature assumption $K_M\le -\phi(\phi-1)/r^2$. Clever idea in his proof was to use a Caccioppoli-type inequality, special type of Young functions F and G, and Young's inequality. Taking certain smooth homeomorphism $H\colon[0,\infty)\to[0,\infty)$ he defined $G(t)=\int_0^t H(s)ds$ and $F(t)=\int_0^t H^{-1}(s)ds$. Then

$$\psi(t)=\int_0^t \frac{ds}{G^{-1}(s)}$$

and $\varphi=\psi^{-1}$ are homeomorphisms so that $G\circ\varphi'=\varphi$. For the functions F and G we have the Young's inequality

$$ab\le F(a)+G(b)$$

and the idea is to reduce the integrability of $\varphi(|u-\theta|)$ to the integrability of $F(|\nabla\theta|w)$ for some Lipschitz weight function w. In order to do this, a Caccioppoli-type inequality (Vähäkangas, 2009, Lemma 2.15)

$$\begin{aligned}\left(\int_U \eta^p\psi'(h)|\nabla u|^p\right)^{1/p} &\le \frac{\beta}{\alpha}\left(\int_U \eta^p\psi'(h)|\nabla\theta|^p\right)^{1/p}\\ &\quad+\frac{p\beta}{\alpha}\left(\int_U \frac{\psi^p}{(\psi')^{p-1}}(h)|\nabla\eta|^p\right)^{1/p},\end{aligned} \tag{3.12}$$

$h=|u-\theta|$, plays a central role.

Refining this idea Casteras, Holopainen and Ripoll proves the L^p-estimate for $\mathcal{A}$-harmonic functions under the curvature assumption (3.7).

Lemma 3.13. *(Casteras, Holopainen, and Ripoll, 2015, Lemma 16) Let M be a Cartan-Hadamard manifold satisfying* (3.7)*. Suppose that $U\subset M$ is an open relatively compact set and that u is an $\mathcal{A}$-harmonic function in U with $u-\theta\in W_0^{1,p}(U)$, where $\mathcal{A}\in\mathcal{A}^p(M)$ with*

$$1<p<\frac{n\alpha}{\beta},$$

and $\theta\in W^{1,\infty}(M)$ is a continuous function with $||\theta||_\infty\le 1$. Then there exists a bounded C^1-function $\mathcal{C}\colon[0,\infty)\to[0,\infty)$ and a constant $c_0\ge 1$, that is independent of θ,U and u, *such that*

$$\int_U \varphi\big(|u-\theta|/c_0\big)^p \big(\log(1+r)+\mathcal{C}(r)\big)$$
$$\le c_0 + c_0\int_U F\left(\frac{c_0|\nabla\theta| r\log(1+r)}{\log(1+r)+\mathcal{C}(r)}\right)\big(\log(1+r)+\mathcal{C}(r)\big).$$

In [A] also the second derivative φ'' appears in the estimates and hence we need also another pair, F_1 and G_1, of Young functions so that $G_1\circ\varphi''\approx\varphi$. Then, with the Caccioppoli-type inequality [A, Lemma 3.1]

$$\int_U \eta^2\varphi'(|u-\theta|/\nu)\frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} \le C_\varepsilon\int_U \eta^2\varphi'(|u-\theta|/\nu)|\nabla\theta|^2 + (4+\varepsilon)\nu^2\int_U \frac{\varphi^2}{\varphi'}(|u-\theta|/\nu)|\nabla\eta|^2, \quad (3.14)$$

we are able to obtain similar estimate if the gradient of θ is bounded in terms of the infimum $j(x)$ of the norms $|V(x)|$ of the Jacobi fields V along geodesic $\gamma^{o,x}$.

Lemma 3.15 ([A, Lemma 3.3]). *Let M be a Cartan-Hadamard manifold satisfying* (3.4) *and* (3.5). *Let $U=B(o,R)$, with $R>0$ big enough, and suppose that $u\in C^2(U)\cap C(\bar U)$ is the unique solution to the minimal graph equation in U, with $u|\partial U=\theta|\partial U$, where $\theta\colon M\to\mathbb{R}$ is a Lipschitz function, with $|\nabla\theta(x)|\le 1/j(x)$ almost everywhere. Then there exists a constant c independent of u such that*

$$\int_U \varphi(|u-\theta|/c)\le c + c\int_U F(r|\nabla\theta|) + c\int_U F_1(r^2|\nabla\theta|^2).$$

The integrability of functions F and F_1 in the previous lemmata follows from their construction and from the assumptions on the curvature and function θ.

3.2.2 *Pointwise estimates*

The last major step is to pass from the integral estimates to pointwise estimates. Together with the Caccioppoli-type inequality (3.12), the Sobolev inequality (see e.g. Hoffman and Spruck (1974))

$$\left(\int_{B(x,r_S)}|\eta|^{n/(n-1)}\right)^{(n-1)/n}\le C_S\int_{B(x,r_S)}|\nabla\eta|, \quad (3.16)$$

$\eta\in C_0^\infty(B(x,r_S))$, and a Moser iteration procedure Vähäkangas obtains the supremum estimate (Vähäkangas, 2009, Lemma 2.20)

$$\operatorname*{ess\,sup}_{B(x,s/2)}\varphi\big(|u-\theta|\big)^{p(n-1)}\le c\int_{B(x,s)}\varphi\big(|u-\theta|\big)^p$$

for $\mathcal{A}$-harmonic functions $u \in W^{1,p}_{\mathrm{loc}}(M)$, with $u - \theta \in W^{1,p}_0(\Omega)$, $\inf_M \theta \le u \le \sup_M \theta$, and $u = \theta$ a.e. in $M \setminus \Omega$.

In [A] we prove a similar estimate for the minimal graphic functions and, again, the Caccioppoli-type inequality (3.14), the Sobolev inequality (3.16) and a Moser iteration procedure are the main tools.

Lemma 3.17 ([A, Lemma 3.4]). *Let $\Omega = B(o,R)$ and suppose that $\theta\colon \Omega \to \mathbb{R}$ is a bounded Lipschitz function with $|\theta|, |\nabla\theta| \le C_1$. Let $u \in C^2(\Omega)$ be a solution of the minimal graph equation in Ω such that u has the boundary values θ and $\inf_\Omega \theta \le u \le \sup_\Omega \theta$. Fix $s \in (0, r_S)$, where r_S is the radius of the Sobolev inequality* (3.16)*, and suppose that $B = B(x,s) \subset \Omega$. Then there exists a positive constant $\nu_0 = \nu_0(\varphi, C_1)$ such that for all fixed $\nu \ge \nu_0$*

$$\sup_{B(x,s/2)} \varphi\big(|u-\theta|/\nu\big)^{n+1} \le c \int_B \varphi\big(|u-\theta|/\nu\big),$$

where c is a positive constant depending only on n, ν, s, C_S, C_1 and φ.

3.2.3 *Further questions*

It remains open whether the curvature upper bound (3.4) could be relaxed to

$$K(P_x) \le -\frac{1+\varepsilon}{r(x)^2 \log r(x)} \tag{3.18}$$

since the methods used in [A] or in Casteras, Holopainen, and Ripoll (To appeara) do not apply to this case. In Casteras, Holopainen, and Ripoll (To appeara) they have the upper bound (3.18) but they also assume a lower bound for the sectional curvatures, which enables to have an a priori gradient estimate. This is needed to obtain (Casteras, Holopainen, and Ripoll, To appeara, Lemma 22).

Another question concerns the condition (3.5). It is a technical assumption coming from the Caccioppoli-type inequality (3.14) and there should not be a deeper reason requiring it.

4 f-MINIMAL GRAPHS

Let M be an n-dimensional Riemannian manifold with a Riemannian metric given by $ds^2 = \sigma_{ij} dx^i dx^j$ in local coordinates. Assume that $f\colon N \to \mathbb{R}$ is a smooth function, where $N = M \times \mathbb{R}$ is equipped with the product metric $ds^2 + dt^2$. Then f-minimal graphs are special type of surfaces with prescribed mean curvature, namely graphs of functions $u\colon \Omega \to \mathbb{R}$ that are solutions to the f-minimal graph equation

$$\begin{cases} \operatorname{div} \dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \langle \bar{\nabla} f, \nu \rangle & \text{in } \Omega; \\ u|\partial\Omega = \varphi, \end{cases} \tag{4.1}$$

where $\Omega \subset M$ is a bounded domain, $\bar{\nabla} f$ is the gradient of f with respect to the product Riemannian metric, and ν denotes the downward unit normal to the graph of u, i.e.

$$\nu = \frac{(\nabla u, -1)}{\sqrt{1+|\nabla u|^2}}. \tag{4.2}$$

More generally an f-minimal hypersurface Σ is an immersed hypersurface of a Riemannian manifold (N, g) whose mean curvature satisfies

$$H = \langle \bar{\nabla} f, \nu \rangle$$

at every point of Σ. To get some interpretation of f-minimal surfaces we mention the following examples:

(a) minimal hypersurfaces if f is identically constant,
(b) self-shrinkers in $\mathbb{R}^{n+1}$ if $f(x) = |x|^2/4$,
(c) minimal hypersurfaces of weighted manifolds $M_f = \big(M, g, e^{-f} d\,\mathrm{vol}_M\big)$, where (M, g) is a complete Riemannian manifold with the Riemannian volume element $d\,\mathrm{vol}_M$.

A reader interested in recent studies on self-shrinkers and f-minimal hypersurfaces should see Wang (2011), Colding and Minicozzi (2012b), Colding and Minicozzi (2012a), Cheng, Mejia, and Zhou (2014), Cheng, Mejia, and Zhou (2015a), Cheng, Mejia, and Zhou (2015b), Impera and Rimoldi (2015), and references therein.

As a remark we point out that we cannot ask for the uniqueness of a solution of (4.1) if the function $f\colon M \times \mathbb{R} \to \mathbb{R}$ depends on the t-variable since comparison principles fail to hold, see (Gilbarg and Trudinger, 2001, Theorem 10.1). A simple counter example is obtained if one considers the function $f\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, $f(x,t) = |(x,t)|^2/4$ and the open disk $B(0,2) \subset \mathbb{R}^2$. Namely, then both the upper and lower hemispheres and the disk $B(0,2)$ itself are f-minimal hypersurfaces with zero boundary values on the circle $\partial B(0,2)$.

4.1 Background

4.1.1 *Barrier method*

A priori estimates and the barrier method goes back to the work Bernstein (1906, 1910) and has been widely used to solve Dirichlet problems for different PDEs. A classical way to construct barriers on bounded domains is to use the distance function to the boundary and combine it with some auxiliary function h satisfying i.a. $h(0) = 0$. For a comprehensive description of the method one should see e.g. Gilbarg and Trudinger (2001). For more recent research, with similar choice of the function h as in [C], see Spruck (2007) and Dajczer, Hinojosa, and de Lira (2008).

To obtain a priori interior gradient estimates, at least for the mean curvature equation, is not as straightforward as the cases of height and boundary gradient estimates. In 1986 Korevaar (1986) introduced two different approaches to obtain the estimate. His idea is to use a carefully chosen cutoff function η and then prove a priori bound for

$$\eta\big(x, u(x)\big)W(x),$$

$W = \sqrt{1 + |\nabla u|^2}$, in a ball $B(0,1) \subset \mathbb{R}^n$. The first approach is "Standard form calculation", suggested by L. Simon, that is based on direct computations at a maximum point of ηW. The second approach is to perturb the surface along its downward normal and then lift the perturbed surface in order to try to get a barrier.

More recently, and in the manifold setting, Korevaar's (also Korevaar-Simon) method has been used for example in Spruck (2007), Rosenberg, Schulze, and Spruck (2013), Dajczer, de Lira, and Ripoll (2016).

4.1.2 ***Barrier at infinity***

The approach of Anderson and Schoen (1985) was based on the idea of extending a continuous boundary value function $\varphi\colon \partial_\infty M \to \mathbb{R}$ radially to the whole $\bar{M}$. Then after a suitable smoothening procedure they obtain sub- and superharmonic functions that can be used as barriers. Holopainen (2002) used similar technique to prove the solvability of the asymptotic Dirichlet problem for p-Laplacian under the same curvature assumption

$$-b^2 \le K \le -a^2$$

for some constants $b \ge a > 0$.

Holopainen and Vähäkangas (2007) generalised this approach to cover the more general curvature conditions (2.3) and (2.4) for the p-Laplacian. In order to allow the more general bounds, their smoothening procedure depends also on the curvature lower bound. This difference to the earlier proofs results to very technical and long computations. However, the barrier function that they obtained has appeared to be very flexible and suit also other PDEs, like the minimal graph equation which was considered in Casteras, Holopainen, and Ripoll (To appearb). We will use their constructions also in [C].

4.2 Article [C]

The article [C] is divided roughly into two parts: In the first part we study the existence of f-minimal graphs over bounded domains Ω with continuous boundary values on $\partial\Omega$ and in the second part we prove the existence of entire f-minimal graphs by solving the asymptotic Dirichlet problem. In the first part, under a technical assumption that $f \in C^2(\bar{\Omega} \times \mathbb{R})$ is of the form

$$f(x,t) = m(x) + r(t), \tag{4.3}$$

we obtain the following existence result.

Theorem 4.4 ([C, Theorem 1.2])**.** *Let $\Omega \subset M$ be a bounded domain with $C^{2,\alpha}$ boundary $\partial\Omega$. Suppose that $f \in C^2(\bar{\Omega} \times \mathbb{R})$ satisfies* (4.3)*, with*

$$F = \sup_{\bar{\Omega}\times\mathbb{R}} |\bar{\nabla} f| < \infty, \quad \operatorname{Ric}_\Omega \geq -\frac{F^2}{n-1}, \quad \text{and} \quad H_{\partial\Omega} \geq F.$$

Then, for all $\varphi \in C(\partial\Omega)$, there exists a solution $u \in C^{2,\alpha}(\Omega) \cap C(\bar{\Omega})$ to the equation (4.1) *with boundary values φ.*

A standard way to obtain solutions for PDEs with $C^{2,\alpha}$ boundary values is to use the Leray-Schauder method (Gilbarg and Trudinger, 2001, Theorem 13.8), that we have chosen, or the continuity method (Gilbarg and Trudinger, 2001, Theorem 5.2, Theorem 17.8). Both of these options reduces the question of the solvability of the Dirichlet problem to the existence of a priori height and gradient (both boundary and interior) estimates. Finally the reduction of the smoothness of boundary data is obtained via similar approximation as in Dajczer, de Lira, and Ripoll (2016). This is possible since the local interior gradient estimate [C, Lemma 2.3] does not depend on the gradient of the boundary data.

In the second part of the article, applying this existence result above, we are able to generalise the result of Holopainen and Vähäkangas (2007) and show that, under the same very general curvature assumptions, the asymptotic Dirichlet problem is solvable for the f-minimal graph equation.

Before stating the main results, we need to give some technical definitions and assumptions on the function f. We assume that there exists an auxiliary smooth function $a_0\colon [0,\infty) \to (0,\infty)$ such that

$$\int_1^\infty \left(\int_r^\infty \frac{ds}{f_a^{n-1}(s)} \right) a_0(r) f_a^{n-1}(r) dr < \infty,$$

for the discussion about the choice of a_0 see [C, Example 4.5] and [C, Example 4.6]. Then we define $g\colon [0,\infty) \to [0,\infty)$ by

$$g(r) = \frac{1}{f_a^{n-1}(r)} \int_0^r a_0(t) f_a^{n-1}(t) dt. \tag{4.5}$$

The function g was introduced in Mastrolia, Monticelli, and Punzo (2015) where they studied elliptic and parabolic equations with asymptotic Dirichlet boundary conditions on Cartan-Hadamard manifolds. In addition to (4.3), we assume that the function $f \in C^2(\bar\Omega \times \mathbb{R})$ satisfies

$$\sup_{\partial B(o,r)\times\mathbb{R}} |\bar\nabla f| \le \min\left\{ \frac{a_0(r) + (n-1)\frac{f_a'(r)}{f_a(r)} g^3(r)}{\big(1+g^2(r)\big)^{3/2}}, (n-1)\frac{f_a'(r)}{f_a(r)} \right\}, \tag{4.6}$$

for every $r > 0$, and

$$\sup_{\partial B(o,r)\times\mathbb{R}} |\bar\nabla f| = o\left(\frac{f_a'(r)}{f_a(r)} r^{-\varepsilon-1}\right) \tag{4.7}$$

for some $\varepsilon > 0$ as $r \to \infty$. Then, as special cases of the main result [C, Theorem 1.3], we obtain the following corollaries.

Corollary 4.8 ([C, Corollary 1.34]). *Let M be a Cartan-Hadamard manifold of dimension $n \ge 2$. Suppose that there are constants $\phi > 1$, $\varepsilon > 0$, and $R_0 > 0$ such that*

$$-\rho(x)^{2(\phi-2)-\varepsilon} \le K(P_x) \le -\frac{\phi(\phi-1)}{\rho(x)^2}, \tag{4.9}$$

for all 2-dimensional subspaces $P_x \subset T_xM$ and for all $x \in M$, with $\rho(x) \ge R_0$. Assume, furthermore, that $f \in C^2(M\times\mathbb{R})$ satisfies (4.3), (4.6), and (4.7), with $f_a(t) = t$ for small $t \ge 0$ and $f_a(t) = c_1 t^{\phi} + c_2 t^{1-\phi}$ for $t \ge R_0$. Then the asymptotic Dirichlet problem for equation (4.1) is solvable for any boundary data $\varphi \in C\big(\partial_\infty M\big)$.

In another special case we assume that sectional curvatures are bounded from above by a negative constant $-k^2$ but allow the lower bound to decrease even exponentially.

Corollary 4.10 ([C, Corollary 1.5]). *Let M be a Cartan-Hadamard manifold of dimension $n \ge 2$. Assume that*

$$-\rho(x)^{-2-\varepsilon} e^{2k\rho(x)} \le K(P_x) \le -k^2 \tag{4.11}$$

for some constants $k > 0$ and $\varepsilon > 0$ and for all 2-dimensional subspaces $P_x \subset T_xM$, with $\rho(x) \ge R_0$. Assume, furthermore, that $f \in C^2(M\times\mathbb{R})$ satisfies (4.3), (4.6), and (4.7), with $f_a(t) = t$ for small $t \ge 0$ and $f_a(t) = c_1\sinh(kt) + c_2\cosh(kt)$ for $t \ge R_0$. Then the asymptotic Dirichlet problem for the equation (4.1) is solvable for any boundary data $\varphi \in C\big(\partial_\infty M\big)$.

The proof of the solvability of the asymptotic Dirichlet problem follows the usual path of solving the problem in a sequence of geodesic balls, hence obtaining a sequence of solutions. Then the last part is to show the existence of a limit that is a solution with correct boundary values on $\partial_\infty M$. In order to extract the converging subsequence, we have to prove a uniform height estimate [C, Lemma 4.4] for the sequence of solutions. The correct behaviour at infinity can be then proved with suitable barrier functions.

4.2.1 *A priori estimates*

The usual way to obtain a priori height and boundary gradient estimates for solutions u in bounded domains Ω is to construct upper and lower barriers using the distance function $d(\cdot) = \operatorname{dist}(\cdot, \partial\Omega)$ to the boundary. Then these barriers, together with the comparison principle, implies the desired estimates. This procedure requires two key assumptions: The (inward) mean curvature of the level sets of d is bounded from below by the prescribed mean curvature of the graph of u in some neighbourhood of $\partial\Omega$ and, of course, that the distance function is smooth enough.

The mean curvature assumption in the neighbourhood of $\partial\Omega$ can be replaced by an assumption on the boundary and by a lower bound for the Ricci curvature. Namely, denoting by $\Omega_0 \subset \Omega$ the open set of points that can be joined to $\partial\Omega$ by unique minimising geodesic, it follows that if

$$H_{\partial\Omega} \geq F \quad \text{and} \quad \operatorname{Ric}_\Omega \geq -F^2/(n-1)$$

then $H(x_0) \geq F$ for all $x_0 \in \Omega_0$. Here $H(x_0)$ denotes the mean curvature of the level set of d passing through x_0. This is done in [C, Lemma 3.1] (see also (Spruck, 2007, Lemma 4.2) and (Dajczer, Hinojosa, and de Lira, 2008, Lemma 5)) and the proof is based on a Riccati equation for the shape operator. The smoothness of the distance function in Ω_0 was proved in Li and Nirenberg (2005), to wit, in Ω_0 d has the same regularity as the boundary $\partial\Omega$.

In order to use the comparison principle we have to "freeze" the mean curvature term $\langle \bar\nabla f, \nu \rangle$ in (4.1). More precisely, if u is a solution of (4.1),

$$Q[u] = \frac{1}{W}\left(\sigma^{ij} - \frac{u^i u^j}{W^2}\right) u_{i;j} - \langle \bar\nabla f, \nu_u \rangle ,$$

we define an operator

$$\tilde{Q}[v] = \frac{1}{W}\left(\sigma^{ij} - \frac{v^i v^j}{W^2}\right) v_{i;j} - b,$$

where $W = \sqrt{1 + |\nabla v|^2}$ and

$$b(x) = \langle \bar\nabla f\big((x, u(x)\big), \nu(x) \rangle .$$

The reason for this is that the operator Q need not satisfy the required assumptions of comparison principles, see e.g. (Gilbarg and Trudinger, 2001, Theorem 10.1), whereas $\tilde{Q}$ does. The desired height estimate is finally obtained in [C, Lemma 2.1] and the boundary gradient estimate in [C, Lemma 2.2].

The interior gradient estimate is obtained in [C, Lemma 2.3] and the proof is based on the method due to Korevaar and Simon Korevaar (1986), see also Dajczer, de Lira, and Ripoll (2016) in the case of Killing graphs. The estimate is localised to balls $B(o,r) \subset \Omega$ and if the solution is $C^1(\bar{\Omega})$ we have also a global gradient estimate with upper bound depending also on the gradient on the boundary. Idea is to have an auxiliary smooth function η vanishing outside $B(o,r)$ and then consider a function

$$h = \eta W$$

with $W = \sqrt{1+|\nabla u|^2}$. It follows that the function h attains its maximum at some point $p \in B(o,r)$ and this permits to prove an upper bound for W, and hence also for $|\nabla u|$. It is in this part of the paper where we need the asumption (4.3), namely, for technical reasons we need to assume that all the "space derivatives"

$$f_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, \dim M$$

are independent of t, i.e. $f_{it} = f_{ti} = 0$.

4.2.2 *Entire f-minimal graphs*

First step of solving the asymptotic Dirichlet problem is to consider an exhaustion of M and obtain a sequence of solutions. A natural exhaustion is, of course, the sequence of geodesic balls $B(o,k)$, $k \in \mathbb{N}$, for which the boundary mean curvature assumption of Theorem 4.4 is satisfied. More precisely, we have

$$H(x) = \Delta r(x) \geq (n-1)\frac{f_a'\big(r(x)\big)}{f_a\big(r(x)\big)} \geq \sup_{\partial B(o,r(x))\times\mathbb{R}} |\bar{\nabla} f|,$$

where $H(x)$ denotes the inward mean curvature of the level set $\{y \in \bar{B}(o,R)\colon d(y) = d(x)\} = \partial B(o,r(x))$ and the last estimate follows from the assumption (4.6). This implies that we can even drop the assumption on the Ricci curvature. This step is done in [C, Lemma 4.7].

In order to obtain the uniform height estimate [C, Lemma 4.4] we use a function V,

$$V(x) = V\big(r(x)\big) = \left(\int_{r(x)}^{\infty} \frac{ds}{f_a^{n-1}(s)}\right)\left(\int_0^{r(x)} a_0(t) f_a^{n-1}(t)dt\right)$$
$$- \int_0^{r(x)} \left(\int_t^{\infty} \frac{ds}{f_a^{n-1}(s)}\right) a_0(t) f_a^{n-1}(t)dt - H + ||\varphi||_\infty,$$

$$H := \limsup_{r\to\infty}\Big\{\int_r^\infty \frac{ds}{f_a^{n-1}(s)}\int_0^r a_0(t)f_a^{n-1}(t)dt - \int_0^r\int_t^\infty \frac{ds}{f_a^{n-1}(s)}a_0(t)f_a^{n-1}(t)dt\Big\} \leq 0,$$

constructed in Mastrolia, Monticelli, and Punzo (2015). There it was used as a supersolution for an elliptic equation but it turns out that under the assumption (4.6) V works also as an upper barrier for the f-minimal equation. Then, replacing V by $-V$, we obtain a lower barrier and together these imply the desired height estimate. Even though (4.6) seems a very technical assumption, it is not more restrictive than (4.7), see [C, Example 4.5] and [C, Example 4.6].

Final crucial step is to prove the correct boundary values on $\partial_\infty M$ and this requires barriers at infinity. It turns out that the barrier function

$$\psi = A(R_3^\delta r^{-\delta} + h)$$

used in Holopainen and Vähäkangas (2007) is very flexible and it suits also the case of f-minimal graphs, see [C, Lemma 4.3]. The assumption (4.7) for the asymptotic behaviour of the gradient $\bar{\nabla} f$ is required in this part of the article.

5 OPTIMALITY OF THE CURVATURE BOUNDS

5.1 Background

The background of the Korevaar-Simon method for obtaining interior gradient estimates was discussed in Section 4.1.1. However, we mention two articles that are closely related to our results. In Rosenberg, Schulze, and Spruck (2013) Rosenberg, Schulze and Spruck proved a gradient estimate for minimal graphic functions $M \times \mathbb{R} \to \mathbb{R}$ assuming non-negative Ricci curvature and negative constant lower bound for the sectional curvatures. This estimate was applied to prove a half-space property for non-negative solutions of the minimal graph equation.

Dajczer and de Lira (2015) extended this result for the Killing graphs in warped products $M \times_\varrho \mathbb{R}$ proving that, under certain assumptions on the manifold, any bounded entire Killing graph with constant mean curvature must be a slice. The key ingredient of their proof was a global gradient estimate that extended the result in Rosenberg, Schulze, and Spruck (2013).

Harnack's inequalities has been studied so widely that it is impossible to give a brief background about the developments in different settings so we just mention the works Grigor'yan and Saloff-Coste (2005), Holopainen (1999), and Li and Tam (1995) that are closely related to [D]. Concerning the background of the asymptotic Dirichlet problems on rotationally symmetric cases, see Section 2.4.

5.2 Article [D]

The motivation for the study of the article [D] was to show that the curvature upper bound

$$K_M \leq -\frac{C}{r^2 \log r} \tag{5.1}$$

really is the best that one can hope in order to show the existence of entire bounded non-constant solutions for the minimal graph equation. The article [D] consists of two parts, namely, the first part deals with non-existence type results and the latter with existence on rotationally symmetric manifolds. In order to prove these non-existence results we assume that the manifold has only one end and asymptotically non-negative sectional curvature, that is

Definition 5.2. Manifold M has asymptotically non-negative sectional curvature (ANSC) if there exists a continuous decreasing function $\lambda\colon [0,\infty) \to [0,\infty)$ such that

$$\int_0^\infty s\lambda(s)\, ds < \infty,$$

and that $K_M(P_x) \geq -\lambda(d(o,x))$ at any point $x \in M$.

The main theorem of the first part is the following.

Theorem 5.3 ([D, Theorem 1.1]). *Let M be a complete Riemannian manifold with asymptotically non-negative sectional curvature and only one end. If $u\colon M \to \mathbb{R}$ is a solution to the minimal graph equation that is bounded from below and has at most linear growth, then it must be a constant. In particular, if M is a Cartan-Hadamard manifold with asymptotically non-negative sectional curvature, the asymptotic Dirichlet problem is not solvable.*

It is worth pointing out that we do not assume, differing from previous results into this direction, the Ricci curvature to be non-negative; see e.g. Rosenberg, Schulze, and Spruck (2013), Ding, Jost, and Xin (2013), Dajczer and de Lira (2015), Dajczer and de Lira (2016). In terms of concrete curvature bounds, our theorem gives immediately the following corollary that answers the question about the optimality of (5.1).

Corollary 5.4 ([D, Corollary 1.2]). *Let M be a complete Riemannian manifold with only one end and assume that the sectional curvatures of M satisfy*

$$K(P_x) \geq -\frac{C}{r(x)^2 \big(\log r(x)\big)^{1+\varepsilon}}$$

for sufficiently large $r(x)$ and for some $C > 0$ and $\varepsilon > 0$. Then any solution $u\colon M \to [a,\infty)$ with at most linear growth to the minimal graph equation must be constant.

The proof of Theorem 5.3 is based on an application of a gradient estimate Proposition 5.11 that enables us to prove a global Harnack's inequality for $u - \inf_M u$.

By well-known methods, see (Heinonen, Kilpeläinen, and Martio, 1993, Theorem 6.6), the global Harnack's inequality can be iterated to yield Hölder continuity estimates and a Liouville (or Bernstein) type result when the solution has controlled growth. More precisely, we obtain the following corollary.

Corollary 5.5 ([D, Corollary 1.3])**.** *Let M be a complete Riemannian manifold with asymptotically non-negative sectional curvature and only one end. Then there exists a constant $\kappa \in (0,1]$, depending only on n and on the function λ in the (ANSC) condition such that every solution $u\colon M \to \mathbb{R}$ to the minimal graph equation with*

$$\lim_{d(x,o)\to\infty} \frac{|u(x)|}{d(x,o)^\kappa} = 0$$

must be constant.

Before turning to the latter part of [D], we point out that our results differ from the theorems of Greene and Wu (1982) (besides the methods) mentioned in Section 2.5 since we do not assume the existence of a pole or the manifold to be simply connected, and the (ANSC) condition allows the sectional curvature to change a sign. Moreover, in Theorems 5.7 and 5.8 we will see that, in order to get the result (Greene and Wu, 1982, Theorem 2), it is necessary to assume $\liminf_{s\to\infty} s^2 k(s) = 0$ for all of the sectional curvatures and not only for the radial ones (recall formula (1.1)).

The goal of the latter part of [D] is to prove the solvability of the asymptotic Dirichlet problem, and hence also the existence of entire bounded non-constant solutions, for the minimal graphic and p-harmonic equations assuming the optimal curvature upper bound (5.1). The main idea is to assume

$$\int_1^\infty \Big(f(s)^\beta \int_s^\infty f(t)^\alpha dt \Big) ds < \infty, \tag{5.6}$$

with an appropriate choice of α and β, and then use this condition to construct barriers at infinity. This results to very elementary proofs when compared to the proofs in the general case that was considered for example in Casteras, Holopainen, and Ripoll (To appearb), Casteras, Holopainen, and Ripoll (To appeara), [A] and [B].

Noticing that, on manifold M_f, the condition (5.6) implies the desired curvature upper bound, we obtain the following results.

Theorem 5.7 ([D, Corollary 4.2])**.** *Let M_f be a rotationally symmetric n-dimensional Cartan-Hadamard manifold whose radial sectional curvatures outside a compact set satisfy the upper bounds*

$$K(P_x) \le -\frac{1+\varepsilon}{r(x)^2 \log r(x)}, \quad \textit{if } n = 2$$

and

$$K(P_x) \le -\frac{1/2+\varepsilon}{r(x)^2 \log r(x)}, \quad \textit{if } n \ge 3.$$

Then the asymptotic Dirichlet problem for the minimal graph equation is solvable with any continuous boundary data on $\partial_\infty M_f$.

Theorem 5.8 ([D, Corollary 4.4]). *Let M_f be a rotationally symmetric n-dimensional Cartan-Hadamard manifold, $n \geq 3$, whose radial sectional curvatures satisfy the upper bound*

$$K(P_x) \leq -\frac{1/2 + \varepsilon}{r(x)^2 \log r(x)}. \tag{5.9}$$

Then the asymptotic Dirichlet problem for the p-Laplace equation, with $p \in (2, n)$, is solvable with any continuous boundary data on $\partial_\infty M_f$.

We point out that the case $p = 2$ in Theorem 5.8 reduces to the case of usual harmonic functions and was covered by March (1986).

Finally, in the last section of [D], we show that in Theorem 5.8 the assumption $p < n$ on the range of p is also optimal. Note also that (ANSC) implies global Harnack's inequality for $\mathcal{A}$-harmonic functions ((Holopainen, 1999, Examples 3.1)).

Theorem 5.10 ([D, Theorem 5.1]). *Let $\alpha > 0$ be a constant and assume that M is a complete n-dimensional Riemannian manifold whose radial sectional curvatures satisfy*

$$K_M(P_x) \geq -\frac{\alpha}{r(x)^2 \log r(x)}$$

for every x outside some compact set and every 2-dimensional subspace $P_x \subset T_xM$ containing $\nabla r(x)$. Then M is p-parabolic

(a) if $p = n$ and $0 < \alpha \leq 1$; or

(b) $p > n$ and $\alpha > 0$.

5.2.1 *Gradient estimate for minimal graphic functions*

It is well-known that the (ANSC) assumption implies a volume doubling condition and a Poincaré inequality (see [D] for short discussion) and these can be used to prove a local Harnack's inequality for uniformly elliptic operators. Then the assumption that M has only one end yields a global Harnack's inequality, see e.g. Abresch (1985), Kasue (1988) and (Holopainen, 1999, Examples 3.1). Therefore the question reduces to interpreting the minimal graph operator as a uniformly elliptic operator

$$\frac{1}{A(x)} \operatorname{div}\big(A(x)\nabla u\big),$$

where

$$A(x) = \frac{1}{\sqrt{1 + |\nabla u|^2}}.$$

Note that if $|\nabla u|$ is uniformly bounded, then there exists a constant c such that $c \leq \sqrt{A} \leq 1$. This uniform gradient bound can be obtained from the following proposition, whose proof is based on the method due to Korevaar.

Proposition 5.11 ([D, Proposition 3.1]). *Assume that the sectional curvature of M has a lower bound $K(P_x) \geq -K_0^2$ for all $x \in B(p,R)$ for some constant $K_0 = K_0(p,R) \geq 0$. Let u be a positive solution to the minimal graph equation in $B(p,R) \subset M$. Then*

$$|\nabla u(p)| \leq \left(\frac{2}{\sqrt{3}} + \frac{32u(p)}{R}\right) \cdot \left(\exp\left[64u(p)^2\left(\frac{2\psi(R)}{R^2} + \sqrt{\frac{4\psi(R)^2}{R^4} + \frac{(n-1)K_0^2}{64u(p)^2}}\right)\right] + 1\right), \tag{5.12}$$

where $\psi(R) = (n-1)K_0 R\coth(K_0 R) + 1$ if $K_0 > 0$ and $\psi(R) = n$ if $K_0 = 0$.

In order to allow at most linear growth for u in [D, Corollary 3.2], we apply Proposition 5.11 to points $p \in M \setminus B(o, R_0)$, for some $R_0 > 0$ large, and use the fact that (ANSC) implies

$$K(P_x) \geq -\frac{c}{d(x,o)^2}$$

for all $x \in M \setminus B(o, R_0/2)$.

5.2.2 *Optimal curvature upper bound on the rotationally symmetric case*

In order to obtain barriers from (5.6) we first define a function

$$\eta(r) = k\int_r^\infty f(t)^\alpha \int_1^t f(s)^\beta \, ds dt, \tag{5.13}$$

$k > 0$, and then consider the function $\eta + B$, where $B \colon M \setminus \{o\}$,

$$B\big(\exp(r\vartheta)\big) = B(r,\vartheta) = b(\vartheta), \quad \vartheta \in \mathbb{S}^{n-1} \subset T_oM,$$

is a radial extension of the boundary data function $b \colon \mathbb{S}^{n-1} \to \mathbb{R}$. In the case of the minimal graph equation we choose $\alpha = -n+1$ and $\beta = n-3$, as in March (1986), and for the p-Laplacian we choose $\alpha = -(n-1)/(p-1)$ and $\beta = (n-2p+1)/(p-1)$. Note that in both cases $\alpha + \beta = -2$ and hence they correspond to the same curvature bound. Then a straightforward computation shows that $\eta + B$ is a supersolution in $M \setminus B(o, R_0)$ for $R_0 > 0$ large enough and we can define global super- and subsolutions that work as barriers.

5.2.3 *p-parabolicity when $p \geq n$*

To be more precise we recall that

Definition 5.14. Riemannian manifold N is p-parabolic, $1 < p < \infty$, if

$$\operatorname{cap}_p(K, N) = 0$$

for every compact set $K \subset N$. Here the p-capacity of the pair (K, N) is

$$\operatorname{cap}_p(K, N) = \inf_{\substack{u \in C_0^\infty(N) \\ u|K \geq 1}} \int_N |\nabla u|^p.$$

In order to prove Theorem 5.10, and to show that the curvature bound (5.9) in Theorem 5.8 is optimal and the upper bound $p < n$ necessary, we apply Bishop-Gromov volume comparison together with (5.9). The proof is a direct application of the following condition that, for $p = 2$, was proved by Varopoulos (1983) and Grigor'yan (1983, 1985). Zorich and Kesel'man (1996) proved the case $p = n$ and their proof applies also to other values of p, see also Holopainen (1999), Coulhon, Holopainen, and Saloff-Coste (2001), Holopainen (2003).

Proposition 5.15. *A complete Riemannian manifold M is p-parabolic if*

$$\int^\infty \left(\frac{t}{V(t)}\right)^{1/(p-1)} dt = \infty. \tag{5.16}$$

We point out that converse of this proposition is not always true, namely, there exists a manifold such that the integral of (5.16) is finite but M is p-parabolic, see Varopoulos (1983).

6 WARPED PRODUCT MANIFOLDS

Let N be a Riemannian manifold of the form $N = M \times_\varrho \mathbb{R}$, where M is a complete n-dimensional Riemannian manifold and $\varrho \in C^\infty(M)$ is a smooth (warping) function. This means that the Riemannian metric $\bar{g}$ in N is of the form

$$\bar{g} = (\varrho \circ \pi_1)^2 \pi_2^* \, \mathrm{d}t^2 + \pi_1^* g,$$

where g denotes the Riemannian metric in M whereas t is the natural coordinate in $\mathbb{R}$ and $\pi_1 : M \times \mathbb{R} \to M$ and $\pi_2 : M \times \mathbb{R} \to \mathbb{R}$ are the standard projections. It follows that the coordinate vector field $X = \partial_t$ is a Killing field and that $\varrho = |X|$ on M. Since the norm of X is preserved along its flow lines, we may extend ϱ to a smooth function $\varrho = |X| \in C^\infty(N)$. From now on, we suppose that $\varrho > 0$ on M.

Killing graphs with prescribed mean curvature were introduced in Dajczer, Hinojosa, and de Lira (2008), where the Dirichlet problem for prescribed mean curvature with $C^{2,\alpha}$ boundary values was solved in bounded domains $\Omega \subset M$ under hypothesis involving data on Ω and the Ricci curvature of the ambient space N. Recall that given a domain $\Omega \subset M$, the Killing graph of a C^2 function $u : \Omega \to \mathbb{R}$ is the hypersurface given by

$$\Sigma_u = \{(x, u(x)) : x \in \Omega\} \subset M \times \mathbb{R}.$$

In other words,

$$\Sigma_u = \{\Psi(x, u(x))\colon x \in \Omega\},$$

where $\Psi\colon \Omega \times \mathbb{R} \to N$ is the flow generated by X. Recently the Killing graphs have been studied studied very actively, for example in Dajczer, de Lira, and Ripoll (2016) the Dirichlet problem was solved with merely continuous boundary data, and in Dajczer and de Lira (2015) and Dajczer and de Lira (2016) Dajczer and Lira studied the entire Killing graphs with constant mean curvature. In particular, it was shown in Dajczer and de Lira (2015) that a bounded entire Killing graph of constant mean curvature must be a slice if $\mathrm{Ric}_M \geq 0,\ K_M \geq -K_0$ for some $K_0 \geq 0$, and if $\varrho \geq \varrho_0 > 0$, with $||\varrho||_{C^2(M)} < \infty$.

6.1 Background

The history of the barrier method and the so-called Korevaar-Simon technique was already dealt in Section 4.1.1. In the article [E] we modify a further development that is due to Wang (1998). In the Korevaar-Simon method, that was mentioned in Section 4.1.1, one does computations at a point where the function ηW attains its maximum value. In Wang's method one uses a similar approach but with a function

$$\chi = \eta\gamma(u)\psi(|\nabla u|^2),$$

where η, γ and ψ are carefully chosen functions. This method allows to obtain quantitative a priori interior gradient estimate.

A very flexible global barrier was introduced in Mastrolia, Monticelli, and Punzo (2015). They used it to study elliptic and parabolic equations with Dirichlet conditions at infinity and later it was used in article [C] in the study of f-minimal surfaces. The background of the barrier at infinity was dealt already in Section 4.1.2

6.2 Article [E]

The article [E] divides roughly into two parts: In the first part we obtain quantitative a priori height and gradient estimates for solutions that are height functions of Killing graphs with prescribed mean curvature H. These estimates can be used to solve the Dirichlet problem

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{W}\right) + \left\langle \nabla \log \varrho, \frac{\nabla u}{W} \right\rangle = nH & \text{in} \quad \Omega \\ u|\partial\Omega = \varphi & \text{in} \quad \partial\Omega \end{cases} \tag{6.1}$$

on bounded geodesic balls $\Omega = B(o,r)$ on the leaf M. In the second part we apply the existence of solutions to an exhaustion of M and solve the asymptotic Dirichlet problem with continuous boundary values on the boundary $\partial_\infty M$.

Applying the local estimates of the first part of [E], the usage of the continuity method (Leray-Schauder method), together with the approximation result from Dajczer, de Lira, and Ripoll (2016) (see also [C]), yields the following existence result.

Theorem 6.2. *Let M be a Cartan-Hadamard manifold, $\Omega = B(o,k) \subset M$, and $\varphi \in C(\partial\Omega)$. Suppose that the prescribed mean curvature function $H \in C^\alpha(\Omega)$ satisfies*

$$|H(x)| < H_{k-d(x)}$$

in $\bar{\Omega}$, where $d(x) = \operatorname{dist}\big(x, \partial B(o,k)\big) = k - r(x)$ and H_{k-d} is the mean curvature of the Killing cylinder $\mathcal{C}_{k-d}$ over the geodesic sphere $\partial B(o,k-d)$. Then there exists a unique solution $u \in C^{2,\alpha}(\Omega) \cap C(\bar{\Omega})$ to (6.1).

Above and in what follows we denote by $r(x) = d(x,o)$ the distance from x to a fixed point $o \in M$. We notice that the mean curvature of the Killing cylinder $\mathcal{C}_r$ over a geodesic sphere $\partial B(o,r)$ is given by

$$H_r = \frac{1}{n}\left(\Delta r + \frac{1}{\varrho}\langle \nabla \varrho, \nabla r\rangle\right)$$

and therefore can be estimated from below in terms of a suitable model manifold $M_{-a^2(r)} \times_{\varrho_+} \mathbb{R}$, where $M_{-a^2(r)}$ is a rotationally symmetric Cartan-Hadamard manifold with radial sectional curvatures equal to $-a^2(r)$ and $\varrho_+ \colon M \to (0,\infty)$ is a positive rotationally symmetric C^1 function such that

$$\frac{1}{\varrho}\langle \nabla \varrho, \nabla r\rangle = \frac{\partial_r \varrho}{\varrho} \geq \frac{\partial_r \varrho_+}{\varrho_+}. \tag{6.3}$$

With the help of these model manifolds we can formulate the following corollary.

Corollary 6.4. *Let M be a Cartan-Hadamard manifold whose radial sectional curvatures are bounded from above by*

$$K(P_x) \leq -a\big(r(x)\big)^2$$

for some smooth function $a \colon [0,\infty) \to [0,\infty)$. Suppose, moreover, that (6.3) holds with some positive rotationally symmetric C^1 function $\varrho_+ = \varrho_+(r)$. If the prescribed mean *curvature function* $H \in C^{\alpha}(\Omega)$ *satisfies*

$$n|H(x)| < \frac{(n-1)f_a'\big(r(x)\big)}{f_a\big(r(x)\big)} + \frac{\varrho_+'\big(r(x)\big)}{\varrho_+\big(r(x)\big)}$$

for all $x \in \bar{\Omega}$*, then there exists a unique solution* $u \in C^{2,\alpha}(\Omega) \cap C(\bar{\Omega})$ *to* (6.1).

The main object in the article [E] is the asymptotic Dirichlet problem for Killing graphs with prescribed mean curvature and behaviour at infinity that are dealt in the second part of the article. We assume that the sectional curvatures of the leaf M satisfies

$$-(b \circ r)^2(x) \le K(P_x) \le -(a \circ r)^2(x) \tag{6.5}$$

for all $x \in M$, where $r(x) = d(o, x)$ is the distance to a fixed point $o \in M$ and P_x is any 2-dimensional subspace of T_xM. The functions $a, b\colon [0, \infty) \to [0, \infty)$ are assumed to be smooth such that $a(t) = 0$ and $b(t)$ is constant for $t \in [0, T_0]$ for some $T_0 > 0$, and that assumptions (A1)–(A7) of Section 5 of [E] hold. Then under suitable assumptions on the mean curvature function and on the warping function, we obtain the following results.

Theorem 6.6. *Let* M *be a Cartan-Hadamard manifold satisfying the curvature assumptions* (6.5) *and (A1)–(A7) in Section 5 of [E]. Furthermore, assume that the prescribed mean curvature function* $H\colon M \to \mathbb{R}$ *satisfies the assumptions* (6.9) *and* (6.14) *with a convex warping function* ϱ *satisfying* (6.10), (6.11), (6.15), *and* (6.16). *Then there exists a unique solution* $u\colon M \to \mathbb{R}$ *to the Dirichlet problem*

$$\begin{cases} \operatorname{div}_{-\log\varrho} \dfrac{\nabla u}{\sqrt{\varrho^{-2} + |\nabla u|^2}} = nH(x) \quad \text{in } M \\ u|\partial_\infty M = \varphi \end{cases} \tag{6.7}$$

for any continuous function $\varphi\colon \partial_\infty M \to \mathbb{R}$.

Theorem 6.8. *Let* M *be a Cartan-Hadamard manifold satisfying the curvature assumptions* (6.5) *and (A1)–(A7) in Section 5 of [E]. Furthermore, assume that the prescribed mean curvature function* $H\colon M \to \mathbb{R}$ *satisfies the assumptions* (6.12) *and* (6.14) *with a convex warping function* ϱ *satisfying* (6.13), (6.15), *and* (6.16). *Then there exists a unique solution* $u\colon M \to \mathbb{R}$ *to the Dirichlet problem* (6.7) *for any continuous function* $\varphi\colon \partial_\infty M \to \mathbb{R}$.

6.2.1 *Entire Killing graphs*

To solve the problem (6.7), we extend the given boundary value function $\varphi \in C(\partial_\infty M)$ to a continuous function $\varphi \in C(\bar{M})$. Then we apply Corollary 6.4 for an exhaustion $\Omega_k = B(o, k)$, $k \in \mathbb{N}$, of M to obtain a sequence of solutions u_k with boundary values $u_k|\partial\Omega_k = \varphi$. In order to obtain a converging subsequence we need to prove that this sequence of solutions is uniformly bounded. This can be done by constructing global barriers, and we provide two different approaches to obtain these functions.

The first approach is motivated by direct computations in the rotationally symmetric case by applying the so-called flux formula. This way we can define an entire function that under suitable assumptions acts as a barrier. For example, we need to assume that the prescribed mean curvature function satisfies

$$n|H(x)| \leq \tag{6.9}$$
$$(1-\varepsilon)\sqrt{\frac{\varrho_+^{-2}\big(r(x)\big)\big(1+(1-\varepsilon)^2/(2\varepsilon-\varepsilon^2)\big)}{\varrho^{-2}(x)+\varrho_+^{-2}\big(r(x)\big)(1-\varepsilon^2)/(2\varepsilon-\varepsilon^2)}}\left(\frac{\varrho_+'\big((r)\big)}{\varrho_+\big(r(x)\big)}+(n-1)\frac{f_a'\big(r(x)\big)}{f_a\big(r(x)\big)}\right),$$

and that the warping function satisfies

$$\frac{\partial_r\varrho(x)}{\varrho(x)} \geq \frac{\varrho_+'\big(r(x)\big)}{\varrho_+\big(r(x)\big)}, \quad \frac{\partial_r\varrho(x)}{\varrho(x)^3} \geq \frac{\varrho_+'\big(r(x)\big)}{\varrho_+\big(r(x)\big)^3}, \tag{6.10}$$

where $\varrho_+\colon [0,\infty] \to (0,\infty)$ is an increasing smooth function with

$$\varrho_+(0) = \varrho(o) \quad \text{and} \quad \int_1^\infty \varrho_+(s)^{-1}\,\mathrm{d}s < \infty. \tag{6.11}$$

For the second approach to obtain a barrier we modify a barrier function that was first obtained in Mastrolia, Monticelli, and Punzo (2015) and later used also in [C]. In order to get this barrier function V to work we need to assume that the mean curvature function satisfies

$$n|H| \leq \frac{\varrho^{-2}\varrho_+^{-2}(r)a_0(r)+\big(-V'(r)\big)^3\left(\frac{(n-1)f_a'(r)}{f_a(r)}+\frac{\varrho_+'(r)}{\varrho_+(r)}\right)}{\Big(\varrho^{-2}+\big(V'(r)\big)^2\Big)^{3/2}}, \tag{6.12}$$

where V is the function defined in Section 4.2 of [E]. In addition we need that the warping functions satisfies

$$\frac{\partial_r\varrho(x)}{\varrho(x)} \geq \frac{\varrho_+'\big(r(x)\big)}{\varrho_+\big(r(x)\big)}. \tag{6.13}$$

These height estimates, consequently together with Schauder estimates, implies that the sequence u_k is uniformly bounded in the $C^{2,\alpha}$-norm. Hence there exists a subsequence that converges in the $C^{2,\alpha}$-norm to a global solution u to the equation

$$\operatorname{div}\Big(\frac{\nabla u}{W}\Big) + \Big\langle \nabla \log\varrho, \frac{\nabla u}{W}\Big\rangle = nH$$

in M.

The final step of solving the asymptotic Dirichlet problem is to show that the obtained solution extends to the boundary at infinity and has the desired behaviour there. For this step we apply the (local) barrier function introduced in Holopainen and Vähäkangas (2007). This part requires the asymptotic assumption

$$\sup_{r(x)=t} n|H(x)| < \frac{C_0 t^{-\delta_1-1}}{\sqrt{\varrho^{-2}(t) + (C_0 t^{-\delta-1})^2}} \left((n-1)\frac{f_a'(t)}{f_a(t)} + \frac{\partial_r \varrho}{\varrho} - \frac{1}{t} \right) \tag{6.14}$$

on the mean curvature function. Moreover, the warping function has to satisfy

$$\max\left(0, -\frac{r\partial_r \varrho}{\varrho}\right) = o\left(\frac{r f_a'(r)}{f_a(r)}\right) \tag{6.15}$$

and

$$|\nabla \varrho| = o\left(\frac{f_a(r)}{r^{\delta+1}} |\partial_r \varrho|\right). \tag{6.16}$$

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APPENDIX 1 – ARTICLE [A]

Solvability of minimal graph equation under pointwise pinching condition for sectional curvatures

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Journal of Geometric Analysis, 27(2):1106-1130, 2017

SOLVABILITY OF MINIMAL GRAPH EQUATION UNDER POINTWISE PINCHING CONDITION FOR SECTIONAL CURVATURES

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ABSTRACT. We study the asymptotic Dirichlet problem for the minimal graph equation on a Cartan-Hadamard manifold M whose radial sectional curvatures outside a compact set satisfy an upper bound

$$K(P) \leq -\frac{\phi(\phi-1)}{r(x)^2}$$

and a pointwise pinching condition

$$|K(P)| \leq C_K |K(P')|$$

for some constants $\phi > 1$ and $C_K \geq 1$, where P and P' are any 2-dimensional subspaces of T_xM containing the (radial) vector $\nabla r(x)$ and $r(x) = d(o,x)$ is the distance to a fixed point $o \in M$. We solve the asymptotic Dirichlet problem with any continuous boundary data for dimensions $n = \dim M > 4/\phi + 1$.

1. Introduction

In this paper we are interested in the asymptotic Dirichlet problem for minimal graph equation

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \tag{1.1}$$

on a Cartan-Hadamard manifold M of dimension $n \geq 2$. We recall that a Cartan-Hadamard manifold is a simply connected complete Riemannian manifold with non-positive sectional curvature. Since the exponential map $\exp_o \colon T_oM \to M$ is a diffeomorphism for every point $o \in M$, it follows that M is diffeomorphic to $\mathbb{R}^n$. One can define an asymptotic boundary $\partial_\infty M$ of M as the set of all equivalence classes of unit speed geodesic rays on M. Then the compactification of M is given by $\bar{M} = M \cup \partial_\infty M$ equipped with the cone topology. We also notice that $\bar{M}$ is homeomorphic to the closed Euclidean unit ball; for details, see [16].

The *asymptotic Dirichlet problem* on M for some operator $\mathcal{Q}$ is the following: Given a function $f \in C(\partial_\infty M)$ does there exist a (unique) function $u \in C(\bar{M})$ such that $\mathcal{Q}[u] = 0$ on M and $u|\partial_\infty M = f$? We will consider this problem for the minimal graph operator (or the mean curvature operator) appearing in (1.1). It is also worth noting that a function u satisfies (1.1) if and only if the graph $\{(x, u(x)) \colon x \in M\}$ is a minimal hypersurface in the product space $M \times \mathbb{R}$.

The asymptotic Dirichlet problem on Cartan-Hadamard manifolds has been solved for various operators and under various assumptions on the manifold. The first result for this problem was due to Choi [7] when he solved the asymptotic Dirichlet problem for the Laplacian assuming that the sectional curvature has a negative upper bound $K_M \leq -a^2 < 0$, and that any two points at infinity can be

2000 *Mathematics Subject Classification.* Primary 58J32; Secondary 53C21.
Key words and phrases. minimal graph equation, Dirichlet problem, Hadamard manifold.
J.-B.C. supported by MIS F.4508.14 (FNRS).
E.H. supported by the Academy of Finland, project 252293 and the Wihuri Foundation.
I.H. supported by the Academy of Finland, project 252293.

separated by convex neighborhoods. Anderson [1] showed that such convex sets exist provided the sectional curvature of the manifold satisfies $-b^2 \leq K_M \leq -a^2 < 0$. We point out that Sullivan [29] solved independently the asymptotic Dirichlet problem for the Laplacian under the same curvature assumptions but using probabilistic arguments. Cheng [6] was the first to solve the problem for the Laplacian under the same type of pointwise pinching assumption for the sectional curvatures as we consider in this paper. Later the asymptotic Dirichlet problem has been generalized for p-harmonic and $\mathcal{A}$-harmonic functions under various curvature assumptions, see [4], [21], [23], [30], [31].

Concerning the mean curvature operator, there has been a growing interest in developing a theory of constant (or prescribed) mean curvature hypersurfaces in Riemannian manifolds. For instance, Guan and Spruck [19] investigated the problem of finding complete hypersurfaces of constant mean curvature with prescribed asymptotic boundaries at infinity in the hyperbolic space (see also the recent [20] and references therein). On the other hand, Dajczer, Hinojosa, and de Lira ([12], [10], [11]) have studied Killing graphs of prescribed mean curvature under curvature conditions on the ambient space. Further studies include so-called half-space theorems in product spaces $M \times \mathbb{R}_+$; see [26], [13], and references therein. In these investigations, *a priori* gradient estimates based on the classical maximum principle for elliptic equations are indispensable. To motivate further the study of the asymptotic Dirichlet problem for the minimal graph equation, we recall the papers [8] and [17] by Collin, Gálvez, and Rosenberg who were able to construct harmonic diffeomorphisms from the complex plane $\mathbb{C}$ onto the hyperbolic plane $\mathbb{H}^2$ and onto any Hadamard surface M whose curvature is bounded from above by a negative constant, respectively, hence disproving a conjecture of Schoen and Yau [27]. The key idea in their constructions was to solve the Dirichlet problem on unbounded ideal polygons with boundary values $\pm\infty$ on the sides of the ideal polygons.

