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ISRAEL DE SOUSA EVANGELISTA

COMPACT ALMOST RICCI SOLITON, CRITICAL METRICS OF THE TOTAL SCALAR CURVATURE FUNCTIONAL AND *P*-FUNDAMENTAL TONE ESTIMATES

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Thesis submitted to the Post-graduate Program of the Mathematical Department of Universidade Federal do Ceará in partial fulfillment of the necessary requirements for the degree of Ph.D. in Mathematics. Area of expertise: Differential Geometry

Advisor: Prof. Dr. Abdênago Alves de Barros

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To my parents, brothers and wife.

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"To humans belong the plans of the heart, but from the LORD comes the proper answer of the tongue."

(Proverbs 16:1)

RESUMO

A presente tese está dividida em três partes diferentes. O objetivo da primeira parte é provar que um quase soliton de Ricci compacto com tensor de Cotton nulo é isométrico a uma esfera canônica desde que uma das seguintes condições associadas ao tensor de Schouten seja válida: a segunda função simétrica é constante e positiva; duas funções simétricas consecutivas são múltiplas, não nulas, ou alguma função simétrica é constante e o tensor de Schouten é positivo.
O objetivo da segunda parte é estudar as métricas críticas do funcional curvatura escalar total em variedades compactas com curvatura escalar constante e volume unitário, por simplicidade, métricas CPE. Foi conjecturado que toda métrica CPE deve ser Einstein. Prova-se que a conjectura é verdadeira para as métricas CPE sob uma condição integral adequada e também se prova que é suficiente que a métrica seja conforme a uma métrica Einstein.
Na terceira parte, estima-se o p-tom fundamental de subvariedades em uma variedade tipo Cartan-Hadamard. Primeiramente, obtém-se estimativas por baixo para o p-tom fundamental de bolas geodésicas e em subvariedades com curvatura média limitada. Além disso, obtém-se estimativas do p-tom fundamental de subvariedades mínimas com certas condições sobre a norma da segunda forma fundamental. Por fim, estudam-se folheações de classe C^2 transversalmente orientadas de codimensão 1 de subconjuntos abertos Ω de variedades riemannianas M e obtêm-se estimativas por baixo para o ínfimo da curvatura média das folhas em termos do p-tom fundamental de Ω.

Palavras-chave: Quase soliton de Ricci. Funcional curvatura escalar total total. P-laplaciano. P-tom fundamental. Métricas de Einstein. Curvatura escalar. Tensor de cotton.

ABSTRACT

The present thesis is divided in three different parts. The aim of the first part is to prove that a compact almost Ricci soliton with null Cotton tensor is isometric to a standard sphere provided one of the following conditions associated to the Schouten tensor holds: the second symmetric function is constant and positive; two consecutive symmetric functions are non null multiple or some symmetric function is constant and the quoted tensor is positive.

The aim of the second part is to study the critical metrics of the total scalar curvature funcional on compact manifolds with constant scalar curvature and unit volume, for simplicity, CPE metrics. It has been conjectured that every CPE metric must be Einstein. We prove that the Conjecture is true for CPE metrics under a suitable integral condition and we also prove that it suffices the metric to be conformal to an Einstein metric.

In the third part we estimate the p-fundamental tone of submanifolds in a Cartan-Hadamard manifold. First we obtain lower bounds for the p-fundamental tone of geodesic balls and submanifolds with bounded mean curvature. Moreover, we provide the p-fundamental tone estimates of minimal submanifolds with certain conditions on the norm of the second fundamental form. Finally, we study transversely oriented codimension one C^2-foliations of open subsets Ω of Riemannian manifolds M and obtain lower bounds estimates for the infimum of the mean curvature of the leaves in terms of the p-fundamental tone of Ω.

Keywords: Almost Ricci soliton. Total scalar curvature functional. P-Laplacian. P-fundamental tone. Einstein metrics. Scalar curvature. Cotton tensor.

CONTENTS

1 INTRODUCTION

This thesis deal with three different problems. First we study compact almost Ricci solitons, we investigate which geometric implication has the assumption of the second symmetric function $S_2(A)$ associated to the Schouten tensor to be constant and positive on a compact almost Ricci soliton. More precisely, we have the following result.

Theorem 1.0.1 *Let (M^n, g, X, λ), $n \geq 3$, be a non-trivial compact oriented almost Ricci soliton such that the Cotton tensor is identically zero. Then, M^n is isometric to a standard sphere $\mathbb{S}^n$ provided that one of the next conditions is satisfied:*

1. *$S_2(A)$ is constant and positive.*
2. *$S_k(A)$ is nowhere zero on M and $S_{k+1}(A) = cS_k(A)$, where $c \in \mathbb{R}\backslash\{0\}$, for some $k = 1, \cdots, n-1$.*
3. *$Ric \geq \frac{R}{n}g$, with $R > 0$, and $\int_M S_k(A)\Delta h \geq 0$ for some $2 \leq k \leq n-1$.*
4. *$S_k(A)$ is constant for some $k = 2, \cdots, n-1$, and $A > 0$.*

We highlight that the symmetric functions associated to the Schouten tensor were used by Hu, Li and Simon (HU *et al.*, 2008) to study locally conformally flat manifolds. By assuming that the Weyl tensor vanishes, the conclusion of item 4 in the above theorem follows directly from Theorem 1 obtained in (HU *et al.*, 2008). In this direction, we point out that item 1 and item 4 of the above theorem improve Theorem 1 in (HU *et al.*, 2008) for compact almost Ricci solitons under the hypothesis of Cotton tensor identically zero.

Second, we study the CPE metric which is a 3-tuple (M^n, g, f) where (M^n, g), $n \geq 3$ is a n-dimensional compact oriented Riemannian manifold with constant Ricci scalar curvature and f is a smooth potential function that satisfies the equation

$$Ric - \frac{R}{n}g = Hess\, f - f\big(Ric - \frac{R}{n-1}g\big),$$

There is a famous conjecture proposed in (BESSE, 2007) which says that a CPE metric is always Einstein.

Recently, by considering the function $h = |\nabla f|^2 + \frac{R}{n(n-1)}f^2$, Leandro (NETO, 2015) was able to show that CPE conjecture is true under the condition that h is a constant. Whereas, Benjamin Filho (FILHO, 2015) improved this result requiring that h is constant along of the flow of ∇f.

Taking into account that height functions are eigenfunctions of the Laplacian on a sphere $\mathbb{S}^n$ with standard metric g, we may conclude that $(\mathbb{S}^n, g, h_v)$ is a CPE metric, where h_v is a height function for an arbitrary fixed vector field $v \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$. Indeed, the existence of a non constant solution is only known in the standard sphere for some height function.

Now we define $\rho_m(f, \nabla f)$, which for simplicity we denote by ρ_m, according to

$$\rho_m = (m-1)\int_M f^{m-2}|\nabla f|^4 dM - \frac{(n+2)R}{n(n-1)}\int_M f^m |\nabla f|^2 dM, \tag{1.0.1}$$

where $m \in \mathbb{N}$. It is easy to check that on $\mathbb{S}^n$ we have $\rho_m = 0$ for every $m = \{1, 2, 3, \cdots\}$. We also recall that Benjamim Filho proved in (FILHO, 2015) that the CPE conjecture is true provided $\rho_1 \leq 0$ and $\rho_2 = 0$.

In this spirit, inspired by the historical development on the study of CPE conjecture, we shall prove that the assumptions considered in (FILHO, 2015) as well as (NETO, 2015) can be replaced by a weaker integral condition. We point out that this integral condition is satisfied in the standard sphere, hence it is a natural hypothesis to consider. In this sense, we have established the following result.

Theorem 1.0.2 *The CPE conjecture is true provided that the function* (1.0.1) *satisfies*

$$\rho_k + \rho_m \leq 0,$$

for $m > k$, where m is even and k is odd.

We now deal another approach. In order to do so, we say that a conformal mapping between two Riemannian manifolds (M, g) and (N, h) is a smooth mapping $F : (M, g) \to (N, h)$ which satisfies the property $F^*h = \alpha^2 g$ for a smooth positive function $\alpha : M \to \mathbb{R}^+$. Here, we ask what happens if a CPE metric is conformal to an Einstein manifold? The answer is the following result:

Theorem 1.0.3 *Let (M^n, g, f) be a CPE metric. If g is conformal to an Einstein metric $\tilde{g}$, then M is isometric to the standard sphere.*

It is important to remark that, if a compact 4-dimension manifold is locally conformal to an Einstein manifold then its Bach tensor vanishes, since the Bach tensor is a conformal invariant in dimension four. On the other hand, Qing and Yuan in (QING; YUAN, 2013) proved that the CPE conjecture is true provided the metric is Bach flat. Therefore, the previous theorem is an extension of this result for any dimension.

Remark 1.0.1 *We remark that the previous theorem is an extension for any dimension of a particular result which is already true in dimension four. In fact, it is well known that if a* *compact 4-dimension manifold is locally conformal to an Einstein manifold then its Bach tensor vanishes. In particular, any 4-dimensional CPE metric conformal to an Eienstein metric is Bach flat. Thus, by Theorem 3.10 in (QING; YUAN, 2013) the CPE conjecture is true in this case.*

In the third part we study with the p-fundamental tone. We denote by $\lambda_p^*(M)$ the p-fundamental tone of M, which is defined by

$$\lambda_p^*(M) = \inf\left\{ \frac{\int_M |\nabla f|^p dM}{\int_M |f|^p dM} : f \in W_0^{1,p}(M), f \neq 0 \right\}.$$

In this part, inspired in (BESSA; MONTENEGRO, 2003), we study lower bounds estimates for the p-fundamental tone on geodesic ball and as a consequence we get lower bounds for the p-fundamental tone of submanifolds with locally bounded mean curvature. We obtain the following results.

Theorem 1.0.4 *Let M be an n-dimensional complete Riemannian manifold. Denote by $B_M(q,r)$ a geodesic ball with radius $r < \mathrm{inj}(q)$. Let $\kappa(q,r) = \sup\{K_M(x) : x \in B_M(q,r)\}$, where $K_M(x)$ denotes the sectional curvature of M at x. Then*

$$\lambda_p^*(B_M(q,r)) \geq \begin{cases} \frac{1}{p^p}\max\{\frac{n^p}{r^p}, [(n-1)k\coth(kr)]^p\} & \text{if} \quad \kappa(q,r) = -k^2, \\ \frac{n^p}{p^p r^p} & \text{if} \quad \kappa(q,r) = 0, \\ \frac{((n-1)k\cot(kr)+1)^p}{p^p r^p} & \text{if} \quad \kappa(q,r) = k^2 \text{ and } r < \frac{\pi}{2k}, \end{cases} \tag{1.0.2}$$

where k is a positive constant.

Theorem 1.0.5 *Let $\varphi : M^n \to N^m$ be an isometric immersion with locally bounded mean curvature and let Ω be any connected component of $\varphi^{-1}(\overline{B_N(q,r)})$ for $q \in N \backslash \varphi(M)$ and $r > 0$. Denote by $\kappa(q,r)$ the supremum of the sectional curvature of M in $B_M(q,r)$ as in Theorem 5.2.2. Then, for a constant $k > 0$, we have the following:*

1. *If $\kappa(q, \mathrm{inj}(q)) = k^2 < +\infty$ and*

$$r < \min\left\{ \mathrm{inj}(q), \frac{\cot^{-1}\left(\frac{h(q,\mathrm{inj}(q))}{(n-1)k}\right)}{k} \right\},$$

then

$$\lambda_p^*(\Omega) \geq \frac{[(n-1)k\cot(kr) - h(q,r)]^p}{p^p}.$$

2. *If $\kappa(q,r) > 0$ for all $r > 0$, $\lim_{r\to\infty}\kappa(q,r) = \infty$, $\mathrm{inj}(q) = \infty$, and*

$$r < r_0 := \max_{s>0}\left\{ \frac{\cot^{-1}\left(\frac{h(q,s)}{(n-1)\sqrt{\kappa(q,s)}}\right)}{\sqrt{\kappa(q,s)}} \right\},$$

then

$$\lambda_p^*(\Omega) \geq \frac{[(n-1)\sqrt{\kappa(q,r)}\cot(\sqrt{\kappa(q,r)}r) - h(q,r)]^p}{p^p}.$$

3. *If* $\kappa(q,\mathrm{inj}(q)) = 0$ *and* $r < \min\{\mathrm{inj}(q), \frac{n}{h(q,\mathrm{inj}(q))}\}$, *where* $\frac{n}{h(q,\mathrm{inj}(q))} = +\infty$ *when* $h(q,\mathrm{inj}(q)) = 0$, *then*

$$\lambda_p^*(\Omega) \geq \frac{[\frac{n}{r} - h(q,r)]^p}{p^p}.$$

4. *If* $\kappa(q,\mathrm{inj}(q)) = -k^2$, $h(q,\mathrm{inj}(q)) < (n-1)k$, *and* $r < \mathrm{inj}(q)$, *then*

$$\lambda_p^*(\Omega) \geq \frac{[(n-1)k - h(q,r)]^p}{p^p}.$$

5. *If* $\kappa(q,\mathrm{inj}(q)) = -k^2$, $h(q,\mathrm{inj}(q)) \geq (n-1)k$, *and*

$$r < \min\left\{\mathrm{inj}(q), \frac{\coth^{-1}\left(\frac{h(q,\mathrm{inj}(q))}{(n-1)k}\right)}{k}\right\},$$

then

$$\lambda_p^*(\Omega) \geq \frac{[(n-1)k\coth(kr) - h(q,r)]^p}{p^p}.$$

Proceeding we study upper bound for the p-fundamental tone which combined with the above lower bound give us the following result.

Theorem 1.0.6 *Let M^n be an n-dimensional complete properly immersed minimal submanifold in a Cartan-Hadamard manifold N of sectional curvature K_N bounded from above by $K_N \leq \kappa \leq 0$. Suppose that*

$$\lim_{R\to\infty} Q(R) < \infty.$$

Then

$$\lambda_p^*(M) = \frac{(n-1)^p\sqrt{-\kappa}^p}{p^p}.$$

As a consequence of the previous theorem , we get the following interesting intrinsic result in the direction of the generalized McKean's Theorem obtained by Lima, Montenegro and Santos in (LIMA *et al.*, 2010).

Theorem 1.0.7 *Let M^n be a complete simply connected manifold with sectional curvature bounded from above $K_M \leq \kappa < 0$. Furthermore, suppose that there exists a point $q \in M$ such that*

$$\sup_{R>0} \frac{\mathrm{Vol}(B_R^q)}{\mathrm{Vol}(B_R^\kappa)} < +\infty, \tag{1.0.3}$$

where B_R^q is the geodesic ball in M centered at q of radius R, and B_R^κ is the geodesic ball in $\mathbb{H}^n(\kappa)$ of the same radius R. Then

$$\lambda_p^*(M) = \frac{(n-1)^p \sqrt{-\kappa}^p}{p^p}. \tag{1.0.4}$$

Finally, we study transversely oriented codimension one C^2-foliations of open subsets Ω of Riemannian manifolds M and obtain lower bounds estimates for the infimum of the mean curvature of the leaves in terms of the p-fundamental $\lambda_p^*(\Omega)$ tone of Ω, which are called Bernstein-Heinz-Chern-Flanders type inequalities. Following Barbosa, Bessa, and Montenegro's idea (BARBOSA *et al.*, 2008), we have the following result.

Theorem 1.0.8 *Let $\mathscr{F}$ be a transversely oriented codimension one C^2-foliation of a connected open set Ω of $(n+1)$-dimensional Riemannian manifold M. Then*

$$p\sqrt[p]{\lambda_p^*(\Omega)} \geq n \inf_{F\in\mathscr{F}} \inf_{x\in F} |H^F(x)|,$$

where H^F denotes the mean curvature function of the leaf F.

As an interesting consequence the above theorem we obtain a Haymann-Makai-Osserman type inequality, which is given in the following result.

Theorem 1.0.9 *Let $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ be a simple smooth curve and $T_\gamma(\rho(t))$ be an embedded tubular neighborhood of γ with variable radius $\rho(t)$ and a smooth boundary $\partial T_\gamma(\rho(t))$. Let $\rho_0 = \sup_t \rho(t) > 0$ be its inradius. Then*

$$\lambda_p^*(T_\gamma(\rho(t))) \geq \frac{(n-1)^p}{p^p \rho_0^p}. \tag{1.0.5}$$

2 PRELIMINARIES

In this chapter we introduce the basic notations necessary to a properly comprehension of the results presents in this work. We recommend for the reader as a complementary reading (BESSE, 2007), (PETERSEN, 2006) and (SCHOEN; YAU, 1994).

2.1 Basic notations

Let (M^n, g) be a smooth, n-dimensional Riemannian manifold with metric g. We denote by $Rm(X,Y)Z$ the Riemann curvature operator defined as follows

$$Rm(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

and we also denote by $Ric(X,Y) = tr(Z \to Rm(Z,X)Y)$ the Ricci tensor, and $R = tr(Ric)$ the scalar curvature. We have the well known formula

$$(div Rm)(X,Y,Z) = \nabla_X Ric(Y,Z) - \nabla_Y Ric(X,Z), \tag{2.1.1}$$

where div means the divergence of the tensor. Let $A = Ric - \frac{R}{2(n-1)} g$ denote the Schouten tensor, which is a $(0,2)$ symmetric tensor. The Weyl tensor is given by

$$Rm = W + \frac{1}{n-2}(A \odot g), \tag{2.1.2}$$

where $\odot$ means the Kulkarni-Nomizu product defined by the following formula

$$(\alpha \odot \beta)_{ijkl} = \alpha_{il}\beta_{jk} + \alpha_{jk}\beta_{il} - \alpha_{ik}\beta_{jl} - \alpha_{jl}\beta_{ik}, \tag{2.1.3}$$

and α, β are $(0,2)$ tensors. Finally, we define the Cotton tensor as follows

$$C_{ijk} = \nabla_i A_{jk} - \nabla_j A_{ik}. \tag{2.1.4}$$

It is well known that

$$\nabla^l W_{ijkl} = \frac{n-3}{n-2} C_{ijk}. \tag{2.1.5}$$

From identities (2.1.4) and (2.1.5) we see that for $n \geq 4$ if the Weyl tensor vanishes, then the Cotton tensor also vanishes. It is not difficult to check that when $n = 3$ the Weyl tensor always vanishes, but the Cotton tensor does not vanish in general. Next, we say that a manifold has harmonic Weyl tensor provided that $divW = 0$. Since $(divW)_{ijk} = \nabla^l W_{ijkl}$, by (2.1.5) we also deduce that for $n \geq 4$, the Cotton tensor is identically zero, if and only if, the Weyl tensor is harmonic.

