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FÉLIX EDUARDO MAPURUNGA DE MELO

MODELING AND LINEAR PARAMETER-VARYING IDENTIFICATION OF A TWO-TANK SYSTEM

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Master's dissertation presented to the graduate program in Electrical Engineering from Federal University of Ceará as part of the requisites to obtain the Master's degree in Electrical Engineering. Concentration Area: Electrical Energy systems.

Supervisor: Prof. Fabrício Gonzalez Nogueira
Co-supervisor: Prof. Arthur P. de Souza Braga

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EXAMINATION BOARD

Prof. Fabrício Gonzalez Nogueira (Supervisor)
Universidade Federal do Ceará

Prof. Arthur P. de Souza Braga (Co-supervisor)
Universidade Federal do Ceará

Prof. Wilkley Bezerra Correia
Universidade Federal do Ceará

Prof. Guilherme de Alencar Barreto
Universidade Federal do Ceará

Prof. José Everardo Bessa Maia
Universidade Estadual do Ceará

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ABSTRACT

This work addresses the modeling and the linear parameter-varying (LPV) system identification of a coupled two-tank system (TTS). The system is a multiple input multiple output (MIMO) with two inputs and two outputs. In order to obtain a suitable model for this system, a first-principle approach based on the mass balance principle is followed. It turns out that the modeling process was driven by the geometrical shape of the tanks. Thus, most of its parameters are based on the tanks' dimensions. When it comes to the LPV identification, several methods are presented ranging from the classical results from the regression approach to the current support vector machines (SVM) based methods. All the identification algorithms presented are extended in order to cope with the MIMO systems. Additionally, a method based on instrumental variables support vector machines was adapted from the general nonlinear case to the LPV case. A new LPV model with two independent scheduling variables is proposed driven by prior knowledge on the process model. The results obtained with this new LPV model have showed a good performance in describing the TTS behavior. Furthermore, they were better than an LPV model considering only a single scheduling variable.

Keywords: two-tank system. LPV system identification.

RESUMO

Este trabalho lida com a modelagem e identificação com abordagem de sistemas com parâmetros variantes (LPV) de um sistema de dois tanques acoplados (TTS). Esse sistema é do tipo múltipla entrada múltipla saída (MIMO) com duas entradas e duas saídas. Com a finalidade de obter um modelo adequado para esse sistema, é feita uma abordagem fenomenológica baseada no princípio do balanço de massa. Descobre-se que o processo de modelagem é dependente da forma geométrica dos tanques. Assim, a maioria dos seus parâmetros são baseados nas dimensões dos tanques. Quando se trata de identificação de sistemas LPV, vários métodos são apresentados desde os resultados clássicos baseados em regressão até os métodos atuais baseados em máquinas de vetor de suporte. Todos os algoritmos de identificação apresentados são estendidos para lidar com sistemas MIMO. Além disso, um método baseado em variáveis instrumentais com máquinas de vetor de suporte foi adaptado do caso não linear geral para o caso LPV. Um novo modelo LPV com duas variáveis de *scheduling* é proposto baseado em conhecimento a priori no modelo do processo. Os resultados obtidos com esse novo modelo LPV mostraram bom desempenho ao descrever o comportamento do sistema de dois tanques. Ademais, eles foram melhores do que um modelo LPV considerando apenas uma variável de *scheduling*.

Palavras-chave: Sistemas de dois tanques. Identificação de modelos LPV.

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LIST OF ABBREVIATIONS AND ACRONYMS

ARMA	Autoregressive-Moving Average
ARX	Autoregressive with Exogenous Output
BFS	Best Fit Score
BJ	Box Jenkins
IO	Input-Output
IR	Impulse Response
IV	Instrumental Variable
IV-SVM	Instrumental Variable Support Vector Machines
IVM	Instrumental Variable Method
LS	Least Squares
LS-SVM	Least Squares Support Vector Machines
LMS	Least Mean Squares
LPV	Linear Parameter-Varying
LTI	Linear Time Invariant
MIMO	Multiple Input Multiple Output
OBF	Orthonormal Basis Functions
OE	Output Error
PE	Prediction Error
PEM	Prediction Error Method
PES	Persistently Exciting Signal
PID	Proportional Integral Derivative
q-LPV	Quasi-Linear Parameter-Varying
QP	Quadratic Programming
RBF	Radial Basis Function

RIV	Refined Instrumental Variable
RLS	Recursive Least Squares
SISO	Single Input Single Output
SS	State-Space
SVM	Support Vector Machines
TTS	Two-Tank System

LIST OF SYMBOLS

$\mathbb{P}$	Scheduling space
$\mathbb{R}$	Real set
$\mathbb{Z}$	Integers set
$\mathfrak{D}$	Data set
$\mathbb{E}$	Mathematical expectation
$\mathscr{L}$	Lagrangian
$\mathscr{M}$	Model
e	White noise
G	Process Filter
h	water height
H	Noise Filter
k	Valve constant (Tank models), discrete time
p	Scheduling variable
q	Volumetric flow-rate, Time shift operator
r	Radius
u	Input of a model
v	Noise process
$\mathfrak{v}$	Nonlinear mapping
V	Volume, Quadratic error cost function
y	Output of a model
α	Learning rate (LMS), Lagrangian multipliers (LS-SVM)
γ	Regularization parameter
ζ	Instrumental variable vector
η	Parameter vector related to the noise filter

θ	Parameters vector
ρ	Specific mass, Parameter vector related to the process filter
φ	Regressors vector
$\Phi_u(\omega)$	Spectrum of the signal u
ψ	Basis function
ω	Mass flow-rate, Parameter vector (LS-SVM)
Ω	Kernel matrix (LS-SVM)

CONTENTS

1 INTRODUCTION

During many years the linear time invariant(LTI) framework has dominated the industry, mainly due to its simplicity and good performance. In fact, the LTI approach has been of such paramount importance to this date that, according to Åström and Hägglund (1995) in the nineties about 90% of the control loops were of PID type, and the use of this approach has never dropped since then. However, the increasing quest for performance requires a framework that is still simple but has better representative behavior of the nonlinear dynamics.

The Linear Parameter-Varying (LPV) systems are inspired in the gain-scheduling strategy (ÅSTRÖM; WITTENMARK, 1994), which consists in the point of view that a nonlinear system can be represented as a collection of LTI systems where each system represents an operation condition. The LPV framework is intended to play the role of the bridge between the lack of general structure in nonlinear systems and the well organized world of linear systems. The LPV systems preserve the linear behavior between input and output for a constant scheduling variable, which is usually an exogenous signal related to the system's working point.

The LPV class cope with nonlinearities with the advantage of a linear formulation. This makes the LPV models an attractive candidate to model nonlinear behavior and time-varying phenomena. The challenges of today's industry are driving the search for more accurate models. Besides, nowadays the industry requires an ability to cope with systems in many industrial scenarios, such as plants working in a wide range of operation points. This usually requires that the controller have adaptive properties. The LPV framework offers a representative system class to deal with these new challenges. In fact, the LPV model class can be viewed as an extension of the linear time varying (LTV) class (TÓTH, 2010).

In order to make clear the paramount importance of the scheduling variable in the LPV framework, consider a model of an aircraft. This system has three inputs, the elevator, canard, and leading edge flaps, in which the pilot can control the pitch movement of the aircraft. These inputs are related to devices in the wings that control the direction of air flow through the wings. Thus, a model that relates these inputs with the pitch rate can be built (or identified) to describe the dynamic behavior of the system. However, the flight dynamics changes according to the altitude of the aircraft. In this way, the model is naturally influenced by the altitude. In this example, the altitude is an exogenous signal that is directly related to the operational condition of the system. Moreover, the pilot can not avoid the influence of the altitude in the system dynamics. The pitch control system of the aircraft can be considered as an LPV model with the altitude playing the role of scheduling variable.

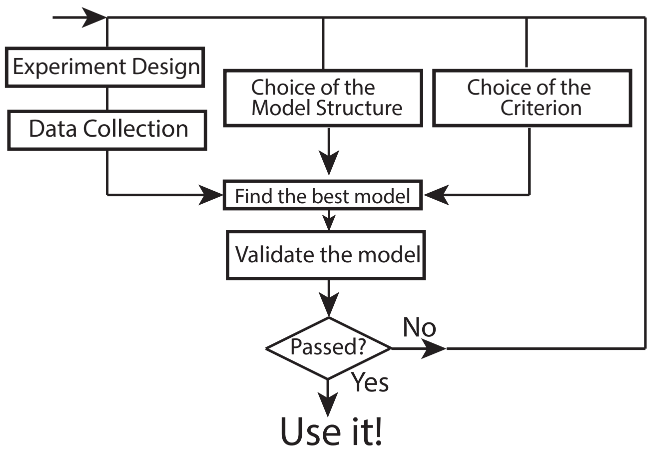
This chapter is organized as follows. In the first section the system identification problem is presented and its steps are given. The second section describes the state-of-the-art of LPV system identification. The third section presents the current work and its objectives.

1.1 The System Identification Problem

The system identification problem can be stated as follows. Given a data set of input-output measurements $\mathscr{D}^N = \{u(k), y(k)\}_{k=1}^{N}$, a model structure and a fitness criterion, typically a cost function based on the model error, find the best model among all feasible models in the collection that best describes the input-output behavior of the system at hand (LJUNG, 1999). These three entities together form the basic core of the system identification itself.

Although identification of dynamical systems has a well-defined logical flow, there are several considerations that one must take in order to successfully identify a representative model. There is a natural logical procedure to the system identification task. Figure 1 illustrates the identification scheme, which has a cycled nature. In the following, a brief overview of each step

Figure 1 – The Identification procedure



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is given.

Experiment Design and Data Collection

In this step the user must answer some questions, such as: which are input and output signals, how to collect these data, when will it be collected, is there any signal conditioning to be made. Additionally, there is the question of selecting an appropriate input in order to give valuable information about the system to be modeled. One input that produces an output with enough information content is said to be a persistently exciting signal (PES). The latter question is known as experiment design. In addition to the information problem, the signal to be used as input generally needs to be feasible due to practical issues, e.g. input constraints. Another focus of the experiment design related to PES is how to produce informative data sets, which must have enough information content to distinguish between different models in the model collection (informativity is a property of a data set and it is dependent on the model structure. PES is a property of a signal and it is independent of the model structure). In some cases the user doesn't have the possibility to interfere in the system variables at hand and there are other cases where the user must design a controller in order to make the experiment, e.g. unstable plants. Data preprocessing is another issue to be dealt with, which focuses on attenuation of disturbances, removal of trends, exclusion of outliers, and noise effects in data.

Choice of the Model Structure

In this part of the identification procedure a model structure must be chosen and it will determine the set of models that one is searching for the best model. In this step, questions concerning the representation form of the model (State-Space (SS), input-output (IO), series-expansion, etc.), parametrization, type of noise modeling, and choice of the model order, must be answered. Here, a priori knowledge and engineering insight must be combined to carefully adjust the model towards its actual behavior. The model structure is directly related to the algorithm that selects the best model in the set. Therefore, questions like existence of local optimal must be considered within the choice of the set of candidates. The complexity of the model (e.g. the number of parameters in the model) is related to the well-known bias-variance trade off. Hence, this question must be considered as well in the choice of the model structure.

Choice of the Identification Criterion

It consists in selecting a performance criteria in order to classify the models in the model set. The assessment of model quality is generally based on the performance of the model when attempting to reproduce the data. The user must look for a criterion that is able to select in the model set the model which best describes the measured data set $\mathscr{D}^N$. Usually, in the system identification literature, a quadratic norm of the error of the output prediction of the model estimate is chosen (LJUNG, 1999).

Selection of the Best Model

In this phase an algorithmic solution (a mathematical expression that delivers the best model) is obtained in terms of model structure and the identification criterion selected. It is in this step, also known as identification method, that the best model is chosen. The method itself can be obtained from an algebraic or a probabilistic point of view among other mathematical perspectives. It must be noticed that the method selects the best model in the model set according to one's chosen identification criterion and that is an important part of the identification cycle, but it is not the most essential one.

Model (in)validation

This step is where one must confront the model with some procedures in order to decide whether the model can be accepted. This question is directly related with the user's purposes for the model. The model should pass some tests that involve how the model relates to observed data, prior knowledge of the system at hand, and its intended use. Such tests are known as model validation. The model that presents poor behavior confronted with the data should be discarded and the identification cycle must run once again from the first step in order to obtain another model.

The System Identification Paths

The system identification procedure has a very clear logical flow. Firstly, collect the data and choose a model set. Secondly, pick the best model in the model set according to a chosen criterion. Finally, perform the tests in order to validate the model. If the model passes, then accept the model, else repeat the procedure using different choices from the beginning. The model may be deficient for many reasons including (LJUNG, 1999):

a) The identification method failed to find the best model according to the identification criterion;

b) The identification criterion was not well-chosen;

c) The model structure was not appropriate, that is, it didn't provide any good description in the model set;

d) The data set was not informative enough.

It is important to understand how the system identification has developed before entering in the LPV identification framework. Basically, when the system descriptor is expressed in the IO setting, the identification problem is formulated through a regression approach. When it comes to SS representation, the system identification is mainly focused on subspace methods, that is, the identification problem is based on certain space projections, see (VAN OVERSCHEE; DE MOOR, 1996) for a detailed overview. The LPV system identification naturally followed the developments of the LTI framework, with the necessary adaptations, of course. The reason lies basically in the fact that LPV models can be seen as an extension of the LTI models, then the most natural way to develop the LPV system identification was to extend the LTI results.

1.2 State-of-the-Art in LPV System Identification

A brief resume of the LPV identification developments is necessary in order to understand what it has done and what may still be coming. Similarly, the LPV identification followed the two basic procedures of the LTI framework, which are dependent on the system representation. In the case of the IO setting, the LPV framework was done exclusively in the regression form, whereas for the SS representation the natural extension was the subspace approach. Due to the lack of transfer function representation in the LPV framework, the LPV identification emerged based on an algorithmic sense, that is, the first works in the LPV identification literature are based on optimization problems, such as (BAMIEH; GIARRÉ, 1999b; BAMIEH; GIARRÉ, 1999a; LEE; POOLLA, 1996; LEE; POOLLA, 1999). In fact, the very first attempt to address the LPV identification problem was done in (NEMANI; RAVIKANTH; BAMIEH, 1995). In that work it was assumed full knowledge of the state sequence and considered only one scheduling variable. In (PREVIDI; LOVERA, 1999) it is attempted to solve the problem by separating the linear and nonlinear parts. The former was performed in a regression form, the latter by using an artificial neural network. Another approach was a robust identification via worst-case identification presented in (MAZZARO; MOVSICHOFF; SÁNCHEZ-PEÑA, 1999).

The Least Mean Squares(LMS) and the Recursive Least Squares(RLS) were introduced in (BAMIEH; GIARRÉ, 1999b; BAMIEH; GIARRÉ, 1999a) for the LPV identification framework. However, differently of the LTI counterpart, for the regression approach in the LPV identification it is necessary to define a parametrization of the scheduling variable. That is, the model output must be linear in parameters(necessary condition for linear regression problems), and each regressor must be defined as a function of the scheduling variable. Here are defined the basis functions that play an important role in the LPV system identification framework. These basis functions are part of the user's choice and give a huge degree of freedom compared to the LTI case, common choices are polynomial and periodic functions such as sine. In (BAMIEH; GIARRÉ, 2002) the identification was investigated using a polynomial basis and an introductory result of persistently excitation signals was presented for the polynomial dependence case.

The first subspace approach for the LPV identification was addressed in (VERDULT; VERHAEGEN, 2001), but still with its roots in the optimization problem. A huge drawback of the subspace approach in the LPV framework is the curse of dimensionality, usually the dimensions of the data matrices involved grow exponentially. In fact, the subspace approach in the LPV case was inspired by the subspace identification procedures developed for bilinear systems. Usually, in the subspace approach for LPV systems, it is commonly assumed that the matrices involved have an affine dependence on the scheduling variable. In (VERDULT; VERHAEGEN, 2002) it was presented an extension of the subspace approach used in the bilinear systems to the LPV case. In that work a first step was taken in order to overcome the curse of dimensionality, a procedure was given to select a subset of the most dominant rows from the data matrices. Another successful approach to avoid the curse of dimensionality in the subspace approach was the use of the Kernel trick to avoid unnecessary matrix computations as introduced in (VERDULT; VERHAEGEN, 2005).

Another approach to represent LTI systems is through orthonormal basis functions (OBF) representation. This kind of representation appeared in the LTI framework, see (HEUBERGER; VAN DEN HOF; WAHLBERG, 2005) for an overview, and it was initially introduced in the

LPV case in (TÓTH; HEUBERGER; VAN DEN HOF, 2006a). In sequence a fuzzy clustering approach was developed to select pole locations for OBFs in the LPV identification problem in (TÓTH; HEUBERGER; VAN DEN HOF, 2006b). As the LPV models can be seen as a collection of LTI models, many of the identification problems are solved as identification of local LTI models and then interpolation is applied in order to obtain the LPV model. In this way, LPV models can be viewed under two perspectives: the global and the local approach. The former consists in identifying an LPV model trying to capture the dynamic relationship of the system with a varying scheduling parameter. The latter considers the identification of many local LTI models (at a constant scheduling variable) and then it applies an interpolation scheme to obtain the global model.

The subspace approach continued giving results based on a convergent sequence of linear deterministic-stochastic state space approximations in (LOPES DOS SANTOS; RAMOS; MARTINS DE CARVALHO, 2007). Many of the subspace methods presented take advantage of the LTI subspace approach, as in (FELICI; VAN WINGERDEN; VERHAEGEN, 2007) where a subspace method was developed capable of determining the deterministic part of an LPV-SS system in the presence of output error, one of the key aspects was to ensure that the scheduling variable must be periodic, it turned out this made the algorithm more computationally efficient. A global and local approach to identify LPV systems based on OBFs representation was introduced in (TÓTH; HEUBERGER; VAN DEN HOF, 2007).

Neither of the identification approaches presented so far dealt with LPV systems in the view of system theory. Moreover, all of these approaches didn't hold any connections. This means that there were no concerns about the differences between IO and SS domains. It was only with the development of LPV system theory initially introduced in (TÓTH et al., 2007) that the IO domain and the SS representation were connected via a realization theory through an affine canonical representation for LPV-SS models similar to the LTV framework. As a matter of fact, many of the control and interpolation based identification papers assumed that the LTI framework intuitively would extend to the LPV case. It turns out, it was demonstrated that this extension didn't follow as straightforward as it seemed. Until 2010, much effort had been made to estimate LPV models at light of estimation problems from LTI system identification, but only few efforts were done to develop a proper LPV system theory. The development of the LPV system theory had crescent interest with the introduced study of optimal design for local experiments in (KHALATE et al., 2009), the investigation of discretization methods for the LPV framework in (TÓTH et al., 2008), state space realization methods in (ABBAS; TÓTH; WERNER, 2010), and finally the introduced behavioral approach, see (WILLEMS, 1991) for a detailed overview, in (TÓTH et al., 2009; TÓTH et al., 2011). These works allowed a strong basis in the LPV system theory and, more importantly, answered questions on how LPV systems should be dealt with in the eye of system theory. An OBF based system identification that gives enough flexibility to LPV models was presented in (TÓTH; HEUBERGER; VAN DEN HOF, 2008). In (BUTCHER; KARIMI; LONGCHAMP, 2008) the instrumental variable method was introduced in the LPV framework from the LTI counterpart. Following such approach, an algorithm also from the LTI case, called refined instrumental variable method was introduced in the LPV framework (LAURAIN et al., 2010a; LAURAIN et al., 2010b). While in the subspace approach an algorithm was developed to cope with LPV and bilinear identification in both open and closed-loop setting in (VAN WINGERDEN; VERHAEGEN, 2009). An instrumental variable method for closed-loop LPV identification was presented in (TÓTH et al., 2011; TÓTH et al., 2012) within the IO setting. The formal introduction of the prediction error method in the LPV case was only made in (TÓTH; HEUBERGER; VAN DEN HOF, 2010). So far, most algorithms presented dealt with discrete time LPV models. In (LAURAIN et al., 2011a; LAURAIN et al., 2011b) the continuous time identification of LPV systems was addressed in the IO settings.

Another strong add in the LPV system identification was the introduction of the least squares support vector machines (LS-SVM) from the machine learning field in (TÓTH et al., 2011). The introduction of LS-SVMs in the LPV framework was very important, mainly due to the learning appeal and the use of Kernels that can learn the underlying parameter's dependency with the scheduling variable. This was a huge advantage over the general methods in regression form, in which one must select an appropriate basis function to define the underlying relationship between regressors and the scheduling variable. As the kernel method became an interesting feature in LPV system identification in IO setting, the subspace approach made important steps toward regularization techniques, such as in (GEBRAAD et al., 2011), where a novel approach using nuclear norm regularization is proposed in the LPV subspace approach. The purpose remained quite the same which is to cope with the curse of dimensionality in data matrices. In fact, the use of kernels dominated the first half of the decade and it is still an active area of research in the LPV identification literature. The LPV LS-SVM identification is investigated under general noise conditions in (LAURAIN et al., 2012). A common assumption in most of the works presented so far is that the scheduling variable is a free noise measured signal. In (LOPES DOS SANTOS et al., 2012) an extension of the algorithm presented in (LOPES DOS SANTOS; RAMOS; MARTINS DE CARVALHO, 2009) is generalized to cope with quasi-stationary scheduling sequences. A separable least squares approach was extended to the LPV case in (LOPES DOS SANTOS et al., 2013). An algorithm that identifies the LPV order in the LS-SVM framework was introduced in (PIGA; TÓTH, 2013). Before that, a study on LPV-ARX order selection had been done in (TÓTH; HJALMARSSON; ROJAS, 2012). The general LS-SVM for the LPV case was extended to cope with noisy scheduling variables in (ABBASI et al., 2014). The separable least squares approach was extended to the LPV LS-SVM framework in (LOPES DOS SANTOS et al., 2014).