Concerning the asymptotic Dirichlet problem for the equation (1.1), Casteras, Holopainen, and Ripoll studied the problem under curvature bounds

$$-b\big(r(x)\big)^2 \leq K(P) \leq -a\big(r(x)\big)^2,$$

where $a, b\colon [0,\infty) \to [0,\infty)$ are smooth functions subject to some growth conditions. Here and throughout the paper $r(x) = d(x,o)$ stands for the distance to a fixed point $o \in M$. As special cases of their main theorem [3, Theorem 1.6] we state here the following two solvability results.

Theorem 1.1. [3, Theorem 1.5, Corollary 1.7] *Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$. Suppose that*

$$-\,r(x)^{2(\phi-2)-\varepsilon} \leq K(P) \leq -\frac{\phi(\phi-1)}{r(x)^2} \tag{1.2}$$

or

$$-\,r(x)^{-2-\varepsilon}e^{2kr(x)} \leq K(P) \leq -k^2 \tag{1.3}$$

for some constants $\varepsilon > 0$, $\phi > 1$, and $k > 0$, and for all 2-dimensional subspaces $P \subset T_xM$, with $x \in M \setminus B(o,R_0)$. Then the asymptotic Dirichlet problem for (1.1) *is uniquely solvable for any boundary data $f \in C(\partial_\infty M)$.*

The solvability of the asymptotic Dirichlet problem for (1.1) under curvature assumptions (1.3) was earlier obtained by Ripoll and Telichevesky in [25]; see also [14] and [15]. Recently, Casteras, Holopainen, and Ripoll [4] were able to weaken the curvature upper bound to an almost optimal one.

Theorem 1.2. [4, Theorem 5] *Let M be a Cartan-Hadamard manifold of dimension $n \geq 3$ satisfying the curvature assumption*

$$-\frac{\big(\log r(x)\big)^{2\bar{\varepsilon}}}{r(x)^2} \leq K(P) \leq -\frac{1+\varepsilon}{r(x)^2 \log r(x)}$$

for some constants $\varepsilon > \bar{\varepsilon} > 0$ and for any 2-dimensional subspace $P \subset T_xM$, with $x \in M \setminus B(o, R_0)$. Then the asymptotic Dirichlet problem for (1.1) *is uniquely solvable for any boundary data $f \in C(\partial_\infty M)$.*

It is worth noting that even a strict negative curvature upper bound alone is not sufficient in dimensions $n \geq 3$ for the solvability of the asymptotic Dirichlet problem for (1.1). Indeed, in [22] Holopainen and Ripoll generalized Borbély's counterexample [2] to cover the minimal graph equation.

Our main theorem is the following. It is worth noticing that no lower bounds for sectional curvatures are needed. Instead we assume a pointwise pinching condition on sectional curvatures.

Theorem 1.3. *Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$ and let $\phi > 1$. Assume that*

$$K(P) \leq -\frac{\phi(\phi-1)}{r(x)^2}, \tag{1.4}$$

where $K(P)$ is the sectional curvature of any two-dimensional subspace $P \subset T_xM$ containing the radial vector $\nabla r(x)$, with $x \in M \setminus B(o, R_0)$. Suppose also that there exists a constant $C_K < \infty$ such that

$$|K(P)| \leq C_K |K(P')| \tag{1.5}$$

whenever $x \in M \setminus B(o, R_0)$ and $P, P' \subset T_xM$ are two-dimensional subspaces containing the radial vector $\nabla r(x)$. Moreover, suppose that the dimension n and the constant ϕ satisfy the relation

$$n > \frac{4}{\phi} + 1. \tag{1.6}$$

Then the asymptotic Dirichlet problem for the minimal graph equation (1.1) *is uniquely solvable for any boundary data $f \in C(\partial_\infty M)$.*

We notice that if we choose the constant ϕ in the curvature assumption to be bigger than 4, then our theorem will hold in every dimension $n \geq 2$. Similarly, if we let the dimension n to be at least 5, we can take the constant ϕ to be as close to 1 as we wish.

In this paper we will proceed as follows. Section 2 is devoted to preliminaries. We will recall some facts about Cartan-Hadamard manifolds, Jacobi equations, the minimal graph equation and Young functions. In Section 3 we will prove our main theorem i.e. the solvability of the minimal graph equation under the curvature assumptions (1.4), (1.5) and (1.6). We will adopt the strategies used in [4], [6], [30] and [31].

2. Preliminaries

2.1. **Cartan-Hadamard manifolds.** Recall that a Cartan-Hadamard manifold is a complete and simply connected Riemannian manifold with non-positive sectional curvature. Let M be a Cartan-Hadamard manifold and $\partial_\infty M$ the sphere at infinity, then we denote $\bar{M} = M \cup \partial_\infty M$. The sphere at infinity is defined as the set of all equivalence classes of unit speed geodesic rays in M; two such rays γ_1 and γ_2 are equivalent if

$$\sup_{t \geq 0} d\big(\gamma_1(t), \gamma_2(t)\big) < \infty.$$

The equivalence class of γ is denoted by $\gamma(\infty)$. For each $x \in M$ and $y \in \bar{M} \setminus \{x\}$ there exists a unique unit speed geodesic $\gamma^{x,y} \colon \mathbb{R} \to M$ such that $\gamma^{x,y}(0) = x$ and $\gamma^{x,y}(t) = y$ for some $t \in (0, \infty]$. For $x \in M$ and $y, z \in \bar{M} \setminus \{x\}$ we denote by

$$\sphericalangle_x(y, z) = \sphericalangle(\dot{\gamma}_0^{x,y}, \dot{\gamma}_0^{x,z})$$

the angle between vectors $\dot{\gamma}_0^{x,y}$ and $\dot{\gamma}_0^{x,z}$ in T_xM. If $v \in T_xM \setminus \{0\}$, $\alpha > 0$, and $R > 0$, we define a cone

$$C(v, \alpha) = \{y \in \bar{M} \setminus \{x\} \colon \sphericalangle(v, \dot{\gamma}_0^{x,y}) < \alpha\}$$

and a truncated cone

$$T(v, \alpha, R) = C(v, \alpha) \setminus \bar{B}(x, R).$$

All cones and open balls in M form a basis for the cone topology in $\bar{M}$. With this topology $\bar{M}$ is homeomorphic to the closed unit ball $\bar{B}^n \subset \mathbb{R}^n$ and $\partial_\infty M$ to the unit sphere $\mathbb{S}^{n-1} = \partial B^n$. For detailed study on the cone topology, see [16].

Let us recall that the local Sobolev inequality holds on any Cartan-Hadamard manifold M. More precisely, there exist constants $r_S > 0$ and $C_S < \infty$ such that

$$\left(\int_B |\eta|^{n/(n-1)} \right)^{(n-1)/n} \le C_S \int_B |\nabla \eta| \tag{2.1}$$

holds for every ball $B = B(x, r_S) \subset M$ and every function $\eta \in C_0^\infty(B)$. This inequality can be obtained e.g. from Croke's estimate of the isoperimetric constant, see [5] and [9].

2.2. **Jacobi equation.** If $k \colon [0, \infty) \to (-\infty, 0]$ is a smooth function, we denote by $f_k \in C^\infty\big([0, \infty)\big)$ the solution to the initial value problem

$$\begin{cases} f_k'' + k f_k = 0 \\ \quad f_k(0) = 0, \\ \quad f_k'(0) = 1. \end{cases} \tag{2.2}$$

The solution is a non-negative smooth function.

In later sections we will need some known results related to Jacobi fields and curvature bounds. The proofs of the following three lemmas are based on the Rauch comparison theorem (see e.g. [18]) and can be found in [30]. Concerning the curvature bounds, we have the following estimates for the growth of Jacobi fields and the Laplacian of the distance function:

Lemma 2.1. [30, Lemma 1] *Let $k, K \colon [0, \infty) \to (-\infty, 0]$ be smooth functions that are constant in some neighborhood of* 0. *Suppose that $v \in T_oM$ is a unit vector and $\gamma = \gamma^v \colon \mathbb{R} \to M$ is the unit speed geodesic with $\dot{\gamma}_0 = v$. Suppose that for every $t > 0$ we have*

$$k(t) \le K_M(P) \le K(t)$$

for every two-dimensional subspace $P \subset T_{\gamma(t)}M$ that contains the radial vector $\dot{\gamma}_t$.

(1) *If W is a Jacobi field along γ with $W_0 = 0$, $|W_0'| = 1$, and $W_0' \perp v$, then*

$$f_K(t) \le |W(t)| \le f_k(t)$$

for every $t \ge 0$.

(2) *For every $t > 0$ we have*

$$(n-1)\frac{f_K'(t)}{f_K(t)} \le \Delta r\big(\gamma(t)\big) \le (n-1)\frac{f_k'(t)}{f_k(t)}.$$

The pinching condition for the sectional curvatures gives a relation between the maximal and minimal moduli of Jacobi fields along a given geodesic that contains the radial vector:

Lemma 2.2. [6, Lemma 3.2][30, Lemma 3] *Let $v \in T_oM$ be a unit vector and $\gamma = \gamma^v$. Suppose that $r_0 > 0$ and $k < 0$ are constants such that $K_M(P) \geq k$ for every two-dimensional subspace $P \subset T_xM$, $x \in B(o, r_0)$. Suppose that there exists a constant $C_K < \infty$ such that*

$$|K_M(P)| \leq C_K |K_M(P')|$$

whenever $t \geq r_0$ and $P, P' \subset T_{\gamma(t)}M$ are two-dimensional subspaces containing the radial vector $\dot{\gamma}_t$. Let V and $\bar{V}$ be two Jacobi fields along γ such that $V_0 = 0 = \bar{V}_0$, $V_0' \perp \dot{\gamma}_0 \perp \bar{V}_0$, and $|V_0'| = 1 = |\bar{V}_0'|$. Then there exists a constant $c_0 = c_0(C_K, r_0, k) > 0$ such that

$$|V_r|^{C_K} \geq c_0 |\bar{V}_r|$$

for every $r \geq r_0$.

To prove the solvability of the minimal graph equation, we will need an estimate for the gradient of a certain angular function. This estimate can be obtained in terms of Jacobi fields:

Lemma 2.3. [30, Lemma 2] *Let $x_0 \in M \setminus \{o\}$, $U = M \setminus \gamma^{o,x_0}(\mathbb{R})$, and define $\theta \colon U \to [0, \pi]$, $\theta(x) = \sphericalangle_o(x_0, x) := \arccos\langle \dot{\gamma}_0^{o,x_0}, \dot{\gamma}_0^{o,x} \rangle$. Let $x \in U$ and $\gamma = \gamma^{o,x}$. Then there exists a Jacobi field W along γ with $W(0) = 0$, $W_0' \perp \dot{\gamma}_0$, and $|W_0'| = 1$ such that*

$$|\nabla \theta(x)| \leq \frac{1}{|W(r(x))|}.$$

2.3. **Young functions.** Let $\phi \colon [0, \infty) \to [0, \infty)$ be a homeomorphism and let $\psi = \phi^{-1}$. Define *Young functions* Φ and Ψ by setting

$$\Phi(t) = \int_0^t \phi(s)\, ds$$

and

$$\Psi(t) = \int_0^t \psi(s)\, ds$$

for each $t \in [0, \infty)$. Then we have the following *Young's inequality*

$$ab \leq \Phi(a) + \Psi(b)$$

for all $a, b \in [0, \infty)$. The functions Φ and Ψ are said to form a *complementary Young pair*. Furthermore, Φ (and similarly Ψ) is a continuous, strictly increasing, and convex function satisfying

$$\lim_{t \to 0+} \frac{\Phi(t)}{t} = 0$$

and

$$\lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty.$$

For a more general definition of Young functions see e.g. [24].

As in [31], we consider complementary Young pairs of a special type. For that, suppose that a homeomorphism $G \colon [0, \infty) \to [0, \infty)$ is a Young function that is a diffeomorphism on $(0, \infty)$ and satisfies

$$\int_0^1 \frac{dt}{G^{-1}(t)} < \infty \tag{2.3}$$

and

$$\lim_{t \to 0} \frac{tG'(t)}{G(t)} = 1. \tag{2.4}$$

Then we define $F \colon [0, \infty) \to [0, \infty)$ so that G and F form a complementary Young pair. The space of such functions F will be denoted by $\mathcal{F}$. Note that if $F \in \mathcal{F}$, then also $\lambda F \in \mathcal{F}$ and $F(\lambda\cdot) \in \mathcal{F}$ for every $\lambda > 0$. In [31] it is proved that for fixed $\varepsilon_0 \in (0,1)$ there exists $F \in \mathcal{F}$ such that

$$F(t) \le t^{1+\varepsilon_0} \exp\left(-\tfrac{1}{t} \Big(\log\Big(e + \tfrac{1}{t}\Big)\Big)^{-1-\varepsilon_0} \right) \tag{2.5}$$

for all $t \in [0,\infty)$. The construction of such F is done by first choosing $\lambda \in (1, 1+\varepsilon_0)$ and a homeomorphism $H\colon [0,\infty) \to [0,\infty)$ that is a diffeomorphism on $(0,\infty)$ and satisfies

$$H(t) = \begin{cases} \big(\log \frac{1}{t}\big)^{-1} \big(\log\log \frac{1}{t}\big)^{-\lambda} & \text{if } t \text{ is small enough,} \\ t^{1/\varepsilon_0} & \text{if } t \text{ is large enough,} \end{cases} \tag{2.6}$$

and then setting $G(t) = \int_0^t H(s)\,ds$ and $F(t) = \int_0^t H^{-1}(s)\,ds$. From now on, G and F will denote the complementary Young pair obtained via this procedure. For details, see [31] and the proof of Proposition 2.5 below.

Since G is convex, we have $G(t) \ge ct$ for all $t \ge 1$. Therefore $G^{-1}(t) \le ct$ for all t large enough and this implies that $\int_0^\infty 1/G^{-1} = \infty$. From this, together with (2.3), we conclude that the function ψ, defined by

$$\psi(t) = \int_0^t \frac{ds}{G^{-1}(s)},$$

is a homeomorphism $[0,\infty) \to [0,\infty)$ that is a diffeomorphism on $(0,\infty)$. Hence the same is true for its inverse

$$\varphi = \psi^{-1}\colon [0,\infty) \to [0,\infty). \tag{2.7}$$

The following lemma collects the properties of φ.

Lemma 2.4. [31, Lemma 4.5] *The function $\varphi\colon [0,\infty) \to [0,\infty)$ is a homeomorphism that is smooth on $(0,\infty)$ and satisfies*

$$G \circ \varphi' = \varphi \tag{2.8}$$

and

$$\lim_{t\to 0+} \frac{\varphi''(t)\varphi(t)}{\varphi'(t)^2} = 1. \tag{2.9}$$

From now on, φ will be the function defined in (2.7) such that the corresponding $F \in \mathcal{F}$ satisfies (2.5). Using the computations done in [31], we obtain a more specific formula for the function φ. Namely, we know that $G^{-1}(t) \approx t/H(t)$ and hence

$$\psi(t) = \int_0^t \frac{ds}{G^{-1}(s)} \approx \int_0^t \frac{1}{s(\log\frac{1}{s})(\log\log\frac{1}{s})^{1+\varepsilon_0}} = \frac{1}{\varepsilon_0}\big(\log\log\tfrac{1}{t}\big)^{-\varepsilon_0}.$$

Here and in what follows $\approx$ means that the ratio of the two sides tends to 1 as $t \to 0^+$. From this it is straightforward to see that

$$\varphi(t) \approx \exp\Big(-\exp\big(\tfrac{1}{\varepsilon_0 t}\big)^{\varepsilon_0}\Big). \tag{2.10}$$

We will also need complementary Young functions G_1 and F_1 to deal with the second derivative of the function φ. The existence of these functions will be proved by the following proposition which is just a modification of [31, Proposition 4.3] since in the construction of the Young functions we will replace the function H in [31] by H^2.

Proposition 2.5. *Let $\varepsilon_0 \in (0,1)$ and $\lambda \in (1, 1+\varepsilon_0)$ be as in* (2.6)*. Then there exist complementary Young functions G_1 and F_1, and a constant $c > 0$ such that G_1 satisfies*

$$G_1\big(\varphi''(t)\big) \approx \varphi(t) \tag{2.11}$$

and F_1 satisfies

$$F_1(t) \le ct \exp\Big(- \tfrac{2^\lambda}{\sqrt{t}}\big(\log \tfrac{1}{t}\big)^{-\lambda}\Big) \tag{2.12}$$

for all sufficiently small $t > 0$.

Proof. Let $H\colon [0,\infty) \to [0,\infty)$ be as in (2.6). We define $G_1(t) = \int_0^t H(s)^2\, ds$. Then G_1 is a Young function and we denote by F_1 its Young conjugate. Notice that $G_1'(t) = H(t)^2$ and that $t(H^2)'(t)/H(t)^2 \to 0$ as $t \to 0$. Hence, by l'Hospital's rule, we have

$$\lim_{t\to 0} \frac{tG_1'(t)}{G_1(t)} = \lim_{t\to 0} \frac{\frac{d}{dt}(tG_1'(t))}{G_1'(t)} = 1$$

and we see that G_1 satisfies (2.4). Next, denote $R(t) = t/H(t)^2$. Then it is easy to see that $R(kt) \approx kR(t)$ for every constant $k > 0$ and we get

$$R\big(G_1(t)\big) \approx R\big(tH(t)^2\big) = \frac{tH(t)^2}{H\big(tH(t)^2\big)^2} \approx t,$$

which gives us $G_1^{-1}(t) \approx R(t)$. It follows that G_1 satisfies (2.3) and hence $F_1 \in \mathcal{F}$. On the other hand $\varphi(t) = \psi^{-1}(t)$ and

$$\psi'(t) = \frac{1}{G^{-1}(t)} \approx \frac{H(t)}{t},$$

and therefore

$$\varphi'(t) = \frac{1}{\psi'\big(\varphi(t)\big)} \approx \frac{\varphi(t)}{H\big(\varphi(t)\big)}.$$

By (2.9) we obtain

$$\varphi''(t) \approx \frac{\varphi(t)}{H\big(\varphi(t)\big)^2} = R\big(\varphi(t)\big) \approx G_1^{-1}\big(\varphi(t)\big),$$

and so

$$G_1\big(\varphi''(t)\big) \approx \varphi(t). \tag{2.13}$$

Thus we are left to estimate F_1 from above.

It is straightforward to check that

$$(H^2)^{-1}(t) = \exp\Big(- \exp\big(\lambda W(\lambda^{-1} t^{-1/(2\lambda)})\big)\Big),$$

for all sufficiently small t, where W is the Lambert W function defined by the identity $W(s)e^{W(s)} = s$. Since $F_1'(t) = (G_1')^{-1}(t) = (H^2)^{-1}(t)$ and $W(s) \ge \log s - \log\log s$ for all $s \ge e$, we get for sufficiently small t

$$\begin{aligned}
F_1(t) &= \int_0^t (H^2)^{-1}(s)\, ds \le t(H^2)^{-1}(t) \\
&= \frac{t}{\exp\Big(\exp\big(\lambda W(\lambda^{-1}t^{-1/2\lambda})\big)\Big)} \\
&\le \frac{t}{\exp\Big(\exp\big(\lambda \log(\lambda^{-1}t^{-1/2\lambda}) - \lambda \log\log(\lambda^{-1}t^{-1/2\lambda})\big)\Big)} \\
&= \frac{t}{\exp\Big((\lambda^{-1}t^{-1/2\lambda})^\lambda\big(\log(\lambda^{-1}t^{-1/2\lambda})\big)^\lambda\Big)} \\
&= t\exp\Big(- \tfrac{1}{\lambda^\lambda\sqrt{t}}\big(\log\tfrac{1}{\lambda} + \tfrac{1}{2\lambda}\log\tfrac{1}{t}\big)^{-\lambda}\Big) \\
&\le ct\exp\Big(- \tfrac{2^\lambda}{\sqrt{t}}\big(\log\tfrac{1}{t}\big)^{-\lambda}\Big).
\end{aligned}$$

□

2.4. **Minimal graph equation.** Let $\Omega \subset M$ be an open set. Then a function $u \in W^{1,1}_{\text{loc}}(\Omega)$ is a *(weak) solution of the minimal graph equation* if

$$\int_\Omega \frac{\langle \nabla u, \nabla \varphi \rangle}{\sqrt{1+|\nabla u|^2}} = 0 \tag{2.14}$$

for every $\varphi \in C_0^\infty(\Omega)$. Note that the integral is well-defined since

$$\sqrt{1+|\nabla u|^2} \geq |\nabla u| \quad \text{a.e.,}$$

and thus

$$\int_\Omega \frac{|\langle \nabla u, \nabla \varphi \rangle|}{\sqrt{1+|\nabla u|^2}} \leq \int_\Omega \frac{|\nabla u||\nabla \varphi|}{\sqrt{1+|\nabla u|^2}} \leq \int_\Omega |\nabla \varphi| < \infty.$$

It is known that under certain conditions there exists a (strong) solution of (1.1) with given boundary values. Namely, let $\Omega \subset\subset M$ be a smooth relatively compact open set whose boundary has positive mean curvature with respect to inwards pointing unit normal. Then for each $f \in C^{2,\alpha}(\bar{\Omega})$ there exists a unique $u \in C^\infty(\Omega) \cap C^{2,\alpha}(\bar{\Omega})$ that solves the minimal graph equation (1.1) in Ω and has the boundary values $u|\partial\Omega = f|\partial\Omega$.

3. Asymptotic Dirichlet problem for minimal graph equation

We begin by the following Caccioppoli-type inequality which will have a crucial role in the proof of the solvability of the minimal graph equation.

Lemma 3.1. *Suppose $\varphi\colon [0,\infty) \to [0,\infty)$ is a homeomorphism that is smooth on $(0,\infty)$ and let $U \subset\subset M$ be open. Suppose that $\eta \geq 0$ is a $C^1(U)$ function and let $u, \theta \in L^\infty(U) \cap W^{1,2}(U)$ be continuous functions such that $u \in C^2(U)$ is a solution to the minimal graph equation* (1.1) *in U. Denote*

$$h = \frac{|u-\theta|}{\nu},$$

where $\nu > 0$ is a constant, and assume that

$$\eta^2 \varphi(h) \in W_0^{1,2}(U).$$

Then we have

$$\int_U \eta^2 \varphi'(h) \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} \leq C_\varepsilon \int_U \eta^2 \varphi'(h) |\nabla \theta|^2 + (4+\varepsilon)\nu^2 \int_U \frac{\varphi^2}{\varphi'}(h) |\nabla \eta|^2 \tag{3.1}$$

for any fixed $\varepsilon > 0$.

Proof. Define an auxiliary function f by

$$f = \eta^2 \varphi\left(\frac{(u-\theta)^+}{\nu}\right) - \eta^2 \varphi\left(\frac{(u-\theta)^-}{\nu}\right).$$

Then it holds that $f \in W_0^{1,2}(U)$ and its gradient is given by

$$\nabla f = \frac{1}{\nu} \eta^2 \varphi'(h)(\nabla u - \nabla \theta) + 2\eta \, sgn(u-\theta) \varphi(h) \nabla \eta.$$

Since u is a solution to the minimal graph equation, we can use f as a test function in

$$\int_U \frac{\langle \nabla u, \nabla f \rangle}{\sqrt{1+|\nabla u|^2}} = 0,$$

and obtain

$$\begin{aligned}\int_U \eta^2\varphi'(h)\frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} &= \int_U \eta^2\varphi'(h)\frac{\langle\nabla u,\nabla\theta\rangle}{\sqrt{1+|\nabla u|^2}}\\ &\quad - 2\nu\int_U \eta\, sgn(u-\theta)\varphi(h)\frac{\langle\nabla u,\nabla\eta\rangle}{\sqrt{1+|\nabla u|^2}}\\ &\le \int_U \eta^2\varphi'(h)\frac{|\nabla u||\nabla\theta|}{\sqrt{1+|\nabla u|^2}} + 2\nu\int_U \eta\varphi(h)\frac{|\nabla u||\nabla\eta|}{\sqrt{1+|\nabla u|^2}}.\end{aligned}$$

Next we use Young's inequality $ab \le (\varepsilon/2)a^2 + 1/(2\varepsilon)b^2$ and $\sqrt{1+|\nabla u|^2} \ge 1$ to estimate the terms on the right hand side as

$$\int_U \eta^2\varphi'(h)\frac{|\nabla u||\nabla\theta|}{\sqrt{1+|\nabla u|^2}} \le \frac{\varepsilon_1}{2}\int_U \eta^2\varphi'(h)\frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} + \frac{1}{2\varepsilon_1}\int_U \eta^2\varphi'(h)|\nabla\theta|^2$$

and

$$2\nu\int_U \eta\varphi(h)\frac{|\nabla u||\nabla\eta|}{\sqrt{1+|\nabla u|^2}} \le \varepsilon_2\int_U \eta^2\varphi'(h)\frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} + \frac{\nu^2}{\varepsilon_2}\int_U \frac{\varphi^2}{\varphi'}(h)|\nabla\eta|^2.$$

Then we choose ε_1 and ε_2 such that ε_1 is small enough and ε_2 minimizes the term

$$\frac{1}{\varepsilon_2(1-\varepsilon_1/2-\varepsilon_2)}$$

i.e. $\varepsilon_2 = (2-\varepsilon_1)/4$. Combining all terms we arrive at

$$\begin{aligned}\int_U \eta^2\varphi'(h)\frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} &\le \frac{2}{\varepsilon_1(2-\varepsilon_1)}\int_U \eta^2\varphi'(h)|\nabla\theta|^2 + \frac{4\nu^2}{1-\varepsilon_1}\int_U \frac{\varphi^2}{\varphi'}(h)|\nabla\eta|^2\\ &= C_\varepsilon\int_U \eta^2\varphi'(h)|\nabla\theta|^2 + (4+\varepsilon)\nu^2\int_U \frac{\varphi^2}{\varphi'}(h)|\nabla\eta|^2.\end{aligned}$$

□

Remark 3.2. *As can be seen in the proof of Lemma 3.3, the second term*

$$(4+\varepsilon)\nu^2\int_U \frac{\varphi^2}{\varphi'}(h)|\nabla\eta|^2$$

on the right hand side of (3.1) *is the only term that affects to the dimension-curvature restriction.*

We notice that the left hand side of (3.1) can be estimated from below by

$$\int_U \eta^2\varphi'(h)\frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} \ge c_1\int_{U_1} \eta^2\varphi'(h)|\nabla u|^2 + c_2\int_{U_2} \eta^2\varphi'(h)|\nabla u| \tag{3.2}$$

where

$$U_1 = \{|\nabla u| \le \sigma\},\quad U_2 = \{|\nabla u| \ge \sigma\},\quad \sigma > 0$$

and

$$c_1 = \frac{1}{\sqrt{1+\sigma^2}},\quad c_2 = \frac{1}{\sqrt{1+(1/\sigma^2)}}.$$

In the following Lemmas we will obtain some estimates using Lipschitz data $\theta\colon M \to \mathbb{R}$. By Rademacher's theorem, Lipschitz functions are differentiable almost everywhere and throughout the computations, the gradient $\nabla\theta$ appears only inside integrals so the points where θ is not differentiable will not be a problem.

Before stating the Lemmas we introduce the following notation. For $x \in M$, we denote by $j(x)$ the infimum of $|V\big(r(x)\big)|$ over Jacobi fields V along the geodesic

$\gamma^{o,x}$ that satisfy $V_0 = 0$, $|V_0'| = 1$ and $V_0' \perp \dot{\gamma}_0^{o,x}$. We also note that since M is a Cartan-Hadamard manifold, we have

$$\Delta r \geq \frac{n-1}{r}$$

in $M \setminus \{o\}$. From the curvature upper bound, Lemma 2.1 and [30, Example 1] it follows that for every $\varepsilon > 0$ there exists $R_1 > R_0$ such that

$$\Delta r \geq \frac{(n-1)\phi}{(1+\varepsilon)r}$$

for $r \geq R_1$ and therefore

$$r\Delta r \geq \begin{cases} n-1, & \text{in } M \setminus \{o\}, \\ \dfrac{(n-1)\phi}{1+\varepsilon}, & \text{in } M \setminus B(o, R_1). \end{cases} \tag{3.3}$$

Lemma 3.3. *Let M be a Cartan-Hadamard manifold satisfying*

$$K(P) \leq -\frac{\phi(\phi-1)}{r(x)^2},$$

where $K(P)$ is the sectional curvature of any plane $P \subset T_xM$ that contains the radial vector field $\nabla r(x)$ and x is any point in $M \setminus B(o, R_0)$. Furthemore, suppose that the dimension of M and the constant ϕ satisfies the relation (1.6)*. Let $U = B(o, R)$, with $R > R_1$, and suppose that $u \in C^2(U) \cap C(\bar{U})$ is the unique solution to the minimal graph equation in U, with $u|\partial U = \theta|\partial U$, where $\theta \colon M \to \mathbb{R}$ is a Lipschitz function, with $|\nabla\theta(x)| \leq 1/j(x)$ almost everywhere. Then there exists a constant c independent of u such that*

$$\int_U \varphi(|u-\theta|/c) \leq c + c\int_U F(r|\nabla\theta|) + c\int_U F_1(r^2|\nabla\theta|^2).$$

Proof. As before, we denote $h = |u-\theta|/\nu$, where $\nu \geq \nu_0$ will be fixed later, and to shorten the notation we denote $(n-1)\phi/(1+\varepsilon) =: C_0$. By splitting the integration domain and using the estimate (3.3), we first obtain

$$\begin{aligned} \int_U \varphi(h) r\Delta r &= \int_{B(o,R_1)} \varphi(h) r\Delta r + \int_{U\setminus B(o,R_1)} \varphi(h) r\Delta r \\ &\geq (n-1)\int_{B(o,R_1)} \varphi(h) + C_0\int_{U\setminus B(o,R_1)} \varphi(h) \\ &\geq (n-1-C_0)\int_{B(o,R_1)} \varphi(h) + C_0\int_U \varphi(h) \\ &\geq -c + C_0\int_U \varphi(h), \end{aligned}$$

where $c \geq 0$ is some constant. Next we use Green's formula to obtain

$$\begin{aligned} -c + C_0\int_U \varphi(h) \leq \int_U \varphi(h) r\Delta r &= -\int_U \langle \nabla(\varphi(h)r), \nabla r\rangle \\ &= -\int_U \varphi(h) - \int_U r\varphi'(h)\langle \nabla h, \nabla r\rangle, \end{aligned}$$

and consequently we have

$$-c + (1+C_0)\int_U \varphi(h) \leq \int_U r\varphi'(h)|\nabla h|.$$

To estimate the right hand side term, we first split the integration domain into two pieces $U = U_1 \cup U_2$, where

$$U_1 = \{x \in U \colon |\nabla u| \leq \sigma\} \quad \text{and} \quad U_2 = \{x \in U \colon |\nabla u| > \sigma\}.$$

Note that $|\nabla h| \le |\nabla u|/\nu + |\nabla \theta|/\nu$, so using the Caccioppoli-type inequality (3.1) and (3.2) we get

$$\begin{aligned}\int_U r\varphi'(h)|\nabla h| &\le \frac{1}{\nu}\int_{U_1} r\varphi'(h)|\nabla u| + \frac{1}{\nu}\int_{U_2} r\varphi'(h)|\nabla u| + \frac{1}{\nu}\int_U r\varphi'(h)|\nabla\theta| \\ &\le \frac{1}{\nu}\int_{U_1} r\varphi'(h)|\nabla u| + \frac{1}{\nu}\int_U r\varphi'(h)|\nabla\theta| \\ &\quad + \frac{C_\varepsilon}{c_2\nu}\int_U r\varphi'(h)|\nabla\theta|^2 + \frac{(4+\varepsilon)\nu}{c_2}\int_U \frac{\varphi^2}{\varphi'}(h)|\nabla\sqrt{r}|^2 \\ &= \frac{1}{\nu}\int_{U_1} r\varphi'(h)|\nabla u| + \frac{1}{\nu}\int_U r\varphi'(h)|\nabla\theta| \\ &\quad + \frac{C_\varepsilon}{c_2\nu}\int_U r\varphi'(h)|\nabla\theta|^2 + \frac{(4+\varepsilon)\nu}{4c_2}\int_U \frac{\varphi^2}{\varphi'}(h)r^{-1}\end{aligned}$$

By (2.8) and the convexity of the Young function G we have $\varphi(h) \le c\varphi'(h)$, and for r large enough, $|\nabla\theta| < 1$, so $|\nabla\theta|^2 \le |\nabla\theta|$. So from the previous estimate, we deduce that

$$\int_U r\varphi'(h)|\nabla h| \le \frac{1}{\nu}\int_{U_1} r\varphi'(h)|\nabla u| + \frac{1 + C_\varepsilon/c_2}{\nu}\int_U r\varphi'(h)|\nabla\theta| + c + \varepsilon'\int_U \varphi(h).$$

We continue again by splitting U_1 into two pieces by $U_1 = U_3 \cup U_4$, where

$$U_3 = \left\{|\nabla u| \le \tilde\sigma\frac{\varphi(h)}{\varphi'(h)r}\right\} \quad\text{and}\quad U_4 = \left\{\tilde\sigma\frac{\varphi(h)}{\varphi'(h)r} < |\nabla u| \le \sigma\right\}$$

and $\tilde\sigma$ is a constant to be determined later. Denote $\Psi(t) := \int_0^t \varphi'(s)^2/\varphi(s)\,ds$. Then using the Caccioppoli-type inequality (3.1) and (3.2) with r and Ψ' instead of η and φ' respectively, we can estimate the integral over U_1 by

$$\begin{aligned}\int_{U_1} r\varphi'(h)|\nabla u| &\le \tilde\sigma\int_{U_3}\varphi(h) + \frac{1}{\tilde\sigma}\int_{U_4} r^2\frac{\varphi'(h)^2}{\varphi(h)}|\nabla u|^2 \\ &\le \tilde\sigma\int_{U_3}\varphi(h) + \frac{1}{\tilde\sigma}\left(\frac{C_\varepsilon}{c_1}\int_U r^2\Psi'(h)|\nabla\theta|^2 + \frac{(4+\varepsilon)\nu^2}{c_1}\int_U\frac{\Psi^2}{\Psi'}(h)\right).\end{aligned}$$

From (2.9) we see that

$$\Psi'(t) = \frac{\varphi'(t)^2}{\varphi(t)} \le \tilde c\varphi''(t)$$

for t small enough, and hence

$$\Psi(t) = \int_0^t \frac{\varphi'(s)^2}{\varphi(s)} \le \tilde c\varphi'(t),$$

which implies that

$$\frac{\Psi^2}{\Psi'}(h) \le \tilde c\frac{\varphi'(h)^2}{\varphi'(h)^2/\varphi(h)} = \tilde c\varphi(h).$$

Notice that $\tilde c$, as well as c_1, can be chosen arbitrarly close to 1. Collecting these estimates together we arrive at

$$\begin{aligned}(1+C_0)\int_U\varphi(h) &\le c + \varepsilon'\int_U\varphi(h) + \frac{\tilde\sigma}{\nu}\int_U\varphi(h) + \frac{1+C_\varepsilon/c_2}{\nu}\int_U r\varphi'(h)|\nabla\theta| \\ &\quad + \frac{C_\varepsilon\tilde c}{c_1\tilde\sigma\nu}\int_U r^2\varphi''(h)|\nabla\theta|^2 + \frac{(4+\varepsilon)\nu\tilde c}{c_1\tilde\sigma}\int_U\varphi(h).\end{aligned}$$

Next we use the complementary Young functions G and F to estimate the term with φ', and G_1 and F_1 to estimate the term with φ''. So all together we have

$$\Big(1 + C_0 - \varepsilon' - \frac{1 + C_\varepsilon/c_2}{\nu} - \frac{\tilde{\sigma}}{\nu} - \frac{C_\varepsilon \tilde{c}}{c_1 \tilde{\sigma}\nu} - \frac{(4+\varepsilon)\nu\tilde{c}}{c_1\tilde{\sigma}}\Big) \int_U \varphi(h)$$
$$\leq c + \frac{1 + C_\varepsilon/c_2}{\nu} \int_U F(r|\nabla\theta|) + \frac{C_\varepsilon}{c_1\tilde{\sigma}\nu} \int_U F_1(r^2|\nabla\theta|^2).$$

For any fixed $\tilde{\varepsilon} > 0$, we can choose first σ and ε small enough, then ν big enough and $\tilde{\sigma} = \nu$ such that the coefficient on the left hand side is positive provided that $C_0 > 4 + \tilde{\varepsilon}$. This last inequality is satisfied thanks to the dimension-curvature restriction (1.6) and hence the claim is proved. □

The next lemma is a modification of [4, Lemma 20] (or originally [31, Lemma 2.20]). The proof is based on the idea of Moser iteration procedure.

Lemma 3.4. *Let $\Omega = B(o, R)$ and suppose that $\theta\colon \Omega \to \mathbb{R}$ is a bounded Lipschitz function with $|\theta|, |\nabla\theta| \leq C_1$. Let $u \in C^2(\Omega)$ be a solution of the minimal graph equation in Ω such that u has the boundary values θ and $\inf_\Omega \theta \leq u \leq \sup_\Omega \theta$. Fix $s \in (0, r_S)$, where r_S is the radius of the Sobolev inequality (2.1), and suppose that $B = B(x, s) \subset \Omega$. Then there exists a positive constant $\nu_0 = \nu_0(\varphi, C_1)$ such that for all fixed $\nu \geq \nu_0$*

$$\sup_{B(x,s/2)} \varphi\big(|u-\theta|/\nu\big)^{n+1} \leq c \int_B \varphi\big(|u-\theta|/\nu\big),$$

where c is a positive constant depending only on n, ν, s, C_S, C_1 and φ.

Remark 3.5. *Before proving the Lemma we note that increasing the constant ν above increases also the constant c. However, it does not cause problems since ν will always be a fixed constant.*

Proof of Lemma 3.4. We denote $\kappa = n/(n-1)$, $B/2 = B(x, s/2)$, and $h = |u-\theta|/\nu$, where $\nu \geq \nu_0 > 0$ will be fixed later. For each $j \in \mathbb{N}$ we denote $s_j = s(1 + \kappa^{-j})/2$ and $B_j = B(x, s_j)$. Note that $s_j \to s/2$ as $j \to \infty$. Let η_j be a Lipschitz function such that $0 \leq \eta_j \leq 1$, $\eta_j|B_{j+1} \equiv 1$, $\eta_j|(M \setminus B_j) \equiv 0$, and that

$$|\nabla\eta_j| \leq \frac{1}{s_j - s_{j+1}} = 2n\kappa^j/s.$$

For every $m \geq 1$, we have

$$|\nabla\eta_j^2\varphi(h)^m| \leq 2\eta_j\varphi(h)^m|\nabla\eta_j| + m\eta_j^2\varphi'(h)\varphi^{m-1}(h)|\nabla h|.$$

First we claim that

$$\Big(\int_{B_{j+1}} \varphi(h)^{\kappa m}\Big)^{1/\kappa} \leq c(\kappa^j + m + \kappa^{2j}/m) \int_{B_j} \varphi^{m-1}. \tag{3.4}$$

We notice that, for every $m, j \geq 1$, $\eta_j^2 \varphi(h)^m$ is a Lipschitz function supported in B_j. Using the Sobolev inequality (2.1), we first have

$$\begin{aligned}
\left(\int_{B_{j+1}} \varphi(h)^{\kappa m}\right)^{1/\kappa} &\leq \left(\int_{B_j} \left(\eta_j^2 \varphi(h)^m\right)^\kappa\right)^{1/\kappa} \leq C_S \int_{B_j} \left|\nabla\left(\eta_j \varphi(h)^m\right)\right| \\
&\leq 2C_S \int_{B_j} \eta_j \varphi(h)^m |\nabla \eta_j| + C_S \int_{B_j} \eta_j^2 (\varphi^m)'(h) |\nabla h| \\
&\leq c\kappa^j \int_{B_j} \varphi(h)^m + \frac{C_S}{\nu} \int_{B_j} (\varphi^m)'(h) |\nabla \theta| \\
&\quad + \frac{C_S}{\nu} \int_{B_j} \eta_j^2 (\varphi^m)'(h) |\nabla u|.
\end{aligned} \tag{3.5}$$

From the assumption

$$-C_1 \leq \inf_\Omega \theta \leq u \leq \sup_\Omega \theta \leq C_1$$

we obtain that $|u - \theta| \leq 2C_1$. We can use this to obtain upper bounds for φ and φ'. Namely, we have $G \circ \varphi' = \varphi$, where $G\colon [0, \infty) \to [0, \infty)$ is the homeomorphic and convex Young function. Consequently there exist constants ν_0 and c such that

$$\varphi(h) \leq 1,\ \varphi'(h) \leq 1 \ \text{ and } \ \varphi(h) \leq c\varphi'(h)$$

whenever $\nu \geq \nu_0$. Thus we get estimates

$$\int_{B_j} \varphi(h)^m \leq \int_{B_j} \varphi(h)^{m-1} \tag{3.6}$$

and

$$\int_{B_j} (\varphi^m)'(h) |\nabla \theta| = m \int_{B_j} \varphi(h)^{m-1} \varphi'(h) |\nabla \theta| \leq mC_1 \int_{B_j} \varphi(h)^{m-1}. \tag{3.7}$$

The third term on the right hand side of (3.5) can be estimated first as

$$\begin{aligned}
\int_{B_j} \eta_j^2 (\varphi^m)'(h) |\nabla u| &\leq \int_{B_j \cap U_1} \eta_j^2 (\varphi^m)'(h) + \int_{B_j \cap U_2} \eta_j^2 (\varphi^m)'(h) |\nabla u| \\
&\leq \int_{B_j} m\varphi(h)^{m-1} + \sqrt{2} \int_{B_j} \eta_j^2 (\varphi^m)'(h) \frac{|\nabla u|^2}{\sqrt{1 + |\nabla u|^2}},
\end{aligned} \tag{3.8}$$

where U_1 is the set where $|\nabla u| < 1$ and U_2 the set where $|\nabla u| \geq 1$. The constant $\sqrt{2}$ comes from (3.2) when we choose $\sigma = 1$.

Next we notice that $\eta_j^2 \varphi(h)^m \in W_0^{1,2}(B_j)$, since supp $\eta_j \subset \bar{B}_j$, and thus we can apply the Caccioppoli-type inequality (3.1) with φ^m instead of φ. We also choose $\varepsilon_1 = \varepsilon_2 = 1/3$ in the proof of (3.1) so the constants become 3 and 6. Hence we obtain

$$\begin{aligned}
\sqrt{2} \int_{B_j} \eta_j^2 (\varphi^m)'(h) \frac{|\nabla u|^2}{\sqrt{1 + |\nabla u|^2}} &\leq 3\sqrt{2} \int_{B_j} \eta_j^2 (\varphi^m)'(h) |\nabla \theta|^2 \\
&\quad + 6\sqrt{2}\nu^2 \int_{B_j} \frac{\varphi^{2m}}{(\varphi^m)'}(h) |\nabla \eta_j|^2 \\
&\leq c(m + \kappa^{2j}/m) \int_{B_j} \varphi(h)^{m-1}.
\end{aligned} \tag{3.9}$$

Now the estimate (3.4) follows by inserting the estimates (3.6)-(3.9) into (3.5). We apply (3.4) with $m = m_j + 1$, where $m_j = (n+1)\kappa^j - n$. Note that $m_{j+1} = \kappa(m_j + 1)$, so we can write (3.4) as

$$\left(\int_{B_{j+1}} \varphi(h)^{m_j}\right)^{1/\kappa} \le C\kappa^j \int_{B_j} \varphi(h)^{m_j}.$$

By denoting

$$I_j = \left(\int_{B_j} \varphi(h)^{m_j}\right)^{1/\kappa^j}$$

we can write the previous inequality as a recursion formula

$$I_{j+1} \le C^{1/\kappa^j} \kappa^{j/\kappa^j} I_j.$$

Since

$$\limsup_{j\to\infty} I_j \ge \lim_{j\to\infty} \left(\int_{B/2} \varphi(h)^{m_j}\right)^{(n+1)/m_j} = \sup_{B/2} \varphi(h)^{n+1},$$

we get

$$\sup_{B/2} \varphi(h)^{n+1} \le \limsup_{j\to\infty} I_j \le C^n \kappa^S I_0 \le c \int_B \varphi(h),$$

where

$$S = \sum_{j=0}^{\infty} j\kappa^{-j} < \infty.$$

□

In order to prove that our solution to the minimal graph equation extends to the boundary $\partial_\infty M$ and has the desired boundary values, we will also need that the right hand side integrals of Lemma 3.3 are finite. The following ensures that the functions F and F_1 decrease fast enough. Recall that $j(x)$ denotes the infimum of $|V\big(r(x)\big)|$ over Jacobi fields V along the geodesic $\gamma^{o,x}$ that satisfy $V_0 = 0$, $|V_0'| = 1$ and $V_0' \perp \dot\gamma_0^{o,x}$.

Lemma 3.6. *Let M be a Cartan-Hadamard manifold satisfying*

$$K(P) \le -\frac{\phi(\phi-1)}{r(x)^2},$$

where $K(P)$ is the sectional curvature of any plane $P \subset T_xM$ that contains the radial vector field $\nabla r(x)$ and x is any point in $M \setminus B(o, R_0)$. Then there exist $F, F_1 \in \mathcal{F}$ such that

$$F\left(\frac{r(x)}{j(x)}\right) j(x)^{C(n-1)} \le r(x)^{-2}$$

and

$$F_1\left(\frac{r(x)^2}{j(x)^2}\right) j(x)^{C(n-1)} \le r(x)^{-2}$$

for any positive constant C and for every $x \in M$ outside a compact set.

Proof. We prove the claim only for function F since the case with F_1 (given by Proposition 2.5) is essentially the same. Let λ be as in Proposition 2.5. By (2.5) there exists $F \in \mathcal{F}$ such that

$$F(t) \le \exp\left(-\tfrac{1}{t}\Big(\log\big(e + \tfrac{1}{t}\big)\Big)^{-\lambda}\right)$$

for all small t. Hence the claim follows if

$$\exp\left(-\tfrac{j(x)}{r(x)}\Big(\log\big(e + \tfrac{j(x)}{r(x)}\big)\Big)^{-\lambda}\right) j(x)^{C(n-1)} \le r(x)^{-2},$$

and taking logarithms, we see that this is equivalent with

$$\frac{j(x)}{r(x)}\Big(\log\Big(e+\tfrac{j(x)}{r(x)}\Big)\Big)^{-\lambda}-C(n-1)\log j(x)-2\log r(x)\geq 0.$$

It follows from the curvature assumptions that $j(x)\geq cr(x)^{\phi}$, $\phi>1$, whenever $r(x)\geq\tilde{R}$ for some $\tilde{R}>0$ (see e.g. Lemma 2.1 and [30, Example 1]), so it is enough to show that

$$f(t):=\frac{t}{a}\Big(\log\big(e+\tfrac{t}{a}\big)\Big)^{-\lambda}-C(n-1)\log t-2\log a\geq 0$$

for all $t\geq ca^{\phi}$ when a is big enough. A straightforward computation gives that

$$f'(t)=\frac{\frac{1}{a}\Big(1-\frac{\lambda}{\log(e+t/a)(ae/t+1)}\Big)}{\Big(\log\big(e+\frac{t}{a}\big)\Big)^{\lambda}}-\frac{C(n-1)}{t},$$

so noticing that $t/a\geq ca^{\phi-1}\geq\tilde{R}^{\phi}$ and $\log(e+t/a)\leq k(t/a)^{\alpha}$, where k is a constant and $\alpha>0$ can be made as small as we wish, we obtain

$$f'(t)\geq\frac{k}{a^{1-\alpha}t^{\alpha}}-\frac{C(n-1)}{t}\geq 0$$

for all $t\geq ca^{\phi}$ and a large enough. Finally we notice that

$$\begin{aligned}f(a^{\phi})&=a^{\phi-1}\big(\log(e+a^{\phi-1})\big)^{-\lambda}-C(n-1)\log a^{\phi-1}-2\log a\\&=a^{\phi-1}\big(\log(e+a^{\phi-1})\big)^{-\lambda}-\big(C(n-1)(\phi-1)+2\big)\log a\end{aligned}$$

which clearly is positive when $a\geq\tilde{R}$ is large enough.

□

3.1. **Solving the asymptotic Dirichlet problem with Lipschitz boundary data.** In order to prove the main theorem we begin by solving the corresponding Dirichlet problem with Lipschitz boundary data. The asymptotic boundary $\partial_{\infty}M$ is homeomorphic to the unit sphere $\mathbb{S}^{n-1}\subset T_oM$ and hence we may interpret the given boundary function $f\in C(\partial_{\infty}M)$ as a continuous function on $\mathbb{S}^{n-1}$. We first solve the asymptotic Dirichlet problem for (1.1) with Lipschitz continuous boundary values $f\in C(\mathbb{S}^{n-1})$. We assume that, for all $x\in M$ and for all 2-planes $P\subset T_xM$,

$$K(P)\leq -a^2\big(r(x)\big), \tag{3.10}$$

where $a\colon[0,\infty)\to[0,\infty)$ is a smooth function that is constant in some neighborhood of 0 and

$$a^2(t)=\frac{\phi(\phi-1)}{t^2},\quad \phi>1,$$

for $t\geq R_0$. Identify $\partial_{\infty}M$ with the unit sphere $\mathbb{S}^{n-1}\subset T_oM$ and assume that $f\colon\mathbb{S}^{n-1}\to\mathbb{R}$ is L-Lipschitz. We extend f radially to a continuous function θ on $M\setminus\{o\}$. The radial extension θ is also a locally Lipschitz function and hence, by Rademacher's theorem, differentiable almost everywhere. The gradient of θ can be estimated in terms of an angle function as follows. Let $x,y\in\bar{M}$ and let $\gamma^{o,x}$ and $\gamma^{o,y}$ be the unique unit speed geodesics joining o to x and y. Denote by $\bar{x}$ and $\bar{y}$ the corresponding points on $\mathbb{S}^{n-1}$ i.e. $\bar{x}=\dot{\gamma}_0^{o,x}$ and $\bar{y}=\dot{\gamma}_0^{o,y}$. Then

$$\begin{aligned}\frac{|\theta(x)-\theta(y)|}{d(x,y)}&=\frac{|\theta(\bar{x})-\theta(\bar{y})|}{d(x,y)}\leq\frac{Ld(\bar{x},\bar{y})}{d(x,y)}\\&=L\frac{\sphericalangle_o(\bar{x},\bar{y})}{d(x,y)}=L\frac{\sphericalangle_o(x,y)}{d(x,y)}\end{aligned}$$

and we obtain $|\nabla\theta| \le L|\nabla\sphericalangle_o(\cdot,\cdot)|$. By Lemma 2.3 this implies

$$|\nabla\theta(x)| \le \frac{L}{j(x)}$$

and we see that θ satisfies the assumptions of Lemmas 3.3 and 3.4.

We are now ready to solve the asymptotic Dirichlet problem with Lipschitz boundary data.

Lemma 3.7. *Let M be a Cartan-Hadamard manifold of dimension $n \ge 2$ satisfying the curvature assumptions* (1.4), (1.5) *and* (1.6) *for all 2-planes $P \subset T_xM$ with $x \in M \setminus B(o,R_0)$. Suppose that $f \in C(\partial_\infty M)$ is L-Lipschitz when interpreted as a function on $\mathbb{S}^{n-1} \subset T_oM$. Then the asymptotic Dirichlet problem for minimal graph equation* (1.1) *is uniquely solvable with boundary data f.*

Proof. Let θ be the radial extension of the given Lipschitz boundary data $f \in C(\partial_\infty M)$ defined above. We exhaust M by an increasing sequence of geodesic balls $B_k = B(o,k)$, $k \in \mathbb{N}$, and show first that there exist smooth solutions $u_k \in C^\infty(B_k) \cap C(\bar{B}_k)$ of the minimal graph equation

$$\begin{cases} \operatorname{div} \dfrac{\nabla u_k}{\sqrt{1+|\nabla u_k|^2}} = 0, & \text{in } B_k, \\ u_k|\partial B_k = \theta|\partial B_k. \end{cases} \tag{3.11}$$

For this, fix $k \in \mathbb{N}$ and let $(\theta_i^k) \subset C^2(\partial B_k)$ be a sequence that converges uniformly to the function θ on ∂B_k. For every i there exists a function $u_i^k \in C^\infty(B_k)$ that solves the minimal graph equation in B_k and has boundary values θ_i^k. By the Maximum principle we have

$$\sup_{B_k}|u_j^k - u_i^k| \le \sup_{\partial B_k}|\theta_j^k - \theta_i^k|$$

so the sequence (u_i^k) converges uniformly to some function $u_k \in C(\bar{B}_k)$. In $\bar{B}_k$ the sectional curvatures are bounded, so we can apply the interior gradient estimate [28, Theorem 1.1] and obtain that $|\nabla u_i^k|$ is locally bounded independent of i. Therefore standard arguments and regularity theory of elliptic PDEs imply that $u_i^k \to u_k$ in $C^2_{\text{loc}}(B_k) \cap C(\bar{B}_k)$ and therefore u_k is also a solution to the minimal graph equation (3.11). Moreover, the comparison principle implies that

$$-\max_{x\in M}|\theta(x)| \le u_k \le \max_{x\in M}|\theta(x)|,$$

so the solutions u_k are bounded in B_k for every $k \in \mathbb{N}$.

Fix a compact set $K \subset M$. Then applying the interior gradient estimate [28, Theorem 1.1], we obtain

$$\sup_K|\nabla u_k| \le c(K),$$

where the constant $c(K)$ is independent of k. The theory of elliptic PDEs implies that there exists a subsequence, still denoted by u_k, that converges in $C^2_{\text{loc}}(M)$ to a solution $u \in C^\infty(M)$. Hence we are left to prove that u extends continuously to the boundary $\partial_\infty M$ and satisfies $u|\partial_\infty M = f$.

Next we will use Lemma 3.3, and in order to estimate the appearing integrals we use geodesic polar coordinates (r,v) for points $x \in M$. Here we denoted $r = r(x)$ and $v = \dot\gamma_0^{o,x} \in S_oM$. Let $\lambda(r,v)$ be the Jacobian for these polar coordinates. Note that then we have $\lambda(r,v) \le J(r,v)^{n-1}$ where $J(x)$ denotes the supremum of $|V\big(r(x)\big)|$ over Jacobi fields V along the geodesic $\gamma^{o,x}$ that satisfy $V_0 = 0$, $|V_0'| = 1$ and $V_0' \perp \dot\gamma_0^{o,x}$.

Let ν be such that it satisfies the assuptions of Lemmas 3.3 and 3.4. Applying Lemma 2.2, Fatou's lemma, and Lemma 3.3 with $U = B_k$ we get

$$\begin{aligned}
\int_M \varphi\big(|u-\theta|/\nu\big) &\le \liminf_{k\to\infty} \int_{B_k} \varphi\big(|u_k-\theta|/\nu\big) \\
&\le c + c\int_M F(r|\nabla\theta|) + c\int_M F_1(r^2|\nabla\theta|^2) \\
&= c + c\int_{R_1}^{\infty}\int_{S_oM} F(r|\nabla\theta(r,v)|)\lambda(r,v)\,dv\,dr \\
&\quad + c\int_{R_1}^{\infty}\int_{S_oM} F_1(r^2|\nabla\theta(r,v)|^2)\lambda(r,v)\,dv\,dr \\
&\le c + c\int_{R_1}^{\infty}\int_{S_oM} F\left(\frac{r}{j(r,v)}\right) j(r,v)^{C_K(n-1)}\,dv\,dr \\
&\quad + c\int_{R_1}^{\infty}\int_{S_oM} F_1\left(\frac{r^2}{j(r,v)^2}\right) j(r,v)^{C_K(n-1)}\,dv\,dr \\
&< \infty. \qquad (3.12)
\end{aligned}$$

Finiteness of the last integrals follows from Lemma 3.6.