2.2 Newton transformations

Let T be a symmetric $(0,2)$ tensor and $\sigma_k(T)$ be the symmetric functions associated to T defined as follows

$$\det(I+sT)=\sum_{k=0}^{n}\sigma_k(T)s^k,$$

where $\sigma_0=1$ and $s\in\mathbb{R}$. Since T is symmetric, then $\binom{n}{k}S_k(T)=\sigma_k(T)$ coincides with the k-th elementary symmetric polynomial of the eigenvalues $\lambda_i(T)$ of T, i.e.,

$$\sigma_k(T)=\sigma(\lambda_1(T),\dots,\lambda_n(T))=\sum_{i_1<\cdots<i_k}\lambda_{i_1}(T)\cdots\lambda_{i_k}(T),\ 1\le k\le n. \tag{2.2.1}$$

For simplicity we do not distinguish between the (0,2) tensor T and the operator $\widetilde{T}:\mathfrak{X}(M)\to\mathfrak{X}(M)$, that is a $(1,1)$ tensor, such that $T(X,Y)=\langle\widetilde{T}X,Y\rangle$. We introduce the Newton transformations $P_k(T):\mathfrak{X}(M)\to\mathfrak{X}(M)$, arising from the operator T, by the following inductive law

$$P_0(T)=I,\ \ P_k(T)=\binom{n}{k}S_k(T)I-TP_{k-1}(T),\ 1\le k\le n \tag{2.2.2}$$

or, equivalently,

$$P_k(T)=\binom{n}{k}S_k(T)I-\binom{n}{k-1}S_{k-1}(T)T+\cdots+(-1)^{k-1}\binom{n}{1}S_1(T)T^{k-1}+(-1)^kT^k.$$

Using the Cayley-Hamilton Theorem we get $P_n(T)=0$.

Note that $P_k(T)$ is a self-adjoint operator that commutes with T for any k. Furthermore, if $\{e_1,\dots,e_n\}$ is an orthonormal frame on T_pM diagonalizing T, then

$$(P_k(T))_p(e_i)=\mu_{i,k}(T)_pe_i, \tag{2.2.3}$$

where

$$\mu_{i,k}(T)=\sum_{i_1<\cdots<i_k,i_j\neq i}\lambda_{i_1}(T)\cdots\lambda_{i_k}(T)=\frac{\partial\sigma_{k+1}}{\partial x_i}(\lambda_1(T),\dots,\lambda_n(T)).$$

Moreover, we have the well known formulae

$$\begin{cases} tr(TP_k(T)) &= c_kS_{k+1}(T)\\ tr(P_k(T)) &= c_kS_k(T),\end{cases} \tag{2.2.4}$$

where

$$c_k=(n-k)\binom{n}{k}=(k+1)\binom{n}{k+1}.$$

The divergence of $P_k(T)$ is defined as follows

$$\mathrm{div}P_k(T) = tr(\nabla P_k(T)) = \sum_{i=1}^{n} \nabla_{e_i} P_k(T)(e_i),$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M. Our aim is to compute the divergence of $P_k(T)$. The following definition is important in the sequel. Define the tensor D by

$$D_{ijk} = \nabla_i T_{jk} - \nabla_j T_{ik}. \tag{2.2.5}$$

Note that when T is the Ricci tensor, then by equation (2.1.1) $D = divRm$, and when T is the Schouten tensor, then D is just the Cotton tensor.

2.3 Algebraic lemmas

In this section we present some results that are essential for our purpose. We prove some useful algebraic results.

2.3.1 Algebraic tools

First of all we show a lemma which concerns to suitable polynomials. Letting $I^j(x) = x^j$, let us consider the polynomials $p_m, q_m, r_m, s_m : \mathbb{R} \to \mathbb{R}$ given by

1. $p_m = \sum_{k=1}^{m-1} (-1)^{k-1} \frac{k(2m+1-k)}{2} I^{m-1-k}$.
2. $q_m = \sum_{j=1}^{m-1} (-1)^{j+1} j(j+1) I^{j-1}$.
3. $r_m = mI^m + \sum_{i=2}^{m+1} (-1)^i I^{m+1-i}$.
4. $s_m = m(I^m + I^{m-1}) - r_m$.

We also set

1. $\tau_m = r_m + \frac{m(m+1)}{2}(I+1)$.
2. $\upsilon_{m,k} = k(k+1)p_m + m(m+1)p_k$.
3. $\mu_m = r_m + \frac{m+1}{m-1} s_m$.
4. $\lambda_{m,k} = k(k+1)r_m + m(m+1)r_k$.

Lemma 2.3.1 *For $m > k$, where m is even and k is odd, the above polynomials satisfy:*

1. $\tau_m = (I+1)^2 p_m$,
2. $\mu_m = \frac{1}{m-1}(I+1)^2 q_m$,
3. $\lambda_{m,k} = (I+1)^2 \upsilon_{m,k}$.

Proof: Since $r_m(-1)=0$ we can decompose $\tau_m(x)=(x+1)\widetilde{\tau}_m(x)$, where

$$\begin{aligned}\widetilde{\tau}_m(x) &= mx^{m-1}-(m-1)x^{m-2}+(m-2)x^{m-3}-\cdots+2x+\frac{(m-1)(m+2)}{2}\\ &= \sum_{i=1}^{m-1}(-1)^{i-1}(m+1-i)x^{m-i}+\frac{(m-1)(m+2)}{2}.\end{aligned}$$

In the same way, $\widetilde{\tau}_m(-1)=0$, enables us to write

$$\begin{aligned}\widetilde{\tau}_m(x) &= (x+1)\Big(mx^{m-2}-(2m-1)x^{m-3}+(3m-3)x^{m-4}-(4m-6)x^{m-5}\\ &+ \cdots-\frac{(m-2)(m+3)}{2}x+\frac{(m-1)(m+2)}{2}\Big)\\ &= (x+1)\sum_{k=1}^{m-1}(-1)^{k-1}\frac{k(2m+1-k)}{2}x^{m-1-k}=(x+1)p_m(x),\end{aligned}$$

which gives $\tau_m(x)=(x+1)^2p_m(x)$ that corresponds to the first item. Proceeding analogously we write $\mu_m(x)=(x+1)\widetilde{\mu}_m(x)$, where

$$\widetilde{\mu}_m(x) = mx^{m-1}+\sum_{k=1}^{m-1}(-1)^{k+1}\frac{2(m-k)}{m-1}x^{m-1-k}.$$

Arguing as in the first item we obtain

$$\begin{aligned}\widetilde{\mu}_m(x) &= (x+1)\Big(mx^{m-2}-(m-2)x^{m-3}+\frac{(m-2)(m-3)}{m-1}x^{m-4}\\ &- \frac{(m-3)(m-4)}{m-1}x^{m-5}+\cdots-\frac{6}{m-1}x+\frac{2}{m-1}\Big)\\ &= (x+1)\big(mx^{m-2}+\sum_{k=1}^{m-2}(-1)^k\frac{(m-k)(m-k-1)}{m-1}x^{m-2-k}\big)\\ &= \frac{1}{m-1}(x+1)\big(m(m-1)x^{m-2}+\sum_{k=1}^{m-2}(-1)^k(m-k)(m-k-1)x^{m-2-k}\big)\\ &= \frac{1}{m-1}(x+1)\sum_{k=0}^{m-2}(-1)^k(m-k)(m-k-1)x^{m-2-k}\\ &= \frac{1}{m-1}(x+1)\sum_{k=1}^{m-1}(-1)^{k+1}k(k+1)x^{k-1}=\frac{1}{m-1}(x+1)q_m(x),\end{aligned}$$

and this completes the proof of the second item. Following the same argument used in the first item we write $\lambda_{m,k}(x)=(x+1)\widetilde{\upsilon}_{m,k}(x)$, where

$$\widetilde{\upsilon}_{m,k}(x) = k(k+1)\sum_{i=1}^{m}(-1)^{i+1}(m+1-i)x^{m-i}+m(m+1)\sum_{j=1}^{k}(-1)^{j+1}(k+1-j)x^{k-j},$$

and $\widetilde{\upsilon}_{m,k}(-1)=0$ enable us to write

$$\widetilde{\upsilon}_{m,k}(x) = (x+1)(k(k+1)p_m(x)+m(m+1)p_k(x))=(x+1)\upsilon_{m,k}(x),$$

which proves the last item. □

Lemma 2.3.2 *Let $p_m(x)$ be the polynomial defined above. If m is even, then $p_m(x) > 0$ for all $x \in \mathbb{R}$.*

Proof: First note that $p_2(x) = 2$. Hence it suffices to show that $p_{m+2} > p_m$. Let us set $J_{m+2} = p_{m+2} - p_m$. Then we claim

$$J_{m+2}(x) = (m+2)x^m + (2m+3)\sum_{i=1}^{m}(-1)^i x^{m-i}. \tag{2.3.1}$$

Indeed, we can write

$$\begin{aligned}
J_{m+2}(x) &= \sum_{i=1}^{m+1}(-1)^{i-1}\frac{i(2m+5-i)}{2}x^{m+1-i} - \sum_{i=1}^{m-1}(-1)^{i-1}\frac{i(2m+1-i)}{2}x^{m-1-i} \\
&= (m+2)x^m - (2m+3)x^{m-1} \\
&+ \sum_{i=3}^{m+1}(-1)^{i-1}\frac{i(2m+5-i)}{2}x^{m+1-i} - \sum_{i=1}^{m-1}(-1)^{i-1}\frac{i(2m+1-i)}{2}x^{m-1-i} \\
&= (m+2)x^m - (2m+3)x^{m-1} \\
&+ \sum_{i=1}^{m-1}(-1)^{i-1}\frac{(i+2)(2m+3-i)}{2}x^{m-1-i} - \sum_{i=1}^{m-1}(-1)^{i-1}\frac{i(2m+1-i)}{2}x^{m-1-i} \\
&= (m+2)x^m + (2m+3)\sum_{i=1}^{m}(-1)^i x^{m-i},
\end{aligned}$$

which gives the claim. Since m is even, by (2.3.1) we deduce that $J_{m+2}(-x) > 0$ if $x \geq 0$, with $J_{m+2}(0) = 2m+3$. Now, we define $u_{m+2}(x) = (x+1)J_{m+2}(x)$ and using once more (2.3.1) we infer

$$\begin{aligned}
(x+1)J_{m+2}(x) &= (m+2)x^m(x+1) + (2m+3)\sum_{i=1}^{m}\left((-1)^i x^{m+1-i} + (-1)^i x^{m-i}\right) \\
&= (m+2)x^{m+1} - (m+1)x^m + 2m+3.
\end{aligned}$$

Next noticing that u_{m+2} achieves its minimum at $x = \frac{m}{m+2}$ we conclude that $J_{m+2}(x) > 0$ for any $x \in \mathbb{R}$, which gives that $p_m(x) \geq 2$ for $x \in \mathbb{R}$ provided that m is even, which finishes the proof. □

Lemma 2.3.3 *Let $q_m(x)$ be the polynomial defined above. If m is even, then $q_m(x) > 0$ for all $x \in \mathbb{R}$.*

Proof: We notice that $q_m(-x) > 0$ for every $x \geq 0$, since m is even. From now on we suppose that $x > 0$. Under this choice we can write

$$q_m(x) = x^{m-2}L_m(x^{-1}),$$

where $L_m(x) = \sum_{k=1}^{m-1} (-1)^{k+1} k(k+1) x^{m-k-1}$. Hence, it suffices to show that $L_m(x)$ is strictly positive for every $x > 0$. Proceeding, it is easy to verify that $L_2(x) = 2$, then it is enough to prove that $L_{m+2}(x) > L_m(x)$ for every $x \in \mathbb{R}$ and m even. Letting $T_{m+2} = L_{m+2} - L_m$ we have

$$\begin{aligned} T_{m+2}(x) &= \sum_{k=1}^{m+1} (-1)^{k+1} k(k+1) x^{m-k+1} - \sum_{k=1}^{m-1} (-1)^{k+1} k(k+1) x^{m-k-1} \\ &= 2x^m - 6x^{m-1} + \sum_{k=1}^{m-1} (-1)^{k+1} 2(2k+3) x^{m-k-1}. \end{aligned}$$

Whence we get

$$\frac{1}{2} T_{m+2}(x)(x+1) = x^{m+1} - 2\big(x^m - x^{m-1} - \ldots + x^2 - x + 1\big) + 2m + 3. \tag{2.3.2}$$

Since m is even we have $x^m - x^{m-1} - \ldots + x^2 - x + 1 = \frac{1}{x+1}\big(x^{m+1} + 1\big)$. Hence, we deduce

$$\frac{1}{2} T_{m+2}(x)(x+1)^2 = x^{m+2} - x^{m+1} + (2m+3)x + 2m + 1. \tag{2.3.3}$$

Since the right hand side of (2.3.3) is strictly positive for $x > 0$ we have the same conclusion for T_{m+2} and we complete the proof of the lemma. □

Lemma 2.3.4 *Let $\upsilon_{m,k}(x)$ be the polynomial defined above. If m is even, k is odd and $m > k$, then, $\upsilon_{m,k}(x) > 0$ for all $x \in \mathbb{R}$.*

Proof: Note that it suffices to prove that $\upsilon_{m+2,k}(x) - \upsilon_{m,k}(x) > 0$, since $\upsilon_{k+1,k} = (k+1)q_{k+1}$ and in the proof of Lemma 2.3.3 we showed that $q_{k+1} > 0$. After a straightforward computation we obtain

$$\begin{aligned} \upsilon_{m,k}(x) &= k(k+1) \sum_{j=k-1}^{m-2} (-1)^j \frac{(m-1-j)(m+2+j)}{2} x^j \\ &+ \sum_{j=0}^{k-2} (-1)^j \frac{(j+1)(j+2)(m-k)(m+k+1)}{2} x^j, \end{aligned}$$

which implies that

$$\begin{aligned} \upsilon_{m+2,k}(x) - \upsilon_{m,k}(x) &= k(k+1)(m+2)x^m + k(k+1)(2m+3) \sum_{j=k-1}^{m-1} (-1)^j x^j \\ &+ (2m+3) \sum_{j=0}^{k-2} (-1)^j (j+1)(j+2) x^j. \end{aligned}$$

By the above expression $\upsilon_{m+2,k}(x) - \upsilon_{m,k}(x) > 0$ for every $x \le 0$. Now it remains to prove that $\upsilon_{m+2,k}(x) - \upsilon_{m,k}(x) > 0$ for every $x > 0$. Defining $Q_{m,k} = (x+1)(\upsilon_{m+2,k} - \upsilon_{m,k})$ a straightforward computation gives

$$Q_{m,k}(x) = k(k+1)(m+2)x^{m+1} - k(k+1)(m+1)x^m + 2(2m+3) \sum_{j=0}^{k-1} (-1)^j (j+1) x^j.$$

Thus, for every $x \geq 1$ we have $\upsilon_{m+2,k}(x) - \upsilon_{m,k}(x) > 0$. Hence, we need to treat only the case $0 < x < 1$. If we define $\eta_{m,k}(x) = x^{m-2}\upsilon_{m,k}(x^{-1})$, we get

$$\begin{aligned}\eta_{m,k}(x) &= k(k+1)\sum_{i=0}^{m-k-1}(-1)^i\frac{(i+1)(2m-i)}{2}x^i \\ &+ (m-k)(m+k+1)\sum_{i=m-k}^{m-2}(-1)^i\frac{(m-i-1)(m-i)}{2}x^i.\end{aligned}$$

Defining $V_{m,k}(x) = \eta_{m+2,k}(x) - \eta_{m,k}(x)$, we obtain after a direct computation that

$$\begin{aligned}V_{m,k}(x) &= 2k(k+1)\sum_{i=0}^{m-k-1}(-1)^i(i+1)x^i \\ &+ (2m+3)\sum_{i=m-k}^{m-2}(-1)^i(m-i-1)(m-i)x^i \\ &+ (m-k+2)(m+k+3)\sum_{i=m-k}^{m}(-1)^i(2m-2i+1)x^i \\ &+ ((k+1)(m+2)(m+3)+k(k+1)^2)x^{m-k}.\end{aligned}$$

We aim to prove that $\eta_{m+2,k}(x) - \eta_{m,k}(x) > 0$ provided that $x > 1$. To do so, we consider $P_{m,k}(x) = (x+1)^3(\eta_{m+2,k}(x) - \eta_{m,k}(x)) = (x+1)^3V_{m,k}(x)$. Whence we get

$$\begin{aligned}P_{m,k}(x) &= 2k(k+1)(x+1)^3\sum_{i=0}^{m-k-1}(-1)^i(i+1)x^i \\ &+ (2m+3)(x+1)^3\sum_{i=m-k}^{m-2}(-1)^i(m-i-1)(m-i)x^i \\ &+ (m-k+2)(m+k+3)(x+1)^3\sum_{i=m-k}^{m}(-1)^i(2m-2i+1)x^i \\ &+ ((k+1)(m+2)(m+3)+k(k+1)^2)x^{m-k}(x+1)^3. \\ &= Z_1+Z_2+Z_3+Z_4.\end{aligned}$$

Now, calculating separately $Z_1, Z_2,$ and Z_3 we get

$$\begin{aligned}Z_1 &= (x+1)^3\sum_{i=0}^{m-k-1}(-1)^i(i+1)x^i \\ &= (m-k)x^{m-k+2}+(2m-2k+1)x^{m-k+1}+(m-k+1)x^{m-k}+x+1,\end{aligned}$$

$$\begin{aligned}Z_2 &= (x+1)^3\sum_{i=m-k}^{m-2}(-1)^i(m-i-1)(m-i)x^i \\ &= 2x^{m+1}-k(k+1)x^{m-k+2}-2(k+1)(k-1)x^{m-k+1}-k(k-1)x^{m-k}\end{aligned}$$

and

$$\begin{aligned} Z_3 &= (x+1)^3 \sum_{i=m-k}^{m} (-1)^i (2m-2i+1) x^i \\ &= x^{m+3} - x^{m+1} - (2k+3)x^{m-k+2} - 4(k+1)x^{m-k+1} - (2k+1)x^{m-k}. \end{aligned}$$

So, it is not difficult to check that

$$\begin{aligned} P_{m,k}(x) &= 2k(k+1)(x+1) + 2(2m+3)x^{m+1} + (m-k+2)(m+k+3)(x^{m+3} - x^{m+1}) \\ &+ (k+1)(m^2+5m+k^2+k+6)x^{m-k+3} + k(m^2+5m+3k^2+6k+9)x^{m-k+2} \\ &+ (k+1)(-m^2-m+3k^2+3k)x^{m-k+1} + k(-m^2-m+k^2+2k+1)x^{m-k}. \end{aligned}$$

Then, by the above expression $P_{m,k}(x) > 0$ for every $x > 1$, which implies $\eta_{m+2,k}(x) \geq \eta_{m,k}(x)$ for all $x > 1$. Now we note that $\upsilon_{k+1,k}(x) = (k+1)q_{k+1}(x) > 0$ for every x. Since $\eta_{k+1,k}(x) = x^{k-1}\upsilon_{k+1,k}(x^{-1})$ we obtain $\eta_{k+1,k}(x) > 0$ for all x. Therefore, $\eta_{m,k}(x) > 0$ for every $x > 1$. Finally, since $\upsilon_{m,k}(x) = x^{m-2}\eta_{m,k}(x^{-1}) > 0$ for all $x^{-1} > 1$, we have $\upsilon_{m,k}(x) > 0$ for $0 < x < 1$, which finishes the proof. $\square$

2.4 CPE lemmas

Let $S, T : \mathscr{H} \to \mathscr{H}$ be operators defined over a finite dimensional Hilbert space $\mathscr{H}$. The Hilbert-Schmidt inner product is defined by

$$\langle S, T \rangle = \mathrm{tr}\big(ST^*\big),$$

where tr and $*$ denote, respectively, the trace and the adjoint operation. Moreover, if I denotes the identity operator on $\mathscr{H}$ of dimension n the traceless of an operator T is given by

$$\mathring{T} = T - \frac{\mathrm{tr}T}{n} I.$$

Using this notation we have the following lemmas.