With all the development in the kernel and regularized parameter estimation, the current LPV system identification is now focusing toward a Bayesian perspective, initially introduced in (GOLABI et al., 2014). An identification procedure based on correlation analysis was given in (COX; TÓTH; PETRECZKY, 2015). A Gaussian process based Bayesian method that accounts for noisy scheduling variables for LPV models in IO setting is presented in (ABBASI et al., 2015).

An algorithm is proposed to correctly estimate LPV models under general noise conditions of Box-Jenkins type in the Bayesian approach in (DARWISH et al., 2015). An instrumental variable scheme is introduced in (PIGA et al., 2015) to cope with noise both in scheduling variable and in system output. A new method that combines the global and local approaches in the identification of LPV systems was given in (TURK; PIPELEERS; SWEVERS, 2015). An instrumental variable based on LS-SVM was introduced in (RIZVI et al., 2015b) for LPV-SS models. Still in the LS-SVM framework, an approach based on kernel was introduced in the SS structure to identify multiple-input multiple-output LPV systems (RIZVI et al., 2015a). A kernel based approach was also introduced in the subspace methodology in (PROIMADIS; BIJL; VAN WINGERDEN, 2015). The Bayesian framework was also extended to the LPV-SS structure in (COX; TÓTH, 2016). A presentation in Kalman style realization theory for LPV-SS with affine dependence on the scheduling variable is given in (PETRECZKY; TÓTH; MERCERE, 2016). A methodology to construct the kernels in the LS-SVM approach for LPV system was presented in (ROMANO et al., 2016). Still in the LS-SVM context, a general approach for identification of partial differential equation-governed by spatially-interconnected LPV systems was given in (LIU et al., 2016). A study of the Bayesian approach in LPV system identification to accurately model nonlinear processes was given in (GOLABI et al., 2017), while in the LPV susbpace approach a predictor-based tensor regressor was introduced in (GUNES; VAN WINGERDEN; VERHAEGEN, 2017).

It is clear that either in the IO setting or in the SS domain the kernel-based and regularization approaches remain the current focus of research in academia. This intense research and the contact with the machine learning community allowed the development of kernel methods and the LS-SVM approach in a wide range of system identification areas. The development of Gaussian processes and the development of new kernel methods that incorporate prior information about the unknown system (PROIMADIS; BIJL; VAN WINGERDEN, 2015) attracted once more the research interest in the Bayesian approach for all system identification branches. Table 1 presents a summary of the works presented in this section.

1.3 The Present Work

This work addresses the identification of LPV systems under the IO approach. From the classical results of the least squares and instrumental variable method to the current state-of-the-art kernels methods. Additionally, in this work a model from first-principles(laws of physics) of a two tank process system and its identification is set under the LPV framework. The latter task is addressed through the simulation of the model obtained from first-principles. This work has the following objectives:

a) To model a nonlinear representation of a multiple input multiple output (MIMO) tank system;

b) Identify an LPV MIMO system for the two tank system.

Table 1 – Summary of the LPV Identification methods

Representation	Method	Work
Regression	RLS-LMS	Bamieh and Giarré, 2002
	IV	Butcher, Karimi and Longchamp, 2008
	RIV	Laurain et al. 2010b
	LS-SVM	Tóth et al. 2011
	CRIV	Piga et al. 2015
	IV-SVM	Laurain et al. 2015
	Bayesian	Abbasi et al. 2015
Subspace	Gradient	Verdult and Verhaegen, 2001
	Row Selection	Verdult and Verhaegen, 2002
	Kernel	Verdult and Verhaegen, 2005
	PBSID	van Wingerden and Verhaegen, 2009
	SLS	Lopes dos Santos et al. 2014
	PBTR	Gunes, van Wingerden and Verhaegen, 2017
Series	OBF	Tóth, Heuberger and Van Den Hof, 2006a
	OBF-Fuzzy	Tóth, Heuberger and Van Den Hof, 2006b
	OBF	Tóth, Heuberger and Van Den Hof, 2009

Source: The Author

This work is organized as follows, in Chapter 2 the nonlinear modeling of the two-tank system (TTS) is discussed. In Chapter 3 the identification methods are presented for the LPV framework under IO approach. In Chapter 4 simulation results are analyzed in an LPV identification sense. The conclusions of this work are presented in Chapter 5.

2 TWO-TANK SYSTEM MODELING

In this chapter, it will be shown the nonlinear modeling of the two-tank system (TTS) with focus on its dynamic behavior, which will be the focus of discussion in this chapter. The organization of this chapter is as follows. The first section deals with TTS process description and its assumptions. In the second section a mathematical model of a cylindrical tank system is given, whereas in the third section a mathematical description of a complex cylindrical-conical tank is shown. The fourth section brings together the results from the previous sections in order to provide a high-fidelity model of the TTS. The fifth section shows the measurement and actuator systems of the TTS and their components. Finally, the sixth section deals with the experimental procedures to obtain some of the involved variables in the modeling of the TTS.

2.1 Overview of the System

The Two-Tank System is a Multiple-Input Multiple-Output (MIMO) system consisting of two coupled tanks. The complete system can be seen in Figure 2. The system is composed

Figure 2 – The Two-Tank System

Source: The Author

by two tanks, one in the left has constant sectional area, whereas the one on the right side is a mixture of a cylindrical and conical shape. For reference reasons the left tank will be named Tank 1, while the right tank will be named Tank 2. Both tanks are connected by a tube with a valve, which regulates the amount of water that passes through one tank to another. Additionally, there are two valves in each exit of the tanks, that allow the liquid to return directly to the collecting reservoir. The different setting of these valves can modify the behavior of the system from two independent SISO systems (the interconnected valve completely closed) to a condition where there is no flow to the collecting reservoir (both exit valves closed, in this condition the system could not).

There is an additional 30 liter tank below the two-tank system which plays the role of collecting reservoir, equipped with two pumps for flowing up the water back to the top of the tanks. Both flow-rates are controllable variables and they are adopted as inputs of the system. The outputs can be chosen as the water heights from each tank.

Assumptions of the System

In order to model the two-tank system based on first-principles some assumptions must be made.

a) It is assumed that water is an incompressible fluid and its specific weight is constant;

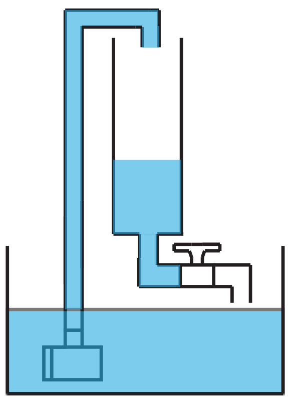
b) There is no pressure drop neither in the tubes nor in the valves.

2.2 Mathematical Model of the Tank 1

Mass balance is the physical principle that governs the tank model. It is based on the conservation of mass, which takes into account the material entering and leaving the system. The mass balance principle states that the mass that enters a system must, by conservation of mass, either leave the system or accumulate within it.

In order to build the model of Tank 1 consider the isolated system in Figure 3.

Figure 3 – Isolated Tank 1 System



Source: The Author

By using the mass balance principle, the difference between mass that enters and mass that leaves must be equal to:

$$\frac{dm}{dt} = \omega_i - \omega_o\,, \tag{2.1}$$

where m is the mass of water in the tank given in Kg, ω_i and ω_o are, respectively, the mass flow rate input and the mass flow rate output, expressed in Kg/s. These mass flow rates can be converted to volumetric flow rate by observing that $m = V\rho$, where V is the volume given in m^3 and ρ is the specific weight of the fluid, expressed as Kg/m^3 . Then, Eq. 2.1 assumes the following form:

$$\frac{dV}{dt} = q_i - q_o \,, \tag{2.2}$$

where q_i is the volumetric flow rate input and q_o is the volumetric flow rate output, both given in m^3/s. In order to complete the task of modeling the dynamical behavior of the system, it is necessary to relate Eq. 2.2 only with the water height h given in meters (output) and the volumetric flow rate of the pump q_i (input). To do this, it is required a relationship between the water volume in the tank and water height. Because the tank has a cylindrical shape, it is well-known that the volume of a cylinder is given by:

$$V = \pi r^2 h \,, \tag{2.3}$$

where r is the radius of the cross-sectional area of the cylinder expressed in m and h is the water height in the tank. However, the model in Eq. 2.2 needs the derivative of water volume in the tank. Then, it is necessary to calculate the derivative of Eq. 2.3 with respect to time. This can be accomplished using the derivative chain rule as following:

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dh}\frac{dh}{dt} \\ &= \pi r^2 \frac{dh}{dt} \,, \end{aligned} \tag{2.4}$$

It is worth to remind that the radius r is constant. Now, it is only necessary to relate the volumetric flow rate output either with the volumetric flow rate input or with water height. This can be done using Bernoulli's equation (HALLIDAY; RESNICK; WALKER, 2013, p. 401). If both water surface and the return pipe are subject to atmospheric pressure and assuming a laminar flow, it can be shown that the speed of water in the pipe is:

$$v_o = \sqrt{2gh} \,, \tag{2.5}$$

where g is the gravitational acceleration given in m/s^2. See Halliday, Resnick and Walker (2013) for details on this result. The volumetric flow rate at the return pipe can be found as $q_o = a v_o$, where a is the cross-sectional area of the pipe expressed in m^2. Generally, the flow rate output has the following form:

$$q_o = k\sqrt{h} \,, \tag{2.6}$$

where k is a constant expressed in m$^{2.5}$/s that depends on the type of flow, cross-sectional area of the pipe, the length of the pipe and the gravitational acceleration, see Garcia (2013) for more information. Putting together Eq.s 2.6 and 2.4 in Eq. 2.2 leads to the final model as:

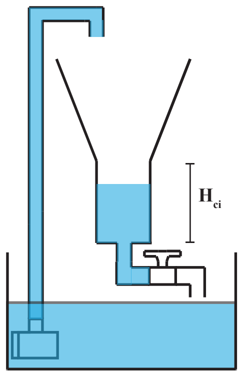
$$\frac{dh}{dt} = \frac{q_i - k\sqrt{h}}{\pi r^2} \,, \tag{2.7}$$

it is clear that this model is a nonlinear process due to the square root involved.

2.3 Mathematical Model of the Tank 2

Now, consider the process of the Tank 2 in Figure 4. The main difficulty is due to the

Figure 4 – Isolated Tank 2 System



Source: The Author

discontinuity of shape in the tank, which has half cylindrical and half conical shape. It is possible to follow the modeling guidelines of the Tank 1. The basic problem is to find the water volume in the tank. Similarly, the model of Tank 2 can be modeled using the mass balance principle as in Eq. 2.1. In the same way, the model can be equally converted to take into account the volumetric flow rate instead of the mass flow rate, then a model of Tank 2 is defined as:

$$\frac{dV}{dt} = q_i - q_o \,, \tag{2.8}$$

which is the same model adopted to Tank 1 in Eq. 2.2. As shown before, the volumetric flow rate output is related to the water height in the tank as pointed in Eq. 2.6. Thus, the only missing component to be modeled is the derivative in Eq. 2.8. In order to calculate that derivative it is necessary to find the expression of water volume, which is defined as the sum of the two different shapes involved as:

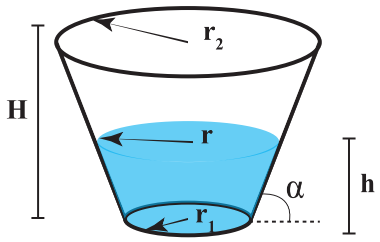
$$V = V_{ci} + V_{co} \,, \tag{2.9}$$

where V_{ci} and V_{co} are, respectively, the volume of the cylindrical and conical parts. The volume of the cylindrical part was previously defined in Eq. 2.3. This leaves only the conical part to be modeled. Actually, this geometric entity is known as truncated cone or conical frustum. An illustration is given in Figure 2.10. The volume of a truncated cone is given by (ZWILLINGER, 2003):

$$V = \frac{1}{3}\pi \left(r_1^2 + r_1 r_2 + r_2^2\right) H \,, \tag{2.10}$$

where r_1 is the lower radius, r_2 is the upper radius and H is the height of the truncated cone. The water volume in the truncated cone depends on the water upper radius r and water height h. To avoid dependency on variables that are not of interest, like the water upper radius, the water

Figure 5 – A truncated cone



Source: The Author

volume in the truncated cone will be put as a function of only the water height h. This can be done by using triangles similarity through the angle α, which relates the upper radius and the water height as follows:

$$\tan\alpha = \frac{H}{r_2 - r_1} = \frac{h}{r - r_1}\,, \tag{2.11}$$

then the water volume can be expressed only as a function of water height as following:

$$V_{co} = \pi h\left(r_1^2 + r_1 h\frac{r_2 - r_1}{H} + h^2\frac{(r_2 - r_1)^2}{3H^2}\right)\,, \tag{2.12}$$

now it is possible to find the derivative of Eq. 2.9. Because the derivative operation is linear, it is possible to write:

$$\frac{dV}{dt} = \frac{dV_{ci}}{dt} + \frac{dV_{co}}{dt}\,, \tag{2.13}$$

it is interesting to make some remarks about this equation. Firstly, when the water level is below the conical part, the volume in the conical part is zero, which means that there is no volume variation on the conical part, thus the last element in the right side of Eq. 2.13 is clearly zero. Secondly, when the water level reaches the conical part, the volume of the cylindrical part becomes constant, again, there is no variation of volume and in conclusion when this happens the first term of the right side of Eq. 2.13 is zero.

These facts reveal the evidences of discontinuous behavior in the system. To finish the model it is just necessary to calculate the derivatives of Eq. 2.13. As pointed out in the previous section, the first derivative is equal to Eq. 2.4. The second derivative can be calculated from Eq. 2.12 as:

$$\frac{dV_{co}}{dt} = \underbrace{\pi\left(r_1^2 + 2r_1\frac{(r_2 - r_1)}{H}h + \frac{(r_2 - r_1)^2}{H^2}h^2\right)}_{A(h)}\frac{dh}{dt}\,, \tag{2.14}$$

finally the model can be described as follows:

$$\frac{dh}{dt} = \frac{q_i - k\sqrt{h}}{\pi r^2} \quad if\ h \le H_{ci}\,, \tag{2.15a}$$

$$\frac{dh}{dt} = \frac{q_i - k\sqrt{h}}{A(h)} \quad if\ h > H_{ci}\,. \tag{2.15b}$$

2.4 Mathematical Model of the TTS

Once the models of Tank 1 and 2 are known, it becomes possible to couple both models in only one model that represents the entire system behavior. The last remaining part to be modeled is the pipe between the two tanks. It can be noticed that this pipe works as an additional exit for both tanks. As shown previously, it is possible to model this additional exit as in Eq. 2.6, so in this way the flow rate between this tank is of the form:

$$q_{12} = k\sqrt{|h_2 - h_1|}\,, \tag{2.16}$$

where q_{12} is the volumetric flow rate between the tanks, k is defined similarly as in Eq. 2.6, h_1 and h_2 are, respectively, the water heights in tanks 1 and 2. It should be remarked that q_{12} acts like an exit in only one tank, depending on which tank has more water. While the one which has lower water, q_{12} works as additional input. In order to determine the water course in q_{12} it is necessary to define a function *sign* which returns the signal of its argument. The complete model then becomes:

$$\frac{dh_1}{dt} = \frac{q_{i1} - k_1\sqrt{h_1} + sign(h_2 - h_1)k_{12}\sqrt{|h_2 - h_1|}}{\pi R_1^2}\,, \tag{2.17a}$$

$$\frac{dh_2}{dt} = \frac{q_{i2} - k_2\sqrt{h_2} + sign(h_1 - h_2)k_{12}\sqrt{|h_2 - h_1|}}{\pi R_2^2}\,, \quad if\ h_2 \leq H_{ci}\,, \tag{2.17b}$$

$$\frac{dh_2}{dt} = \frac{q_{i2} - k_2\sqrt{h_2} + sign(h_1 - h_2)k_{12}\sqrt{|h_2 - h_1|}}{A(h_2 - H_{ci})}\,, \quad if\ h_2 > H_{ci}\,, \tag{2.17c}$$

where $qi1, qi2$ are the volumetric flow rate inputs, k_1, k_2, k_{12} are, respectively, the constant which relates the resistance of the valves 1, 2 and the valve in the pipe between the tanks. h_1 is the water height in Tank 1 and h_2 is the water height in Tank 2. R_1 and R_2 are, respectively, the sectional area of the cylindrical parts of Tank 1 and Tank 2.

It must be mentioned that the parameters k_1, k_2 and k_{12} can be computed if one knows the cross-sectional area, the kind of flow type and the dynamics of the valve. However, they regard dependence on the Reynolds number as pointed out in Garcia (2013). To know the exact value of these constants it is necessary to perform dedicated experiments. For this reason, it is preferable to perform an experimental setup to estimate the values of these constants.

2.5 The Measurement and Actuator Systems

In this section it will be described the sensing elements and the actuators elements of the Two-Tank System.

The Pumps

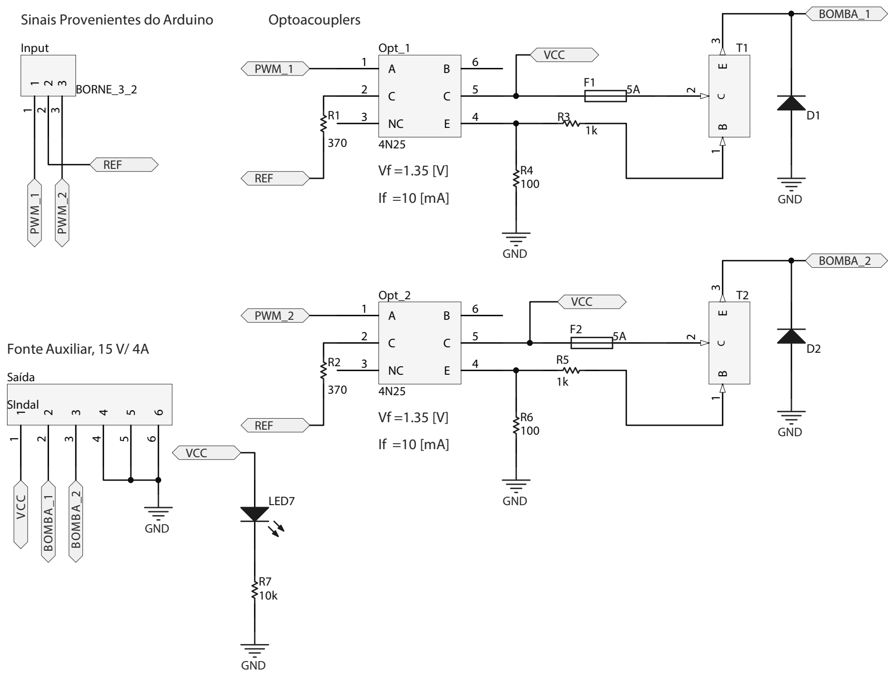
The Pumps used in the system are from SHURFLO 1000 Gallons per hour Bilge Pumps. These pumps are powered by a 12V DC source. In order to control the flow rate at the pump outlet, an electronic driver was built to control the power supply, and consequently at the terminals of the pump. Figure 6 shows one pump used in the system.

Figure 6 – SHURFLO Pump

Source: The Author

An Arduino board is responsible for controlling the electronic driver. The Arduino board uses a pulse width modulation (PWM) signal to control voltage at the terminals of the pump. A schematic showing how these elements are connected can be seem in Figure 7.

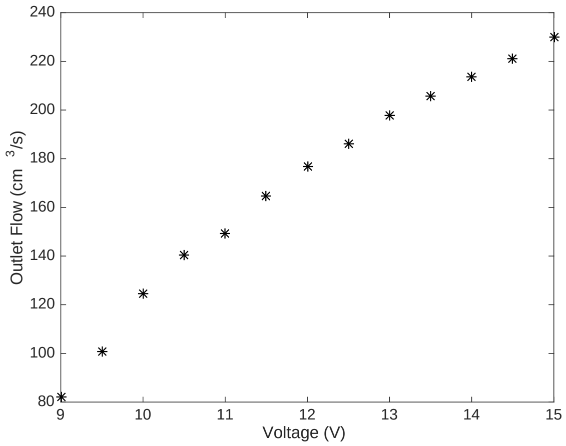
Figure 7 – Schematic of the pump driver



Source: The Author

The electronic driver uses a TIP120, which is an integrated circuit transistor based component, to switch the pump on and off following the PWM signal. The effect of continuously switching on and off at the PWM frequency acts like a power supply divisor. The PWM signal is defined by two times, the first being T_{on}, which is the amount of time in which the signal is in the upper level, while the second, T_{off}, is the amount of time in the lower level. The relation between T_{on} and the PWM period defines a percentage of which the average of the signal is directly dependent. For further details on PWM see (UMANAND, 2009). Figure 8 illustrates the curve voltage versus outlet flow of the pumps.