Let $x \in M$ and fix $s \in (0, r_S)$. For k large enough, u_k satisfies the assumptions of Lemma 3.4, and hence

$$\sup_{B(x,s/2)} \varphi\big(|u_k-\theta|/\nu\big)^{n+1} \le c\int_{B(x,s)} \varphi\big(|u_k-\theta|/\nu\big).$$

This and the dominated convergence theorem implies that

$$\begin{aligned}
\sup_{B(x,s/2)} \varphi\big(|u-\theta|/\nu\big)^{n+1} &= \sup_{B(x,s/2)} \lim_{k\to\infty} \varphi\big(|u_k-\theta|/\nu\big)^{n+1} \\
&\le \limsup_{k\to\infty} \sup_{B(x,s/2)} \varphi\big(|u_k-\theta|/\nu\big)^{n+1} \qquad (3.13) \\
&\le c\limsup_{k\to\infty} \int_{B(x,s)} \varphi\big(|u_k-\theta|/\nu\big) = c\int_{B(x,s)} \varphi\big(|u-\theta|/\nu\big).
\end{aligned}$$

Let $\xi \in \partial_\infty M$ and (x_i) be a sequence of points in M with $x_i \to \xi$ as $i \to \infty$. Applying the estimate (3.13) with $x = x_i$ and fixed $s \in (0, r_S)$ we obtain, by (3.12), that

$$\lim_{i\to\infty} \sup_{B(x_i,s/2)} \varphi\big(|u-\theta|/\nu\big)^{n+1} \le c\lim_{i\to\infty} \int_{B(x_i,s)} \varphi\big(|u-\theta|/\nu\big) = 0$$

and hence $|u(x_i) - \theta(x_i)| \to 0$ as $i \to \infty$. Since $\xi \in \partial_\infty M$ was arbitrary, it follows that u extends continuously to $\partial_\infty M$ and satisfies $u|\partial_\infty M = f$.

□

3.2. **Proof of the main theorem.** Let $f \in C(\partial_\infty M)$. As in the case of Lipschitz functions, we identify $\partial_\infty M$ with the unit sphere $\mathbb{S}^{n-1} \subset T_oM$. Let (f_i) be a sequence of Lipschitz functions such that $f_i \to f$ uniformly as $i \to \infty$. By Lemma 3.7 there exist solutions $u_i \in C^\infty(M) \cap C(\bar{M})$ of the minimal graph equation (1.1) with the desired boundary values $u_i = f_i$ on $\partial_\infty M$. It follows from the Maximum principle that

$$\sup_M |u_i - u_j| = \max_{\partial_\infty M} |f_i - f_j|$$

and consequently the sequence u_i converges uniformly to a function $u \in C(\bar{M})$. Applying the interior gradient estimate [28, Theorem 1.1] in compact subsets of M we conclude that the convergence takes place in $C(\bar{M}) \cap C^2_{\text{loc}}(M)$ and therefore u is also a solution to (1.1) in M and $u = f$ on $\partial_\infty M$. Regularity theory implies that $u \in C^\infty(M)$.

For the proof of uniqueness, suppose that u and v are both solutions of (1.1) in M, continuous in $\bar{M}$ and $u = v$ on the boundary $\partial_\infty M$. By symmetry we can assume that $u(y) > v(y)$ for some $y \in M$. Denote $\delta = \big(u(y) - v(y)\big)/2$ and let $U \subset \{x \in M : u(x) > v(x) + \delta\}$ be the component that contains y. Then U is a relatively compact open domain since both u and v are continuous and coincide on $\partial_\infty M$. Furthemore $u = v + \delta$ on ∂U and it follows that $u = v + \delta$ in U which is a contradiction since we have $y \in U$.

□

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APPENDIX 2 – ARTICLE [B]

Asymptotic Dirichlet problem for $\mathcal{A}$-harmonic functions on manifolds with pinched curvature

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Potential Analysis, 46(1):63-74, 2017

ASYMPTOTIC DIRICHLET PROBLEM FOR $\mathcal{A}$-HARMONIC FUNCTIONS ON MANIFOLDS WITH PINCHED CURVATURE

ESKO HEINONEN

Abstract. We study the asymptotic Dirichlet problem for $\mathcal{A}$-harmonic functions on a Cartan-Hadamard manifold whose radial sectional curvatures outside a compact set satisfy an upper bound

$$K(P) \leq -\frac{1+\varepsilon}{r(x)^2 \log r(x)}$$

and a pointwise pinching condition

$$|K(P)| \leq C_K |K(P')|$$

for some constants $\varepsilon > 0$ and $C_K \geq 1$, where P and P' are any 2-dimensional subspaces of T_xM containing the (radial) vector $\nabla r(x)$ and $r(x) = d(o,x)$ is the distance to a fixed point $o \in M$. We solve the asymptotic Dirichlet problem with any continuous boundary data $f \in C(\partial_\infty M)$. The results apply also to the Laplacian and p-Laplacian, $1 < p < \infty$, as special cases.

1. Introduction

In this paper we are interested in the asymptotic Dirichlet problem for $\mathcal{A}$-harmonic functions on a Cartan-Hadamard manifold M of dimension $n \geq 2$. We recall that a Cartan-Hadamard manifold is a simply connected complete Riemannian manifold with non-positive sectional curvature. Since the exponential map $\exp_o\colon T_oM \to M$ is a diffeomorphism for every point $o \in M$, it follows that M is diffeomorphic to $\mathbb{R}^n$. One can define an asymptotic boundary $\partial_\infty M$ of M as the set of all equivalence classes of unit speed geodesic rays on M. Then the compactification of M is given by $\bar{M} = M \cup \partial_\infty M$ equipped with the *cone topology*. We also notice that $\bar{M}$ is homeomorphic to the closed Euclidean unit ball; for details, see Section 2 and [8].

The *asymptotic Dirichlet problem* on M for some operator $\mathcal{Q}$ is the following: Given a function $f \in C(\partial_\infty M)$ does there exist a (unique) function $u \in C(\bar{M})$ such that $\mathcal{Q}[u] = 0$ on M and $u|\partial_\infty M = f$? We will consider this problem for the $\mathcal{A}$-harmonic operator (of type p)

$$\mathcal{Q}[u] = -\mathrm{div}\, \mathcal{A}_x(\nabla u), \tag{1.1}$$

where $\mathcal{A}\colon TM \to TM$ is subject to certain conditions; for instance $\langle \mathcal{A}(V), V\rangle \approx |V|^p$, $1 < p < \infty$, and $\mathcal{A}(\lambda V) = \lambda|\lambda|^{p-2}\mathcal{A}(V)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ (see Section 2.3 for the precise definition). A function u is said to be $\mathcal{A}$-*harmonic* if it satisfies the equation

$$-\,\mathrm{div}\, \mathcal{A}_x(\nabla u) = 0. \tag{1.2}$$

The asymptotic Dirichlet problem on Cartan-Hadamard manifolds has been solved for various operators and under various assumptions on the manifold. The first result for this problem was due to Choi [6] when he solved the asymptotic Dirichlet problem for the Laplacian assuming that the sectional curvature has a negative upper bound $K_M \leq -a^2 < 0$, and that any two points at infinity can be

2010 *Mathematics Subject Classification.* 58J32, 53C21, 31C45.
Key words and phrases. $\mathcal{A}$-harmonic functions, Dirichlet problem, Hadamard manifold.

separated by convex neighborhoods. Anderson [1] showed that such convex sets exist provided the sectional curvature of the manifold satisfies $-b^2 \le K_M \le -a^2 < 0$. We point out that Sullivan [13] solved independently the asymptotic Dirichlet problem for the Laplacian under the same curvature assumptions but using probabilistic arguments. Cheng [5] was the first to solve the problem for the Laplacian under the same type of pointwise pinching assumption for the sectional curvatures as we consider in this paper. Later the asymptotic Dirichlet problem has been generalized for p-harmonic and $\mathcal{A}$-harmonic functions and for minimal graph equation under various curvature assumptions, see [2], [3], [10], [11], [14], [15].

In [14] Vähäkangas had exactly the same pinching condition but with weaker upper bound for the sectional curvatures. Namely, he solved the asymptotic Dirichlet problem assuming the pointwise pinching condition and

$$K(P) \le -\frac{\phi(\phi-1)}{r(x)^2},$$

where $\phi > 1$ is constant. In [2] the authors showed that, with these stronger assumptions, the solvability result holds also for the minimal graph equation.

In this paper we will use similar techniques as in [2], [3] and [15]. Our main theorem is the following.

Theorem 1.1. *Let M be a Cartan-Hadamard manifold of dimension $n \ge 2$. Assume that*

$$K(P) \le -\frac{1+\varepsilon}{r(x)^2 \log r(x)}, \tag{1.3}$$

for some constant $\varepsilon > 0$, where $K(P)$ is the sectional curvature of any two-dimensional subspace $P \subset T_xM$ containing the radial vector $\nabla r(x)$, with $x \in M \setminus B(o, R_0)$. Suppose also that there exists a constant $C_K < \infty$ such that

$$|K(P)| \le C_K |K(P')| \tag{1.4}$$

whenever $x \in M \setminus B(o, R_0)$ and $P, P' \subset T_xM$ are two-dimensional subspaces containing the radial vector $\nabla r(x)$. Then the asymptotic Dirichlet problem for the $\mathcal{A}$-harmonic equation (1.2) *is uniquely solvable for any boundary data $f \in C(\partial_\infty M)$ provided that $1 < p < n\alpha/\beta$.*

In the case of usual Laplacian we have $\alpha = \beta = 1$ and $p = 2$. Hence we obtain the following special case.

Corollary 1.2. *Let M be a Cartan-Hadamard manifold of dimension $n \ge 3$ and assume that the assumptions* (1.3) *and* (1.4) *are satisfied. Then the asymptotic Dirichlet problem for the Laplace operator is uniquely solvable for any boundary data $f \in C(\partial_\infty M)$.*

We close this introduction by commenting that our upper bound (1.3) is in a sense optimal, since assuming

$$K(P) \ge -\frac{1}{r(x)^2 \log r(x)}$$

and considering $\mathcal{A}$-harmonic operator of type $p \ge n$, implies that M is p-parabolic i.e. every bounded $\mathcal{A}$-harmonic function (of type p) is constant. For more detailed discussion, see e.g. [3].

2. Preliminaries

2.1. **Cartan-Hadamard manifolds.** Recall that a Cartan-Hadamard manifold is a complete and simply connected Riemannian manifold with non-positive sectional curvature. Let M be a Cartan-Hadamard manifold and $\partial_\infty M$ the sphere at infinity, then we denote $\bar{M} = M \cup \partial_\infty M$. The sphere at infinity is defined as the set of all equivalence classes of unit speed geodesic rays in M; two such rays γ_1 and γ_2 are equivalent if

$$\sup_{t\ge 0} d\big(\gamma_1(t),\gamma_2(t)\big) < \infty.$$

The equivalence class of γ is denoted by $\gamma(\infty)$. For each $x\in M$ and $y\in \bar M\setminus\{x\}$ there exists a unique unit speed geodesic $\gamma^{x,y}\colon \mathbb{R}\to M$ such that $\gamma^{x,y}(0)=x$ and $\gamma^{x,y}(t)=y$ for some $t\in(0,\infty]$. For $x\in M$ and $y,z\in\bar M\setminus\{x\}$ we denote by

$$\sphericalangle_x(y,z) = \sphericalangle(\dot\gamma_0^{x,y},\dot\gamma_0^{x,z})$$

the angle between vectors $\dot\gamma_0^{x,y}$ and $\dot\gamma_0^{x,z}$ in T_xM. If $v\in T_xM\setminus\{0\}$, $\alpha>0$, and $R>0$, we define a cone

$$C(v,\alpha)=\{y\in\bar M\setminus\{x\}\colon \sphericalangle(v,\dot\gamma_0^{x,y})<\alpha\}$$

and a truncated cone

$$T(v,\alpha,R)=C(v,\alpha)\setminus \bar B(x,R).$$

All cones and open balls in M form a basis for the cone topology in $\bar M$. With this topology $\bar M$ is homeomorphic to the closed unit ball $\bar B^n\subset\mathbb{R}^n$ and $\partial_\infty M$ to the unit sphere $\mathbb{S}^{n-1}=\partial B^n$. For detailed study on the cone topology, see [8].

Let us recall that the local Sobolev inequality holds on any Cartan-Hadamard manifold M. More precisely, there exist constants $r_S>0$ and $C_S<\infty$ such that

$$\left(\int_B |\eta|^{n/(n-1)}\right)^{(n-1)/n} \le C_S\int_B|\nabla\eta| \tag{2.1}$$

holds for every ball $B=B(x,r_S)\subset M$ and every function $\eta\in C_0^\infty(B)$. This inequality can be obtained e.g. from Croke's estimate of the isoperimetric constant, see [4] and [7].

2.2. **Jacobi equation.** If $k\colon[0,\infty)\to[0,\infty)$ is a smooth function, we denote by $f_k\in C^\infty\big([0,\infty)\big)$ the solution to the initial value problem

$$\begin{cases} f_k''=k^2f_k\\ f_k(0)=0,\\ f_k'(0)=1.\end{cases} \tag{2.2}$$

The solution is a non-negative smooth function. Concerning the curvature upper bound (1.3), we have the following estimate by Choi.

Proposition 2.1. [6, Prop. 3.4] *Suppose that $f\colon[R_0,\infty)\to\mathbb{R}$, $R_0>0$, is a positive strictly increasing function satisfying the equation $f''(r)=a^2(r)f(r)$, where*

$$a^2(r)\ge\frac{1+\varepsilon}{r^2\log r},$$

for some $\varepsilon>0$ on $[R_0,\infty)$. Then for any $0<\tilde\varepsilon<\varepsilon$, there exists $R_1>R_0$ such that, for all $r\ge R_1$,

$$f(r)\ge r(\log r)^{1+\tilde\varepsilon},\qquad \frac{f'(r)}{f(r)}\ge\frac1r+\frac{1+\tilde\varepsilon}{r\log r}.$$

The pinching condition for the sectional curvatures gives a relation between the maximal and minimal moduli of Jacobi fields along a given geodesic that contains the radial vector:

Lemma 2.2. [5, Lemma 3.2][14, Lemma 3] *Let $v\in T_oM$ be a unit vector and $\gamma=\gamma^v$. Suppose that $r_0>0$ and $k<0$ are constants such that $K_M(P)\ge k$ for every two-dimensional subspace $P\subset T_xM$, $x\in B(o,r_0)$. Suppose that there exists a constant $C_K<\infty$ such that*

$$|K_M(P)|\le C_K|K_M(P')|$$

whenever $t \geq r_0$ and $P, P' \subset T_{\gamma(t)}M$ are two-dimensional subspaces containing the radial vector $\dot\gamma_t$. Let V and $\bar V$ be two Jacobi fields along γ such that $V_0 = 0 = \bar V_0$, $V_0' \perp \dot\gamma_0 \perp \bar V_0$, and $|V_0'| = 1 = |\bar V_0'|$. Then there exists a constant $C_0 = C_0(C_K, r_0, k) > 0$ such that

$$|V_r|^{C_K} \geq C_0 |\bar V_r|$$

for every $r \geq r_0$.

To prove the solvability of the $\mathcal{A}$-harmonic equation, we will need an estimate for the gradient of a certain angular function. This estimate can be obtained in terms of Jacobi fields:

Lemma 2.3. [14, Lemma 2] *Let $x_0 \in M \setminus \{o\}$, $U = M \setminus \gamma^{o,x_0}(\mathbb{R})$, and define $\theta\colon U \to [0,\pi]$, $\theta(x) = \sphericalangle_o(x_0, x) := \arccos\langle \dot\gamma_0^{o,x_0}, \dot\gamma_0^{o,x}\rangle$. Let $x \in U$ and $\gamma = \gamma^{o,x}$. Then there exists a Jacobi field W along γ with $W(0) = 0$, $W_0' \perp \dot\gamma_0$, and $|W_0'| = 1$ such that*

$$|\nabla \theta(x)| \leq \frac{1}{|W(r(x))|}.$$

2.3. **$\mathcal{A}$-harmonic functions.** Let M be a Riemannian manifold and $1 < p < \infty$. Suppose that $\mathcal{A}\colon TM \to TM$ is an operator that satisfies the following assumptions for some constants $0 < \alpha \leq \beta < \infty$: the mapping $\mathcal{A}_x = \mathcal{A}|T_xM\colon T_xM \to T_xM$ is continuous for almost every $x \in M$ and the mapping $x \mapsto \mathcal{A}_x(V_x)$ is measurable for all measurable vector fields V on M; for almost every $x \in M$ and every $v \in T_xM$ we have

$$\begin{aligned} \langle \mathcal{A}_x(v), v\rangle &\geq \alpha |v|^p, \\ |\mathcal{A}_x(v)| &\leq \beta |v|^{p-1}, \\ \langle \mathcal{A}_x(v) - \mathcal{A}_x(w), v - w\rangle &> 0, \end{aligned}$$

whenever $w \in T_xM \setminus \{v\}$, and

$$\mathcal{A}_x(\lambda v) = \lambda |\lambda|^{p-2} \mathcal{A}_x(v)$$

for all $\lambda \in \mathbb{R} \setminus \{0\}$. The set of all such operators is denoted by $\mathcal{A}^p(M)$ and we say that $\mathcal{A}$ is of type p. The constants α and β are called the *structure constants* of $\mathcal{A}$.

Let $\Omega \subset M$ be an open set and $\mathcal{A} \in \mathcal{A}^p(M)$. A function $u \in C(\Omega) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$ is *$\mathcal{A}$-harmonic* in Ω if it is a weak solution of the equation

$$-\operatorname{div} \mathcal{A}(\nabla u) = 0. \tag{2.3}$$

In other words, if

$$\int_\Omega \langle \mathcal{A}(\nabla u), \nabla \varphi\rangle = 0 \tag{2.4}$$

for every test function $\varphi \in C_0^\infty(\Omega)$. If $|\nabla u| \in L^p(\Omega)$, then it is equivalent to require (2.4) for all $\varphi \in W_0^{1,p}(\Omega)$ by approximation.

In the special case $\mathcal{A}(v) = |v|^{p-2}v$, $\mathcal{A}$-harmonic functions are called *p-harmonic* and, in particular, if $p = 2$, we obtain the usual harmonic functions.

A lower semicontinuous function $u\colon \Omega \to (-\infty, \infty]$ is called *$\mathcal{A}$-superharmonic* if $u \not\equiv \infty$ in each component of Ω, and for each open $D \subset\subset \Omega$ and for every $h \in C(\bar D)$, $\mathcal{A}$-harmonic in D, $h \leq u$ on ∂D implies $h \leq u$ in D.

The *asymptotic Dirichlet problem* (for $\mathcal{A}$-harmonic functions) is the following: for given function $f \in C(\partial_\infty M)$, find a function $u \in C(\bar M)$ such that $\mathcal{A}(u) = 0$ in M and $u|\partial_\infty M = f$. The asymptotic Dirichlet problem can be solved using the so called Perron's method which we will recall next. The definitions follow [9].

Fix $p \in (1, \infty)$ and let $\mathcal{A} \in \mathcal{A}^p(M)$.

Definition 1. A function $u\colon M \to (-\infty, \infty]$ belongs to the upper class $\mathcal{U}_f$ of $f\colon \partial_\infty M \to [-\infty, \infty]$ if

(1) u is $\mathcal{A}$-superharmonic in M,
(2) u is bounded from below, and
(3) $\liminf_{x\to x_0} u(x) \ge f(x_0)$ for all $x_0 \in \partial_\infty M$.

The function

$$\overline{H}_f = \inf\{u\colon u \in \mathcal{U}_f\}$$

is called the *upper Perron solution* and $\underline{H}_f = -\overline{H}_{-f}$ the *lower Perron solution.*

Theorem 2.4. *One of the following is true:*

(1) $\overline{H}_f$ *is* $\mathcal{A}$*-harmonic in* M*,*
(2) $\overline{H}_f \equiv \infty$ *in* M*,*
(3) $\overline{H}_f \equiv -\infty$ *in* M*.*

We define $\mathcal{A}$-regular points as follows.

Definition 2. A point $x_0 \in \partial_\infty M$ is called $\mathcal{A}$-regular if

$$\lim_{x\to x_0} \overline{H}_f(x) = f(x_0)$$

for all $f \in C(\partial_\infty M)$.

Regularity and solvability of the Dirichlet problem are related. Namely, the asymptotic Dirichlet problem for $\mathcal{A}$-harmonic functions is uniquely solvable if and only if every point at infinity is $\mathcal{A}$-regular.

2.4. **Young functions.** Let $\phi\colon [0,\infty) \to [0,\infty)$ be a homeomorphism and let $\psi = \phi^{-1}$. Define *Young functions* Φ and Ψ by setting

$$\Phi(t) = \int_0^t \phi(s)\,ds$$

and

$$\Psi(t) = \int_0^t \psi(s)\,ds$$

for each $t \in [0,\infty)$. Then we have the following *Young's inequality*

$$ab \le \Phi(a) + \Psi(b)$$

for all $a, b \in [0,\infty)$. The functions Φ and Ψ are said to form a *complementary Young pair.* Furthermore, Φ (and similarly Ψ) is a continuous, strictly increasing, and convex function satisfying

$$\lim_{t\to 0^+} \frac{\Phi(t)}{t} = 0$$

and

$$\lim_{t\to \infty} \frac{\Phi(t)}{t} = \infty.$$

For a more general definition of Young functions see e.g. [12].

As in [15], we consider complementary Young pairs of a special type. For that, suppose that a homeomorphism $G\colon [0,\infty) \to [0,\infty)$ is a Young function that is a diffeomorphism on $(0,\infty)$ and satisfies

$$\int_0^1 \frac{dt}{G^{-1}(t)} < \infty \tag{2.5}$$

and

$$\lim_{t\to 0} \frac{tG'(t)}{G(t)} = 1. \tag{2.6}$$

Then $G(\cdot^{1/p})^p$, $p > 1$, is also a Young function and we define $F\colon [0,\infty) \to [0,\infty)$ so that $G(\cdot^{1/p})^p$ and $F(\cdot^{1/p})$ form a complementary Young pair. The space of such functions F will be denoted by $\mathcal{F}_p$. Note that if $F \in \mathcal{F}_p$, then also $\lambda F \in \mathcal{F}_p$ and

$F(\lambda\cdot) \in \mathcal{F}_p$ for every $\lambda > 0$. In [15] it is proved that for fixed $\varepsilon_0 \in (0,1)$ there exists $F \in \mathcal{F}_p$ such that

$$F(t) \le t^{p+\varepsilon_0} \exp\left(-\tfrac{1}{t} \Big(\log\big(e + \tfrac{1}{t}\big)\Big)^{-1-\varepsilon_0} \right) \tag{2.7}$$

for all $t \in [0,\infty)$.

3. Solving the asymptotic Dirichlet problem

In order to solve the asymptotic Dirichlet problem for the $\mathcal{A}$-harmonic equation, we need the following two lemmas, which we state without proofs. Their proofs can be found from the original papers. The first lemma allows us to estimate the supremum of a function in a ball by the integral over a bigger ball. The second lemma shows that we can estimate the previous integral up to another integral, which will be uniformly bounded provided the sectional curvatures of M satisfy (1.3) and (1.4).

Lemma 3.1. [15, Lemma 2.20] *Suppose that $||\theta||_{L^\infty} \le 1$. Suppose that $s \in (0, r_S)$ is a constant and $x \in M$. Assume also that $u \in W^{1,p}_{loc}(M)$ is a function that is $\mathcal{A}$-harmonic in the open set $\Omega \cap B(x,s)$, satisfies $u - \theta \in W^{1,p}_0(\Omega)$, $\inf_M \theta \le u \le \sup_M \theta$, and $u = \theta$ a.e. in $M \setminus \Omega$. Then*

$$\operatorname*{ess\,sup}_{B(x,s/2)} \varphi\big(|u-\theta|\big)^{p(n-1)} \le c \int_{B(x,s)} \varphi\big(|u-\theta|\big)^p,$$

where the constant c is independent of x.

Lemma 3.2. [3, Lemma 16] *Let M be a Cartan-Hadamard manifold of dimension $n \ge 2$. Suppose that*

$$K(P) \le -\frac{1+\varepsilon}{r(x)^2 \log r(x)},$$

for some constant $\varepsilon > 0$, where $K(P)$ is the sectional curvature of any plane $P \subset T_xM$ that contains the radial vector $\nabla r(x)$ and x is any point in $M \setminus B(o, R_0)$. Suppose that $U \subset M$ is an open relatively compact set and that u is an $\mathcal{A}$-harmonic function in U, with $u - \theta \in W^{1,p}_0(U)$, where $\mathcal{A} \in \mathcal{A}^p(M)$ with

$$1 < p < \frac{n\alpha}{\beta},$$

and $\theta \in W^{1,\infty}(M)$ is a continuous function with $||\theta||_\infty \le 1$. Then there exists a bounded C^1-function $\mathcal{C}\colon [0,\infty) \to [0,\infty)$ and a constant $c_0 \ge 1$, that is independent of θ, U and u, such that

$$\begin{aligned}\int_U \varphi\big(|u-\theta|/c_0\big)^p \big(\log(1+r) + \mathcal{C}(r)\big) \\ \le c_0 + c_0 \int_U F\left(\frac{c_0 |\nabla\theta| r \log(1+r)}{\log(1+r) + \mathcal{C}(r)}\right) \big(\log(1+r) + \mathcal{C}(r)\big).\end{aligned} \tag{3.1}$$

In what follows, we will denote by $j(x)$ the infimum, and by $J(x)$ the supremum, of $|V\big(r(x)\big)|$ over Jacobi fields V along the geodesic $\gamma^{o,x}$ that satisfy $V_0 = 0$, $|V_0'| = 1$ and $V_0' \perp \dot\gamma_0^{o,x}$.

Next we show that the integral appearing in Lemma 3.2 is finite provided the upper bound (1.3) and the pinching condition (1.4) for the sectional curvatures.

Lemma 3.3. *Let M be a Cartan-Hadamard manifold satisfying*

$$K(P) \le -\frac{1+\varepsilon}{r(x)^2 \log r(x)},$$

where $K(P)$ is the sectional curvature of any plane $P \subset T_xM$ that contains the radial vector field $\nabla r(x)$ and x is any point in $M \setminus B(o,R_0)$. Then there exists $F \in \mathcal{F}_p$ such that

$$F\left(\frac{r(x)}{c_1 j(x)}\right)\Big(\log(1+r)+\mathcal{C}(r)\Big)j(x)^{C(n-1)} \le r(x)^{-2}$$

for any positive constants C and c_1, and for every $x \in M$ outside a compact set.

Proof. Fix $\varepsilon_0 \in (0,1)$ and denote $\lambda := 1+\varepsilon_0$. Then by (2.7) there exists $F \in \mathcal{F}_p$ such that

$$F(t) \le \exp\left(-\tfrac{1}{t}\Big(\log\left(e+\tfrac{1}{t}\right)\Big)^{-\lambda}\right)$$

for all small t. Hence the claim follows if we show that

$$\exp\left(-\frac{c_1 j(x)}{r(x)}\left(\log\left(e+\frac{c_1 j(x)}{r(x)}\right)\right)^{-\lambda}\right)\Big(\log\big(1+r(x)\big)+\mathcal{C}(r)\Big)j(x)^{C(n-1)} \le r(x)^{-2},$$

which, by taking logarithms, is equivalent with

$$\begin{aligned}\frac{c_1 j(x)}{r(x)}\left(\log\left(e+\frac{c_1 j(x)}{r(x)}\right)\right)^{-\lambda} &- \log\Big(\log\big(1+r(x)\big)+\mathcal{C}(r)\Big)\\ &- C(n-1)\log j(x) - 2\log r(x) \ge 0.\end{aligned}$$

Let $\tilde\varepsilon \in (0,\varepsilon)$. Then the curvature upper bound and Proposition 2.1 implies that $j(x) \ge r(x)\big(\log r(x)\big)^{1+\tilde\varepsilon}$ for $r(x) \ge R_1 > R_0$, so it is enough to show that

$$f(t) := \frac{c_1 t}{a}\left(\log\left(e+\frac{c_1 t}{a}\right)\right)^{-\lambda} - \log\Big(\log(1+a)+\mathcal{C}(a)\Big) - C(n-1)\log t - 2\log a \ge 0$$

for all $t \ge a\big(\log a\big)^{1+\tilde\varepsilon}$ when a is big enough. By straight computation we get

$$\begin{aligned} f'(t) &= \left(\log\left(e+\frac{c_1 t}{a}\right)\right)^{-\lambda}\left(\frac{-\lambda c_1^2 t}{a^2\log(e+c_1t/a)(e+c_1t/a)}+\frac{c_1}{a}\right) - \frac{C(n-1)}{t}\\ &= \left[\frac{c_1}{a}\left(1-\frac{\lambda}{\log(e+c_1t/a)(\frac{ea}{c_1t}+1)}\right)\Big/\left(\log\left(e+\frac{c_1t}{a}\right)\right)^{\lambda}\right] - \frac{C(n-1)}{t}.\end{aligned}$$

Then we notice that $c_1t/a \ge c_1(\log a)^{1+\tilde\varepsilon}$, which can be made big by increasing a, and $(\log(e+c_1t/a))^\lambda \le k(t/a)^\nu$, where k is some constant and $\nu>0$ can be made as small as we wish. Hence we obtain

$$f'(t) \ge \frac{k_1}{a^{1-\nu}t^{\nu}} - \frac{C(n-1)}{t} \ge 0$$

for all $t \ge a(\log a)^{1+\tilde\varepsilon}$ and some constant k_1 when a is big enough.

Finally we have to check that f is positive at least when t is big enough. To see this, we notice that

$$\begin{aligned} f\big(a(\log a)^{1+\tilde\varepsilon}\big) &= c_1(\log a)^{1+\tilde\varepsilon}\big(\log\big(e+c_1(\log a)^{1+\tilde\varepsilon}\big)\big)^{-\lambda}\\ &\quad - \log\big(\log(1+a)-\mathcal{C}(a)\big) - C(n-1)\log\big((a\log a)^{1+\tilde\varepsilon}\big) - 2\log a,\end{aligned}$$

and this being positive is equivalent to

$$\begin{aligned} c_1(\log a)^{1+\tilde\varepsilon} \ge \big(\log(e+c_1(\log a)^{1+\tilde\varepsilon})\big)^{\lambda}\Big(&\big(C(n-1)+2\big)\log a\\ &+\log\big(\log(1+a)+\mathcal{C}(a)\big)+C(n-1)\log(\log a)^{1+\tilde\varepsilon}\Big),\end{aligned}$$

which holds true for a big enough, since $(\log a)^{1+\tilde\varepsilon}$ increases faster than $(\log a)\big(\log(e+c_1(\log a)^{1+\tilde\varepsilon})^{\lambda}\big)$. □

To prove the Theorem 1.1, we give a proof for the following localized version that shows the $\mathcal{A}$-regularity of a point $x_0 \in \partial_\infty M$. That, in turn, implies Theorem 1.1 since the uniqueness follows from the comparison principle.

The proof of the following theorem is the same as the proof of [3, Theorem 17] except that to prove

$$\int_\Omega F\left(\frac{c_0|\nabla\theta| r\log(1+r)}{L(r)}\right)L(r) < \infty,$$

where $L(r) = \log(1+r) + \mathcal{C}(r)$, we use Lemma 3.3 instead of some estimates involving the curvature lower bound. For convenience, we will also write down the proof.

Theorem 3.4. *Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$ and let $x_0 \in \partial_\infty M$. Assume that x_0 has a cone neighborhood U such that*

$$K(P) \leq -\frac{1+\varepsilon}{r(x)^2 \log r(x)}, \tag{3.2}$$

for some constant $\varepsilon > 0$, where $K(P)$ is the sectional curvature of any two-dimensional subspace $P \subset T_xM$ containing the radial vector $\nabla r(x)$, with $x \in U \cap M$. Suppose also that there exists a constant $C_K < \infty$ such that

$$|K(P)| \leq C_K |K(P')| \tag{3.3}$$

whenever $x \in U \cap M$ and $P, P' \subset T_xM$ are two-dimensional subspaces containing the radial vector $\nabla r(x)$. Then x_0 is $\mathcal{A}$-regular for every $\mathcal{A} \in \mathcal{A}^p(M)$ with $1 < p < n\alpha/\beta$.

Proof. Let $f\colon \partial_\infty M \to \mathbb{R}$ be a continuous function. To prove the $\mathcal{A}$-regularity of x_0, we need to show that

$$\lim_{x\to x_0} \overline{H}_f(x) = f(x_0).$$

Fix $\varepsilon' > 0$ and let $v_0 = \dot\gamma_0^{o,x_0}$ be the initial vector of the geodesic ray from o to x_0. Furthermore, let $\delta \in (0,\pi)$ and $R_0 > 0$ be such that $T(v_0,\delta,R_0) \subset U$ and that $|f(x_1) - f(x_0)| < \varepsilon'$ for all $x_1 \in C(v_0,\delta) \cap \partial_\infty M$. Fix also $\tilde\varepsilon \in (0,\varepsilon)$, where ε is the constant in (3.2), and let $r_1 > \max(2, R_1)$, where $R_1 \geq R_0$ is given by Proposition 2.1.

We denote $\Omega = T(v_0,\delta,r_1) \cap M$ and define $\theta \in C(\bar M)$ by setting

$$\theta(x) = \min\Big(1, \max\big(r_1 + 1 - r(x), \delta^{-1}\sphericalangle_o(x_0,x)\big)\Big).$$

Let $\Omega_j = \Omega \cap B(o,j)$ for integers $j > r_1$ and let u_j be the unique $\mathcal{A}$-harmonic function in Ω_j with $u_j - \theta \in W_0^{1,p}(\Omega_j)$. Each $y \in \partial\Omega_j$ is $\mathcal{A}$-regular and hence functions u_j can be continuously extended to $\partial\Omega_j$ by setting $u_j = \theta$ on $\partial\Omega_j$. Next we notice that $0 \leq u_j \leq 1$, so the sequence (u_j) is equicontinuous, and hence, by Arzelá-Ascoli, we obtain a subsequence (still denoted by (u_j)) that converges locally uniformly to a continuous function $u\colon \bar\Omega \to [0,1]$. It follows that u is $\mathcal{A}$-harmonic in Ω; see e.g. [9, Chapter 6].

Next we aim to prove that

$$\lim_{\substack{x\to x_0\\ x\in\Omega}} u(x) = 0, \tag{3.4}$$

and for that we use geodesic polar coordinates (r,v) for points $x \in \Omega$. Here $r = r(x) \in [r_1,\infty)$ and $v = \dot\gamma_0^{o,x}$, and we denote by $\lambda(r,v)$ the Jacobian of these polar coordinates. Denote $\tilde\theta = \theta/c_0$, $\tilde u_j = u_j/c_0$ and $\tilde u = u/c_0$, where $c_0 \geq 1$ is a constant given by Lemma 3.2. Then applying Fatou's lemma and Lemma 3.2 to Ω_j we obtain

$$\begin{aligned}\int_\Omega \varphi\big(|\tilde u-\tilde\theta|\big)^p &= \int_\Omega \varphi\big(|u-\theta|/c_0\big)^p \le \liminf_{j\to\infty}\int_{\Omega_j}\varphi\big(|u_j-\theta|/c_0\big)^p\\
&\le \liminf_{j\to\infty}\int_{\Omega_j}\varphi\big(|u_j-\theta|/c_0\big)^p L(r)\\
&\le c_0+c_0\int_\Omega F\left(\frac{c_0|\nabla\theta| r\log(1+r)}{L(r)}\right)L(r)\\
&= c_0+c_0\int_{r_1}^\infty\int_{S_oM}F\left(\frac{c_0|\nabla\theta(r,v)|r\log(1+r)}{\log(1+r)+\mathcal{C}(r)}\right)\big(\log(1+r)\\
&\quad+\mathcal{C}(r)\big)\lambda(r,v)\,dv\,dr\\
&\le c_0+c_0\int_{r_1}^\infty\int_{S_oM}F\left(\frac{r}{c_1 j(r,v)}\right)\Big(\log(1+r)+\mathcal{C}(r)\Big)\frac{1}{C_0}j(r,v)^{C(n-1)}\,dv\,dr\\
&<\infty.\end{aligned}\tag{3.5}$$

At the end we applied also Lemmas 2.2, 2.3, and 3.3.

Next, we extend each u_j to a function $u_j\in W^{1,p}_{\text{loc}}(M)\cap C(M)$ by setting $u_j(y)=\theta(y)$ for every $y\in M\setminus\Omega_j$. Let $x\in\Omega$ and fix $s\in(0,r_S)$. For j large enough, we obtain by Lemma 3.1

$$\sup_{B(x,s/2)}\varphi\big(|\tilde u_j-\tilde\theta|\big)^{p(n+1)}\le c\int_{B(x,s)}\varphi\big(|\tilde u_j-\tilde\theta|\big)^p.$$

Applying this with dominated convergence theorem, we get

$$\begin{aligned}\sup_{B(x,s/2)}\varphi\big(|\tilde u-\tilde\theta|\big)^{p(n+1)} &= \sup_{B(x,s/2)}\lim_{j\to\infty}\varphi\big(|\tilde u_j-\tilde\theta|\big)^{p(n+1)}\\
&\le \limsup_{j\to\infty}\sup_{B(x,s/2)}\varphi\big(|\tilde u_j-\tilde\theta|\big)^{p(n+1)}\\
&\le c\limsup_{j\to\infty}\int_{B(x,s)}\varphi\big(|\tilde u_j-\tilde\theta|\big)^p = c\int_{B(x,s)}\varphi\big(|\tilde u-\tilde\theta|\big)^p.\end{aligned}\tag{3.6}$$

Let $(x_k)\subset\Omega$ be a sequence such that $x_k\to x_0$ as $k\to\infty$. We apply the estimate (3.6) with $x=x_k$ and a fixed $s\in(0,r_S)$, together with (3.5), to obtain

$$\lim_{k\to\infty}\sup_{B(x_k,s/2)}\varphi\big(|\tilde u-\tilde\theta|\big)^{p(n+1)}\le c\lim_{k\to\infty}\int_{B(x_k,s)}\varphi\big(|\tilde u-\tilde\theta|\big)^p=0.$$

It follows that

$$\lim_{k\to\infty}|\tilde u(x_k)-\tilde\theta(x_k)|=0,$$

which, in turn, implies (3.4).

Define a function $w\colon M\to\mathbb{R}$ by

$$w(x)=\begin{cases}\min\big(1,2u(x)\big) & \text{if } x\in\Omega;\\ 1, & \text{if } x\in M\setminus\Omega.\end{cases}$$

The minimum of two $\mathcal{A}$-superharmonic functions is $\mathcal{A}$-superharmonic and hence w is $\mathcal{A}$-superharmonic. The definition of $\overline{H}_f$ implies that

$$\overline{H}_f\le f(x_0)+\varepsilon'+2(\sup|f|)w,$$

and therefore, by (3.4), we have

$$\limsup_{x\to x_0}\overline{H}_f(x)\le f(x_0)+\varepsilon'.$$

Similarly one can prove that

$$\liminf_{x\to x_0}\underline{H}_f(x)\ge f(x_0)-\varepsilon',$$

and because $\overline{H}_f \geq \underline{H}_f$ and ε' was arbitrary, we conclude that

$$\lim_{x \to x_0} \overline{H}_f(x) = f(x_0).$$

Therefore x_0 is $\mathcal{A}$-regular point. □

Acknowledgements. The author would like to thank his advisor Ilkka Holopainen for reading the manuscript and making helpful suggestions.

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APPENDIX 3 – ARTICLE [C]

Dirichlet problem for f-minimal graphs

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Journal d'Analyse Mathématique, to appear

Preprint: arXiv:1605.01935 (2016)

DIRICHLET PROBLEM FOR f-MINIMAL GRAPHS

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ABSTRACT. We study the asymptotic Dirichlet problem for f-minimal graphs in Cartan-Hadamard manifolds M. f-minimal hypersurfaces are natural generalizations of self-shrinkers which play a crucial role in the study of mean curvature flow. In the first part of this paper, we prove the existence of f-minimal graphs with prescribed boundary behavior on a bounded domain $\Omega \subset M$ under suitable assumptions on f and the boundary of Ω. In the second part, we consider the asymptotic Dirichlet problem. Provided that f decays fast enough, we construct solutions to the problem. Our assumption on the decay of f is linked with the sectional curvatures of M. In view of a result of Pigola, Rigoli and Setti, our results are almost sharp.

CONTENTS

1. INTRODUCTION

In this paper we study the Dirichlet problem for the so-called f-minimal graph equation on a complete non-compact n-dimensional Riemannian manifold M with the Riemannian metric given by $ds^2 = \sigma_{ij}dx^i dx^j$ in local coordinates. We equip $N = M \times \mathbb{R}$ with the product metric $ds^2 + dt^2$ and assume that $f\colon N \to \mathbb{R}$ is a smooth function. The Dirichlet problem for f-minimal graphs is to find a solution u to the equation

$$\begin{cases} \operatorname{div} \dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \langle \bar{\nabla} f, \nu \rangle & \text{in } \Omega \\ u|\partial\Omega = \varphi, \end{cases} \tag{1.1}$$

where $\Omega \subset M$ is a bounded domain, $\bar{\nabla} f$ is the gradient of f with respect to the product Riemannian metric, and ν denotes the downward unit normal to the graph

2000 *Mathematics Subject Classification.* Primary 58J32; Secondary 53C21.
Key words and phrases. Mean curvature equation, Dirichlet problem, Hadamard manifold.
J.-B.C. supported by MIS F.4508.14 (FNRS).
E.H. supported by Jenny and Antti Wihuri Foundation.

of u, i.e.

$$\nu = \frac{(\nabla u, -1)}{\sqrt{1+|\nabla u|^2}}. \tag{1.2}$$

The regularity assumptions on f, $\partial\Omega$, and on φ will be specified in due course.

The equation (1.1) can be written in non-divergence form as

$$\frac{1}{W}\left(\sigma^{ij} - \frac{u^i u^j}{W^2}\right) u_{i;j} = \langle \bar\nabla f, \nu\rangle, \tag{1.3}$$

where $W = \sqrt{1+|\nabla u|^2}$, (σ^{ij}) stands for the inverse matrix of (σ_{ij}), $u^i = \sigma^{ij}u_j$, with $u_j = \partial u/\partial x^j$, and $u_{i;j} = u_{ij} - \Gamma^k_{ij}u_k$ denotes the second order covariant derivative of u.

We recall that an immersed hypersurface Σ of a Riemannian manifold (N, g) is called an f-minimal hypersurface if its (scalar) mean curvature H satisfies an equation

$$H = \langle \bar\nabla f, \nu\rangle$$

at every point of Σ. Here, too, ν is a unit normal vector field along Σ, f is a smooth function on N, and $\bar\nabla f$ denotes its gradient with respect to the Riemannian metric g. Hence the graph of a solution u of (1.1) is an f-minimal hypersurface in $M\times\mathbb{R}$. Note that we define the mean curvature as the trace of the second fundamental form. Other examples of f-minimal hypersurfaces are

(a) minimal hypersurfaces if f is identically constant,
(b) self-shrinkers in $\mathbb{R}^{n+1}$ if $f(x) = |x|^2/4$,
(c) minimal hypersurfaces of weighted manifolds $M_f = \big(M, g, e^{-f} d\,\mathrm{vol}_M\big)$, where (M, g) is a complete Riemannian manifold with the Riemannian volume element $d\,\mathrm{vol}_M$.

We refer to [7], [6], [3], [4], [5], [16], and references therein for recent studies on self-shrinkers and f-minimal hypersurfaces. Let us just point out a recent result relevant to our paper. Wang in [23] investigated graphical self-shrinkers in $\mathbb{R}^n$ by studying the equation (1.1) in the whole $\mathbb{R}^n$ when $f(x) = |x|^2/4$. She proved that any smooth solution to this equation has to be a hyperplane improving an ealier result of Ecker and Huisken [12], where they made the extra assumption that the solution has polynomial growth. We will show that the situation is quite different when $\mathbb{R}^n$ is replaced by a Cartan-Hadamard manifold with strictly negative sectional curvatures and for more general f satisfying some suitable assumptions. In particular, we impose that $\sup_{\bar\Omega\times\mathbb{R}}|\bar\nabla f| < \infty$ which is not valid for $f(x) = |x|^2/4$.

In our existence results we always assume that $f \in C^2(\bar\Omega\times\mathbb{R})$ is of the form

$$f(x,t) = m(x) + r(t), \tag{1.4}$$

for discussion about this, see Section 2.3. Our first result is the following:

Theorem 1.1. *Let $\Omega \subset M$ be a bounded domain with $C^{2,\alpha}$ boundary $\partial\Omega$. Suppose that $f \in C^2(\bar\Omega\times\mathbb{R})$ satisfies* (1.4)*, with*

$$F = \sup_{\bar\Omega\times\mathbb{R}}|\bar\nabla f| < \infty, \quad \mathrm{Ric}_\Omega \ge -\frac{F^2}{n-1}, \quad \text{and} \quad H_{\partial\Omega} \ge F,$$

where Ric_Ω stands for the Ricci curvature of Ω and $H_{\partial\Omega}$ for the inward mean curvature of $\partial\Omega$. Then, for all $\varphi \in C^{2,\alpha}(\partial\Omega)$, there exists a solution $u \in C^{2,\alpha}(\bar\Omega)$ to the equation (1.1) *with boundary values φ.*

The proof of Theorem 1.1 is based on the Leray-Schauder method (see [13, Theorem 13.8]), and hence requires a priori height and gradient (both interior and boundary) estimates for solutions. It is worth noting already at this point that we cannot ask for the uniqueness of a solution if the function $f\colon M\times\mathbb{R}\to\mathbb{R}$ depends on the t-variable since comparison principles fail to hold. Indeed, an easy computation shows that for the open disk $B(0,2) \subset \mathbb{R}^2$ and $f\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, $f(x,t) = |(x,t)|^2/4$, both the upper and lower hemispheres and the disk $B(0,2)$ itself are f-minimal hypersurfaces with zero boundary values on the circle $\partial B(0,2)$.

Thanks to the interior gradient estimate Lemma 2.3 we can weaken the regularity assumption on the boundary value function.

Theorem 1.2. *Let $\Omega \subset M$ be a bounded domain with $C^{2,\alpha}$ boundary $\partial\Omega$. Suppose that $f \in C^2(\bar\Omega \times \mathbb{R})$ satisfies* (1.4)*, with*

$$F = \sup_{\bar\Omega\times\mathbb{R}} |\bar\nabla f| < \infty, \quad \mathrm{Ric}_\Omega \geq -\frac{F^2}{n-1}, \quad \text{and} \quad H_{\partial\Omega} \geq F.$$

Then, for all $\varphi \in C(\partial\Omega)$, there exists a solution $u \in C^{2,\alpha}(\Omega) \cap C(\bar\Omega)$ to the equation (1.1) *with boundary values φ.*

Let us point out that the assumption $H_{\partial\Omega} \geq F$ is necessary. Indeed, Serrin [21] has proved that the constant mean curvature equation

$$\operatorname{div} \frac{\nabla u}{W} = H_0$$

is solvable on a bounded domain $\Omega \subset \mathbb{R}^n$ if and only if $H_{\partial\Omega} \geq |H_0|$; see also [14] for a related result. If the function f were a density function defined only on M, it might be possible to refine the assumptions in terms of mean convexity of the boundary with respect to the weighted mean curvature and Bakry-Emery-Ricci tensor. However, in our case the function f depends also on the $\mathbb{R}$-variable and for the a priori estimates it is necessary to have a control of the full gradient.

Finally in Section 4, we consider the Dirichlet problem at infinity. Here we suppose that M is a Cartan-Hadamard manifold, i.e. a complete, simply connected Riemannian manifold with non-positive sectional curvature. We denote by $\bar M$ the compactification of M in the cone topology (see [11]) and by $\partial_\infty M$ the asymptotic boundary of M. The Dirichlet problem at infinity consists in finding solutions to (1.1) in the case where $\Omega = M$ and $\partial\Omega = \partial_\infty M$. In order to formulate the assumptions on sectional curvatures of M and on the function $f\colon M \times \mathbb{R} \to \mathbb{R}$, we first denote by $\rho(\cdot) = d(o,\cdot)$ the (Riemannian) distance to a fixed point $o \in M$. Then we assume that sectional curvatures of M satisfy

$$-(b\circ\rho)^2(x) \leq K(P_x) \leq -(a\circ\rho)^2(x) \tag{1.5}$$

for all $x \in M$ and all 2-dimensional subspaces $P_x \subset T_xM$, where a and b are smooth functions subject to conditions (A1)-(A7); see Section 4. Given a smooth function $k\colon [0,\infty) \to [0,\infty)$, we denote by $f_k\colon [0,\infty) \to \mathbb{R}$ the smooth non-negative solution to the initial value problem

$$\begin{cases} f_k(0) = 0, \\ f_k'(0) = 1, \\ \quad f_k'' = k^2 f_k. \end{cases} \tag{1.6}$$

To state the main result on the solvability of the asymptotic Dirichlet problem requires a number of definitions. First of all we assume that there exists an auxiliary smooth function $a_0\colon [0,\infty) \to (0,\infty)$ such that

$$\int_1^\infty \left(\int_r^\infty \frac{ds}{f_a^{n-1}(s)} \right) a_0(r) f_a^{n-1}(r)\,dr < \infty.$$

Then we define $g\colon [0,\infty) \to [0,\infty)$ by

$$g(r) = \frac{1}{f_a^{n-1}(r)} \int_0^r a_0(t) f_a^{n-1}(t)\,dt. \tag{1.7}$$

The function g was introduced in [19] where they studied some elliptic and parabolic equations with asymptotic Dirichlet boundary conditions on Cartan-Hadamard manifolds. In addition to (1.4), we assume that the function $f \in C^2(\bar{\Omega}\times\mathbb{R})$ satisfies

$$\sup_{\partial B(o,r)\times\mathbb{R}} |\bar{\nabla} f| \le \min\left\{ \frac{a_0(r)+(n-1)\frac{f_a'(r)}{f_a(r)}g^3(r)}{\big(1+g^2(r)\big)^{3/2}}, (n-1)\frac{f_a'(r)}{f_a(r)} \right\}, \tag{1.8}$$

for every $r>0$, and

$$\sup_{\partial B(o,r)\times\mathbb{R}} |\bar{\nabla} f| = o\left(\frac{f_a'(r)}{f_a(r)} r^{-\varepsilon-1}\right) \tag{1.9}$$

for some $\epsilon > 0$ as $r\to\infty$.

The general solvability result for the asymptotic Dirichlet problem is the following.

Theorem 1.3. *Let M be a Cartan-Hadamard manifold of dimension $n\ge 2$. Assume that*

$$-(b\circ\rho)^2(x) \le K(P_x) \le -(a\circ\rho)^2(x)$$

for all $x\in M$ and all 2-dimensional subspaces $P_x\subset T_xM$ where a and b satisfy assumptions (A1)-(A7) *and that the function $f\in C^2(M\times\mathbb{R})$ on the right side of* (1.1) *satisfies* (1.4), (1.8), *and* (1.9). *Then the asymptotic Dirichlet problem for the equation* (1.1) *is solvable for any boundary data $\varphi\in C\big(\partial_\infty M\big)$.*

As a special case of the above theorem, we have:

Corollary 1.4. *Let M be a Cartan-Hadamard manifold of dimension $n\ge 2$. Suppose that there are constants $\phi>1$, $\varepsilon>0$, and $R_0>0$ such that*

$$-\rho(x)^{2(\phi-2)-\varepsilon} \le K(P_x) \le -\frac{\phi(\phi-1)}{\rho(x)^2}, \tag{1.10}$$

for all 2-dimensional subspaces $P_x\subset T_xM$ and for all $x\in M$, with $\rho(x)\ge R_0$. Assume, furthermore, that $f\in C^2(M\times\mathbb{R})$ satisfies (1.4), (1.8), *and* (1.9), *with $f_a(t)=t$ for small $t\ge 0$ and $f_a(t)=c_1t^{\phi}+c_2t^{1-\phi}$ for $t\ge R_0$. Then the asymptotic Dirichlet problem for equation* (1.1) *is solvable for any boundary data $\varphi\in C\big(\partial_\infty M\big)$.*

In another special case we assume that sectional curvatures are bounded from above by a negative constant $-k^2$.

Corollary 1.5. *Let M be a Cartan-Hadamard manifold of dimension $n\ge 2$. Assume that*

$$-\rho(x)^{-2-\varepsilon}e^{2k\rho(x)} \le K(P_x) \le -k^2 \tag{1.11}$$

for some constants $k>0$ and $\varepsilon>0$ and for all 2-dimensional subspaces $P_x\subset T_xM$, with $\rho(x)\ge R_0$. Assume, furthermore, that $f\in C^2(M\times\mathbb{R})$ satisfies (1.4), (1.8), *and* (1.9), *with $f_a(t)=t$ for small $t\ge 0$ and $f_a(t)=c_1\sinh(kt)+c_2\cosh(kt)$ for $t\ge R_0$. Then the asymptotic Dirichlet problem for the equation* (1.1) *is solvable for any boundary data $\varphi\in C\big(\partial_\infty M\big)$.*

We refer to [15, Ex. 2.1, Cor. 3.22] and to [15, Cor. 3.23] for the verification of the assumptions (A1)-(A7) for the curvature bounds (1.10) and (1.11), respectively. We point out that, thanks to Examples 4.5 and 4.6, the assumption (1.8) in the above corollaries is weaker than (1.9) when $r\to\infty$.

Let us discuss where the assumptions (1.8) and (1.9) will be used in our paper. First of all, we prove Theorem 1.3 by extending the boundary value function φ to M, exhausting M by geodesic balls and solving the Dirichlet problem (1.1) in each ball. In this step, the assumption

$$\sup_{\partial B(o,r)\times\mathbb{R}} |\bar{\nabla} f| \le (n-1)\frac{f_a'(r)}{f_a(r)}$$

is used. Secondly, the other assumption in (1.8),

$$\sup_{\partial B(o,r)\times\mathbb{R}} |\bar\nabla f| \le \frac{a_0(r) + (n-1)\frac{f_a'(r)}{f_a(r)} g^3(r)}{\big(1+g^2(r)\big)^{3/2}},$$

is used to prove that the sequence of solutions above is uniformly bounded, thus allowing us to extract a subsequence converging towards a global solution. Finally, we apply (1.9) to prove that this global solution has proper boundary values at infinity. Furthermore, concerning (1.9), let us mention a result of Pigola, Rigoli, and Setti in [20]. There they considered the equation

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = h(x),$$

for a function $h \in C^\infty(M)$. They proved that if $\max_M |u| < \infty$, h has a constant sign, and M satisfies one of the following growth assumptions:

$$\operatorname{vol}\big(\partial B(o,r)\big) \le Cr^\alpha, \text{ for some } \alpha \ge 0 \tag{1.12}$$

or

$$\operatorname{vol}\big(\partial B(o,r)\big) \le Ce^{\alpha r}, \text{ for some } \alpha \ge 0, \tag{1.13}$$

then necessarily we have

$$\liminf_{\rho(x)\to\infty} \frac{|h(x)|}{\rho^{-2}(x)\big(\log\rho(x)\big)^{-1}} = 0,$$

and

$$\liminf_{\rho(x)\to\infty} \frac{|h(x)|}{\rho^{-1}(x)\big(\log r(x)\big)^{-1}} = 0,$$

respectively. We notice that condition (1.12) (resp. (1.13)) is implied by (1.10) (resp. (1.11)). On the other hand, assuming (1.10) (resp. (1.11)), we notice (using Examples 4.5 and 4.6) that (1.9) reduces to $\sup_{\partial B(o,r)\times\mathbb{R}} |\bar\nabla f| = o(r^{-2-\varepsilon})$ (resp. $\sup_{\partial B(o,r)\times\mathbb{R}} |\bar\nabla f| = o(r^{-1-\varepsilon})$) when $r\to\infty$. Therefore, in these cases, (1.9) is almost sharp.