Lemma 2.4.1 *(FILHO, 2015) Let* (M^n, g, f) *be a* CPE metric. Then we have:

1. $(f+1)\mathring{Ric} = \mathring{\nabla}^2 f$. In particular, (M^n, g, f) is Einstein if and only if ∇f is a conformal vector field.
2. $\int_M f^m \langle \mathring{Ric}, \mathring{\nabla}^2 f \rangle dM = -\int_M m f^{m-1} \mathring{Ric}(\nabla f, \nabla f) dM$.
3. $\int_M \big(f+1\big) |\mathring{\nabla}^2 f|^2 dM = -2 \int_M \mathring{\nabla}^2 f(\nabla f, \nabla f) dM$.
4. $\int_M f^m \langle \mathring{Ric}, \mathring{\nabla}^2 f \rangle = \sum_{i=1}^{m} (-1)^{i+1} \int_M f^{m-i} |\mathring{\nabla}^2 f|^2$.

Lemma 2.4.2 *(FILHO, 2015) Let (M^n, g) be a Riemannian manifold and f, φ smooth functions on M such that $\Delta f + \frac{R}{n-1} f = 0$ and let $h = |\nabla f|^2 + \frac{R}{n(n-1)} f^2$. Then we have:*

1. $\frac{1}{2}\Delta h = |\mathring{\nabla}^2 f|^2 + \mathring{Ric}(\nabla f, \nabla f)$,
2. $\frac{1}{2}\langle \nabla \varphi, \nabla f\rangle = \mathring{\nabla}^2 f(\nabla \varphi, \nabla f)$.

2.5 Conformal geometry

A conformal mapping between two Riemannian manifolds (M,g) and (N,h) is a smooth mapping $F : (M,g) \to (N,h)$ which satisfies the property $F^*h = \alpha^2 g$ for a smooth positive function $\alpha : M \to \mathbb{R}^+$. We look to conformal variations of M, that is, those variations of the form $\phi^{-2}g$, for a smooth positive function $\phi : M \to \mathbb{R}$. In the approach of conformal geometry we have the following lemma, which is a well known result of the conformal geometry theory whose proof is standard.

Lemma 2.5.1 *Let (M^n, g) be a Riemannian manifold and $\tilde{g} = \phi^{-2}g$ a metric conformal to g. Then, the following geometric data associated to the metric $\tilde{g}$ are given in terms of the metric g by the following expressions:*

$$
\begin{aligned}
(i) \qquad \widetilde{\nabla}_X Y &= \nabla_X Y - (X\phi)Y - (Y\phi)X + \langle X, Y\rangle \nabla\phi \\
(ii) \quad \widetilde{Rm}(X,Y)Z &= Rm(X,Y)Z - \langle X,Z\rangle \nabla^2\phi(Y) + \langle Y,Z\rangle \nabla^2\phi(X) \\
&+ (\nabla^2\phi(Y,Z) + (Y\phi)(Z\phi) - \langle Y,Z\rangle \|\nabla\phi\|^2)X \\
&- (\nabla^2\phi(X,Z) + (X\phi)(Z\phi) - \langle X,Z\rangle \|\nabla\phi\|^2)Y \\
&+ (X\phi\langle Y,Z\rangle - Y\phi\langle X,Z\rangle)\nabla\phi \\
(iii) \qquad \widetilde{Ric} &= Ric + \phi^{-1}\Big((n-2)\nabla^2\phi - (n-1)\frac{|\nabla\phi|^2}{\phi} g + \Delta\phi g\Big), \\
(iv) \qquad \tilde{R} &= \phi^2\Big(R + \phi^{-1}\Big(2(n-1)\Delta\phi - (n-1)n\frac{|\nabla\phi|^2}{\phi}\Big)\Big), \\
(v) \qquad \widetilde{\mathring{Ric}} &= \mathring{Ric} + (n-2)\phi^{-1}\mathring{\nabla}^2\phi
\end{aligned}
$$

Proof: The proof of (i) and (ii) are straightforward, see (KÜHNEL, 1988) . To obtain (iii) we take trace in (ii) to get

$$\widetilde{R_{jk}} = R_{jk} + \phi^{-1}((n-2)\phi_{jk} - (n-1)\frac{\phi_i\phi^i}{\phi} g_{jk} + \Delta\phi g_{jk}).$$

Taking the trace again we obtain

$$\tilde{R} = \phi^2(R + \phi^{-1}(2(n-1)\Delta\phi - (n-1)n\frac{\phi_i\phi^i}{\phi})).$$

Finally we deduce

$$\begin{aligned}\widetilde{R_{jk}}-\frac{\tilde{R}}{n}\tilde{g}_{jk} &= R_{jk}+\phi^{-1}((n-2)\phi_{jk}-(n-1)\frac{\phi_i\phi^i}{\phi}g_{jk}+\Delta\phi g_{jk})\\ &- \frac{\phi^2}{n}(R+\phi^{-1}(2(n-1)\Delta\phi-(n-1)n\frac{\phi_i\phi^i}{\phi}))\phi^{-2}g_{jk}\\ &= \mathring{R}_{jk}+(n-2)\phi^{-1}(\phi_{jk}-\frac{1}{n}\Delta\phi g_{jk}).\end{aligned}$$

□

2.6 On the p-fundamental tone

Let (M,g) be a Riemannian manifold. For any $p\in(1,\infty)$ and any function $u\in W^{1,p}_{loc}(M)$, the p-Laplacian is the differential operator defined by

$$\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

The p-Laplacian appears naturally on the variational problems associated to the energy functional

$$\begin{aligned}&E_p : W^{1,p}_0(M)\to\mathbb{R}\\ &E_p(u)=\int_M|\nabla u|^p dM.\end{aligned}$$

In particular, if $p=2$, the p-Laplacian Δ_p is the usual Laplace operator Δ. We denote by $\lambda^*_p(M)$ the p-fundamental tone of M, which is defined by

$$\lambda^*_p(M)=\inf\left\{\frac{\int_M|\nabla f|^p dM}{\int_M|f|^p dM} : f\in W^{1,p}_0(M), f\neq 0\right\}. \qquad (2.6.1)$$

Let M be an n-dimensional complete noncompact manifold. Let $\{\Omega_i\}$ be an exhaustion of M by compact domains, i.e., $\{\Omega_i\}$ are compact domains such that $\cup_{i=1}^{\infty}\Omega_i=M$ and $\Omega_i\subset\Omega_{i+1}$ for all $i\in\mathbb{N}$. Consider the first eigenvalue $\lambda_{1,p}(\Omega_i)$ of the following Dirichlet boundary value problem:

$$\begin{cases}\Delta_p u = \lambda|u|^{p-2}u & \text{in } \Omega_i,\\ u = 0 & \text{on } \partial\Omega_i.\end{cases}$$

In (VÉRON, 1991), Veron showed the existence of the above eigenvalue problem and the variational characterization as in (2.6.1). Lindqvist (LINDQVIST, 1990) proved that $\lambda_{1,p}(\Omega_i)$ is simple for each compact domain Ω_i, $i\in\mathbb{N}$ (see also (BELLONI; KAWOHL, 2002)). By definition, we see that $\lambda^*_p(\Omega_i)=\lambda_{1,p}(\Omega_i)$ for each compact domain Ω_i, $i\in\mathbb{N}$. Using the domain monotonicity of $\lambda_{1,p}(\Omega_i)$, we deduce that $\lambda_{1,p}(\Omega_i)$ is non-increasing in $i \in \mathbb{N}$ and has a limit which is independent of the choice of the exhaustion of M. Therefore

$$\lambda_p^*(M) = \lim_{i\to\infty} \lambda_p^*(\Omega_i). \tag{2.6.2}$$

2.7 Isometric immersions

Consider an isometric immersion $\varphi : M \hookrightarrow N$, where M^n and N^m are complete Riemannian manifolds. Denote by $\overline{\nabla}$, $\overline{\nabla}^2$ and $\overline{\Delta}$ the Riemannian connection, the Hessian and the Laplacian on N, respectively, while by ∇, ∇^2 and Δ the Riemannian connection, the Hessian and the Laplacian on M, respectively.

When M is an orientable hypersurface, consider η a unit normal vector field along M. Thereby, the Gauss and Weingarten formulae for the hypersurfaces in N are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y - \langle \alpha X, Y \rangle \eta \tag{2.7.1}$$

and

$$\alpha(x) = -\overline{\nabla}_X \eta, \tag{2.7.2}$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M)$. Here $\alpha : \mathfrak{X}(M) \to \mathfrak{X}(M)$ defines the shape operator of M with respect to η. For simplicty we identify the operator α with the symetric (0,2) tensor $\widetilde{\alpha}$ associated to α defined by $\widetilde{\alpha}(X,Y) = \langle \alpha(X), Y \rangle$. We also define the mean curvature H by $H = \operatorname{tr} \alpha$. Thereby, using this notation we have the following well known theorem (for a proof see for instance (MANFREDO, 1992)).

Theorem 2.7.1 *(Gauss) Let $p \in M$ and X, Y orthonormal vectors on the tangent space T_pM. Then,*

$$K(X,Y) - \bar{K}(X,Y) = \langle \alpha(X,X), \alpha(Y,Y) \rangle - |\alpha(X,Y)|^2. \tag{2.7.3}$$

Definition 2.7.1 *Let $\varphi : M \hookrightarrow N$ be a isometric immersion. We say that φ is a proper immersion if φ is a proper map, i.e., for every compact set $K \subset N$, the inverse image $\varphi^{-1}(K)$ is compact.*

2.7.1 Newton transformations for the shape operator

Let α be the shape operator, since α is a symmetric $(0,2)$ tensor we may consider the symmetric functions associated to α

$$\det(I+s\alpha)=\sum_{k=0}^{m}\sigma_k(\alpha)s^k,$$

where $\sigma_0=1$. Now we define H_k called the k-th mean curvature by

$$H_k=\binom{m}{k}^{-1}\sigma_k(\alpha),$$

which are precisely the average of the k-th elementary symmetric polynomial of the eigenvalues $\lambda_i(\lambda)$. Moreover, the Newton transformations associated to the shape operator are given by

$$P_k(T)=\binom{m}{k}H_kI-\binom{m}{k-1}H_{k-1}\alpha+\cdots+(-1)^{k-1}\binom{m}{1}H_1\alpha^{k-1}+(-1)^k\alpha^k.$$

Note that our choice of the mean curvature H of N^m satisfies $H=mH_1$.

2.7.2 Volume growth of submanifolds

Denote by $\mathbb{M}^n(\kappa)$ the n-dimensional simply connected real space form of constant sectional curvature $\kappa\leq 0$. Recall that the volume of the geodesic sphere S_R^κ and the geodesic ball B_R^κ of radius R in $\mathbb{M}^n(\kappa)$ are given by

$$\mathrm{Vol}(S_R^\kappa)=\omega_{n-1}S_\kappa(R)^{n-1}\quad\text{and}\quad\mathrm{Vol}(B_R^\kappa)=\int_0^R\mathrm{Vol}(S_t^\kappa)dt,$$

where ω_{n-1} stands for the volume of the unit sphere in $\mathbb{R}^n$ and $S_\kappa(t)$ is

$$S_\kappa(t)=\begin{cases} t & \text{if}\quad \kappa=0,\\ \frac{\sinh(\sqrt{-\kappa}t)}{\sqrt{-\kappa}} & \text{if}\quad \kappa<0.\end{cases}$$

The mean curvature $\bar{H}(t)$ of the geodesic spheres of radius t in $\mathbb{M}^n(\kappa)$ is $\bar{H}(t)=(n-1)H_\kappa(t)$, where

$$H_\kappa(t)=\frac{S'_\kappa(t)}{S_\kappa(t)}. \tag{2.7.4}$$

Let $\varphi:M\to N$ be an immersion from a manifold M to a Cartan-Hadamard manifold of sectional curvature K_N bounded from above by $K_N\leq\kappa\leq 0$. Given a point $q\in M$, the extrinsic distance function $r_q:M\to\mathbb{R}^+$ is defined by

$$r_q(x)=\mathrm{dist}_N(\varphi(q),\varphi(x)),$$

where dist_N denotes the distance function in N. The extrinsic ball M_q^R centered at $q \in M$ of radius R is given by

$$M_q^R := \{x \in M : r_q(x) < R\}.$$

The volume growth function $Q : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$Q(R) := \frac{\mathrm{Vol}(M_q^R)}{\mathrm{Vol}(B_R^\kappa)},$$

where B_R^κ is the geodesic ball of radius R in $\mathbb{M}^n(\kappa)$. It is well-known that the volume growth function $Q(R)$ of a minimal submanifold M in a Cartan-Hadamard manifold is non-decreasing for $0 < R < \mathrm{dist}_N(q, \partial M)$ (see (ALLARD, 1972; ANDERSON, 1984; PALMER, 1999; SIMON *et al.*, 1983)). Using this monotonicity property of minimal submanifolds, Gimeno (GIMENO, 2014) proved the following useful fact:

Lemma 2.7.1 *(GIMENO, 2014) Let $\varphi : M \to N$ be an isometric minimal immersion from a manifold M to a Cartan-Hadamard manifold of sectional curvature K_N bounded from above by $K_N \leq \kappa \leq 0$. Then*

$$Q(t)\mathrm{Vol}(S_t^\kappa) \leq \mathrm{Vol}(M_q^t)' = (\ln Q(t))'\mathrm{Vol}(B_t^\kappa)Q(t) + Q(t)\mathrm{Vol}(S_t^\kappa). \tag{2.7.5}$$

Now we present a couple of definitions which will be necessary for a better understand of the next three theorems presented in this section.

Definition 2.7.2 *A noncompact manifold M is of finite topological type if there is a compact domain Ω such that $M \backslash \Omega$ is homeomorphic to $\partial\Omega \times [1, \infty)$.*

Definition 2.7.3 *Let M be a complete non-compact Riemannian manifold. Let $K \subset M$ be a compact set with non-empty interior and smooth boundary. We denote by $\mathscr{E}_K(M)$ the number of connected components $U_1, \ldots, U_{\mathscr{E}_K(M)}$ of $M \backslash K$ with non-compact closure. Then M has $\mathscr{E}_K(M)$ ends with respect to K, and the global number of ends $\mathscr{E}(M)$ is given by*

$$\mathscr{E}(M) = \sup_{K \subset M} \mathscr{E}_K(M),$$

where K ranges on the compact sets of M with non-empty interior and smooth boundary.

In the sequel, we have the following results which relate some properties of the extrinsic geometry of the submanifold with its volume growth.

Theorem 2.7.2 *(ANDERSON, 1984), (CHEN, 1995) and (GIMENO; PALMER, 2013) Let M^n be a minimal submanifold immersed in the Euclidean space $\mathbb{R}^m$. If M^n has finite total scalar curvature*

$$\int_M |\alpha|^n dM < \infty.$$

Then

$$\sup_{R>0} Q(R) < \infty.$$

Theorem 2.7.3 *(QING; YI, 2000) Let M^2 be a minimal surface immersed in the hyperbolic space $\mathbb{H}^n$ of has constant sectional curvature $\kappa < 0$ or in the Euclidean space $\mathbb{R}^n$. If M has finite total extrinsic curvature, namely $\int_M |\alpha|^2 dM < \infty$, then M has finite topological type, and*

$$\sup_{R>0} Q(R) \leq \frac{1}{4}\int_M |\alpha|^2 dM + \chi(M),$$

being $\chi(M)$ the Euler characteristic of M.

Theorem 2.7.4 *(GIMENO; PALMER, 2014) Let M^n be a minimal $n-$dimensional submanifold properly immersed in the hyperbolic space $\mathbb{H}^m$ of constant sectional curvature $\kappa < 0$. If $n > 2$ and the submanifold is of faster than exponential decay of its extrinsic curvature, namely, there exists a point $p \in M$ such that*

$$|\alpha|(x) \leq \frac{\delta(r_p(x))}{e^{2\sqrt{-\kappa} r_p(x)}},$$

where $\delta(r)$ is a function such that $\delta(r) \to 0$ when $r \to \infty$. Then the submanifold has finite topological type, and

$$\sup_{R>0} Q(R) \leq \mathscr{E}(M),$$

being $\mathscr{E}(M)$ the (finite) number of ends of M.

2.8 Foliations

In this section we define and give a few important definition concerning to foliations.