Figure 8 – Operation range of the pumps



Source: The Author

__The Flow Measurement__

The flow measurement is taken by a flow sensor model YF-S201, shown in Figure 9, whose principle is based on the hall effect. This sensor gives a square wave as output with

Figure 9 – Flow Sensor Model YF-S201

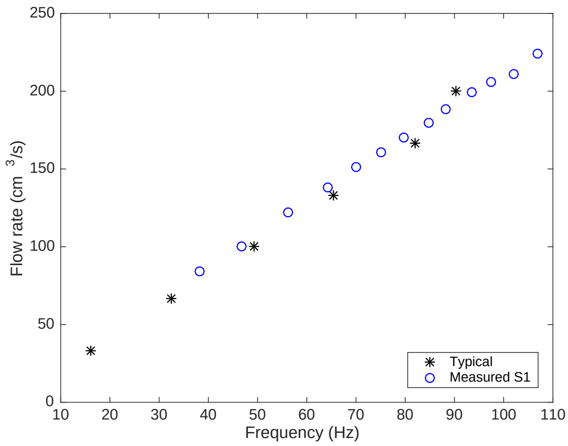
Source: The Author

frequency proportional to the flow rate passing by. A typical response of this sensor as well as the measurements taken from one of the sensors are given in Figure 10.

__The Height Measurement__

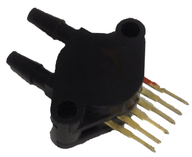
The height measurement is performed by a differential pressure sensor model MPX5010 illustrated in Figure 11. The principle behind height measurement using differential pressure is

Figure 10 – Output of the flow sensors



Source: The Author

Figure 11 – Differential pressure sensor



Source: The Author

the fact that independently of the object's shape the pressure is given by (HALLIDAY; RESNICK; WALKER, 2013):

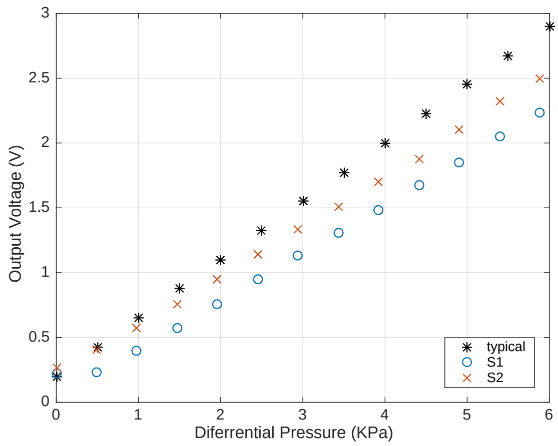
$$\Delta P = \rho g h \tag{2.18}$$

where ΔP is the differential pressure, ρ is the specific weight of the liquid, g is the gravitational acceleration and h is the liquid's height. If g, ρ and P are known then the height h is found from Eq. 2.18 accordingly. Both Earth's gravitational acceleration and water's specific weight are known, while the sensor gives the differential pressure. This makes possible the liquid's height measurement in the tanks. The sensor gives a linear voltage output according to the pressure level. Figure 12 shows a typical pressure versus voltage curve of a MPX5010 and an actual curve of both sensors (Tanks 1 and 2). The real curve was obtained as the average of ten experiments. The calibration was done using a ruler to measure the real liquid height.

The Valve Opening Measurement

The valve between the two-tank system can only be operated manually. The position of this valve changes the system behavior, and because of that it is important to measure the valve's position. This is done by an auxiliary system based on a potentiometer connected to the valve.

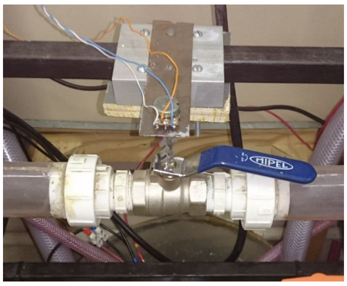
Figure 12 – Curve voltage versus differential pressures



Source: The Author

Every time that the valve changes its position the resistance of the potentiometer also changes. Thus, a voltage divider is used to measure the valve's aperture. It is important to notice that this method can only map the percentage of the valve's aperture when the minimal and maximum voltage are known. Figure 13 illustrates the real system.

Figure 13 – Valve's aperture measurement



Source: The Author

2.6 The Experimental setup

As mentioned previously, in order to estimate the constant k for each valve, it is necessary to perform dedicated experiments. In this section these experiments will be the focus of discussion.

The experiment to estimate the value of k is performed in the following way: for each known flow rate of the pump, the height of equilibrium is measured after a long period, enough for the system to reach its steady state. It is known through Bernoulli's equation that the equilibrium water height and the flow rate of the pump are related as pointed in Eq. 2.6. This process is accomplished for both exit valves of each tank. For such experiment, a data set $\{q_j, h_j\}_{j=1}^{N}$ will be collected, and based on it the value of k can be estimated .

Table 2 and 3 show, respectively, the data set collected from the experiment using the valves from Tank 1 and 2. Each k from both valves were estimated using the Least Squares (LS) algorithm. See Sec. 3.4 for further details.

Table 2 – Data set from Tank 1's valve

Flow (cm^3/s)	216.2	202.6	192.1	186.8	176.3	165.8	157.4	149	141.7	133.3
Height (cm)	51.5	44.5	38.5	34.5	30	24.5	20	15	12.5	9.5

Source: The Author

Table 3 – Data set from Tank 2's valve

Flow (cm^3/s)	220.4	209.9	199.4	190	180.5	170	159.5	150.1	139.6	129.1
Height (cm)	55	48.5	43	37.5	32.5	28	22	17	12.5	9

Source: The Author

2.7 Summary of the Chapter

In this chapter, the nonlinear dynamic model for the TTS was derived according to the mass balance principle. This model will be useful as a data generator for LPV identification purposes. The parameters of the TTS model depends on its geometrical shape. The measurement and actuator systems were described and an experimental procedure to obtain the value of the constant k was given.

3 THE LINEAR PARAMETER-VARYING SYSTEM IDENTIFICATION

In this chapter are shown the methods used to identify models in the framework of Linear Parameter-Varying (LPV) systems. In order to understand the dynamic behavior and what an LPV system is, it will be first described the basic properties of LPV systems in section 3.1,i.e., its input-output relationship in association with the scheduling signal. In this context will be presented the model structures of LPV systems in section 3.2, such as input-output (IO) and state space(SS) representation. Following, methods to parametrically estimate these models are examined in sections 3.3-3.5. In sequence, it will be shown a non-parametric strategy to estimate LPV Models in an input-output setting in section 3.6. An extension of the previous method that delivers unbiased estimates regardless of the noise structure is given in section 3.7.

3.1 LPV Systems

The LPV system framework was originally introduced by Shamma (1988). The idea was to extend the gain-scheduling technique (ÅSTRÖM; WITTENMARK, 1994), which is a design approach that constructs nonlinear controllers considering a nonlinear plant as an array of linear plants in many operational conditions. In the LPV framework, the so-called scheduling variable p, usually an external signal, plays an important role, it represents a dynamic mapping between input u and output y. In this way, both have parameters that are p-dependent. As a matter of fact, LPV systems can describe both nonlinear behavior and time-varying phenomena, while keeping the attractive structure of a linear system. In fact, for a constant signal p an LPV system behaves exactly as an LTI system. Regarding the LTI system theory, an LPV system can be seen as a collection of LTI systems interpolated by a scheduling function (based on p).

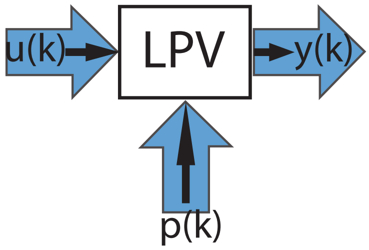
The LPV systems can be represented as a convolution depending on u and p, which in discrete time is represented as (TÓTH, 2010):

$$y(k) = \sum_{i=0}^{\infty} g_i(p)q^{-i}u(k)\,, \tag{3.1}$$

where q denotes the forward/backward time shift operator, i.e. $q^{-i}u(k) = u(k-i)$, $u : \mathbb{Z} \to \mathbb{R}^{n_u}$ is the discrete input, $y : \mathbb{Z} \to \mathbb{R}^{n_y}$ is the discrete output, and $p : \mathbb{Z} \to \mathbb{P}$ is the scheduling variable of the system with scheduling space $\mathbb{P} \subseteq \mathbb{R}^{n_p}$. The coefficients g_i in Eq. 3.1 are functions of the scheduling variable and they define the varying dynamical relation between u and y (TÓTH, 2010). Additionally, there are two types of dependence related to time on p: the static and dynamic dependence. The former is when the coefficients g_i depend only on instantaneous values of p, i.e. $y(k) = g_0(p(k))u(k) + g_1(p(k))u(k-1)...$, whereas the latter is defined by coefficients that depend on time-shifted versions of p, i.e. $y(k) = g_0(p(k), p(k-1))u(k) + g_1(p(k), p(k-1), p(k-2))u(k-1)....$ As pointed out previously, for a constant p the convolution form of an

LPV system described by Eq. 3.1 is equivalent to an LTI system, where the coefficients g_i are constants. See (OPPENHEIM; WILLSKY; HAMID, 1996) for the definition of convolution in LTI systems. Figure 14 shows the relationship between aforementioned variables.

Figure 14 – LPV System



Source: The Author

3.2 LPV Model Structures

There are two basic types for LPV model representation: the IO and SS structures. These are based on the well-established LTI framework, see (OPPENHEIM; WILLSKY; HAMID, 1996) for details. Again, the similarity between LPV and LTI systems are advantageous in terms of application. The equivalence between model structures IO and SS in the LPV framework is, in general, more complicated than the LTI counterpart, as in the LPV case usually involves dynamic dependence on the scheduling variable(TÓTH, 2010). In this thesis, it will be explored the LPV-IO representation. Although the SS structure allows the insertion of noise in the model, the IO setting allows a clear separation between process and noise structures. In this way, IO representation gives a better understanding of the model stochastic properties. Moreover, the IO representation is relatively easier to parametrize and doesn't suffer from explosions of data (curse of dimensionality), contrary to the SS representation (VAN WINGERDEN; VERHAEGEN, 2009).

The LPV-IO Representation

This particular representation originates from the difference equation (discrete time) and is well-established in the LTI framework (OPPENHEIM; WILLSKY; HAMID, 1996). The LPV-IO representation describes the system input-output behavior by using polynomial equations in terms of the forward/backward time-shift operator. The model is generally described in a filter form:

$$y(k) = -\sum_{i=1}^{n_a} a_i(p)q^{-i}y(k) + \sum_{j=0}^{n_b} b_j(p)q^{-j}u(k)\,, \tag{3.2}$$

where the coefficients $\{a_i\}_{i=1}^{n_a}, \{b_j\}_{j=0}^{n_b}$ are the parameters of the model, which are functions of the scheduling p, with $n_a \geq 0$ and $n_b \geq 0$. The model represented by Eq. 3.2 is usually referred as process model. As mentioned previously, these coefficients are considered with static dependence on p. The case where $n_a = 0$, meaning that there is no output dynamic involved is known as Finite Impulse Response (FIR) model.

Usually in real world applications, the model represented in Eq. 3.2 is just an abstraction of the deterministic behavior of the system and, in general, it can barely represent the system behavior. Therefore, a noise must be regarded in order to take into account uncertainties of the system. Typically, the noise added is white noise or a filtered version of it. Hence, the model in Eq. 3.2 is described as following:

$$y(k) = -\sum_{i=1}^{n_a} a_i(p)q^{-i}y(k) + q^{-nk}\sum_{j=0}^{n_b} b_j(p)q^{-j}u(k) + e(k)\,, \tag{3.3}$$

where $e(k)$ is a zero-mean white noise process and n_k is the dead time. This model is the LPV version of the well-known ARX (Autoregressive with exogenous input) model from the LTI framework. Such model is part of a more general transfer function family. See (LJUNG, 1999) for further details on the LTI models.

The LPV-SS Representation

Similar to the LTI case, the LPV models have a state space representation, see (OPPENHEIM; WILLSKY; HAMID, 1996) for more details on LTI-SS structure. An LPV-SS model is generally described as:

$$qx = A(p)x + B(p)u\,, \tag{3.4a}$$

$$y = C(p)x + D(p)u\,, \tag{3.4b}$$

where $x : \mathbb{Z} \to \mathbb{R}^{n_x}$ is the state-variable and $\big(A(p) \in \mathbb{R}^{n_x \times n_x}$, $B(p) \in \mathbb{R}^{n_x \times n_u}$, $C(p) \in \mathbb{R}^{n_y \times n_x}$, $D(p) \in \mathbb{R}^{n_y \times n_u}\big)$ are matrix functions with static dependence on p. In the LPV framework, most of the control synthesis assumes a state space representation as model.

3.3 LPV Identification Approaches

The first difference on the identification procedure of LPV systems is the necessity to measure a third signal entity, which is the scheduling variable. Usually for identification of LTI systems, a data set in the form $\{u_k, y_k\}_{k=1}^{N}$ must be collected, whereas in the LPV framework this data set must include the scheduling variable. Hence, the data set must be in the form

$$\mathfrak{D}_N = \{u_k, y_k, p_k\}_{k=1}^{N}\,. \tag{3.5}$$

Basically, there are two approaches for identification of LPV systems: the local and global approaches. The former is based on the concept of LPV systems viewed as a collection of local LTI systems in an operational space. Whereas in the latter, data collection is done with a varying p, ranging possibly all feasible scheduling space in order to provide a unique global model. The procedure in the local approach is to estimate LTI models from several working points. This can be achieved by maintaining a constant p while collecting the data. Recall that for a constant scheduling $p(k) = \bar{p}$ for any k the LPV model is equivalent to an LTI model. Subsequently, the LPV system is obtained by an interpolation method, such as polynomial, radial basis functions, sigmoidal (TÓTH, 2010). When it comes to the global approach, a unique global structure assumption in the model within the scheduling space is considered, conversely of the local approach. However, the estimation problem using a local approach can be solved similar to the global approach, by considering only one data set ranging many sub-data sets in many working points. In (NOGUEIRA, 2012) was shown a local estimation approach using a global structure. One main drawback of this approach is the explosion of data (curse of dimensionality) as many working points are added. Such issue is unlikely to happen within the scope of a global approach, because of the global nature of the data collection.

In this thesis, it will be explored the methods within the scope of the global approach under the LPV-IO setting.

3.4 The Regression Approach

The regression approach is based on considering the one-step ahead predictor of the system model in regression form. This method lies on the prediction error method (PEM) (LJUNG, 1999), which includes the maximum likelihood method, as well. In order to extend the PEM for LPV systems, it is necessary to define the concept of LPV system description. Then, it is possible to obtain a general one-step-ahead predictor to formulate the identification under the mean-square error framework (MOHAMMADPOUR; SCHERER, 2012).

General LPV System Description

The general LPV system description can be extended from the LTI framework as a process filter with additive disturbance as:

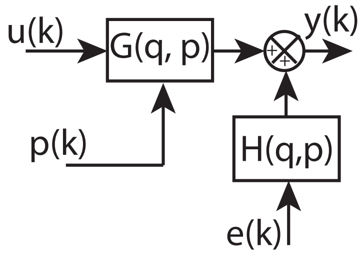
$$y(k) = G(q, p)u(k) + v(k), \tag{3.6}$$

where $G(q, p)$ is a $p-$dependent filter defined similarly as in Eq. 3.1. In (TÓTH, 2010) it was shown that this filter can be equivalently defined as a convolution between u and p. This definition is the LPV form of the impulse response (IR) in the LTI framework, where each $g_i(p)$ is the LPV equivalent of the impulse response coefficients. It is assumed that v is a quasi-stationary noise process with a bounded power spectral density $\Phi_v(\omega)$, and can be described by the following relationship:

$$v(k) = H(q, p)e(k), \tag{3.7}$$

where H is a monic LPV filter as in Eq. 3.1. Similar to the LTI case, the filter H must be asymptotically stable in order for the identification problem to be well-posed under the prediction error (PE) setting, see (MOHAMMADPOUR; SCHERER, 2012; LJUNG, 1999). Figure 15 describes the signal flow of the general LPV system descriptor.

Figure 15 – Signal flow of the general LPV system descriptor



Source: The Author

General Assumptions Under LPV Identification Framework

A first assumption of the LPV framework stands the fact that the scheduling variable must be a measurable entity. Another common assumption in literature involving the scheduling variable is that measurements of p are noise free, see (BAMIEH; GIARRÉ, 2002; DANKERS et al., 2011; LOPES DOS SANTOS; RAMOS; MARTINS DE CARVALHO, 2007; LOPES DOS SANTOS et al., 2013; LAURAIN et al., 2011c; LAURAIN et al., 2012; LAURAIN et al., 2010a; TÓTH et al., 2011; LAURAIN et al., 2010b), exceptions are (BUTCHER; KARIMI; LONGCHAMP, 2008; PIGA et al., 2015; ABBASI et al., 2014). The main reason for the assumption of the noise free observations of the scheduling variable lies on issues regarding the conditional expectation of $v(k)$ when the true observation of p is not available, as each coefficient of the filter defined in Eq. 3.7 can be a nonlinear function with dynamic dependence on p (MOHAMMADPOUR; SCHERER, 2012). This is a quite non-realistic scenario since, generally, observations of the scheduling variable are subject to uncertainties, such as, noise measurements due to sensors and experimental conditions.

In this thesis, it will be considered the case where true p, which is the noise-free version of the scheduling variable, is available.

The One-Step Ahead Prediction of v

In order to formulate the estimation of parametric LPV models in the PE setting, it is necessary to characterize the one-step ahead predictor of y (MOHAMMADPOUR; SCHERER, 2012). Consequently, a one-step ahead prediction of the noise process is necessary to formulate the prediction error. To do so, the filter $H(q,p)$ must be stable and it must have a stable inverse, which means that, there exists a monic convergent filter denoted as $H^{\dagger}(q,p)$ such that $\left(H^{\dagger}(q,p)H(q,p)\right) = 1$ (TÓTH, 2010). This implies that Eq. 3.7 can be rewritten as:

$$e(k) = H^{\dagger}(q,p)v(k), \tag{3.8}$$

and similarly to the LTI case, it can be shown that the one-step ahead predictor of $v(k)$ is the following (TÓTH, 2010):

$$v(k|k-1) = \left(1 - H^{\dagger}(q,p)\right) v(k). \tag{3.9}$$

The One-Step Ahead Prediction of y

To address the problem of estimation parametric LPV models minimizing the prediction error, which is the difference between the actual output and the predicted model output, it is necessary to define the one-step ahead predictor of the model output y.