The paper is organised as follows: in Section 2, we prove a priori height and gradient estimates that are needed in Section 3 where we apply the Leray-Schauder method and prove Theorem 1.1 and 1.2. Section 4 is devoted to the asymptotic Dirichlet problem and proofs of Theorem 1.3 and Corollaries 1.4 and 1.5.

2. Height and gradient estimates

In this section we adapt methods from [10], [9], [17], and [22] to obtain a priori height and gradient estimates.

2.1. **Height estimate.** We begin by giving an a priori height estimate for solutions of the equation (1.1) in a bounded open set $\Omega \subset M$ with a C^2-smooth boundary assuming the estimate (2.3) on the function f. First we construct an upper barrier for a solution u of (1.1) of the form

$$\psi(x) = \sup_{\partial\Omega}\varphi + h\big(d(x)\big),$$

where $d = \operatorname{dist}(\cdot,\partial\Omega)$ is the distance from $\partial\Omega$ and h is a real valued function that will be determined later. Denote by Ω_0 the open set of all points $x\in\Omega$ that can be joined to $\partial\Omega$ by a *unique* minimizing geodesic. It was shown in [18] that in Ω_0 the distance function d has the same regularity as $\partial\Omega$.

In particular, now $d\in C^2(\Omega_0)$ and straightforward computations give

$$\psi_i = h'd_i \quad\text{and}\quad \psi_{i;j} = h''d_id_j + h'd_{i;j}.$$

Moreover, $|\nabla d|^2 = d^i d_i = 1$ and hence $d^i d_{i;j} = 0$. We also have that

$$\sigma^{ij} d_{i;j} = \Delta d = -H,$$

where $H = H(x)$ is the (inward) mean curvature of the level set $\{y \in \Omega_0 \colon d(y) = d(x)\}$.

Given a solution $u \in C^2(\Omega)$ of (1.1),

$$Q[u] = \frac{1}{W}\left(\sigma^{ij} - \frac{u^i u^j}{W^2}\right) u_{i;j} - \langle \bar{\nabla} f, \nu \rangle = 0,$$

we define $b \colon \Omega \to \mathbb{R}$ by

$$b(x) = \left\langle \bar{\nabla} f\big(x, u(x)\big), \nu(x) \right\rangle, \tag{2.1}$$

where $\nu(x)$ is the downward pointing unit normal to the graph of u at $\big(x, u(x)\big)$. Next we define an operator

$$\tilde{Q}[v] = \frac{1}{W}\left(\sigma^{ij} - \frac{v^i v^j}{W^2}\right) v_{i;j} - b,$$

where $W = \sqrt{1 + |\nabla v|^2}$ and b does not depend on v. The reason to define such an operator is that it allows us to use the comparison principle whereas the operator Q need not satisfy the required assumptions, see e.g. [13, Theorem 10.1]. Then for a point $x \in \Omega_0$ we obtain

$$\begin{aligned}
\tilde{Q}[\psi] + b &= \frac{1}{W}\left(\sigma^{ij} - \frac{(h')^2 d^i d^j}{W^2}\right)(h'' d_i d_j + h' d_{i;j}) \\
&= \frac{1}{W}\left(h'' + h'\Delta d - \frac{(h')^2 h''}{W^2}\right) \\
&= \frac{1}{W}\left(\frac{h''}{W^2} - h' H(x)\right) \\
&= \frac{h''}{W^3} - \frac{h'}{W} H(x),
\end{aligned} \tag{2.2}$$

where we used that $W^2 = 1 + (h')^2$.

Next we impose an extra condition on the function $f \colon M \times \mathbb{R} \to \mathbb{R}$ by assuming that

$$\sup_{s \in \mathbb{R}} |\bar{\nabla} f(x, s)| \le H(x) \tag{2.3}$$

for all $x \in \Omega_0$. Hence $|b(x)| \le H(x)$ for all $x \in \Omega_0$. By choosing

$$h = \frac{e^{AC}}{C}\big(1 - e^{-Cd}\big),$$

where $A = \operatorname{diam}(\Omega)$ and

$$C > \sup_{\Omega_0 \times \mathbb{R}} |\bar{\nabla} f|$$

is a constant, we obtain

$$h' = e^{C(A-d)} \ge 1 \quad \text{and} \quad h'' = -Ch',$$

and so

$$\begin{aligned}
\tilde{Q}[\psi] &= -\frac{Ch'}{W^3} - \frac{h' H}{W} - b \\
&< -|b| \left(\frac{h'}{W^3} + \frac{h'}{W} - 1\right) \\
&\le 0.
\end{aligned}$$

Therefore we have

$$\begin{cases} \tilde{Q}[\psi] < 0 = \tilde{Q}[u] = Q[u] & \text{in } \Omega_0 \\ \psi | \partial\Omega \ge u | \partial\Omega = \varphi | \partial\Omega. \end{cases}$$

Next we observe that $\psi \ge u$ in $\bar\Omega$. Assume on the contrary that the continuous function $u-\psi$ attains its positive maximum at an interior point $x_0 \in \Omega$. As in [22, p. 795] (see also [10, pp. 239-240]), we conclude that, in fact, x_0 is an interior point of Ω_0 that leads to a contradiction with the comparison principle [13, Theorem 10.1] which states that $u-\psi$ can not attain its maximum in the open set Ω_0.

Similarly we deduce that ψ^-,

$$\psi^-(x) = \inf_{\partial\Omega} \varphi - h\big(d(x)\big),$$

is a lower barrier for u, i.e. $\psi^- \le u$ in $\bar\Omega$. These barriers imply the following height estimate for u.

Lemma 2.1. *Let $\Omega \subset M$ be a bounded open set with a C^2-smooth boundary and suppose that*

$$\sup_{s\in\mathbb{R}} |\bar\nabla f(x,s)| \le H(x) \tag{2.4}$$

in Ω_0. Let $u \in C^2(\Omega)\cap C(\bar\Omega)$ be a solution of $Q[u]=0$ with $u|\partial\Omega = \varphi$. Then there exists a constant $C=C(\Omega)$ such that

$$\sup_{\Omega} |u| \le C + \sup_{\partial\Omega} |\varphi|.$$

2.2. **Boundary gradient estimate.** In this subsection we will obtain an a priori boundary gradient estimate for the Dirichlet problem (1.1). We assume that $\Omega \subset M$ is a bounded open set with a C^2-smooth boundary and that Ω_ε is a sufficiently small tubular neighborhood of $\partial\Omega$ so that the distance function d from $\partial\Omega$ is C^2 in $\Omega_\varepsilon \cap \bar\Omega$. Furthermore, we assume that the (inward) mean curvature $H=H(x)$ of the level set $\{y \in \bar\Omega_0 \colon d(y)=d(x)\}$ satisfies

$$H(x) \ge \sup_{s\in\mathbb{R}} |\bar\nabla f(x,s)| := F(x) \tag{2.5}$$

for all $x \in \Omega_\varepsilon \cap \bar\Omega$. Next we extend the boundary function φ, which is assumed to be C^2-smooth, to Ω_ε by setting $\varphi\big(\exp_y t\nabla d(y)\big) = \varphi(y)$, for $y \in \partial\Omega$, where $\nabla d(y)$ is the unit *inward* normal to $\partial\Omega$ at $y \in \partial\Omega$. We will construct barriers of the form $w+\varphi$, where $w = \psi \circ d$ and ψ is a real function that will be determined later.

We denote

$$a^{ij} = a^{ij}(x,\nabla v) = \frac{1}{W}\left(\sigma^{ij} - \frac{v^i v^j}{W^2}\right), \quad W = \sqrt{1+|\nabla v|^2}, \tag{2.6}$$

and, given a solution $u \in C^2(\Omega)\cap C^1(\bar\Omega)$ of (1.1), we define an operator

$$\tilde Q[v] = a^{ij}(x,\nabla v) v_{i;j} - b,$$

with b as in (2.1).

The matrix $a^{ij}(x,\nabla v)$ is positive definite with eigenvalues

$$\lambda = \frac{1}{W^3} \quad \text{and} \quad \Lambda = \frac{1}{W} \tag{2.7}$$

with multiplicities 1 and $n-1$ corresponding respectively to the directions parallel and orthogonal to ∇v. Hence a simple estimate gives

$$\tilde Q[w+\varphi] = a^{ij}(w_{i;j}+\varphi_{i;j}) - b \le a^{ij} w_{i;j} + \Lambda \|\varphi\|_{C^2} - b, \tag{2.8}$$

where $a^{ij} = a^{ij}(x,\nabla w + \nabla\varphi)$, $\Lambda = (1+|\nabla w+\nabla\varphi|^2)^{-1/2}$, and $\|\varphi\|_{C^2}$ denotes the $C^2(\Omega_\varepsilon)$-norm of φ. Since in $\Omega_\varepsilon \cap \bar\Omega$ we have $|\nabla d|^2 = d^i d_i = 1$, $d^i d_{i;j} = 0$, and $\langle \nabla d, \nabla\varphi\rangle = 0$, straightforward computations give that

$$\begin{aligned}
\Delta w &= \psi'' + \psi' \Delta d,\\
w^i w^j w_{i;j} &= (\psi')^2 \psi'',\\
w^i \varphi^j w_{i;j} &= \psi'\psi'' \langle \nabla d, \nabla\varphi\rangle = 0,
\end{aligned}$$

and also

$$\varphi^i\varphi^j w_{i;j} = \psi''\langle\nabla\varphi,\nabla d\rangle^2 + \psi'\varphi^i\varphi^j d_{i;j} = \psi'\varphi^i\varphi^j d_{i;j}.$$

With these, and noticing that now $W^2 = 1 + (\psi')^2 + |\nabla\varphi|^2$, we obtain

$$a^{ij}w_{i;j} = \frac{\psi'\Delta d}{W} + \frac{\psi''(1+|\nabla\varphi|^2)}{W^3} - \frac{\psi'\varphi^i\varphi^j d_{i;j}}{W^3}. \tag{2.9}$$

Putting (2.8) and (2.9) together, we arrive at

$$\tilde{Q}[w+\varphi] \le \frac{\psi'\Delta d}{W} + \frac{\psi''(1+|\nabla\varphi|^2)}{W^3} - \frac{\psi'\varphi^i\varphi^j d_{i;j}}{W^3} + \Lambda\|\varphi\|_{C^2} + F. \tag{2.10}$$

Next we define

$$\psi(t) = \frac{C\log(1+Kt)}{\log(1+K)},$$

where the constants

$$C \ge 2\big(\max_{\bar\Omega}|u| + \max_{\bar\Omega}|\varphi|\big),$$

$K \ge (1-2\varepsilon)\varepsilon^{-2}$, and $\varepsilon \in (0,1/2)$ will be chosen later. Then

$$\psi(\varepsilon) = \frac{C\log(1+K\varepsilon)}{\log(1+K)} \ge C/2$$

and we have

$$(w+\varphi)|\Gamma_\varepsilon = \psi(\varepsilon) + \varphi|\Gamma_\varepsilon \ge u|\Gamma_\varepsilon \tag{2.11}$$

on the "inner boundary" $\Gamma_\varepsilon = \{x \in \Omega \colon d(x) = \varepsilon\}$ of Ω_ε. On the other hand,

$$(w+\varphi)|\partial\Omega = u|\partial\Omega. \tag{2.12}$$

We claim that $\tilde{Q}[w+\varphi] \le 0$ in $\Omega_\varepsilon \cap \Omega$ if C, K, and ε are properly chosen. All the computations below will be done in $\Omega_\varepsilon \cap \Omega$ without further notice. We first observe that

$$\psi'(t) = \frac{CK}{(1+Kt)\log(1+K)} \quad \text{and} \quad \psi''(t) = -\frac{\log(1+K)\psi'(d)^2}{C},$$

and therefore we have

$$\begin{aligned} W\tilde{Q}[w+\varphi] \le{}& \big(W-\psi'\big)H - \frac{\log(1+K)}{C}\left(\frac{\psi'}{W}\right)^2\big(1+|\nabla\varphi|^2\big) \\ &+ \|\varphi\|_{C^2} + |\nabla\varphi|^2 H \end{aligned} \tag{2.13}$$

by (2.5), (2.7), and (2.10). We estimate

$$\psi' \ge \frac{CK}{(1+K\varepsilon)\log(1+K)} = \frac{C}{(\varepsilon+1/K)\log(1+K)} = 1$$

and consequently,

$$\frac{\psi'}{W} \ge c_1 = c_1\big(\max_{\bar\Omega}|\nabla\varphi|\big) > 0$$

and

$$W - \psi' \le c_2 = c_2\big(\max_{\bar\Omega}|\nabla\varphi|\big)$$

by choosing $C = (\varepsilon + 1/K)\log(1+K)$. The claim $\tilde{Q}[w+\varphi] \le 0$ now follows from (2.13) since

$$\frac{\log(1+K)}{C} = \frac{1}{\varepsilon+1/K} \ge \frac{c_2 H + \|\varphi\|_{C^2} + |\nabla\varphi|^2 H}{c_1^2\big(1+|\nabla\varphi|^2\big)}$$

by choosing sufficiently small ε and large K depending only on $\max_{\bar\Omega}|u|$, $\|\varphi\|_{C^2}$, and $H_{\partial\Omega}$.

Hence

$$\tilde{Q}[w+\varphi] \le 0 = \tilde{Q}[u],$$

and therefore $w+\varphi$ is an upper barrier in $\Omega_\varepsilon \cap \Omega$. Similarly, $-w+\varphi$ is a lower barrier. Together these barriers imply that

$$|\nabla u| \le |\nabla w| + |\nabla \varphi| = \psi'(0) + |\nabla \varphi| = \frac{CK}{\log(1+K)} + |\nabla \varphi|$$

on $\partial\Omega$.

We have proven the following boundary gradient estimate.

Lemma 2.2. *Let $\Omega \subset M$ be a bounded open set with a C^2-smooth boundary and suppose that*

$$\sup_{s\in\mathbb{R}} |\bar\nabla f(x,s)| \le H(x) \tag{2.14}$$

in some tubular neighborhood of $\partial\Omega$. Let $u \in C^2(\Omega)\cap C^1(\bar\Omega)$ be a solution to $Q[u]=0$ with $u|\partial\Omega = \varphi \in C^2(\partial\Omega)$. Then

$$\max_{\partial\Omega} |\nabla u| \le C,$$

where C is a constant depending only on $\sup_{\bar\Omega}|u|$, $H_{\partial\Omega}$, and $\|\varphi\|_{C^2(\partial\Omega)}$.

2.3. **Interior gradient estimate.** In this subsection we will assume that u is a C^3 function. The elliptic regularity theory will guarantee that the estimate holds also for $C^{2,\alpha}$ solutions. We also assume that $f\colon M\times\mathbb{R}\to\mathbb{R}$ is of the form

$$f(x,t) = m(x) + r(t).$$

In particular, all "space" derivatives

$$f_i = \frac{\partial f}{\partial x_i}, \quad i = 1,\dots,\dim M,$$

are independent of t; $f_{it} = f_{ti} = 0$. Since we are dealing with the Riemannian product $M\times\mathbb{R}$ carrying the parallel vector field ∂_t, this assumption is not so unnatural. Also, due to the fact that f depends also on the $\mathbb{R}$-variable, it seems to be hard, if not impossible, to adapt other known approaches (see e.g. [8]) to get rid of this assumption.

For an open set $\Omega\subset M$, we denote $i(\Omega) = \inf_{x\in\Omega} i(x)$, where $i(x)$ is the injectivity radius at x. Thus $i(\Omega) > 0$ if $\Omega \Subset M$ is relatively compact. Furthermore, we denote by R_Ω the Riemannian curvature tensor in Ω.

Lemma 2.3. *Let $u\in C^3(\Omega)$ be a solution of* (1.1) *with $u < m_u$ for some constant $m_u < \infty$.*

(a) *For every ball $B(o,r)\subset\Omega$, there exists a constant*

$$L = L\big(u(o), m_u, r, R_\Omega, \|f\|_{C^2(\Omega\times(-\infty,m_u))}\big)$$

such that $|\nabla u(o)| \le L$.

(b) *If, furthermore, $u\in C^1(\bar\Omega)$, we have a global gradient bound*

$$|\nabla u(o)| \le L$$

for every $o\in\bar\Omega$, with

$$L = L\big(u(o), m_u, i(\Omega), \operatorname{diam}(\Omega), R_\Omega, \|f\|_{C^2(\Omega\times(-\infty,m_u))}, \max_{\partial\Omega}|\nabla u|\big) < \infty.$$

Proof. We apply the method due to Korevaar and Simon [17]; see also [9]. Let $0 < r \le \min\{i(\Omega), \operatorname{diam}(\Omega)\}$, $o\in\Omega$, and let η be a continuous non-negative function on M, vanishing outside $B(o,r)$ and smooth whenever positive. The function η will be specified later. Define

$$h = \eta W$$

and assume first that h attains its maximum at an interior point $p\in B(o,r)\cap\Omega$. The case $p\in B(o,r)\cap\partial\Omega$ and $u\in C^1(\bar\Omega)$ will be commented at the end of the proof.

We will first prove an upper bound for $|\nabla u(p)|$. Therefore we may assume that $|\nabla u(p)| \neq 0$. We choose normal coordinates at p so that $\partial_1 = \nabla u/|\nabla u|$ at p. All the computations below will be made at p without further notice. Thus we have $\sigma_{ij} = \sigma^{ij} = \delta^{ij}$, $u_1 = u^1 = |\nabla u|$, and $u_j = u^j = 0$ for $j > 1$. Furthermore,

$$a^{ij} = \frac{1}{W}\left(\delta^{ij} - \frac{|\nabla u|^2\delta^{1i}\delta^{1j}}{W^2}\right),$$

and therefore $a^{11} = W^{-3}$, $a^{ii} = W^{-1}$ for $i > 1$, and $a^{ij} = 0$ if $i \neq j$. At the maximum point p, we have $h_i = 0$ and $h_{i;i} \leq 0$ for all i. Hence

$$\eta_i W = -\eta W_i \tag{2.15}$$

and

$$a^{ij}h_{i;j} = a^{ii}h_{i;i} = a^{ii}\left(W\eta_{i;i} + 2\eta_i W_i + \eta W_{i;i}\right) \leq 0.$$

With (2.15) we can write this as

$$Wa^{ii}\eta_{i;i} + \frac{\eta a^{ii}}{W}\left(WW_{i;i} - 2(W_i)^2\right) \leq 0. \tag{2.16}$$

We have

$$W_i = \frac{u^k u_{k;i}}{W} = \frac{|\nabla u| u_{1;i}}{W}$$

and from (1.2) we see that the k^{th} component of the unit normal is

$$\nu^k = \frac{u^k}{W} = \frac{|\nabla u|\delta^{k1}}{W}.$$

To scrutinize the second order differential inequality (2.16), we first compute

$$\begin{aligned} a^{ii}W_{i;i} &= a^{ii}\left(W^{-1}u^k u_{k;i}\right)_{;i} \\ &= -\frac{a^{ii}|\nabla u|u_{1;i}W_i}{W^2} + \frac{a^{ii}u^k_{;i}u_{k;i}}{W} + \frac{a^{ii}|\nabla u|u_{1;ii}}{W} \\ &= -\frac{a^{ii}|\nabla u|^2(u_{1;i})^2}{W^3} + \frac{a^{ii}u^k_{;i}u_{k;i}}{W} + \frac{a^{ii}|\nabla u|u_{1;ii}}{W} \\ &= \frac{a^{ii}(u_{1;i})^2}{W^3} + \frac{a^{ii}\sum_{k\neq 1}(u_{k;i})^2}{W} + \frac{a^{ii}|\nabla u|u_{1;ii}}{W}. \end{aligned}$$

Hence

$$Wa^{ii}W_{i;i} = A + a^{ii}|\nabla u|u_{1;ii}, \tag{2.17}$$

where

$$A = a^{ii}(u_{1;i})^2W^{-2} + a^{ii}\sum_{k\neq 1}(u_{k;i})^2 \geq 0.$$

Using the Ricci identities for the Hessian of u we get

$$u_{k;ij} = u_{i;kj} = u_{i;jk} + R^{\ell}_{kji}u_{\ell},$$

where R is the curvature tensor in M. This yields

$$|\nabla u|a^{ii}u_{1;ii} = |\nabla u|a^{ii}u_{i;i1} + |\nabla u|^2a^{ii}R^1_{1ii}. \tag{2.18}$$

To compute $|\nabla u|a^{ii}u_{i;i1}$, we first observe that

$$Wa^{ij}u_{i;j} = Wa^{ii}u_{i;i} = \langle\bar\nabla f, (\nabla u, -1)\rangle = f_i u^i - f_t.$$

Since

$$\begin{aligned}\nu^1\big(Wa^{ij}\big)_{;1}u_{i;j} &= \nu^1\big(\sigma^{ij}-u^iu^jW^{-2}\big)_{;1}u_{i;j}\\ &= -\frac{|\nabla u|}{W}\left(\frac{2u^iu^j_{;1}}{W^2}-\frac{2u^iu^jW_1}{W^3}\right)u_{i;j}\\ &= -\frac{2|\nabla u|^2}{W^3}u^j_{;1}u_{1;j}+\frac{2|\nabla u|^4(u_{1;1})^2}{W^5}\\ &= -\frac{2|\nabla u|^2}{W^3}\left(\sum_i(u_{1;i})^2-\frac{|\nabla u|^2}{W^2}(u_{1;1})^2\right)\\ &= -\frac{2|\nabla u|^2a^{ii}(u_{1;i})^2}{W^2}\\ &= -2a^{ii}(W_i)^2,\end{aligned}$$

we obtain

$$\begin{aligned}|\nabla u|a^{ii}u_{i;i1} &= |\nabla u|a^{ij}u_{i;j1}=\nu^1Wa^{ij}u_{i;j1}\\ &= \nu^1\big(Wa^{ij}u_{i;j}\big)_{;1}-\nu^1\big(Wa^{ij}\big)_{;1}u_{i;j}\\ &= \nu^1\big(f_iu^i-f_t\big)_{;1}+2a^{ii}(W_i)^2\\ &= \nu^1\big(f_iu^i_{;1}+(f_\ell)_{;1}u^\ell-(f_t)_{;1}\big)+2a^{ii}(W_i)^2\\ &= \frac{|\nabla u|}{W}\big(f_iu^i_{;1}+(f_1)_{;1}u^1-f_{tt}u^1\big)+2a^{ii}(W_i)^2\\ &= W_if^i+\frac{f_{11}|\nabla u|^2}{W}-\frac{f_{tt}|\nabla u|^2}{W}+2a^{ii}(W_i)^2,\end{aligned} \tag{2.19}$$

where we have denoted $(f_j)_{;1}=\big(x\mapsto f_j(x,u(x))\big)_{;1}$ and used the assumption $f_{it}=f_{ti}=0$. Putting together (2.15), (2.17), (2.18), and (2.19) we can estimate the inequality (2.16) as

$$\begin{aligned}0 &\ge Wa^{ii}\eta_{i;i}+\frac{\eta a^{ii}}{W}\big(WW_{i;i}-2(W_i)^2\big)\\ &= Wa^{ii}\eta_{i;i}+\frac{\eta}{W}\left(A+|\nabla u|a^{ii}u_{1;ii}-|\nabla u|a^{ii}u_{i;i1}+W_if^i+\frac{|\nabla u|^2\,(f_{11}-f_{tt})}{W}\right)\\ &= Wa^{ii}\eta_{i;i}+\eta\left(\frac{A}{W}+\frac{|\nabla u|^2a^{ii}R^1_{1ii}}{W}+\frac{|\nabla u|^2(f_{11}-f_{tt})}{W^2}\right)-f^i\eta_i\\ &\ge Wa^{ii}\eta_{i;i}-f^i\eta_i-N\eta,\end{aligned} \tag{2.20}$$

where N is a positive constant depending only on the curvature tensor in Ω and the C^2-norm of f in the cylinder $\Omega\times(-\infty,m_u)$. Note that $A\ge 0$, $a^{11}=W^{-3}$, and $a^{ii}=W^{-1}$ for $i\ne 1$.

Now we are ready to choose the function η as

$$\eta(x)=g\big(\phi(x)\big),$$

where

$$g(t)=e^{C_1t}-1$$

with a positive constant C_1 to be specified later and

$$\phi(x)=\big(1-r^{-2}d^2(x)+C\big(u(x)-m_u\big)\big)^+.$$

Here $d(x)=d(x,o)$ is the geodesic distance to o and

$$C=\frac{-1}{2\big(u(o)-m_u\big)}>0.$$

It follows that η fulfils the requirements and, moreover, $\eta(o) = e^{C_1/2} - 1 > 0$. We have

$$\eta_i = \big(-r^{-2}(d^2)_i + Cu_i\big)\, g' \tag{2.21}$$

and

$$\eta_{i;j} = \big(-r^{-2}(d^2)_{i;j} + Cu_{i;j}\big)\, g' + \big(-r^{-2}(d^2)_i + Cu_i\big)\big(-r^{-2}(d^2)_j + Cu_j\big)\, g''. \tag{2.22}$$

A straightforward computation gives the estimate

$$\begin{aligned}
Wa^{ii}\big(r^{-2}(d^2)_i - Cu_i\big)^2 &= Wa^{ii}\big(r^{-4}(d^2)_i^2 - 2Cr^{-2}(d^2)_i u_i + C^2(u_i)^2\big)\\
&= r^{-4}|\nabla d^2|^2 - 2Cr^{-2}\langle \nabla d^2, \nabla u\rangle + C^2|\nabla u|^2\\
&\quad - \frac{\langle \nabla d^2, \nabla u\rangle^2}{r^4W^2} + \frac{2C|\nabla u|^2\langle \nabla d^2, \nabla u\rangle}{r^2W^2} - \frac{C^2|\nabla u|^4}{W^2}\\
&= \frac{C^2|\nabla u|^2}{W^2} - \frac{2C\langle \nabla d^2, \nabla u\rangle}{r^2W^2} + \frac{1}{r^4}\left(|\nabla d^2|^2 - \frac{\langle \nabla d^2, \nabla u\rangle^2}{W^2}\right)\\
&\ge \frac{C^2|\nabla u|^2}{W^2} - \frac{2C\langle \nabla d^2, \nabla u\rangle}{r^2W^2}.
\end{aligned} \tag{2.23}$$

Next we observe that

$$\begin{aligned}
Wa^{ii}\big(-r^{-2}(d^2)_{i;i} + Cu_{i;i}\big) &= -r^{-2}Wa^{ii}(d^2)_{i;i} + CWa^{ii}u_{i;i}\\
&= -r^{-2}\Delta d^2 + \frac{|\nabla u|^2}{r^2W^2}(d^2)_{1;1} + CW\langle \bar\nabla f, \nu\rangle\\
&= -r^{-2}\Delta d^2 + \frac{|\nabla u|^2}{r^2W^2}\operatorname{Hess} d^2(\partial_1, \partial_1) + CW\langle \bar\nabla f, \nu\rangle.
\end{aligned} \tag{2.24}$$

Putting together (2.21), (2.22), (2.23), and (2.24) we obtain

$$\begin{aligned}
Wa^{ii}\eta_{i;i} \ge g'&\left(-r^{-2}\Delta d^2 + \frac{|\nabla u|^2}{r^2W^2}\operatorname{Hess} d^2(\partial_1, \partial_1) + CW\langle \bar\nabla f, \nu\rangle\right)\\
&+ g''\left(\frac{C^2|\nabla u|^2}{W^2} - \frac{2C}{r^2W^2}\langle \nabla u, \nabla d^2\rangle\right).
\end{aligned}$$

Hence, by (2.20), we have

$$g''\left(\frac{C^2|\nabla u|^2}{W^2} - \frac{2C}{r^2W^2}\langle \nabla u, \nabla d^2\rangle\right) + g'P - Ng \le 0, \tag{2.25}$$

where

$$P = \frac{|\nabla u|^2}{r^2W^2}\operatorname{Hess} d^2(\partial_1, \partial_1) - \frac{\Delta d^2}{r^2} + \frac{f^i(d^2)_i}{r^2} - Cf_t.$$

It is easy to see that

$$|P| \le \frac{|\operatorname{Hess} d^2(\partial_1, \partial_1)| + |\Delta d^2|}{r^2} + \frac{2d|f^i d_i|}{r^2} + C|f_t| \le C_0,$$

with a constant $C_0 = C_0(u(o) - m_u, r, R_\Omega, \|f\|_{C^1})$.

In order to obtain an upper bound for $|\nabla u(p)|$, we suppose that

$$|\nabla u(p)| \ge \frac{16\big(m_u - u(o)\big)}{r}$$

and derive a contradiction. Since $|\nabla d^2(p)| \le 2r$, we see that

$$|\nabla u(p)| \ge \frac{4|\nabla d^2(p)|}{Cr^2}$$

and hence we have

$$|\nabla u|^2 - \frac{2}{Cr^2}\langle \nabla u, \nabla d^2\rangle \ge \frac{1}{2}|\nabla u|^2$$

at p. Therefore there exists a constant D depending only on $m_u - u(o)$ and r such that

$$\frac{C^2}{W^2}\left(|\nabla u|^2 - \frac{2}{Cr^2}\langle \nabla u, \nabla d^2\rangle\right) \geq D > 0.$$

But now, taking $C_1 = C_1(C_0, D, N)$ large enough, we obtain

$$Dg''(\phi(p)) - C_0 g'(\phi(p)) - Ng(\phi(p)) = (DC_1^2 - C_1C_0 - N)e^{C_1\phi(p)} + N > 0$$

which is a contradiction with (2.25). Hence we have

$$|\nabla u(p)| < \frac{16\big(m_u - u(o)\big)}{r}$$

which implies

$$W(p) \leq C_2 = 1 + \frac{16\big(m_u - u(o)\big)}{r}.$$

Since p is a maximum point of $h = \eta W$, we have

$$\left(e^{C_1/2} - 1\right) W(o) = \eta(o)W(o) \leq \eta(p)W(p) \leq C_2\left(e^{C_1} - 1\right).$$

This proves the case (a).

For the case (b), we assume, in addition, that $u \in C^1(\bar{\Omega})$ and we fix $r = \min\{i(\Omega), \operatorname{diam}(\Omega)\} > 0$. Let $o \in \bar{\Omega}$ and $h = \eta W$ be as above with the same constant C_1. If a maximum point p of h is an interior point of Ω, the proof for the case (a) applies and we have a desired upper bound for $|\nabla u(o)|$. On the other hand, if $p \in \partial\Omega$ we have an upper bound

$$|\nabla u(p)| \leq \max_{\partial\Omega}|\nabla u|$$

and again we are done. □

3. Existence of f-minimal graphs

In this section we will prove Theorem 1.1 and 1.2. Throughout this section we assume that $\Omega \subset M$ is a bounded open set with $C^{2,\alpha}$ boundary $\partial\Omega$. As in Subsection 2.1 we denote by Ω_0 the open set of all those points of Ω that can be joined to $\partial\Omega$ by a unique minimizing geodesic. We start with the following lemma from [22, Lemma 4.2]; see also [10, Lemma 5]. Since our definition of the mean curvature differs by a multiple constant from the one used in [22] and [10], we sketch the proof.

Lemma 3.1. *Let $F = \sup\{|\bar{\nabla} f(x,s)| \colon (x,s) \in \bar{\Omega} \times \mathbb{R}\} < \infty$ and suppose that $\operatorname{Ric}_\Omega \geq -F^2/(n-1)$ and $H_{\partial\Omega} \geq F$. Then for all $x_0 \in \Omega_0$ the inward mean curvature $H(x_0)$ of the level set $\{x \in \Omega \colon d(x) = d(x_0)\}$ passing through x_0 has a lower bound $H(x_0) \geq F$.*

Proof. Denote by $H(t)$ the inward mean curvature of the level set $\Gamma_t = \{x \in \Omega \colon d(x) = t\}$ at the point which lies on the unit speed minimizing geodesic γ joining $\gamma(0) \in \partial\Omega$ to x_0. Denote by $N = \dot{\gamma}_t$ the inward unit normal to Γ_t and by S_t the shape operator, $S_t(X) = -\nabla_X N$, of the level set Γ_t. As in [10] we obtain the Riccati equation

$$S_t' = S_t^2 + R_t,$$

where $R_t = R(\cdot, \dot{\gamma}_t)\dot{\gamma}_t$. Trace and derivative commute, but because of the term S_t^2, we need to substitute $s = \operatorname{tr} S_t/(n-1)$ in order to get similar differential equation for the traces. Hence we have

$$s' = s^2 + r,$$

where r satisfies $r \geq \operatorname{Ric}(\dot\gamma_t, \dot\gamma_t)/(n-1)$. In other words,

$$\frac{\operatorname{tr} S_t'}{n-1} \geq \left(\frac{\operatorname{tr} S_t}{n-1}\right)^2 + \frac{1}{n-1}\operatorname{Ric}(\dot\gamma_t, \dot\gamma_t).$$

Since $H(t) = \operatorname{tr} S_t$, we obtain the estimate

$$\frac{H'(t)}{n-1} \geq \left(\frac{H(t)}{n-1}\right)^2 + \frac{1}{n-1}\operatorname{Ric}\left(\dot\gamma_t, \dot\gamma_t\right) \geq \frac{H^2(t)}{(n-1)^2} - \frac{F^2}{(n-1)^2}.$$

On the boundary we have $H(0) = H_{\partial\Omega} \geq F$ which implies that $H'(t) \geq 0$ and hence the claim follows. □

Proof of Theorem 1.1. In order to prove Theorem 1.1 we assume that the given boundary value function is extended to a function $\varphi \in C^{2,\alpha}(\bar\Omega)$ and we consider a family of Dirichlet problems

$$\begin{cases} \operatorname{div} \dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} - \tau\langle\bar\nabla f, \nu\rangle = 0 \quad \text{in } \Omega, \\ u = \tau\varphi \quad \text{in } \partial\Omega,\ 0 \leq \tau \leq 1. \end{cases} \tag{3.1}$$

By Lemma 3.1,

$$H(x) \geq F \geq \sup_{\bar\Omega\times\mathbb{R}} |\bar\nabla(\tau f)|$$

for all $x \in \Omega_0$ and for all $\tau \in [0,1]$. Hence *if* $u \in C^{2,\alpha}(\bar\Omega)$ is a solution of (3.1) for some $\tau \in [0,1]$, it follows from Lemmata 2.1, 2.2, and 2.3 that

$$\|u\|_{C^1(\bar\Omega)} \leq C$$

with a constant C that is independent of τ. The Leray-Schauder method [13, Theorem 13.8] then yields a solution to the Dirichlet problem (3.1) for all $\tau \in [0,1]$. In particular, with $\tau = 1$ we obtain a solution to the original Dirichlet problem. □

Proof of Theorem 1.2. Let $\varphi \in C(\partial\Omega)$ and let $\varphi_k^\pm \in C^{2,\alpha}(\partial\Omega)$ be two monotonic sequence converging uniformly on $\partial\Omega$ to φ from above and from below, respectively. Denote

$$F^+ = \sup_{\bar\Omega\times\mathbb{R}} |\bar\nabla f| \quad \text{and} \quad F^- = -F^+.$$

By Theorem 1.1 there are functions $u_k^\pm, v_k^\pm \in C^{2,\alpha}(\bar\Omega)$ such that $u_k^\pm|\partial\Omega = v_k^\pm|\partial\Omega = \varphi_k^\pm$ and

$$\begin{aligned} &a^{ij}(x, \nabla u_k^\pm)(u_k^\pm)_{i;j} - \langle\bar\nabla f, \nu_k^\pm\rangle = 0 \\ &a^{ij}(x, \nabla v_k^\pm)(v_k^\pm)_{i;j} + F^\pm = 0 \end{aligned}$$

in Ω, where a^{ij} is as in (2.6) and $\nu_k^\pm$ is the downward unit normal to the graph of $u_k^\pm$. Since

$$\begin{aligned} a^{ij}(x, \nabla v_k^+)(v_k^+)_{i;j} + F^- &\leq a^{ij}(x, \nabla v_k^+)(v_k^+)_{i;j} + F^+ = 0 \\ &= a^{ij}(x, \nabla v_\ell^-)(v_\ell^-)_{i;j} + F^- \end{aligned}$$

and $v_k^+|\partial\Omega \geq v_\ell^-|\partial\Omega$ for all k, ℓ, we obtain from the comparison principle [13, Theorem 10.1] applied to the operator $a^{ij} + F^-$ that

$$v_\ell^- \leq v_k^+ \quad \text{in } \bar\Omega.$$

On the other hand, since $\varphi_{k+1}^+ \leq \varphi_k^+$ and $\varphi_\ell^- \leq \varphi_{\ell+1}^-$ on $\partial\Omega$, we have again by the comparison principle that

$$v_1^- \leq \cdots \leq v_\ell^- \leq v_{\ell+1}^- \cdots \leq v_{k+1}^+ \leq v_k^+ \cdots \leq v_1^+. \tag{3.2}$$

Similarly, since

$$\begin{aligned} a^{ij}(x,\nabla v_k^+)(v_k^+)_{i;j} - \langle \bar\nabla f, \nu_k^+\rangle &\le a^{ij}(x,\nabla v_k^+)(v_k^+)_{i;j} - F^- = 0 \\ &= a^{ij}(x,\nabla u_k^+)(u_k^+)_{i;j} - \langle \bar\nabla f, \nu_k^+\rangle \end{aligned}$$

and $v_k^+|\partial\Omega = u_k^+|\partial\Omega$, we get

$$u_k^+ \le v_k^+ \quad \text{in } \bar\Omega.$$

Similar reasoning implies that $v_k^- \le u_k^-$, and therefore

$$v_k^- \le u_k^\pm \le v_k^+ \quad \text{in } \bar\Omega. \tag{3.3}$$

Hence the sequences $u_k^\pm, v_k^\pm$ have uniformly bounded C^0 norms and the local interior gradient estimate (Lemma 2.3) together with [13, Corollary 6.3] imply that the sequences $u_k^\pm, v_k^\pm$ have equicontinuous $C^{2,\alpha}$ norms on compact subsets $K \subset \Omega$. Taking an exhaustion of Ω by compact sets we obtain, with a diagonal argument, that $u_k^\pm$ and $v_k^\pm$ contain subsequences that converge uniformly in compact subsets to functions $u, v^\pm \in C^2(\Omega)$ with respect to the C^2 norm. Moreover, we have

$$a^{ij}(x,\nabla u)u_{i;j} - \langle \bar\nabla f, \nu\rangle = 0 \quad \text{and} \quad a^{ij}(x,\nabla v^\pm)v^\pm_{i;j} + F^\pm = 0.$$

Since $v_k^\pm|\partial\Omega = \varphi_k^\pm$ convergences to φ, (3.2) implies that $v^\pm$ extends continuously to the boundary $\partial\Omega$ and $v^\pm|\partial\Omega = \varphi$. In turn, this and (3.3) give that u extends continuously to $\partial\Omega$ with $u|\partial\Omega = \varphi$. Furthermore, because $f \in C^2(M\times\mathbb{R})$, it follows that $u \in C^{2,\alpha}(\Omega) \cap C(\bar\Omega)$ ([13, Theorem 6.17]). □

4. Dirichlet problem at infinity

In this section we assume that M is a Cartan-Hadamard manifold of dimension $n \ge 2$, $\partial_\infty M$ is the asymptotic boundary of M, and $\bar M = M \cup \partial_\infty M$ the compactification of M in the cone topology. Recall that the asymptotic boundary is defined as the set of all equivalence classes of unit speed geodesic rays in M; two such rays γ_1 and γ_2 are equivalent if $\sup_{t\ge 0} d\big(\gamma_1(t), \gamma_2(t)\big) < \infty$. The equivalence class of γ is denoted by $\gamma(\infty)$. For each $x \in M$ and $y \in \bar M \setminus \{x\}$ there exists a unique unit speed geodesic $\gamma^{x,y}\colon \mathbb{R} \to M$ such that $\gamma_0^{x,y} = x$ and $\gamma_t^{x,y} = y$ for some $t \in (0,\infty]$. If $v \in T_xM \setminus \{0\}$, $\alpha > 0$, and $r > 0$, we define a cone

$$C(v,\alpha) = \{y \in \bar M \setminus \{x\} : \sphericalangle(v, \dot\gamma_0^{x,y}) < \alpha\}$$

and a truncated cone

$$T(v,\alpha,r) = C(v,\alpha) \setminus \bar B(x,r),$$

where $\sphericalangle(v, \dot\gamma_0^{x,y})$ is the angle between vectors v and $\dot\gamma_0^{x,y}$ in T_xM. All cones and open balls in M form a basis for the cone topology on $\bar M$.

Throughout this section, we assume that the sectional curvatures of M are bounded from below and above by

$$-(b\circ\rho)^2(x) \le K(P_x) \le -(a\circ\rho)^2(x) \tag{4.1}$$

for all $x \in M$, where $\rho(x) = d(o,x)$ is the distance to a fixed point $o \in M$ and P_x is any 2-dimensional subspace of T_xM. The functions $a, b\colon [0,\infty) \to [0,\infty)$ are assumed to be smooth such that $a(t) = 0$ and $b(t)$ is constant for $t \in [0, T_0]$ for some $T_0 > 0$, and $b \ge a$. Furthermore, we assume that b is monotonic and that there exist positive constants T_1, C_1, C_2, C_3, and $Q \in (0,1)$ such that

$$a(t) \begin{cases} = C_1 t^{-1} & \text{if } b \text{ is decreasing,} \\ \ge C_1 t^{-1} & \text{if } b \text{ is increasing} \end{cases} \tag{A1}$$

for all $t \geq T_1$ and

$$a(t) \leq C_2, \tag{A2}$$

$$b(t+1) \leq C_2 b(t), \tag{A3}$$

$$b(t/2) \leq C_2 b(t), \tag{A4}$$

$$b(t) \geq C_3(1+t)^{-Q} \tag{A5}$$

for all $t \geq 0$. In addition, we assume that

$$\lim_{t\to\infty} \frac{b'(t)}{b(t)^2} = 0 \tag{A6}$$

and that there exists a constant $C_4 > 0$ such that

$$\lim_{t\to\infty} \frac{t^{1+C_4} b(t)}{f_a'(t)} = 0. \tag{A7}$$

It can be checked from [15] or from [2] that the curvature bounds in Corollary 1.4 and Corollary 1.5 satisfy the assumptions (A1)-(A7).

4.1. **Construction of a barrier.** The curvature bounds (4.1) are needed to control the first two derivatives of the "barrier" functions that we will construct in this subsection. Recall from the introduction that for a smooth function $k\colon [0,\infty) \to [0,\infty)$, we denote by $f_k\colon [0,\infty) \to \mathbb{R}$ the smooth non-negative solution to the initial value problem

$$\begin{cases} f_k(0) = 0, \\ f_k'(0) = 1, \\ \quad f_k'' = k^2 f_k. \end{cases}$$

Following [15], we construct a barrier function for each boundary point $x_0 \in \partial_\infty M$. Towards this end let $v_0 = \dot\gamma_0^{o,x_0}$ be the initial (unit) vector of the geodesic ray γ^{o,x_0} from a fixed point $o \in M$ and define a function $h : \partial_\infty M \to \mathbb{R}$,

$$h(x) = \min\big(1, L\sphericalangle(v_0, \dot\gamma_0^{o,x})\big), \tag{4.2}$$

where $L \in (8/\pi, \infty)$ is a constant. Then we define a crude extension $\tilde h \in C(\bar M)$, with $\tilde h|\partial_\infty M = h$, by setting

$$\tilde h(x) = \min\Big(1, \max\big(2 - 2\rho(x), L\sphericalangle(v_0, \dot\gamma_0^{o,x})\big)\Big). \tag{4.3}$$

Finally, we smooth out $\tilde h$ to get an extension $h \in C^\infty(M) \cap C(\bar M)$ with controlled first and second order derivatives. For that purpose, we fix $\chi \in C^\infty(\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\operatorname{spt}\chi \subset [-2,2]$, and $\chi|[-1,1] \equiv 1$. Then for any function $\varphi \in C(M)$ we define functions $F_\varphi\colon M \times M \to \mathbb{R}$, $\mathcal{R}(\varphi)\colon M \to M$, and $\mathcal{P}(\varphi)\colon M \to \mathbb{R}$ by

$$\begin{aligned} F_\varphi(x,y) &= \chi\big(b(\rho(y))d(x,y)\big)\varphi(y), \\ \mathcal{R}(\varphi)(x) &= \int_M F_\varphi(x,y)\,dm(y), \quad \text{and} \\ \mathcal{P}(\varphi) &= \frac{\mathcal{R}(\varphi)}{\mathcal{R}(1)}, \end{aligned}$$

where

$$\mathcal{R}(1)(x) = \int_M \chi\big(b(\rho(y))d(x,y)\big)\,dm(y) > 0.$$

If $\varphi \in C(\bar M)$, we extend $\mathcal{P}(\varphi)\colon M \to \mathbb{R}$ to a function $\bar M \to \mathbb{R}$ by setting $\mathcal{P}(\varphi)(x) = \varphi(x)$ whenever $x \in M(\infty)$. Then the extended function $\mathcal{P}(\varphi)$ is C^∞-smooth in

M and continuous in $\bar{M}$; see [15, Lemma 3.13]. In particular, applying $\mathcal{P}$ to the function $\tilde{h}$ yields an appropriate smooth extension

$$h := \mathcal{P}(\tilde{h}) \tag{4.4}$$

of the original function $h \in C\big(\partial_\infty M\big)$ that was defined in (4.2).

We denote

$$\Omega = C(v_0, 1/L) \cap M \quad \text{and} \quad \ell\Omega = C(v_0, \ell/L) \cap M$$

for $\ell > 0$ and collect various constants and functions together to a data

$$C = (a, b, T_1, C_1, C_2, C_3, C_4, Q, n, L).$$

Furthermore, we denote by $\|\operatorname{Hess}_x u\|$ the norm of the Hessian of a smooth function u at x, that is

$$\|\operatorname{Hess}_x u\| = \sup_{\substack{X \in T_x M \\ |X| \le 1}} |\operatorname{Hess} u(X, X)|.$$

The following lemma gives the desired estimates for derivatives of h. We refer to [15] for the proofs of these estimates; see also [1].

Lemma 4.1. [15, Lemma 3.16] *There exist constants $R_1 = R_1(C)$ and $c_1 = c_1(C)$ such that the extended function $h \in C^\infty(M) \cap C(\bar{M})$ in (4.4) satisfies*

$$\begin{aligned} |\nabla h(x)| &\le c_1 \frac{1}{(f_a \circ \rho)(x)}, \\ \|\operatorname{Hess}_x h\| &\le c_1 \frac{(b \circ \rho)(x)}{(f_a \circ \rho)(x)}, \end{aligned} \tag{4.5}$$

for all $x \in 3\Omega \setminus B(o, R_1)$. In addition,

$$h(x) = 1$$

for every $x \in M \setminus \big(2\Omega \cup B(o, R_1)\big)$.

We define a function $F \colon M \to [0, \infty)$ and an elliptic operator $\tilde{Q}$ by setting

$$F(x) = \sup_{t \in \mathbb{R}} |\bar{\nabla} f(x, t)| \tag{4.6}$$

and

$$\tilde{Q}[v] = \operatorname{div} \frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} + F(x). \tag{4.7}$$

Let then $A > 0$ be a fixed constant. We aim to show that

$$\psi = A(R_3^\delta \rho^{-\delta} + h) \tag{4.8}$$

is a supersolution $\tilde{Q}[\psi] < 0$ in the set $3\Omega \setminus \bar{B}(o, R_3)$, where $\delta > 0$ and $R_3 > 0$ are constants that will be specified later and h is the extended function defined in (4.4). We shall make use of the following estimates obtained in [15]:

Lemma 4.2. [15, Lemma 3.17] *There exist constants $R_2 = R_2(C)$ and $c_2 = c_2(C)$ with the following property. If $\delta \in (0, 1)$, then*

$$\begin{aligned} |\nabla h| &\le c_2/(f_a \circ \rho), \\ \|\operatorname{Hess} h\| &\le c_2 \rho^{-C_4 - 1} (f_a' \circ \rho)/(f_a \circ \rho), \\ |\nabla \langle \nabla h, \nabla h \rangle| &\le c_2 \rho^{-C_4 - 2} (f_a' \circ \rho)/(f_a \circ \rho), \\ |\nabla \langle \nabla h, \nabla(\rho^{-\delta}) \rangle| &\le c_2 \rho^{-C_4 - 2} (f_a' \circ \rho)/(f_a \circ \rho), \\ \nabla \big\langle \nabla(\rho^{-\delta}), \nabla(\rho^{-\delta}) \big\rangle &= -2\delta^2 (\delta + 1) \rho^{-2\delta - 3} \nabla \rho \end{aligned}$$

in the set $3\Omega \setminus B(o, R_2)$.

As in [15] we denote

$$\phi_1 = \frac{1+\sqrt{1+4C_1^2}}{2} > 1, \quad \text{and} \quad \delta_1 = \min\left\{C_4, \frac{-1+(n-1)\phi_1}{1+(n-1)\phi_1}\right\} \in (0,1),$$

where C_1 and C_4 are constants defined in (A1) and (A7), respectively.

Lemma 4.3. *Let $A>0$ be a fixed constant and h the function defined in* (4.4)*. Assume that the function F defined in* (4.6) *satisfies*

$$\sup_{\rho(x)=t} F(x) = o\left(\frac{f_a'(t)}{f_a(t)} t^{-\varepsilon-1}\right) \tag{4.9}$$

for some $\epsilon>0$ as $t\to\infty$. Then there exist two positive constants $\delta\in(0,\min(\delta_1,\varepsilon))$ and R_3 depending on C and ε such that the function $\psi = A(R_3^\delta\rho^{-\delta}+h)$ satisfies $\tilde{Q}[\psi]<0$ in the set $3\Omega\setminus\bar{B}(o,R_3)$.

Proof. In the proof c will denote a positive constant whose actual value may vary even within a line. Since

$$\begin{aligned}\tilde{Q}[\psi] &= \frac{\Delta\psi}{\sqrt{1+|\nabla\psi|^2}} - \frac{1}{2}\frac{\langle\nabla|\nabla\psi|^2,\nabla\psi\rangle}{(1+|\nabla\psi|^2)^{3/2}} + F(x)\\ &= \frac{(1+|\nabla\psi|^2)\Delta\psi + (1+|\nabla\psi|^2)^{3/2}F(x) - \frac{1}{2}\langle\nabla|\nabla\psi|^2,\nabla\psi\rangle}{(1+|\nabla\psi|^2)^{3/2}},\end{aligned}$$

it is enough to show that there exist $\delta>0$ and R_3 such that

$$(1+|\nabla\psi|^2)\Delta\psi + (1+|\nabla\psi|^2)^{3/2}F(x) - \frac{1}{2}\langle\nabla|\nabla\psi|^2,\nabla\psi\rangle < 0 \tag{4.10}$$

in the set $3\Omega\setminus\bar{B}(o,R_3)$.

First we notice that ψ is C^∞-smooth and

$$\nabla\psi = A\big(-R_3^\delta\delta\rho^{-\delta-1}\nabla\rho + \nabla h\big)$$

in $M\setminus\{o\}$. Lemma 4.2 and our curvature assumption imply that $|\nabla h|\le c/\rho$ for ρ large enough, and therefore

$$|\nabla\psi|^2 = (AR_3^\delta)^2\delta^2\rho^{-2\delta-2} + A^2|\nabla h|^2 - 2A^2R_3^\delta\delta\rho^{-\delta-1}\langle\nabla\rho,\nabla h\rangle \le c\rho^{-2}$$

in $3\Omega\setminus\bar{B}(o,R_3)$ for sufficiently large R_3. Then, to estimate the term with $\Delta\psi$ in (4.10), we first note that

$$\Delta\psi = AR_3^\delta\big(\delta(\delta+1)\rho^{-\delta-2} - \delta\rho^{-\delta-1}\Delta\rho\big) + A\Delta h.$$

Furthermore, for every $\delta\in(0,\delta_1)$, there exists $R_3 = R_3(C,\delta)$ such that

$$\Delta\rho \ge (n-1)\frac{f_a'\circ\rho}{f_a\circ\rho} \ge \frac{(n-1)(1-\delta)\phi_1}{\rho} > 0$$

whenever $\rho \geq R_3$; see [15, (3.25)]. Therefore, using Lemma 4.2, we obtain

$$\begin{aligned}
(1+|\nabla\psi|^2)\Delta\psi &\leq (1+|\nabla\psi|^2)AR_3^\delta\delta\left(\delta+1-(n-1)\frac{\rho f_a'\circ\rho}{f_a\circ\rho}\right)\rho^{-\delta-2} \\
&\quad + (1+|\nabla\psi|^2)Anc_2\left(\frac{f_a'\circ\rho}{f_a\circ\rho}\right)\rho^{-C_4-1} \\
&\leq AR_3^\delta\delta\left(\delta+1-(n-1)\frac{\rho f_a'\circ\rho}{f_a\circ\rho}\right)\rho^{-\delta-2} \\
&\quad + \left(1+c\rho^{-2}\right)Anc_2\left(\frac{\rho f_a'\circ\rho}{f_a\circ\rho}\right)\rho^{-C_4-2} \\
&= -\left(\frac{\rho f_a'\circ\rho}{f_a\circ\rho}\right)\rho^{-\delta-2}\left(AR_3^\delta\delta(n-1)-(1+c\rho^{-2})Anc_2\rho^{\delta-C_4}\right) \\
&\quad + AR_3^\delta\delta(\delta+1)\rho^{-\delta-2} \\
&\leq -c\left(\frac{\rho f_a'\circ\rho}{f_a\circ\rho}\right)\rho^{-\delta-2}
\end{aligned}$$

whenever $\delta \in (0, \delta_1)$ is small enough and $\rho \geq R_3(C, \delta)$. These estimates hold since

$$\delta+1-(n-1)\frac{\rho f_a'\circ\rho}{f_a\circ\rho} \leq \delta+1-(n-1)(1-\delta)\phi_1 \leq 0$$

for a sufficiently small $\delta \in (0, \delta_1)$. Now taking into account our assumption (4.9) we obtain

$$\begin{aligned}
(1+|\nabla\psi|^2)\Delta\psi + (1+|\nabla\psi|^2)^{3/2}F &\leq -c\left(\frac{\rho f_a'\circ\rho}{f_a\circ\rho}\right)\rho^{-\delta-2} + (1+c\rho^{-2})F \\
&\leq -c\left(\frac{\rho f_a'\circ\rho}{f_a\circ\rho}\right)\rho^{-\delta-2}
\end{aligned} \tag{4.11}$$

whenever $\delta \in (0, \min(\varepsilon, \delta_1))$ is small enough and $\rho \geq R_3(C, \delta)$.

It remains to estimate $|\langle\nabla|\nabla\psi|^2, \nabla\psi\rangle|$ from above. Since

$$\nabla\psi = AR_3^\delta\nabla(\rho^{-\delta}) + A\nabla h,$$

we have

$$\begin{aligned}
\nabla|\nabla\psi|^2 &= A^2\nabla\langle R_3^\delta\nabla(\rho^{-\delta}) + \nabla h, R_3^\delta\nabla(\rho^{-\delta}) + \nabla h\rangle \\
&= (AR_3^\delta)^2\nabla\langle\nabla(\rho^{-\delta}), \nabla(\rho^{-\delta})\rangle + 2A^2R_3^\delta\nabla\langle\nabla(\rho^{-\delta}), \nabla h\rangle + A^2\nabla\langle\nabla h, \nabla h\rangle.
\end{aligned}$$

By Lemma 4.2 we then get

$$\begin{aligned}
|\langle\nabla|\nabla\psi|^2, \nabla\psi\rangle| &\leq c\rho^{-1}\left(2(\delta AR_3^\delta)^2(\delta+1)\rho^{-2\delta-3} + A^2c_2(2R_3^\delta+1)\left(\frac{f_a'\circ\rho}{f_a\circ\rho}\right)\rho^{-C_4-2}\right) \\
&\leq c\delta^2(\delta+1)\rho^{-2\delta-4} + c\left(\frac{\rho f_a'\circ\rho}{f_a\circ\rho}\right)\rho^{-C_4-4} \\
&\leq c\left(\rho^{-2\delta-4}+\rho^{-C_4-4}\right)\frac{\rho f_a'\circ\rho}{f_a\circ\rho}.
\end{aligned} \tag{4.12}$$

Putting together (4.11) and (4.12) we finally obtain

$$(1+|\nabla\psi|^2)\Delta\psi + (1+|\nabla\psi|^2)^{3/2}F(x) - \frac{1}{2}\langle\nabla|\nabla\psi|^2, \nabla\psi\rangle \leq -c\left(\frac{\rho f_a'\circ\rho}{f_a\circ\rho}\right)\rho^{-\delta-2} < 0$$

in $3\Omega \setminus \bar{B}(o, R_3)$ for a sufficiently small $\delta > 0$ and large R_3. □

Similarly, we have

$$\operatorname{div}\frac{\nabla(-\psi)}{\sqrt{1+|\nabla(-\psi)|^2}} - F(x) > 0 \tag{4.13}$$

in $3\Omega \setminus \bar{B}(o, R_3)$.