Definition 2.8.1 *A family $\mathscr{F} = \{F_\gamma\}_{\gamma \in A}$ of connected subsets of a manifold M^n is said to be an m-dimensional C^r foliation, if*

1. $\bigcup_{\gamma \in A} F_\gamma = M$,

2. $\gamma \neq \lambda \Rightarrow F_\gamma \cap F_\lambda = \emptyset$,
3. *For any point* $q \in M$ *there is a* C^r *chart (local coordinate system)* $\varphi_q : U_q \to R^n$ *such that* $q \in U_q$, $\varphi_q(q) = 0$, *and if* $U_q \cap F_\gamma \neq \emptyset$ *the connected components of the sets* $\varphi_q(U_q \cap F_\gamma)$ *are given by equations* $x_{m+1} = c_{m+1}, \ldots, x_n = c_n$, *where* c_j*'s are constants. The sets* F_γ *are immersed submanifolds of* M *called the leaves of* $\mathscr{F}$.

The family of all the vectors tangent to the leaves is the integrable subbundle of TM denoted by $T\mathscr{F}$. If M carries a Riemannian structure, $T\mathscr{F}^\perp$ denotes the subbundle of all the vectors orthogonal to the leaves. A foliation $\mathscr{F}$ is said to be orientable (respectively, transversely orientable) if the bundle $T\mathscr{F}$ (respectively, $T\mathscr{F}^\perp$) is orientable.

3 ALMOST RICCI SOLITONS

The results in this chapter can be found in (BARROS; EVANGELISTA, 2016), which is a joint work with Professor Abdênago Barros. Here we study almost Ricci solitons with null Cotton tensor.

3.1 Introduction

The concept of almost Ricci soliton was introduced by Pigola et al. in (PIGOLA *et al.*, 2010), where essentially they modified the definition of a Ricci soliton by permitting to the parameter λ to be a variable function. More precisely,

Definition 3.1.1 *We say that a Riemannian manifold (M^n, g) is an almost Ricci soliton if there exist a complete vector field X and a smooth soliton function $\lambda : M^n \to \mathbb{R}$ satisfying*

$$Ric + \frac{1}{2}\mathscr{L}_X g = \lambda g, \tag{3.1.1}$$

where Ric and $\mathscr{L}$ stand for the Ricci curvature tensor and the Lie derivative, respectively.

We shall refer to this equation as the fundamental equation of an almost Ricci soliton (M^n, g, X, λ). We say that an almost Ricci soliton is shrinking, steady or expanding provided $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively, otherwise we say that it is indefinite. When $X = \nabla f$ for some smooth function f on M^n, we say that it is a gradient almost Ricci soliton. In this case identity (3.1.1) becomes

$$Ric + \nabla^2 f = \lambda g, \tag{3.1.2}$$

where $\nabla^2 f$ stands for the Hessian of f. Further, an almost Ricci soliton is trivial provided X is a Killing vector field, otherwise it will be called a non-trivial almost Ricci soliton. We point out that when X is a Killing vector field and $n \geq 3$, we have that M is an Einstein manifold since Schur's lemma ensures that λ is constant.

We highlight that Ricci solitons also correspond to self-similar solutions of Hamilton's Ricci flow, for more details about Ricci soliton see e.g. (CAO, 2009). In this perspective Brozos-Vázquez, García-Río and Valle-Regueiro (BROZOS-VÁZQUEZ *et al.*, 2016) observed that some proper gradient almost Ricci solitons correspond to self-similar solutions of the Ricci-Bourguignon flow, which is a geometric flow given by

$$\frac{\partial}{\partial t} g(t) = -2(Ric(t) - kR(t)g(t)),$$

where $k \in \mathbb{R}$ and R stands for the scalar curvature. This flow can be seen as an interpolation between the flows of Ricci and Yamabe. For more details on Ricci-Bourguignon flow we recommend (GIOVANNI *et al.*, 2015).

It is important to emphasize that the round sphere does not admit a (nontrivial) Ricci soliton structure. However, Barros and Ribeiro Jr (BARROS; JR, 2012) showed an explicit example of an almost Ricci soliton on the standard sphere. From this, it is interesting to know if, in compact case, this example is the unique with soliton function λ non constant. In this sense, Barros and Ribeiro Jr (BARROS; JR, 2012) proved that a compact gradient almost Ricci soliton with constant scalar curvature must be isometric to a standard sphere. Afterward, Barros, Batista and Ribeiro Jr (BARROS *et al.*, 2014) proved that every compact almost Ricci soliton with constant scalar curvature is gradient. In (BRASIL *et al.*, 2014), Costa, Brasil and Ribeiro Jr showed that under a suitable integral condition, a 4-dimensional compact almost Ricci soliton is isometric to standard sphere $\mathbb{S}^4$. While Ghosh (GHOSH, 2014) was able to prove that if a compact K-contact metric is a gradient almost Ricci soliton, then it is isometric to a unit sphere. We also remark that Barros, Batista and Ribeiro Jr (BARROS *et al.*, 2012) proved that under a suitable integral condition a locally conformally flat compact almost Ricci soliton is isometric to a standard sphere $\mathbb{S}^n$. For more details see, for instance, (BARROS *et al.*, 2014), (BARROS *et al.*, 2012), (GHOSH, 2014), (MASCHLER, 2015) and (SHARMA, 2014).

When M is a compact manifold the Hodge-de Rham decomposition theorem (see for instance (WARNER, 2013)) asserts that X can be decomposed as a sum of a gradient of a function h and a divergence-free vector field Y, i.e.

$$X = \nabla h + Y,$$

where $divY = 0$. From now on we consider h the function given by this decomposition. Henceforth, in this chapter, we denote by M^n, $n \geq 3$, a compact connected oriented manifold without boundary.

3.2 Auxiliaries lemmas

In this section we calculate the divergence of the Newton transformations of a general symmetric (0,2) tensor and obtain an explicit relation with the tensor D defined by $D_{ijk} = \nabla_i T_{jk} - \nabla_j T_{ik}$. This result will be useful when T is the Schouten tensor A. In this case the divergence of the Newton transformations will be related to the Cotton tensor.

Lemma 3.2.1 *Let $P_k(T)$ be the Newton transformations associated with T defined Section 2.2 and let $\{e_1,\ldots,e_n\}$ be a local orthonormal frame on M. Then, for all $Z \in \mathfrak{X}(M)$, the divergence of $P_k(T)$ are given recursively as*

$$\begin{aligned} &\mathrm{div}P_0(T) = 0 \\ &\langle \mathrm{div}P_k(T), Z\rangle = -\langle T(\mathrm{div}P_{k-1}(T)), Z\rangle - \sum_{i=1}^{n} D(e_i, Z, P_{k-1}(T)e_i), \end{aligned} \tag{3.2.1}$$

or equivalently

$$\langle \mathrm{div}P_k(T), Z\rangle = \sum_{j=1}^{k}\sum_{i=1}^{n}(-1)^j D(e_i, T^{j-1}Z, P_{k-j}(T)e_i). \tag{3.2.2}$$

Proof: Since $P_0(T) = I$, then $\mathrm{div}P_0(T) = 0$. By the inductive definition of $P_k(T)$ we have

$$\begin{aligned} \nabla_Z P_k(T)Y &= \langle \nabla\sigma_k(T), Z\rangle Y - \nabla_Z(T\circ P_{k-1}(T))Y \\ &= \langle \nabla\sigma_k(T), Z\rangle Y - (\nabla_Z T\circ P_{k-1}(T))Y - (T\circ \nabla_Z P_{k-1}(T))Y, \end{aligned}$$

so that

$$\mathrm{div}P_k(T) = \sum_{i=1}^{n}(\nabla_{e_i}P_k(T))e_i = \nabla\sigma_k(T) - \sum_{i=1}^{n}(\nabla_{e_i}T)(P_{k-1}(T)e_i) - (T\,\mathrm{div}P_{k-1}(T)).$$

Now, by using (2.2.5) we get

$$\begin{aligned} \langle(\nabla_{e_i}T)(P_{k-1}(T)e_i), Z\rangle &= \langle(\nabla_{e_i}T)Z, P_{k-1}(T)e_i\rangle \\ &= (\nabla_{e_i}T)(Z, P_{k-1}(T)e_i) \\ &= D(e_i, Z, P_{k-1}(T)e_i) + \nabla_Z T(e_i, P_{k-1}(T)e_i) \\ &= D(e_i, Z, P_{k-1}(T)e_i) + \langle(\nabla_Z T)e_i, P_{k-1}(T)e_i\rangle \\ &= D(e_i, Z, P_{k-1}(T)e_i) + \langle(P_{k-1}(T)\circ\nabla_Z T)(e_i), e_i\rangle. \end{aligned}$$

Therefore, letting $\rho = \sum_{i=1}^{n} D(e_i, Z, P_{k-1}(T)e_i)$, we deduce

$$\langle \mathrm{div}P_k(T), Z\rangle = \langle \nabla\sigma_k(T), Z\rangle - tr(P_{k-1}(T)\circ\nabla_Z T) - (T\,\mathrm{div}P_{k-1}(T)) - \rho. \tag{3.2.3}$$

Now we just need to prove that

$$tr(P_{k-1}(T)\circ\nabla_Z T) = \langle \nabla\sigma_k(T), Z\rangle. \tag{3.2.4}$$

We prove the above equation using a local orthonormal frame that diagonalizes T. We point out that such a frame does not always exist, since the multiplicity of the eigenvalues may changes. Therefore, we will work in a subset $M_T \subset M$ consisting of points at which the multiplicity of the eigenvalues is locally constant. We recall that such subset is open and dense in M, and in every connected component of M_T the eigenvalues form mutually smooth distinct eigenfunctions and, for such a function λ, the assignment $p \to V_{\lambda(p)}(p) \subset T_pM$ defines a smooth eigenspace distribution V_λ of T (consult (BESSE, 2007), Paragraph 16.10). Therefore, for every $p \in M_T$ there exists a local orthonormal frame defined on a neighborhood of p that diagonalizes T, i.e.,

$$(\nabla_Z T)e_i = Z(\lambda_i)e_i + \sum_{j \neq i}(\lambda_i - \lambda_j)\omega_i^j(Z)e_j,$$

where we use the standard notation $\omega_i^j(Z) = \langle \nabla_Z e_i, e_j \rangle$. Using (2.2.3) we get

$$\begin{aligned} tr(P_{k-1}(T) \circ \nabla_Z T) &= \sum_{i=1}^n \mu_{i,k-1} Z(\lambda_i) \\ &= \sum_{i=1}^n Z(\lambda_i) \sum_{i_1 < \cdots < i_k, i_j \neq i} \lambda_{i_1} \cdots \lambda_{i_{k-1}} \\ &= Z\Big(\sum_{i_1 < \cdots < i_k} \lambda_{i_1}(T) \cdots \lambda_{i_k}(T)\Big) = \langle \nabla \sigma_k(T), Z \rangle. \end{aligned}$$

This proves the statement on M_T, and by continuity on M. Substituting (3.2.4) into (3.2.3), we get (3.2.1). In order to arrive at (3.2.2) it suffices to use an inductive argument. □

In particular, when $D \equiv 0$ all the Newton transformations are divergence free. In general we have the following.

Corollary 3.2.1 *If $D = 0$, then the Newton transformations are divergence free:* $\mathrm{div} P_k(T) = 0$ *for each k.*

3.3 Integral formulae

In this section we take T to be the Schouten tensor, i.e., $T = A = Ric - \frac{R}{2(n-1)}g$. Now we obtain some integral formulae for the symmetric functions associated to the Schouten tensor A, which will be used to prove our main result. Thereby, we have the following lemma.

Lemma 3.3.1 *Let (M, g, X, λ) be a compact oriented almost Ricci soliton. For each k, the following integral formula holds:*

$$\int_M \langle \mathrm{div} P_k(A), X \rangle dM + c_k \int_M \Big(\big(S_1(A) + \frac{1}{n}\Delta h\big)S_k(A) - S_{k+1}(A)\Big) dM = 0. \tag{3.3.1}$$

Proof: Note that $\nabla_Y P_k(A)$ is self-adjoint for all $Y \in \mathfrak{X}(M)$. A straightforward computation shows that

$$\mathrm{div}(P_k(A)X) = \langle \mathrm{div} P_k(A), X \rangle + \sum_{i=1}^n \langle \nabla_{e_i} X, P_k(A)e_i \rangle, \tag{3.3.2}$$

where $\{e_1,\ldots,e_n\}$ is a local orthonormal frame. If we take such a local orthonormal frame that diagonalizes A, then by (2.2.3) we have

$$\langle\nabla_{e_i}X,P_k(A)e_i\rangle=\mu_{i,k}\langle\nabla_{e_i}X,e_i\rangle=\langle\nabla_{P_k(A)e_i}X,e_i\rangle.$$

By the almost Ricci soliton equation (3.1.1) we get

$$\begin{aligned}\langle\nabla_{P_k(A)e_i}X,e_i\rangle &=\lambda\langle P_k(A)e_i,e_i\rangle-Ric(P_k(A)e_i,e_i)\\ &=\Big(\lambda-\tfrac{R}{2(n-1)}\Big)\langle P_k(A)e_i,e_i\rangle-\langle AP_k(A)e_i,e_i\rangle,\end{aligned}\tag{3.3.3}$$

hence, using equations (3.3.3) and (2.2.4), equation (3.3.2) becomes

$$\begin{aligned}\mathrm{div}(P_k(A)X) &=\langle\mathrm{div}P_k(A),X\rangle+\Big(\lambda-\tfrac{R}{2(n-1)}\Big)trP_k(A)-tr(AP_k(A))\\ &=\langle\mathrm{div}P_k(A),X\rangle+\Big(\lambda-\tfrac{R}{2(n-1)}\Big)c_kS_k(A)-c_kS_{k+1}(A),\end{aligned}$$

Taking trace in (3.1.1) we get $R+divX=n\lambda$. Since $S_1(A)=\frac{(n-2)R}{2n(n-1)}$ we obtain

$$\mathrm{div}(P_k(A)X)=\langle\mathrm{div}P_k(A),X\rangle+\Big(S_1(A)+\frac{1}{n}divX\Big)c_kS_k(A)-c_kS_{k+1}(A).\tag{3.3.4}$$

When M is a compact manifold and h is the function given by the Hodge-de Rham decomposition theorem it is easy to see that identity (3.3.4) becomes

$$\mathrm{div}(P_k(A)X)=\langle\mathrm{div}P_k(A),X\rangle+\Big(S_1(A)+\frac{1}{n}\Delta h\Big)c_kS_k(A)-c_kS_{k+1}(A).\tag{3.3.5}$$

Integrating (3.3.5) we get the desired result. □

Note that when the Cotton tensor vanishes Corollary 3.2.1 implies that

$$\int_M\langle\mathrm{div}P_k(A),X\rangle dM=0.\tag{3.3.6}$$

Therefore, we obtain the next corollary.

Corollary 3.3.1 *Let (M,g,X,λ) be a compact oriented almost Ricci soliton such that the Cotton tensor vanishes. Then,*

$$\int_M\Big(\big(S_1(A)+\frac{1}{n}\Delta h\big)S_k(A)-S_{k+1}(A)\Big)dM=0.\tag{3.3.7}$$

3.4 Rigidity result

In this section we present our main result which concern to hypothesis on the symmetric functions $S_k(A)$ associated to the Schouten tensor A. Thereby, since $S_1(A)=\frac{(n-2)R}{2n(n-1)}$, we may conclude that $S_1(A)$ is constant if and only if R is constant, in this case Barros and others in (BARROS *et al.*, 2014) proved that the almost Ricci soliton is isometric to a round sphere. In the next theorem we consider the other symmetric functions $S_k(A)$, $k = 2, ..., n-1$ and under the additional hypothesis on the nullity of the cotton tensor we obtain a rigidity result for the almost Ricci soliton. We also note that $n^2(S_1(A))^2 = n(n-1)S_2(A) + |A|^2$, hence $S_2(A)$ constant, in general, does not implies that $S_1(A)$ constant.

We remark that Hu and others in (HU *et al.*, 2008) used the symmetric functions of the Schouten tensor to study locally conformally flat manifolds. They proved that under conditions on item 1 and item 4 of the next theorem, a compact locally conformally flat manifold with semi-positive definite Schouten tensor is isometric to a space form of constant sectional curvature. In this direction, the next theorem improves the results obtained in (HU *et al.*, 2008) for compact almost Ricci solions with null cotton tensor, since null cotton tensor does not imply locally conformally flat in general. For example, the complex projective space $\mathbb{CP}^n$ with Fubini–Study metric is Einstein, hence has null cotton tensor, however $\mathbb{CP}^n$ is not locally conformally flat.

Remark 3.4.1 *Before presenting the proofs of the results, we recall that the symmetric functions satisfy Newton's inequalities:*

$$S_k(A)S_{k+2}(A) \leq S_{k+1}^2(A) \text{ for } 0 \leq k < n-1, \tag{3.4.1}$$

which is a generalized Cauchy-Schwarz inequality. Moreover, if equality occurs for $k = 0$ *or* $1 \leq k < n$ *with* $S_{k+2}(A) \neq 0$*, then* $\lambda_1(A) = \lambda_2(A) = \ldots = \lambda_n(A)$*. As an application, provided that* $\lambda_k(A) > 0$ *for* $1 \leq k \leq n$*, we obtain Gårding's inequalities*

$$S_1 \geq S_2^{\frac{1}{2}} \geq S_3^{\frac{1}{3}} \geq \cdots \geq S_n^{\frac{1}{n}}. \tag{3.4.2}$$

Here equality holds, for some $1 \leq k < n$*, if and only if,* $\lambda_1(A) = \lambda_2(A) = \ldots = \lambda_n(A)$*. Note that* (3.4.2) *implies that* $S_k^{\frac{k+1}{k}} \geq S_{k+1}$ *for* $1 \leq k < n$*. For a proof see for instance (HARDY et al., 1952) Theorem 51, p. 52 or Proposition* 1 *in (CAMINHA, 2006).*

Theorem 3.4.1 *Let* (M^n, g, X, λ)*,* $n \geq 3$*, be a non-trivial compact oriented almost Ricci soliton such that the Cotton tensor is identically zero. Then,* M^n *is isometric to a standard sphere* $\mathbb{S}^n$ *provided that one of the next condition is satisfied:*

1. $S_2(A)$ *is constant and positive.*

2. *$S_k(A)$ is nowhere zero on M and $S_{k+1}(A) = cS_k(A)$, where $c \in \mathbb{R}\backslash\{0\}$, for some $k = 1,\cdots,n-1$.*
3. *$Ric \geq \frac{R}{n}g$, with $R > 0$, and $\int_M S_k(A)\Delta h \geq 0$ for some $2 \leq k \leq n-1$.*
4. *$S_k(A)$ is constant for some $k = 2,\cdots,n-1$, and $A > 0$.*

Proof: In item 1 we suppose that $S_2(A)$ is constant and positive. Thereby, choosing $k = 2$ in (3.3.7) we obtain

$$\int_M \Big(\big(S_1(A) + \frac{1}{n}\Delta h\big)S_2(A) - S_3(A)\Big)dM = 0. \tag{3.4.3}$$

Since $S_2(A)$ is constant we deduce

$$\int_M \big(S_2(A)S_1(A) - S_3(A)\big)dM = 0. \tag{3.4.4}$$

On the other hand,

$$S_1^2(A) - S_2(A) \geq 0, \tag{3.4.5}$$

by Newton's inequality (3.4.1). Moreover, equality in (3.4.5) holds, if and only if, $\lambda_1(A) = \cdots = \lambda_n(A)$, which means that A is umbilical (a multiple of g). In this case it is easy to check that

$$A = \frac{(n-2)R}{2n(n-1)}g. \tag{3.4.6}$$

We know from (3.4.5) that $S_1^2(A) \geq S_2(A) > 0$, then $S_1(A)$ does not vanish, this means that either $S_1(A) < 0$ or $S_1(A) > 0$. Now we prove that $S_2(A)S_1(A) - S_3(A)$ is positive or negative, according to the sign of $S_1(A)$.