As an extension of the LTI case (LJUNG, 1999), it was shown that under the p true case with information about $y^{k-1} = \{y(\tau)\}_{\tau \leq k-1}$, $u^k = \{u(\tau)\}_{\tau \leq k}$, and $p^k = \{p(\tau)\}_{\tau \leq k}$, the one-step ahead output predictor is (TÓTH, 2010):

$$y(k|k-1) = \left(H^{\dagger}(q,p)G(q,p)\right) u(k) + \left(1 - H^{\dagger}(q,p)\right) y(k). \tag{3.10}$$

Parametrization of LPV Models

In order to write the LPV-IO model in the regression, the scheduling variable dependencies must be well-defined. The main requirement is that the model must be linear in parameters. To parametrize the model it will be considered that each parameter of the filter $A(p)$ and $B(p)$ can be decomposed in terms of a priori selected basis set $\psi_{ij} : \mathbb{P} \to \mathbb{R}$. Then for $i = 1, ...n_a$ each element of $A(p)$ in Eq. 3.2 can be defined as:

$$a_i(p(k)) = \theta_{i0} + \theta_{i1}\psi_{i1}(p(k)) + \cdots + \theta_{il}\psi_{is_i}(p(k)), \tag{3.11}$$

where $\theta_{ij} \in \mathbb{R}$ are the unknown parameters to be identified. Similarly, each element of $B(p)$ in Eq. 3.2 can be defined for $i = 0, ...n_b$ as:

$$b_i(p(k)) = \theta_{i0} + \theta_{i1}\psi_{i1}(p(k)) + \cdots + \theta_{il}\psi_{is_i}(p(k)). \tag{3.12}$$

It is possible to select a function $\phi(\cdot)$ that generalizes the dependency on $p(k)$ for each parameter of the LPV-IO model. In this way, the process part is fully characterized by $\{\phi_i(\cdot)\}_{i=1}^{n_a+n_b+1}$. Therefore, the LPV model in a regression form must be linearly parametrized as follows:

$$\phi_i(\cdot) = \theta_{i0} + \sum_{j=1}^{s_i} \theta_{ij}\psi_{ij}(\cdot), \tag{3.13}$$

once this parametrization is chosen, it becomes possible to pose the estimation problem in a regression form. As pointed in (BAMIEH; GIARRÉ, 2002) there are many possibilities for the choice of the basis functions, such as, monomials, periodic functions like sine and cosine, and sigmoidal. It is a common assumption in literature that this parameter dependency is of polynomial form, see (BAMIEH; GIARRÉ, 2002; BUTCHER; KARIMI; LONGCHAMP, 2008; LAURAIN et al., 2010b). This corresponds to the following parametrization:

$$\psi_{i,j}(p(k)) = p^j(k). \tag{3.14}$$

MIMO LPV-IO Models

In this thesis, it will be dealt mainly with multiple inputs multiple outputs (MIMO) systems. For this reason, a formal characterization of these models in an IO setting is necessary. The extension to MIMO LPV-IO is accomplished similarly for the LTI case (LJUNG, 1999). The MIMO LTI-ARX model can be described as in Eq. 3.3 by analogy (LJUNG, 1999):

$$y(k) = -\sum_{i=1}^{n_a} \boldsymbol{A}_i q^{-i} y(k) + \sum_{j=0}^{n_b} \boldsymbol{B}_j q^{-n_k - j} u(k) + e(k), \tag{3.15}$$

with $y \in \mathbb{R}^{n_y}$ the vector output, $u \in \mathbb{R}^{n_u}$ the vector input, and the coefficient matrices A_i, B_j defined as:

$$y(k) = \left[y_1(k) \cdots y_{n_y}(k)\right]^T, \quad u(k) = \left[u_1(k) \cdots u_{n_u}(k)\right]^T, \tag{3.16}$$

$$\boldsymbol{A}_i = \begin{bmatrix} a_{i,1,1} & \cdots & a_{i,1,n_y} \\ \vdots & \ddots & \vdots \\ a_{i,n_y,1} & \cdots & a_{i,n_y,n_y} \end{bmatrix}, \quad \boldsymbol{B}_j = \begin{bmatrix} b_{i,1,1} & \cdots & b_{i,1,n_u} \\ \vdots & \ddots & \vdots \\ b_{i,n_y,1} & \cdots & b_{i,n_y,n_u} \end{bmatrix}, \tag{3.17}$$

where $e(k) = \left[e_1(k) \cdots e_{n_y}(k)\right]^T$ is a white noise stochastic vector. However, this model assumes that each output channel has influence over each other, leading to dynamics of each output influencing each other. A common approach in dealing with MIMO systems is to assume that there is no influence in dynamics from other outputs. In this way, the matrix $\boldsymbol{A}_i$ in Eq. 3.17 assumes the following form:

$$\boldsymbol{A}_i = \begin{bmatrix} a_{i,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{i,n_y} \end{bmatrix}, \tag{3.18}$$

Using this parametrization, identification of a MIMO LTI system can be performed separately for each output via the estimation of multiple input single output (MISO) systems (LJUNG, 1999). In fact, it is preferable to cope with the estimation of n_y MISO systems than a MIMO system, mainly to avoid overparametrization due to the presence of zeros on the regressors. To extend this model to the LPV-ARX case it is just necessary to allow the coefficients in matrices to depend on p, by using the LPV matrices $\boldsymbol{A}_i(p(k))$ and $\boldsymbol{B}_j(p(k))$:

$$\boldsymbol{A}_i(p(k)) = \begin{bmatrix} a_{i,1,1}(p(k)) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{i,n_y,n_y}(p(k)) \end{bmatrix}, \boldsymbol{B}_j(p(k)) = \begin{bmatrix} b_{i,1,1}(p(k)) & \cdots & b_{i,1,n_u}(p(k)) \\ \vdots & \ddots & \vdots \\ b_{i,n_y,1}(p(k)) & \cdots & b_{i,n_y,n_u}(p(k)) \end{bmatrix}, \tag{3.19}$$

with the following equation defining the MISO LPV-ARX Model:

$$y(k) = \sum_{i=1}^{n_a} \boldsymbol{A}_i(p(k)) q^{-i} y(k) + \sum_{j=0}^{n_b} \boldsymbol{B}_j(p(k)) q^{-nk-j} u(k). \tag{3.20}$$

Where $y(k)$ is a scalar corresponding to one specific output of the MIMO system, $u(k)$ is defined in Eq.3.16 and $\boldsymbol{A}_i$, $\boldsymbol{B}_j$ are defined as in Eq.3.19. In the case of the MIMO LPV-ARX system, the set $\left\{\phi_{i,j}\right\}_{i=1;j=1}^{n_y,n_t}$ fully characterizes the dependence on the scheduling variable, with each basis function as in Eq. 3.13:

$$\phi_{i,j}(\cdot) = \theta_{i,j,0} + \sum_{k=1}^{s_{ij}} \theta_{i,j,k}\psi_{i,j,k}(\cdot), \tag{3.21}$$

with $n_t = n_a + n_u\,(n_b+1)$, and each $\phi_{i,j}$ is a real function with static dependence on $p(k)$.

Estimation via the LS Criterion

It is possible to write the LPV-IO model as a linear regression form using Eqs. 3.3 and 3.13. Firstly, it is necessary to write the LPV model in a linear regression form as:

$$y(k) = \varphi^T(k)\theta + e(k), \tag{3.22}$$

where y is the output, φ is the regression vector, θ is the parameter of the model and e is a gaussian white noise process. For the LPV-ARX model defined in Eq. 3.3, this transformation is straightforward. However, in order to write this model in regression form, it is first required to choose appropriate basis functions that properly parametrize the model in p as in Eq. 3.13. Choosing a linear in parameters basis function leads to coefficients of filter A and B in the form of Eqs. 3.11-3.12. Then, the parameter and regression vectors in Eq. 3.22 for the LPV-ARX model will be the following:

$$\theta^T = \left[\theta_{10}\cdots\theta_{1s_1}\;\theta_{20}\cdots\theta_{2s_2}\cdots\theta_{n_as_{na}}\;\theta_{n_a+10}\cdots\theta_{n_a+1s_{na+1}}\cdots\theta_{n_gs_{ng}}\right] \tag{3.23a}$$

$$\begin{aligned}\varphi^T(k) = \Big[&-y(k-1)\;-y(k-1)\psi_{i1}(p(k))\cdots-y(k-n_a)\psi_{n_as_{na}}(p(k))\cdots\\ &u(k-n_k)\;\;u(k-n_k)\psi_{n_a+11}(p(k))\cdots\;\;u(k-n_k-n_b+1)\psi_{n_gs_{ng}}(p(k))\Big]\end{aligned} \tag{3.23b}$$

where $n_g = n_a + n_b + 1$. The MISO LPV-ARX can be parametrized in a similar fashion. It is only necessary to add the additional inputs and their respective parameters in the regression equation. The only requirement is that the whole model must be linear in parameters. Then, the regressors and the parameter vectors of the MISO LPV-ARX can be defined as follows:

$$\theta^T = \left[\theta_{10}\cdots\theta_{1s_1}\;\theta_{20}\cdots\theta_{2s_2}\cdots\theta_{n_as_{na}}\;\theta_{n_a+10}\cdots\theta_{n_a+1s_{na+1}}\cdots\theta_{n_ts_{n_t}}\right] \tag{3.24a}$$

$$\begin{aligned}\varphi^T(k) = \Big[&-y(k-1)\;-y(k-1)\psi_{i1}(p(k))\cdots-y(k-n_a)\psi_{n_as_{na}}(p(k))\cdots\\ u_1(k-n_k)\;\;&u_1(k-n_k)\psi_{1,n_a+1,1}(p(k))\cdots\;\;u_1(k-n_k-n_b+1)\psi_{1,n_g,s_{ng}}(p(k))\cdots\\ u_{n_u}(k-n_k)\;\;&u_{n_u}(k-n_k)\psi_{n_u,n_a+1,1}(p(k))\cdots\;\;u_{n_u}(k-n_k-n_b+1)\psi_{n_u,n_g,s_{ng}}(p(k))\Big]\end{aligned} \tag{3.24b}$$

with the first index of $\theta_{i,j,k}$, and $\psi_{i,j,k}$ not used because there is only one output for MISO systems. The Least Squares criteria is based on the minimization of the mean square error, which is summarized in the following cost function:

$$V_N = \frac{1}{N}\sum_{i=1}^{N} e^2(i), \tag{3.25}$$

with the LS estimates as a minimizer defined as follows:

$$\theta_{LS} = \arg\min V_N, \tag{3.26}$$

it is well-known that for the linear regression case the minimum of Eq. 3.25 can be found as follows, by defining $\boldsymbol{Y} = [y(1)\, y(2)\, \cdots Y(N)]^T$ and $\boldsymbol{\Phi} = [\varphi(1)\, \varphi(2) \cdots \varphi(N)]^T$ (LJUNG, 1999):

$$\theta_{LS} = \boldsymbol{\Phi}^{\dagger}\boldsymbol{Y}, \tag{3.27}$$

where $\boldsymbol{\Phi}^{\dagger} = \left(\frac{1}{N}\boldsymbol{\Phi}^T\boldsymbol{\Phi}\right)^{-1}\frac{1}{N}\boldsymbol{\Phi}^T$ is the regularized version of the Penroe-Monrose pseudoinverse (LJUNG, 1999; MOHAMMADPOUR; SCHERER, 2012). This solution is identical to the LTI case (LJUNG, 1999), which is possible because an LPV system can be viewed as a multiple input single output (MISO) LTI model with virtual inputs as pointed in (MOHAMMADPOUR; SCHERER, 2012; LAURAIN et al., 2010a; LAURAIN et al., 2010b). However, it is well-known that the minimizer in Eq. 3.27 leads to an unbiased estimate if the noise e in Eq. 3.22 is white, which is only true for the ARX model structure (LJUNG, 1999; MOHAMMADPOUR; SCHERER, 2012).

Recursive Versions of the LS Algorithm

The adopted LS approach herein is based on the classical approach, which considers the regression vector stacked as column. As the LPV-ARX model can be viewed as an LTI MISO system, other type of parametrizations arise. In (BAMIEH; GIARRÉ, 2002) it was shown that the least mean squares (LMS) and recursive least squares (RLS) algorithms can be extended to the LPV case. A parametrization similar to the MISO LTI models was chosen (LJUNG, 1999), where the regression and parameter vectors in Eqs. 3.23a, 3.23b are matrices, instead of vectors.

The parametrization is then performed in the following way. Define the parameter matrix $\boldsymbol{\Theta} \in \mathbb{R}^{n_g \times n_p}$, with n_p being the number of basis functions, as:

$$\boldsymbol{\Theta} := \begin{bmatrix} a_1^1 & \cdots & a_1^{n_p} \\ a_2^1 & \cdots & a_2^{n_p} \\ \vdots & \vdots & \vdots \\ a_{n_a}^1 & \cdots & a_{n_a}^{n_p} \\ b_0^1 & \cdots & b_0^{n_p} \\ \vdots & \vdots & \vdots \\ b_{nb}^1 & \cdots & b_{n_b}^{n_p} \end{bmatrix}. \tag{3.28}$$

Herein it will be considered the subscript as an indicator of sample, e.g. $y_k = y(k)$. Define the extended regression matrix as:

$$\mathbf{\Psi}_k := \varphi_k \pi_k := \begin{bmatrix} -y_{k-1} \\ \vdots \\ -y_{k-na} \\ u_k \\ \vdots \\ u_{k-n_b} \end{bmatrix} \begin{bmatrix} \psi_1(p(k)) & \psi_2(p(k)) & \cdots \psi_{n_p}(p(k)) \end{bmatrix}, \tag{3.29}$$

based on this, it is possible to infer that the model output of the LPV model is as follows:

$$y_k =< \mathbf{\Theta}, \mathbf{\Psi}_k >, \tag{3.30}$$

which is referred in (BAMIEH; GIARRÉ, 2002) as the notation of inner product, defined as $< A,B >:= trace(A^*B) = trace(BA^*)$ with A and B matrices of same dimension and A^* the transposed conjugate of matrix A. Also, the generalization of outer product is defined as $(U \otimes V)(X) := U < V,X >$ with $U \in \mathbb{R}^{n\times m}$ and $V \in \mathbb{R}^{p\times q}$ and the outer product is an operator $(U \otimes V) : \mathbb{R}^{p\times q} \to \mathbb{R}^{n\times m}$, see (BAMIEH; GIARRÉ, 2002) for further details.

As Eq. 3.30 defines a linear relationship as in 3.22, it can be similarly solved using Eq. 3.27. However, another approach will be taken in this work, instead of solving the cost function analytically, steps will be taken towards to the minimum of the mean square cost function in 3.25. To minimize the cost function iteratively the steepest descent strategy will be used for updating the parameters as follows:

$$\delta\hat{\mathbf{\Theta}} = \hat{\mathbf{\Theta}}_k - \hat{\mathbf{\Theta}}_{k-1} = -\frac{1}{2}\alpha g(\hat{\mathbf{\Theta}}_{k-1}), \tag{3.31}$$

where $g(\hat{\mathbf{\Theta}}_{k-1}) = \frac{dV(\mathbf{\Theta})}{d\mathbf{\Theta}}$ is the mean square error gradient and α is the step size. An instantaneous approximation of the cost function in Eq.3.25 is:

$$V(\mathbf{\Theta}) = e(k,\mathbf{\Theta})^2, \tag{3.32}$$

which is the LMS equivalent. The updating of the parameters is done according to the following recursion:

$$\begin{aligned} \mathbf{\Theta}_{k+1} &= \mathbf{\Theta}_k - \alpha \left(e_k \frac{d}{d\mathbf{\Theta}} \left(y_k - trace(\mathbf{\Theta}^T \mathbf{\Psi}_k) \right) \right) \\ &= \mathbf{\Theta}_k + \alpha e_k \mathbf{\Psi}_k, \end{aligned} \tag{3.33}$$

recalling that $e_k = y_k - trace(\mathbf{\Theta}^T \mathbf{\Psi}_k)$ and the derivative property $(d/dX)trace(X^T B) = B$. In (BAMIEH; GIARRÉ, 2002) a recursive version of the LS algorithm was shown over a general finite dimensional inner product space, rather than the Euclidian space. This is done in order to provide a proof on results related to the persistency of excitation condition. It was pointed out that the RLS version in this inner product space is similar to the euclidean version (LJUNG, 1999) and it has the following form:

$$\hat{\boldsymbol{\Theta}}_k = \hat{\boldsymbol{\Theta}}_{k-1} + \boldsymbol{K}_k e_k \tag{3.34a}$$

$$\boldsymbol{K}_k = \boldsymbol{P}_k \boldsymbol{\Psi}_k \tag{3.34b}$$

$$\boldsymbol{P}_k = \boldsymbol{P}_{k-1} - \boldsymbol{P}_{k-1} \frac{\boldsymbol{\Psi}_k \otimes \boldsymbol{\Psi}_k}{1 + < \boldsymbol{\Psi}_k, \boldsymbol{P}_{k-1}\boldsymbol{\Psi}_k >} \boldsymbol{P}_{k-1}, \tag{3.34c}$$

it is worth to remind that these versions of LMS and RLS can be transformed on the usual form (LJUNG, 1999) by stacking the components of the extended regression matrix $\boldsymbol{\Psi}$ and the parameter matrix $\boldsymbol{\Theta}$ in a column vector as in Eqs. 3.23a, 3.23b.

3.5 The Correlation Approach

The correlation approach is based on a regression form as well. However, in this approach, it is not assumed that the noise stochastic process is white. In typical cases the θ_{LS} estimated via LS do not tend to the true model, giving an unbiased estimate. The reason is, typically, the correlation between $v(k)$ and $\varphi(k)$(LJUNG, 1999). One alternative is to introduce a general correlation vector $\zeta(k)$ in the least squares problem. In identification literature this method is called instrumental variable method (IVM), and $\zeta(k)$ the instruments. It's required that these instruments must be derived from past data and they must be transformed in a certain way that they are uncorrelated with the prediction error sequence(LJUNG, 1999). Resulting in

$$\frac{1}{N}\sum_{k=1}^{N} \zeta(k) e(k,\theta) = 0, \tag{3.35}$$

the solution θ_{IV} that satisfies this equation gives the best estimate based on the observed data and it is of the form:

$$\theta_{IV} = \left[\frac{1}{N}\sum_{k=1}^{N} \zeta(k)\varphi^T(k)\right]^{-1} \frac{1}{N}\sum_{k=1}^{N} \zeta(k) y(k). \tag{3.36}$$

A complete treatment of instrumental variable methods is given in (SÖDERSTRÖM; STOICA, 1983). In this section, it will be shown how the instrumental variable method is extended to the LPV framework.

Instrumental Variable Method for LPV Systems

The IVM was introduced and its consistency analyzed in the LPV framework in (BUTCHER; KARIMI; LONGCHAMP, 2008). While in the LTI case, the instrumental vector $\zeta(k)$ depends on the regressors $\varphi(k)$, in the LPV case, this instrumental vector also depends on the scheduling variable $p(k)$. As shown in (BUTCHER; KARIMI; LONGCHAMP, 2008) the consistency of the IVM depends on two conditions (Similar to the LS case):

$$\mathbb{E}\left\{\zeta(k)\varphi^T(k)\right\} > 0 \quad \text{and} \quad \mathbb{E}\{\zeta(k)e(k)\} = 0, \tag{3.37}$$

where $\mathbb{E}$ denotes mathematical expectation. As in the LS case, basis functions must be chosen in order to parametrize the problem in a linear in parameters form as in Eqs. 3.13, 3.21. An interesting choice of the instruments $\zeta(k)$ are, under the ARX structure assumption, the variance of the IVM is minimal if the instruments is chosen as the noise-free version of the regressor (SÖDERSTRÖM; STOICA, 1983). In this way, the instruments $\zeta(k)$ must be in the form:

$$\begin{aligned}\zeta^T(k) = \Big[& -\breve{y}(k-1)\ -\breve{y}(k-1)\psi_{i1}(p(k))\cdots -\breve{y}(k-n_a)\psi_{n_a s_{na}}(p(k))\cdots \\ & u(k-n_k)\ u(k-n_k)\psi_{n_a+11}(p(k))\cdots\ u(k-n_k-n_b+1)\psi_{n_g s_{ng}}(p(k))\Big],\end{aligned} \tag{3.38}$$

in order for the estimates be optimum, and $\breve{y}$ represents the noise-free output of the system, which is normally unknown in practice. For MISO LPV systems the instruments can be constructed as in Eq. 3.24b, but instead of the measured output, the output regressors are replaced by an estimate as in Eq. 3.38. The IVM algorithm is as follows:

a) Provided a data set $\mathfrak{D}_N$ as in Eq.3.5, estimate via LS an LPV-ARX model with estimate θ_{LS};

b) Simulate the output of the LPV-ARX model using θ_{LS} and then construct the instruments based on the simulated output;

c) Once the instruments are obtained, estimate the parameters of the LPV-ARX using the IVM as in Eq. 3.36.

This procedure can be further improved by using a multi-step algorithm, which is known in literature as IV4 method (LJUNG, 1999), see (SÖDERSTRÖM; STOICA, 1983) for details.

Refined Instrumental Variable Method

This method was first proposed in (YOUNG; JAKEMAN, 1979) for the LTI case. A complete treatment is given in (YOUNG, 1984). A main characteristic of this method is the separation between the deterministic and the stochastic part of the model. Since the process part can always be estimated independent of the noise $v(k)$ using the IVM, this method estimates the noise $\hat{v}(k)$ as an ARMA process using the simulated output from the parameter obtained using the IVM. The extension to the LPV framework was introduced in (LAURAIN et al., 2010a; LAURAIN et al., 2010b), which considers LPV-OE (Output Error) and LPV-BJ (Box Jenkins) structures in the estimation problem.

Consider the following LPV-BJ Model:

$$\mathfrak{S}_\theta = \begin{cases} A(p_k, q^{-1}, \rho)\chi(k) = B(p_k, q^{-1}, \rho)u(k-n_k) \\ v(k) = \dfrac{C(q^{-1},\eta)}{D(q^{-1},\eta)}e(k) = H(q^{-1},\eta)e(k) \\ y(k) = \chi(k) + v(k), \end{cases} \tag{3.39}$$

with $\eta = [c_1 \cdots c_{n_c} d_1 \cdots d_{n_d}]^T$ and $\rho = [a_1 \cdots a_{n_a} b_0 \cdots b_{n_b}]^T$ with the coefficients defined as in Eqs. 3.11, 3.12. For the LPV-OE case the noise model is $H(q^{-1},\eta) = 1$. Notice that for both

LPV-OE and LPV-BJ structures, the scheduling variable $p(k)$ only affects the process part. Based on the PEM approach, it can be shown that the prediction error with respect to Eq. 3.39 is (LAURAIN et al., 2010b):

$$e_\theta(k) = \frac{D(q^{-1},\eta)}{C(q^{-1},\eta)} A^\dagger(p_k,q^{-1},\rho) \times \left(A(p_k,q^{-1},\rho)y(k) - B(p_k,q^{-1},\rho)u(k)\right), \tag{3.40}$$

where $D(q^{-1},\eta)/C(q^{-1},\eta)$ can be recognized as the inverse of the ARMA noise model. Although the prediction error of the LPV case is similar to the LTI case, the polynomial operators do not commute. Hence, no filter can be chosen such that both conditions hold simultaneously (LAURAIN et al., 2010b):

$$A(p_k,q^{-1},\rho)y_f(k) = \frac{D(q^{-1},\eta)}{C(q^{-1},\eta)} A^\dagger(p_k,q^{-1},\rho)A(p_k,q^{-1},\rho)y(k), \tag{3.41a}$$

$$B(p_k,q^{-1},\rho)u_f(k) = \frac{D(q^{-1},\eta)}{C(q^{-1},\eta)} A^\dagger(p_k,q^{-1},\rho)B(p_k,q^{-1},\rho)u(k), \tag{3.41b}$$

which means that no filtering of data can lead a regression equation with a white noise process. In order to introduce a model which provides a solution the identification problem, rewrite the signals' relations in Eq. 3.39 as (LAURAIN et al., 2010b):

$$\mathfrak{S}_\theta = \begin{cases} \underbrace{\chi(k) + \sum_{i=1}^{n_a} a_{i0}\chi(k-i)}_{F(q^{-1})\chi(k)} + \sum_{i=1}^{n_a}\sum_{l=1}^{n_\alpha} a_{il} \underbrace{f_l(p_k)\chi(k-i)}_{\chi_{il}(k)} \\ \qquad = \sum_{j=0}^{n_b}\sum_{l=0}^{n_\beta} b_{jl} \underbrace{g_l(p_k)u(k-nk-j)}_{u_{jl}(k)} \\ v(k) = \frac{C(q^{-1},\eta)}{D(q^{-1},\eta)} e(k) = H(q^{-1},\eta)e(k) \\ y(k) = \chi(k) + v(k), \end{cases} \tag{3.42}$$

here, it is assumed that each $a_i, i = 1, \ldots n_a$ has the same number of elements, that is, each a_i has n_α elements in Eq.3.11. The same assumption is valid to the number of elements of each $b_j, j = 1, \ldots n_b$, that is n_β. Also, the basis functions in Eqs. 3.11 and 3.12 are denoted by f_l and g_l, with $g_0 = 1$ and $F(q^{-1}) = 1 + \sum_{i=1}^{n_a} a_{i0}q^{-i}$. In this form, the LPV-BJ represents a MISO system with $(n_b+1)(n_\beta+1)$ inputs $\{\chi_{il}\}_{i=1,l=1}^{n_a,n_\alpha}$ and $\{u_{jl}\}_{j=0,l=0}^{n_b,n_\beta}$ with the scheme represented in Figure 16.