4.2. **Uniform height estimate.** We will solve the asymptotic Dirichlet problem by solving the problem first in a sequence of balls with increasing radii. In order to obtain a converging subsequence of solutions, we need to have a uniform height estimate. This subsection is devoted to the construction of a barrier function that will guarantee the height estimate.

Since $f_a'' - a^2 f_a = 0$, where $a(t) = 0$ for $t \in [0, T_0]$ and

$$a(t) \geq \frac{\sqrt{\phi(\phi-1)}}{t}$$

for $t \geq T_1$ and some $\phi > 1$, we have $f_a(t) \geq ct^\phi$ for $t \geq T_1$. Therefore

$$\int_1^\infty \frac{dr}{f_a^{n-1}(r)} < \infty. \tag{4.14}$$

Let $\varphi \colon M \to \mathbb{R}$ be a bounded function. We aim to show the existence of a barrier function V such that $\tilde{Q}[V] \leq 0$ and $V(x) > ||\varphi||_\infty$ in M. In order to define such a function V, we need an auxiliary function $a_0 > 0$, so that

$$\int_1^\infty \left(\int_r^\infty \frac{ds}{f_a^{n-1}(s)} \right) a_0(r) f_a^{n-1}(r) dr < \infty. \tag{4.15}$$

We will discuss about the choice of a_0 in Examples 4.5 and 4.6. Now, following [19], we can define

$$\begin{aligned} V(x) = V\big(\rho(x)\big) = &\left(\int_{\rho(x)}^\infty \frac{ds}{f_a^{n-1}(s)} \right) \left(\int_0^{\rho(x)} a_0(t) f_a^{n-1}(t) dt \right) \\ &- \int_0^{\rho(x)} \left(\int_t^\infty \frac{ds}{f_a^{n-1}(s)} \right) a_0(t) f_a^{n-1}(t) dt - H + ||\varphi||_\infty, \end{aligned} \tag{4.16}$$

where

$$\begin{aligned} H := \limsup_{r\to\infty} \Big\{ &\int_r^\infty \frac{ds}{f_a^{n-1}(s)} \int_0^r a_0(t) f_a^{n-1}(t) dt \\ &- \int_0^r \int_t^\infty \frac{ds}{f_a^{n-1}(s)} a_0(t) f_a^{n-1}(t) dt \Big\} \leq 0; \end{aligned} \tag{4.17}$$

see [19, (4.5)]. From (4.14) and (4.15) we see that H is finite and hence V is well defined.

As in the proof of Lemma 4.3, we write

$$\tilde{Q}[V] = \frac{(1+|\nabla V|^2)\Delta V + (1+|\nabla V|^2)^{3/2} F(x) - \frac{1}{2}\langle \nabla |\nabla V|^2, \nabla V\rangle}{(1+|\nabla V|^2)^{3/2}}, \tag{4.18}$$

where $F(x)$ is as in (4.6), and estimate the terms of the numerator. To begin, we notice that

$$\begin{aligned} V'(r) &= -\frac{1}{f_a^{n-1}(r)} \int_0^r a_0(t) f_a^{n-1}(t) dt < 0, \\ V''(r) &= (n-1)\frac{f_a'(r)}{f_a^n(r)} \int_0^r a_0(t) f_a^{n-1}(t) dt - a_0(r), \end{aligned}$$

and

$$\big|\nabla V\big(\rho(x)\big)\big| = \big|V'\big(\rho(x)\big)\nabla\rho(x)\big| = \big|V'\big(\rho(x)\big)\big|.$$

Note that $-V(r) = g(r)$, the function (1.7) in Introduction. The Laplace comparison theorem implies that

$$\Delta\rho \geq (n-1)\frac{f_a' \circ \rho}{f_a \circ \rho}.$$

Hence we can estimate the Laplacian of V as

$$\begin{aligned}\Delta V &= V''(\rho) + \Delta\rho V'(\rho)\\ &\le V''(\rho) + (n-1)\frac{f_a'(\rho)}{f_a(\rho)}V'(\rho)\\ &= (n-1)\frac{f_a'(\rho)}{f_a^n(\rho)}\int_0^\rho a_0(t)f_a^{n-1}(t)dt - a_0(\rho) - (n-1)\frac{f_a'(\rho)}{f_a^n(\rho)}\int_0^\rho a_0(t)f_a^{n-1}(t)dt\\ &= -a_0(\rho),\end{aligned}$$

and thus the first term of (4.18) can be estimated as

$$\big(1+|\nabla V|^2\big)\Delta V \le -\big(1+|\nabla V|^2\big)a_0(\rho) \le -\big(1+V'(\rho)^2\big)a_0(\rho).$$

Then, for the last term of (4.18) we have

$$\begin{aligned}-\frac{1}{2}\langle\nabla|\nabla V|^2,\nabla V\rangle &= -\frac{1}{2}\langle\nabla(V'(\rho))^2, V'(\rho)\nabla\rho\rangle = -\frac{1}{2}\langle 2V'(\rho)V''(\rho)\nabla\rho, V'(\rho)\nabla\rho\rangle\\ &= -\big(V'(\rho)\big)^2V''(\rho)\\ &= \frac{-1}{f_a^{2n-2}(\rho)}\left(\int_0^\rho a_0(t)f_a^{n-1}(t)dt\right)^2\\ &\quad\cdot\left((n-1)\frac{f_a'(\rho)}{f_a^n(\rho)}\int_0^\rho a_0(t)f_a^{n-1}(t)dt - a_0(\rho)\right)\\ &= \frac{a_0(\rho)}{f_a^{2n-2}(\rho)}\left(\int_0^\rho a_0(t)f_a^{n-1}(t)dt\right)^2\\ &\quad - \frac{(n-1)f_a'(\rho)}{f_a^{3n-2}(\rho)}\left(\int_0^\rho a_0(t)f_a^{n-1}(t)dt\right)^3\\ &= a_0(\rho)V'(\rho)^2 - (n-1)\frac{f_a'(\rho)}{f_a(\rho)}\big(-V'(\rho)\big)^3.\end{aligned}$$

Collecting everything together, we obtain that $\tilde{Q}[V] \le 0$ if

$$\sup_{\partial B(o,r)\times\mathbb{R}}|\bar{\nabla}f| \le \frac{a_0(r)+(n-1)\frac{f_a'(r)}{f_a(r)}\big(-V'(r)\big)^3}{\big(1+V'(r)^2\big)^{3/2}}.$$

Finally it is easy to check that, since H is finite and V is decreasing, we have $V(x) > ||\varphi||_\infty$ for all $x \in M$ and $V(x) \to ||\varphi||_\infty$ as $\rho(x) \to \infty$. Altogether, we have obtained the following.

Lemma 4.4. *Let $\varphi\colon M \to \mathbb{R}$ be a bounded function and assume that the function V defined in* (4.16) *satisfies*

$$\sup_{\partial B(o,r)\times\mathbb{R}}|\bar{\nabla}f| \le \frac{a_0(r)+(n-1)\frac{f_a'(r)}{f_a(r)}\big(-V'(r)\big)^3}{\big(1+V'(r)^2\big)^{3/2}}. \tag{4.19}$$

Then the function V is an upper barrier for the Dirichlet problem such that

$$\tilde{Q}[V] = \operatorname{div}\frac{\nabla V}{\sqrt{1+|\nabla V|^2}} + F(x) \le 0 \quad \text{in } M, \tag{4.20}$$

$$V(x) > ||\varphi||_\infty \quad \text{for all } x \in M \tag{4.21}$$

and

$$\lim_{r(x)\to\infty} V(x) = ||\varphi||_\infty. \tag{4.22}$$

Furthermore,

$$\operatorname{div}\frac{\nabla(-V)}{\sqrt{1+|\nabla(-V)|^2}} - F(x) \ge 0 \quad \text{in } M. \tag{4.23}$$

Next we show by examples that in the situation of Corollaries 1.4 and 1.5 the condition (4.19) is not a stronger restriction than the assumption (4.9) in Lemma 4.3. First note that $V'(r) \to 0$ as $r \to \infty$, and hence the upper bound (4.19) for $|\bar{\nabla} f|$ is asymptotically the function a_0.

Example 4.5. Assume that the sectional curvatures of M satisfy

$$K(P_x) \le -a\big(\rho(x)\big)^2 = -\frac{\phi(\phi-1)}{\rho(x)^2}, \quad \phi > 1,$$

for $\rho(x) \ge T_1$. We need to choose the function a_0 such that (4.15) holds, and since this is a question about its asymptotical behaviour, it is enough to consider the integral

$$\int_{T_1}^{\infty} \left(\int_r^{\infty} \frac{ds}{f_a^{n-1}(s)} \right) a_0(r) f_a^{n-1}(r) dr.$$

For $t \ge T_1$, $f_a(t) = c_1 t^{\phi} + c_2 t^{1-\phi}$, and hence, by a straightforward computation, we have (4.15) if

$$\int_{T_1}^{\infty} a_0(r) r \, dr < \infty.$$

So it is enough to choose for example

$$a_0(r) = O\left(\frac{1}{r^2 (\log r)^{\alpha}} \right)$$

as $r \to \infty$ for some $\alpha > 1$. On the other hand, with this curvature upper bound, the assumption (4.9) requires decreasing of order $o\big(r^{-2-\varepsilon}\big)$.

Example 4.6. Assume that the sectional curvatures of M satisfy

$$K \le -k^2,$$

for $\rho(x) \ge T_1$ and some constant $k > 0$. Then, for large t, $f_a(t) = c_1 \sinh kt + c_2 \cosh kt \approx e^{kt}$. Therefore it is straightforward to see that we have (4.15) if

$$\int_{T_1}^{\infty} a_0(r) \, dr < \infty,$$

which holds by choosing, for example,

$$a_0(r) = O\left(\frac{1}{r (\log r)^{\alpha}} \right), \ \alpha > 1,$$

as $r \to \infty$. On the other hand, with this curvature upper bound, the assumption (4.9) requires decreasing of order $o\big(r^{-1-\varepsilon}\big)$.

4.3. **Proof of Theorem 1.3.** We start with solving the Dirichlet problem in geodesic balls $B(o,R)$.

Lemma 4.7. *Suppose that $f \in C^2(M \times \mathbb{R})$ is of the form $f(x,t) = m(x) + r(t)$ and satisfies*

$$\sup_{\partial B(o,r) \times \mathbb{R}} |\bar{\nabla} f| \le (n-1) \frac{f_a'(r)}{f_a(r)}$$

for all $r > 0$. Then for every $R > 0$ and $\varphi \in C(\partial B(o,R))$ there exists a solution $u \in C^{2,\alpha}(B(o,R)) \cap C(\bar{B}(o,R))$ of the Dirichlet problem

$$\begin{cases} \operatorname{div} \dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \langle \bar{\nabla} f, \nu \rangle & \textit{in } B(o,R) \\ u | \partial B(o,R) = \varphi. \end{cases}$$

Proof. Assuming first that $\varphi \in C^{2,\alpha}(\partial B(o,R))$ the claim follows by the Leray-Schauder method. Indeed, for each $x \in \bar{B}(o,R) \setminus \{o\}$ the inward mean curvature $H(x)$ of the level set $\{y \in \bar{B}(o,R) \colon d(y) = d(x)\} = \partial B(o,\rho(x))$ satisfies

$$H(x) = \Delta\rho(x) \geq (n-1)\frac{f_a'\big(\rho(x)\big)}{f_a\big(\rho(x)\big)} \geq \sup_{\partial B(o,\rho(x))\times\mathbb{R}} |\bar{\nabla} f|.$$

In other words, (2.4) and (2.14) hold and therefore we can apply the Leray-Schauder method as in the proof of Theorem 1.1. The general case $\varphi \in C(\partial B(o,R))$ follows by approximation as in the proof of Theorem 1.2. □

Proof of Theorem 1.3. We extend the boundary data function $\varphi \in C(\partial_\infty M)$ to a function $\varphi \in C(\bar{M})$. Let $\Omega_k = B(o,k)$, $k \in \mathbb{N}$, be an exhaustion of M. By Lemma 4.7, there exist solutions $u_k \in C^{2,\alpha}(\Omega_k) \cap C(\bar{\Omega}_k)$ to

$$\begin{cases} Q[u_k] = \operatorname{div} \dfrac{\nabla u_k}{\sqrt{1+|\nabla u_k|^2}} - \langle \bar{\nabla} f, \nu_k \rangle & \text{in } \Omega_k \\ u_k|\partial\Omega_k = \varphi, \end{cases}$$

where ν_k is the downward pointing unit normal to the graph of u_k. Applying the uniform height estimate, Lemma 4.4, we see that the sequence (u_k) is uniformly bounded and hence the interior gradient estimate (Lemma 2.3), together with the diagonal argument, implies that there exists a subsequence, still denoted by u_k, that converges locally uniformly with respect to C^2-norm to a solution u. Therefore we are left to prove that u extends continuously to $\partial_\infty M$ and satisfies $u|\partial_\infty M = \varphi$.

Towards that end let us fix $x_0 \in \partial_\infty M$ and $\varepsilon > 0$. Since the boundary data function φ is continuous, we find $L \in (8/\pi, \infty)$ such that

$$|\varphi(y) - \varphi(x_0)| < \varepsilon/2$$

for all $y \in C(v_0, 4/L) \cap \partial_\infty M$, where $v_0 = \dot{\gamma}_0^{o,x_0}$ is the initial vector of the geodesic ray representing x_0. Moreover, by (4.22) we can choose R_3 in Lemma 4.3 so large that $V(r) \leq \max_{\bar{M}} |\varphi| + \varepsilon/2$ for $r \geq R_3$.

We claim that

$$w^-(x) := -\psi(x) + \varphi(x_0) - \varepsilon \leq u(x) \leq w^+(x) := \psi(x) + \varphi(x_0) + \varepsilon \tag{4.24}$$

in the set $U := 3\Omega \setminus \bar{B}(o,R_3)$, where $\psi = A(R_3^\delta \rho^{-\delta} + h)$ is the supersolution $\tilde{Q}[\psi] < 0$ in Lemma 4.3 and $A = 2\max_{\bar{M}} |\tilde{\varphi}|$. Recall the notation $\Omega = C(v_0, 1/L) \cap M$ and $\ell\Omega = C(v_0, \ell/L) \cap M$, $\ell > 0$, from Subsection 4.1.

The function φ is continuous in $\bar{M}$ so there exists k_0 such that $\partial\Omega_{k_0} \cap U \neq \emptyset$, and

$$|\varphi(x) - \varphi(x_0)| < \varepsilon/2 \tag{4.25}$$

for all $x \in \partial\Omega_k \cap U$ when $k \geq k_0$. Denote $V_k = \Omega_k \cap U$ for $k \geq k_0$. We will conclude that

$$w^- \leq u_k \leq w^+ \tag{4.26}$$

in V_k by using the comparison principle for the operator $\tilde{Q}_k$,

$$\tilde{Q}_k[v] = \operatorname{div} \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} - \langle \bar{\nabla} f, \nu_k \rangle,$$

where ν_k is the downward pointing unit normal to the graph of the solution u_k. Notice that

$$\partial V_k = (\partial\Omega_k \cap \bar{U}) \cup (\partial U \cap \bar{\Omega}_k).$$

Let $x \in \partial\Omega_k \cap \bar{U}$ and $k \geq k_0$. Then (4.25) and $u_k|\partial\Omega_k = \varphi|\partial\Omega_k$ imply that

$$w^-(x) \leq \varphi(x_0) - \varepsilon/2 \leq \varphi(x) = u_k(x) \leq \varphi(x_0) + \varepsilon/2 \leq w^+(x).$$

Moreover, by Lemma 4.1, we have

$$h|M \setminus \big(2\Omega \cup B(o,R_1)\big) = 1$$

and $R_3^\delta \rho^{-\delta} = 1$ on $\partial B(o,R_3)$, so

$$\psi \geq A = 2\max_{\bar{M}} |\varphi|$$

on $\partial U \cap \bar{\Omega}_k$. By Lemma 4.4, V is a supersolution $\tilde{Q}[V] \leq 0$ and hence

$$\begin{aligned}\operatorname{div} \frac{\nabla V}{\sqrt{1+|\nabla V|^2}} - \langle \bar{\nabla} f, \nu_k\rangle &\leq \operatorname{div} \frac{\nabla V}{\sqrt{1+|\nabla V|^2}} + F(x)\\ &= \tilde{Q}[V] \leq 0\\ &= \operatorname{div} \frac{\nabla u_k}{\sqrt{1+|\nabla u_k|^2}} - \langle \bar{\nabla} f, \nu_k\rangle.\end{aligned}$$

Since $V \geq \max_{\bar{M}} |\varphi|$ on $\partial\Omega_k$, the comparison principle yields $u_k|\Omega_k \leq V|\Omega_k$, and by the choice of R_3, we have

$$u_k \leq \max_{\bar{M}} |\varphi| + \varepsilon/2$$

in $\Omega_k \setminus B(o,R_3)$.

Altogether, it follows that

$$w^+ = \psi + \varphi(x_0) + \varepsilon \geq 2\max_{\bar{M}} |\varphi| + \varphi(x_0) + \varepsilon \geq \max_{\bar{M}} |\varphi| + \varepsilon \geq u_k$$

on $\partial U \cap \bar{\Omega}_k$, and similarly $u_k \geq w^-$ on $\partial U \cap \bar{\Omega}_k$. Consequently $w^- \leq u_k \leq w^+$ on ∂V_k. By Lemma 4.3, $\tilde{Q}[\psi] < 0$, and therefore

$$\begin{aligned}\tilde{Q}_k[w^+] &= \operatorname{div} \frac{\nabla w^+}{\sqrt{1+|\nabla w^+|^2}} - \langle \bar{\nabla} f, \nu_k\rangle\\ &= \operatorname{div} \frac{\nabla \psi}{\sqrt{1+|\nabla \psi|^2}} - \langle \bar{\nabla} f, \nu_k\rangle\\ &\leq \operatorname{div} \frac{\nabla \psi}{\sqrt{1+|\nabla \psi|^2}} + F(x)\\ &= \tilde{Q}[\psi] < 0\\ &= \operatorname{div} \frac{\nabla u_k}{\sqrt{1+|\nabla u_k|^2}} - \langle \bar{\nabla} f, \nu_k\rangle\end{aligned}$$

in U. By the comparison principle, $u_k \leq w^+$ in U. Similarly, using (4.13) we conclude that

$$\operatorname{div} \frac{\nabla w^-}{\sqrt{1+|\nabla w^-|^2}} - \langle \bar{\nabla} f, \nu_k\rangle > \operatorname{div} \frac{\nabla u_k}{\sqrt{1+|\nabla u_k|^2}} - \langle \bar{\nabla} f, \nu_k\rangle$$

in U. Hence $u_k \geq w^-$ in U and we obtain (4.26). This holds for every $k \geq k_0$ and hence (4.24) follows. Finally,

$$\limsup_{x\to x_0} |u(x) - \varphi(x_0)| \leq \varepsilon$$

since $\lim_{x\to x_0} \psi(x) = 0$. Because $x_0 \in \partial_\infty M$ and $\varepsilon > 0$ were arbitrary, this shows that u extends continuously to $C(\bar{M})$ and $u|\partial_\infty M = \varphi$. □

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APPENDIX 4 – ARTICLE [D]

Existence and non-existence of minimal graphic and p-harmonic functions

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Preprint: arXiv:1510.05789 (2015)

EXISTENCE AND NON-EXISTENCE OF MINIMAL GRAPHIC AND p-HARMONIC FUNCTIONS

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Abstract. We prove that every entire solution of the minimal graph equation that is bounded from below and has at most linear growth must be constant on a complete Riemannian manifold M with only one end if M has asymptotically non-negative sectional curvature. On the other hand, we prove the existence of bounded non-constant minimal graphic and p-harmonic functions on rotationally symmetric Cartan-Hadamard manifolds under optimal assumptions on the sectional curvatures.

1. Introduction

It is an interesting question to ask under which conditions on the underlying space M there exist entire non-constant bounded solutions $u\colon M \to \mathbb{R}$ to the minimal graph equation

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \tag{1.1}$$

or to the p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0. \tag{1.2}$$

Namely, in $\mathbb{R}^n$ there is the famous Bernstein theorem which states that entire solutions of (1.1) are affine for dimensions $n \le 7$. Moreover, entire positive solutions in $\mathbb{R}^n$ are constant in all dimensions by the celebrated result due to Bombieri, De Giorgi, and Miranda [2]. For the p-harmonic equation (1.2) the situation is the same as for the harmonic functions, i.e. entire positive solutions in $\mathbb{R}^n$ are constants, the reason being the validity of a global Harnack's inequality.

If the underlying space is changed from $\mathbb{R}^n$ to a Cartan-Hadamard manifold with sufficiently negative curvature, the situation changes for the both equations. The existence results have been proved by studying the so-called *asymptotic Dirichlet problem*. If M is an n-dimensional Cartan-Hadamard manifold, it can be compactified by adding a *sphere at infinity*, $\partial_\infty M$, and equipping the resulting space $\bar{M} := M \cup \partial_\infty M$ by the *cone topology*. With this compactification $\bar{M}$ is homeomorphic to the closed unit ball and $\partial_\infty M$ is homeomorphic to the unit sphere $\mathbb{S}^{n-1}$. For details, see [12]. The asymptotic Dirichlet problem can then be stated as follows: Given a continuous function $\theta\colon \partial_\infty M \to \mathbb{R}$, find a function $u \in C(\bar{M})$ that is a solution to the desired equation in M and has "boundary values" θ on $\partial_\infty M$.

Recently the asymptotic Dirichlet problem for minimal graph, f-minimal graph, p-harmonic and $\mathcal{A}$-harmonic equations has been studied for example in [23], [31], [32], [7], [22], [8], [5], [6], and [18], where the existence of solutions was studied under various curvature assumptions and via different methods. In [8] the existence of

2000 *Mathematics Subject Classification.* Primary 58J32; Secondary 53C21, 31C45.

Key words and phrases. Mean curvature equation, p-Laplace equation, Dirichlet problem, Hadamard manifold.

J.-B.C. supported by MIS F.4508.14 (FNRS).

E.H. supported by Jenny and Antti Wihuri Foundation.

solutions to the minimal graph equation and to the $\mathcal{A}$-harmonic equation was proved in dimensions $n \ge 3$ under curvature assumptions

$$-\frac{\big(\log r(x)\big)^{2\bar\varepsilon}}{r(x)^2} \le K(P_x) \le -\frac{1+\varepsilon}{r(x)^2\log r(x)}, \tag{1.3}$$

where $\varepsilon > \bar\varepsilon > 0$, $P_x \subset T_xM$ is a 2-dimensional subspace, $x \in M \setminus B(o,R_0)$, and $r(x) = d(o,x)$ is the distance to a fixed point $o \in M$. In [18] it was shown that in the case of $\mathcal{A}$-harmonic functions the curvature lower bound can be replaced by a so-called pinching condition

$$|K(P_x)| \le C|K(P_x')|,$$

where C is some constant and $P_x, P_x' \subset T_xM$. One of our main theorems shows that in the above result the upper bound for the curvatures is (almost) optimal, namely we prove the following.

Theorem 1.1. *Let M be a complete Riemannian manifold with asymptotically non-negative sectional curvature and only one end. If $u\colon M \to \mathbb{R}$ is a solution to the minimal graph equation* (1.1) *that is bounded from below and has at most linear growth, then it must be a constant. In particular, if M is a Cartan-Hadamard manifold with asymptotically non-negative sectional curvature, the asymptotic Dirichlet problem is not solvable.*

The notion of asymptotically non-negative sectional curvature (ANSC) is defined in Definition 2.1. It is worth pointing out that we do not assume, differing from previous results into this direction, the Ricci curvature to be non-negative; see e.g. [29], [11], [9], [10].

Our theorem gives immediately the following corollary.

Corollary 1.2. *Let M be a complete Riemannian manifold with only one end and assume that the sectional curvatures of M satisfy*

$$K(P_x) \ge -\frac{C}{r(x)^2\big(\log r(x)\big)^{1+\varepsilon}}$$

for sufficiently large $r(x)$ and for any $C > 0$ and $\varepsilon > 0$. Then any solution $u\colon M \to [a,\infty)$ with at most linear growth to the minimal graph equation (1.1) *must be constant.*

The main tool in the proof of Theorem 1.1 is the gradient estimate in Proposition 3.1, where we obtain an upper bound for the gradient of a solution u of the minimal graph equation in terms of an appropriate lower bound for the sectional curvature of M and the growth of u. Under the assumptions in Theorem 1.1 we obtain a uniform gradient upper bound that enables us to prove a global Harnack's inequality for $u - \inf_M u$.

It is well-known that a global Harnack's inequality (for positive solutions) can be iterated to yield Hölder continuity estimates for all solutions and, furthermore, a Liouville (or Bernstein) type result for solutions with controlled growth.

Corollary 1.3. *Let M be a complete Riemannian manifold with asymptotically non-negative sectional curvature and only one end. Then there exists a constant $\kappa \in (0,1]$, depending only on n and on the function λ in the (ANSC) condition such that every solution $u\colon M \to \mathbb{R}$ to the minimal graph equation* (1.1) *with*

$$\lim_{d(x,o)\to\infty} \frac{|u(x)|}{d(x,o)^\kappa} = 0$$

must be constant.

Before turning to the existence results, we mention two closely related results by Greene and Wu [15]. Firstly, in [15, Theorem 2 and Theorem 4] they show that an n-dimensional, $n \neq 2$, Cartan-Hadamard manifold with asymptotically non-negative sectional curvature is isometric to $\mathbb{R}^n$. Secondly, in [15, Theorem 2] they show that an odd dimensional Riemannian manifold with a pole $o \in M$ and everywhere non-positive or everywhere non-negative sectional curvature is isometric to $\mathbb{R}^n$ if $\liminf_{s\to\infty} s^2 k(s) = 0$, where $k(s) = \sup\{|K(P_x)| \colon x \in M, d(o,x) = s, P_x \in T_xM \text{ two-plane}\}$.

We point out that our results differ from these theorems of [15] (besides the methods) since we do not assume the existence of a pole or the manifold to be simply connected, and the (ANSC) condition allows the sectional curvature to change a sign. Moreover, in the following theorems we will see that, in order to get the result of Greene and Wu, it is necessary to assume $\liminf_{s\to\infty} s^2 k(s) = 0$ for all of the sectional curvatures and not only for the radial ones.

Concerning the existence results, we prove that, at least in the rotationally symmetric case, the curvature upper bound can be slightly improved from (1.3). We also point out that the proof of Theorem 1.4 is very elementary compared to the ones in [8] concerning the general cases.

Theorem 1.4 (= Corollary 4.2)**.** *Let M be a rotationally symmetric n-dimensional Cartan-Hadamard manifold whose radial sectional curvatures outside a compact set satisfy the upper bounds*

$$K(P_x) \le -\frac{1+\varepsilon}{r(x)^2 \log r(x)}, \quad \text{if } n = 2 \tag{1.4}$$

and

$$K(P_x) \le -\frac{1/2+\varepsilon}{r(x)^2 \log r(x)}, \quad \text{if } n \ge 3. \tag{1.5}$$

Then the asymptotic Dirichlet problem for the minimal graph equation (1.1) *is solvable with any continuous boundary data on $\partial_\infty M$. In particular, there are non-constant bounded entire solutions of* (1.1) *in M.*

The rotationally symmetric 2-dimensional case was previously considered in [28], where the solvability of the asymptotic Dirichlet problem was proved under the curvature assumption (1.4).

In Section 4 we consider the existence of bounded non-constant p-harmonic functions and prove the following.

Theorem 1.5 (= Corollary 4.4)**.** *Let M be a rotationally symmetric n-dimensional Cartan-Hadamard manifold, $n \ge 3$, whose radial sectional curvatures satisfy the upper bound*

$$K(P_x) \le -\frac{1/2+\varepsilon}{r(x)^2 \log r(x)}.$$

Then the asymptotic Dirichlet problem for the p-Laplace equation (1.2)*, with $p \in (2,n)$, is solvable with any continuous boundary data on $\partial_\infty M$.*

We point out that the case $p = 2$ reduces to the case of usual harmonic functions, which were considered under the same curvature assumptions in [27]. It is also worth noting that our curvature upper bound is optimal in a sense that asymptotically non-negative sectional curvature would imply a global Harnack's inequality for the $\mathcal{A}$-harmonic functions and hence also for the p-harmonic functions, see e.g. [20, Example 3.1]. Also the upper bound of p is optimal for this curvature bound, namely in Theorem 5.1 we show that if

$$K_M(P_x) \ge -\frac{\alpha}{r(x)^2 \log r(x)}$$

and $p = n$, the manifold M is p-parabolic for all $0 < \alpha \leq 1$, and if $p > n$, then M is p-parabolic for all $\alpha > 0$. We want to point out that all entire positive p-harmonic functions or, more generally, positive $\mathcal{A}$-harmonic functions (of type p) on M must be constant if M is p-parabolic.

2. Preliminaries and definitions

We begin by giving some definitions that are needed in later sections. For the terminology in this section, we mainly follow [16], [25], and [20].

Let (M, g) be a complete smooth Riemannian manifold. If $C \subset M$ is a compact set, then an unbounded component of $M \setminus C$ is called an *end* with respect to C. We say that M has finitely many ends if the number of ends with respect to any compact set has a uniform finite upper bound.

If σ is a smooth positive function on M, we define a measure μ by $d\mu = \sigma^2 d\mu_0$, where μ_0 is the Riemannian measure of the metric g. We will use the notation (M, μ) for the weighted manifold. The *weighted Laplace operator* Δ_μ is a second order differential operator on M defined as

$$\Delta_\mu f = \sigma^{-2} \operatorname{div}(\sigma^2 \nabla f) = \operatorname{div}_\mu(\nabla f), \tag{2.1}$$

where ∇ is the gradient and div the divergence with respect to the Riemannian metric g. We call div_μ the *weighted divergence*.

Definition 2.1. We say that

(ANSC) M has asymptotically non-negative sectional curvature if there exists a continuous decreasing function $\lambda\colon [0, \infty) \to [0, \infty)$ such that

$$\int_0^\infty s\lambda(s)\, ds < \infty,$$

and that $K_M(P_x) \geq -\lambda\big(d(o, x)\big)$ at any point $x \in M$;

(EHI) the weighted manifold (M, μ) satisfies the elliptic Harnack inequality if there exists a constant C_H such that, for any ball $B(x, r)$, any positive weighted harmonic function u in $B(x, 2r)$ satisfies

$$\sup_{B(x,r)} u \leq C_H \inf_{B(x,r)} u;$$

(PHI) the weighted manifold (M, μ) satisfies the parabolic Harnack inequality if there exists a constant C_H such that, for any ball $B(x, r)$, any positive solution u to the weighted heat equation in the cylinder $Q := (0, t) \times B(x, r)$ with $t = r^2$ satisfies

$$\sup_{Q^-} u \leq C_H \inf_{Q^+} u,$$

where

$$Q^- = (t/4, t/2) \times B(x, r/2), \quad \text{and} \quad Q^+(3t/4, t) \times B(x, r/2).$$

Using the previous definitions we can now state the following main result [16, Theorem 1.1] due Grigor'yan and Saloff-Coste, although we do not need it in its full strength.

Theorem 2.2. *Let M be a complete non-compact Riemannian manifold having either (a) asymptotically non-negative sectional curvature or (b) non-negative Ricci curvature outside a compact set and finite first Betti number. Then M satisfies (PHI) if and only if it satisfies (EHI). Moreover, (PHI) and (EHI) hold if and only if either M has only one end or M has more that one end and the functions V and V_i satisfy for large enough r the conditions $V_i(r) \approx V(r)$ (for all indices i) and*

$$\int_1^r \frac{s\, ds}{V(s)} \approx \frac{r^2}{V(r)}.$$

Above $V(r) = \mu\big(B(o,r)\big)$ and $V_i(r) = \mu\big(B(o,r) \cap E_i\big)$ for an end E_i.

We will briefly sketch the rather well-known proof of the validity of (EHI) in the case where M has only one end. For that purpose, we need additional definitions.

Definition 2.3. We say that

(VD) a family $\mathcal{F}$ of balls in (M,μ) satisfies the volume doubling property if there exists a constant C_D such that for any ball $B(x,r) \in \mathcal{F}$ we have
$$\mu\big(B(x,r)\big) \le C_D \mu\big(B(x,r/2)\big);$$

(PI) a family $\mathcal{F}$ of balls in (M,μ) satisfies the Poincaré inequality if there exists a constant C_P such that for any ball $B(x,r) \in \mathcal{F}$ and any $f \in C^1(B(x,r))$ we have
$$\inf_{\xi\in\mathbb{R}} \int_{B(x,r)} (f-\xi)^2\, d\mu \le C_P r^2 \int_{B(x,r)} |\nabla f|^2\, d\mu;$$

(BC) a set $A \subset \partial B(o,t)$ has a ball-covering property if, for each $0 < \varepsilon < 1$, A can be covered by k balls of radius εt with centres in A, where k depends on ε and possibly on some other parameters, but is independent of t.

From the curvature assumptions ((a) or (b)) in Theorem 2.2 it follows that (VD) and (PI) hold for all “remote” balls, that is for balls $B(x,r)$, where $r \le \frac{\varepsilon}{2} d(o,x)$ and $\varepsilon \in (0,1]$ is a suitable remote parameter. The familiar Moser iteration then yields local (EHI) for such remote balls. Furthermore, if E is an end of M and $E(t)$ denotes the unbounded component of $E \setminus \bar{B}(o,t)$, then set $\partial E(t)$ is connected and has the ball-covering property (BC) for all sufficiently large t. Iterating the local (EHI) k times, one obtains Harnack's inequality
$$\sup_{\partial E(t)} u \le C \inf_{\partial E(t)} u$$
with C independent of t. Finally, if M has only one end, the global (EHI) follows from the maximum principle. We will give a bit more details in Section 3 and refer to [16], [25], and [20] for more details, and to [1], [4], [24], [25], and [26] for the connectivity and the covering properties mentioned above.

3. Non-existence for minimal graph equation

In order to prove Theorem 1.1 we need a uniform gradient estimate for the solutions of the minimal graph equation (1.1). Our proof follows closely the computations in [9] and [29]. We begin by introducing some notation.

We assume that M is a complete non-compact n-dimensional Riemannian manifold whose Riemannian metric is given by $ds^2 = \sigma_{ij} dx^i dx^j$ in local coordinates. Let $u\colon M \to \mathbb{R}$ be a solution to the minimal graph equation, i.e.
$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0,$$
where the gradient and divergence are taken with respect to the Riemannian metric of M. We denote by
$$S = \big\{(x,u(x))\colon x \in M\big\}$$
the graph of u in the product manifold $M \times \mathbb{R}$ and by
$$N = \frac{-u^j \partial_j + \partial_t}{W}$$
the upward pointing unit normal to the graph of u expressed in terms of a local coordinate frame $\{\partial_1, \dots, \partial_n\}$ and $\partial_t = e_{n+1}$. Here $W = \sqrt{1+|\nabla u|^2}$ and $u^i =$

$\sigma^{ij}D_ju$, D_j being the covariant derivative on M. The components of the induced metric on the graph are given by $g_{ij} = \sigma_{ij} + u_iu_j$ with inverse

$$g^{ij} = \sigma^{ij} - \frac{u^iu^j}{W^2}.$$

We denote by ∇^S and Δ^S the gradient and, respectively, the Laplace-Beltrami operator on the graph S. For the Laplacian on the graph we have the Bochner-type formula (see e.g. [13])

$$\Delta^S \langle e_{n+1}, N\rangle = -\big(|A|^2 + \overline{\mathrm{Ric}}(N,N)\big)\langle e_{n+1}, N\rangle, \tag{3.1}$$

where $|A|$ is the norm of the second fundamental form and $\overline{\mathrm{Ric}}$ is the Ricci curvature of $M\times\mathbb{R}$. From (3.1) we obtain

$$\Delta^S W = 2\frac{|\nabla^S W|^2}{W} + \big(|A|^2 + \overline{\mathrm{Ric}}(N,N)\big)W. \tag{3.2}$$

Here and in what follows we extend, without further notice, functions h defined on M to $M\times\mathbb{R}$ by setting $h(x,t)=h(x)$. The Laplace-Beltrami operator of the graph can be expressed in local coordinates as

$$\Delta^S = g^{ij}D_iD_j.$$

Now we are ready to prove the following gradient estimate.

Proposition 3.1. *Assume that the sectional curvature of M has a lower bound $K(P_x)\ge -K_0^2$ for all $x\in B(p,R)$ for some constant $K_0 = K_0(p,R)\ge 0$. Let u be a positive solution to the minimal graph equation in $B(p,R)\subset M$. Then*

$$\begin{aligned}&|\nabla u(p)| \qquad (3.3)\\ &\le \left(\frac{2}{\sqrt{3}} + \frac{32u(p)}{R}\right)\left(\exp\left[64u(p)^2\left(\frac{2\psi(R)}{R^2} + \sqrt{\frac{4\psi(R)^2}{R^4} + \frac{(n-1)K_0^2}{64u(p)^2}}\right)\right] + 1\right),\end{aligned}$$

where $\psi(R) = (n-1)K_0R\coth(K_0R)+1$ if $K_0>0$ and $\psi(R)=n$ if $K_0=0$.

Proof. Define a function $h=\eta W$, where $\eta(x) = g(\varphi(x))$ with $g(t) = e^{Kt}-1$,

$$\varphi(x) = \left(1 - \frac{u(x)}{4u(p)} - \frac{d(x,p)^2}{R^2}\right)^+,$$

and a constant K that will be specified in (3.10). Denote by $C(p)$ the cut-locus of p and let $U(p) = B(p,R)\setminus C(p)$. Then it is well known that $d(x,p)$ is smooth in the open set $U(p)$. We assume that the function h attains its maximum at a point $q\in U(p)$, and for the case $q\notin U(p)$ we refer to [29].

In all the following, the computations will be done at the maximum point q of h. We have

$$\nabla^S h = \eta\nabla^S W + W\nabla^S\eta = 0 \tag{3.4}$$

and since the Hessian of h is non-positive, we obtain, using (3.4) and (3.2),

$$\begin{aligned}0 \ge \Delta^S h &= W\Delta^S\eta + 2\left\langle\nabla^S\eta, \nabla^S W\right\rangle + \eta\Delta^S W\\ &= W\Delta^S\eta + \left(\Delta^S W - \frac{2}{W}|\nabla^S W|^2\right)\eta\\ &= W\big(\Delta^S\eta + (|A|^2 + \overline{\mathrm{Ric}}(N,N))\eta\big),\end{aligned} \tag{3.5}$$

where $\overline{\mathrm{Ric}}$ is the Ricci curvature of $M\times\mathbb{R}$. Since the Ricci curvature of $M\times\mathbb{R}$ in $B(p,R)\times\mathbb{R}$ has a lower bound $\overline{\mathrm{Ric}}(N,N)\ge -(n-1)K_0^2$, we obtain from (3.5) that $\Delta^S\eta \le (n-1)K_0^2\eta$ and hence, from the definition of η, we get

$$\Delta^S\varphi + K|\nabla^S\varphi|^2 \le \frac{(n-1)K_0^2}{K}. \tag{3.6}$$

Next we want to estimate $\Delta^S\varphi$ from below by using the lower bound for the sectional curvature and the Hessian comparison theorem. For this, let $\{e_i\}$ be a local orthonormal frame on S. Since u is a solution to the minimal graph equation, we have $\Delta^S u = 0$ and

$$\sum_{i=1}^n \left\langle \bar\nabla_{e_i} N, e_i \right\rangle = 0,$$

where $\bar\nabla$ denotes the Riemannian connection of the ambient space $M\times\mathbb{R}$. Hence

$$\begin{aligned}
\Delta^S\varphi = \Delta^S\left(-\frac{d^2}{R^2}\right) &= -\frac{1}{R^2}\sum_{i=1}^n \left\langle \nabla^S_{e_i}\nabla^S d^2, e_i\right\rangle\\
&= -\frac{1}{R^2}\sum_{i=1}^n \left\langle \bar\nabla_{e_i}\left(\bar\nabla d^2 - \left\langle \bar\nabla d^2, N\right\rangle N\right), e_i\right\rangle\\
&= -\frac{1}{R^2}\sum_{i=1}^n \left\langle \bar\nabla_{e_i}\bar\nabla d^2, e_i\right\rangle\\
&= -\frac{2d}{R^2}\sum_{i=1}^n \left\langle \bar\nabla_{e_i}\bar\nabla d, e_i\right\rangle - \frac{2}{R^2}\sum_{i=1}^n (e_i d)\left\langle \bar\nabla d, e_i\right\rangle\\
&\geq -\frac{2d}{R^2}\sum_{i=1}^n \left\langle \bar\nabla_{e_i}\bar\nabla d, e_i\right\rangle - \frac{2}{R^2}.
\end{aligned}$$

Now decompose e_i as $e_i = (e_i - \langle \partial_t, e_i\rangle \partial_t) + \langle \partial_t, e_i\rangle \partial_t =: \hat e_i + \langle \partial_t, e_i\rangle \partial_t$. Then

$$\begin{aligned}
\left\langle \bar\nabla_{e_i}\bar\nabla d, e_i\right\rangle &= \left\langle \bar\nabla_{\hat e_i + \langle\partial_t, e_i\rangle\partial_t}\bar\nabla d, \hat e_i + \langle \partial_t, e_i\rangle \partial_t\right\rangle\\
&= \left\langle \bar\nabla_{\hat e_i}\bar\nabla d, \hat e_i\right\rangle = \operatorname{Hess} d(\hat e_i, \hat e_i)
\end{aligned}$$

and by the Hessian comparison (e.g. [14, Theorem A]) we have

$$\operatorname{Hess} d(\hat e_i, \hat e_i) \leq \frac{f'(d)}{f(d)}\left(|\hat e_i|^2 - \langle \nabla d, \hat e_i\rangle\right),$$

where $f(t) = K_0^{-1}\sinh(K_0 t)$ if $K_0 > 0$ and $f(t) = t$ if $K_0 = 0$. Choosing $\hat e_n$ parallel to ∇d at q we have

$$\operatorname{Hess} d(\hat e_i, \hat e_i) \leq \begin{cases} 0, & \text{if } i = n;\\ \frac{f'(d)}{f(d)}, & \text{if } i \in \{1,\ldots,n-1\}.\end{cases}$$

Hence

$$\begin{aligned}
\sum_{i=1}^n \left\langle \bar\nabla_{e_i}\bar\nabla d, e_i\right\rangle &= \sum_{i=1}^n \operatorname{Hess} d(\hat e_i, \hat e_i)\\
&\leq \begin{cases} (n-1)K_0\coth(K_0 d), & \text{if } K_0 > 0;\\ \frac{n-1}{d}, & \text{if } K_0 = 0.\end{cases}
\end{aligned}$$

Therefore

$$\Delta^S\varphi \geq -\frac{2d}{R^2}\sum_{i=1}^n \left\langle \bar\nabla_{e_i}\bar\nabla d, e_i\right\rangle - \frac{2}{R^2} \geq -\frac{2\psi(R)}{R^2}, \tag{3.7}$$

where ψ is as in the claim.

A straightforward computation gives also

$$\begin{aligned}|\nabla^S \varphi|^2 = g^{ij} D_i\varphi D_j\varphi &= \frac{|\nabla u|^2}{16u(p)^2W^2} + \frac{4d(x,p)^2}{R^4}\left(1 - \left\langle \frac{\nabla u}{W}, \nabla d(x,p)\right\rangle^2\right) \\ &\quad + \frac{d(x,p)}{u(p)R^2W^2}\langle \nabla u, \nabla d(x,p)\rangle \\ &\geq \frac{|\nabla u|^2}{16u(p)^2W^2} + \frac{4d(x,p)^2}{R^4}\left(1 - \frac{|\nabla u|^2}{W^2}\right) - \frac{d(x,p)|\nabla u|}{u(p)R^2W^2} \\ &= \left(\frac{|\nabla u|}{4u(p)W} - \frac{2d(x,p)}{R^2W}\right)^2.\end{aligned}$$

Note that

$$\left(\frac{|\nabla u|}{4u(p)W} - \frac{2d(x,p)}{R^2W}\right)^2 > \frac{1}{16u(p)^2\alpha^2} \tag{3.8}$$

with some constant $\alpha > 2$ if and only if

$$\left(\frac{|\nabla u|}{4u(p)W} - \frac{2d(x,p)}{R^2W} - \frac{1}{4u(p)\alpha}\right)\left(\frac{|\nabla u|}{4u(p)W} - \frac{2d(x,p)}{R^2W} + \frac{1}{4u(p)\alpha}\right) > 0.$$

This is clearly true if the first factor is positive, i.e. if

$$\alpha|\nabla u| - W > \frac{\alpha 8d(x,p)u(p)}{R^2}.$$

On the other hand,

$$\alpha|\nabla u| - W > W$$

if

$$W^2 > \frac{\alpha^2}{\alpha^2 - 4}.$$

Therefore assuming

$$W(q) > \max\left\{\frac{\alpha}{\sqrt{\alpha^2-4}}, \frac{\alpha 8u(p)}{R}\right\}$$

we see that also (3.8) holds and thus we have the estimate

$$|\nabla^S\varphi|^2 > \frac{1}{16u(p)^2\alpha^2}. \tag{3.9}$$

Plugging (3.7) and (3.9) into (3.6) we obtain

$$-\frac{2\psi(R)}{R^2} + \frac{K}{16u(p)^2\alpha^2} < \frac{(n-1)K_0^2}{K}.$$

But choosing

$$K = 8u(p)^2\alpha^2\left(\frac{2\psi(R)}{R^2} + \sqrt{\frac{4\psi(R)^2}{R^4} + \frac{(n-1)K_0^2}{4u(p)^2\alpha^2}}\right) \tag{3.10}$$

with $\alpha = 4$ we get a contradiction and hence we must have

$$W(q) \leq \max\left\{\frac{2}{\sqrt{3}}, \frac{32u(p)}{R}\right\}.$$

This implies

$$\begin{aligned}h(p) = (e^{K\varphi(p)} - 1)W(p) = (e^{\frac{3}{4}K} - 1)W(p) &\leq h(q) \\ &\leq (e^K - 1)\max\left\{\frac{2}{\sqrt{3}}, \frac{32u(p)}{R}\right\} \\ &\leq (e^K - 1)\left(\frac{2}{\sqrt{3}} + \frac{32u(p)}{R}\right)\end{aligned}$$

and noting that $e^{\frac{3}{4}K} - 1 \geq e^{\frac{K}{2}} - 1$ we obtain the desired estimate

$$|\nabla u(p)| \leq (e^{\frac{K}{2}} + 1)\left(\frac{2}{\sqrt{3}} + \frac{32u(p)}{R}\right).$$

□

Next we apply Proposition 3.1 to the setting of Theorem 1.1 to obtain a uniform gradient estimate.

Corollary 3.2. *Let M be a complete Riemannian manifold with asymptotically non-negative sectional curvature. If $u\colon M \to \mathbb{R}$ is a solution to the minimal graph equation* (1.1) *that is bounded from below and has at most linear growth, then there exist positive constants C and R_0 such that*

$$|\nabla u(x)| \leq C \tag{3.11}$$

for all $x \in M \setminus B(o, R_0)$.

Proof. We may assume, without loss of generality, that $u > 0$. Then the assumptions on the growth of u and on the curvature of M imply that there exist constants c and R_0 such that

$$u(x) \leq c\, d(x, o) \tag{3.12}$$

and

$$K(P_x) \geq -\frac{c}{d(x, o)^2}$$

for all $x \in M \setminus B(o, R_0/2)$. Next we apply Proposition 3.1 to points $p \in M \setminus B(o, R_0)$ with the radius $R = d(p, o)/2 \geq R_0/2$. Noticing that $B(p, R) \subset M \setminus B(o, R) \subset M \setminus B(o, R_0/2)$, we obtain an upper bound

$$K_0^2 = K_0(p, R)^2 \leq c^2/R^2 \tag{3.13}$$

for the constant K_0 in the sectional curvature bound in $B(p, R)$. It follows now from (3.12) and (3.13) that

$$\frac{u(p)}{R} \leq 2c,$$
$$\psi(R) \leq (n-1)c \coth(c) + 1,$$

and

$$u(p)^2 K_0^2 \leq 4c^3.$$

Plugging these upper bounds into (3.3) gives the estimate (3.11). □

We are now ready to prove the Theorem 1.1.

Proof of Theorem 1.1. Denoting

$$A(x) = \frac{1}{\sqrt{1 + |\nabla u|^2}}$$

we see that

$$\operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = \operatorname{div}\big(A(x)\nabla u\big) = 0$$

is equivalent to

$$\frac{1}{A(x)} \operatorname{div}\big(A(x)\nabla u\big) = 0.$$

Now we can interpret the minimal graph equation as a weighted Laplace equation Δ_σ with the weight

$$\sigma = \sqrt{A}.$$

Note that due to the uniform gradient estimate (3.11) of Corollary 3.2 there exists a constant $c > 0$ such that $c \leq \sigma \leq 1$ in $M \setminus B(o, R_0)$ and hence the operator Δ_σ is uniformly elliptic there. On the other hand, the assumption (ANSC) implies that the (unweighted) volume doubling condition (VD) and the (unweighted) Poincaré inequality (PI) hold for balls inside $B(p,R)$, with $R = d(o,p)/2 \geq R_0$. More precisely, the Ricci curvature of M satisfies

$$\mathrm{Ric}(x) \geq -\frac{(n-1)K^2}{d(x,o)^2} \tag{3.14}$$

for some constant $K \geq 0$ if $d(x,o) \geq R_0$ and R_0 is large enough. Then for each $x \in B(p,R)$, with $d(o,p) \geq 2R \geq 2R_0$, we have $\mathrm{Ric}(x) \geq -(n-1)\tilde{K}^2$, where $\tilde{K} = KR^{-1}$. Then the well-known Bishop-Gromov comparison theorem (see [17, 5.3.bis Lemma]) implies that

$$\frac{\mu_0\big(B(x,2r)\big)}{\mu_0\big(B(x,r)\big)} \leq 2^n \exp\big(2r(n-1)\tilde{K}\big) \leq 2^n \exp\big((n-1)K\big) \tag{3.15}$$

for all balls $B(x,2r) \subset B(p,R) \subset M \setminus B(o,R_0)$. On the other hand, it follows from Buser's isoperimetric inequality [3] that

$$\int_{B(x,r)} |f - f_{B(x,r)}|\, d\mu_0 \leq r \exp\big(c_n(1+\tilde{K}r)\big) \int_B |\nabla f|\, d\mu_0 \leq cr \int_B |\nabla f|\, d\mu_0, \tag{3.16}$$

for every $f \in C^1\big(B(x,r)\big)$, where

$$f_{B(x,r)} = \frac{1}{\mu_0\big(B(x,r)\big)} \int_{B(x,r)} f\, d\mu_0$$

and the constant c also has an upper bound that depends only on n and K. Since Δ_σ is uniformly elliptic in $M \setminus B(o,R_0)$, the Moser iteration method gives a local Harnack's inequality

$$\sup_{B(p,R/2)} u \leq c \inf_{B(p,R/2)} u \tag{3.17}$$

for all $p \in \partial B(o,2R)$, with the constant c independent of p and R. Since we assume that M has only one end, the boundary of the unbounded component of $M \setminus \bar{B}(o,2R)$ is connected for all sufficiently large R and can be covered by k balls $B(x,R/2)$, with $x \in \partial B(o,2R)$ and k independent of R; see [1] and [24]. Iterating the Harnack inequality (3.17) k times and applying the maximum principle we obtain

$$\sup_{B(o,2R)} u \leq C \inf_{B(o,2R)} u. \tag{3.18}$$

Finally, we may suppose, without loss of generality, that $\inf_M u = 0$. Letting then $R \to \infty$, we get

$$\sup_{B(o,2R)} u \leq C \inf_{B(o,2R)} u \to 0$$

as $R \to \infty$, and therefore u must be constant. □

Proof of Corollary 1.3. In the proof below, the constants c, C, C_0, Λ, and κ depend only on n and on the function λ in the (ANSC) assumption.

We may assume that $u(o) = 0$. Suppose first that $u\colon M \to \mathbb{R}$ is a solution to the minimal graph equation (1.1) such that

$$\lim_{d(x,o)\to\infty} \frac{|u(x)|}{d(x,o)} = 0. \tag{3.19}$$

Then there exists a sufficiently large R_0 such that $|u(x)| \leq d(x,o)$ for all $x \in M \setminus B(o,R_0/2)$ and that (3.13) holds, i.e. $K_0^2 = K_0(p,R)^2 \leq c^2/R^2$ for all $p \in M \setminus B(o,R_0)$ and $R = d(p,o)/2 \geq R_0/2$. Denote

$$M(t) = \sup_{B(o,t)} u \quad \text{and} \quad m(t) = \inf_{B(o,t)} u$$

for $t > 0$. Then $u - m(2t)$ is a positive solution in $B(o, 2t)$ and, moreover, $u(x) - m(2t) \le 4t$ for all $x \in \partial B(o, 3t/2)$ and $t \ge R_0$. Applying Corollary 3.2 to $u - m(2t)$ in balls $B(x, t/2)$, where $x \in \partial B(o, 3t/2)$ and $t \ge R_0$, we obtain a uniform gradient bound

$$|\nabla u(x)| \le C$$

for all $x \in M \setminus B(o, 3R_0/2)$. Therefore, we may apply the Harnack inequality (3.18) to functions $u - m(2t)$, for all sufficiently large t, to obtain

$$M(t) - m(2t) \le C_0\big(m(t) - m(2t)\big). \tag{3.20}$$

Then we proceed as in the proof of the Hölder continuity estimate for $\mathcal{A}$-harmonic functions in [19, 6.6. Theorem] to obtain

$$M(t) - m(t) \le \Lambda\big(M(2t) - m(2t)\big), \tag{3.21}$$

where $\Lambda = (C_0 - 1)/C_0$. For reader's convenience we give the short proof of (3.21). To obtain (3.21) suppose first that

$$m(t) - m(2t) \le C_0^{-1}\big(M(2t) - m(2t)\big). \tag{3.22}$$

Then

$$\begin{aligned} M(t) - m(t) &= M(t) - m(2t) + m(2t) - m(t) \\ &\le (C_0 - 1)\big(m(t) - m(2t)\big) \\ &\le \Lambda\big(M(2t) - m(2t)\big) \end{aligned}$$

by (3.20) and (3.22). On the other hand, if

$$m(t) - m(2t) \ge C_0^{-1}\big(M(2t) - m(2t)\big),$$

then

$$\begin{aligned} M(t) - m(t) &\le M(2t) - m(t) - \big(m(t) - m(2t)\big) \\ &\le \Lambda\big(M(2t) - m(2t)\big). \end{aligned}$$

Thus (3.21) always holds. Suppose then that $R \ge r$, with r sufficiently large. Choose the integer $m \ge 1$ such that $2^{m-1} \le R/r \le 2^m$. Then

$$\begin{aligned} M(r) - m(r) &\le \Lambda^{m-1}\big(M(2^{m-1}) - m(2^{m-1})\big) \\ &\le \Lambda\big(M(R) - m(R)\big). \end{aligned}$$

Setting $\kappa = (-\log \Lambda)/\log 2$, we get $(r/R)^\kappa \ge 2^{-\kappa}\Lambda^{m-1}$, and therefore

$$M(r) - m(r) \le 2^\kappa \left(\frac{r}{R}\right)^\kappa \big(M(R) - m(R)\big) \tag{3.23}$$

for every $R \ge r$, with r sufficiently large. Notice that (3.23) holds for all entire solutions satisfying (3.19). Finally, if u is an entire solution to (1.1) such that

$$\lim_{d(x,o)\to\infty} \frac{|u(x)|}{d(x,o)^\kappa} = 0,$$

the estimate (3.23) holds for u. Letting $R \to \infty$ in (3.23), we obtain $M(r) - m(r) = 0$ for all r and, consequently, u must be constant. □

4. Existence results on rotationally symmetric manifolds

In this section we assume that M is a rotationally symmetric Cartan-Hadamard manifold with the Riemannian metric given by

$$ds^2 = dr^2 + f(r)^2 d\vartheta^2$$

where $r(x) = d(o,x)$ is the distance to a fixed point $o \in M$ and $f\colon (0,\infty) \to (0,\infty)$ is a smooth function with $f'' \geq 0$. Then the (radial) sectional curvature of M is given by $K(r) = -f''(r)/f(r)$.