Indeed, from (3.4.1) we get $S_2^2(A) - S_1(A)S_3(A) \geq 0$. Supposing $S_1(A) > 0$ we obtain

$$S_2(A)S_1(A) - S_3(A) \geq S_2(A)S_1(A) - \frac{S_2^2(A)}{S_1(A)} = \frac{S_2(A)}{S_1(A)}\big(S_1^2(A) - S_2(A)\big) \geq 0. \tag{3.4.7}$$

On the other hand, if $S_1(A) < 0$ we have

$$S_2(A)S_1(A) - S_3(A) \leq S_2(A)S_1(A) - \frac{S_2^2(A)}{S_1(A)} = \frac{S_2(A)}{S_1(A)}\big(S_1^2(A) - S_2(A)\big) \leq 0. \tag{3.4.8}$$

In both cases $S_2(A)S_1(A) - S_3(A)$ has a sign. Using this fact together with equation (3.4.4), we get $S_2(A)S_1(A) - S_3(A) = 0$, and hence equality in (3.4.1), obtaining identity (3.4.6). Therefore (M^n, g) is an Einstein manifold, by Schur's Lemma (M, g) has constant scalar curvature and by Corollary 1 in (BARROS *et al.*, 2014) we conclude that (M^n, g) is isometric to a standard sphere $\mathbb{S}^n$, which concludes the proof of the first item of Theorem 3.4.1.

Proceeding, in item 2 we assume that $S_k(A)$ is nowhere zero on M and $S_{k+1}(A) = cS_k(A)$, where $c \in \mathbb{R}\backslash\{0\}$, for some $k = 1,\cdots,n-1$. Thus we can use Corollary 3.3.1 to infer

$$\int_M \Big(\big(S_1(A)+\frac{1}{n}\Delta h\big)cS_k(A)-cS_{k+1}(A)\Big)dM = 0 \tag{3.4.9}$$

and

$$\int_M \Big(\big(S_1(A)+\frac{1}{n}\Delta h\big)S_{k+1}(A)-S_{k+2}(A)\Big)dM = 0. \tag{3.4.10}$$

By hypothesis $S_{k+1}(A) = cS_k(A)$, whence using (3.4.10) and (3.4.9) we deduce

$$\int_M \big(cS_{k+1}(A)-S_{k+2}(A)\big)dM = 0. \tag{3.4.11}$$

Using once more that $S_{k+1}(A) = cS_k(A)$ we invoke inequality (3.4.1) to get

$$S_k(A)(cS_{k+1}(A)-S_{k+2}(A)) \geq 0.$$

We recall that $S_k(A)$ is nowhere zero on M by hypothesis, and by connectedness it does not change sign on M. Hence $cS_{k+1}(A)-S_{k+2}(A)$ also does not change sign on M. This fact together with equation (3.4.11) gives $cS_{k+1}(A)-S_{k+2}(A) = 0$, which implies $S^2_{k+1}(A)-S_k(A)S_{k+2}(A) = 0$ and consequently $S_{k+2}(A) \neq 0$ on M. Then equality in the inequality (3.4.1) implies that $A = \frac{(n-2)R}{2n(n-1)}g$ and g is an Einstein metric on M and we conclude the proof reasoning as in the previous case.

We suppose in item 3 that $Ric \geq \frac{R}{n}g$, , hence $A \geq \frac{(n-2)R}{2n(n-1)}g$, which implies that $A > 0$, since $R > 0$. This allows us to use all inequalities presented in Remark 3.4.1. By identity (3.3.7) we have

$$\int_M \Big(\big(S_1(A)+\frac{1}{n}\Delta h\big)S_k(A)-S_{k+1}(A)\Big)dM = 0. \tag{3.4.12}$$

Taking into account that $\int_M S_k(A)\Delta h dM \geq 0$, we deduce from the last identity

$$\int_M \Big(S_1(A)S_k(A)-S_{k+1}(A)\Big)dM \leq 0. \tag{3.4.13}$$

Next we make use of (3.4.2) to arrive at

$$S_1(A)S_k(A)-S_{k+1}(A) \geq S_1(A)S_k(A)-S_k(A)^{\frac{k+1}{k}} = S_k(A)(S_1(A)-S_k(A)^{\frac{1}{k}}) \geq 0. \tag{3.4.14}$$

Hence, using (3.4.13) and (3.4.14) we obtain $S_1(A)S_k(A)-S_{k+1}(A) = 0$. Since $A > 0$ and hence $S_k(A), S_{k+2}(A) > 0$, using (3.4.14) again we obtain $S_1(A) = S_k(A)^{\frac{1}{k}}$. By equality in (3.4.2) we get $A = \frac{(n-2)R}{2n(n-1)}g$ and g is an Einstein metric on M. Now, in order to complete the proof it suffices to use once more Corollary 1 in (BARROS *et al.*, 2014).

Finally, we assume in item 4 that $S_k(A)$ is constant for some $k = 2, \cdots, n-1$, and $A > 0$. Hence applying the same argument used to prove item 3 we conclude that $S_1(A) = S_k(A)^{\frac{1}{k}} > 0$ and $S_{k+2}(A) > 0$, which implies that g is an Einstein metric on M. Since $S_1(A) = \frac{(n-2)}{2n(n-1)}R$, we get $R > 0$. The result follows by the same arguments used to conclude item 3, completing the proof of the theorem. $\square$

4 CPE CONJECTURE

The results in this chapter can be found in (BARROS; EVANGELISTA, 2018) which is a joint work with Professor Abdênago Barros. Here we study the CPE conjecture.

4.1 Introduction

Let (M^n, g) be a compact oriented manifold and $\mathscr{M}$ the set of smooth Riemannian structures on M^n of volume 1. Given a metric $g \in \mathscr{M}$ we define the total scalar curvature functional $\mathscr{R} : \mathscr{M} \to \mathbb{R}$ by

$$\mathscr{R}(g) = \int_{M^n} R_g dM, \tag{4.1.1}$$

where R_g and dM stand, respectively, for the scalar curvature and the volume form of the metric g. It is well-known that the critical metrics of the functional $\mathscr{R}$ restricted to $\mathscr{M}$ are Einstein, for more details see Chapter 4 in (BESSE, 2007).

We recall that the Yamabe problem guarantees that there exists a constant scalar curvature metric in every conformal class of Riemannian metrics on a compact manifold M^n. From this, we may consider the set

$$\mathscr{C} = \{g \in \mathscr{M}; R_g \text{ is constant}\}.$$

In (KOISO, 1979), Koiso showed that, under generic condition, $\mathscr{C}$ is an infinite dimensional manifold. Moreover, we recall that the linearization $\mathfrak{L}_g$ of the scalar curvature operator is given by

$$\mathfrak{L}_g(h) = -\Delta_g(tr_g(h)) + div(div(h)) - g(h, Ric_g),$$

where h is any symmetric bilinear form on M^n. Moreover, the formal L^2-adjoint $\mathfrak{L}_g^*$ of $\mathfrak{L}_g$ is given by

$$\mathfrak{L}_g^*(f) = -(\Delta_g f)g + Hess f - fRic_g, \tag{4.1.2}$$

where f is a smooth function on M^n.

It has been conjectured that the critical points of the total scalar curvature functional $\mathscr{R}$ restricted to $\mathscr{C}$ are Einstein. More precisely, in (BESSE, 2007) the authors wrote:

> *"When restricting the total scalar curvature functional to $\mathscr{C}$, are there other critical points besides the Einstein metric?"*[see (BESSE, 2007), p. 128.]

When restricting the total scalar curvature to a pointwise conformal class of metrics a large critical set is obtained. Formally the Euler-Lagrangian equation of Hilbert-Einstein action on the space of Riemannian metrics g with unit volume and constant Ricci scalar curvature is given by

$$Ric - \frac{1}{n}Rg = Hess\,f - f\big(Ric - \frac{R}{n-1}g\big).$$

Definition 4.1.1 *A* CPE metric *is a 3-tuple (M^n, g, f) where (M^n, g), $n \geq 3$ is a n-dimensional compact oriented Riemannian manifold with constant Ricci scalar curvature and f is a smooth potential function that satisfies the equation*

$$Ric - \frac{R}{n}g = Hess\,f - f\big(Ric - \frac{R}{n-1}g\big), \tag{4.1.3}$$

where Ric and $Hess\,f$ stand, respectively, for the Ricci tensor and the Hessian of f.

In order to proceed we notice that, computing the trace in (4.1.3), we obtain

$$\Delta f + \frac{R}{n-1}f = 0. \tag{4.1.4}$$

Therefore, R lies on the spectrum of M^n, thus it must be positive.

The conjecture proposed in (BESSE, 2007) in the middle of 1980's can be announced in terms of CPE definition, see also (BARROS; RIBEIRO, 2014), (QING; YUAN, 2013) and (HWANG, 2003). More precisely, the authors proposed the following conjecture.

Conjecture 4.1.1 *A CPE metric is always Einstein.*

It should be emphasized that Einstein metrics are recovered when $f = 0$. In the last years many mathematicians have contributed to the proof of the CPE Conjecture. However, none has obtained its complete proof. Among its partial answers, Lafontaine (LAFONTAINE, 1983) proved that the CPE Conjecture is true under locally conformally flat assumption. Recently, Ribeiro Jr and Barros (BARROS; RIBEIRO, 2014) showed that Conjecture 4.1.1 is also true for 4-dimensional half conformally flat manifolds. While Qing and Yuan (QING; YUAN, 2013) obtained a positive answer for Bach-flat manifolds in any dimension. In 2014 Chang, Hwang and Yun (YUN *et al.*, 2014) proved that the conjecture is true if the manifold has harmonic curvature.

In (BARROS *et al.*, 2015), Barros et al. showed that a 4-dimensional CPE metric with harmonic tensor W^+ must be isometric to a round sphere $\mathbb{S}^4$.

4.2 Integral Formulae

Proceeding we focus our attention to a smooth function f defined on a Riemannian manifold M^n such that $\Delta f = -\frac{R}{n-1}f$, where R is constant. Then we have

$$\frac{1}{m}\Delta f^m = -\frac{R}{n-1}f^m + (m-1)f^{m-2}|\nabla f|^2. \tag{4.2.1}$$

Whence, for M^n compact, we immediately obtain from (4.2.1)

$$\frac{R}{n-1}\int_M f^m dM = (m-1)\int_M f^{m-2}|\nabla f|^2 dM \tag{4.2.2}$$

as well as $(m-1)\int_M f^{m-2}|\nabla f|^4 dM = \frac{R}{n-1}\int_M f^m|\nabla f|^2 dM + \frac{1}{m}\int_M |\nabla f|^2 \Delta f^m dM$, which gives

$$(m-1)\int_M f^{m-2}|\nabla f|^4 dM = \frac{R}{n-1}\int_M f^m|\nabla f|^2 dM - 2\int_M f^{m-1}\nabla^2 f(\nabla f,\nabla f)dM. \tag{4.2.3}$$

On the other hand we remember that for operators $S,T : \mathscr{H} \to \mathscr{H}$ defined over a finite dimensional Hilbert space $\mathscr{H}$ the Hilbert-Schmidt inner product is defined according to

$$\langle S,T\rangle = \mathrm{tr}\big(ST^*\big), \tag{4.2.4}$$

where tr and $*$ denote, respectively, the trace and the adjoint operation. Moreover, if I denotes the identity operator on $\mathscr{H}$ of dimension n the traceless of an operator T is given by

$$\mathring{T} = T - \frac{\mathrm{tr}T}{n}I. \tag{4.2.5}$$

We notice that identity (4.1.3) becomes

$$(f+1)\mathring{Ric} = \mathring{\nabla}^2 f. \tag{4.2.6}$$

Now we use (4.2.3) to write

$$\rho_m = -2\int_M f^{m-1}\mathring{\nabla}^2 f(\nabla f,\nabla f)dM. \tag{4.2.7}$$

Proceeding, given a $(0,2)$ symmetric tensor field T and any vector field X on a Riemannian manifold M^n we have

$$div\big(\psi T(X)\big) = \psi(divT)(X) + \psi\langle T,\nabla X\rangle + T(\nabla\psi,X), \tag{4.2.8}$$

where ψ is a smooth function on M^n. In particular, choosing $T = \mathring{Ric}, X = \nabla u$ where u in any smooth function on M^n and using the second contracted Bianchi identity we derive

$$div\big(\psi \mathring{Ric}(\nabla u)\big) = \frac{(n-2)}{2n}\psi\langle \nabla R, \nabla u\rangle + \psi\langle \mathring{Ric}, \mathring{\nabla}^2 u\rangle + \mathring{Ric}(\nabla\psi, \nabla u). \tag{4.2.9}$$

On the other hand, for any smooth function u on M^n the Ricci-Bochner identity in tensorial language says $div(\nabla^2 u) = Ric(\nabla u, .) + \nabla\Delta u$. In particular, when $\Delta u = -\frac{R}{n-1}u$ we obtain $div(\mathring{\nabla}^2 u) = \mathring{Ric}(\nabla u, .)$. Using this in (4.2.8) we have

$$div\big(\psi\mathring{\nabla}^2 f(\nabla f)\big) = \psi\big(\mathring{Ric}(\nabla f, \nabla f) + |\mathring{\nabla}^2 f|^2\big) + \mathring{\nabla}^2 f(\nabla\psi, \nabla f). \tag{4.2.10}$$

Now we choose $\psi = f^m$ in (4.2.10) to obtain

$$\int_M f^m\big(\mathring{Ric}(\nabla f, \nabla f)dM + |\mathring{\nabla}^2 f|^2\big)dM = -m\int_M f^{m-1}\mathring{\nabla}^2 f(\nabla f, \nabla f)dM. \tag{4.2.11}$$

Using (4.2.7) we deduce

$$\rho_m = \frac{2}{m}\int_M f^m\big(\mathring{Ric}(\nabla f, \nabla f) + |\mathring{\nabla}^2 f|^2\big)dM. \tag{4.2.12}$$

4.3 Integral condition for the CPE conjecture

For a better understanding of the reader, we prove two particular cases of the main result of this section which will help the reader to understand how we arrived at the main theorem. Using the previous integral formulae we have the following results.

Theorem 4.3.1 *Conjecture 4.1.1 is true for CPE metrics (M^n, g, f), provided that the functions* (1.0.1) *satisfy*

$$\rho_1 + \rho_m \leq 0, \tag{4.3.1}$$

where m is even.

Proof: To begin with, taking $\psi = f^{m+1}$ and $u = f$ in (4.2.9) (or see for instance item (2) of Lemma 2.4.1), we obtain from (4.2.12)

$$\rho_m = \frac{2}{m}\int_M f^m|\mathring{\nabla}^2 f|^2 dM - \frac{2}{m(m+1)}\int_M f^{m+1}\langle\mathring{Ric}, \mathring{\nabla}^2 f\rangle dM. \tag{4.3.2}$$

Next we claim that

$$\rho_m = \frac{2}{m(m+1)}\int_M r_m(f)|\mathring{\nabla}^2 f|^2 dM, \tag{4.3.3}$$

where $r_m(x) = mx^m + \sum_{i=2}^{m+1}(-1)^i x^{m+1-i}$ is one of the polynomials given before in Section 2.3. Indeed, we use item (3) of Lemma 2.4.1 to deduce

$$\rho_m = \frac{2}{m}\int_M f^m|\mathring{\nabla}^2 f|^2 dM - \frac{2}{m(m+1)}\sum_{i=1}^{m+1}(-1)^{i+1}\int_M f^{m+1-i}|\mathring{\nabla}^2 f|^2 dM.$$

Jointing the first two terms of the above identity it becomes

$$\rho_m = \frac{2}{m(m+1)}\int_M \Big(mf^m + \sum_{i=2}^{m+1}(-1)^i f^{m+1-i}\Big)|\mathring{\nabla}^2 f|^2 dM, \tag{4.3.4}$$

which gives our claim. Thus, we have

$$\begin{aligned}
\rho_m + \rho_1 &= \frac{2}{m(m+1)}\int_M r_m(f)|\mathring{\nabla}^2 f|^2 dM + \int_M (f+1)|\mathring{\nabla}^2 f|^2 dM \\
&= \frac{2}{m(m+1)}\int_M (r_m(f) + \frac{m(m+1)}{2}(f+1))|\mathring{\nabla}^2 f|^2 dM \\
&= \frac{2}{m(m+1)}\int_M \tau_m(f)|\mathring{\nabla}^2 f|^2 dM,
\end{aligned}$$

applying Lemma 2.3.1 we get

$$\rho_1 + \rho_m = \frac{2}{m(m+1)}\int_M (f+1)^2 p_m(f)|\mathring{\nabla}^2 f|^2 dM. \tag{4.3.5}$$

In particular, for m even we deduce that $\int_M (f+1)^2 p_m(f)|\mathring{\nabla}^2 f|^2 dM_g = 0$. In fact, we are supposing that $\rho_m + \rho_1 \le 0$, and according to Lemma 2.3.4 $p_m(f) > 0$. On the other hand, since $f^{-1}(-1)$ has measure zero, we conclude that $\mathring{\nabla}^2 f = 0$. Now the result follows by Lemma 2 in (FILHO, 2015) or equation 4.2.6. □

Remark 4.3.1 *Note that* $\int_M \mathring{\nabla}^2 f(\nabla f, \nabla f) dM = \int_M \langle \frac{1}{2}\nabla|\nabla f|^2, \nabla f\rangle dM + \int_M \frac{R}{n(n-1)} f|\nabla f|^2 dM$. *Integrating by parts and using* (4.1.4) *we get* $\int_M \mathring{\nabla}^2 f(\nabla f, \nabla f) dM = \frac{(n+2)R}{2n(n-1)}\int_M f|\nabla f|^2 dM$. *Since* $\int_M f|\nabla f|^2 dM = \frac{R}{2(n-1)}\int_M f^3 dM$, *we may apply Lemma 2.4.1 to deduce*

$$\rho_1 = -2\int_M \mathring{\nabla}^2 f(\nabla f, \nabla f) dM = -\frac{(n+2)R^2}{2n(n-1)^2}\int_M f^3 dM. \tag{4.3.6}$$

Thus, we may conclude that if $\int_M f^3 = 0$ *then* $\rho_m \ge 0$ *for every m even.*

Proceeding, now we give a similar condition for the summation of two consecutive ρ_m.