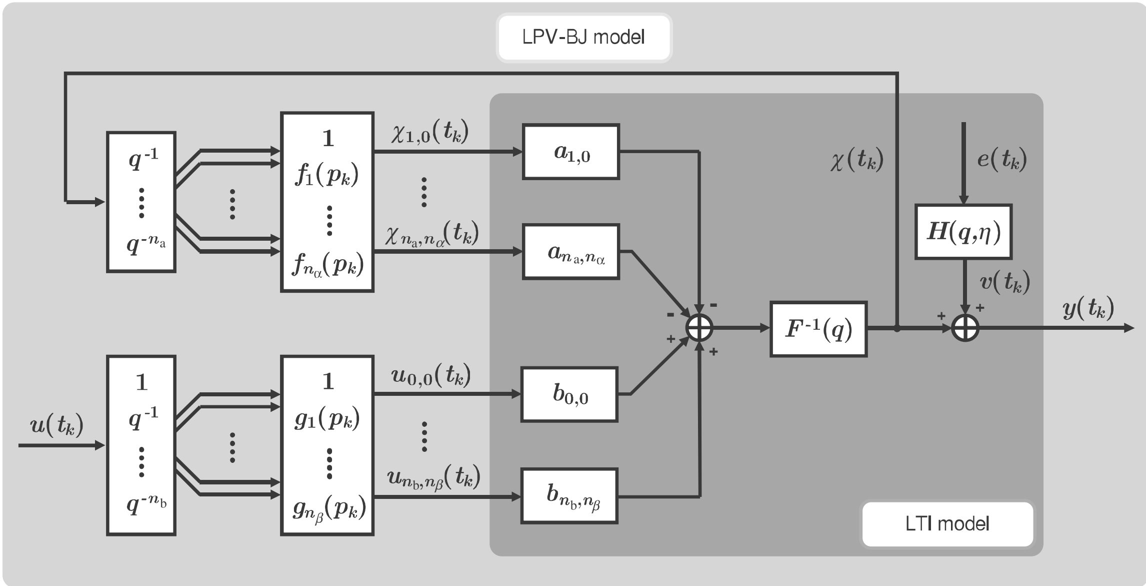
Since now the operator commutes (there is no dependence on $p(k)$), Eq. 3.42 can be rewritten as:

$$y(k) = -\sum_{i=1}^{n_a}\sum_{l=1}^{n_\alpha} \frac{a_{il}}{F(q^{-1})}\chi_{il}(k) + \sum_{j=0}^{n_b}\sum_{l=0}^{n_\beta} \frac{b_{jl}}{F(q^{-1})} u_{jl}(k) + H(q^{-1},\eta)e(k), \tag{3.43}$$

which is an LTI interpretation of Eq. 3.42. Using Eq. 3.43, $y(k)$ can now be written in the regression form as:

$$y(k) = \varphi^T(k)\rho + \tilde{v}(k), \tag{3.44}$$

Figure 16 – MISO interpretation of an LPV system



Source: See (LAURAIN et al., 2010b)

where,

$$\varphi^T(k) = \left[-y(k-1)\cdots - y(k-n_a) \;\; -\chi_{11}(k)\cdots - \chi_{n_a n_\alpha}(k) \;\; u_{00}(k)\cdots u_{n_b n_\beta}(k)\right] \tag{3.45a}$$

$$\rho^T = \left[a_{10}\cdots a_{n_a 0} \;\; a_{11}\cdots a n_a n_\alpha \;\; b_{00}\cdots b_{n_b n_\beta}\right] \tag{3.45b}$$

$$\tilde{v}(k) = F(q^{-1},\rho)v(k). \tag{3.45c}$$

In the LPV-BJ MISO case, it is necessary to add a third index which refers to the $k-th$ input. In this way, the regressors in Eq. 3.45a will also have elements of this type: $\left\{u_{k,j,l}\right\}_{k=1,j=0,l=0}^{n_u,n_b,n_\beta}$, while the parameters in Eq. 3.45b will have $\left\{b_{k,j,l}\right\}_{k=1,j=0,l=0}^{n_u,n_b,n_\beta}$. Now, based on the LTI representation in Eq. 3.43 the prediction error is given as:

$$e_\theta(k) = \frac{D(q^{-1},\eta)}{C(q^{-1},\eta)F(q^{-1},\rho)}\left(F(q^{-1},\rho)y(k) + \sum_{i=1}^{n_a}\sum_{l=1}^{n_\alpha} a_{il}\chi_{il}(k) - \sum_{j=0}^{n_b}\sum_{l=0}^{n_\beta} b_{jl}u_{jl}(k)\right), \tag{3.46}$$

where $D(q^{-1},\eta)/C(q^{-1},\eta)$ can be recognized again as the inverse of the ARMA noise model. However, since this system represents an LTI model (the polynomial operator commutes), the prediction error can be rewritten as:

$$F(q^{-1},\rho)y_f(k) + \sum_{i=1}^{n_a}\sum_{l=1}^{n_\alpha} a_{il}\chi_{il}^f(k) - \sum_{j=0}^{n_b}\sum_{l=0}^{n_\beta} b_{jl}u_{jl}^f(k), \tag{3.47}$$

where $y_f(k)$, $u_{jl}^f(k)$ and $\chi_{il}^f(k)$ represent the outputs of the prefiltering operation, using the following filter:

$$Q(q^{-1},\theta) = \frac{D(q^{-1},\eta)}{C(q^{-1},\eta)F(q^{-1},\rho)}. \tag{3.48}$$

Based on Eq. 3.47, the associated linear in parameters model takes the form (YOUNG, 1984):

$$y(k) = \varphi_f^T(k)\rho + \tilde{v}_f(k), \tag{3.49}$$

where,

$$\varphi_f^T(k) = \left[-y_f(k-1)\cdots - y_f(k-n_a) \;\; -\chi_{11}^f(k)\cdots - \chi_{n_a n_\alpha}^f(k) \;\; u_{00}^f(k)\cdots u_{n_b n_\beta}^f(k)\right] \tag{3.50a}$$

$$\begin{aligned}\tilde{v}_f(k) &= F(q^{-1},\rho)v_f(k)\\ &= F(q^{-1},\rho)\frac{D(q^{-1},\eta)}{C(q^{-1},\eta)F(q^{-1},\rho)}v(k) = e(k),\end{aligned} \tag{3.50b}$$

this result means that the stochastic part of the regression in Eq. 3.49 is equivalent to a white noise process as pointed in Eq. 3.50b. Hence, the optimal solution can be obtained by the LS approach. However, the filter in Eq.3.48 is usually not known a priori. The RIV algorithm is then based on two main steps. First, a reliable estimate of ρ must be used in the presence of colored noise. The main alternative is the use of the IVM as pointed in (YOUNG, 1984), (SÖDERSTRÖM; STOICA, 1983). Second, after obtained a reliable estimate of the process part, the prediction error can be obtained using the simulation output of the model previously estimated. In this way the filter in Eq. 3.49 can be estimated by using an algorithm that estimates ARMA processes, like the same IVM as pointed in (YOUNG, 1984). For the LPV-BJ case, the RIV algorithm is as follows (LAURAIN et al., 2010b):

a) Based on a data set as in Eq. 3.5, estimate an LPV-ARX model using the LS approach and denote by θ_0 its estimate. Assume that η is some initial value. Set $\tau = 0$;

b) Compute an estimate of $\chi(k)$ via $A(p(k),q^{-1},\hat{\rho}^{(\tau)})\hat{\chi}(k) = B(p(k),q^{-1},\hat{\rho}^{(\tau)})u(k-n_k)$, where $\hat{\rho}^{(\tau)}$ is estimated in the previous iteration. Based on Eq. 3.42, deduce $\{\hat{\chi}_{il}(k)\}_{i=1,l=0}^{n_a,n_\alpha}$.

c) Compute the estimated filter:

$$\hat{Q}(q^{-1},\theta) = \frac{\hat{D}(q^{-1},\eta)}{\hat{C}(q^{-1},\eta)\hat{F}(q^{-1},\rho)},$$

and the associated filtered signals $y_f(k)$, $\{\chi_{il}^f(k)\}_{i=1,l=0}^{n_a,n_\alpha}$ and $\{u_{jl}^f\}_{j=0,l=0}^{n_b,n_\beta}$;

d) Build the filtered estimated regressor $\hat{\varphi}_f(k)$ and the filtered instruments $\hat{\zeta}_f(k)$ as:

$$\hat{\varphi}_f^T(k) = \left[-y_f(k-1)\cdots - y_f(k-n_a) \;\; -\hat{\chi}_{11}^f(k)\cdots - \hat{\chi}_{n_a n_\alpha}^f(k) \;\; u_{00}^f(k)\cdots u_{n_b n_\beta}^f(k)\right]$$
$$\hat{\zeta}_f^T(k) = \left[-\hat{\chi}_f(k-1)\cdots - \hat{\chi}_f(k-n_a) \;\; -\hat{\chi}_{11}^f(k)\cdots - \hat{\chi}_{n_a n_\alpha}^f(k) \;\; u_{00}^f(k)\cdots u_{n_b n_\beta}^f(k);\right]$$

e) Now solve the IVM as:

$$\hat{\rho}^{(\tau+1)} = \left[\sum_{k=1}^{N}\hat{\zeta}_f(k)\hat{\varphi}_f^T(k)\right]^{-1}\sum_{k=1}^{N}\hat{\zeta}_f(k)y_f(k);$$

f) obtain an estimate of the noise signal $v(k)$ as:

$$\hat{v}(k) = y(k) - \hat{\chi}(k, \hat{\rho}^{(\tau)});$$

based on $\hat{v}$ the estimation of the noise parameter vector $\eta^{(\tau+1)}$ follows; if $\theta^{(\tau+1)}$ has converged or the maximum number of iterations is reached, then stop, else increase τ by 1 and go back to item b) and repeat the procedure.

This procedure can be used for LPV-OE models, being only necessary to skip item f), since the LPV-OE structure does not assume an ARMA filter on the noise model. This algorithm receives the name of simplified refined instrumental variable (SRIV) algorithm (LAURAIN et al., 2010a; LAURAIN et al., 2010b; YOUNG, 1984).

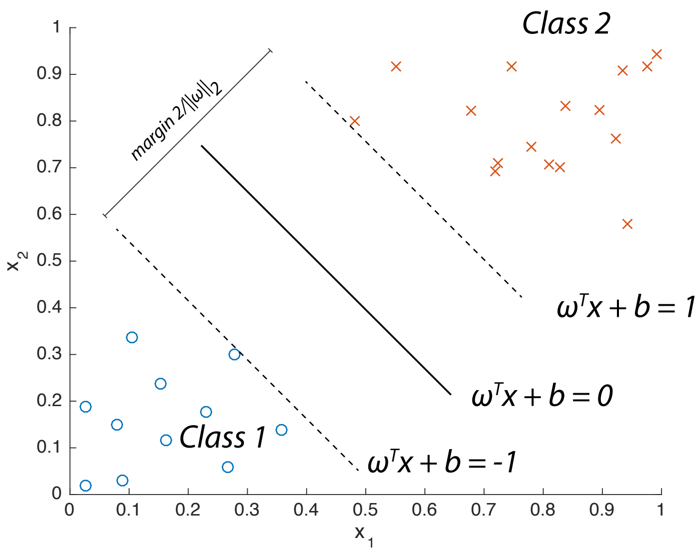
3.6 The LS-SVM approach

The methods presented so far depend on an additional user choice, the adequate selection of the basis functions ψ in Eq. 3.13 to parametrize the LPV model. Besides the inherent identification questions like, the polynomial orders n_a, n_b, n_k. The basis set selection is of paramount importance for the identification task. Generally, in order to capture the underlying dependence of a_i and b_j on p, which can range from simple polynomial to rational and even discontinuous functions, a large set of basis is adopted. However, in many cases, only a few might be needed for an accurate approximation. This means that methods which were shown previously may face an over-parametrization problem (TÓTH et al., 2011). As a solution to this problem, a support vector machine (SVM) approach was introduced in (TÓTH et al., 2011). In this section, it will be presented how the LPV-IO can be modeled in the LS-SVM framework and how to identify the feature map involved with the unknown dependence on the scheduling variable.

Support Vector Machines

The SVMs were originally introduced by Valpnik and they have their roots on statistical learning theory (VAPNIK, 1999). The original SVM aims to separate two classes in the best possible way. The best solution in the SVM framework is related to the margin concept, which plays an important role in the SVM theory. To illustrate the problem which the SVM was introduced to solve, consider the case in Figure 17, where there are two classes. The problem here is to find a linear classifier $\omega^T x + b$ that is able to solve the separation problem, that is, capable of classifying correctly the two classes. An optimal classifier will separate the mapped data set by the highest margin, which can be shown that is equivalent to $2/||\omega||_2$ (VAPNIK, 1999; SCHÖLKOPF; SMOLA, 2002; SUYKENS; VAN GESTEL; DE BRABANTER, 2002). Thus, the optimal classifier can be achieved with the maximization of the margin, but notice that this is equivalent to the minimization of $||\omega||$ subject to the correct classification $y_k \left[\omega^T x + b\right] \geq 1$ with $y_k \in \mathbb{R}$ labeled as $y_k \in \{-1, 1\}$. Although this form of SVM can solve many problems,

Figure 17 – The margin in the SVM framework



Source: The Author

they are restricted to the case where data can be linearly separated. In order to address the nonlinear separable case, it is necessary to introduce some slack variables ξ, which act as a relaxation allowing tolerance in cases of misclassification (SUYKENS; VANDEWALLE, 1999). Additionally, in order to deal with the nonlinear separable case, it is required to map (commonly a nonlinear mapping) the input x to a high dimensional space (feature space) by introducing a feature space $\mathfrak{v} : \mathbb{R}^d \to \mathbb{R}^{n_h}$, with n_h possibly infinity. Moreover, no explicit construction of the nonlinear mapping is needed. This is motivated by the following result. For any symmetric, continuous function $K(x,z)$ satisfying the Mercer's condition, there exists a Hilbert space $\mathscr{H}$, a map $\sigma : \mathbb{R}^d \to \mathscr{H}$ and positive numbers λ_i such that (SUYKENS; VAN GESTEL; DE BRABANTER, 2002; SCHÖLKOPF; SMOLA, 2002):

$$K(x,z) = \sum_{i=1}^{n_h} \lambda_i \varsigma_i(x) \varsigma_i(z), \tag{3.51}$$

where $x, z \in \mathbb{R}^d$ and n_h is the dimension of $\mathscr{H}$. The Mercer's condition requires that:

$$\int K(x,z)h(x)h(z)\mathrm{d}x\mathrm{d}z \geq 0, \tag{3.52}$$

for any square integrable function $h(x)$ (SUYKENS; VAN GESTEL; DE BRABANTER, 2002). It's possible to write the Kernel function in Eq. 3.51 as $K(x,z) = \sum_1^{n_h} \sqrt{\lambda_i}\varsigma_i(x)\sqrt{\lambda_i}\varsigma_i(z)$ and define $\sigma(x) = \sqrt{\lambda_i}\varsigma_i(x)$ and $\sigma(z) = \sqrt{\lambda_i}\varsigma_i(z)$ such that the kernel function can be expressed as the inner product (dot product):

$$K(x,z) = \sigma^T(x)\sigma(z). \tag{3.53}$$

The application of this property is often called kernel trick (SUYKENS; VAN GESTEL; DE BRABANTER, 2002; VAPNIK, 1999; SCHÖLKOPF; SMOLA, 2002; TÓTH et al., 2011). This strategy allows one to work in the feature space without having to make computations on that space. The problem then becomes (SUYKENS; VAN GESTEL; DE BRABANTER, 2002):

$$\begin{aligned} \min_{\omega,b,\xi} \mathfrak{J}_P(\omega,\xi) =& \frac{1}{2}\omega^T\omega + c\sum_{k=1}^{N}\xi_k \\ \text{such that} \quad & y_k\left[\omega^T \mathfrak{v}(x_k)+b\right] \geq 1-\xi_k, \quad k=1,...N, \\ & \xi_k \geq 0, \quad k=1,...N, \end{aligned} \tag{3.54}$$

where c is a positive constant. The optimization problem in Eq. 3.54 is usually called the primal form of the SVM. The Lagragian for this problem is:

$$\mathscr{L}(\omega,b,\xi;\alpha,\nu) = \mathfrak{J}(\omega,\xi) - \sum_{k=1}^{N}\alpha_k\left(y_k\left[\omega^T\mathfrak{v}(x_k)+b\right]-1+\xi_k\right) - \sum_{k=1}^{N}\nu_k\xi_k, \tag{3.55}$$

with the Lagrangian multipliers $\alpha_k \geq 0, \nu_k \geq 0$ for $k = 1,...,N$. The optimal solution of this problem can be obtained at the saddle point of the Lagragian (SUYKENS; VAN GESTEL; DE BRABANTER, 2002):

$$\max_{\alpha,\nu}\min_{\omega,b,\xi}\mathscr{L}(\omega,b,\xi;\alpha,\nu). \tag{3.56}$$

Then, it is possible to eliminate the primal variables by setting the partial derivatives of the Lagrangian with respect to the primal variable ω, b, ξ_k to be equal to zero:

$$\begin{cases} \dfrac{\partial\mathscr{L}}{\partial\omega} = 0 \rightarrow \omega = \displaystyle\sum_{k=1}^{N}\alpha_k y_k \mathfrak{v}(x_k) & \text{(3.57a)} \\ \dfrac{\partial\mathscr{L}}{\partial b} = 0 \rightarrow \displaystyle\sum_{k=1}^{N}\alpha_k y_k = 0 & \text{(3.57b)} \\ \dfrac{\partial\mathscr{L}}{\partial\xi_k} = 0 \rightarrow 0 \leq \alpha_k \leq c,\ k=1,\ldots,N. & \text{(3.57c)} \end{cases}$$

Finally, the dual problem of the SVM can be posed as:

$$\begin{aligned} \max_{\alpha}\mathfrak{J}_D(\alpha) &= -\frac{1}{2}\sum_{k,l=1}^{N} y_k y_l \underbrace{\mathfrak{v}^T(x_k)\mathfrak{v}(x_l)}_{K(x_k,x_l)}\alpha_k\alpha_l + \sum_{k=1}^{N}\alpha_k \\ \text{such that} \quad & \sum_{k=1}^{N}\alpha_k y_k = 0;\ 0 \leq \alpha_k \leq c, k=1,\ldots,N. \end{aligned} \tag{3.58}$$

The dual problem is more attractive to the LPV identification due to the fact that it is only necessary to define a type of kernel, instead of characterizing the explicit dependencies of the nonlinear mapping. Moreover, in this form, it is possible to formulate the problem without explicitly describing the parameter vector ω.