On such manifold the Laplace operator can be written as

$$\Delta = \frac{\partial^2}{\partial r^2} + (n-1)\frac{f' \circ r}{f \circ r}\frac{\partial}{\partial r} + \frac{1}{(f\circ r)^2}\Delta^{\mathbb{S}}, \tag{4.1}$$

where $\Delta^{\mathbb{S}}$ is the Laplacian on the unit sphere $\mathbb{S}^{n-1} \subset T_oM$. For the gradient of a function φ we have

$$\nabla\varphi = \frac{\partial\varphi}{\partial r}\frac{\partial}{\partial r} + \frac{1}{f(r)^2}\nabla^{\mathbb{S}}\varphi \tag{4.2}$$

and

$$|\nabla\varphi|^2 = \varphi_r^2 + f^{-2}|\nabla^{\mathbb{S}}\varphi|^2.$$

Here $\nabla^{\mathbb{S}}$ is the gradient on $\mathbb{S}^{n-1}$, $|\nabla^{\mathbb{S}}\varphi|$ denotes the norm of $\nabla^{\mathbb{S}}\varphi$ with respect to the Euclidean metric on $\mathbb{S}^{n-1}$, and $\varphi_r = \partial\varphi/\partial r$. More precisely, in geodesic polar coordinates (r,ϑ),

$$\begin{aligned}\Delta\varphi(r,\vartheta) &= \frac{\partial^2\varphi(r,\vartheta)}{\partial r^2} + (n-1)\frac{f'(r)}{f(r)}\frac{\partial\varphi(r,\vartheta)}{\partial r} + \frac{1}{f(r)^2}\Delta^{\mathbb{S}}\tilde{\varphi}(\vartheta),\\ \nabla\varphi(r,\vartheta) &= \frac{\partial\varphi(r,\vartheta)}{\partial r}\frac{\partial}{\partial r} + \frac{1}{f(r)^2}\nabla^{\mathbb{S}}\tilde{\varphi}(\vartheta) \in \mathbb{R}\oplus T_\vartheta\mathbb{S}^{n-1},\end{aligned}$$

where $\tilde{\varphi}\colon \mathbb{S}^{n-1} \to \mathbb{R}$, $\tilde{\varphi}(\vartheta) = \varphi(r,\vartheta)$ for each fixed $r > 0$.

Existence of non-constant bounded harmonic functions on rotationally symmetric manifolds was considered in [27], where March proved, with probabilistic arguments, that such functions exist if and only if

$$J(f) := \int_1^\infty \Big(f^{n-3}(r)\int_r^\infty f^{1-n}(\rho)d\rho\Big)dr < \infty.$$

In terms of radial sectional curvature we have (for the proof see [27])

$$J(f) < \infty \quad \text{if } K(r) \leq -\frac{c}{r^2\log r} \text{ for } c > c_n \text{ and large } r,$$

and

$$J(f) = \infty \quad \text{if } K(r) \geq -\frac{c}{r^2\log r} \text{ for } c < c_n \text{ and large } r,$$

where $K(r) = -f''(r)/f(r)$ and $c_2 = 1$, $c_n = 1/2$ for $n \geq 3$. Another proof for the existence was given in [30] and our approach in this section is similar to that one.

4.1. Minimal graph equation.

First we consider the minimal graph equation and prove the following existence result.

Theorem 4.1. *Assume that*

$$\int_1^\infty \Big(f(s)^{n-3}\int_s^\infty f(t)^{1-n}dt\Big)ds < \infty. \tag{4.3}$$

Then there exist non-constant bounded solutions of the minimal graph equation and, moreover, the asymptotic Dirichlet problem for the minimal graph equation is uniquely solvable for any continuous boundary data on $\partial_\infty M$.

Proof. First, changing the order of integration, the condition (4.3) reads

$$\int_1^\infty \frac{\int_1^t f(s)^{n-3}ds}{f(t)^{n-1}}dt < \infty. \tag{4.4}$$

Now we interpret $\partial_\infty M$ as $\mathbb{S}^{n-1}$ and let $b\colon \mathbb{S}^{n-1} \to \mathbb{R}$ be a smooth non-constant function and define $B\colon M \setminus \{o\} \to \mathbb{R}$,

$$B(\exp(r\vartheta)) = B(r,\vartheta) = b(\vartheta),\ \vartheta \in \mathbb{S}^{n-1} \subset T_oM.$$

Define also

$$\eta(r) = k\int_r^\infty f(t)^{-n+1}\int_1^t f(s)^{n-3}dsdt,$$

with $k > 0$ to be determined later, and note that by the assumption (4.4) $\eta(r) \to 0$ as $r \to \infty$.

The idea in the proof is to use the functions η and B, and condition (4.4) to construct barrier functions for the minimal graph equation to show the existence of solutions that extends continuously to the asymptotic boundary $\partial_\infty M$ with prescribed asymptotic behaviour.

Begin by noticing that

$$\eta'(r) = -kf(r)^{-n+1}\int_1^r f(s)^{n-3}ds < 0,$$

$$\eta''(r) = k(n-1)f'(r)f(r)^{-n}\int_1^r f(s)^{n-3}ds - kf^{-2}(r),$$

and

$$\Delta\eta = -kf^{-2}$$

where $\eta(x) := \eta\big(r(x)\big)$. The minimal graph equation for $\eta + B$ can be written as

$$\begin{aligned}\operatorname{div}\frac{\nabla(\eta+B)}{\sqrt{1+|\nabla(\eta+B)|^2}} &= \frac{\Delta(\eta+B)}{\sqrt{1+|\nabla(\eta+B)|^2}}\\ &\quad+\left\langle\nabla(\eta+B),\nabla\Big(\frac{1}{\sqrt{1+|\nabla(\eta+B)|^2}}\Big)\right\rangle,\end{aligned} \tag{4.5}$$

and we want to estimate the terms on the right hand side. First note that

$$\Delta(\eta+B)(r,\vartheta) = -kf(r)^{-2} + f(r)^{-2}\Delta^{\mathbb{S}}b(\vartheta) \tag{4.6}$$

and

$$|\nabla(\eta+B)(r,\vartheta)|^2 = \eta_r(r)^2 + f(r)^{-2}|\nabla^{\mathbb{S}}b(\vartheta)|^2.$$

Hence the second term on the right hand side of (4.5) becomes

$$\begin{aligned}\left\langle\nabla(\eta+B),\nabla\Big(\frac{1}{\sqrt{1+|\nabla(\eta+B)|^2}}\Big)\right\rangle &= \big(1+\eta_r^2+f^{-2}|\nabla^{\mathbb{S}}b|^2\big)^{-3/2}\\ &\cdot\Big(-\eta_r^2\eta_{rr}+\eta_rf_r|\nabla^{\mathbb{S}}b|^2f^{-3}-f^{-4}\big\langle\nabla^{\mathbb{S}}b,\nabla^{\mathbb{S}}\big(|\nabla^{\mathbb{S}}b|^2\big)\big\rangle_{\mathbb{S}}/2\Big)\\ &= \big(1+\eta_r^2+f^{-2}|\nabla^{\mathbb{S}}b|^2\big)^{-3/2}\Big(-\eta_r^2\eta_{rr}+\eta_rf_r|\nabla^{\mathbb{S}}b|^2f^{-3}\\ &\quad-f^{-4}\operatorname{Hess}^{\mathbb{S}}b(\nabla^{\mathbb{S}}b,\nabla^{\mathbb{S}}b)\Big),\end{aligned}$$

where $\operatorname{Hess}^{\mathbb{S}}$ is the Hessian on $\mathbb{S}^{n-1}$. Using (4.5) and (4.6) we get

$$\begin{aligned}
\operatorname{div}\frac{\nabla(\eta+B)}{\sqrt{1+|\nabla(\eta+B)|^2}} &= \big(1+\eta_r^2+f^{-2}|\nabla^{\mathbb{S}}b|^2\big)^{-3/2}\Big(-\frac{k}{f^2}+\frac{\Delta^{\mathbb{S}}b}{f^2}-\frac{k\eta_r^2}{f^2} \\
&\quad +\frac{\eta_r^2\Delta^{\mathbb{S}}b}{f^2}-\frac{k|\nabla^{\mathbb{S}}b|^2}{f^4}+\frac{|\nabla^{\mathbb{S}}b|^2\Delta^{\mathbb{S}}b}{f^4}-\eta_r^2\eta_{rr}+\frac{\eta_r f_r|\nabla^{\mathbb{S}}b|^2}{f^3} \\
&\quad -\frac{\operatorname{Hess}^{\mathbb{S}} b\big(\nabla^{\mathbb{S}}b,\nabla^{\mathbb{S}}b\big)}{f^4}\Big) \\
&= \big(1+\eta_r^2+f^{-2}|\nabla^{\mathbb{S}}b|^2\big)^{-3/2}\Big(f^{-2}\big(-k+\Delta^{\mathbb{S}}b-k\eta_r^2+\eta_r^2\Delta^{\mathbb{S}}b\big) \\
&\quad +f^{-4}\Big(-k|\nabla^{\mathbb{S}}b|^2+|\nabla^{\mathbb{S}}b|^2\Delta^{\mathbb{S}}b-\operatorname{Hess}^{\mathbb{S}} b\big(\nabla^{\mathbb{S}}b,\nabla^{\mathbb{S}}b\big)\Big) \qquad (4.7)\\
&\quad -\eta_r^2\Big(k(n-1)\frac{f_r}{f^n}\int_1^r f(s)^{n-3}ds-kf^{-2}\Big)+\eta_r f_r|\nabla^{\mathbb{S}}b|^2f^{-3}\Big) \\
&= \big(1+\eta_r^2+f^{-2}|\nabla^{\mathbb{S}}b|^2\big)^{-3/2}\Big(f^{-2}\big(-k+\Delta^{\mathbb{S}}b+\eta_r^2\Delta^{\mathbb{S}}b\big) \\
&\quad +f^{-4}\Big(-k|\nabla^{\mathbb{S}}b|^2+|\nabla^{\mathbb{S}}b|^2\Delta^{\mathbb{S}}b-\operatorname{Hess}^{\mathbb{S}} b\big(\nabla^{\mathbb{S}}b,\nabla^{\mathbb{S}}b\big)\Big) \\
&\quad -\eta_r^2\Big(k(n-1)\frac{f_r}{f^n}\int_1^r f(s)^{n-3}ds\Big)+\eta_r f_r|\nabla^{\mathbb{S}}b|^2f^{-3}\Big) \\
&\le \big(1+\eta_r^2+f^{-2}|\nabla^{\mathbb{S}}b|^2\big)^{-3/2}\Big(f^{-2}\big(-k+\Delta^{\mathbb{S}}b+\eta_r^2\Delta^{\mathbb{S}}b\big) \\
&\quad +f^{-4}|\nabla^{\mathbb{S}}b|^2\big(-k+\Delta^{\mathbb{S}}b+|\operatorname{Hess}^{\mathbb{S}} b|\big) \\
&\quad -\eta_r^2\Big(k(n-1)\frac{f_r}{f^n}\int_1^r f(s)^{n-3}ds\Big)+\eta_r f_r|\nabla^{\mathbb{S}}b|^2f^{-3}\Big) \\
&\le 0
\end{aligned}$$

when we choose r large enough and then $k \ge ||b||_{C^2}$ large enough. Note that $\mathbb{S}^{n-1}$ is compact so $||b||_{C^2}$ is bounded. Then the computation above shows that

$$\operatorname{div}\frac{\nabla(\eta+B)}{\sqrt{1+|\nabla(\eta+B)|^2}} \le 0$$

for r and k large enough. In particular, $\eta+B$ is a supersolution to the minimal graph equation in $M\setminus B(o,r_0)$ for some r_0.

Choose k so that (4.7) holds and $\eta > 2\max|B|$ on the geodesic sphere $\partial B(o,r_0)$. Then $a := \min_{\partial B(o,r_0)}(\eta+B) > \max B$. Since $\eta(r)\to 0$ as $r\to\infty$, the function

$$w(x) := \begin{cases} \min\{(\eta+B)(x),a\} & \text{if } x\in M\setminus B(o,r_0);\\ a & \text{if } x\in B(o,r_0)\end{cases}$$

is continuous in $\bar M$ and coincide with b on $\partial_\infty M$. Moreover, w is a global upper barrier for the asymptotic Dirichlet problem with the boundary values b on $\partial_\infty M$. By replacing η with $-\eta$ we obtain the global lower barrier v,

$$v(x) := \begin{cases} \max\{(-\eta+B)(x),d\} & \text{if } x\in M\setminus B(o,r_0);\\ d & \text{if } x\in B(o,r_0),\end{cases}$$

where $d=\max_{\partial B(o,r_0)}(-\eta+B)$. Notice that $v\le B\le w$ by construction.

Next we solve the Dirichlet problem

$$\begin{cases} \operatorname{div} \dfrac{\nabla u_\ell}{\sqrt{1+|\nabla u_\ell|^2}} = 0 & \text{in } B(o,\ell); \\ u|\partial B(o,\ell) = B|\partial B(o,\ell) \end{cases}$$

in geodesic balls $B(o,\ell)$, with $\ell \geq r_0$. The existence of barrier functions implies that

$$v \leq u_\ell \leq w$$

on $\partial B(o,\ell)$ for all $\ell \geq r_0$. Hence, by the maximum principle, (u_ℓ) is a bounded sequence and we may apply gradient estimates in compact subsets of M to find a subsequence, still denoted by (u_ℓ), that converges uniformly on compact subsets in the C^2-norm to an entire solution u. The PDE regularity theory implies that $u \in C^\infty(M)$. Moreover, $v \leq u \leq w$ and hence it follows that u extends continuously to the boundary $\partial_\infty M$ and has the boundary values b.

Suppose then that $\theta \in C(\partial_\infty M)$. Again we interpret $\partial_\infty M$ as $\mathbb{S}^{n-1} \subset T_oM$. Let b_i be a sequence of smooth functions converging uniformly to θ. For each i, let $u_i \in C(\bar{M})$ be a solution to (1.1) in M with $u_i|\partial_\infty M = b_i$. Then the sequence (u_i) is uniformly bounded and consequently their gradients $|\nabla u_i|$ are uniformly bounded. By a diagonal argument we find a subsequence that converges locally uniformly with respect to C^2-norm to an entire C^∞-smooth solution u of (1.1) that is continuous in $\bar{M}$ with $u|\partial_\infty M = \theta$.

For the uniqueness, assume that u and $\tilde{u}$ are solutions to the minimal graph equation, continuous up to the boundary, and $u = \tilde{u}$ on $\partial_\infty M$. Assume that there exists $y \in M$ with $u(y) > \tilde{u}(y)$. Now denote $\delta = (u(y) - \tilde{u}(y))/2$ and let $U \subset \{x \in M : u(x) > \tilde{u}(x) + \delta\}$ be the component containing the point y. Since u and $\tilde{u}$ are continuous functions that coincides on the boundary $\partial_\infty M$, it follows that U is relatively compact open subset of M. Moreover, $u = \tilde{u} + \delta$ on ∂U, which implies $u = \tilde{u} + \delta$ in U. This is a contradiction since $y \in U$. □

In terms of the curvature bounds, we obtain the following corollary; see [27, Theorem 2] or the proof of Corollary 4.4.

Corollary 4.2. *Let M be a rotationally symmetric n-dimensional Cartan-Hadamard manifold whose radial sectional curvatures outside a compact set satisfy the upper bounds*

$$K(P_x) \leq -\frac{1+\varepsilon}{r(x)^2 \log r(x)}, \quad \text{if } n = 2 \tag{4.8}$$

and

$$K(P_x) \leq -\frac{1/2+\varepsilon}{r(x)^2 \log r(x)}, \quad \text{if } n \geq 3. \tag{4.9}$$

Then the asymptotic Dirichlet problem for the minimal graph equation (1.1) *is solvable with any continuous boundary data on $\partial_\infty M$. In particular, there are non-constant bounded entire solutions of* (1.1) *in M.*

Indeed, the radial curvature assumptions (4.8) and (4.9) imply the integral condition (4.3).

4.2. **p-Laplacian.** Similar approach works also for the p-Laplacian and we prove the following existence result for $p \in (2,n)$. The case $p = 2$ equals to the case of usual harmonic functions, which is already known, and the case $p \geq n$ is discussed in Section 5. The case $1 < p < 2$ remains open.

Theorem 4.3. *Let $p \in (2,n)$ and assume that*

$$\int_1^\infty \Big(f(s)^\beta \int_s^\infty f(t)^\alpha dt \Big) ds < \infty, \tag{4.10}$$

$\alpha = -(n-1)/(p-1)$ and $\beta = (n-2p+1)/(p-1)$, i.e. $\alpha+\beta = -2$. Then the asymptotic Dirichlet problem for the p-Laplacian is uniquely solvable for any continuous boundary data on $\partial_\infty M$, in particular, there exist entire non-constant bounded p-harmonic functions.

Proof. Again we interpret $\partial_\infty M$ as $\mathbb{S}^{n-1}$. Let $b\colon \mathbb{S}^{n-1} \to \mathbb{R}$ be a smooth non-constant function such that $|\operatorname{Hess}^{\mathbb{S}} b| < \varepsilon$, where $\varepsilon > 0$ will be specified later. Define $B\colon M\setminus\{o\} \to \mathbb{R}$, $B(\exp(r\vartheta)) = B(r,\vartheta) = b(\vartheta)$, $\vartheta \in \mathbb{S}^{n-1} \subset T_oM$. Similarly as in the proof of Theorem 4.1 we define a function

$$\eta(r) = \int_r^\infty f^\alpha(t) \int_1^t f^\beta(s)dsdt,$$

where α and β are constants to be determined later. We show that the function $\eta + B,\ \eta(x) := \eta\big(r(x)\big)$, is a supersolution for the p-Laplace equation, i.e.

$$\Delta_p(\eta+B) := \operatorname{div}\big(|\nabla(\eta+B)|^{p-2}\nabla(\eta+B)\big) \le 0.$$

Since $\eta_r < 0$ and $B_r = 0$, we have $|\nabla(\eta+B)| > 0$ in $M\setminus\{o\}$. First we compute

$$\begin{aligned}\Delta_p(\eta+B) &= \operatorname{div}(|\nabla(\eta+B)|^{p-2}\nabla(\eta+B)) \\ &= |\nabla(\eta+B)|^{p-2}\Delta(\eta+B) \\ &\quad + \frac{p-2}{2}|\nabla(\eta+B)|^{p-4}\left\langle \nabla(\eta+B), \nabla\big(|\nabla(\eta+B)|^2\big)\right\rangle \\ &= |\nabla(\eta+B)|^{p-4}\Bigg[\left(\eta_r^2 + \frac{|\nabla^{\mathbb{S}} b|^2}{f^2}\right)\left(\eta_{rr} + (n-1)\frac{f_r\eta_r}{f} + \frac{\Delta^{\mathbb{S}} b}{f^2}\right) \\ &\quad + (p-2)\left(\eta_r^2\eta_{rr} - \frac{\eta_r f_r |\nabla^{\mathbb{S}} b|^2}{f^3} + \frac{\operatorname{Hess}^{\mathbb{S}} b\big(\nabla^{\mathbb{S}} b, \nabla^{\mathbb{S}} b\big)}{f^4}\right)\Bigg].\end{aligned}$$

Since we are interested in the sign of $\Delta_p(\eta+B)$, we may just consider the term inside the brackets. Again, by straightforward computation, we obtain

$$\begin{aligned}\frac{\Delta_p(\eta+B)}{|\nabla(\eta+B)|^{p-4}} &= \left(\eta_r^2 + \frac{|\nabla^{\mathbb{S}} b|^2}{f^2}\right)\left(\eta_{rr} + (n-1)\frac{f_r\eta_r}{f} + \frac{\Delta^{\mathbb{S}} b}{f^2}\right) \\ &\quad + (p-2)\left(\eta_r^2\eta_{rr} - \frac{\eta_r f_r |\nabla^{\mathbb{S}} b|^2}{f^3} + \frac{\operatorname{Hess}^{\mathbb{S}} b\big(\nabla^{\mathbb{S}} b, \nabla^{\mathbb{S}} b\big)}{f^4}\right) \\ &= \eta_r^2\left((p-1)\eta_{rr} + (n-1)\frac{f_r\eta_r}{f} + \frac{\Delta^{\mathbb{S}} b}{f^2}\right) \\ &\quad + \frac{|\nabla^{\mathbb{S}} b|^2}{f^2}\left(\eta_{rr} + (n-p+1)\frac{f_r\eta_r}{f} + \frac{\Delta^{\mathbb{S}} b}{f^2}\right) + (p-2)f^{-4}\operatorname{Hess}^{\mathbb{S}} b\big(\nabla^{\mathbb{S}} b, \nabla^{\mathbb{S}} b\big) \\ &= \eta_r^2\left(-\big((p-1)\alpha + n - 1\big)f^{\alpha-1}f_r\int_1^r f^\beta(s)ds - (p-1)f^{\alpha+\beta} + \frac{\Delta^{\mathbb{S}} b}{f^2}\right) \\ &\quad + \frac{|\nabla^{\mathbb{S}} b|^2}{f^2}\left((\alpha+n-p+1)\frac{f_r\eta_r}{f} - f^{\alpha+\beta} + \frac{\Delta^{\mathbb{S}} b}{f^2}\right) \\ &\quad + (p-2)f^{-4}\operatorname{Hess}^{\mathbb{S}} b\big(\nabla^{\mathbb{S}} b, \nabla^{\mathbb{S}} b\big).\end{aligned}$$

Then choosing $\alpha = -(n-1)/(p-1)$ and $\beta = (n-2p+1)/(p-1)$, i.e. such that $\alpha+\beta=-2$, and recalling that $p \in (2,n)$ and $\eta_r < 0$ we see that

$$\begin{aligned}\frac{\Delta_p(\eta+B)}{|\nabla(\eta+B)|^{p-4}} &= \frac{\eta_r^2}{f^2}(\Delta^{\mathbb{S}} b - p + 1) + \frac{|\nabla^{\mathbb{S}} b|^2}{f^4}\left(\frac{(n-p)(p-2)ff_r\eta_r}{p-1} - 1 + \Delta^{\mathbb{S}} b\right)\\ &\quad + (p-2)f^{-4}\operatorname{Hess}^{\mathbb{S}} b\big(\nabla^{\mathbb{S}} b, \nabla^{\mathbb{S}} b\big)\\ &\le \frac{\eta_r^2}{f^2}\big(-p+1+\Delta^{\mathbb{S}} b\big)\\ &\quad + \frac{|\nabla^{\mathbb{S}} b|^2}{f^4}\left(\frac{(n-p)(p-2)ff_r\eta_r}{p-1} - 1 + \Delta^{\mathbb{S}} b + (p-2)|\operatorname{Hess}^{\mathbb{S}} b|\right)\\ &\le 0\end{aligned}$$

when $|\operatorname{Hess}^{\mathbb{S}} b| < \varepsilon$, with $\varepsilon > 0$ small enough, e.g. $\varepsilon < \min\big(\frac{p-1}{n-1}, \frac{1}{n+p-3}\big)$. Hence $\eta + B$ is a p-supersolution in $M\setminus\{o\}$. Similarly, we obtain an estimate $\Delta_p(-\eta+B) \ge 0$, and therefore $-\eta+B$ is a p-subsolution in $M\setminus\{o\}$. Notice that $k(\eta+B)$ is a p-supersolution in $M\setminus\{o\}$ for all $k\ge 0$ and similarly $k(-\eta+B)$ is a p-subsolution in $M\setminus\{o\}$. Hence the assumption $|\operatorname{Hess}^{\mathbb{S}} b| < \varepsilon$ is not a restriction. The asymptotic Dirichlet problem with any continuous boundary data $\varphi \in C(\partial_\infty N)$ can then be uniquely solved either by Perron's method with a suitable choice of the function b or approximating the given $\varphi \in C(\partial_\infty N)$ by functions $b_i \in C^\infty$. We omit the details and refer to [31] and [21]. □

In terms of curvature bounds, we obtain the following corollary.

Corollary 4.4. *Let M be a rotationally symmetric n-dimensional Cartan-Hadamard manifold, with $n \ge 3$, whose radial sectional curvatures outside a compact set satisfy*

$$K(P_x) \le -\frac{1/2+\varepsilon}{r(x)^2 \log r(x)}. \tag{4.11}$$

Then the asymptotic Dirichlet problem for the p-Laplacian, with $p \in (2,n)$, is uniquely solvable for any continuous boundary data on $\partial_\infty M$. In particular, there exist non-constant bounded p-harmonic functions on M.

Proof. It is enough to show that the curvature assumption (4.11) implies finiteness of the integral

$$\int_1^\infty \Big(f(s)^\beta \int_s^\infty f(t)^\alpha dt\Big) ds < \infty,$$

where $\alpha = -(n-1)/(p-1)$ and $\beta = (n-2p+1)/(p-1)$, i.e. $\alpha+\beta=-2$. Although this seems more complicated than the situation with (4.3), it is essentially the same because α and β are chosen so that $\alpha+\beta=-2$ which is same as the sum of the exponents in (4.3). For the sake of convenience, we give some details.

As in [27], define $\phi(r) = r(\log r)^c$, $c>0$. Choose $a>1$ such that $\phi'(a)>0$, $\phi''(a)>0$ and let $g(r) = (\phi(r+a)-\phi(a))/\phi'(a)$. Then $g(0)=0$, $g'(0)=1$, and $-g''(r)/g(r) \le 0$ behaves asymptotically as

$$-\frac{g''}{g}(r) \approx -\frac{\phi''}{\phi}(r) = -\frac{c}{r^2\log r}\left(1+\frac{c-1}{\log r}\right)$$

as $r\to\infty$. Applying [27, Lemma 5], we see that (4.10) is equivalent to the finiteness of the similar integral condition for g. Moreover, $g(r)$ behaves asymptotically as $\phi(r)$, so it is enough to show

$$\int_2^\infty \Big(\phi(s)^\beta \int_s^\infty \phi(t)^\alpha dt\Big) ds < \infty. \tag{4.12}$$

But $\int_s^\infty \phi(t)^\alpha dt$ behaves asymptotically as

$$\frac{s^{\alpha+1}(\log s)^{c\alpha}}{-\alpha-1},$$

and therefore

$$\phi(s)^\beta \int_s^\infty \phi(t)^\alpha dt \approx \frac{p-1}{n-p}\frac{1}{s(\log s)^{2c}}$$

as $s \to \infty$. Hence (4.12) holds if and only if $c > 1/2$. □

5. p-parabolicity when $p \geq n$

In this section we show that the upper bound $p < n$ in Theorem 4.3 cannot be improved. Namely there exist manifolds that satisfy the curvature assumption (4.10) and are p-parabolic when $p \geq n$. Recall that a Riemannian manifold N is called *p-parabolic*, $1 < p < \infty$, if

$$\operatorname{cap}_p(K, N) = 0$$

for every compact set $K \subset N$. Here the *p-capacity* of the pair (K, N) is defined as

$$\operatorname{cap}_p(K, N) = \inf_u \int_N |\nabla u|^p d\mu_0,$$

where the infimum is taken over all $u \in C_0^\infty(N)$, with $u|K \geq 1$. In [20, Proposition 1.7] it was shown that a complete Riemannian manifold is p-parabolic if

$$\int^\infty \left(\frac{t}{V(t)}\right)^{1/(p-1)} dt = \infty,$$

where $V(t) = \mu_0(B(o,t))$ and $o \in N$ is a fixed point. We apply this to get the following result.

Theorem 5.1. *Let $\alpha > 0$ be a constant and assume that M is a complete n-dimensional Riemannian manifold whose radial sectional curvatures satisfy*

$$K_M(P_x) \geq -\frac{\alpha}{r(x)^2 \log r(x)} \tag{5.1}$$

for every x outside some compact set and every 2-dimensional subspace $P_x \subset T_xM$ containing $\nabla r(x)$. Then M is p-parabolic

(a) *if $p = n$ and $0 < \alpha \leq 1$; or*
(b) *$p > n$ and $\alpha > 0$.*

Proof. Let $R > 1$ be so large that the curvature assumption (5.1) holds in $M \setminus B(o, R)$ and denote

$$B = \inf\left\{K_M(P_x) \colon x \in \bar{B}(o, R-1)\right\} > -\infty.$$

Let $k\colon [o,\infty) \to (-\infty, 0]$ be a smooth function that is constant in some neighborhood of 0, $k(t) \leq B$ for $t \in [0, R-1]$, $k(t) \leq -\alpha/(t^2\log t)$ for $t \in [R-1, R]$ and $k(t) = -\alpha/(t^2 \log t)$ for all $t \geq R$. Then the sectional curvatures of M are bounded from below by $k \circ r$. Applying the Bishop-Gromov volume comparison theorem we obtain

$$V(r) = \mu_0\big(B(o,r)\big) \leq Cr^n(\log r)^{\alpha(n-1)}$$

for some constant C and for $r \geq R$ large enough.

Consider first the case $p = n$. Then

$$\begin{aligned}\int_R^\infty \left(\frac{t}{V(t)}\right)^{1/(n-1)} dt &\geq c\int_R^\infty \left(\frac{t}{t^n(\log t)^{\alpha(n-1)}}\right)^{1/(n-1)} dt \\ &= c\int_R^\infty \frac{1}{t(\log t)^\alpha} dt = \infty\end{aligned}$$

if $0 < \alpha \leq 1$. This proves the first case. On the other hand, if $p > n$, we have $t^{n-1}(\log t)^{\alpha(n-1)} \leq t^{n-1}(\log t)^{\alpha(p-1)}$ and

$$\frac{t^{(n-1)/(p-1)}(\log t)^{\alpha}}{t} \longrightarrow 0$$

for any $\alpha > 0$ as $t \to \infty$, and hence

$$\begin{aligned} \int_R^\infty \left(\frac{t}{V(t)}\right)^{1/(p-1)} dt &\geq c \int_R^\infty \left(\frac{1}{t^{n-1}(\log t)^{\alpha(p-1)}}\right)^{1/(p-1)} dt \\ &= c \int^\infty \frac{1}{t^{(n-1)/(p-1)}(\log t)^{\alpha}} dt = \infty \end{aligned}$$

for any $\alpha > 0$. This proves the second case. □

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APPENDIX 5 – ARTICLE [E]

Asymptotic Dirichlet problems in warped products

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Preprint: arXiv:1801.04210 (2018)

ASYMPTOTIC DIRICHLET PROBLEMS IN WARPED PRODUCTS

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Abstract. We study the asymptotic Dirichlet problem for Killing graphs with prescribed mean curvature H in warped product manifolds $M \times_\varrho \mathbb{R}$. In the first part of the paper, we prove the existence of Killing graphs with prescribed boundary on geodesic balls under suitable assumptions on H and the mean curvature of the Killing cylinders over geodesic spheres. In the process we obtain a uniform interior gradient estimate improving previous results by Dajczer and de Lira. In the second part we solve the asymptotic Dirichlet problem in a large class of manifolds whose sectional curvatures are allowed to go to 0 or to $-\infty$ provided that H satisfies certain bounds with respect to the sectional curvatures of M and the norm of the Killing vector field. Finally we obtain non-existence results if the prescribed mean curvature function H grows too fast.

Contents

1. Introduction

Let N be a Riemannian manifold of the form $N = M \times_\varrho \mathbb{R}$, where M is a complete n-dimensional Riemannian manifold and $\varrho \in C^\infty(M)$ is a smooth (warping)

2000 *Mathematics Subject Classification.* Primary 58J32; Secondary 53C21.
Key words and phrases. Mean curvature equation, Killing graph, Dirichlet problem, Hadamard manifold, warped product.
J.-B.C. supported by MIS F.4508.14 (FNRS).
E.H. supported by Jenny and Antti Wihuri Foundation and CNPq.
I.H. supported by the Faculty of Science, University of Helsinki, and FUNCAP.
J.L. supported by CNPq and FUNCAP.

function. This means that the Riemannian metric $\bar{g}$ in N is of the form

$$\bar{g} = (\varrho \circ \pi_1)^2 \pi_2^* \,\mathrm{d}t^2 + \pi_1^* g, \tag{1.1}$$

where g denotes the Riemannian metric in M whereas t is the natural coordinate in $\mathbb{R}$ and $\pi_1 : M \times \mathbb{R} \to M$ and $\pi_2 : M \times \mathbb{R} \to \mathbb{R}$ are the standard projections. It follows that the coordinate vector field $X = \partial_t$ is a Killing field and that $\varrho = |X|$ on M. Since the norm of X is preserved along its flow lines, we may extend ϱ to a smooth function $\varrho = |X| \in C^\infty(N)$. From now on, we suppose that $\varrho > 0$ on M.

In this paper we study Killing graphs with prescribed mean curvature. Such graphs were introduced in [10], where the Dirichlet problem for prescribed mean curvature with $C^{2,\alpha}$ boundary values was solved in bounded domains $\Omega \subset M$ under hypothesis involving data on Ω and the Ricci curvature of the ambient space N. Recall that given a domain $\Omega \subset M$, the Killing graph of a C^2 function $u : \Omega \to \mathbb{R}$ is the hypersurface given by

$$\Sigma_u = \{(x, u(x)) : x \in \Omega\} \subset M \times \mathbb{R}.$$

In other words,

$$\Sigma_u = \{\Psi(x, u(x)) \colon x \in \Omega\},$$

where $\Psi \colon \Omega \times \mathbb{R} \to N$ is the flow generated by X. In [7] the Dirichlet problem was solved with merely continuous boundary data. Furthermore, the authors proved the existence and uniqueness of so-called radial graphs in the hyperbolic space $\mathbb{H}^{n+1}$ with prescribed mean curvature and asymptotic boundary data at infinity thus solving the asymptotic Dirichlet problem in $\mathbb{H}^n \times_{\cosh r} \mathbb{R}$. One of our goals in the current paper is to solve the asymptotic Dirichlet problem with prescribed mean curvature in a large class of negatively curved manifolds.

On the other hand, it is an interesting question under which conditions on a Riemannian manifold M every entire constant mean curvature graph over M is a slice, i.e. a graph of a constant function. The first such result is the celebrated theorem due to Bombieri, De Giorgi, and Miranda [3] that an entire minimal positive graph over $\mathbb{R}^n$ is a totally geodesic slice. Their result was extended by Rosenberg, Schulze, and Spruck [16] to a complete Riemannian manifold M with nonnegative Ricci curvature and the sectional curvature bounded from below by a negative constant. Ding, Jost, and Xin considered in [11] complete, noncompact Riemannian manifolds with nonnegative Ricci curvature, Euclidean volume growth, and quadratic decay of the curvature tensor. They proved that an entire minimal graph over such a manifold M must be a slice if its height function has at most linear growth on one side unless M is isometric to Euclidean space. In the recent paper [4] Casteras, Heinonen, and Holopainen showed that a minimal positive graph over a complete Riemannian manifold with asymptotically nonnegative sectional curvature and only one end is a slice if its height function has at most linear growth. Entire Killing graphs in $M \times_\varrho \mathbb{R}$ with constant mean curvature were studied in [8] and [9]. In particular, it was shown in [8] that a bounded entire Killing graph of constant mean curvature must be a slice if $\mathrm{Ric}_M \geq 0$, $K_M \geq -K_0$ for some $K_0 \geq 0$, and if $\varrho \geq \varrho_0 > 0$, with $||\varrho||_{C^2(M)} < \infty$.

Our current paper is inspired by the above mentioned research [10], [7], [8], and [9] on Killing graphs with prescribed mean curvature as well as by the recent paper [5]. In the latter, the asymptotic Dirichlet problem for f-minimal graphs in Cartan-Hadamard manifolds M has been studied. Recall that f-minimal hypersurfaces are natural generalizations of self-shrinkers which play a crucial role in the study of mean curvature flow. Moreover, they are minimal hypersurfaces of weighted manifolds $M_f = \big(M, g, e^{-f}\mathrm{d}\,\mathrm{vol}_M\big)$, where (M, g) is a complete Riemannian manifold with the Riemannian volume element $\mathrm{d}\,\mathrm{vol}_M$.

Returning to the Killing graph Σ_u of a function u, we note that the induced metric in Σ_u has components

$$g_{ij} + \varrho^2(x) u_i u_j, \tag{1.2}$$

where g_{ij} are local components of the metric g. The induced volume element in Σ_u (or equivalently, on the domain $\Omega \subset M$) is given by

$$\mathrm{d}\Sigma = \varrho\sqrt{\varrho^{-2} + |\nabla u|^2}\,\mathrm{d}M.$$

We consider the constrained area functional

$$\mathcal{A}_H[u] = \int_\Omega \varrho\sqrt{\varrho^{-2} + |\nabla u|^2}\,\mathrm{d}M + \mathcal{V}_H[u],$$

where

$$\mathcal{V}_H[u] = \int_\Omega \int_0^{u(x)\varrho(x)} nH\mathrm{d}M = \int_\Omega nH\varrho u\,\mathrm{d}M$$

and H is a smooth function on Ω. Given an arbitrary compactly supported function $v \in C_0^\infty(\Omega)$ we have the first variation formula

$$\delta\mathcal{A}_H[u]\cdot v = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\mathcal{A}_H[u+sv] = -\int_\Omega \Big(\mathrm{div}\Big(\frac{\nabla u}{W}\Big) + \Big\langle \nabla \log\varrho, \frac{\nabla u}{W}\Big\rangle - nH\Big) v\varrho\,\mathrm{d}M,$$

where

$$W = \sqrt{\varrho^{-2} + |\nabla u|^2}$$

and the differential operators ∇ and div are taken with respect to the metric g in M. Then the Euler-Lagrange equation of this functional is

$$\mathrm{div}\Big(\frac{\nabla u}{W}\Big) + \Big\langle \nabla\log\varrho, \frac{\nabla u}{W}\Big\rangle = nH \tag{1.3}$$

and $H(x)$ is the mean curvature of the graph $\Sigma_u \subset M \times_\varrho \mathbb{R}$ at $(x, u(x))$. The equation (1.3) can be rewritten as

$$\mathrm{div}_{-\log\varrho}\Big(\frac{\nabla u}{W}\Big) = nH,$$

where the weighted divergence operator corresponding to a smooth density function $f \in C^\infty(M)$ is defined by

$$\mathrm{div}_f Z = e^f\,\mathrm{div}(e^{-f}Z) = \mathrm{div}Z - \langle \nabla f, Z\rangle.$$

Note that this is the divergence-form operator that fits well with the weighted measure $\varrho\,\mathrm{d}M$ in the sense that a suitable version of the divergence theorem is still valid in this context. Reasoning another way around, since Σ is oriented by the normal vector field

$$N = \frac{1}{W}\big(\varrho^{-2}X - \nabla u|_{(x,u(x))}\big)$$

and

$$\Big\langle \nabla\log\varrho, \frac{\nabla u}{W}\Big\rangle = -\langle\bar\nabla\log\varrho, N\rangle,$$

where $\bar\nabla$ is the Riemannian connection in N, we can interpret

$$H_{\log\varrho} = H + \frac{1}{n}\langle\nabla\log\varrho, N\rangle$$

as a weighted mean curvature of the submanifold Σ_u in the *Riemannian* product $M\times\mathbb{R}$ in the sense that the Euler-Lagrange PDE may be rewritten as

$$\mathrm{div}\Big(\frac{\nabla u}{W}\Big) = nH_{\log\varrho}.$$

More generally, if f is an arbitrary density in M we consider a weighted area functional of the form

$$\mathcal{A}_{H,f}[u] = \int_\Omega e^{-f}\varrho\sqrt{\varrho^{-2}+|\nabla u|^2}\,\mathrm{d}M + \int_\Omega nHe^{-f}\varrho u\,\mathrm{d}M.$$

In this case, the Euler-Lagrange equation is

$$\mathrm{div}_f\Big(\frac{\nabla u}{W}\Big) + \Big\langle \nabla\log\varrho, \frac{\nabla u}{W}\Big\rangle = nH. \tag{1.4}$$

As before, this equation may be rewritten either in terms of a modified weighted divergence

$$\mathrm{div}_{f-\log\varrho}\Big(\frac{\nabla u}{W}\Big) = nH$$

or as a prescribed weighted mean curvature equation

$$\mathrm{div}_f\Big(\frac{\nabla u}{W}\Big) := \mathrm{div}\Big(\frac{\nabla u}{W}\Big) + \langle\bar\nabla f, N\rangle = nH_{\log\varrho}.$$

For the time being, we restrict ourselves to the case where $f=0$. Intrinsically, given a hypersurface $\Sigma\subset N$ and denoting $u = t|_\Sigma$, the parametric counterpart of (1.3) is

$$\Delta_\Sigma u = nH\langle N,\partial_t\rangle - 2\langle\nabla^\Sigma\log\varrho, \nabla^\Sigma u\rangle, \tag{1.5}$$

where Δ_Σ is the Laplace-Beltrami operator in Σ. Indeed if ∇^Σ denotes the intrinsic covariant derivative in Σ, we have

$$\nabla^\Sigma u = (\bar\nabla t)^T = \varrho^{-2}\partial_t^T,$$

where T denotes tangential projection onto $T\Sigma$. Hence we obtain

$$\Delta_\Sigma u = nH\varrho^{-2}\langle\partial_t, N\rangle + \langle\nabla^\Sigma\varrho^{-2}, \partial_t^T\rangle,$$

from where the formula (1.5) above follows.

In particular, minimal graphs in $N = M\times_\varrho\mathbb{R}$ have height function that satisfies the weighted harmonic equation

$$\Delta_\Sigma u + 2\langle\nabla^\Sigma\log\varrho, \nabla^\Sigma u\rangle = 0. \tag{1.6}$$

This may be considered as a PDE in Ω if we replace the metric g by the induced metric with components given by (1.2).

Denoting

$$\sigma^{ij} = g^{ij} - \frac{u^iu^j}{W^2}$$

we can write (1.3) in non-divergence form as

$$\sigma^{ij}u_{i;j} + (\log\varrho)^i u_i\left(1+\frac{1}{\varrho^2W^2}\right) = nHW. \tag{1.7}$$

2. Main results

The existence of Killing graphs with prescribed mean curvature H over bounded domains $\Omega\subset M$ with continuous boundary data on $\partial\Omega$ was established in [7, Theorem 2] under suitable conditions on the Ricci curvature on Ω, the mean curvature function H, and on the mean curvature of the Killing cylinder over $\partial\Omega$; see also [10].

In this paper we mainly focus on the setting where M is a Cartan-Hadamard manifold with sectional curvatures controlled from above and below by some radial functions. We prove *quantitative* a priori height and gradient estimates for solutions of (1.3) on geodesic balls $\Omega = B(o,k)\subset M$ under natural conditions on the prescribed mean curvature function in terms of sectional curvatures K_M and the warping function ϱ. These estimates allow us to use the continuity method (the Leray-Schauder method) and hence are enough to guarantee the existence of solutions to the following Dirichlet problem

$$\begin{cases} \operatorname{div}\big(\frac{\nabla u}{W}\big) + \big\langle \nabla \log \varrho, \frac{\nabla u}{W} \big\rangle = nH & \text{in} \quad \Omega \\ u|\partial\Omega = \varphi & \text{in} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where $\varphi \in C(\partial\Omega)$. We formulate the (local) existence result in geodesic balls on Cartan-Hadamard manifolds.

Theorem 2.1. *Let M be a Cartan-Hadamard manifold, $\Omega = B(o,k) \subset M$, and $\varphi \in C(\partial\Omega)$. Suppose that the prescribed mean curvature function $H \in C^\alpha(\Omega)$ satisfies*

$$|H(x)| < H_{k-d(x)}$$

in $\bar\Omega$, where $d(x) = \operatorname{dist}\big(x, \partial B(o,k)\big) = k - r(x)$ and H_{k-d} is the mean curvature of the Killing cylinder $\mathcal{C}_{k-d}$ over the geodesic sphere $\partial B(o, k-d)$. Then there exists a unique solution $u \in C^{2,\alpha}(\Omega) \cap C(\bar\Omega)$ to (2.1).

Above and in what follows we denote by $r(x) = d(x,o)$ the distance from x to a fixed point $o \in M$. We notice that the mean curvature of the Killing cylinder $\mathcal{C}_r$ over a geodesic sphere $\partial B(o,r)$ is given by

$$H_r = \frac{1}{n}\left(\Delta r + \frac{1}{\varrho}\langle \nabla\varrho, \nabla r\rangle\right)$$

and therefore can be estimated from below in terms of a suitable model manifold $M_{-a^2(r)} \times_{\varrho_+} \mathbb{R}$, where $M_{-a^2(r)}$ is a rotationally symmetric Cartan-Hadamard manifold with radial sectional curvatures equal to $-a^2(r)$ and $\varrho_+ \colon M \to (0,\infty)$ is a positive rotationally symmetric C^1 function such that

$$\frac{1}{\varrho}\langle \nabla\varrho, \nabla r\rangle = \frac{\partial_r \varrho}{\varrho} \ge \frac{\partial_r \varrho_+}{\varrho_+}. \tag{2.2}$$

To formulate the next corollary and for later purposes we denote by $f_\kappa \in C^\infty([0,\infty))$ the solution of the Jacobi equation

$$\begin{cases} f_\kappa'' - \kappa^2 f_\kappa = 0 \\ f_\kappa(0) = 0 \\ f_\kappa'(0) = 1, \end{cases} \tag{2.3}$$

whenever $\kappa\colon [0,\infty) \to [0,\infty)$ is a smooth function.

Corollary 2.2. *Let M be a Cartan-Hadamard manifold whose radial sectional curvatures are bounded from above by*

$$K(P_x) \le -a\big(r(x)\big)^2$$

for some smooth function $a\colon [0,\infty) \to [0,\infty)$. Suppose, moreover, that (2.2) *holds with some positive rotationally symmetric C^1 function $\varrho_+ = \varrho_+(r)$. If the prescribed mean curvature function $H \in C^\alpha(\Omega)$, $\Omega = B(o,k)$, satisfies*

$$n|H(x)| < \frac{(n-1) f_a'\big(r(x)\big)}{f_a\big(r(x)\big)} + \frac{\varrho_+'\big(r(x)\big)}{\varrho_+\big(r(x)\big)}$$

for all $x \in \bar\Omega$, then there exists a unique solution $u \in C^{2,\alpha}(\Omega) \cap C(\bar\Omega)$ to (2.1).

As mentioned above the proofs of Theorem 2.1 and Corollary 2.2 for boundary data $\varphi \in C^{2,\alpha}(\partial\Omega)$ follow from the well-known continuity method once the a priori height and gradient estimates are at our disposal. The case of a continuous boundary values $\varphi \in C(\partial\Omega)$ can be treated as in [7]; see also [5].

Our main object in this paper is the asymptotic Dirichlet problem for Killing graphs with prescribed mean curvature and behaviour at infinity. To solve the problem, we extend the given boundary value function $\varphi \in C(\partial_\infty M)$ to a continuous function $\varphi \in C(\bar{M})$; see Section 5 for the notation. Then we apply Corollary 2.2 for an exhaustion $\Omega_k = B(o,k),\ k \in \mathbb{N}$, of M to obtain a sequence of solutions u_k with boundary values $u_k|\partial\Omega_k = \varphi$. Under a suitable bound on $|H|$ in terms of a comparison manifold $M_{-a^2(r)} \times_{\varrho_+} \mathbb{R}$ we obtain a global height estimate and, consequently together with Schauder estimates, the sequence is uniformly bounded in the $C^{2,\alpha}$-norm. Hence there exists a subsequence that converges in the $C^{2,\alpha}$-norm to a global solution u to the equation

$$\operatorname{div}\Big(\frac{\nabla u}{W}\Big) + \Big\langle \nabla \log \varrho, \frac{\nabla u}{W}\Big\rangle = nH$$

in M. Finally, under suitable curvature upper and lower bounds as well as conditions on $|H|$ we are able to construct (local) barriers at infinity and prove that the solution u extends continuously to $\partial_\infty M$ and attains the given boundary values φ there.

The following two solvability theorems will be proven in Section 6.

Theorem 2.3. *Let M be a Cartan-Hadamard manifold satisfying the curvature assumptions* (5.1) *and* (A1)–(A7) *in Section 5. Furthermore, assume that the prescribed mean curvature function $H\colon M \to \mathbb{R}$ satisfies the assumptions* (4.17) *and* (5.7) *with a convex warping function ϱ satisfying* (4.12), (4.13), (5.8), *and* (5.9). *Then there exists a unique solution $u\colon M \to \mathbb{R}$ to the Dirichlet problem*

$$\begin{cases} \operatorname{div}_{-\log\varrho} \dfrac{\nabla u}{\sqrt{\varrho^{-2} + |\nabla u|^2}} = nH(x) & \text{in } M \\ u|\partial_\infty M = \varphi \end{cases} \tag{2.4}$$

for any continuous function $\varphi\colon \partial_\infty M \to \mathbb{R}$.

Theorem 2.4. *Let M be a Cartan-Hadamard manifold satisfying the curvature assumptions* (5.1) *and* (A1)–(A7) *in Section 5. Furthermore, assume that the prescribed mean curvature function $H\colon M \to \mathbb{R}$ satisfies the assumptions* (4.24) *and* (5.7) *with a convex warping function ϱ satisfying* (4.18), (5.8), *and* (5.9). *Then there exists a unique solution $u\colon M \to \mathbb{R}$ to the Dirichlet problem* (2.4) *for any continuous function $\varphi\colon \partial_\infty M \to \mathbb{R}$.*

3. A priori height and gradient estimates

Throughout this section we denote by $\Omega_k = B(o,k)$ the geodesic ball centered at a given point $o \in M$ with radius $k \in \mathbb{N}$, and by $d(\cdot) = \operatorname{dist}(\cdot, \partial\Omega_k)$ the distance function to the boundary of Ω_k.

3.1. Height estimate.

Fix $k \in \mathbb{N}$ and suppose that $u_k \in C^2(\Omega_k)$ is a solution of the Dirichlet problem (2.1). We aim to show that the function

$$v_k(x) = \sup_{\partial\Omega_k} \varphi_k + h(d(x)), \tag{3.1}$$

where h will be determined later, is an upper barrier for the solution u_k. It suffices to show (see [17, p. 795] or [10, pp. 239-240]) that v_k is a barrier in an open neighbourhood of $\partial\Omega_k$ in which the points can be joined to $\partial\Omega_k$ by unique geodesics. In this neighbourhood the distance function d has the same regularity as $\partial\Omega_k$ and therefore the derivatives of d in the following computations are well-defined.

Since X is Killing field, we have

$$\langle \nabla \log \varrho, \nabla d\rangle = \frac{1}{\varrho}\langle \nabla \varrho, \nabla d\rangle = -\frac{1}{\varrho^2}\langle \bar{\nabla}_X X, \nabla d\rangle =: -\kappa(d), \tag{3.2}$$

where κ is the principal curvature of the Killing cylinder $\mathcal{C}_{k-d}$ over the geodesic sphere $\partial B(o, k-d)$. This implies that

$$\begin{aligned}\mathcal{Q}[v_k] &= \operatorname{div}\left(\frac{h'\nabla d}{\sqrt{\varrho^{-2}+h'^2}}\right) - \kappa\frac{h'}{\sqrt{\varrho^{-2}+h'^2}} \\ &= \frac{h'}{\sqrt{\varrho^{-2}+h'^2}}(\Delta d - \kappa) + \partial_d\left(\frac{h'}{\sqrt{\varrho^{-2}+h'^2}}\right),\end{aligned}$$

where ∂_d denotes the derivative to the direction ∇d. However,

$$\Delta d - \kappa = -nH_{k-d},$$

where H_{k-d} is the mean curvature of the cylinder $\mathcal{C}_{k-d}$, and we have

$$\begin{aligned}\partial_d\left(\frac{h'}{\sqrt{\varrho^{-2}+h'^2}}\right) &= \frac{h''}{\sqrt{\varrho^{-2}+h'^2}} - \frac{h'}{(\varrho^{-2}+h'^2)^{3/2}}(\varrho^{-2}\kappa + h'h'') \\ &= \frac{\varrho^{-2}}{(\varrho^{-2}+h'^2)^{3/2}}(h'' - \kappa h').\end{aligned}$$

Hence it follows that

$$\mathcal{Q}[v_k] = -\frac{h'}{\sqrt{\varrho^{-2}+h'^2}}nH_{k-d} + \frac{\varrho^{-2}}{(\varrho^{-2}+h'^2)^{3/2}}(h'' - \kappa h').$$

Suppose that the principal curvature of the Killing cylinder $\mathcal{C}_{k-d}$ satisfies

$$\kappa(d) \geq -\frac{\varrho_0'(d)}{\varrho_0(d)},$$

where ϱ_0 is a smooth positive increasing function on $[0,\infty)$. We note already at this point that, in the case of Cartan-Hadamard manifolds, $\nabla d = -\nabla r$ and this agrees with the assumption (4.12). Then define the function h as

$$h(d) = C\int_0^d \varrho_0^{-1}(t)\mathrm{d}t \tag{3.3}$$

for some constant $C > 0$ to be fixed later. Now, since $h' > 0$, we have

$$h'' - \kappa h' \leq 0$$

and

$$\mathcal{Q}[v_k] \leq -\frac{h'}{\sqrt{\varrho^{-2}+h'^2}}nH_{k-d} = -\frac{C\varrho}{\sqrt{\varrho_0^2 + C^2\varrho^2}}nH_{k-d}.$$

Assuming that

$$|H| < H_{k-d}$$

in $\bar{\Omega}_k$ and choosing the constant C as

$$C^2 > \frac{H^2/H_{k-d}^2}{1 - H^2/H_{k-d}^2}\frac{\sup \varrho_0^2}{\inf \varrho^2}$$

we see that

$$\mathcal{Q}[v_k] - nH \leq 0$$

and hence v_k is an upper barrier for u_k.

Similarly we see that the function

$$v_k^- = \inf_{\partial\Omega_k} \varphi_k - h(d)$$

is a lower barrier for u_k and together these barriers give the following height estimate.