Theorem 4.3.2 *Conjecture 4.1.1 is true for CPE metrics (M^n, g, f), provided that the functions* (1.0.1) *satisfy*

$$\rho_m + \rho_{m-1} \leq 0, \tag{4.3.7}$$

where m is even.

Proof: Proceeding analogously to the previous theorem, we have

$$\begin{aligned}
\rho_m + \rho_{m-1} &= \frac{2}{m(m+1)} \int_M r_m(f)|\mathring{\nabla}^2 f|^2 + \frac{2}{m(m-1)} \int_M r_{m-1}(f)|\mathring{\nabla}^2 f|^2 \\
&= \frac{2}{m(m-1)(m+1)} \int_M ((m-1)r_m(f) + m r_{m-1}(f))|\mathring{\nabla}^2 f|^2 \\
&= \frac{2}{m(m+1)} \int_M \mu_m(f)|\mathring{\nabla}^2 f|^2,
\end{aligned}$$

where $\mu_m = r_m + \frac{(m+1)}{m-1} s_m = \frac{1}{m-1}((m-1)r_m + m r_{m-1})$, since $s_m = r_{m-1}$. By item (2) in Lemma 2.3.1, we have

$$\rho_m + \rho_{m-1} = \frac{2}{m(m-1)(m+1)} \int_M (f+1)^2 q_m(f)|\mathring{\nabla}^2 f|^2.$$

Hence, for m even we may apply Lemma 2.3.3 and the hypothesis $\rho_m + \rho_{m-1} \leq 0$ to deduce that $\int_M (f+1)^2 q_m(f)|\mathring{\nabla}^2 f|^2 = 0$, and the result follows by the same argument used in the previous theorem. □

Now we state and prove our main result of this section, which concerns to a general condition on the summation of two different ρ_m.

Theorem 4.3.3 *Conjecture 4.1.1 is true for CPE metrics (M^n, g, f), provided that the functions* (1.0.1) *satisfy*

$$\rho_k + \rho_m \leq 0, \tag{4.3.8}$$

for $m > k$, where m is even and k is odd.

Proof: Note that

$$\begin{aligned}
\rho_m + \rho_k &= \frac{2}{m(m+1)} \int_M r_m(f)|\mathring{\nabla}^2 f|^2 dM + \frac{2}{k(k+1)} \int_M r_k(f)|\mathring{\nabla}^2 f|^2 dM \\
&= \frac{2}{m(m+1)k(k+1)} \int_M (k(k+1)r_m(f) + m(m+1)r_k(f))|\mathring{\nabla}^2 f|^2 dM \\
&= \frac{2}{m(m+1)k(k+1)} \int_M \lambda_{m,k}(f)|\mathring{\nabla}^2 f|^2 dM.
\end{aligned}$$

By item 3 in Lemma 2.3.1 we have

$$\rho_m + \rho_k \quad = \quad \frac{2}{m(m+1)k(k+1)} \int_M (f+1)^2 \upsilon_{m,k}(f) |\mathring{\nabla}^2 f|^2 dM.$$

Therefore, for $m > k$, m even and k odd, we may use Lemma 2.3.4 and the hyothesis $\rho_m + \rho_k \leq 0$ to conclude that $\int_M (f+1)^2 \upsilon_{m,k}(f) |\mathring{\nabla}^2 f|^2 dM = 0$. Then, the result follows by applying the same argument used in Theorem 4.3.1. □

Recalling that $h = |\nabla f|^2 + \frac{R}{n(n-1)} f^2$, and choosing $\psi = 1$ in identity (4.2.10) we obtain $\Delta h = 2\mathring{Ric}(\nabla f, \nabla f) + 2|\mathring{\nabla}^2 f|^2$ (see also Lemma 2.4.2). Thereby, (4.2.12) becomes

$$\rho_m = -\frac{1}{m} \int_M \langle \nabla f^m, \nabla h \rangle dM. \tag{4.3.9}$$

Therefore, (4.3.9) enable us to conclude that the hypotheses in (NETO, 2015) as well as in (FILHO, 2015) imply that $\rho_m = \rho_k = 0$ for every m and every k, this shows that the conditions in Theorem 4.3.3 are weaker than that one of (NETO, 2015) and (FILHO, 2015).

4.3.1 The conformal case

We already know that if a non-trival CPE metric is Einstein then it is isometric to the round sphere, which is a straightforward consequence of Theorem 2 in (OBATA, 1962). Lafontaine in (LAFONTAINE, 1983) proved that the CPE conjecture is true provided the metric is locally conformally flat. In this direction, a natural question arises:

Question 4.3.1 *Let (M, g, f) be CPE metric which is conformal to an Einstein metric, can we conclude that it is isometric to a round sphere?*

It is well known that if a compact 4-dimension manifold is locally conformal to an Einstein manifold then its Bach tensor vanishes. In particular, any 4-dimensional CPE metric conformal to an Eienstein metric is Bach flat. Thus, by Theorem 3.10 in (QING; YUAN, 2013) the CPE conjecture is true in this case.

In this section we give a positive answer for Question 4.3.1 in the general case.

Theorem 4.3.4 *Let (M^n, g, f) be a CPE metric. If g is conformal to an Einstein metric $\tilde{g}$, then M is isometric to the standard sphere.*

Proof: Considering the CPE metric $\tilde{g}$ as "background"metric on M, we can write $\tilde{g} = \phi^{-2} g$, where $\phi \in C^\infty(M)$ is strictly positive. Then, by Lemma 2.5.1 we have

$$\widetilde{Ric} = Ric + \phi^{-1}\Big((n-2)\nabla^2\phi - (n-1)\frac{|\nabla\phi|^2}{\phi} g + \Delta\phi g\Big),$$

in which the covariant derivatives and Laplacian are to be taken with respect to g, not with respect to $\tilde{g}$. Since $\tilde{g}$ is Einstein, we have

$$0 = \widetilde{\mathring{Ric}} = \mathring{Ric} + (n-2)\phi^{-1}\mathring{\nabla}^2\phi.$$

Using the last equation we get

$$\begin{aligned}\int_M \phi|\mathring{Ric}|^2 dM &= \int_M \phi\langle \mathring{Ric}, \mathring{Ric}\rangle dM \\ &= -(n-2)\int_M \langle \mathring{Ric}, \mathring{\nabla}^2\phi\rangle dM,\end{aligned}$$

taking $\psi = 1$ and $u = \phi$ in (4.2.9) we obtain

$$\int_M \phi|\mathring{Ric}|^2 dM = -(n-2)\int_M \langle \mathring{Ric}, \mathring{\nabla}^2\phi\rangle dM = 0.$$

Then, $\mathring{Ric}$ is identically zero which implies that g is Einstein and this finishes the proof of the theorem. □

5 ON THE *P*-FUNDAMENTAL TONE ESTIMATES

The results in this chapter can be found in (EVANGELISTA; SEO, 2017), which is a joint work with Professor Keomkyo Seo. Here we study estimates for the p-fundamental tone on geodesic ball and submanifolds with bounded mean curvature. Moreover we provide the p-fundamental tone estimates of minimal submanifolds with certain conditions on the norm of the second fundamental form. Finally, we study transversely oriented codimension one C^2-foliations of open subsets of Riemannian manifolds.

5.1 introduction

Let (M,g) be a Riemannian manifold. We denote by $\lambda_p^*(M)$ the p-fundamental tone of M, which is defined by

$$\lambda_p^*(M) = \inf\left\{ \frac{\int_M |\nabla f|^p dM}{\int_M |f|^p dM} : f \in W_0^{1,p}(M), f \neq 0 \right\}. \tag{5.1.1}$$

Bessa and Montenegro (BESSA; MONTENEGRO, 2003) introduced the following interesting constant $c(\Omega)$, which was used to get lower bounds for the fundamental tone and later for the p-fundamental tone as we will see soon.

Definition 5.1.1 *(BESSA; MONTENEGRO, 2003) Let Ω be a domain in a smooth Riemannian manifold (M^n,g). Let $\mathfrak{X}(\Omega)$ be the set of all smooth vector fields X on Ω such that $\|X\|_\infty = \sup_\Omega \|X\| < \infty$ and $\inf_\Omega \operatorname{div} X > 0$, where $\|X\| = g(X,X)^{\frac{1}{2}}$. Define $c(\Omega)$ by*

$$c(\Omega) = \sup\left\{ \frac{\inf_\Omega \operatorname{div} X}{\|X\|_\infty} : X \in \mathfrak{X}(\Omega) \right\}. \tag{5.1.2}$$

Note that $\mathfrak{X}(\Omega) \neq \emptyset$. In fact, consider the boundary value problem

$$\begin{cases} \Delta u &= 0 \ \textit{in} \ \ \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

and set $X = \nabla u$, then $divX = 1$ and $||X||_\infty < \infty$. In terms of the constant $c(\Omega)$, Lima, Montenegro and Santos (LIMA *et al.*, 2010) obtained a lower bound of the p-fundamental tone of the domain Ω. This lower bound will play an important role to obtain lower bounds for the p-fundamental tone of submanifolds in a Riemannian manifold with bounded mean curvature.

Theorem 5.1.1 *(LIMA et al., 2010) Let Ω be a domain in a Riemannian manifold M such that $\partial\Omega \neq \emptyset$. Then*

$$\lambda_p^*(\Omega) \geq \frac{c(\Omega)^p}{p^p}. \tag{5.1.3}$$

Remark 5.1.1 *The proof of Theorem 5.1.1 in (LIMA et al., 2010) uses the divergence theorem as follows:*

$$\int_\Omega \operatorname{div}(f^p X) = 0$$

for all positive $f \in C_0^\infty(\Omega)$ and $X \in \mathfrak{X}(\Omega)$. However, the condition on X can be weakened by considering almost everywhere smooth vector fields in Ω, which was also mentioned in (BESSA; MONTENEGRO, 2003). This observation will be useful to obtain some lower bounds for the p-fundamental tone, since it allows us to choose a suitable vector field X on a larger set of vector fields.

When $p = 2$, Theorem 5.1.1 was obtained by Bessa and Montenegro (BESSA; MONTENEGRO, 2003). Applying Theorem 5.1.1, they were able to obtain eigenvalue estimates for submanifolds with locally bounded mean curvature. As a consequence, they proved the following:

Theorem 5.1.2 *(BESSA; MONTENEGRO, 2003) Let M be an n-dimensional complete noncompact submanifold with bounded mean curvature H in a Cartan-Hadamard manifold N with sectional curvature K_N satisfying $K_N \leq \kappa < 0$ for some negative constant κ. If $\|H\| \leq \beta < (n-1)\sqrt{-\kappa}$, then*

$$\lambda_2^*(M) \geq \frac{((n-1)\sqrt{-\kappa} - \beta)^2}{4}.$$

We remark that Theorem 5.1.2 was obtained by Cheung and Leung (CHEUNG; LEUNG, 2001) when the ambient space N is a hyperbolic space $\mathbb{H}^m(\kappa)$ of constant sectional curvature $\kappa < 0$ (see also (SEO, 2012)). In Section 5.2, we obtain lower bounds for the p-fundamental tone of geodesic balls and submanifolds with bounded mean curvature, following Bessa and Montenegro's idea (BESSA; MONTENEGRO, 2003).

Let M be a complete properly immersed minimal submanifold in a Cartan-Hadamard manifold N of sectional curvature K_N bounded from above by a negative constant κ. Thereby, Theorem 5.1.2 shows that

$$\lambda_2^*(M) \geq -\frac{(n-1)^2\kappa}{4}. \tag{5.1.4}$$

It would be interesting to characterize the equality case in the above inequality. Gimeno (GIMENO, 2014) proved that the equality in (5.1.4) is attained under the assumption that M has a finite volume growth (see Section 5.4 for definition). In Section 5.4, we extend Gimeno's result to the cases where $1 < p < \infty$. Indeed, we obtain that the p-fundamental tone $\lambda_p^*(M)$ is exactly $\frac{(n-1)^p\sqrt{-\kappa}^p}{p^p}$ under the same assumption as above (see Theorem 5.4.1). As applications, we determine the p-fundamental tone for complete immersed minimal submanifolds in a Euclidean or hyperbolic space under certain conditions on the norm of the second fundamental form.

Finally, we study the infimum of the mean curvature of the leaves of transversely oriented C^2 foliation of codimension one. The motivation of this study is the following. Let $z(x,y)$ be a C^2 function defined on $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 < r\}$. Denote by $H(x,y)$ and $K(x,y)$ the mean and Gauss curvatures of the graph of $z = z(x,y)$. Heinz (HEINZ, 1955) proved that if $|H| \geq \beta > 0$, then $r \leq \frac{1}{\beta}$ and if $K \geq \beta > 0$, then $r \leq \frac{1}{\sqrt{\beta}}$, which improves the result of Bernstein (BERNSTEIN, 1910a; BERNSTEIN, 1910b) and Efimov (EFIMOV, 1953). Later, Chern (CHERN, 1965) and Flanders (FLANDERS, 1966) obtained independently a natural generalization for graphs of C^2 functions $z = z(x_1,\ldots,x_n)$ defined on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. More precisely, they obtained that if $|H| \geq \beta > 0$, then $n\beta \leq \frac{\mathrm{Vol}(\partial\Omega)}{\mathrm{Vol}(\Omega)}$ and if $S \geq \beta > 0$, then $\sqrt{n(n-1)\beta} \leq \frac{\mathrm{Vol}(\partial\Omega)}{\mathrm{Vol}(\Omega)}$, where H and S denote the mean and scalar curvature, respectively. The above inequalities are known as the Chern-Heinz inequalities for graphs. From these inequalities, it follows that if an entire graph of C^2 function defined on $\mathbb{R}^n$ has constant mean curvature H, then $H = 0$ and if it has a constant scalar curvature $S \geq 0$, then $S = 0$. Salavessa (SALAVESSA, 1989) generalized the Chern-Heinz inequality to graphs of smooth functions defined on a Riemannian manifold. On the other hand, given a Riemannian manifold M, a graph of a C^∞ function $f : \Omega \subset M \to \mathbb{R}$ can be regarded as a leaf of transversely oriented smooth foliation of codimension one of $\Omega \times \mathbb{R}$, which is obtained by vertical translation of the graph. In this point of view, Barbosa, Bessa, and Montenegro (BARBOSA *et al.*, 2008) gave another generalization of the Chern-Heinz inequalities. They were able to estimate the infimum of the mean curvature H^F of the leaves of a C^2 foliation of an open set Ω of a Riemannian manifold M in terms of the fundamental tone of the open set Ω. We extend their results into the case where $1 < p < \infty$ in Theorem 5.5.1.

5.2 Lower bounds for the p-fundamental tone on geodesic balls

Let M be an n-dimensional complete noncompact Riemannian manifold with sectional curvature bounded from above by a constant. In this section, we obtain lower bounds for the p-fundamental tone of geodesic balls $B_M(q,r)$ of radius r centered at $q \in M$, where $r < \operatorname{inj}(q)$. Following Bessa and Montenegro's idea in (BESSA; MONTENEGRO, 2003), we obtain lower bounds for the p-fundamental tone of geodesic balls and submanifolds with locally bounded mean curvature. For this purpose, we begin with the well-known Hessian comparison theorem (see e.g (BESSA; MONTENEGRO, 2003; CHEEGER; EBIN, 1975; JORGE; KOUTROUFIOTIS, 1981; SCHOEN; YAU, 1994)).

Theorem 5.2.1 *Let M be an n-dimensional complete Riemannian manifold and $x_0, x_1 \in M$. Let $\gamma : [0,\rho(x)] \to M$ be a minimizing geodesic joining x_0 and x_1 where $\rho(x)$ is the distance function* $\operatorname{dist}_M(x_0,x)$. *Let K be the sectional curvature of M. Define $\mu(\rho)$ by*

$$\mu(\rho) = \begin{cases} k\coth(k\rho(x)) & \text{if} \quad \sup_\gamma K = -k^2, \\ \frac{1}{\rho(x)} & \text{if} \quad \sup_\gamma K = 0, \\ k\cot(kr) & \text{if} \quad \sup_\gamma K = k^2 \text{ and } \rho(x) < \frac{\pi}{2k}. \end{cases} \tag{5.2.1}$$

Then the Hessian of ρ and ρ^2 satisfies

$$\begin{aligned} \nabla^2\rho(x)(X,X) \geq& \mu(\rho(x))\|X\|^2 \text{ and } \nabla^2\rho(x)(\gamma',\gamma') = 0 \\ \nabla^2\rho^2(x)(X,X) \geq& 2\rho(x)\mu(\rho(x))\|X\|^2 \text{ and } \nabla^2\rho^2(x)(\gamma',\gamma') = 2, \end{aligned}$$

where X is any vector in T_xM perpendicular to $\gamma'(\rho(x))$.

From Theorem 5.2.1, it follows that

$$\begin{aligned} \Delta\rho(x) &\geq (n-1)\mu(\rho(x)), \\ \Delta\rho^2(x) &\geq 2(n-1)\rho(x)\mu(\rho(x)) + 2. \end{aligned} \tag{5.2.2}$$

We now state our result on lower bounds for the p-fundamental tone on geodesic balls.