Leasts Squares Support Vector Machines

The formulation in Eq. 3.58 defines a quadratic programming (QP) problem. Although this formulation leads to a convex optimization problem and to sparseness on the solution, it is generally more complicated to solve than a simple LS problem. Motived by that, in (SUYKENS; VANDEWALLE, 1999) was proposed a modification in the original SVM problem in order to simplify its solution. The QP problem was transformed in a linear regression problem with an analytic solution. This method was called least squares support vector machine (LS-SVM). The LS-SVM problem is written as:

$$\begin{aligned} \min_{\omega,b,e} \mathfrak{J}_P(\omega,b,e) = & \frac{1}{2}\omega^T\omega + \frac{\gamma}{2}\sum_{k=1}^{N} e_k^2 \\ \text{such that} \quad & y_k\left[\omega^T \mathfrak{v}(x_k) + b\right] = 1 - e_k, \quad k = 1,\ldots,N. \end{aligned} \tag{3.59}$$

This formulation modifies the original SVM problem in two aspects. The first one is, instead of the inequality constraints used in the original SVM, it is used equality constraints where the value of the right hand side is considered as a target rather than a threshold value. As in the original SVM case, an error variable e is introduced to play a similar role of the slack variables and allow misclassifications. Second, a squared loss function is taken for this error variable (SUYKENS; VAN GESTEL; DE BRABANTER, 2002). As in the original SVM, it is possible to derive the problem in the dual form using the Lagrangian once again as:

$$\mathscr{L}(\omega,b,e;\alpha) = \mathfrak{J}(\omega,b,e) - \sum_{k=1}^{N} \alpha_k \left(y_k\left[\omega^T \mathfrak{v}(x_k) + b\right] - 1 + e_k\right), \tag{3.60}$$

where α_k are the Lagrangian multipliers, which can be either positives or negatives due to the equality constraints. The Karush-Kuhn-Tucker (KKT) conditions for optimality yield:

$$\begin{cases} \dfrac{\partial \mathscr{L}}{\partial \omega} = 0 \rightarrow \omega = \sum_{k=1}^{N} \alpha_k y_k \mathfrak{v}(x_k) & \text{(3.61a)} \\ \dfrac{\partial \mathscr{L}}{\partial b} = 0 \rightarrow \sum_{k=1}^{N} \alpha_k y_k = 0 & \text{(3.61b)} \\ \dfrac{\partial \mathscr{L}}{\partial e_k} = 0 \rightarrow \alpha_k = \gamma e_k, \qquad k = 1,\ldots,N & \text{(3.61c)} \\ \dfrac{\partial \mathscr{L}}{\partial \alpha_k} = 0 \rightarrow y_k\left[\omega^T \mathfrak{v}(x_k) + b\right] - 1 + e_k = 0, \qquad k = 1,\ldots,N. & \text{(3.61d)} \end{cases}$$

The value of α and b can be obtained by formulating a linear KKT system as:

$$\left[\begin{array}{c|c} 0 & y^T \\ \hline y & \Omega + I/\gamma \end{array}\right] \begin{bmatrix} b \\ \hline \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ \hline 1_v \end{bmatrix}, \tag{3.62}$$

where $y = [y_1 \ \ldots \ y_N]^T$, $1_v = [1 \ \ldots \ 1]^T$, $e = [e_1 \ \ldots \ e_N]^T$, $\alpha = [\alpha_1 \ \ldots \ \alpha_N]^T$. The variable Ω is known as the kernel matrix and is defined as (SUYKENS; VAN GESTEL; DE BRABANTER, 2002):

$$\begin{aligned} \Omega_{kl} &= y_k y_l \mathfrak{v}^T(x_k)\mathfrak{v}(x_l) \\ &= y_k y_l K(x_k,x_l), \quad k = 1,\ldots,N. \end{aligned} \tag{3.63}$$

Although the LS-SVM offers a unique and global solution, a main drawback related to the original SVM is the lack of sparseness (SUYKENS; VAN GESTEL; DE BRABANTER, 2002).

LPV formulation in the LS-SVM Setting

The extension to the LPV framework was introduced in (TÓTH et al., 2011) and the MISO extension will be presented in this section. The first assumption here is that structural dependencies of the model coefficients in Eq. 3.13 is not known a priori. Then, the LPV model in Eq.3.3 can be formulated in a manner similar to the one used for the LS-SVM:

$$\mathscr{M}_{\omega,\mathfrak{v}} : y(k) = \sum_{i=1}^{n_t} \omega_i^T \mathfrak{v}_i(p(k)) x_i(k) + e(k), \tag{3.64}$$

where $\mathfrak{v} : \mathbb{R} \to \mathbb{R}^{n_h}$ denotes an undefined, possibly infinite dimensional feature space, and ω_i is the i-th parameter vector. Notice that the model coefficient function is now described as $\phi_i(p(k)) = \omega_i^T \mathfrak{v}(p(k))$. Here, the variables denoted by $\{x_i\}_{i=1}^{n_h}$ represent the LPV dynamical variables, which are the past outputs and inputs, as:

$$x_i(k) = y(k-i), \quad \text{for} \quad i = 1, \dots, n_a, \tag{3.65a}$$

$$x_{n_a+1+j}(k) = u_1(k-j), \quad \text{for} \quad j = 0, \dots, n_b, \tag{3.65b}$$

$$x_{n_t}(k) = u_{n_u}(k-j), \quad \text{for} \quad j = 0, \dots, n_b, \tag{3.65c}$$

with $n_t = n_a + n_u(n_b + 1)$. It's possible to write this model in the regression form by introducing:

$$\omega = \begin{bmatrix} \omega_1 & \dots & \omega_{n_t}^T \end{bmatrix}^T, \tag{3.66}$$

and the regressor:

$$\varphi(k) = \begin{bmatrix} \mathfrak{v}_1^T(p(k)) x_1(k) & \dots & \mathfrak{v}_{n_t}^T(p(k)) x_{n_t}(k) \end{bmatrix}^T, \tag{3.67}$$

in view of this notation, the model can be written in the linear regression as:

$$y(k) = \omega^T \varphi(k) + e(k). \tag{3.68}$$

The regression form allows to employ the technique available from the LS-SVM approach and achieve a non-parametric estimation of the model coefficients. Then, the LPV-LS-SVM can be defined by the following optimization problem:

$$\begin{aligned} \min_{\omega,e} \mathfrak{J}_P(\omega, e) = & \frac{1}{2} \sum_{i=1}^{n_t} \omega_i^T \omega_i + \frac{\gamma}{2} \sum_{k=1}^{N} e_k^2 \\ \text{such that} \quad & e(k) = y(k) - \sum_{i=1}^{n_t} \omega_i^T \mathfrak{v}_i(p(k)) x_i(k), \quad k = 1, \dots, N. \end{aligned} \tag{3.69}$$

where $\gamma \in \mathbb{R}^+$ is the regularization parameter. Once again, this problem can be solved on the dual form by using the Lagrangian defined as:

$$\mathscr{L}(\omega, e; \alpha) = \mathfrak{J}_P(\omega, e) - \sum_{k=1}^{N} \alpha_k \left(\sum_{i=1}^{n_t} \omega_i^T \mathfrak{v}_i(p(k)) x_i(k) + e(k) - y(k) \right), \tag{3.70}$$

with α_k being the Lagrange multipliers again. The global optimum can be obtained for the KKT conditions as:

$$\begin{cases} \dfrac{\partial \mathscr{L}}{\partial \omega_i} = 0 \rightarrow \omega_i = \sum_{k=1}^{N} \alpha_k \mathfrak{v}(p(k)) x_k, & (3.71a) \\ \dfrac{\partial \mathscr{L}}{\partial e} = 0 \rightarrow \alpha_k = \gamma e(k) & (3.71b) \\ \dfrac{\partial \mathscr{L}}{\partial \alpha_k} = 0 \rightarrow e(k) = y(k) - \sum_{i=1}^{n_t} \omega_i^T \mathfrak{v}_i(p(k)) x_i(k). & (3.71c) \end{cases}$$

Now, substituting Eqs. 3.71a and 3.71b into Eq. 3.71c yields to the set of equations:

$$y(k) = \sum_{i=1}^{n_t} \underbrace{\left(\sum_{j=1}^{N} \alpha_j x_i(j) \mathfrak{v}_i^T(p(j)) \right)}_{\omega_i^T} \mathfrak{v}_i(p(k)) x_i(k) + \underbrace{\gamma^{-1} \alpha_k}_{e(k)} \tag{3.72}$$

for $k = 1, \ldots, N$. The dual form is the non-parametric version of the primal form problem. In this form, it is not necessary to compute directly neither the parameter vector ω nor the feature space $\mathfrak{v}$. This becomes possible thanks to the kernel trick , where the dot product $\mathfrak{v}_i^T(p(j))\mathfrak{v}_i(p(k))$ in Eq. 3.72 can be substituted with a kernel $K^i(p(j), p(k))$. This is a key feature in the LPV-LS-SVM identification problem. In this way, the problem can be seen from a function estimation perspective, instead of the traditional over-parametrization as in the LS framework, where an a priori basis must be chosen to perform the identification task.

The kernel works as a measure of similarity between $\mathfrak{v}_i(k)$ at the instant k with respect to all other $\mathfrak{v}_i(j)$ in the data set. Besides, the kernel allows working on a high dimensional feature space without making any explicit computations in the space (SCHÖLKOPF; SMOLA, 2002). Eq. 3.72 can be rewritten in a vector form as:

$$Y = \left(\Omega + \gamma^{-1} I_N \right) \alpha, \tag{3.73}$$

where $\alpha = [\alpha_1 \; \ldots \; \alpha_N]^T$ and the kernel matrix Ω is defined as:

$$[\Omega]_{j,k} = \sum_{i=1}^{n_t} \left[\Omega^i \right]_{jk}, \tag{3.74}$$

with

$$\begin{aligned} \left[\Omega^i \right]_{jk} &= x_i(j) \mathfrak{v}_i^T(p(j)) \mathfrak{v}_i(p(k)) x_i(k) \\ &= x_i(j) K^i(p(j), p(k)) x_i(k) \end{aligned} \tag{3.75}$$

Thus, the kernels K^i define the kernel matrix Ω and characterizes the feature maps $\{\mathfrak{v}\}_{i=1}^{n_t}$ in an efficient manner. There are many types of kernels such as: polynomial, radial basis functions (RBF), sigmoid, linear, and so on, see (SCHÖLKOPF; SMOLA, 2002) for further discussion on kernels. The choice of the appropriate kernel must be driven by the application at hand. For instance, RBF kernels represent circles (hyper-spheres), whereas linear kernels represent lines (hyperplanes). Each kernel has its own parameters to be tuned. In most cases the tuning process can be cumbersome and is driven by trial and error. As the main kernel used in literature for LPV identification is the RBF (LAURAIN et al., 2012; TÓTH et al., 2011; RIZVI et al., 2015b), it will be the focus here. A comparison under the LPV identification framework among kernels is given in (ABBASI et al., 2014). The RBF kernel is defined as follows:

$$K(p(j),p(k)) = \exp\left(-\frac{||p(j)-p(k)||_2^2}{\sigma^2}\right), \tag{3.76}$$

here σ specifies the width of the RBF kernel, and $p(j)$, $p(k)$ are the scheduling variables at time j and k, respectively. The γ and σ parameters both play the roles of tuning parameters and they will determine the bias-variance trade-off in the LPV-LS-SVM scheme. Once the kernel is chosen, the next step is to estimate the Lagrangian multipliers using Eq. 3.73 as:

$$\alpha = \left(\Omega + \gamma^{-1} I_N\right)^{-1} Y. \tag{3.77}$$

Afterwards, the underlying model coefficients can be computed using the Lagrangian multipliers as:

$$\phi_i(\cdot) = \omega_i^T \mathfrak{v}_i(\cdot) = \sum_{j=1}^{N} \alpha_j x_i(j) K^i(p(j),\cdot). \tag{3.78}$$

As the LS based methods in the previous sections, this LS-SVM only produces unbiased estimates with the ARX noise structure. Some strategies were presented in (LAURAIN et al., 2012) in order to overcome this problem, including an IV strategy introduced in the LPV-SVM framework in (LAURAIN; ZHENG; TÓTH, 2011).

3.7 The IV Method in the LS-SVM Framework

As pointed out in the previous section there were works that have extended the IV method to the LS-SVM framework, see for instance (LAURAIN; ZHENG; TÓTH, 2011; LAURAIN et al., 2015). In these, a general treatment regarding nonlinear systems was covered, mainly because of the capabilities of the SVM framework in representing nonlinear models. Thus, the instrumental variable method was introduced for the general nonlinear case in the IO setting. Here, the method proposed in (LAURAIN et al., 2015) was adapted to the LPV case and its derivation is given. This method is generally called IV-SVM. An extension of this method to the LPV-SS representation is presented in (RIZVI et al., 2015b; RIZVI et al., 2015a).

The idea is pretty much the same as the IV method in the linear regression. To do so, introduce an instrumental variable ζ in the following cost function:

$$\mathfrak{J}(\omega,e) = \frac{1}{2}||\omega||_{l_2}^2 + \frac{\gamma}{2N^2}||\Gamma^T E||_2^2 \tag{3.79}$$

$$\text{such that } e(k) = y(k) - \sum_{i=1}^{n_t} \omega_i^T \mathfrak{v}_i(p(k)) x_i(k), \quad k = 1,\dots,N.$$

where

$$\Gamma = [\zeta(1) \ \ldots \ \zeta(N)]^T \tag{3.80a}$$

$$E = [e(1) \ \ldots \ e(N)]^T \tag{3.80b}$$

Similarly to the LS-SVM case it is possible to solve this minimization problem in the dual space by using the Lagrangian as follows:

$$\mathscr{L}(\omega, e, \alpha) = \mathfrak{J}(\omega, e) - \sum_{k=1}^{N} \alpha_k \left(\sum_{i=1}^{n_t} \omega_i^T \mathfrak{v}_i(p(k)) x_i(k) + e(k) - y(k) \right) \tag{3.81}$$

with the α_k being the Lagrangian multipliers again. Notice that the last term in parentheses in Eq. 3.81 can be simply written as $\left(\omega^T \varphi(k) + e(k) - y(k)\right)$ using the regression form in Eq. 3.68. The global solution of this problem can be achieved when the KKT conditions are fulfilled:

$$\begin{cases} \dfrac{\partial \mathscr{L}}{\partial \omega} = 0 \rightarrow \omega = \sum_{k=1}^{N} \alpha_k \varphi(k), & (3.82a) \\ \dfrac{\partial \mathscr{L}}{\partial \alpha_k} = 0 \rightarrow y(k) = \omega^T \varphi(k) + e(k), & (3.82b) \\ \dfrac{\partial \mathscr{L}}{\partial e} = 0 \rightarrow \alpha_k = \dfrac{\gamma}{N^2} \Gamma \Gamma^T e(k). & (3.82c) \end{cases}$$

By substituting Eq. 3.82a into Eq. 3.82b yields:

$$y(k) = \left(\sum_{k=1}^{N} \alpha_k \varphi^T(k) \right) \varphi(k) + e(k), \tag{3.83}$$

which can be written in matrix form:

$$Y = \Phi \Phi^T \alpha + E. \tag{3.84}$$

Finally, substitution of Eq. 3.84 into Eq. 3.82c leads to

$$\alpha = \frac{\gamma}{N^2} \Gamma \Gamma^T \left(Y - \Phi \Phi^T \right), \tag{3.85}$$

which has the solution

$$\alpha = \left(\frac{1}{N^2} H \Omega + \gamma^{-1} I_N \right)^{-1} \frac{1}{N^2} H Y, \tag{3.86}$$

where $H = \Gamma \Gamma^T$. Recall that Ω can be constructed as pointed in Eq 3.75. The matrix H is called the grammian matrix and it can be constructed in similar way as Ω. Thus, the grammian matrix can be obtained using the kernel trick as follows:

$$[H]_{j,k} = \sum_{i=1}^{n_t} \left[H^i \right]_{jk}, \tag{3.87}$$

with

$$\begin{aligned} \left[H^i \right]_{jk} &= \zeta_i(j) \mathfrak{v}_i^T(p(j)) \mathfrak{v}_i(p(k)) \zeta_i(k) \\ &= \zeta_i(j) K^i(p(j), p(k)) \zeta_i(k) \end{aligned} \tag{3.88}$$

leading to the kernels K^i to define the matrix H as in the LS-SVM case. So far the choice of the instruments was not discussed yet. In the linear identification framework the optimal instruments are given by the noise-free input and output samples (SÖDERSTRÖM; STOICA, 1983). However, this is a non realistic scenario since the measurements are generally susceptible to noise. Then, the natural choice of the instruments is an approximation of the noise-free output using the LS-SVM simulated output. Thus, the choice of the instruments is as follows:

$$\zeta_i(k) = \breve{y}(k-i), \;\; i = 1, \dots n_a, \tag{3.89a}$$

$$\zeta_{n_a+1+j}(k) = u_1(k-j), \;\; j = 0, \dots n_{b1}, s \tag{3.89b}$$

$$\zeta_{n_{bn_u}+1+j}(k) = u_{n_u}(k-j), \;\; j = 0, \dots n_{bn_u}, \tag{3.89c}$$

where $\breve{y}$ is the LS-SVM simulated output. This way of choosing the instruments is similar to the widely used IV method for linear regression (LJUNG, 1999; SÖDERSTRÖM; STOICA, 1983).

3.8 Summary of the Chapter

In this chapter were presented the identification methods in the LPV framework. As the aim of this work is to model a MIMO system, all the methods were extended to the MISO case. Recall that a MIMO model can be represented by n_y MISO models, with n_y being the number of outputs. It was shown that in the case of methods based on regression, an appropriate basis function must be chosen in order to properly parametrize the LPV model. However, the choice of basis functions is commonly a nontrivial task and usually leads to over-parametrization. A solution to overcome this problem was presented based on LS-SVM that allow a solution using the Kernel's trick without the implicit definition of a basis function. Although the LS-SVM provides a good solution for the parametrization problem in the presence of white noise, the results for a general noise scenario does not give consistent estimates. Therefore, an instrumental variable scheme was adapted for the LPV-SVM approach. Once the identification methods are known, it is possible to obtain an LPV model for the two-tank system.

4 SIMULATION RESULTS

In this chapter, it will be presented simulation results of the two-tank system described in Chapter 2. The organization of this chapter is as follows. Section 4.1 presents the simulation aspects that will be considered in the identification experiment, i. e., how to simulate the model from the two-tank system. Additionally, it will be seen the input design for identification purposes in section 4.2. How the data is generated and how the input is designed based on the model characteristics are presented in section 4.3. The performance of the algorithms described in chapter 3 will be assessed in section 4.5. In order to understand the relationship among the scheduling variable and the system's input and output, a single tank model obtained in section 2.2 is considered in the LPV framework. Based on the nonlinear model for the tanks, section 4.7 presents a new LPV model with two scheduling variables, each one related either to the input or to the output. In section 4.8, a model with three scheduling variables is proposed based on the model from the previous section for the two-tank system. Section 4.9 presents the results regarding an independent data set for model validation purposes.

4.1 Simulation Setting

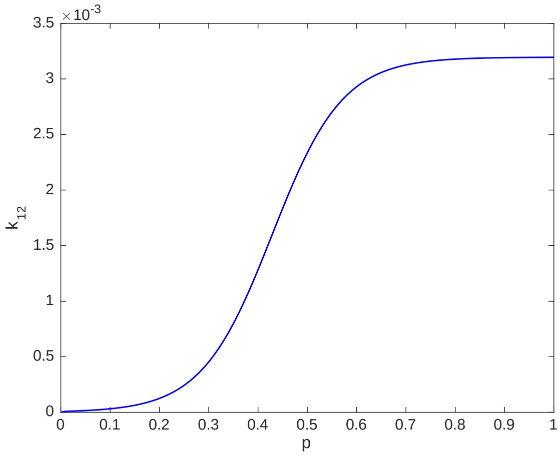
In this section will be presented how the TTS was simulated in order to perform the identification experiments. The simulation experiment is carried out in Matlab® environment and it consists of simulating the model equations in Eqs. 2.17a-2.17c. However, in this model, some parameters are unknown such as the coefficients k_1, k_2 of each output valves and the coefficient k_{12} from the valve between the tanks, as well.

In order to simulate the continuous differential equation that defines the TTS model, it was used a fourth order Runge Kutta (RK4) method. See (CHAPRA; CANALE, 2006) for a general treatment on numerical methods. As the main objective of the system modeling in Chapter 2 was to build a high fidelity model, it was considered a sampling time of 0.5 seconds and a RK4 step time was chosen as one-five hundredth of the sampling time. The two output valves' coefficients were estimated as pointed in sec. 2.6, while the coefficient of the interconnected valve was considered as the following function:

$$k_{12} = f(p) = \begin{cases} 10k_1\left(0.5\tanh\left(7p-3\right)+1\right) & , if \quad p \neq 0, \\ 0 & , if \quad p = 0, \end{cases} \tag{4.1}$$

it is considered that both output valves have the same flow coefficient $k_1 = k_2 = 3.4 * 10^{-4}$, and $p \in [0,1]$. Figure 18 illustrates the function values for the acceptable interval of the scheduling variable p. Notice that the interconnected valve coefficient is variable and depends on the external signal p, which is the interconnected valve control. Then, if $p = 0$ it means that the valve is closed, and when $p = 1$ the valve is totally opened. For the other parameters of the TTS model,

Figure 18 – k_{12} in the range of p



Source: The Author

e.g the radius R_1, the height H_{ci} etc., they were taken using the dimension measurements of the system. The flow rates of each pump, which were used as inputs of the system, have been considered to be within the measured range in Chapter 2, between 1.1e−4 and 1.9e−4 m^3/s, which are the minimum and maximum working points of the pumps, respectively. The choice of this function was driven by the typical operation of hydraulic valves (EMERSON PROCESS MANAGEMENT, 2005), which have a linear behavior in typical cases. However, in some cases, there are two common nonlinearities close to the operation boundaries of the valve. Both act in a similar manner as the $tanh$ in Eq. 4.1.

4.2 Input Design

The major issue of the input design is that an appropriate input must be chosen in order to stimulate all frequencies of the system under experiment. That is, the input must not be arbitrarily selected, instead the input must be designed to give as much information as possible about the system in question. It was shown in (LANDAU; ZITO, 2007) that even if the prediction error is null, when the input is a constant signal $u(k) = \bar{u}$, it is only possible to identify the steady state gain of the system. This fact illustrates the importance of a good choice for the input in order to correctly identify the system dynamics.