Lemma 3.1. *Assume that*

$$|H| < H_{k-d} \tag{3.4}$$

in $\bar{\Omega}_k$ *and that* u_k *is a solution to the Dirichlet problem* (2.1)*. Then there exists a constant* $C = C(\Omega_k)$ *such that*

$$\sup_{\Omega_k} |u_k| \le C + \sup_{\partial\Omega_k} |\varphi|.$$

3.2. **Boundary gradient estimate.** For given $\varepsilon > 0$ we define an annulus

$$U_k(\varepsilon) = \{x \in \Omega_k \colon d(x) < \varepsilon\}.$$

In order to obtain a boundary gradient estimate, we aim to show that a function of the form

$$w(x) = g(d(x)) + \psi(x)$$

is an upper barrier in the set $U_k(\varepsilon)$ for a fixed $\varepsilon \in (0, 1/2)$ chosen so that d is smooth in $U_k(\varepsilon)$. Here we denote by ψ the extension of the boundary data that is constant along geodesics issuing perpendicularly from $\partial\Omega_k$, i.e. $\psi(\exp_y t\nabla d(y)) = \varphi(y)$, where $y \in \partial\Omega_k$ and $\nabla d(y)$ is the unit inward normal to $\partial\Omega_k$ at y. From (1.7) we have that

$$\mathcal{Q}[w] = \frac{1}{W}\Delta w - \frac{1}{W^3}\langle \nabla_{\nabla w}\nabla w, \nabla w\rangle - \frac{1}{\varrho^2}\left(1 + \frac{1}{\varrho^2 W^2}\right)\left\langle \bar{\nabla}_X X, \frac{\nabla w}{W}\right\rangle, \tag{3.5}$$

where

$$W = \sqrt{\varrho^{-2} + |\nabla w|^2} = \sqrt{\varrho^{-2} + g'^2 + |\nabla\psi|^2}.$$

Since

$$\nabla w = g'\nabla d + \nabla\psi,$$

with $\langle \nabla d, \nabla\psi\rangle = 0$, it follows that

$$\Delta w = g'\Delta d + g'' + \Delta\psi$$

and

$$\langle \nabla_{\nabla w}\nabla w, \nabla w\rangle = g'^2 g'' - g'\langle \nabla_{\nabla\psi}\nabla d, \nabla\psi\rangle + \langle \nabla_{\nabla\psi}\nabla\psi, \nabla\psi\rangle.$$

Moreover, by (3.2)

$$\langle \nabla w, \bar{\nabla}_X X\rangle = g'\langle \nabla d, \bar{\nabla}_X X\rangle + \langle \nabla\psi, \bar{\nabla}_X X\rangle = g'\varrho^2\kappa + \langle \nabla\psi, \bar{\nabla}_X X\rangle.$$

Using the expression (3.5) we obtain that

$$\begin{aligned}\mathcal{Q}[w] = &\frac{1}{W}(g'' + g'\Delta d + \Delta\psi) - \frac{1}{W^3}(g'^2 g'' - g'\langle \nabla_{\nabla\psi}\nabla d, \nabla\psi\rangle + \langle \nabla_{\nabla\psi}\nabla\psi, \nabla\psi\rangle)\\ &- \frac{1}{W}\left(1 + \frac{1}{\varrho^2 W^2}\right)\left(g'\kappa + \left\langle \bar{\nabla}_{\frac{X}{|X|}}\frac{X}{|X|}, \nabla\psi\right\rangle\right)\end{aligned}$$

and combining with the previous reasoning, this results to

$$\begin{aligned}\mathcal{Q}[w] = &\frac{g''}{W^3}(\varrho^{-2} + |\nabla\psi|^2) + \frac{g'}{W}\left(\Delta d - \left(1 + \frac{1}{\varrho^2 W^2}\right)\kappa\right) + \frac{1}{W}\Delta\psi\\ &- \frac{1}{W^3}\langle \nabla_{\nabla\psi}\nabla\psi, \nabla\psi\rangle + \frac{g'}{W^3}\langle \nabla_{\nabla\psi}\nabla d, \nabla\psi\rangle\\ &- \frac{1}{W}\left(1 + \frac{1}{\varrho^2 W^2}\right)\left\langle \bar{\nabla}_{\frac{X}{|X|}}\frac{X}{|X|}, \nabla\psi\right\rangle.\end{aligned}$$

We note that

$$\frac{1}{W}\Delta\psi - \frac{1}{W^3}\langle \nabla_{\nabla\psi}\nabla\psi, \nabla\psi\rangle = \frac{1}{W}\left(g^{ij} - \frac{\psi^i\psi^j}{W^2}\right)\psi_{i;j}$$

and, on the other hand,

$$\frac{1}{W}\left(g^{ij}-\frac{(g'd^i+\psi^i)(g'd^j+\psi^j)}{W^2}\right)\psi_{i;j}=\frac{1}{W}\left(g^{ij}-\frac{\psi^i\psi^j}{W^2}\right)\psi_{i;j}-\frac{2g'd^i\psi^j}{W^3}\psi_{i;j}$$
$$=\frac{1}{W}\left(g^{ij}-\frac{\psi^i\psi^j}{W^2}\right)\psi_{i;j}-\frac{2g'}{W^3}\left\langle\nabla_{\nabla d}\nabla\psi,\nabla\psi\right\rangle.$$

Moreover, the matrix (σ^{ij}) has eigenvalues $1/(\varrho^2W^3)$ and $1/W$ which can be estimated as

$$\max\left(\frac{1}{\varrho^2W^3},\frac{1}{W}\right)\le\frac{1}{\varrho^2W^3}+\frac{1}{W}\le\frac{1}{W}(1+\varrho^2). \tag{3.6}$$

When $\varrho\ge 1$, this is trivial, and when $\varrho<1$ we can choose the constant K in the definition (3.7) of g such that this holds. Therefore we are able to estimate

$$\begin{aligned}\mathcal{Q}[w]&\le\frac{g''}{W^3}(\varrho^{-2}+|\nabla\psi|^2)+\frac{g'}{W}\left(\Delta d-\left(1+\frac{1}{\varrho^2W^2}\right)\kappa\right)+\frac{1}{W}(1+\varrho^2)||\nabla^2\psi||\\&\quad+\frac{3g'}{W^3}\langle\nabla_{\nabla\psi}\nabla d,\nabla\psi\rangle+\frac{1}{W}\left(1+\frac{1}{\varrho^2W^2}\right)\langle\nabla\log\varrho,\nabla\psi\rangle\\&\le\frac{g''}{W^3}(\varrho^{-2}+|\nabla\psi|^2)-\frac{g'}{W}\left(nH_{k-d}+\frac{1}{\varrho^2W^2}\kappa\right)+\frac{1}{W}(1+\varrho^2)||\nabla^2\psi||\\&\quad+\frac{3g'}{W^3}|II_{k-d}(\nabla\psi,\nabla\psi)|+\frac{1}{W}\left(1+\frac{1}{\varrho^2W^2}\right)\langle\nabla\log\varrho,\nabla\psi\rangle.\end{aligned}$$

Now we choose

$$g(d)=\frac{C}{\log(1+K)}\log(1+Kd), \tag{3.7}$$

where

$$C=K(1+K\varepsilon)\log(1+K)$$

and $K\ge(1-2\varepsilon)\varepsilon^{-2}$ so large that

$$C\ge 2\big(\max_{\bar\Omega_k}|u_k|+\max_{\bar\Omega_k}|\psi|\big). \tag{3.8}$$

Note that this choice yields $g\ge u_k$ on the "inner" boundary $\{x\in\Omega_k\colon d(x)=\varepsilon\}$ of $U_k(\varepsilon)$. Then, for K large, we have

$$1\ge\frac{g'^2}{W^2}\ge\frac{K^4(1+K\varepsilon)^2}{(1+Kd)^2L+K^4(1+K\varepsilon)^2}=\frac{(1+K\varepsilon)^2}{\frac{(1+Kd)^2}{K^4}L+(1+K\varepsilon)^2}\ge c_1^2>0,$$

where

$$c_1^2\le\frac{(1+K\varepsilon)^2}{(1+K\varepsilon)^2+L}$$

with

$$L=\sup_{\Omega}(\varrho^{-2}+|\nabla\psi|^2).$$

We also have

$$g''(d)=-\frac{g'^2}{K(1+K\varepsilon)},$$

which implies that

$$\frac{g''}{W^2}=-\frac{1}{K(1+K\varepsilon)}\frac{g'^2}{W^2}\le-\frac{1}{K(1+K\varepsilon)}c_1^2.$$

Hence we obtain

$$\begin{aligned}\mathcal{Q}[w]&\le-\frac{1}{K(1+K\varepsilon)}c_1^2\frac{1}{W}(\varrho^{-2}+|\nabla\psi|^2)-c_1\left(nH_{k-d}+\frac{1}{\varrho^2W^2}\kappa\right)\\&\quad+\frac{1}{W}(1+\varrho^2)||\nabla^2\psi||+\frac{g'}{W}\frac{|\nabla\psi|^2}{W^2}||II_{k-d}||+\frac{|\nabla\psi|}{W}|\nabla\log\varrho|\left(1+\frac{1}{\varrho^2W^2}\right).\end{aligned}$$

However,

$$\frac{1}{\varrho^2 W^2} \geq \frac{c_1^2}{\varrho^2 g'^2} \geq \left(\frac{1+Kd}{K^2(1+K\varepsilon)}\right)^2 \frac{c_1^2}{\varrho^2} \geq \frac{1}{K^4(1+K\varepsilon)^2}\frac{c_1^2}{\varrho^2}$$

and

$$\frac{1}{W} \geq \frac{1+Kd}{K^2(1+K\varepsilon)} c_1 \geq \frac{1}{K^2(1+K\varepsilon)} c_1.$$

Combining these with the fact that $W \geq K^2$, we obtain the estimate

$$\begin{aligned}
\mathcal{Q}[w] &\leq -c_1 \frac{K+\kappa}{K^4(1+K\varepsilon)^2}\frac{c_1^2}{\varrho^2} - c_1 n H_{k-d} + \frac{1}{W}(1+\varrho^2)||\nabla^2\psi|| \\
&\quad + \frac{g'}{W}\frac{|\nabla\psi|^2}{W^2}||II_{k-d}|| + \left(1 + \frac{1}{\varrho^2}\frac{1}{K^2}\right)\frac{|\nabla\psi|}{W}|\nabla \log \varrho| \\
&\leq -c_1 \frac{K+\kappa}{K^4(1+K\varepsilon)^2}\frac{c_1^2}{\varrho^2} - c_1 n H_{k-d} + \frac{1}{K^2}(1+\varrho^2)||\nabla^2\psi|| \\
&\quad + |\nabla\psi|^2 ||II_{k-d}||\frac{1}{K^4} + \left(1 + \frac{1}{\varrho^2 K^4}\right)|\nabla\psi||\nabla \log \varrho|\frac{1}{K^2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathcal{Q}[w] - nH \leq -n(c_1 H_{k-d} + H) - c_1 \frac{K+\kappa}{K^4(1+K\varepsilon)^2}\frac{c_1^2}{\varrho^2} + \frac{1}{K^2}(1+\varrho^2)||\nabla^2\psi|| \\
+ |\nabla\psi|^2 ||II_{k-d}||\frac{1}{K^4} + \left(1 + \frac{1}{\varrho^2 K^4}\right)|\nabla\psi||\nabla \log \varrho|\frac{1}{K^2}. \qquad (3.9)
\end{aligned}$$

Finally observe that

$$c_1 H_{k-d} \geq |H|$$

if we choose K such that

$$\frac{H^2}{H_{k-d}^2} \leq c_1^2 \leq \frac{(1+K\varepsilon)^2}{(1+K\varepsilon)^2 + L},$$

that is

$$(1+K\varepsilon)^2 \geq L \frac{H^2/H_{k-d}^2}{1 - H^2/H_{k-d}^2}. \qquad (3.10)$$

Taking (3.6), (3.8), (3.9) and (3.10) into account, we can choose

$$K = K(\Omega_k, H, ||\psi||_{C^2}, \varepsilon, \sup_{\Omega_k} |u|)$$

so large that

$$\mathcal{Q}[w] - nH \leq 0$$

holds in $U_k(\varepsilon)$. This suffices for the following boundary gradient estimate.

Lemma 3.2. *Assume that*

$$|H| < H_{k-d}$$

in $\bar{\Omega}_k$ *and that* u_k *is a solution to the Dirichlet problem* (2.1)*. Then there exists a constant* $C = C(\Omega_k, H, ||\psi||_{C^2}, \varepsilon, \sup_{\Omega_k} |u|)$ *such that*

$$\max_{\partial\Omega_k} |\nabla u_k| \leq C.$$

3.3. **Interior gradient estimate.** In this subsection we prove a quantitative interior gradient estimate that is interesting on its own. The proof is based on the technique due to Korevaar and Simon [13], and further developed by Wang [18]. We will perform the computations in a coordinate free way.

Let u be a (C^3-smooth) positive solution of the equation (1.3) in a ball $B(p,R)\subset M$. Suppose that sectional curvatures in $B(p,R)$ are bounded from below by $-K_0^2$ for some constant $K_0 = K_0(p,R)\ge 0$. We consider a nonnegative and smooth function η with $\eta=0$ in $M\setminus B(p,R)$ and define a function χ in $B(p,R)$ of the form

$$\chi = \eta\gamma(u)\psi(|\nabla u|^2), \tag{3.11}$$

where the functions η, γ and ψ will be specified later.

Suppose that χ attains its maximum at $x_0\in B(p,R)$, and without loss of generality, that $\eta(x_0)\neq 0$. Then at x_0

$$(\log\chi)_j = \frac{\eta_j}{\eta}+\frac{\gamma'}{\gamma}u_j+2\frac{\psi'}{\psi}u^k u_{k;j}=0 \tag{3.12}$$

and therefore

$$2\frac{\psi'}{\psi}u^k u_{k;j} = -\left(\frac{\eta_j}{\eta}+\frac{\gamma'}{\gamma}u_j\right). \tag{3.13}$$

Moreover, the matrix

$$\begin{aligned}(\log\chi)_{i;j} = (\log\eta)_{i;j} + \left(\frac{\gamma'}{\gamma}\right)' u_i u_j + \frac{\gamma'}{\gamma}u_{i;j} + 2\frac{\psi'}{\psi}(u^k u_{k;ij}+u^k_{;i}u_{k;j})\\ + 4\left(\frac{\psi'}{\psi}\right)' u^k u_{k;i}u^\ell u_{\ell;j}\end{aligned}$$

is non-positive at x_0. Applying the Ricci identities for the Hessian of u we have

$$u_{k;ij} = u_{i;kj} = u_{i;jk} + R^\ell_{kji}u_\ell,$$

and this yields

$$\begin{aligned}(\log\chi)_{i;j} &= \frac{\eta_{i;j}}{\eta} + \frac{\gamma''}{\gamma}u_i u_j + \frac{\gamma'}{\gamma}u_{i;j} + \frac{\gamma'}{\gamma}\left(\frac{\eta_i}{\eta}u_j + \frac{\eta_j}{\eta}u_i\right)\\ &+ 2\frac{\psi'}{\psi}(u^k u_{i;jk} + u^k_{;i}u_{k;j}) - 2\frac{\psi'}{\psi}R^\ell_{jki}u^k u_\ell + 4\left(\left(\frac{\psi'}{\psi}\right)' - \frac{\psi'^2}{\psi^2}\right)u^k u_{k;i}u^\ell u_{\ell;j}.\end{aligned}$$

On the other hand, denoting

$$f(x) = nHW - \langle\nabla\log\varrho,\nabla u\rangle\left(1+\frac{1}{\varrho^2W^2}\right),$$

and differentiating both sides in (1.7) we have

$$\sigma^{ij}u_{i;jk} = f_k - \sigma^{ij}_{;k}u_{i;j}. \tag{3.14}$$

Contracting (3.14) with u^k, we get

$$\begin{aligned}\sigma^{ij}u^k u_{i;jk} = f_k u^k + \frac{1}{W^2}u^k(u^i_{;k}u^j + u^i u^j_{;k})u_{i;j}\\ - \frac{2}{W^4}u^i u^j u_{i;j}(-\varrho^{-2}(\log\varrho)_k u^k + u^k u^\ell u_{\ell;k}).\end{aligned}$$

Using the previous identity, (3.13) and noticing that

$$\sigma^{ij}R^\ell_{jki}u^k u_\ell = -\mathrm{Ric}_g(\nabla u,\nabla u),$$

lenghty computations give

$$\begin{aligned}
0 \geq \sigma^{ij}(\log\chi)_{i;j} &= 2n\frac{\psi'}{\psi}\langle\nabla H,\nabla u\rangle W + nH\frac{\gamma'}{\gamma}\frac{1}{\varrho^2 W} - nH\frac{1}{W}\left\langle\frac{\nabla\eta}{\eta},\nabla u\right\rangle \\
&- 2nH\frac{1}{\varrho^2 W}\frac{\psi'}{\psi}\langle\nabla\log\varrho,\nabla u\rangle + 4\left(\left(\frac{\psi'}{\psi}\right)' - \frac{\psi'^2}{\psi^2} + \frac{3}{2}\frac{\psi'}{\psi}\frac{1}{W^2}\right)\sigma^{i\ell}u^j u^k u_{k;i}u_{j;\ell} \\
&+ \sigma^{ij}\frac{\eta_{i;j}}{\eta} + 2\frac{\gamma'}{\gamma}\frac{1}{\varrho^2 W^2}\left\langle\frac{\nabla\eta}{\eta},\nabla u\right\rangle + 2\frac{\psi'}{\psi}\big(\mathrm{Ric}_g(\nabla u,\nabla u) - \nabla^2\log\varrho(\nabla u,\nabla u)\big) \\
&+ \frac{4|\nabla u|^2}{\varrho^2 W^4}\frac{\psi'}{\psi}\langle\nabla\log\varrho,\nabla u\rangle^2 - \frac{2}{\varrho^2 W^4}\langle\nabla\log\varrho,\nabla u\rangle\left\langle\frac{\nabla\eta}{\eta} + \frac{\gamma'}{\gamma}\nabla u,\nabla u\right\rangle \\
&- 2\frac{\psi'}{\psi}\frac{1}{\varrho^2 W^2}\nabla^2\log\varrho(\nabla u,\nabla u) + \left\langle\nabla\log\varrho,\frac{\nabla\eta}{\eta}\right\rangle\left(1 + \frac{1}{\varrho^2 W^2}\right) \\
&- \frac{2}{\varrho^2 W^4}\left\langle\frac{\nabla\eta}{\eta} + \frac{\gamma'}{\gamma}\nabla u,\nabla u\right\rangle\langle\nabla\log\varrho,\nabla u\rangle + \frac{\gamma''}{\gamma}\sigma^{ij}u_i u_j + 2\frac{\psi'}{\psi}\sigma^{i\ell}\sigma^{jk}u_{k;i}u_{j;\ell}.
\end{aligned}$$

Notice that (3.13) yields to

$$\begin{aligned}
4\frac{\psi'^2}{\psi^2}\sigma^{i\ell}u^j u^k u_{k;i}u_{j;\ell} &= \left|\frac{\nabla\eta}{\eta} + \frac{\gamma'}{\gamma}\nabla u\right|_\sigma^2 \geq \frac{1}{\varrho^2 W^2}\left|\frac{\nabla\eta}{\eta} + \frac{\gamma'}{\gamma}\nabla u\right|_g^2 \\
&= \frac{|\nabla u|^2}{\varrho^2 W^2}\left|\frac{\nabla\eta}{|\nabla u|\eta} + \frac{\gamma'}{\gamma}\frac{\nabla u}{|\nabla u|}\right|_g^2.
\end{aligned}$$

Plugging this into the previous estimate, we get

$$\begin{aligned}
&\frac{\psi^2}{\psi'^2}\left(\left(\frac{\psi'}{\psi}\right)' - \frac{\psi'^2}{\psi^2} + \frac{3}{2}\frac{\psi'}{\psi}\frac{1}{W^2}\right)\frac{|\nabla u|^2}{\varrho^2 W^2}\left|\frac{\nabla\eta}{|\nabla u|\eta} + \frac{\gamma'}{\gamma}\frac{\nabla u}{|\nabla u|}\right|_g^2 \\
&\qquad + \frac{\gamma''}{\gamma}\sigma^{ij}u_i u_j + 2\frac{\psi'}{\psi}\sigma^{i\ell}\sigma^{jk}u_{k;i}u_{j;\ell} \\
&\leq 2n\frac{\psi'}{\psi}|\nabla H||\nabla u|W + 2n\frac{\psi'}{\psi}|H|\frac{|\nabla u|}{W}\frac{|\nabla\log\varrho|}{\varrho^2} \\
&- 2\frac{\psi'}{\psi}\big(\mathrm{Ric}_g(\nabla u,\nabla u) - \nabla^2\log\varrho(\nabla u,\nabla u)\big) + 4\frac{\psi'}{\psi}\frac{|\nabla\log\varrho|^2}{\varrho^2}\frac{|\nabla u|^4}{W^4} \\
&+ 2\frac{\psi'}{\psi}\frac{|\nabla^2\log\varrho|}{\varrho^2}\frac{|\nabla u|^2}{W^2} + n|H|\frac{\gamma'}{\gamma}\frac{1}{\varrho^2 W} + n|H|\left|\frac{\nabla\eta}{\eta}\right|\frac{|\nabla u|}{W} - \sigma^{ij}\frac{\eta_{i;j}}{\eta} \\
&+ 2\frac{\gamma'}{\gamma}\frac{1}{\varrho^2 W}\frac{|\nabla u|}{W}\left|\frac{\nabla\eta}{\eta}\right| + 4\frac{|\nabla\log\varrho|}{\varrho^2}\left(\left|\frac{\nabla\eta}{\eta}\right|\frac{|\nabla u|^2}{W^4} + \frac{\gamma'}{\gamma}\frac{|\nabla u|^3}{W^4}\right) \\
&+ |\nabla\log\varrho|\left|\frac{\nabla\eta}{\eta}\right|\left(1 + \frac{1}{\varrho^2 W^2}\right).
\end{aligned}$$

Suppose that $|\nabla u|(x_0) > 1$. Otherwise we are done. Hence, following [18], we set

$$\psi(t) = \log t, \tag{3.15}$$

where $t = |\nabla u|^2$. Then we have

$$\frac{|\nabla u|^2}{W^2}\frac{\psi^2}{\psi'^2}\left(\left(\frac{\psi'}{\psi}\right)' - \frac{\psi'^2}{\psi^2} + \frac{3}{2}\frac{\psi'}{\psi}\frac{1}{W^2}\right) = \frac{t}{W^2}\left(\log t\,\frac{\frac{1}{2}t - \varrho^{-2}}{t + \varrho^{-2}} - 2\right).$$

Now we fix a constant

$$\max\left\{\frac{2}{3},\frac{\varrho^2}{1+\varrho^2}\right\} < \beta < 1$$

and suppose that

$$\frac{t}{W^2} = \frac{|\nabla u|^2}{W^2} \geq \beta. \tag{3.16}$$

Setting $\frac{1}{\varrho^2}\frac{\beta}{1-\beta} =: e^{\delta'}$, $\delta = \frac{3}{2}\beta - 1$, and $\mu := 2\beta\frac{\delta\delta'-2}{\delta'}$, we get

$$
\begin{aligned}
&\mu \log|\nabla u| \frac{1}{\varrho^2}\left|\frac{\nabla\eta}{|\nabla u|\eta} + \frac{\gamma'}{\gamma}\frac{\nabla u}{|\nabla u|}\right|_g^2 + \frac{\gamma''}{\gamma}\frac{|\nabla u|^2}{\varrho^2 W^2}\\
&\qquad + 2\frac{\psi'}{\psi}|\nabla u|^2\bigg(\mathrm{Ric}_g\left(\frac{\nabla u}{|\nabla u|},\frac{\nabla u}{|\nabla u|}\right) - \nabla^2\log\varrho\bigg(\frac{\nabla u}{|\nabla u|},\frac{\nabla u}{|\nabla u|}\bigg)\bigg)\\
&\le \frac{2}{\sqrt{\beta}\delta'}\big(n|\nabla H| + (1-\beta)n|H||\nabla\log\varrho| + 2(1-\beta)|\nabla\log\varrho|^2 + (1-\beta)|\nabla^2\log\varrho|\big)\\
&\quad + \sqrt{1-\beta}\frac{1}{\varrho}\frac{\gamma'}{\gamma}\big(n|H| + 4|\nabla\log\varrho|\big) + 2\sqrt{1-\beta}\frac{1}{\varrho}\frac{\gamma'}{\gamma}\left|\frac{\nabla\eta}{\eta}\right|\\
&\quad + \left|\frac{\nabla\eta}{\eta}\right|\big(n|H| + (6-5\beta)|\nabla\log\varrho|\big) - \sigma^{ij}\frac{\eta_{i;j}}{\eta}.
\end{aligned}
$$

By modifying the argument in [16, Proof of Theorem 4.1, Case 2] we may assume that the maximum point x_0 is not in the cut-locus $C(p)$ of p. Then we choose η as

$$\eta = \hat\eta^2 \tag{3.17}$$

where

$$\hat\eta = 1 - \frac{1}{C_R}\int_0^r \xi(\tau)\,\mathrm{d}\tau, \quad r = d(\cdot,p), \tag{3.18}$$

with

$$C_R = \int_0^R \xi(\tau)\,\mathrm{d}\tau$$

and $\xi(\tau) = K_0^{-1}\sinh(K_0\tau)$ if $K_0 > 0$ and $\xi(\tau) = \tau$ if $K_0 = 0$. Denoting

$$\kappa = \varrho^{-2}\langle\bar\nabla_X\bar\nabla r, X\rangle = \langle\nabla r, \nabla\log\varrho\rangle,$$

one can show that $|\nabla\eta| = 2\hat\eta\frac{\xi(r)}{C_R}$ and

$$
\begin{aligned}
\Delta_\Sigma\eta &= 2\hat\eta\Delta_\Sigma\hat\eta + 2|\nabla^\Sigma\hat\eta|^2\\
&\le 2\hat\eta(r)\frac{\xi(r)}{C_R}\left|(n-1)\frac{\xi'(r)}{\xi(r)} + \kappa + n|H| + (1-\beta)\left|\frac{\xi'(r)}{\xi(r)} - \kappa\right|\right| + 2\frac{\xi^2(r)}{C_R^2}.
\end{aligned}
$$

As in [18], we set

$$\gamma(u) = 1 + \frac{1}{M}\big(\min_{\bar B(p,R)}\varrho\big)u$$

where $M > 0$ is a constant to be fixed later. Then $\gamma'' = 0$ and hence

$$
\begin{aligned}
&\mu\log|\nabla u|\frac{1}{\varrho^2}\left|\frac{\nabla\eta}{|\nabla u|\eta} + \frac{\gamma'}{\gamma}\frac{\nabla u}{|\nabla u|}\right|_g^2\\
&\quad + 2\frac{\psi'}{\psi}|\nabla u|^2\bigg(\mathrm{Ric}_g\left(\frac{\nabla u}{|\nabla u|},\frac{\nabla u}{|\nabla u|}\right) - \nabla^2\log\varrho\bigg(\frac{\nabla u}{|\nabla u|},\frac{\nabla u}{|\nabla u|}\bigg)\bigg) \le \widetilde{M}\frac{1}{M\eta},
\end{aligned}
\tag{3.19}
$$

where

$$
\begin{aligned}
\widetilde{M} &= \frac{2}{\sqrt{\beta}\delta'}\big(n|\nabla H| + (1-\beta)n|H||\nabla\log\varrho| + 2(1-\beta)|\nabla\log\varrho|^2\\
&\quad + (1-\beta)|\nabla^2\log\varrho|\big)M\eta + \sqrt{1-\beta}\big(n|H| + 4|\nabla\log\varrho|\big)\eta + 4\sqrt{1-\beta}\frac{\xi(r)}{C_R}\hat\eta\\
&\quad + 2\frac{\xi(r)}{C_R}\big(n|H| + (6-5\beta)|\nabla\log\varrho|\big)M\hat\eta\\
&\quad + M\bigg(2\hat\eta(r)\frac{\xi(r)}{C_R}\left|(n-1)\frac{\xi'(r)}{\xi(r)} + \kappa + n|H| + (1-\beta)\left|\frac{\xi'(r)}{\xi(r)} - \kappa\right|\right| + 2\frac{\xi^2(r)}{C_R^2}\bigg).
\end{aligned}
\tag{3.20}
$$

Let $L = L(p,R) \geq 0$ be chosen in such a way that

$$\mathrm{Ric}_g + \nabla^2 \log \varrho \geq -Lg \tag{3.21}$$

in $B(p,R)$. Then we obtain

$$\mu \log |\nabla u| \frac{1}{\varrho^2} \left| \frac{\nabla \eta}{|\nabla u| \eta} + \frac{\gamma'}{\gamma} \frac{\nabla u}{|\nabla u|} \right|_g^2 - 2L \frac{1}{\delta'} \leq \widetilde{M} \frac{1}{M\eta}.$$

Set $M = \max_{\bar{B}(p,R)} u$. We consider first the case

$$\left| \frac{\nabla \eta}{|\nabla u| \eta} \right| \leq \frac{\gamma'}{2\gamma}.$$

Then we have

$$\eta \log |\nabla u| \leq \frac{4\gamma^2 \varrho^2}{\mu \min_{\bar{B}(p,R)} \varrho^2} \left(\widetilde{M} M + 2LM^2 \frac{\eta}{\delta'} \right).$$

On the other hand, when

$$\frac{\gamma'}{2\gamma} \leq \left| \frac{\nabla \eta}{|\nabla u| \eta} \right|$$

we have

$$\eta |\nabla u| \leq \frac{4\gamma}{\gamma'} \frac{\xi(r)}{C_R}.$$

which implies that

$$\eta \log |\nabla u| \leq \frac{4\gamma}{\gamma'} \frac{\xi(r)}{C_R}.$$

Hence at x_0

$$\eta \log |\nabla u| \leq \max \left\{ \frac{4\gamma(u(x_0))\xi(r(x_0))}{\gamma'(u(x_0)) C_R}, \frac{4\gamma^2(u(x_0))\varrho^2(x_0)}{\mu \min_{\bar{B}(p,R)} \varrho^2} \left(\widetilde{M} M + 2LM^2 \frac{1}{\delta'} \right) \right\}. \tag{3.22}$$

Since $\eta(p) = 1$ and $\gamma(p) \geq 1$ we conclude that

$$\begin{aligned} \log |\nabla u(p)| &\leq \eta(p)\gamma(p) \log |\nabla u(p)| \leq \eta(x_0)\gamma(x_0) \log |\nabla u(x_0)| \qquad (3.23) \\ &\leq \frac{4M(1+\min_{\bar{B}(p,R)} \varrho)^2}{\min_{\bar{B}(p,R)} \varrho} \max \left\{ \frac{\xi(r(x_0))}{C_R}, \frac{(1+\min_{\bar{B}(p,R)} \varrho)\varrho^2(x_0)}{\mu \min_{\bar{B}(p,R)} \varrho} \left(\widetilde{M} + 2LM \frac{1}{\delta'} \right) \right\} \end{aligned}$$

unless $|\nabla u(x_0)| \leq 1$.

We have proven the following quantitative gradient estimate. Here we denote by $\mathcal{R}_B$ the Riemannian curvature tensor in a set B.

Lemma 3.3. *Let u be a positive solution of (1.3) in an open set Ω and let $B = B(p,R) \subset \Omega$. Then there exists a constant $C = C(\mathcal{R}_B, \varrho|B, H|B, u(p), \max_{\bar{B}} u, R)$ such that*

$$|\nabla u(p)| \leq C.$$

If the gradient of u is continuous up to the boundary of Ω and Ω is bounded, we obtain the following quantitative global estimate.

Lemma 3.4. *Let u be a positive solution of (1.3) in a bounded open set Ω and suppose, moreover, that $u \in C^1(\bar{\Omega})$. Then there exists a constant*

$$C = C(\mathcal{R}_\Omega, \varrho|\Omega, H|\Omega, u(p), \max_{\bar{\Omega}} u, \mathrm{diam}(\Omega), \max_{\partial\Omega} |\nabla u|)$$

such that

$$|\nabla u(p)| \leq C$$

for every $p \in \bar{\Omega}$.

Proof. Let $p \in \Omega$ and $R = \mathrm{diam}(\Omega)$. Define in $\bar{\Omega} \cap B(p,R)$ a function

$$\chi = \eta\gamma(u)\psi(|\nabla u|^2),$$

where η, γ, and ψ are as in the previous proof. If χ attains its maximum in an interior point $x_0 \in B(p,R) \cap \Omega$, the proof of Lemma 3.3 applies and we have a desired upper bound. Otherwise, χ attains its maximum at $x_0 \in \partial\Omega$, but then $|\nabla u(x_0)| \le \max_{\partial\Omega} |\nabla u|$ and again we are done. □

We remark that a global gradient estimate for bounded Killing graphs follows immediately from (3.23), (3.20), and (3.21) in the case of bounded warping functions under some assumptions on the curvature.

Corollary 3.5. *Suppose that the sectional curvatures in M satisfy $K_M \ge -K_0$ for some positive constant K_0. Suppose also that $\inf_M \varrho > 0$ and that $||\varrho||_{C^2(M)} < +\infty$. If a function $u : M \to \mathbb{R}$ is uniformly bounded and the mean curvature of its graph satisfies $||H||_{C^1(M)} < +\infty$ then the gradient of u is uniformly bounded.*

4. Global barriers

In this section we present two methods to obtain global (upper and lower) barriers for solutions to (2.1).

In the case when $\widetilde{H}$ is constant along flow lines of X, that is, when $\widetilde{H}$ is a function in M, there is a conservation law (a flux formula) corresponding to the invariance of $\mathcal{A}_{\widetilde{H}}$ with respect to the flow generated by X. This flux formula for graphs is stated as

$$\int_\Gamma \Big\langle \frac{\nabla u}{W}, \nu \Big\rangle \varrho \,\mathrm{d}\Gamma = \int_\Omega n\widetilde{H} \varrho \,\mathrm{d}M, \tag{4.1}$$

where $\Gamma = \partial\Omega$ and ν is the outward unit normal vector field along $\Gamma \subset M$.

Suppose for a while that M is a model manifold with respect to a fixed pole $o \in M$ and that $\varrho = |X|$ is a radial function. In terms of polar coordinates $(r, \vartheta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$ centered at o the metric in M is of the form

$$g = \mathrm{d}r^2 + \xi^2(r)\,\mathrm{d}\vartheta^2,$$

where $\mathrm{d}\vartheta^2$ stands for the canonical metric in $\mathbb{S}^{n-1}$. Suppose that $\widetilde{H}$ and u are also radial functions. Applying (4.1) to $\Omega = B(o,r)$, the geodesic ball centered at o with radius r, we obtain

$$\frac{u'(r)}{\sqrt{\varrho^{-2}(r) + u'^2(r)}} \varrho(r)\xi^{n-1}(r) = \int_0^r n\widetilde{H}(\tau)\varrho(\tau)\xi^{n-1}(\tau)\,\mathrm{d}\tau \tag{4.2}$$

This is a first integral of (1.3) in this rotationally invariant setting. Indeed, taking derivatives on both sides of (4.2) with respect to r we get

$$n\widetilde{H}(r) = \bigg(\frac{u'(r)}{\sqrt{\varrho^{-2}(r) + u'^2(r)}}\bigg)' + \frac{u'(r)}{\sqrt{\varrho^{-2}(r) + u'^2(r)}}\bigg(\frac{\varrho'(r)}{\varrho(r)} + (n-1)\frac{\xi'(r)}{\xi(r)}\bigg).$$

On the other hand in this particular setting (1.3) becomes

$$\begin{aligned} n\widetilde{H}(r) &= \mathrm{div}\bigg(\frac{u'(r)}{\sqrt{\varrho^{-2}(r) + u'^2(r)}}\partial_r\bigg) + \bigg(\frac{u'(r)}{\sqrt{\varrho^{-2}(r) + u'^2(r)}}\bigg)\frac{\varrho'(r)}{\varrho(r)} \\ &= \bigg(\frac{u'(r)}{\sqrt{\varrho^{-2}(r) + u'^2(r)}}\bigg)' + \frac{u'(r)}{\sqrt{\varrho^{-2}(r) + u'^2(r)}}\,\mathrm{div}\,\partial_r \\ &\quad + \bigg(\frac{u'(r)}{\sqrt{\varrho^{-2}(r) + u'^2(r)}}\bigg)\frac{\varrho'(r)}{\varrho(r)} \\ &= \bigg(\frac{u'(r)}{\sqrt{\varrho^{-2}(r) + u'^2(r)}}\bigg)' + \frac{u'(r)}{\sqrt{\varrho^{-2}(r) + u'^2(r)}}\bigg((n-1)\frac{\xi'(r)}{\xi(r)} + \frac{\varrho'(r)}{\varrho(r)}\bigg). \end{aligned}$$

It is convenient to write (4.2) in a "quadrature" form as follows

$$u'^2(r) = \frac{I^2(r)\varrho^{-2}(r)}{\varrho^2(r)\xi^{2(n-1)}(r) - I^2(r)}, \tag{4.3}$$

where

$$I(r) = \int_0^r n\widetilde{H}(\tau)\varrho(\tau)\xi^{n-1}(\tau)\,\mathrm{d}\tau.$$

For instance, in the case when $\widetilde{H}$ is constant we have to impose a condition such as

$$n|\widetilde{H}| \le \liminf_{r\to\infty} \frac{\varrho(r)\xi^{n-1}(r)}{\int_0^r \varrho(\tau)\xi^{n-1}(\tau)\,\mathrm{d}\tau} \tag{4.4}$$

in order to guarantee the existence of radial solutions $u = u(r)$ to (1.3) for model manifolds. Note that the right-hand side in (4.4) is a sort of weighted isoperimetric ratio in M with respect to the density $\varrho(r(x)) = |X(x)|$. By l'Hospital's rule we see that (4.4) is equivalent to the requirement

$$n|\widetilde{H}| \le \liminf_{r\to\infty} (n-1)\frac{\xi'(r)}{\xi(r)} + \frac{\varrho'(r)}{\varrho(r)}. \tag{4.5}$$

This discussion motivates us to define in the general case a function of the form

$$\begin{aligned} u_+(x) &= u_+\big(r(x)\big) \\ &= \int_{r(x)}^{+\infty} \frac{\int_0^\tau n\widetilde{H}(s)\varrho_+(s)\xi_+^{n-1}(s)\,\mathrm{d}s}{\varrho_+(\tau)\sqrt{\varrho_+^2(\tau)\xi_+^{2(n-1)}(\tau) - \big(\int_0^\tau n\widetilde{H}(s)\varrho_+(s)\xi_+^{n-1}(s)\,\mathrm{d}s\big)^2}}\,\mathrm{d}\tau \quad (4.6)\\ &\quad + ||\varphi||_{C^0(\partial_\infty M)} \quad (4.7) \end{aligned}$$

for some nonnegative functions $\varrho_+(r(x))$, $\xi_+(r(x))$ and $\widetilde{H}(r(x))$ to be chosen later.

Plugging $u_+(x) = u_+(r(x))$ into the differential operator

$$\mathcal{Q}[u] = \operatorname{div}\Big(\frac{\nabla u}{W}\Big) + \Big\langle \nabla \log\varrho, \frac{\nabla u}{W}\Big\rangle - nH$$

yields

$$\begin{aligned} \mathcal{Q}[u_+] &= \Big\langle \nabla \frac{u_+'(r)}{(\varrho^{-2}(x) + u_+'^2(r))^{1/2}}, \partial_r \Big\rangle \\ &\quad + \frac{u_+'(r)}{(\varrho^{-2}(x) + u_+'^2(r))^{1/2}}\Big(\operatorname{div}\partial_r + \frac{1}{\varrho}\langle\nabla\varrho,\partial_r\rangle\Big) - nH \\ &= \partial_r\Big(\frac{u_+'(r)}{(\varrho^{-2}(x) + u_+'^2(r))^{1/2}}\Big) + \frac{u_+'(r)}{(\varrho^{-2}(x) + u_+'^2(r))^{1/2}}\Big(\Delta r + \frac{1}{\varrho}\langle\nabla\varrho,\partial_r\rangle\Big) - nH \\ &= \partial_r\Big(\frac{u_+'(r)}{(\varrho_+^{-2}(r) + u_+'^2(r))^{1/2}}\frac{(\varrho_+^{-2}(r) + u_+'^2(r))^{1/2}}{(\varrho^{-2}(x) + u_+'^2(r))^{1/2}}\Big) \\ &\quad + \frac{u_+'(r)}{(\varrho^{-2}(x) + u_+'^2(r))^{1/2}}\Big(\Delta r + \frac{1}{\varrho}\langle\nabla\varrho,\partial_r\rangle\Big) - nH \\ &= \frac{(\varrho_+^{-2}(r) + u_+'^2(r))^{1/2}}{(\varrho^{-2}(x) + u_+'^2(r))^{1/2}}\Bigg[\frac{u_+'(r)}{(\varrho_+^{-2}(r) + u_+'^2(r))^{1/2}}\Big(\Delta r + \frac{1}{\varrho}\langle\nabla\varrho,\partial_r\rangle\Big) \\ &\quad + \partial_r\Big(\frac{u_+'(r)}{(\varrho_+^{-2}(r) + u_+'^2(r))^{1/2}}\Big)\Bigg] \\ &\quad + \frac{u_+'(r)}{(\varrho_+^{-2}(r) + u_+'^2(r))^{1/2}}\partial_r\Big(\frac{(\varrho_+^{-2}(r) + u_+'^2(r))^{1/2}}{(\varrho^{-2}(x) + u_+'^2(r))^{1/2}}\Big) - nH. \end{aligned}$$

Moreover, suppose that

$$\frac{\partial_r \varrho(x)}{\varrho(x)} \geq \frac{\varrho_+'\big(r(x)\big)}{\varrho_+\big(r(x)\big)} \tag{4.8}$$

for some positive and increasing C^1-function $\varrho_+\colon [0,\infty) \to (0,\infty)$ such that $\varrho_+(0) = \varrho(o)$. By our choice of u_+,

$$u_+'(r) = -\frac{\int_0^r n\widetilde{H}(s)\varrho_+(s)\xi_+^{n-1}(s)\,\mathrm{d}s}{\varrho_+(r)\sqrt{\varrho_+^2(r)\xi_+^{2(n-1)}(r) - \big(\int_0^r n\widetilde{H}(s)\varrho_+(s)\xi_+^{n-1}(s)\,\mathrm{d}s\big)^2}},$$

and therefore

$$-n\widetilde{H} = \left(\frac{u_+'(r)}{(\varrho_+^{-2}(r) + u_+'^2(r))^{1/2}}\right)' + \frac{u_+'(r)}{(\varrho_+^{-2}(r) + u_+'^2(r))^{1/2}}\left(\frac{\varrho_+'(r)}{\varrho_+(r)} + (n-1)\frac{\xi_+'(r)}{\xi_+(r)}\right).$$

Hence we obtain

$$\begin{aligned}
\mathfrak{Q}[u_+] &= \frac{(\varrho_+^{-2}(r) + u'^2_+(r))^{1/2}}{(\varrho^{-2}(x) + u'^2_+(r))^{1/2}}\left[\frac{u_+'(r)}{(\varrho_+^{-2}(r) + u'^2_+(r))^{1/2}}\left(\Delta r + \frac{1}{\varrho}\langle\nabla\varrho, \partial_r\rangle\right)\right.\\
&\quad \left. + \partial_r\left(\frac{u_+'(r)}{(\varrho_+^{-2}(r) + u'^2_+(r))^{1/2}}\right)\right] + \frac{u_+'(r)}{(\varrho_+^{-2}(r) + u'^2_+(r))^{1/2}}\partial_r\left(\frac{(\varrho_+^{-2}(r) + u'^2_+(r))^{1/2}}{(\varrho^{-2}(x) + u'^2_+(r))^{1/2}}\right) - nH\\
&\leq -\frac{(\varrho_+^{-2}(r) + u'^2_+(r))^{1/2}}{(\varrho^{-2}(x) + u'^2_+(r))^{1/2}}n\widetilde{H} + \frac{u_+'(r)}{(\varrho_+^{-2}(r) + u'^2_+(r))^{1/2}}\partial_r\left(\frac{(\varrho_+^{-2}(r) + u'^2_+(r))^{1/2}}{(\varrho^{-2}(x) + u'^2_+(r))^{1/2}}\right) - nH.
\end{aligned}$$

In order to prove that u_+ is indeed an upper barrier we next check that

$$\partial_r\left(\frac{(\varrho_+^{-2}(r) + u'^2_+(r))^{1/2}}{(\varrho^{-2}(x) + u'^2_+(r))^{1/2}}\right) \geq 0. \tag{4.9}$$

Note that $u_+' \leq 0$. We observe that

$$\frac{\partial}{\partial r}\left(\sqrt{\frac{\varrho_+(r)^{-2} + (u_+'(r))^2}{\varrho(x)^{-2} + (u_+'(r))^2}}\right) \geq 0$$

if and only if

$$(\varrho_+^{-2} + (u_+')^2)\Big(\frac{\partial_r\varrho}{\varrho^3} - u_+'u_+''\Big) \geq (\varrho^{-2} + (u_+')^2)\Big(\frac{\varrho_+'}{\varrho_+^3} - u_+'u_+''\Big). \tag{4.10}$$

But now integrating (4.8) we get

$$\log\varrho(x) \geq \log\varrho_+\big(r(x)\big)$$

which implies

$$\frac{1}{\varrho(x)} \leq \frac{1}{\varrho_+\big(r(x)\big)}$$

and furthermore assuming

$$\frac{\partial_r\varrho(x)}{\varrho(x)^3} \geq \frac{\varrho_+'\big(r(x)\big)}{\varrho_+\big(r(x)\big)^3}$$

we see that (4.10) holds.

Therefore we are left to show that

$$-nH \leq \sqrt{\frac{\varrho_+^{-2}(r) + u'^2_+(r)}{\varrho^{-2}(x) + u'^2_+(r)}}\,n\widetilde{H}.$$

The conditions (4.4) and (4.5) in our mind, we choose $\widetilde{H}$ as

$$n\widetilde{H}(r) = (1-\varepsilon)\left(\frac{\varrho_+'(r)}{\varrho_+(r)} + (n-1)\frac{\xi_+'(r)}{\xi_+(r)}\right)$$

with some $\varepsilon \in (0,1)$. Note that then

$$\int_0^r n\widetilde{H}(s)\varrho_+(s)\xi_+^{n-1}(s)\,\mathrm{d}s = (1-\varepsilon)\varrho_+(r)\xi_+^{n-1}(r)$$

and we see that with this choice the denominator in the definition of u_+ stays bounded from 0. Moreover, we have

$$u_+'(r) = -\frac{1-\varepsilon}{\varrho_+(r)\sqrt{2\varepsilon-\varepsilon^2}}$$

and therefore u_+ is well defined, positive and decreasing function if

$$\int_1^\infty \frac{1}{\varrho_+(r)}\,\mathrm{d}r < \infty. \tag{4.11}$$

Now we can compute

$$\begin{aligned}\frac{\varrho_+^{-2}(r) + u'^2_+(r)}{\varrho^{-2}(x) + u'^2_+(r)} &= \frac{\varrho_+^{-2}(r) + \big((1-\varepsilon)/(\varrho_+(r)\sqrt{2\varepsilon-\varepsilon^2})\big)^2}{\varrho^{-2}(x) + \big((1-\varepsilon)/(\varrho_+(r)\sqrt{2\varepsilon-\varepsilon^2})\big)^2} \\ &= \frac{\varrho_+^{-2}(r)\big(1+(1-\varepsilon)^2/(2\varepsilon-\varepsilon^2)\big)}{\varrho^{-2}(x) + \varrho_+^{-2}(r)(1-\varepsilon^2)/(2\varepsilon-\varepsilon^2)},\end{aligned}$$

and for example, taking $\varepsilon = 1-\sqrt{2}/2$ we have

$$\frac{\varrho_+^{-2}(r) + u'^2_+(r)}{\varrho^{-2}(x) + u'^2_+(r)} = \frac{2\varrho_+^{-2}(r)}{\varrho^{-2}(x) + \varrho_+^{-2}(r)}.$$

For the prescribed mean curvature we obtain the bound

$$-nH(x) \le (1-\varepsilon)\sqrt{\frac{\varrho_+^{-2}(r)\big(1+(1-\varepsilon)^2/(2\varepsilon-\varepsilon^2)\big)}{\varrho^{-2}(x) + \varrho_+^{-2}(r)(1-\varepsilon^2)/(2\varepsilon-\varepsilon^2)}}\left(\frac{\varrho_+'(r)}{\varrho_+(r)} + (n-1)\frac{\xi_+'(r)}{\xi_+(r)}\right)$$

which implies that $\mathcal{Q}[u_+] \le 0$. Similarly, $\mathcal{Q}[-u_+] \ge 0$ if

$$nH(x) \le (1-\varepsilon)\sqrt{\frac{\varrho_+^{-2}(r)\big(1+(1-\varepsilon)^2/(2\varepsilon-\varepsilon^2)\big)}{\varrho^{-2}(x) + \varrho_+^{-2}(r)(1-\varepsilon^2)/(2\varepsilon-\varepsilon^2)}}\left(\frac{\varrho_+'(r)}{\varrho_+(r)} + (n-1)\frac{\xi_+'(r)}{\xi_+(r)}\right).$$

All together, we have obtained the following.

Lemma 4.1. *Let M be a complete Riemannian manifold with a pole o and consider the warped product manifold $M \times_\varrho \mathbb{R}$, where ϱ satisfies*

$$\frac{\partial_r\varrho(x)}{\varrho(x)} \ge \frac{\varrho_+'\big(r(x)\big)}{\varrho_+\big(r(x)\big)}, \quad \frac{\partial_r\varrho(x)}{\varrho(x)^3} \ge \frac{\varrho_+'\big(r(x)\big)}{\varrho_+\big(r(x)\big)^3} \tag{4.12}$$

for some positive and increasing C^1-function $\varrho_+\colon [0,\infty) \to (0,\infty)$ such that

$$\varrho_+(0) = \varrho(o) \quad \textit{and} \quad \int_1^\infty \varrho_+(s)^{-1}\,\mathrm{d}s < \infty. \tag{4.13}$$

Furthermore, assume that the radial sectional curvatures of M are bounded from above by

$$K_M(P_x) \le -\frac{\xi_+''\big(r(x)\big)}{\xi_+\big(r(x)\big)}$$

and that the prescribed mean curvature function satisfies

$$n|H(x)| \le \tag{4.14}$$

$$(1-\varepsilon)\sqrt{\frac{\varrho_+^{-2}\big(r(x)\big)\big(1+(1-\varepsilon)^2/(2\varepsilon-\varepsilon^2)\big)}{\varrho^{-2}(x) + \varrho_+^{-2}\big(r(x)\big)(1-\varepsilon^2)/(2\varepsilon-\varepsilon^2)}}\left(\frac{\varrho_+'\big((r)\big)}{\varrho_+\big(r(x)\big)} + (n-1)\frac{\xi_+'\big(r(x)\big)}{\xi_+\big(r(x)\big)}\right)$$

for some $\varepsilon \in (0,1)$*. Then the function* u_+ *defined by* (4.6) *satisfies* $\mathcal{Q}[u_+] \le 0$ *and* $u_+ \ge ||\varphi||_{C^0}$ *in* M *with*

$$u_+(r) \to ||\varphi||_{C^0} \quad \text{as } r \to \infty. \tag{4.15}$$

Furthermore $\mathcal{Q}[-u_+] \ge 0$ *and* $-u_+ \le -||\varphi||_{C^0}$ *in* M*.*

Remark 4.2. In particular, if the sectional curvatures of a Cartan-Hadamard manifold M are bounded from above as

$$K_M(P_x) \le -a\big(r(x)\big)^2 \tag{4.16}$$

for some smooth function $a\colon [0,\infty) \to [0,\infty)$, the condition (4.14) reads as

$$n|H(x)| \le \tag{4.17}$$
$$(1-\varepsilon)\sqrt{\frac{\varrho_+^{-2}\big(r(x)\big)\big(1+(1-\varepsilon)^2/(2\varepsilon-\varepsilon^2)\big)}{\varrho^{-2}(x)+\varrho_+^{-2}\big(r(x)\big)(1-\varepsilon^2)/(2\varepsilon-\varepsilon^2)}}\left(\frac{\varrho_+'\big((r)\big)}{\varrho_+\big(r(x)\big)}+(n-1)\frac{f_a'\big(r(x)\big)}{f_a\big(r(x)\big)}\right),$$

with f_a as in (2.3).

In a rotationally symmetric case if $\varrho = \varrho_+(r)$ (and (4.11) holds), we see that the bound for the mean curvature is

$$n|H(x)| \le (1-\varepsilon)\left(\frac{\varrho_+'\big(r(x)\big)}{\varrho_+\big(r(x)\big)}+(n-1)\frac{\xi_+'\big((r(x)\big)}{\xi_+\big(r(x)\big)}\right).$$

4.1. **Example: hyperbolic space.** We consider the warped model of $\mathbb{H}^{n+1}$ given by $\mathbb{H}^n \times_{\cosh r} \mathbb{R}$, where r is a radial coordinate in $\mathbb{H}^n$ defined with respect to a fixed reference point $o \in \mathbb{H}^n$. Then the hyperbolic metric is expressed as

$$\cosh^2 \mathrm{d}t^2 + \mathrm{d}r^2 + \sinh^2 r\, \mathrm{d}\vartheta^2,$$

where $\mathrm{d}\vartheta^2$ stands for the standard metric in $\mathbb{S}^{n-1} \subset T_o\mathbb{H}^n$. The flow of the Killing field $X = \partial_t$ is given by the hyperbolic translations generated by a geodesic γ orthogonal to $\mathbb{H}^n$ through o. Since $\varrho(r) = \cosh r$ and $\xi(r) = \sinh r$ in this case, we obtain

$$\begin{aligned}\lim_{r\to\infty}\frac{\varrho(r)\xi^{n-1}(r)}{\int_0^r \varrho(\tau)\xi^{n-1}(\tau)\,\mathrm{d}\tau} &= \lim_{r\to\infty}\frac{\sinh^n r + (n-1)\cosh^2 r\sinh^{n-2} r}{\cosh r \sinh^{n-1} r}\\ &= \lim_{r\to\infty}\left(\frac{\sinh r}{\cosh r}+(n-1)\frac{\cosh r}{\sinh r}\right) \ge n.\end{aligned}$$

Therefore a natural bound to the mean curvature function according (4.4) is

$$|H| < 1,$$

that is, below the mean curvature of horospheres.