Theorem 5.2.2 *Let M be an n-dimensional complete Riemannian manifold. Denote by $B_M(q,r)$ a geodesic ball with radius $r < \operatorname{inj}(q)$. Let $\kappa(q,r) = \sup\{K_M(x) : x \in B_M(q,r)\}$, where $K_M(x)$ denotes the sectional curvature of M at x. Then*

$$\lambda_p^*(B_M(q,r)) \geq \begin{cases} \frac{1}{p^p}\max\{\frac{n^p}{r^p}, [(n-1)k\coth(kr)]^p\} & \text{if} \quad \kappa(q,r) = -k^2, \\ \frac{n^p}{p^p r^p} & \text{if} \quad \kappa(q,r) = 0, \\ \frac{((n-1)k\cot(kr)+1)^p}{p^p r^p} & \text{if} \quad \kappa(q,r) = k^2 \text{ and } r < \frac{\pi}{2k}, \end{cases} \tag{5.2.3}$$

where k is a positive constant.

Proof: Applying Theorem 5.1.1, we have

$$\lambda_p^*(B_M(q,r)) \geq \frac{c(B_M(q,r))^p}{p^p}. \tag{5.2.4}$$

Let $\rho(x) = \mathrm{dist}_M(q,x)$. Consider $X = \nabla\rho^2$. From (5.2.2), it follows

$$\mathrm{div}X = \Delta\rho^2(x) \geq \begin{cases} 2(n-1)k\rho\coth(k\rho)+2 \geq 2n & \text{if} \quad \kappa(q,r) = -k^2, \\ 2n & \text{if} \quad \kappa(q,r) = 0, \\ 2(n-1)k\rho\cot(k\rho)+2 & \text{if} \quad \kappa(q,r) = k^2 \text{ and } r < \frac{\pi}{2k}. \end{cases} \tag{5.2.5}$$

Since $\|X\|_\infty = 2r$, using (5.2.4) and (5.2.5) we get the desired results in the cases where $\kappa(q,r) = 0$ and $\kappa(q,r) = k^2$. For the case $\kappa(q,r) = -k^2$, we have an additional estimate of the lower bound as follows.

Consider a vector field $X = \nabla\rho^s$ on $B_M(q,r)$, for $1 < s < 2$. Note that X is smooth in $B_M(q,r)\setminus\{q\}$ and continuous in $B_M(q,r)$. Thus by the divergence theorem and the dominated convergence theorem, we still have

$$\int_{B_M(q,r)} \mathrm{div}(f^pX) = 0 \tag{5.2.6}$$

for all positive $f \in C_0^\infty(B_M(q,r))$ (for details on the proof of (5.2.6) see e.g. (BESSA; MONTENEGRO, 2003) p. 286). Therefore, by Remark 5.1.1, equation (5.2.4) and the definition of $c(\Omega)$ we have

$$\lambda_p^*(B_M(q,r)) \geq \frac{1}{p^p}\left[\frac{\inf\{s(s-1)\rho^{s-2}+s(n-1)k\rho^{s-1}\coth(k\rho)\}}{sr^{s-1}}\right]^p.$$

Letting $s \to 1$, we obtain

$$\lambda_p^*(B_M(q,r)) \geq \frac{1}{p^p}\left[(n-1)k\coth(kr)\right]^p,$$

which completes the proof. □

5.3 Submanifolds with locally bounded mean curvature

Definition 5.3.1 *Let $\varphi : M^n \to N^m$ be an isometric immersion. Let H be the mean curvature vector field along M. We say that the immersion has locally bounded mean curvature if for every $q \in N$ and $r > 0$, the number $h(q,r) = \sup\{|H(x)| : x \in \varphi(M)\cap B_N(q,r)\}$ is finite.*

In the sequel, we obtain lower bounds for the p-fundamental tone for domains on submanifolds with locally bounded mean curvature. Consider an isometric immersion $\varphi : M \hookrightarrow N$, where M^n and N^m are complete Riemannian manifolds. We denote by ∇ and $\overline{\nabla}$ the Riemannian connections on M and N, respectively, while by ∇^2 and $\overline{\nabla}^2$ the Hessian on M and N, respectively. First we need to establish a relationship between the Hessian of a smooth function $g : N \to \mathbb{R}$ and the Hessian of $f = g \circ \varphi : M \to \mathbb{R}$, which is well-known in the literature (see (BESSA; MONTENEGRO, 2003; CHEUNG; LEUNG, 2001; JORGE; KOUTROUFIOTIS, 1981) for example). Note that $d\varphi(X)$ can be identified with X. For every $q \in M$ and for every $X \in T_qM$,

$$\langle \nabla f, X \rangle = df(X) = dg(X) = \langle \overline{\nabla} g, X \rangle.$$

Thus

$$\overline{\nabla} g = \nabla f + (\overline{\nabla} g)^{\perp}, \tag{5.3.1}$$

where $(\overline{\nabla} g)^{\perp}$ is perpendicular to T_qM. Applying the Gauss equation, we see

$$\nabla^2 f(q)(X,Y) = \overline{\nabla}^2 g(\varphi(q))(X,Y) + \langle \overline{\nabla} g, \alpha(X,Y) \rangle_{\varphi(q)}, \tag{5.3.2}$$

where $\alpha(q)(X,Y)$ and $\nabla^2 f(q)(X,Y)$ stand for the second fundamental form of the immersion φ and the Hessian of f at $q \in M$, $X,Y \in T_qM$, respectively. Choosing an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ for T_qM and taking trace in (5.3.2) with respect to this basis we get

$$\begin{aligned} \Delta f(q) &= \sum_{i=1}^{n} \nabla^2 f(q)(e_i, e_i) \\ &= \sum_{i=1}^{n} \overline{\nabla}^2 g(\varphi(q))(e_i, e_i) + \langle \overline{\nabla} g, \sum_{i=1}^{n} \alpha(e_i, e_i) \rangle. \end{aligned} \tag{5.3.3}$$

Applying Theorem 5.1.1 and Theorem 5.2.1, we give the following lower bounds estimates for the p-fundamental tone for domains on submanifolds with locally bounded mean curvature.

Theorem 5.3.1 *Let $\varphi : M^n \to N^m$ be an isometric immersion with locally bounded mean curvature and let Ω be any connected component of $\varphi^{-1}(\overline{B_N(q,r)})$ for $q \in N \backslash \varphi(M)$ and $r > 0$. Denote by $\kappa(q,r)$ the supremum of the sectional curvature of N in $B_N(q,r)$ as in Theorem 5.2.2. Then, for a constant $k > 0$, we have the following:*

1. *If $\kappa(q, \mathrm{inj}(q)) = k^2 < +\infty$ and*

$$r < \min \left\{ \mathrm{inj}(q), \frac{\cot^{-1}\left(\frac{h(q,\mathrm{inj}(q))}{(n-1)k}\right)}{k} \right\},$$

then

$$\lambda_p^*(\Omega) \geq \frac{[(n-1)k\cot(kr) - h(q,r)]^p}{p^p}.$$

2. *If* $\kappa(q,r) > 0$ *for all* $r > 0$, $\lim_{r\to\infty} \kappa(q,r) = \infty$, $\mathrm{inj}(q) = \infty$, *and*

$$r < r_0 := \max_{s>0} \left\{ \frac{\cot^{-1}\left(\frac{h(q,s)}{(n-1)\sqrt{\kappa(q,s)}}\right)}{\sqrt{\kappa(q,s)}} \right\},$$

then

$$\lambda_p^*(\Omega) \geq \frac{[(n-1)\sqrt{\kappa(q,r)}\cot(\sqrt{\kappa(q,r)}r) - h(q,r)]^p}{p^p}.$$

3. *If* $\kappa(q,\mathrm{inj}(q)) = 0$ *and* $r < \min\{\mathrm{inj}(q), \frac{n}{h(q,\mathrm{inj}(q))}\}$, *where* $\frac{n}{h(q,\mathrm{inj}(q))} = +\infty$ *when* $h(q,\mathrm{inj}(q)) = 0$, *then*

$$\lambda_p^*(\Omega) \geq \frac{[\frac{n}{r} - h(q,r)]^p}{p^p}.$$

4. *If* $\kappa(q,\mathrm{inj}(q)) = -k^2$, $h(q,\mathrm{inj}(q)) < (n-1)k$, *and* $r < \mathrm{inj}(q)$, *then*

$$\lambda_p^*(\Omega) \geq \frac{[(n-1)k - h(q,r)]^p}{p^p}.$$

5. *If* $\kappa(q,\mathrm{inj}(q)) = -k^2$, $h(q,\mathrm{inj}(q)) \geq (n-1)k$, *and*

$$r < \min\left\{ \mathrm{inj}(q), \frac{\coth^{-1}\left(\frac{h(q,\mathrm{inj}(q))}{(n-1)k}\right)}{k} \right\},$$

then

$$\lambda_p^*(\Omega) \geq \frac{[(n-1)k\coth(kr) - h(q,r)]^p}{p^p}.$$

Proof: Let Ω be a connected component of $\varphi^{-1}(\overline{B_N(q,\mathrm{inj}(q))})$ and in this component let $X_i = \nabla f_i$, where $\rho(x) = \mathrm{dist}_N(q,x)$ is the distance function on N and $f_i = \rho^i \circ \varphi : M \to \mathbb{R}$, $i = 1,2$. Note that f_i is smooth on $\varphi^{-1}(B_N(q,\mathrm{inj}(q)))$. By (5.3.3) we have

$$\mathrm{div} X_i(x) = \sum_{j=1}^{n} \overline{\nabla}^2 \rho^i(\varphi(x))(e_j,e_j) + \langle \overline{\nabla}\rho^i, H\rangle(\varphi(x)),$$

where $\{e_1,\dots,e_n\}$ is an orthonormal basis of T_xM. Rewriting the orthonormal basis $\{e_1,\dots,e_n\}$ in terms of an orthonormal basis $\{E_1,\dots,E_m\}$ of $T_{\varphi(x)}N$ such that $E_1 = \overline{\nabla}\rho$, we get

$$e_j = \sum_{i=1}^{m} a_k^j E_k, \quad j = 1,\dots,n.$$

Since $|\overline{\nabla}\rho|^2 = 1$ we have $\overline{\nabla}^2\rho(\varphi(x))(E_1, X) = 0$ for every $X \in T_{\varphi(x)}N$. Thereby, we obtain

$$\begin{aligned}
\operatorname{div} X_1(x) &= \sum_{j=1}^{n} \overline{\nabla}^2\rho(\varphi(x))(e_j, e_j) + \langle\overline{\nabla}\rho, H\rangle(\varphi(x)) \\
&= \sum_{j=1}^{n} \overline{\nabla}^2\rho(\varphi(x))\left(\sum_{i=1}^{m} a_k^j E_k, \sum_{i=1}^{m} a_k^j E_k\right) + \langle\overline{\nabla}\rho, H\rangle(\varphi(x)) \\
&= \sum_{j=1}^{n} \overline{\nabla}^2\rho(\varphi(x))\left(\sum_{i=2}^{m} a_k^j E_k, \sum_{i=2}^{m} a_k^j E_k\right) + \langle\overline{\nabla}\rho, H\rangle(\varphi(x)).
\end{aligned}$$

Now we are on the hypothesis of Theorem 5.2.1, hence

$$\begin{aligned}
\operatorname{div} X_1(x) &\geq \sum_{j=1}^{n} \mu(\rho(x)) \left|\sum_{i=2}^{m} a_k^j E_k\right|^2 + \langle\overline{\nabla}\rho, H\rangle(\varphi(x)) \\
&= \mu(\rho(x)) \sum_{j=1}^{n}\sum_{i=2}^{m} (a_k^j)^2 + \langle\overline{\nabla}\rho, H\rangle(\varphi(x)) \\
&\geq \mu(\rho(x)) \sum_{j=1}^{n}\sum_{i=2}^{m} (a_k^j)^2 - h(q, r).
\end{aligned}$$

Since $\{e_k\}_{k=1}^{n}$ is an orthonormal basis, we have

$$n = \sum_{j=1}^{n} |e_j|^2 = \sum_{j=1}^{n} |\sum_{k=1}^{m} a_k^j E_k|^2 = \sum_{j=1}^{n}\sum_{k=1}^{m} (a_k^j)^2,$$

i.e,

$$\sum_{j=1}^{n}\sum_{k=2}^{m} (a_k^j)^2 = n - \sum_{j=1}^{n} (a_1^j)^2.$$

On the other hand, since $\overline{\nabla}\rho = E_1$ using (5.3.1) we deduce

$$\sum_{j=1}^{n} (a_1^j)^2 = \sum_{j=1}^{n} \langle E_1, e_j\rangle^2 = \sum_{j=1}^{n} \langle\nabla f, e_j\rangle^2 = |\nabla f|^2,$$

again by (5.3.1) we have $1 = |\overline{\nabla}\rho|^2 = |\nabla f|^2 + |\overline{\nabla}\rho^{\perp}|^2$, which implies that $|\nabla f|^2 \leq 1$. Whence

$$\sum_{j=1}^{n}\sum_{k=2}^{m} (a_k^j)^2 \geq n - 1.$$

Therefore,

$$\operatorname{div} X_1(x) \geq \mu(\rho(x))(n-1) - h(q, r).$$

On the other hand, if $i = 2$ we have $\overline{\nabla}^2\rho(\varphi(x))^2(E_1, X) = 0$ for every $X \in T_{\varphi(x)}N$ such that $X \perp E_1$, and $\overline{\nabla}^2\rho(\varphi(x))^2(E_1, E_1) = 2$. Proceeding analogously to the previous case we get

$$\operatorname{div} X_2(x) \geq 2\rho(x)\mu(\rho(x))(n-1) + 2 - h(q, r).$$

Then, we obtain

(1) If $\kappa(q,r)=k^2$ and $r<\min\left\{\text{inj}(q),\frac{\pi}{2k},\frac{\cot^{-1}\left(\frac{h(q,\text{inj}(q))}{(n-1)k}\right)}{k}\right\}=\min\left\{\text{inj}(q),\frac{\cot^{-1}\left(\frac{h(q,\text{inj}(q))}{(n-1)k}\right)}{k}\right\}$, since $\frac{h(q,\text{inj}(q))}{(n-1)k}\geq 0$ and the cotangent function is decreasing. Then,

$$\text{div}X_1(x)\geq (n-1)k\cot(kr)-h(q,r)>0.$$

(2) The proof of Item (2) is analogous to that of Item (1).

(3) If $\kappa(q,r)=0$ and $r<\min\{\text{inj}(q),\frac{n}{h(q,\text{inj}(q))}\}$, then

$$\text{div}X_2(x)\geq 2n-2rh(q,r)>0.$$

(4) If $\kappa(q,r)=-k^2$ and $h(q,\text{inj}(q))<(n-1)k$, then

$$\text{div}X_1(x)\geq (n-1)k\coth(kr)-h(q,r)>0.$$

(5) If $\kappa(q,r)=-k^2$, $h(q,\text{inj}(q))\geq (n-1)k$ and $r<\min\left\{\text{inj}(q),\frac{\coth^{-1}\left(\frac{h(q,\text{inj}(q))}{(n-1)k}\right)}{k}\right\}$, then

$$\text{div}X_1(x)\geq (n-1)k\coth(kr)-h(q,r)>0.$$

Since $\|X_1\|\leq 1$ and $\|X_2\|\leq 2r$, the conclusion follows from Theorem 5.1.1. □

We remark that the dimension of the ambient space N does not appear in the lower bound of the p-fundamental tone in Theorem 5.3.1. Furthermore, we obtain the following straightforward consequence.

Corollary 5.3.1 *Let $\varphi: M^n\to\mathbb{R}^m$ be an isometric minimal immersion of a complete submanifold. Suppose that $\varphi(M)\subset B_{\mathbb{R}^n}(0,r)$, then $\lambda_p^*(M)\geq\frac{n^p}{2^p p^p r^p}$. In particular, if M is a complete minimal surface in $\mathbb{R}^3$ such that $\varphi(M)\subset B_{\mathbb{R}^3}(0,r)$, then $\lambda_p^*(M)\geq\frac{1}{p^p r^p}$.*

Proof: Let $q\in B_{\mathbb{R}^n}(0,r)\setminus\varphi(M)$. Taking the smallest ball $B_{\mathbb{R}^n}(q,2r)$ such that $\varphi(M)\subset B_{\mathbb{R}^n}(q,2r)$ and applying Item (3) of Theorem 5.3.1, we get the conclusion. □

As an interesting consequence of Theorem 5.3.1, one can obtain a lower bound of a complete noncompact submanifold with bounded mean curvature in a Cartan-Hadamard manifold.

Corollary 5.3.2 *(DUNG; SEO, 2016) Let $\varphi: M\hookrightarrow N$ be an isometric immersion, where M is an n-dimensional complete noncompact Riemannian manifold and N is a complete simply connected Riemannian manifold sectional curvature K_N satisfying $K_N\leq\kappa<0$ for some negative constant κ. If $\|H\|\leq\beta<(n-1)\sqrt{-\kappa}$, then*

$$\lambda_p^*(M)\geq\frac{((n-1)\sqrt{-\kappa}-\beta)^p}{p^p}. \tag{5.3.4}$$

Proof: Let q be an arbitrary point of N such that $q \in N \setminus \varphi(M)$. Since N is simply connected and negatively curved, we see that $\mathrm{inj}(q) = \infty$. That is, q is a pole of N. Consider $\rho = \mathrm{dist}_N(q,\cdot)$ and $X = \nabla(\rho \circ \varphi)$. By Item (4) of Theorem 5.3.1, we have that $\lambda_p^*(\Omega) \geq \frac{((n-1)\sqrt{-\kappa}-\beta)^p}{p^p}$ for any connected component of $\varphi^{-1}(\overline{B_N(q,r)})$. Thus

$$\lim_{i\to\infty} \lambda_p^*(\Omega_i) \geq \frac{((n-1)\sqrt{-\kappa}-\beta)^p}{p^p}$$

for any exhaustion $\{\Omega_i\}$ of M. The conclusion follows from (2.6.2). □

Remark 5.3.1 *In particular, if $p = 2$ and N is a hyperbolic space, Corollary 5.3.2 was originally obtained by Cheung and Leung (CHEUNG; LEUNG, 2001). Moreover, if $p = 2$ and N is a Cartan-Hadamard manifold, Bessa and Montenegro (BESSA; MONTENEGRO, 2003) obtained the same lower bound for the fundamental tone (see also (SEO, 2012)). Very recently, Dung and the second author obtained Corollary 5.3.2 in (DUNG; SEO, 2016).*

5.4 The p-fundamental tone of minimal submanifolds with controlled extrinsic curvature

As mentioned in the introduction, adapting Gimeno's idea (GIMENO, 2014), we give an upper bound for the p-fundamental tone of minimal submanifolds of a Cartan-Hadamard manifold with controlled extrinsic curvature in this section. Combining this upper bound with Theorem 5.3.1, we obtain the p-fundamental tone of such minimal submanifolds. Moreover, we provide an intrinsic result from which the fundamental tone of a Cartan-Hadamard manifold with finite volume growth is determined.