There are many choices for input when it comes to informative experiments. Among those, signals such as white noise, filtered white noise, pseudo random binary signal (PRBS), chirp signals, and sum of sinusoids (LJUNG, 1999). They are attractive choices because of their frequency contents. A common feature in the system identification literature is the definition of persistency of excitation or persistently exciting signals. According to (LJUNG, 1999) a signal $u(k)$ with spectrum $\Phi_u(\omega)$ is said to be persistently exciting if $\Phi_u(\omega) > 0$ for almost all ω. The definition of spectrum is as follows (LJUNG, 1999):

$$\Phi_u(\omega) = \sum_{\tau=-\infty}^{\infty} R_u(\tau) e^{-i\tau\omega}, \tag{4.2}$$

with $R_u(\tau) = \bar{E}\{u(k)u(k-\tau)\}$. Depending on the model structure at hand, for the case of an LTI system, a data set is said to be informative if the input is persistently exciting (LJUNG, 1999). Regarding the LPV case, some caution must be taken mainly because of the extra signal involved, which is the scheduling variable p. One of the first papers to address this problem was (BAMIEH; GIARRÉ, 2002), where it was shown that in order to obtain an informative data set for LPV models the two following conditions must hold.

a) There exists N distinct points $\bar{p}_i \in \mathbb{R}$, $i = 1, \ldots, N$ that are limit points of $p(k)$.

b) The system must be excited by a persistently exciting input in every point of the scheduling variable.

A recent paper that addresses the problem of informative data in the LPV framework is (DANKERS et al., 2011). Due to the lack of transfer function representation in the LPV case, in (DANKERS et al., 2011) it was considered equality in terms of predicted output and identifiability of the LPV-ARX structure is studied. However, the problem of how to choose an optimal input that delivers an informative experiment regarding practical applications is still subject for further research.

For the experiments made in this thesis, it was chosen a combination of PRBS type for the input u and a sinusoidal signal for the scheduling variable p. Such a choice is adequate regarding the conditions in a) -b) . The experiments are carried out with this setting. The PRBS is defined as follows (LJUNG, 1999):

$$u(k) = rem\left(A(q)u(k), 2\right) = rem\left(a_1 u(k-1) + \cdots + a_n u(t-n), 2\right), \tag{4.3}$$

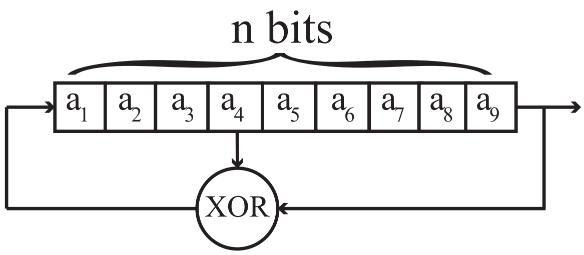
here $rem(x,2)$ refers to the remainder as x divided by 2 which is equivalent to a modulo-2 operation. Thus, $u(k)$ only assumes 0 and 1 values. After u is generated, it is possible to convert to any two desired levels. The vector of past inputs can only assume 2^n values (LJUNG, 1999). In fact, since n consecutive zeros would further lead u to the same state, u must be periodic with a maximum period of $M = 2^n - 1$. It can be shown that for any n, there are choices of the polynomial $A(q)$ that yield to a maximum period length (LJUNG, 1999). Table 4 summarizes such choices. There is another interpretation for the PRBS design. The modulo-2 operation is equivalent to a logical xor operation. In this way, a vector of n bits can be created and the xor operation is carried out only with some selected bits for each n, similar to the choice of the nonzero parameters a_k in Eq.4.3. Both procedures lead to the same sequence for the same choice of n. The latter procedure can be seen in (LANDAU; ZITO, 2007). Figure 19 illustrates the PRBS design using the xor procedure. The interest in the PRBS signal is due to its frequency contents. In (LJUNG, 1999) is shown that any maximum length PRBS behaves like a periodic

Table 4 – Choice of the nonzero elements in $A(q)$ for the PRBS design

Order n	length M	k for $a_k = 1$
2	3	1,2
3	7	2,3
4	15	1,4
5	31	2,5
6	63	1,6
7	127	3,7
8	255	1,8
9	511	4,9
10	1023	7,10
11	2047	9,11

Source: (LJUNG, 1999)

Figure 19 – PRBS design using the logic xor



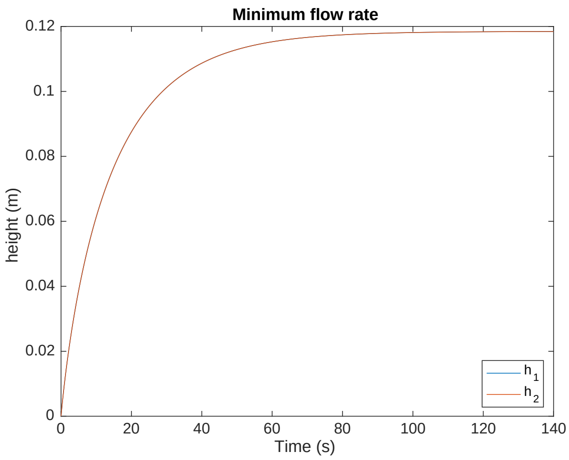
Source: The Author

white noise. The comparison with white noise is made here because of its properties, which is a uniform contribution to the power spectrum density for all ω (TANGIRALA, 2014).

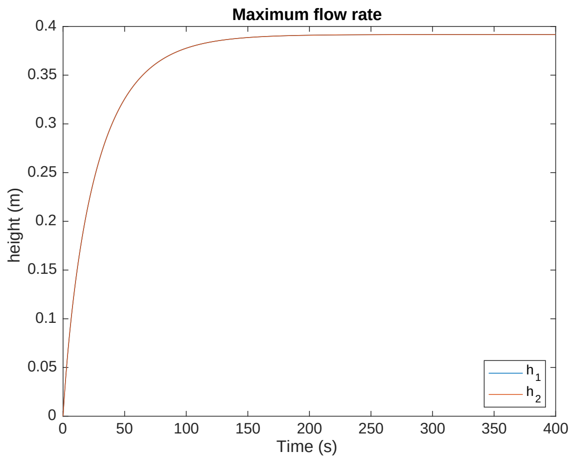
4.3 Identification Experiment in the Simulation Setting

In this section the data generator system is presented and an appropriate input is designed based on the previous section for the identification experiment. To the best of the author's knowledge, the problem of optimal input design for the LPV case was not directly addressed yet. Therefore, a general approach for LTI systems was chosen to be followed in this thesis. The selected input was two PRBS signals, with one being a shifted version of the other. The procedure of the PRBS design was chosen as the one indicated in (LANDAU; ZITO, 2007). The amplitude of the PRBS was chosen in order to maximize the entire operation region of the system, that is, the first value was chosen as the minimum flow rate of the pumps, while the second was chosen as the maximum flow rate of the pumps. Recall that the PRBS has similar properties to the white noise, but the latter in practice is usually not feasible. In order to correctly identify the dominant poles of the system, a step response experiment was done for the minimum and maximum output of the pumps. Figure 20 illustrates the step response of the two-tank systems. There are some

Figure 20 – The step response of the system



(a) Minimum flow rate step response



(b) Maximum flow rate step response

Source: The Author

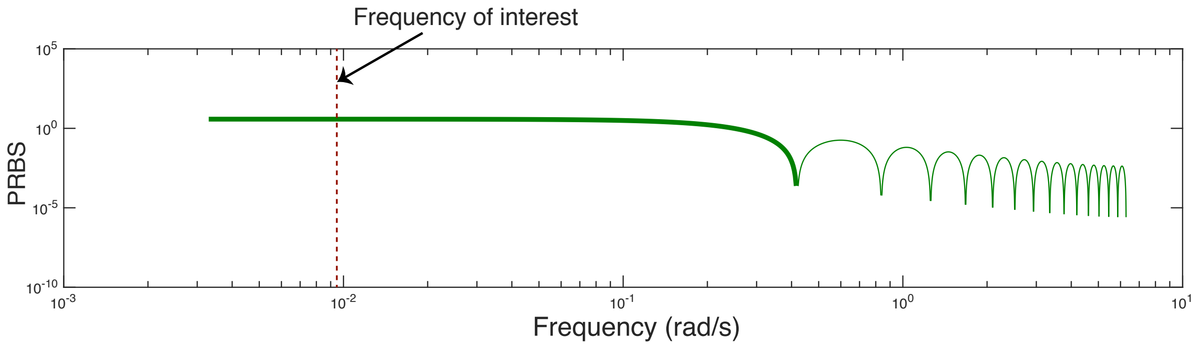
conditions that a PRBS must satisfy in order to maximize the information about the system at hand. The first condition is that the PRBS spectrum must be constant at the frequency of the dominant poles. A second condition is that in order to identify the steady state gain the maximum pulse length (when all bits of the PRBS are equal to one) must be greater than the system's rise time (LANDAU; ZITO, 2007). Again, it is worthy to remind that there is no general approach to input design in the LPV framework. Therefore, it will be followed the general approach for LTI systems.

As Figure 20 indicates, there is a difference between the step responses either in the static gain and time constant for each input. This illustrates the well-known nonlinear behavior of the system. However, an important feature of the system is its first order behavior. For any flow-rate used as input, there are no changes of concavity in the step response, thus characterizing a plant with only real poles, which can be successfully approximated by a first order transfer function in a determined working point. The slowest rise time is about 100 seconds, whereas the fastest is about 42 seconds. The latter rise time can be discarded as the slow pole plays a more important role in the dynamics of the system.

The first parameter that must be chosen in the experiment setup is the sampling rate of the process. Regarding the fastest time constant is equal to 59 seconds, a general guideline is to choose the sampling time between $T_s/5$ and $T_s/10$ (TANGIRALA, 2014) with T_s being the dominant time constant. The sampling time was chosen as 0.5 seconds considering that in the real system the output valves may vary with time. Depending on their outlet constant the time constant could be fewer than 5 seconds. In order to efficiently design a PRBS signal to excite the dominant poles of the plant it is necessary to choose a number of bits n and a PRBS time sampling (T_{PRBS}) to excite the dominant poles of the system (LANDAU; ZITO, 2007). The

PRBS time sampling corresponds to the time in which the bits in the PRBS register are shifted. As mentioned previously, the maximum length pulse of a PRBS signal is equal to the number of bits n. If the PRBS sampling frequency is chosen the same as the process's sampling time, the maximum pulse will be equal to 5.5 seconds considering n equals to 11, which is far from the slowest rise time. Recall that the maximum pulse length is $n * T_{PBRS}$. Instead of increasing the n value and consequently increasing the total length of the experiment, this was avoided by changing the clock frequency of the PRBS signal with a $T_{PRBS} = 15$ seconds and choosing a number of bits $n = 7$. Figure 21 illustrates the periodogram of the designed PRBS. In order

Figure 21 – Periodogram of the designed PRBS



Source: The Author

to provide a meaningful simulation experiment, an additive white noise was added to the TTS outputs, both independent from each other. Moreover, in order to quantify how much noise is added in each output channel, it was considered the signal-to-noise ratio(SNR) defined as (TANGIRALA, 2014):

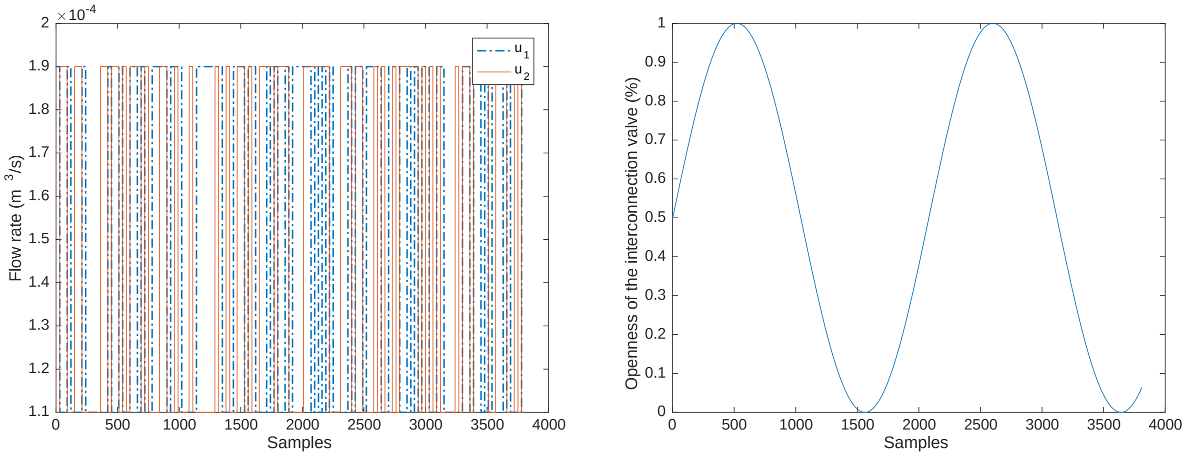
$$SNR = \frac{var(signal)}{var(noise)} \tag{4.4}$$

where $var(u)$ stands for the variance of the signal u. Usually, in the system identification framework, *signal* in Eq. 4.4 refers to the noise free output(true response) of the system at hand. This measure must be viewed as the ratio between the effects of a known variable versus the uncertainties. Several scenarios with different SNRs will be analyzed in this experimental setup investigation. Thus the data generator system is as follows:

$$y_i(k) = h_i(k) + e_i(k), \quad \text{for } i = 1, 2, \tag{4.5}$$

where the index i refers to each tank, e_i is a white noise process, h_i is the output of the i-th tank, y_i is the noisy simulated output of the i-th tank. The height of each tank can be calculated through Eqs. 2.17a-2.17c from Chapter 2 in the continuous domain. Regarding $h_i(k)$, it can be simply obtained by sampling the continuous version using the specified sampling rate. Thus, $e_i(k)$ can be regarded as a specific noise due to the sampling measurements. Figure 22 illustrates the input and the scheduling variable used in the simulated data.

Figure 22 – D_E - Input and Scheduling variable

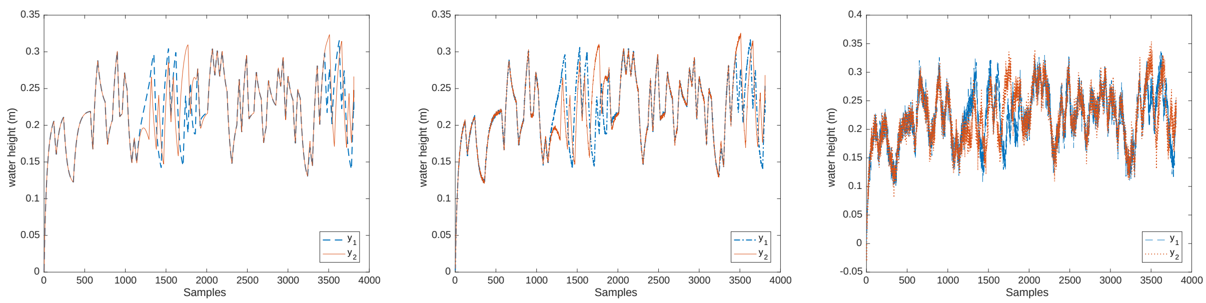


(a) Input (b) Scheduling Variable

Source: The Author

Figure 23 shows the outputs produced by the inputs from Fig. 22 for different noise scenarios.

Figure 23 – D_E - Outputs



(a) Noise-free (b) Low Noise (SNR = 30dB) (c) Heavy Noise (SNR = 10 dB)

Source: The Author

4.4 Model Structure

The nonlinear modeling allowed some prior knowledge about the system at hand. They are listed as follows:

a) Since the equations that describe the system behavior (Eqs. 2.17a-2.17c) involve only first order derivatives, the model orders (n_a and n_b) are both likely to be equal to one;

b) The nonlinearity involved in the system process is known and it depends on the square root of the water height in both tanks;

c) Since the derivatives depend only in terms up to the same time instant, there is no dead time in this process;

d) The stochastic effect in the plant is a nonlinear output error (NOE) type.

Regarding this prior knowledge, it was considered the polynomial orders (n_a, n_b and n_k) as equal to one. As it comes to the scheduling dependence, for the methods that need a parametrization in the scheduling function, it was considered a third order polynomial dependence on the scheduling variable.

The model structures considered herein for identification purposes are ARX and OE type. Both structures considered here were motivated by the fact that the system is in output error form.

4.5 Results

In this section, it will be shown the results of the methods presented in Chapter 3 for the simulation experiment. As mentioned in the previous section, the scheduling function was chosen to be of polynomial dependence. The models will be denoted as the model structure and its orders, e.g. a model *ARX(NA, NB, NK, [NP])* represents a model with ARX structure with the following orders in the operator q, $n_a = NA$, $n_b = NB$, $n_k = NK$ and polynomial dependence on the scheduling $n_p = NP$. In order to access the quality of a model, the Best Fit Score (BFS) is used, which is defined as (LJUNG, 1999):

$$BFS = 100\% \left(1 - \frac{||y - \hat{y}||_2}{||y - \bar{y}||_2}\right), \tag{4.6}$$

where $\bar{y}$ refers to the average of signal y, $\hat{y}$ is the predicted (or simulated) output from the model and y is the actual system response. This measure is related to how good the model is in explaining the measured signal. It works as a fitness function, lower values indicate that the model failed to describe the system behavior, while a higher value shows a good fit. This way of analyzing the goodness of a model is widely used, including in the System Identification toolbox for Matlab (LJUNG, 2007).

Recall that the model tries to explain the input-output behavior. In this context it was considered the water height as output and the inlet flow-rate as the input of the model. In the LPV framework the scheduling variable plays an important role at explaining the system behavior in different working points. In this model it was considered the scheduling variable as the constant k_{12} of the valve between the two tanks. The difference equation that describes the IO behavior of the two tanks is given as:

$$y_i(k) = -a_{i1}(p(k))y_i(k-1) + b_{i01}(p(k))u_1(k-1) + b_{i02}(p(k))u_2(k-1) \quad \text{for } i = 1,2, \tag{4.7}$$

where the index i refers to each tank. The data was generated using a PRBS signal specified in Sec. 4.2. For each input a shifted version of the same PRBS was used. The data set, which has a total of 3880 samples, was split into two data sets; the first, denoted by D_E, is used for estimation purposes, whereas the second, denoted by D_V, is used for the validation task. The data set was divided into two equal parts of same length. Results are summarized in Table 5.

Table 5 – Results on the Validation data set

	Method		SNR (dB) 40	30	20	10
BFS(%)	LS	Tank 1	92.25	90.99	66.04	17.61
		Tank 2	87.81	85.19	64.82	16.70
	RLS	Tank 1	89.81	88.33	69.42	27.62
		Tank 2	90.06	89.14	69.52	26.87
	LMS	Tank 1	89.78	88.43	69.09	27.25
		Tank 2	90.00	89.23	69.39	26.61
	IV	Tank 1	92.49	91.58	86.18	66.60
		Tank 2	87.20	86.42	84.15	65.23
	RIV	Tank 1	92.15	91.79	87.02	66.92
		Tank 2	90.42	89.77	85.33	66.43
	LS-SVM	Tank 1	94.22	89.73	55.36	08.42
		Tank 2	90.98	89.02	55.60	08.42
	IV-SVM	Tank 1	90.85	89.20	82.25	53.05
		Tank 2	90.48	89.39	81.85	47.57

Source: The Author

The results show similar performance between the methods for a low noise scenario (SNR = 40dB). As for the severe noise case, the method based on the ARX structure (LS, RLS, LMS) are clearly poor at describing the system behavior. The same does not happen to the methods based on the correlation approach (IV, RIV). To provide a fair comparison, all methods presented here are obtained minimizing the prediction error. However, the results presented in Table 5 show the BFS for the validation data set D_V using the model simulated output. The LMS and RLS methods require initialization of the parameter vector, in both cases, an ARX model estimated via LS is used as an initial estimate. Additionally, the covariance matrix P required in the RLS algorithm is initialized as the identity matrix I. It can be seem from Table 5 that both RLS and LMS have a performance slightly better than their batch version (LS). This is due to the strategy adopted here, the LS guarantees the least sum of the squared prediction errors, that in a nonlinear system framework doesn't always coincide with the best simulation error. Thus, the RLS and LMS algorithm were run 50 times in the estimation data searching for the best BFS regarding the simulation errors. Another interesting strategy used here is the elitism, which consists in choosing the best model of the training data set considering a determined criterion, here it was considered BFS. Notice that this strategy can only be used in recursive schemes. Although in the RLS and LMS methods were selected the best models using the simulation error as a selector parameter, the RIV method performs better in all noise scenarios. The main reason for that is that the IV method performs well independently of the noise structure. A more realistic scenario in the TTS is the 30dB SNR case. Regarding the (IV)LS-SVM case , RBF Kernels were chosen (see Eq. 3.76) for the regressors and their hyper-parameters are tuned using cross-validation. The hyper-parameters obtained are $\sigma_1 = 0.7$, $\sigma_2 = 0.5, \sigma_3 = 2.2, \gamma_1 = 6.2 * 10^{11}$ for tank 1 and $\sigma_1 = 0.5$, $\sigma_2 = 0.4, \sigma_3 = 2, \gamma_1 = 4.5 * 10^{11}$ for tank 2. This procedure was done considering the noise-free version of the outputs. This is why the LS-SVM performs so badly in a heavy noise scenario compared to the other methods, the same does not happen with its IV counterpart (IV-SVM). However, in a low noise scenario the LS-SVM outperforms all the other methods. It should be remarked that a proper selection of the hyper-parameters may improve the results for the SVM-based methodology.