We also have for $|H| < 1$

$$\frac{I^2(r)\varrho^{-2}(r)}{\varrho^2(r)\xi^{2(n-1)}(r) - I^2(r)} \le \frac{\sinh^{2n} r \cosh^{-2}(r)}{\cosh^2 r \sinh^{2(n-1)} r - \sinh^{2n} r} = \frac{\sinh^2 r}{\cosh^2 r}.$$

Therefore we have

$$u'^2(r) \le 1.$$

If $|H| = \text{cte.} < 1$ we have an explicit expression

$$u'^2(r) = \frac{H^2}{\cosh^2 r - H^2\sinh^2 r}\frac{\cosh^2 r}{\sinh^2 r}.$$

4.2. **Global barrier V.** In this subsection we construct a global barrier using an idea of Mastrolia, Monticelli, and Punzo [14]; see also [5]. Recall that $\varrho_+ \colon [0,\infty) \to (0,\infty)$ is an increasing smooth function satisfying $\varrho_+(0) = \varrho(o)$ and

$$\frac{\partial_r \varrho(x)}{\varrho(x)} \geq \frac{\varrho_+'\big(r(x)\big)}{\varrho_+\big(r(x)\big)} \tag{4.18}$$

for all $x \in M$. Then we have an estimate

$$\Delta_{-\log \varrho} r(x) \geq (n-1)\frac{f_a'(r(x))}{f_a(r(x))} + \frac{\varrho_+'(r(x))}{\varrho_+(r(x))} \tag{4.19}$$

for the weighted Laplacian of the distance function r. Let a_0 be a positive function such that

$$\int_0^\infty \left(\int_t^\infty \frac{\mathrm{d}s}{\varrho_+^2(s) f_a^{n-1}(s)} \right) a_0(t) f_a^{n-1}(t) \mathrm{d}t < \infty. \tag{4.20}$$

We define

$$\begin{aligned} V(x) = &\left(\int_{r(x)}^\infty \frac{\mathrm{d}s}{\varrho_+^2(s) f_a^{n-1}(s)} \right) \left(\int_0^{r(x)} a_0(t) f_a^{n-1}(t) \mathrm{d}t \right) \\ &- \int_0^{r(x)} \left(\int_t^\infty \frac{\mathrm{d}s}{\varrho_+^2(s) f_a^{n-1}(s)} \right) a_0(t) f_a^{n-1}(t) \mathrm{d}t - D + ||\varphi||_\infty, \end{aligned} \tag{4.21}$$

where D is the constant given by (4.22). Denoting $V(r) = V(r(x))$, we observe that

$$V'(r) = -\frac{1}{\varrho_+^2(r) f_a^{n-1}(r)} \int_0^r a_0(t) f_a^{n-1}(t) \mathrm{d}t < 0$$

and

$$V''(r) = \frac{1}{\varrho_+^2(r) f_a^{n-1}(r)} \left(\frac{(n-1) f_a'(r)}{f_a(r)} + \frac{2\varrho_+'(r)}{\varrho_+(r)} \right) \int_0^r a_0(t) f_a^{n-1}(t) \mathrm{d}t - \frac{a_0(r)}{\varrho_+^2(r)}.$$

Since $V'(r) < 0$, the limit

$$\begin{aligned} D = \lim_{r \to \infty} \Big\{ &\int_r^\infty \frac{\mathrm{d}s}{\varrho_+^2(s) f_a^{n-1}(s)} \int_0^r a_0(t) f_a^{n-1}(t) \mathrm{d}t \\ &- \int_0^r \int_t^\infty \frac{\mathrm{d}s}{\varrho_+^2(s) f_a^{n-1}(s)} a_0(t) f_a^{n-1}(t) \mathrm{d}t \Big\} \end{aligned} \tag{4.22}$$

exists. Furthermore, $D \leq 0$ (see [14, (4.5)]) and finite by (4.20) and therefore V is well defined. Next we write

$$\begin{aligned} &\mathcal{Q}[V] \\ &= \frac{(\varrho^{-2} + |\nabla V|^2)\Delta_{-\log \varrho} V - (\varrho^{-2} + |\nabla V|^2)^{3/2} n H(x) - \frac{1}{2}\left\langle \nabla(\varrho^{-2} + |\nabla V|^2), \nabla V \right\rangle}{(\varrho^{-2} + |\nabla V|^2)^{3/2}} \end{aligned} \tag{4.23}$$

and aim to prove that $\mathcal{Q}[V] \le 0$. First we estimate the weighted Laplacian of V by using (4.19)

$$\begin{aligned}
\Delta_{-\log\varrho} V &= V''(r) + V'(r)\Delta_{-\log\varrho} r \\
&\le V''(r) + \left((n-1)\frac{f_a'(r)}{f_a(r)} + \frac{\varrho_+'(r)}{\varrho_+(r)}\right) V'(r) \\
&= \frac{1}{\varrho_+^2(r) f_a^{n-1}(r)} \left(\frac{(n-1)f_a'(r)}{f_a(r)} + \frac{2\varrho_+'(r)}{\varrho_+(r)}\right) \int_0^r a_0(t) f_a^{n-1}(t)\mathrm{d}t \\
&- \frac{a_0(r)}{\varrho_+^2(r)} - \frac{1}{\varrho_+^2(r) f_a^{n-1}(r)} \left(\frac{(n-1)f_a'(r)}{f_a(r)} + \frac{\varrho_+'(r)}{\varrho_+(r)}\right) \int_0^r a_0(t) f_a^{n-1}(t)\mathrm{d}t \\
&= -\frac{a_0(r)}{\varrho_+^2(r)} + \frac{\varrho_+'(r)}{\varrho_+^3(r) f_a^{n-1}(r)} \int_0^r a_0(t) f_a^{n-1}(t)\mathrm{d}t \\
&= -\frac{a_0(r)}{\varrho_+^2(r)} - \frac{\varrho_+'(r)}{\varrho_+(r)} V'(r),
\end{aligned}$$

and thus the first term of (4.23) can be estimated as

$$\big(\varrho^{-2} + |\nabla V|^2\big)\Delta_{-\log\varrho} V \le -\big(\varrho^{-2} + (V'(r))^2\big) \left(\frac{a_0(r)}{\varrho_+^2(r)} + \frac{\varrho_+'(r)}{\varrho_+(r)} V'(r)\right).$$

Then, for the last term of (4.23) we have

$$\begin{aligned}
-\frac{1}{2}\left\langle \nabla(\varrho^{-2} + |\nabla V|^2), \nabla V\right\rangle &= -\big(V'(r)\big)^2 V''(r) + \frac{\partial_r \varrho}{\varrho^3} V'(r) \\
&= -(V'(r))^2 \left(\left(\frac{(n-1)f_a'(r)}{f_a(r)} + \frac{2\varrho_+'(r)}{\varrho_+(r)}\right) V'(r) - \frac{a_0(r)}{\varrho_+^2(r)}\right) + \frac{\partial_r \varrho}{\varrho^3} V'(r).
\end{aligned}$$

Hence

$$\begin{aligned}
&\varrho_+^2(r)\big(\varrho^{-2} + |\nabla V|^2\big)\Delta_{-\log\varrho} V - \frac{1}{2}\varrho_+^2(r)\left\langle \nabla(\varrho^{-2} + |\nabla V|^2), \nabla V\right\rangle \\
&\quad \le -\varrho^{-2} a_0(r) - \varrho^{-2}\varrho_+^2(r) \left(\frac{\varrho_+'(r)}{\varrho_+(r)} - \frac{\partial_r \varrho}{\varrho}\right) V'(r) \\
&\qquad - \varrho_+^2(r)\big(V'(r)\big)^3 \left(\frac{(n-1)f_a'(r)}{f_a(r)} + \frac{\varrho_+'(r)}{\varrho_+(r)}\right) \\
&\quad \le -\varrho^{-2} a_0(r) - \varrho_+^2(r)\big(V'(r)\big)^3 \left(\frac{(n-1)f_a'(r)}{f_a(r)} + \frac{\varrho_+'(r)}{\varrho_+(r)}\right).
\end{aligned}$$

Finally, if the prescribed mean curvature function satisfies

$$-nH \le \frac{\varrho^{-2}\varrho_+^{-2}(r) a_0(r) + \big(-V'(r)\big)^3 \left(\frac{(n-1)f_a'(r)}{f_a(r)} + \frac{\varrho_+'(r)}{\varrho_+(r)}\right)}{\left(\varrho^{-2} + \big(V'(r)\big)^2\right)^{3/2}}$$

in M, we obtain $\mathcal{Q}[V] \le 0$ as desired. Similarly, we see that $\mathcal{Q}[-V] \ge 0$ if

$$nH \le \frac{\varrho^{-2}\varrho_+^{-2}(r) a_0(r) + \big(-V'(r)\big)^3 \left(\frac{(n-1)f_a'(r)}{f_a(r)} + \frac{\varrho_+'(r)}{\varrho_+(r)}\right)}{\left(\varrho^{-2} + \big(V'(r)\big)^2\right)^{3/2}}.$$

Hence we have proved the following uniform height estimate.

Lemma 4.3. *Let $\varphi\colon M \to \mathbb{R}$ be a bounded function and assume that the prescribed mean curvature function H and the function V defined in (4.21) satisfy*

$$n|H| \le \frac{\varrho^{-2}\varrho_+^{-2}(r)a_0(r) + \big(-V'(r)\big)^3\left(\frac{(n-1)f_a'(r)}{f_a(r)} + \frac{\varrho_+'(r)}{\varrho_+(r)}\right)}{\Big(\varrho^{-2} + \big(V'(r)\big)^2\Big)^{3/2}}, \tag{4.24}$$

with some positive functions ϱ_+ and a_0 satifying (4.18) and (4.20), respectively. Then

$$\mathcal{Q}[V] = \operatorname{div}_{-\log\varrho} \frac{\nabla V}{\sqrt{\varrho^{-2} + |\nabla V|^2}} - nH \le 0 \quad \text{in } M, \tag{4.25}$$

$$V(x) > ||\varphi||_\infty \quad \text{for all } x \in M, \tag{4.26}$$

and

$$\lim_{r(x)\to\infty} V(x) = ||\varphi||_\infty. \tag{4.27}$$

Furthermore, $\mathcal{Q}[-V] \ge 0$ in M.

Next we discuss possible choices of the functions ϱ_+ and a_0 and their influence on the bound of $|H|$. Notice that the right hand side of (4.24) can be written as

$$\frac{\frac{\varrho\varrho_+^{-2}(r)a_0(r)}{\left(-V'(r)\varrho\right)^3} + \frac{(n-1)f_a'(r)}{f_a(r)} + \frac{\varrho_+'(r)}{\varrho_+(r)}}{\Big(1 + \big(-V'(r)\varrho\big)^{-2}\Big)^{3/2}}. \tag{4.28}$$

Hence if we can choose the comparison manifold $M_{-a^2(r)} \times_{\varrho_+} \mathbb{R}$ and a_0 such that $V'(r)\varrho \to -\infty$ and

$$\frac{\varrho\varrho_+^{-2}(r)a_0(r)}{\big(-V'(r)\varrho\big)^3} \to 0$$

as $r \to \infty$, we obtain

$$n|H| \le \frac{(n-1)f_a'(r)}{f_a(r)} + \frac{\varrho_+'(r)}{\varrho_+(r)} \tag{4.29}$$

asymptotically as $r \to \infty$.

Example 4.4. In the hyperbolic case $\mathbb{H}^{n+1} = \mathbb{H}^n \times_{\cosh r} \mathbb{R}$ we may take $\varrho_+(r) = \varrho = \cosh$. Choosing $a_0(r) = \sinh^\alpha r$ for some $\alpha \in (1,2)$ yields to the natural asymptotic bound $|H| < 1$ as $r \to \infty$.

Example 4.5. More generally, if $N = M \times_\varrho \mathbb{R}$, where the sectional curvatures of M have a negative upper bound $-k^2$ and if the warping function ϱ satisfies (4.18) with $\varrho_+(r) \ge c_1 e^{\alpha r}$ for some $\alpha > 0$, then $f_a(r) \approx e^{kr}$ and (4.20) holds if

$$\int_0^\infty a_0(t)e^{-2\alpha t}\mathrm{d}t < \infty.$$

Moreover, if $\varrho_+(r) \le c_2 e^{\beta r}$ for some $0 < \beta < 2\alpha$, then by choosing $a_0(t) = e^{\kappa t}$, $\beta < \kappa < 2\alpha$, we get (4.29) asymptotically as $r \to \infty$.

Example 4.6. If $N = M \times_\varrho \mathbb{R}$, where the sectional curvatures of M have a negative upper bound

$$K(P_x) \le -\frac{\phi(\phi-1)}{r(x)^2}, \quad \phi > 1,$$

and if the warping function ϱ satisfies (4.18) with $\varrho_+(r) = cr^\alpha$, $\alpha > 1$, then $f_a(r) \approx r^\phi$ and (4.20) holds if

$$\int_0^\infty a_0(r)r^{-2\alpha+1}\mathrm{d}r < \infty.$$

Choosing $a_0(r) = r^\kappa$, for some $\alpha - 1 < \kappa < 2(\alpha - 1)$, we get (4.29) asymptotically as $r \to \infty$.

5. Barrier at infinity

In this section we assume that M is a Cartan-Hadamard manifold of dimension $n \geq 2$, $\partial_\infty M$ is the asymptotic boundary of M, and $\bar{M} = M \cup \partial_\infty M$ the compactification of M in the cone topology. Recall that the asymptotic boundary is defined as the set of all equivalence classes of unit speed geodesic rays in M; two such rays γ_1 and γ_2 are equivalent if $\sup_{t\geq 0} d\big(\gamma_1(t), \gamma_2(t)\big) < \infty$. The equivalence class of γ is denoted by $\gamma(\infty)$. For each $x \in M$ and $y \in \bar{M} \setminus \{x\}$ there exists a unique unit speed geodesic $\gamma^{x,y} \colon \mathbb{R} \to M$ such that $\gamma_0^{x,y} = x$ and $\gamma_t^{x,y} = y$ for some $t \in (0, \infty]$. If $v \in T_x M \setminus \{0\}$, $\alpha > 0$, and $r > 0$, we define a cone

$$C(v, \alpha) = \{y \in \bar{M} \setminus \{x\} : \sphericalangle(v, \dot{\gamma}_0^{x,y}) < \alpha\}$$

and a truncated cone

$$T(v, \alpha, r) = C(v, \alpha) \setminus \bar{B}(x, r),$$

where $\sphericalangle(v, \dot{\gamma}_0^{x,y})$ is the angle between vectors v and $\dot{\gamma}_0^{x,y}$ in $T_x M$. All cones and open balls in M form a basis for the cone topology on $\bar{M}$.

Throughout this section, we assume that the sectional curvatures of M are bounded from below and above by

$$-(b \circ r)^2(x) \leq K(P_x) \leq -(a \circ r)^2(x) \tag{5.1}$$

for all $x \in M$, where $r(x) = d(o, x)$ is the distance to a fixed point $o \in M$ and P_x is any 2-dimensional subspace of $T_x M$. The functions $a, b \colon [0, \infty) \to [0, \infty)$ are assumed to be smooth such that $a(t) = 0$ and $b(t)$ is constant for $t \in [0, T_0]$ for some $T_0 > 0$, and that assumptions (A1)–(A7) hold. These curvature bounds are needed to control the first two derivatives of "barrier" functions that we will construct in the next subsection. We assume that function b in (5.1) is monotonic and that there exist positive constants $T_1 \geq T_0, C_1, C_2, C_3$, and $Q \in (0, 1)$ such that

$$a(t) \begin{cases} = C_1 t^{-1} & \text{if } b \text{ is decreasing,} \\ \geq C_1 t^{-1} & \text{if } b \text{ is increasing} \end{cases} \tag{A1}$$

for all $t \geq T_1$ and

$$a(t) \leq C_2, \tag{A2}$$

$$b(t+1) \leq C_2 b(t), \tag{A3}$$

$$b(t/2) \leq C_2 b(t), \tag{A4}$$

$$b(t) \geq C_3 (1+t)^{-Q} \tag{A5}$$

for all $t \geq 0$. In addition, we assume that

$$\lim_{t\to\infty} \frac{b'(t)}{b(t)^2} = 0 \tag{A6}$$

and that there exists a constant $C_4 > 0$ such that

$$\lim_{t\to\infty} \frac{t^{1+C_4} b(t)}{f_a'(t)} = 0; \tag{A7}$$

see (2.3) for the definition of f_a.

We recall from [12] the following two examples of functions a and b.

Example 5.1. Let $C_1 = \sqrt{\phi(\phi - 1)}$, where $\phi > 1$ is a constant. For $t \geq R_0$ let

$$a(t) = \frac{C_1}{t}$$

and

$$b(t) = t^{\phi - 2 - \varepsilon/2},$$

where $0 < \varepsilon < 2\phi - 2$, and extend them to smooth functions $a\colon [0,\infty) \to (0,\infty)$ and $b\colon [0,\infty) \to (0,\infty)$ such that they are constants in some neighborhood of 0, b is monotonic and $b \geq a$. Then a and b satisfy (A1)-(A7) with constants $T_1 = R_0$, C_1, some $C_2 > 0$, some $C_3 > 0$, $Q = \max\{1/2, -\phi + 2 + \varepsilon/2\}$, and any $C_4 \in (0, \varepsilon/2)$. It is easy to verify that then

$$f_a(t) = c_1 t^{\phi} + c_2 t^{1-\phi}$$

for all $t \geq R_0$, where

$$c_1 = R_0^{-\phi} \frac{f_a(R_0)(\phi - 1) + R_0 f_a'(R_0)}{2\phi - 1} > 0,$$

and

$$c_2 = R_0^{\phi-1} \frac{f_a(R_0)\phi - R_0 f_a'(R_0)}{2\phi - 1}.$$

We then have

$$\lim_{t\to\infty} \frac{t f_a'(t)}{f_a(t)} = \phi$$

and, for all $C_4 \in (0, \varepsilon/2)$

$$\lim_{t\to\infty} \frac{t^{1+C_4} b(t)}{f_a'(t)} = 0.$$

It follows that a and b satisfy (A1)-(A7) with constants $T_1 = R_0$, C_1, some $C_2 > 0$, some $C_3 > 0$, $Q = \max\{1/2, -\phi + 2 + \varepsilon/2\}$, and any $C_4 \in (0, \varepsilon/2)$.

Example 5.2. Let $k > 0$ and $\varepsilon > 0$ be constants and define $a(t) = k$ for all $t \geq 0$. Define

$$b(t) = t^{-1-\varepsilon/2} e^{kt}$$

for $t \geq R_0 = r_0 + 1$, where $r_0 > 0$ is so large that $t \mapsto t^{-1-\varepsilon/2} e^{kt}$ is increasing and greater than k for all $t \geq r_0$. Extend b to an increasing smooth function $b\colon [0,\infty) \to [k,\infty)$ that is constant in some neighborhood of 0. We can choose $C_1 > 0$ in (A1) as large as we wish. Then a and b satisfy (A1)-(A7) with constants C_1, $T_1 = C_1/k$, some $C_2 > 0$, some $C_3 > 0$, $Q = 1/2$, and any $C_4 \in (0, \varepsilon/2)$.

5.1. **Construction of a barrier.** Following [12], we construct a barrier function for each boundary point $x_0 \in \partial_\infty M$. Towards this end let $v_0 = \dot\gamma_0^{o,x_0}$ be the initial (unit) vector of the geodesic ray γ^{o,x_0} from a fixed point $o \in M$ and define a function $h : \partial_\infty M \to \mathbb{R}$,

$$h(x) = \min\big(1, L\sphericalangle(v_0, \dot\gamma_0^{o,x})\big), \tag{5.2}$$

where $L \in (8/\pi, \infty)$ is a constant. Then we define a crude extension $\tilde h \in C(\bar M)$, with $\tilde h | \partial_\infty M = h$, by setting

$$\tilde h(x) = \min\Big(1, \max\big(2 - 2r(x), L\sphericalangle(v_0, \dot\gamma_0^{o,x})\big)\Big). \tag{5.3}$$

Finally, we smooth out $\tilde h$ to get an extension $h \in C^\infty(M) \cap C(\bar M)$ with controlled first and second order derivatives. For that purpose, we fix $\chi \in C^\infty(\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\operatorname{supp} \chi \subset [-2, 2]$, and $\chi | [-1, 1] \equiv 1$. Then for any function $\varphi \in C(M)$ we define functions $F_\varphi\colon M \times M \to \mathbb{R}$, $\mathcal{R}(\varphi)\colon M \to M$, and $\mathcal{P}(\varphi)\colon M \to \mathbb{R}$ by

$$\begin{aligned} F_\varphi(x,y) &= \chi\big(b(r(y)) d(x,y)\big)\varphi(y), \\ \mathcal{R}(\varphi)(x) &= \int_M F_\varphi(x,y) \mathrm{d}m(y), \quad \text{and} \\ \mathcal{P}(\varphi) &= \frac{\mathcal{R}(\varphi)}{\mathcal{R}(1)}, \end{aligned}$$

where

$$\mathcal{R}(1)(x) = \int_M \chi\big(b(r(y))d(x,y)\big)\mathrm{d}m(y) > 0.$$

Thus $\mathcal{P}(\varphi)$ is an integral average of φ with respect to χ similar to that in [1, p. 436] except that here the function b is taken into account explicitly. If $\varphi \in C(\bar{M})$, we extend $\mathcal{P}(\varphi)\colon M \to \mathbb{R}$ to a function $\bar{M} \to \mathbb{R}$ by setting $\mathcal{P}(\varphi)(x) = \varphi(x)$ whenever $x \in M(\infty)$. Then the extended function $\mathcal{P}(\varphi)$ is C^∞-smooth in M and continuous in $\bar{M}$; see [12, Lemma 3.13]. In particular, applying $\mathcal{P}$ to the function $\tilde{h}$ yields an appropriate smooth extension

$$h := \mathcal{P}(\tilde{h}) \tag{5.4}$$

of the original function $h \in C\big(\partial_\infty M\big)$ that was defined in (5.2).

We denote

$$\Omega = C(v_0, 1/L) \cap M \quad \text{and} \quad \ell\Omega = C(v_0, \ell/L) \cap M$$

for $\ell > 0$. We collect together all these constants and functions and denote

$$C = (a, b, T_1, C_1, C_2, C_3, C_4, Q, n, L).$$

Furthermore, we denote by $\|\operatorname{Hess}_x u\|$ the norm of the Hessian of a smooth function u at x, that is

$$\|\operatorname{Hess}_x u\| = \sup_{\substack{X \in T_x M \\ |X| \le 1}} |\operatorname{Hess} u(X,X)|.$$

The following lemma gives the desired estimates for derivatives of h. We refer to [12] for the proofs of these estimates; see also [6].

Lemma 5.3. [12, Lemma 3.16] *There exist constants $R_1 = R_1(C)$ and $c_1 = c_1(C)$ such that the extended function $h \in C^\infty(M) \cap C(\bar{M})$ in (5.4) satisfies*

$$\begin{aligned} |\nabla h(x)| &\le c_1 \frac{1}{(f_a \circ r)(x)}, \\ \|\operatorname{Hess}_x h\| &\le c_1 \frac{(b \circ r)(x)}{(f_a \circ r)(x)}, \end{aligned} \tag{5.5}$$

for all $x \in 3\Omega \setminus B(o, R_1)$. In addition,

$$h(x) = 1$$

for every $x \in M \setminus \big(2\Omega \cup B(o, R_1)\big)$.

Let $A > 0$ be a fixed constant, and $R_3 > 0$ and $\delta > 0$ constants that will be determined later, and h the function defined in (5.4). We will show that a function

$$\psi = A(R_3^\delta r^{-\delta} + h) \tag{5.6}$$

is a supersolution

$$\begin{aligned} \mathcal{Q}[\psi] &= \operatorname{div}_{-\log \varrho} \frac{\nabla \psi}{\sqrt{\varrho^{-2} + |\nabla \psi|^2}} - nH \\ &= \operatorname{div} \frac{\nabla \psi}{W} + \left\langle \nabla \log \varrho, \frac{\nabla \psi}{W} \right\rangle - nH < 0 \end{aligned}$$

in the $3\Omega \setminus \bar{B}(o, R_3)$. In the proof we shall use the following estimates obtained in [12]:

Lemma 5.4. [12, Lemma 3.17] *There exist constants $R_2 = R_2(C)$ and $c_2 = c_2(C)$ with the following property. If $\delta \in (0,1)$, then*

$$\begin{aligned}|\nabla h| &\le c_2/(f_a \circ r),\\ \|\operatorname{Hess} h\| &\le c_2 r^{-C_4-1}(f_a' \circ r)/(f_a \circ r),\\ |\nabla\langle\nabla h, \nabla h\rangle| &\le c_2 r^{-C_4-2}(f_a' \circ r)/(f_a \circ r),\\ |\nabla\langle\nabla h, \nabla(r^{-\delta})\rangle| &\le c_2 r^{-C_4-2}(f_a' \circ r)/(f_a \circ r),\\ \nabla\big\langle\nabla(r^{-\delta}), \nabla(r^{-\delta})\big\rangle &= -2\delta^2(\delta+1)r^{-2\delta-3}\nabla r\end{aligned}$$

in the set $3\Omega \setminus B(o, R_2)$.

Let us denote

$$\phi = \frac{1+\sqrt{1+4C_1^2}}{2} > 1, \quad \text{and} \quad \delta_1 = \min\left\{C_4/2, \frac{-1+(n-1)\phi}{1+(n-1)\phi}\right\} \in (0,1),$$

where C_1 and C_4 are constants defined in (A1) and (A7), respectively.

Lemma 5.5. *Assume that the prescribed mean curvature function H satisfies*

$$\sup_{r(x)=t} n|H(x)| < \frac{C_0 t^{-\delta_1-1}}{\sqrt{\varrho^{-2}(t) + (C_0 t^{-\delta-1})^2}}\left((n-1)\frac{f_a'(t)}{f_a(t)} + \frac{\partial_r \varrho}{\varrho} - \frac{1}{t}\right) \tag{5.7}$$

for some positive constants $C_0 > 1$ and $\delta < \min\{\delta_1, \phi - 1\}$, and that the warping function ϱ satisfies

$$\max\left(0, -\frac{r\partial_r \varrho}{\varrho}\right) = o\left(\frac{r f_a'(r)}{f_a(r)}\right) \tag{5.8}$$

and

$$|\nabla \varrho| = o\left(\frac{f_a(r)}{r^{\delta+1}}|\partial_r \varrho|\right) \tag{5.9}$$

as $r \to \infty$. Then there exists a constant $R_3 = R_3(C, C_0, \delta) \ge R_2$ such that the function ψ defined in (5.6) satisfies $\mathcal{Q}[\psi] < 0$ in the set $3\Omega \setminus \bar{B}(o, R_3)$.

Proof. In the proof we will denote by c those positive constants whose actual value is irrelevant and may vary even within a line. Furthermore, the estimates will be done in $3\Omega \setminus \bar{B}(o, R_3)$, with R_3 large enough. Note that

$$\begin{aligned}\mathcal{Q}[\psi] &= \frac{\Delta_{-\log\varrho}\psi}{\sqrt{\varrho^{-2} + |\nabla\psi|^2}} - \frac{1}{2}\frac{\big\langle\nabla(\varrho^{-2} + |\nabla\psi|^2), \nabla\psi\big\rangle}{(\varrho^{-2} + |\nabla\psi|^2)^{3/2}} - nH\\ &= \frac{(\varrho^{-2} + |\nabla\psi|^2)\Delta_{-\log\varrho}\psi - \frac{1}{2}\big\langle\nabla(\varrho^{-2} + |\nabla\psi|^2), \nabla\psi\big\rangle - (\varrho^{-2} + |\nabla\psi|^2)^{3/2} nH}{(\varrho^{-2} + |\nabla\psi|^2)^{3/2}}\end{aligned}$$

and hence we only need to find $R_3 = R_3(C, C_0, \delta) \ge R_2$ so that

$$\begin{aligned}&(\varrho^{-2} + |\nabla\psi|^2)^{3/2}\mathcal{Q}[\psi] \qquad (5.10)\\ &= (\varrho^{-2} + |\nabla\psi|^2)\Delta_{-\log\varrho}\psi - \frac{1}{2}\big\langle\nabla(\varrho^{-2} + |\nabla\psi|^2), \nabla\psi\big\rangle - (\varrho^{-2} + |\nabla\psi|^2)^{3/2} nH < 0\end{aligned}$$

holds in the set $3\Omega \setminus \bar{B}(o, R_3)$.

The function ψ is C^∞-smooth and, in $M \setminus \{o\}$, we have

$$\nabla\psi = A(-R_3^\delta \delta r^{-\delta-1}\nabla r + \nabla h).$$

By Lemma 5.3, $|\nabla h| \le c_1/f_a(r) \le \delta r^{-\delta-1}$ when r is large enough and $0 < \delta < \min\{\delta_1, \phi - 1\}$; see [12, (3.30)]. Hence, for any fixed $\varepsilon > 0$, we have

$$\begin{aligned}|\nabla\psi|^2 &= (AR_3^\delta\delta)^2 r^{-2\delta-2} + A^2|\nabla h|^2 - 2A^2 R_3^\delta \delta r^{-\delta-1}\langle\nabla r, \nabla h\rangle\\ &\le A^2\delta^2\big(R_3^{2\delta} + 2R_3^\delta + 1\big)r^{-2\delta-2}\\ &\le (1+\varepsilon)(AR_3^\delta\delta)^2 r^{-2\delta-2}\end{aligned}$$

and

$$|\nabla\psi|^2 \ge A^2\delta^2\big(R_3^{2\delta} - 2R_3^\delta\big)r^{-2\delta-2} \ge (1-\varepsilon)(AR_3^\delta\delta)^2 r^{-2\delta-2}$$

in $3\Omega \setminus \bar{B}(o,R_3)$ for R_3 large enough.

Next we fix $\varepsilon > 0$ so that

$$\varepsilon < 1 - \frac{\delta+1}{(n-1)(1-\delta)\phi}, \tag{5.11}$$

which is possible since $\delta < \delta_1$. To simplify the notation below, we denote $\tilde{\varepsilon} = \varepsilon \operatorname{sgn}(\partial_r\varrho)$. In order to estimate the first term in the right-hand side of (5.10), we first observe that

$$-(n-1)\frac{rf_a'(r)}{f_a(r)} - \frac{r\partial_r\varrho}{\varrho} + \frac{\delta+1}{1-\varepsilon} < 0 \tag{5.12}$$

for $r \ge R_3$ by (5.8) and (5.11); see [12, (3.25)]. Then we can estimate the weighted Laplacian of ψ as

$$\begin{aligned}
\Delta_{-\log\varrho}\psi &= AR_3^\delta\Delta_{-\log\varrho}r^{-\delta} + A\Delta_{-\log\varrho}h \\
&= AR_3^\delta\left(\Delta r^{-\delta} + \frac{1}{\varrho}\left\langle\nabla\varrho,\nabla(r^{-\delta})\right\rangle\right) + A\left(\Delta h + \frac{1}{\varrho}\left\langle\nabla\varrho,\nabla h\right\rangle\right) \\
&= AR_3^\delta\left(-\delta r^{-\delta-1}\Delta r - \delta r^{-\delta-1}\frac{1}{\varrho}\left\langle\nabla\varrho,\nabla r\right\rangle + \delta(\delta+1)r^{-\delta-2}\right) \\
&\quad + A\left(\Delta h + \frac{1}{\varrho}\left\langle\nabla\varrho,\nabla h\right\rangle\right) \\
&\le AR_3^\delta\delta\left(-(n-1)\frac{rf_a'(r)}{f_a(r)} - \frac{r\partial_r\varrho}{\varrho} + \delta + 1\right)r^{-\delta-2} \\
&\quad + A\left(nc_2r^{-C_4-1}\frac{f_a'(r)}{f_a(r)} + \frac{c_2|\nabla\varrho|}{\varrho f_a(r)}\right) \\
&\le AR_3^\delta\delta\left(\frac{-(1-\varepsilon)(n-1)rf_a'(r)}{f_a(r)} - \frac{(1-\tilde{\varepsilon})r\partial_r\varrho}{\varrho} + \delta + 1\right)r^{-\delta-2} < 0
\end{aligned}$$

for $r \ge R_3$. In the last step we used (5.8), (5.9), and the fact that $C_4 > \delta$. Hence

$$\begin{aligned}
&(\varrho^{-2} + |\nabla\psi|^2)\Delta_{-\log\varrho}\psi \\
&\le -\big(\varrho^{-2} + (1-\varepsilon)(AR_3^\delta\delta)^2r^{-2\delta-2}\big)AR_3^\delta\delta\bigg(\frac{(1-\varepsilon)(n-1)rf_a'(r)}{f_a(r)} \\
&\quad + \frac{(1-\tilde{\varepsilon})r\partial_r\varrho}{\varrho} - 1 - \delta\bigg)r^{-\delta-2}.
\end{aligned} \tag{5.13}$$

To estimate the second term of (5.10) we split it into two parts as

$$-\frac{1}{2}\left\langle\nabla(\varrho^{-2} + |\nabla\psi|^2),\nabla\psi\right\rangle = -\frac{1}{2}\left\langle\nabla(\varrho^{-2}),\nabla\psi\right\rangle - \frac{1}{2}\left\langle\nabla|\nabla\psi|^2,\nabla\psi\right\rangle.$$

For the first term, by (5.9) and Lemma 5.4, we have

$$\begin{aligned}
-\frac{1}{2}\left\langle\nabla(\varrho^{-2}),\nabla\psi\right\rangle &= \left\langle\frac{\nabla\varrho}{\varrho^3},\nabla\psi\right\rangle = \left\langle\frac{\nabla\varrho}{\varrho^3},-AR_3^\delta\delta r^{-\delta-1}\nabla r\right\rangle + \left\langle\frac{\nabla\varrho}{\varrho^3},A\nabla h\right\rangle \\
&\le -AR_3^\delta\delta r^{-\delta-1}\frac{\partial_r\varrho}{\varrho^3} + c_2A\frac{|\nabla\varrho|}{\varrho^3f_a(r)} \\
&\le -(1-\tilde{\varepsilon})AR_3^\delta\delta r^{-\delta-1}\frac{\partial_r\varrho}{\varrho^3}.
\end{aligned} \tag{5.14}$$

To estimate the second term we note that

$$\begin{aligned}
\nabla|\nabla\psi|^2 &= A^2\nabla\left\langle R_3^\delta\nabla(r^{-\delta}) + \nabla h, R_3^\delta\nabla(r^{-\delta}) + \nabla h\right\rangle \\
&= (AR_3^\delta)^2\nabla\left\langle\nabla(r^{-\delta}),\nabla(r^{-\delta})\right\rangle + 2A^2R_3^\delta\nabla\left\langle\nabla(r^{-\delta}),\nabla h\right\rangle + A^2\nabla\left\langle\nabla h,\nabla h\right\rangle
\end{aligned}$$

and hence, by a straightforward computation using the estimates of Lemma 5.4, we get

$$
\begin{aligned}
-\frac{1}{2}\left\langle \nabla|\nabla\psi|^2, \nabla\psi\right\rangle &= -\frac{1}{2}(AR_3^\delta)^2 \left\langle \nabla\left\langle \nabla(r^{-\delta}), \nabla(r^{-\delta})\right\rangle, \nabla\psi\right\rangle \\
&\quad - A^2R_3^\delta \left\langle \nabla\left\langle \nabla(r^{-\delta}), \nabla h\right\rangle, \nabla\psi\right\rangle - \frac{1}{2}A^2\left\langle \nabla\left\langle \nabla h, \nabla h\right\rangle, \nabla\psi\right\rangle \\
&\le (AR_3^\delta\delta)^2(\delta+1)r^{-2\delta-3}\left\langle \nabla r, \nabla\psi\right\rangle + A^2R_3^\delta c_2 r^{-C_4-2}\frac{f_a'(r)}{f_a(r)}|\nabla\psi| \\
&\quad + \frac{1}{2}A^2c_2r^{-C_4-2}\frac{f_a'(r)}{f_a(r)}|\nabla\psi| \qquad (5.15)\\
&\le (AR_3^\delta\delta)^2(\delta+1)r^{-2\delta-3}\left\langle \nabla r, -AR_3^\delta\delta r^{-\delta-1}\nabla r + A\nabla h\right\rangle \\
&\quad + cr^{-C_4-\delta-3}\frac{f_a'(r)}{f_a(r)} \\
&\le -cr^{-3\delta-4} + cr^{-2\delta-3}\frac{1}{f_a(r)} + cr^{-C_4-\delta-3}\frac{f_a'(r)}{f_a(r)} \\
&\le -cr^{-3\delta-4} + cr^{-C_4-\delta-3}\frac{f_a'(r)}{f_a(r)},
\end{aligned}
$$

where in the last step we have absorbed the term $cr^{-2\delta-3}\frac{1}{f_a(r)}$ into the first by using the fact that $f_a(r) \ge cr^\phi$ and the choice of $\delta < \phi - 1$. Putting together (5.14) and (5.15) we get

$$
-\frac{1}{2}\left\langle \nabla(\varrho^{-2} + |\nabla\psi|^2), \nabla\psi\right\rangle \le -(1-\tilde{\varepsilon})AR_3^\delta\delta r^{-\delta-1}\frac{\partial_r\varrho}{\varrho^3} - cr^{-3\delta-4} + cr^{-C_4-\delta-3}\frac{f_a'(r)}{f_a(r)},
$$

and combining this with (5.13) yields

$$
\begin{aligned}
&(\varrho^{-2} + |\nabla\psi|^2)\Delta_{-\log\varrho}\psi - \frac{1}{2}\left\langle \nabla(\varrho^{-2} + |\nabla\psi|^2), \nabla\psi\right\rangle \\
&\quad \le -\frac{AR_3^\delta\delta}{\varrho^2}\left(\frac{(1-\varepsilon)(n-1)rf_a'(r)}{f_a(r)} + \frac{2(1-\tilde{\varepsilon})r\partial_r\varrho}{\varrho} - \delta - 1\right)r^{-\delta-2} \qquad (5.16)\\
&\qquad - (1-\varepsilon)(AR_3^\delta\delta)^3\left(\frac{(1-\varepsilon)(n-1)rf_a'(r)}{f_a(r)} + \frac{(1-\tilde{\varepsilon})r\partial_r\varrho}{\varrho} - 1 - \delta + c\right)r^{-3\delta-4},
\end{aligned}
$$

where we have absorbed the positive term $cr^{-C_4-\delta-3}f_a'(r)/f_a(r)$ by using the assumption $\delta < C_4/2$. Finally, using the assumption (5.7) we can estimate the term involving the mean curvature as

$$
\begin{aligned}
&-(\varrho^{-2} + |\nabla\psi|^2)^{3/2}nH \\
&\le (1+\varepsilon)^{3/2}(\varrho^{-2} + (AR_3^\delta\delta)^2r^{-2\delta-2})^{3/2}n|H| \qquad (5.17)\\
&\le \frac{c}{\varrho^2}\left(\frac{(n-1)rf_a'(r)}{f_a(r)} + \frac{r\partial_r\varrho}{\varrho} - 1\right)r^{-\delta_1-2} \\
&\quad + c\left(\frac{(n-1)rf_a'(r)}{f_a(r)} + \frac{r\partial_r\varrho}{\varrho} - 1\right)r^{-2\delta-\delta_1-4}.
\end{aligned}
$$

Combining (5.16) and (5.17) and noting that $\delta_1 > \delta$ we obtain (5.10) and the claim follows. □

Remark 5.6. In the case of the hyperbolic (ambient) space $\mathbb{H}^{n+1} = \mathbb{H}^n \times_{\cosh r} \mathbb{R}$ we have $\varrho = \varrho_+(r) = \cosh r$ and $f_a(r) = \sinh r$ on $\mathbb{H}^n$ for any reference point $o \in \mathbb{H}^n$. Hence (5.8) and (5.9) hold trivially. Moreover, we may choose $\phi > 1$ as large as we wish by increasing R_3 and therefore (5.11) and (5.12) hold even with $\delta = \delta_1$.

Finally,

$$-(\varrho^{-2}+|\nabla\psi|^2)^{3/2}nH \le (1+\varepsilon)(AR_3^\delta\delta)^3r^{-3\delta-3})n|H|$$

for r large enough, and consequently we may assume $\delta=\delta_1$ in (5.7) thus reducing it to an asymptotically sharp assumption.

Similarly, if the sectional curvatures of M have estimates

$$-r(x)^{-2-\varepsilon}e^{2kr(x)} \le K(P_x) \le -k^2$$

for $r(x)\ge R_0$ as in Example 5.2 and if the warping function ϱ satisfies (5.8), (5.9), and

$$\varrho(x)\ge cr(x)^2$$

for $r(x)\ge R_0$, we may take $\delta=\delta_1$ in (5.7).

6. Solving the asymptotic Dirichlet problem

In this section we solve the asymptotic Dirichlet problem (6.1) on a Cartan-Hadamard manifold M with given boundary data $\varphi\in C(\partial_\infty M)$. If the ambient manifold $N=M\times_\varrho\mathbb{R}$ is a Cartan-Hadamard manifold, too, we will interpret the graph $S=\{(x,u(x))\colon x\in M\}$ of the solution u as a Killing graph with prescribed mean curvature H and continuous boundary values at infinity. We recall from [2, 7.7] that N is a Cartan-Hadamard manifold if and only if the warping function ϱ is convex. In that case we may consider $\partial_\infty M$ as a subset of $\partial_\infty N$ in the sense that a representative γ of a boundary point $x_0\in\partial_\infty M$ is also a representative of a point $\tilde{x}_0\in\partial_\infty N$ since M is a totally geodesic submanifold of N. Given $\varphi\in C(\partial_\infty M)$ we define its Killing graph on $\partial_\infty N$ as follows. For $x\in\partial_\infty M$, take the (totally geodesic) leaf

$$M_{\varphi(x)}=\Psi(M,\varphi(x))=\{(y,\varphi(x))\colon y\in M\}\subset M\times\mathbb{R},$$

where Ψ is the flow generated by X. Let γ^x be any geodesic on M representing x. Then $\tilde{\gamma}^x\colon t\mapsto\Psi(\gamma^x(t),\varphi(x))$ is a geodesic on $M_{\varphi(x)}$ and also on N since $\Psi(\cdot,\varphi(x))$ is an isometry. Hence $\tilde{\gamma}^x$ defines a point in $\partial_\infty N$ which we, by abusing the notation, denote by $(x,\varphi(x))$. Using this notation, we call the set

$$\Gamma=\{(x,\varphi(x))\colon x\in\partial_\infty M\}\subset\partial_\infty N$$

the Killing graph of φ. Note that, in general, $\partial_\infty N$ has no canonical smooth structure.

Lemma 6.1. *Let u be the solution to* (6.1) *with boundary data φ and let S be the graph of u. If $\partial_\infty S=\bar{S}\setminus S$, where $\bar{S}$ is the closure of S in the cone topology $\bar{N}$, we have $\partial_\infty S=\Gamma$.*

Proof. Suppose first that $x\in\partial_\infty S$ and let $(x_i,u(x_i))$ be a sequence in S converging to x in the cone topology of $\bar{N}$. Since $\bar{M}$ is compact, there exist $x_0\in\partial_\infty M$ and a subsequence $(x_{i_j},u(x_{i_j}))$ such that $x_{i_j}\to x_0\in\partial_\infty M$ in the cone topology of $\bar{M}$. Hence $u(x_{i_j})\to\varphi(x_0)$, and consequently $(x_{i_j},u(x_{i_j}))\to(x_0,\varphi(x_0))$ in the product topology of $\bar{M}\times\mathbb{R}$. On the other hand, $\Psi(x_{i_j},\varphi(x_0))\to(x_0,\varphi(x_0))$ in the cone topology of $M_{\varphi(x_0)}$. We need to verify that $\Psi(x_{i_j},u(x_{i_j}))\to(x_0,\varphi(x_0))$ in the cone topology of $\bar{N}$ which then implies that $x=(x_0,\varphi(x_0))\in\Gamma$. Towards this end, let V be an arbitrary cone neighborhood in $\bar{N}$ of $(x_0,\varphi(x_0))$ and let σ be a geodesic ray emanating from $(o,\varphi(x_0))$ representing $(x_0,\varphi(x_0))$. It is a geodesic ray both in N and in $M_{\varphi(x_0)}$. Let $T(\dot{\sigma}_0,2\alpha,r)\subset V$ be a truncated cone in $\bar{N}$ and $T:=T^M(\dot{\sigma}_0,\alpha,2r)$ a truncated cone in $\bar{M}_{\varphi(x_0)}$. Then $\Psi(T,(\varphi(x_0)-\delta,\varphi(x_0)+\delta))\subset V$ for sufficiently small $\delta>0$. It follows that $\Psi(x_{i_j},u(x_{i_j}))\in V$ for all i_j large enough, and therefore $x=(x_0,\varphi(x_0))\in\Gamma$.

Conversely, if $(x_0,\varphi(x_0))\in\Gamma$, let $x_i\in M$ be a sequence such that $x_i\to x_0$ in the cone topology of $\bar{M}$. Then $\Psi(x_i,u(x_i))\in S$ and $(x_i,u(x_i))\to(x_0,\varphi(x_0))$ in the product topology of $\bar{M} \times \mathbb{R}$. We need to show that $\Psi(x_i, u(x_i)) \to (x_0, \varphi(x_0)) \in \Gamma$ in the cone topology of $\bar{N}$. To prove this, fix $o = \Psi(x, \varphi(x_0)) \in M_{\varphi(x_0)}$ and let σ be a geodesic ray in N (and in $M_{\varphi(x_0)}$) representing $(x_0, \varphi(x_0))$. Let $V = T(\dot\sigma_0, 2\alpha, r)$ be an arbitrary truncated cone neighborhood in $\bar{N}$ of $(x_0, \varphi(x_0))$. Furthermore, let $\delta > 0$ be so small that $U := \Psi(\tilde{V}, (\varphi(x_0)-\delta, \varphi(x_0)+\delta)) \subset V$, where $\tilde{V} = T(\dot\sigma_0, \alpha, 2r)$ is a truncated cone neighborhood in $M_{\varphi(x_0)}$ of $(x_0, \varphi(x_0))$. Since $x_i \to x_0$ and $u(x_i) \to \varphi(x_0)$, we obtain $\Psi(x_i, u(x_i)) \in U$ for all sufficiently large i. Hence $\Psi(x_i, u(x_i)) \to (x_0, \varphi(x_0)) \in \Gamma$ in the cone topology of $\bar{N}$. □

We formulate our global existence results in the following two theorems depending on the assumption on the prescribed mean curvature function H.

Theorem 6.2. *Let M be a Cartan-Hadamard manifold satisfying the curvature assumptions* (5.1) *and* (A1)–(A7) *in Section 5. Furthermore, assume that the prescribed mean curvature function $H\colon M \to \mathbb{R}$ satisfies the assumptions* (4.17) *and* (5.7) *with a convex warping function ϱ satisfying* (4.12), (4.13), (5.8), *and* (5.9). *Then there exists a unique solution $u\colon M \to \mathbb{R}$ to the Dirichlet problem*

$$\begin{cases} \operatorname{div}_{-\log\varrho} \dfrac{\nabla u}{\sqrt{\varrho^{-2} + |\nabla u|^2}} = nH(x) \quad \text{in } M \\ u|\partial_\infty M = \varphi \end{cases} \tag{6.1}$$

for any continuous function $\varphi\colon \partial_\infty M \to \mathbb{R}$.

Theorem 6.3. *Let M be a Cartan-Hadamard manifold satisfying the curvature assumptions* (5.1) *and* (A1)–(A7) *in Section 5. Furthermore, assume that the prescribed mean curvature function $H\colon M \to \mathbb{R}$ satisfies the assumptions* (4.24) *and* (5.7) *with a convex warping function ϱ satisfying* (4.18), (5.8), *and* (5.9). *Then there exists a unique solution $u\colon M \to \mathbb{R}$ to the Dirichlet problem* (6.1) *for any continuous function $\varphi\colon \partial_\infty M \to \mathbb{R}$.*

Proof. The proofs of Theorems 6.2 and 6.3 are similar. The only difference is to use the global barrier u_+ in Lemma 4.1 for 6.2 relative to V in Lemma 4.3 for 6.3.

Extend the boundary data function $\varphi \in C(\partial_\infty M)$ to a function $\varphi \in C(\bar{M})$ and let $B_k = B(o, k)$, $k \in \mathbb{N}$ be an exhaustion of M. Then by Corollary 2.2 there exist solutions $u_k \in C^{2,\alpha}(B_k) \cap C(\bar{B}_k)$ to the Dirichlet problem

$$\begin{cases} \operatorname{div}_{-\log\varrho} \dfrac{\nabla u_k}{\sqrt{\varrho^{-2} + |\nabla u_k|^2}} = nH(x) \quad \text{in } B_k \\ u_k|\partial B_k = \varphi. \end{cases}$$

By Lemma 4.1, we see that the sequence (u_k) is uniformly bounded. Applying the gradient estimates in compact domains and then the diagonal argument, we obtain a subsequence converging locally uniformly with respect to C^2-norm to a solution u. Next we show that u extends continuously to the boundary $\partial_\infty M$ with $u|\partial_\infty M = \varphi$.

Let $x_0 \in \partial_\infty M$ and $\varepsilon > 0$ be fixed. By the continuity of the function φ we find a constant $L \in (8/\pi, \infty)$ so that

$$|\varphi(y) - \varphi(x_0)| < \varepsilon/2$$

whenever $y \in C(v_0, 4/L) \cap \partial_\infty M$, where $v_0 = \dot\gamma_0^{o,x_0}$ is the initial direction of the geodesic ray representing x_0. Taking (4.15) into account, we can choose R_3 in Lemma 5.5 so big that $u_+(r) \le ||\varphi||_\infty + \varepsilon/2$ when $r \ge R_3$.

We will show that

$$w^-(x) \coloneqq -\psi(x) + \varphi(x_0) - \varepsilon \le u(x) \le w^+(x) \coloneqq \psi(x) + \varphi(x_0) + \varepsilon \tag{6.2}$$

in the set $U \coloneqq 3\Omega \setminus \bar{B}(o, R_3)$. Here $\psi = A(R_3^\delta r^{-\delta} + h)$ is the supersolution from the Lemma 5.5 and $A = 2||\varphi||_\infty$.

Again, by the continuity of the function φ in $\bar{M}$, we can choose k_0 such that $\partial B_k \cap U \neq \emptyset$ and

$$|\varphi(x) - \varphi(x_0)| < \varepsilon/2 \tag{6.3}$$

for every $x \in \partial B_k \cap U$ when $k \geq k_0$. We denote $V_k = B_k \cap U$ for $k \geq k_0$ and note that

$$\partial V_k = (B_k \cap \bar{U}) \cup (\partial U \cap \bar{B}_k).$$

We prove (6.2) by showing that

$$w^- \leq u_k \leq w^+ \tag{6.4}$$

holds in V_k for every $k \geq k_0$.

Let $k \geq k_0$ and $x \in \partial B_k \cap \bar{U}$. Since $u_k|\partial B_k = \varphi|\partial B_k$, (6.3) implies

$$w^-(x) \leq \varphi(x_0) - \varepsilon/2 \leq \varphi(x) = u_k(x) \leq \varphi(x_0) + \varepsilon/2 \leq w^+(x).$$

By Lemma 5.3

$$h|M \setminus \big(2\Omega \cup B(o, R_1)\big) = 1$$

and since $R_3^\delta r^{-\delta} = 1$ on $\partial B(o, R_3)$ we have

$$\psi \geq A = 2||\varphi||_\infty$$

on $\partial U \cap B_k$. Since u_+ from Lemma 4.1 is global supersolution with $u_+ \geq ||\varphi||_\infty$ on ∂B_k, the comparison principle gives $u_k|B_k \leq u_+|B_k$ and by the choice of R_3, we have

$$u_k \leq ||\varphi||_\infty + \varepsilon/2$$

in the set $B_k \setminus B(o, R_3)$.

Putting all together, it follows that

$$w^+ = \psi + \varphi(x_0) + \varepsilon \geq 2||\varphi||_\infty + \varphi(x_0) + \varepsilon \geq ||\varphi||_\infty + \varepsilon \geq u_k$$

on $\partial U \cap \bar{B}_k$. Similarly we have $u_k \geq w^-$ on $\partial U \cap \bar{B}_k$ and therefore $w^- \leq u_k \leq w^+$ on ∂V_k. By Lemma 5.5 ψ is a supersolution in U and hence the comparison principle yields $u_k \leq w^+$ in U. On the other hand, $-\psi$ is a subsolution in U, so $u_k \geq w^-$ in U, and (6.4) follows. This is true for every $k \geq k_0$ so we have (6.2). Since $\lim_{x \to x_0} = 0$, we have

$$\limsup_{x \to x_0} |u(x) - \varphi(x_0)| \leq \varepsilon.$$

The point $x_0 \in \partial_\infty M$ and constant $\varepsilon > 0$ were arbitrary so this shows that u extends continuously to $C(\bar{M})$ and $u|\partial_\infty M = \varphi$. Finally, the uniqueness follows from the comparison principle.

□

7. Non-existence result

In the following, we state a non-existence result for the prescribed weighted mean curvature graph equation by adapting the approach of Pigola, Rigoli and Setti in [15]. We denote by $A(r)$ the area of the geodesic sphere $\partial B(o, r)$ centred at a fixed point $o \in M$.

Proposition 7.1. *Let $p\colon [0, \infty) \to [0, \infty)$ be a continuous function such that for some $\bar{R} > 0$ and for all $r \geq \bar{R}$ at least one of the following conditions is satisfied:*

$$\frac{\exp\left(D\left(\int_0^r \sqrt{p(s)}\mathrm{d}s\right)^2\right)}{\varrho_0(r)^2 A(r)} \notin L^1(+\infty) \tag{7.1}$$

for some constant $D>0$ and a smooth function ϱ_0, so that $\varrho(x)\le\varrho_0(r(x))$, or

$$\frac{\left(\int_r^{3r/2}\sqrt{p(s)}\mathrm{d}s\right)^2}{r\log\left(\varrho_0(2r)^2\operatorname{vol}(B(o,2r))\right)}\ge h(r)\notin L^1(+\infty) \tag{7.2}$$

with some continuous and monotonically non-increasing $h\colon[\bar{R},\infty)\to(0,\infty)$. Let $u,v\in C^2(M)$ satisfy

$$\begin{aligned}\operatorname{div}_{-\log\varrho}\frac{\nabla u}{\sqrt{\varrho^{-2}+|\nabla u|^2}}-\operatorname{div}_{-\log\varrho}\frac{\nabla v}{\sqrt{\varrho^{-2}+|\nabla v|^2}}&=q(x)\\ &\ge p\big(r(x)\big)\varrho_0\big(r(x)\big)\ge 0,\end{aligned} \tag{7.3}$$

and

$$\sup_M(u-v)<+\infty.$$

Then, if $q\not\equiv 0$, there are no solutions to (7.3).

Proof. The proof is very similar to that in [15], the only differences being our use of the divergence operator with respect to the weighted volume form $\varrho\mathrm{d}M$ and a suitable form of the Mikljukov-Hwang-Collin-Krust inequality which in our setting reads as follows

$$\begin{aligned}&\left\langle\frac{\nabla u}{\sqrt{\varrho^{-2}+|\nabla u|^2}}-\frac{\nabla v}{\sqrt{\varrho^{-2}+|\nabla v|^2}},\nabla u-\nabla v\right\rangle\\ &\ge\frac{1}{2}\left(\sqrt{\varrho^{-2}+|\nabla u|^2}+\sqrt{\varrho^{-2}+|\nabla v|^2}\right)\left|\frac{\nabla u}{\sqrt{\varrho^{-2}+|\nabla u|^2}}-\frac{\nabla v}{\sqrt{\varrho^{-2}+|\nabla v|^2}}\right|^2\\ &\ge\varrho^{-1}\left|\frac{\nabla u}{\sqrt{\varrho^{-2}+|\nabla u|^2}}-\frac{\nabla v}{\sqrt{\varrho^{-2}+|\nabla v|^2}}\right|^2.\end{aligned}$$

Together these result in the extra factors of ϱ_0 in (7.1), (7.2), and on the right hand side of (7.3). Taking into account these differences the proof in [15] applies almost verbatim. □

As direct corollaries of the previous theorem, we have

Corollary 7.2. *Let u be a bounded solution to*

$$\operatorname{div}_{-\log\varrho}\frac{\nabla u}{\sqrt{\varrho^{-2}+|\nabla u|^2}}=nH(x)\quad\text{in } M,$$

with $H\ge 0$.

(i) *Suppose that $\varrho(x)\le\varrho_0\big(r(x)\big)\le r(x)^{\beta_1}$, $\beta_1>0$, and that $A(r)\le r^{\beta_2}$, $\beta_2>0$, for large values of $r=r(x)$. Then*

$$\liminf_{r(x)\to\infty}H(x)\cdot\frac{r(x)^2\log r(x)}{\varrho_0\big(r(x)\big)}=0.$$

(ii) *Suppose that $\varrho(x)\le\varrho_0\big(r(x)\big)\le e^{\beta_1 r(x)}$, $\beta_1>0$, and that $A(r)\le e^{\beta_2 r}$, $\beta_2>0$, for large values of $r=r(x)$. Then*

$$\liminf_{r(x)\to\infty}H(x)\cdot\frac{r(x)\log r(x)}{\varrho_0\big(r(x)\big)}=0.$$

(iii) *Suppose that $\varrho(x)\le\varrho_0\big(r(x)\big)\le e^{\beta_1 r(x)^2}$, $\beta_1>0$, and that $A(r)\le e^{\beta_2 r^2}$, $\beta_2>0$, for large values of $r=r(x)$. Then*

$$\liminf_{r(x)\to\infty}H(x)\cdot\frac{\log r(x)}{\varrho_0\big(r(x)\big)}=0.$$

Proof. By choosing $p(s) = (s^2 \log s)^{-1}$ in (i), we see that (7.1) holds, and therefore the claim follows. Similarly, choosing $p(s) = (s \log s)^{-1}$ in (ii) or $p(s) = (\log s)^{-1}$ in (iii), the condition (7.2) holds and the claim follows. □

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