4.6 A Single Tank in the LPV Framework

The models obtained so far have a representative system behavior regarding the measurements from data. In this section, a single tank model will be derived in the LPV framework. This is done here for a better understanding of the tank model in the LPV form. Here, the noise effects will not be considered as the main interest is the ability of an LPV model in describing a nonlinear system. Recall the nonlinear model of a single tank from Sec. 2.2. Because no exogenous signal is present in this scheme, the scheduling variable will be considered as the water height, which is also the system output. In this case, the model is called quasi-LPV (q-LPV), which is motivated by the fact that the scheduling variable is an endogenous signal. The system's input is the inlet flow-rate, thus the system is a SISO. If needed the reader should revisit Chapter 2 for the modeling part, the system scheme can be seem in Figure 3. The nonlinear dynamic was described in Eq. 2.7.

In order to describe this system in the LPV framework, it will be considered a SISO-LPV Model with polynomial dependence on the scheduling variable. Additionally, the scheduling dependence is static as the entire thesis only deals with this kind of dependence, there is no reason to do otherwise. The polynomial dependence was chosen as third order degree. Again, the BFS defined in Eq. 4.6 will be used as a measure of models' goodness using the simulation error. The model structure used is the following:

$$y(k) = -a_1(p)y(k-1) + b_0(p)u(k-1), \tag{4.8}$$

with $a_1(p) = a_{10} + a_{11}p(k) + a_{12}p^2(k) + a_{13}p^3(k)$ and $b_0(p) = b_{00} + b_{01}p(k) + b_{02}p^2(k) + b_{03}p^3(k)$. As there is no noise involved, it will be used the LS method to estimate an LPV model for the nonlinear process. Using a similar PRBS signal adopted in the previous section an LPV model was obtained using the LS method and it was achieved a BSF equivalent to 99.56%, which clearly denotes that the estimated model reproduces the nonlinear behavior. Although this model responds well enough, it has eight parameters. Looking for the first-principle model obtained in Eq. 2.7, it is clear that the scheduling variable chosen here is not directly related to the input. Hence, the terms in the $B(p,q)$ polynomial that depend on p are possibly not relevant for the LPV model. This means that only the $A(p,q)$ depends on the scheduling chosen, which reveals an important question about the proper choice of the scheduling variable. The problem of choosing the scheduling variable p is not addressed in this work, but the signal p could be chosen as the nonlinearities involved in the process making the overall system linear. Now it was chosen an LPV-ARX(1, 1, 1, [4, 1]) structure, where the polynomial dependence on p in the B term was chosen as 0. For comparison reasons an LTI-ARX Model was identified as well. Table 6 shows the BFS achieved by these models. As expected, the influence of p in the polynomial $B(q,p)$

Table 6 – Results of the models for a single tank

Model	BFS
LPV-ARX(1,1,1, [4 4])	99.56%
LPV-ARX(1,1,1, [4 1])	99.51%
LTI-ARX(1,1,1)	57.23%

Source: The Author

is irrelevant. Notice that the BFS achieved by a simpler LPV-ARX model, which eliminates the dependence of the scheduling variable in the numerator term, is only marginally decreased. The BFS from the LTI-ARX model reveals the superiority of the LPV model in reproducing a nonlinear system.

4.7 An LPV Model with 2 Scheduling Variables

Motivated by the results in the previous section it was proposed an LPV model with two scheduling variables to identify the two-tank system. In the LPV system identification literature only few papers have dealt with more than one scheduling variable, more specifically with two. In (HUANG et al., 2012) an LPV system with two scheduling variables is identified using a multi-model approach. This approach is very close to the concept of the LPV system. In few words, it is an approach that considers a system with a response related to many LTI systems in different working points, each LTI system has a specific weight that depends on the working point. Notice that considering only one scheduling variable gives a wide amount of possibilities. In fact, many function structures can be used, e.g. affine, polynomial, irrational, etc. In the case where two variables are considered, the combination of them make the possibilities become even wider. Here it is proposed an LPV model with two scheduling variables as follows:

$$y(k) = -\sum_{i=1}^{n_a} (a_i \diamond p_1)\, y(k-i) + \sum_{i=0}^{n_b} q^{-n_k} (b_i \diamond p_2)\, u(k-i) \tag{4.9}$$

where $\diamond$ stands for a specific operation in the respective scheduling space. Notice that in this model there is no relationship between the scheduling variables, more precisely, each scheduling is only related either to the output dynamics or to the input dynamics. This kind of structure is similar to the model structures used so far in this thesis. As there is no connection between the scheduling of input and output, then there are no connection possibilities between the two scheduling variables. This preserves the model's complexity in the same way as the models discussed so far. The only difference is that there is an additional scheduling involving only the input, whereas the other one is related to the output. To the best of the author's knowledge, there is no such treatment for models with two scheduling variables, and even the academic literature assumes that there is a combination between the scheduling variables.

4.8 The TTS with 3 Scheduling Variables

In this section it was used the model structure from the previous section to describe the two-tank system behavior. Notice that each MISO system that describes the TTS outputs will have only two scheduling variables, while the overall system will have dependence on three scheduling variables. The choice of the first scheduling variable is still the constant k_{12} between the two tanks. Motivated by the results from Sec. 4.6 , the two remaining scheduling variables are the two water heights from each tank. Recall from Eqs. 2.17a-2.17c that the scheduling variable k_{12} only has relationship with the input, whereas the water height from each tank have relationship with their respective output. It should be remarked that this model is q-LPV due to the use of an endogenous signal (water height). It was also considered the prior knowledge of the model structure. Thus, it was chosen the following structure to describe the TTS behavior:

$$y_1(k) = -a_{11}(p_2(k))y_1(k-1) + b_{101}(p_1(k))u_1(k-1) + b_{102}(p_1(k))u_2(k-1) \tag{4.10}$$

$$y_2(k) = -a_{21}(p_3(k))y_2(k-1) + b_{201}(p_1(k))u_1(k-1) + b_{202}(p_1(k))u_2(k-1) \tag{4.11}$$

with p_1 being the constant k_{12}, p_2, p_3 the water's height in tank 1 and 2, respectively. Exactly the same procedure was done for the identification setting using this new model. In order to compare the results of this new model with the results from Sec. 4.5 it was chosen the SNR of 30dB to compute the estimates and analyze the BFS for both models. Table 7 summarizes the results in the validation data. In order to perform the identification experiments, exactly the same procedure of Sec. 4.5 was done. Notice that in all methods the proposed model performs better than the model using only one scheduling variable. It should be remarked that in some methods a significant increase of quality is obtained for the model that describes the tank 2 system. Besides, the model complexity is not increased since the additional scheduling variables are endogenous and they have been already measured as outputs. Notice that in all identification methods the TTS with three scheduling variables has a better performance in terms of BFS.

4.9 The Final Choice

In order to provide an assessment of model quality from the estimation methods adopted, it was considered a third data set D_{IV} regarding the step response in different working points. This is naturally justified by the LPV system's nature. This data set is completely independent

Table 7 – Results on the Validation data set with SNR = 30dB

	Method		TTS	TTS-3S
BFS(%)	LS	Tank 1	90.99	91.97
		Tank 2	85.19	93.56
	RLS	Tank 1	88.33	91.66
		Tank 2	89.14	91.61
	LMS	Tank 1	88.43	91.63
		Tank 2	89.23	91.58
	IV	Tank 1	91.58	92.85
		Tank 2	86.42	93.94
	RIV	Tank 1	91.79	92.78
		Tank 2	89.77	90.66
	LS-SVM	Tank 1	89.73	92.27
		Tank 2	89.02	90.21
	IV-SVM	Tank 1	89.20	92.37
		Tank 2	89.39	90.20

Source: The Author

from those used so far. At this point, the analysis will be carried out graphically, instead of using the numerical results to assess the model's goodness.

Recall that there are two model structures in this thesis. Basically, one that has only one scheduling variable, whereas the other has three scheduling variables. The former will be denoted simply by LPV-TTS, while the latter will be denoted as LPV3S-TTS. Both have a similar structure, which is ARX(1,1,1, [4, 4, 4]) and OE(1, 1, 1, [4, 4, 4]) for the RIV method. As in the 30dB noise scenario all methods had a similar performance for the LPV3S-TSS, the estimates from the IVM were chosen due to its greater BFS. While for the LPV-TSS the estimates from the RIV method were chosen. Additionally, an LTI system with ARX structure was identified using the system identification toolbox for Matlab®. Figure 24 and 25 illustrate the best models of each model structure. Table 8 shows the parameters from the models obtained through the identification methods.

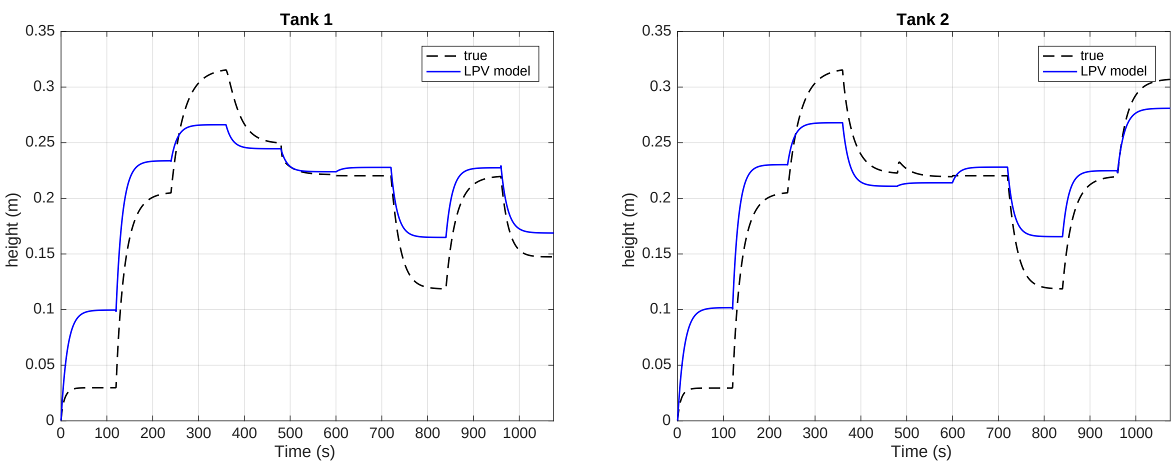
This data set is particularly interesting because it is made of many step levels in both inputs and at the same time the scheduling variable also changes. Thus, the model is tested in a wide variety of working points. Figure 26 shows the inputs and the scheduling variable used to generate the data set D_{IV}. It can be seem from the results that the model LPV-TTS, despite its good performance on the identification data set, can't successfully describe the IO behavior of the system. The model fails in reproducing the appropriate gain at many working points, even if its dynamics is well captured. The same doesn't happen to the LPV3S-TTS model, which maintains the same performance in all working points. From this viewpoint, the choice of the LPV3S-TTS as the model that better describes the system behavior is clear. For comparison purposes, an LTI model is identified using the system identification toolbox for Matlab®. It is

Table 8 – Parameters of the models obtained

Model	Method		a_{10}	a_{11}	a_{12}	a_{13}
LPV-TTS	RIV	Tank 1	-0.9678	0.0221	-0.0438	0.0256
		Tank 2	-0.9692	0.0114	0.0068	-0.0144
LPV3S-TTS	IV	Tank 1	-0.9678	0.3069	-2.1022	2.1437
		Tank 2	-1.1210	0.7045	-0.8021	0.1005
			b_{100}	b_{101}	b_{102}	b_{103}
LPV-TTS	RIV	Tank 1	62.57	-155.33	163.39	-37.95
		Tank 2	-13.67	182.63	-294.94	157.96
LPV3S-TTS	IV	Tank 1	4.88	-114.88	208.19	-106.82
		Tank 2	7.14	199.67	-281.63	117.51
			b_{200}	b_{201}	b_{202}	b_{203}
LPV-TTS	RIV	Tank 1	-12.70	163.22	-163.10	32.13
		Tank 2	61.59	-187.68	362.71	-218.57
LPV3S-TTS	IV	Tank 1	-13.45	23.62	143.40	-173.72
		Tank 2	73.74	-142.44	165.01	-48.33

Source: The Author

Figure 24 – Results for the LPV-TTS with the RIV method on D_{IV}

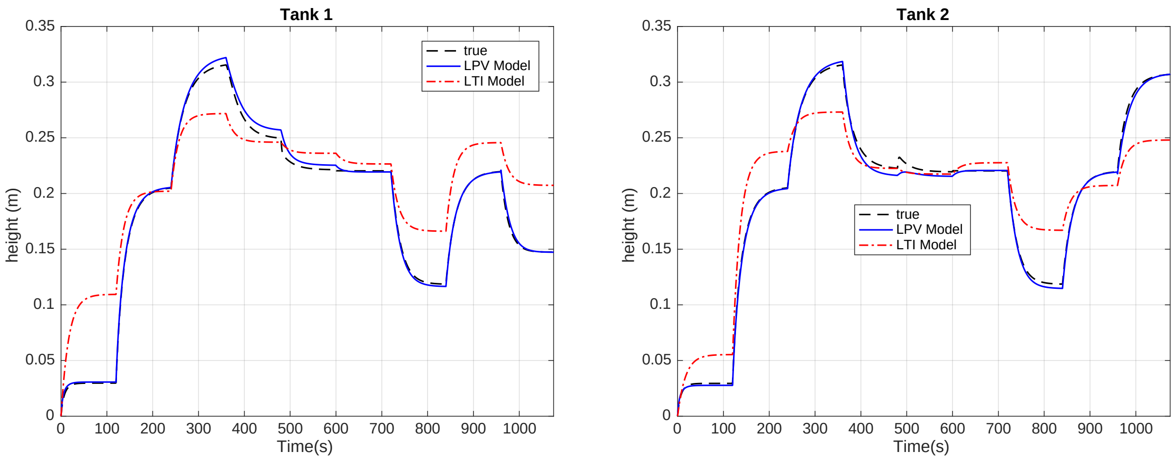


(a) Tank 1 (b) Tank 2

Source: The Author

clear the improvement of the LPV framework related to all working points of the system. Table 9 summarizes the results for the models obtained regarding the BFS performance on the validation data set. A important remark must be mentioned when the scheduling variable p_1 comes close to zero (the interconnected valve is virtually closed). When this happens, the TTS becomes two independent SISO systems as there is no coupling between the tanks. Naturally, in the regression approach there are terms of both inputs in the model. Ideally, the terms b_{20} and b_{10} in the models of Tanks 1 and 2, respectively, should be zero when p_1 is zero.

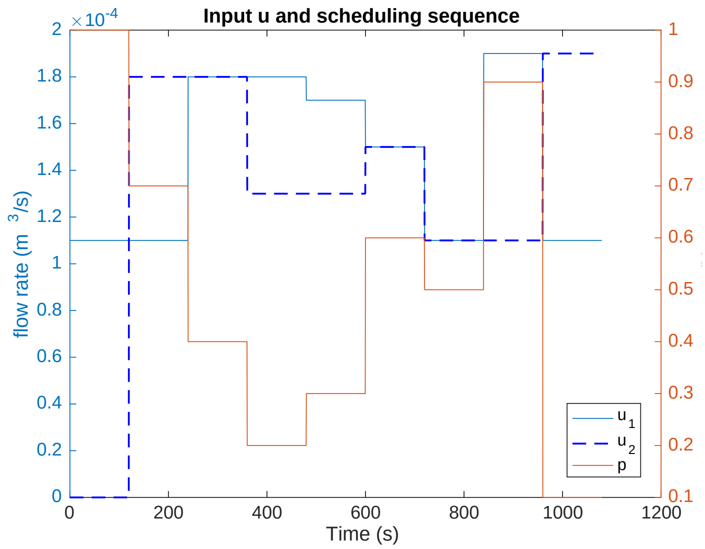
Figure 25 – Results for the LPV3S-TTS with the IV method on D_{IV}



(a) Tank 1 (b) Tank 2

Source: The Author

Figure 26 – Input and scheduling sequence of D_{IV}



Source: The Author

4.10 Summary of the Chapter

In this chapter it was shown the simulation results for the identification of a nonlinear two tanks process. It was presented the simulation aspects of the experiment and how to obtain an appropriate choice of the input in order to excite all the modes of the system at hand. The identification methods described in Chapter 3 were performed in order to obtain a suitable model for the TTS in the LPV framework. Additionally, an LPV model was identified for one single tank process and its scheduling dependence analyzed. It was shown that the scheduling variable for a single tank does not affect the input. This result is not surprising according to the model

Table 9 – BFS for the obtained models

		LTI-TTS	LPV-TTS	LPV3S-TTS
BFS(%)	Tank 1	49.53	56.63	96.73
	Tank 2	62.23	55.48	95.32

Source: The Author

obtained for the TTS. Motivated by the previous result, it was proposed a new LPV model with two scheduling variables associated either with the input or with the output. Afterwards, a model with three scheduling variables for the TTS was proposed. The quality of the models identified were assessed through an independent data set. The proposed model outperformed all the other models identified in this chapter.

5 CONCLUSION

In this chapter the overall conclusions about the subject of this thesis are pointed. This thesis can be outlined as follows, in chapter 2 a modeling procedure was done in order to obtain a first-principles model based on mass conservation. Chapter 3 presents the LPV framework and the methods used for the identification of LPV models used in this thesis. In chapter 4 a simulation study is carried out in which is showed how an LPV model can successfully represent nonlinear system behavior. Besides, a new LPV model based on two scheduling variables independently associated with the input and output was presented. It was shown that this method had a better performance in describing the TTS behavior.

The main objective of this thesis was to provide a MIMO LPV model capable of representing the two-tank system. For this reason, the techniques developed in the academic literature were extended to the MIMO case. Throughout this thesis, the identification task was performed using the global approach and no investigation was done with respect to appropriate choice of basis functions. The motivation behind that lies in the good properties of polynomial basis in approximating functions as well as it is desired in any finite interval. Furthermore, it was used a non-parametric approach driven by the use of support vector machines. The use of the LS-SVM strategy provides the identification of the underlying dependencies without defining an appropriate function basis a priori. However, as presented in the simulation study, the LS-SVM had a similar result compared to the regression approach. In order to cope with a more general noise scenario, an instrumental variable approach adapted to the LPV-SVM case from the nonlinear general case. All the methods presented in this thesis had a similar performance, with the IV based methods providing slightly better results. Based on prior knowledge of the system at hand a new LPV model was developed in order to provide meaningful scheduling dependence in the parameters. This new model has had much better performance in describing the IO behavior in a new independently generated data set.

A secondary goal was to provide a first-principles model for the two-tank system. This was done using the mass conservation law. This kind of modeling in tank systems was widely used in academic research. However, to the best of the author's knowledge, no procedure relating different shapes on the geometry of the tank was addressed. The procedure presented in this thesis follows directly from differential calculus and it is directly related to the volume and its derivatives. Complex shapes may turn the problem of building a first-principles model infeasible within this procedure.

This work has generated the following publication:

a) MELO, F. E. M. ; NOGUEIRA, F. G. ; BARRA JR., W. ; DA COSTA, CARLOS T. ; LANA, J. J. ; RIOS, C. S. ; BRAGA, A. P. S. . *Least Squares and LS-SVM LPV*

Identification Methods with Application on a Generating Units. In: IEEE Argencon Proceeedings, Buenos Aires, 2016.

Recommendations

There were many open problems in this thesis. Here it is showed some recommendations for future research.

a) In this thesis is assumed that the scheduling variable is a noise-free signal. This is a non-realistic scenario in practice due to the measuring sensors. Thus, one can not access the noise free version of the scheduling variable. Only few works in academic research have coped with this issue. Additionally, some of them have selected a proper basis to analyze this problem. An interesting area of research would be to analyze the LPV identification with the eyes of an errors-in-variables approach in order to deal with the noise in the scheduling variables.

b) The instrumental variable method provides good estimates in the presence of colored noise. However, only few works dealt with a scheduling dependent noise structure. Investigation of p-dependent moments (standard deviation that depends on p) in the noise structure would be interesting for the local approach.

c) The choice of adequate instruments in the IV method is basically driven by the linear case. Some works had shown that this approach can be successfully extended to the LPV case for a specific scheduling function dependence. The choice of the instruments in the nonlinear case is still an open problem. With respect to the LPV case, to the best of the author's knowledge, a general approach regarding the scheduling structure was not addressed yet.

d) In the recent kernel methods (LS-SVM), the choice of an adequate kernel has paramount importance on the performance of the method. Thus, the quest for kernels that provide better estimates is now an active research area. A comparison among kernel structures would be beneficial to the LPV system identification area.

e) With the development of the kernel methods, the identification task using the Bayesian perspective is currently a research topic in many works. Naturally, this is a good direction for future works.

f) Few works had dealt with the problem of input design in the LPV framework. Hence, a very practical task is how to get informative data sets for LPV systems. Specially, regarding the two approaches (local and global) for data gathering in the LPV framework.

g) The new LPV model presented here was inspired by an observation of the process variables involved in the nonlinear tank model. An analysis in the point of view of system theory would benefit the application of this model. Moreover, it could be verified for what other systems this model can be successfully applied.

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