The following technical lemma will give us an appropriate interval to construct a suitable test function to get an upper bound for the p-fundamental tone.

Lemma 5.4.1 *Given $n \in \mathbb{N}$, $p \in (1,\infty)$, and $\kappa \leq 0$, there exists an interval $[aR, bR] \subset [\frac{R}{2}, R]$ such that for all $t \in [aR, bR]$*

$$\frac{(n-1)}{p} H_\kappa(t) \sin\left(\frac{2\pi(t-\frac{R}{2})}{R}\right) > \frac{2\pi}{R}\left|\cos\left(\frac{2\pi(t-\frac{R}{2})}{R}\right)\right|, \tag{5.4.1}$$

where $R > 0$ and $H_\kappa(t)$ is defined as in (2.7.4).

Proof: Suppose that $\kappa < 0$. Since $\sin\left(\frac{2\pi(t-\frac{R}{2})}{R}\right) \geq \left|\cos\left(\frac{2\pi(t-\frac{R}{2})}{R}\right)\right|$ for $t \in [\frac{5R}{8}, \frac{7R}{8}]$ and $\frac{(n-1)}{p}H_\kappa(t) \geq \frac{(n-1)\sqrt{-\kappa}}{p} > \frac{2\pi}{R}$, we have

$$\frac{(n-1)}{p}H_\kappa(t)\sin\left(\frac{2\pi(t-\frac{R}{2})}{R}\right) > \frac{2\pi}{R}\left|\cos\left(\frac{2\pi(t-\frac{R}{2})}{R}\right)\right|$$

for any $t \in [\frac{5R}{8}, \frac{7R}{8}]$. This proves the case $\kappa < 0$.

Suppose that $\kappa = 0$. In this case, (5.4.1) is equivalent to the inequality

$$\frac{(n-1)}{p}\frac{1}{s}\sin\left(2\pi(s-\frac{1}{2})\right) > 2\pi\left|\cos\left(2\pi(s-\frac{1}{2})\right)\right|, \tag{5.4.2}$$

where $s \in [\frac{1}{2}, 1]$. Thus it suffices to find an inteveral $[\bar{a}, \bar{b}] \subset [\frac{1}{2}, 1]$ such that (5.4.2) holds for every $s \in [\bar{a}, \bar{b}]$. Note that (5.4.2) is obviously satisfied for $s = \frac{3}{4}$. Then, by continuity, there exist constants $\bar{a}, \bar{b} \in (\frac{1}{2}, 1)$ depending only on p and n satisfying the desired conditions.

We remark that, in both cases above where $\kappa = 0$ and $\kappa < 0$, we obtain an interval which contians the same point $\frac{3R}{4}$. This fact allows us to choose an interval $[aR, bR]$ which is a subset of both and contains $\frac{3R}{4}$. The behavior of the cosine function allows a suitable choice of aR and bR in a such way that

$$\max_{[aR,bR]}\left\{\left|\cos\left(\frac{2\pi(t-\frac{R}{2})}{R}\right)\right|\right\} = |\cos(2\pi(b-1/2))| = |\cos(2\pi(a-1/2))|.$$

□

Remark 5.4.1 *Define* $\binom{p}{j} := \frac{p(p-1)\dots(p-j+1)}{j!}$, *where p is a real number and j is a nonnegative integer. Then*

$$(x+y)^p = x^p + \sum_{j=1}^{\infty}\binom{p}{j}x^{p-j}y^j,$$

where the above power series converges absolutely provided $|x| > |y|$. The proof of this fact is a straightforward application of the ratio test for convergence of power series. Furthermore, if $\bar{\Omega} \subset \mathbb{R}^2$ is a compact set such that $|x| > |y|$ for all, $(x,y) \in \bar{\Omega}$, then the series converges absolutely and uniformly on $\bar{\Omega}$.

We construct a suitable test function as follows. Consider the interval $[aR, bR]$ obtained in Lemma 5.4.1. Define

$$\phi(t) = \begin{cases} f(t) & \text{if} \quad t \in [aR, bR], \\ 0 & \text{otherwise}, \end{cases} \tag{5.4.3}$$

where $f:\mathbb{R}\to\mathbb{R}$ is defined as

$$f(t)=\frac{\sin\left(\frac{2\pi(t-\frac{R}{2})}{R}\right)}{\mathrm{Vol}(S_t^k)^{\frac{1}{p}}}.$$

We now consider a function $\Phi: M\to\mathbb{R}$ defined by

$$\Phi(x)=\phi(r_q(x)). \tag{5.4.4}$$

Then we have the following technical lemma, which will be used to obtain an upper bound for the p-fundamental tone of minimal submanifolds in a Cartan-Hadamard manifold.

Lemma 5.4.2 *Let ϕ defined as in* (5.4.3) *for $\kappa\le 0$. Then there exists an upper bound function $\theta_\kappa:\mathbb{R}^+\to\mathbb{R}^+$ such that*

$$\frac{\int_0^R|\phi'(t)|^p\mathrm{Vol}(S_t^\kappa)dt}{\int_0^R|\phi(t)|^p\mathrm{Vol}(S_t^\kappa)dt}\le\theta_\kappa(R) \tag{5.4.5}$$

and

$$\lim_{R\to\infty}\theta_\kappa(R)=\frac{(n-1)^p\sqrt{-\kappa}^p}{p^p}. \tag{5.4.6}$$

Proof: First suppose that $\kappa<0$. Then

$$\frac{\int_0^R|\phi'(t)|^p\mathrm{Vol}(S_t^\kappa)dt}{\int_0^R|\phi(t)|^p\mathrm{Vol}(S_t^\kappa)dt}=\frac{\int_{aR}^{bR}|f'(t)|^p\mathrm{Vol}(S_t^\kappa)dt}{\int_{aR}^{bR}|f(t)|^p\mathrm{Vol}(S_t^\kappa)dt}.$$

Moreover

$$\begin{aligned}|f'(t)|^p &= \frac{\left|-\frac{(n-1)}{p}H_\kappa(t)\sin\left(\frac{2\pi(t-\frac{R}{2})}{R}\right)+\frac{2\pi}{R}\cos\left(\frac{2\pi(t-\frac{R}{2})}{R}\right)\right|^p}{\mathrm{Vol}(S_t^\kappa)}\\ &\le \frac{\left(\frac{(n-1)}{p}H_\kappa(t)\sin\left(\frac{2\pi(t-\frac{R}{2})}{R}\right)+\frac{2\pi}{R}\left|\cos\left(\frac{2\pi(t-\frac{R}{2})}{R}\right)\right|\right)^p}{\mathrm{Vol}(S_t^\kappa)}.\end{aligned}$$

Using Remark 5.4.1 and our choice of the interval $[aR,bR]$ in Lemma 5.4.1 we deduce

$$|f'(t)|^p \le \frac{\frac{(n-1)^p}{p^p}H_\kappa(t)^p\sin^p\left(\frac{2\pi(t-\frac{R}{2})}{R}\right)+\sum_{j=1}^{\infty}\binom{p}{j}\frac{(n-1)^{p-j}H_\kappa(t)^{p-j}}{p^{p-j}}\frac{2^j\pi^j}{R^j}}{\mathrm{Vol}(S_t^\kappa)}.$$

The power series on the right hand side of the above inequality is convergent, since $H_\kappa(k)$ is bounded on $[aR,bR]$, $H_\kappa(t)\ge\sqrt{-\kappa}$ and R was chosen in Lemma 5.4.1 such that $R>\frac{2p\pi}{(n-1)\sqrt{-\kappa}}$.

Using that $H_\kappa(t)$ is a non-increasing function and $\int_{aR}^{bR}\sin^p\left(\frac{2\pi(t-\frac{R}{2})}{R}\right)dt=\frac{R\lambda(p)}{2\pi}$, where $\lambda(p)=\int_{2\pi(b-\frac{1}{2})}^{2\pi(a-\frac{1}{2})}\sin^p(y)dy$, we obtain

$$\begin{aligned}\frac{\int_{aR}^{bR}|f'(t)|^p\mathrm{Vol}(S_t^\kappa)dt}{\int_{aR}^{bR}|f(t)|^p\mathrm{Vol}(S_t^\kappa)dt} &\leq \frac{(n-1)^p}{p^p}H_\kappa(aR)^p+\frac{2\pi(b-a)}{\lambda(p)}\sum_{j\leq p}\binom{p}{j}\frac{(n-1)^{p-j}H_\kappa(aR)^{p-j}}{p^{p-j}}\frac{2^j\pi^j}{R^j}\\ &+ \frac{2\pi(b-a)}{\lambda(p)}\sum_{j\in N_1}\binom{p}{j}\frac{(n-1)^{p-j}H_\kappa(aR)^{p-j}}{p^{p-j}}\frac{2^j\pi^j}{R^j}\\ &+ \frac{2\pi(b-a)}{\lambda(p)}\sum_{j\in N_2}\binom{p}{j}\frac{(n-1)^{p-j}H_\kappa(bR)^{p-j}}{p^{p-j}}\frac{2^j\pi^j}{R^j}\\ &\leq \frac{(n-1)^p}{p^p}H_\kappa(aR)^p+\frac{2\pi(b-a)}{\lambda(p)}\sum_{j\leq p}\binom{p}{j}\frac{(n-1)^{p-j}H_\kappa(aR)^{p-j}}{p^{p-j}}\frac{2^j\pi^j}{R^j}\\ &+ \frac{2\pi(b-a)}{\lambda(p)}\sum_{j\in N_2}\binom{p}{j}\frac{(n-1)^{p-j}H_\kappa(bR)^{p-j}}{p^{p-j}}\frac{2^j\pi^j}{R^j},\end{aligned}$$

where $N_1=\{j_0+2l+1:l\in\mathbb{N}\}$, $N_2=\{j_0+2l+2:l\in\mathbb{N}\}$ and $j_0=\min\{j\in\mathbb{N}:j>p\}$. In the last inequality we used that $\binom{p}{j}\leq 0$ for all $j\in N_1$. Define

$$\begin{aligned}\theta_\kappa(R) &:= \tfrac{(n-1)^p}{p^p}H_\kappa(aR)^p+\tfrac{2\pi(b-a)}{\lambda(p)}\sum_{j\leq p}\binom{p}{j}\frac{(n-1)^{p-j}H_\kappa(aR)^{p-j}}{p^{p-j}}\frac{2^j\pi^j}{R^j}\\ &+ \tfrac{2\pi(b-a)}{\lambda(p)}\sum_{j\in N_2}\binom{p}{j}\frac{(n-1)^{p-j}H_\kappa(bR)^{p-j}}{p^{p-j}}\frac{2^j\pi^j}{R^j}.\end{aligned}$$

Note also that

$$\begin{aligned}&\lim_{R\to\infty}H_\kappa(bR)=\lim_{R\to\infty}H_\kappa(aR)=\sqrt{-\kappa},\\ &\lim_{R\to\infty}\sum_{j\in N_2}^{\infty}\binom{p}{j}\frac{(n-1)^{p-j}H_\kappa(bR)^{p-j}}{p^{p-j}}\frac{2^j\pi^j}{R^j}=0,\end{aligned}$$

where the last limit follows from the fact that the general term of the above power series converges uniform to zero as R goes to infinity. Therefore we get the desired result for this case.

Suppose that $\kappa=0$. Then we have

$$|f'(t)|^p\leq\frac{\frac{(n-1)^p}{p^p}H_0(t)^p\sin^p\left(\frac{2\pi(t-\frac{R}{2})}{R}\right)+\sum_{j=1}^{\infty}\binom{p}{j}\frac{(n-1)^{p-j}H_0(t)^{p-j}}{p^{p-j}}\frac{2^j\pi^j}{R^j}\nu(t)^j}{\mathrm{Vol}(S_t^0)},$$

where $\nu(t)=\left|\cos\left(\frac{2\pi(t-\frac{R}{2})}{R}\right)\right|$. Again we use Remark 5.4.1 and our choice of the interval $[aR,bR]$ to ensure that the power series on the right hand side of the above inequality converges absolutely and uniformly on $[aR, bR]$, since $H_0(t)\frac{(n-1)}{p} \geq \frac{2\pi}{R}\left|\cos\left(\frac{2\pi(t-\frac{R}{2})}{R}\right)\right|$. As before,

$$\begin{aligned}
\frac{\int_{aR}^{bR}|f'(t)|^p \mathrm{Vol}(S_t^0)dt}{\int_{aR}^{bR}|f(t)|^p \mathrm{Vol}(S_t^0)dt} &\leq \frac{(n-1)^p}{p^p}H_0(aR)^p + \frac{1}{\lambda(p)}\sum_{j\leq p}\binom{p}{j}\frac{(n-1)^{p-j}H_0(aR)^{p-j}}{p^{p-j}}\frac{2^j\pi^j}{R^j}\tau(j) \\
&+ \frac{1}{\lambda(p)}\sum_{j\in N_1}\binom{p}{j}\frac{(n-1)^{p-j}H_0(aR)^{p-j}}{p^{p-j}}\frac{2^j\pi^j}{R^j}\tau(j) \\
&+ \frac{1}{\lambda(p)}\sum_{j\in N_2}\binom{p}{j}\frac{(n-1)^{p-j}H_0(bR)^{p-j}}{p^{p-j}}\frac{2^j\pi^j}{R^j}\tau(j) \\
&\leq \frac{(n-1)^p}{p^p}\frac{1}{a^pR^p} + \frac{1}{R^p\lambda(p)}\sum_{j\leq p}\binom{p}{j}\frac{(n-1)^{p-j}}{p^{p-j}}\frac{2^j\pi^j}{a^{p-j}}\tau(j) \\
&+ \frac{1}{R^p\lambda(p)}\sum_{j\in N_2}\binom{p}{j}\frac{(n-1)^{p-j}}{p^{p-j}}\frac{2^j\pi^j}{b^{p-j}}\tau(j),
\end{aligned}$$

where $\tau(j) = \int_{2\pi(a-\frac{1}{2})}^{2\pi(b-\frac{1}{2})}|\cos(y)|^j dy$. Define

$$\begin{aligned}
\theta_0(R) &:= \frac{(n-1)^p}{p^p}\frac{1}{a^pR^p} + \frac{1}{R^p\lambda(p)}\sum_{j\leq p}\binom{p}{j}\frac{(n-1)^{p-j}}{p^{p-j}}\frac{2^j\pi^j}{a^{p-j}}\tau(j) \\
&+ \frac{1}{R^p\lambda(p)}\sum_{j\in N_2}\binom{p}{j}\frac{(n-1)^{p-j}}{p^{p-j}}\frac{2^j\pi^j}{b^{p-j}}\tau(j).
\end{aligned}$$

Since the above power series in $\theta_0(R)$ is convergent, it is easy to see that $\lim_{R\to\infty}\theta_0(R) = 0$, which completes the proof. □

Proceeding, we use the suitable test function constructed above and the previous lemma to get an upper bound for the p-fundamental and using the lower bound on Corollary 5.3.2 we get the main result of this section.

Theorem 5.4.1 *Let M^n be an n-dimensional complete properly immersed minimal submanifold in a Cartan-Hadamard manifold N of sectional curvature K_N bounded from above by $K_N \leq \kappa \leq 0$. Suppose that*

$$\lim_{R\to\infty} Q(R) < \infty.$$

Then

$$\lambda_p^*(M) = \frac{(n-1)^p\sqrt{-\kappa}^p}{p^p}.$$

Proof: The lower bound for the p-fundamental tone is given in Corollary 5.3.2. To get the upper bound, we will make use of the definition (5.1.1) of the p-fundamental tone and a suitable test function defined as in (5.4.4). Then the coarea formula gives

$$\begin{aligned}
\lambda_p^*(M) &\le \frac{\int_M |\nabla\Phi|^p dM}{\int_M |\Phi|^p dM} = \frac{\int_M |\phi'|^p |\nabla r|^p dM}{\int_M |\phi|^p dM} \\
&\le \frac{\int_M |\phi'|^p dM}{\int_M |\phi|^p dM} = \frac{\int_0^R \left[\int_{\partial M_q^t} \frac{|\phi'|^p}{|\nabla r|}\right] dt}{\int_0^R \left[\int_{\partial M_q^t} \frac{|\phi|^p}{|\nabla r|}\right] dt} \\
&= \frac{\int_0^R |\phi'(t)|^p \left[\int_{\partial M_q^t} \frac{1}{|\nabla r|}\right] dt}{\int_0^R |\phi(t)|^p \left[\int_{\partial M_q^t} \frac{1}{|\nabla r|}\right] dt} \\
&= \frac{\int_{aR}^{bR} |\phi'(t)|^p \mathrm{Vol}(M_q^t)' dt}{\int_{aR}^{bR} |\phi(t)|^p \mathrm{Vol}(M_q^t)' dt}.
\end{aligned}$$

By Lemma 2.7.1 we have

$$\lambda_p^*(M) \le \frac{\int_{aR}^{bR} |\phi'(t)|^p (\ln Q(t))' \mathrm{Vol}(B_t^\kappa) Q(t) dt}{\int_{aR}^{bR} |\phi(t)|^p Q(t) \mathrm{Vol}(S_t^\kappa) dt} + \frac{\int_{aR}^{bR} |\phi'(t)|^p Q(t) \mathrm{Vol}(S_t^\kappa) dt}{\int_{aR}^{bR} |\phi(t)|^p Q(t) \mathrm{Vol}(S_t^\kappa) dt}.$$

We define two functions g_κ and h as follows:

For $\kappa < 0$,

$$\begin{aligned}
g_\kappa(R) &:= \tfrac{(n-1)^p}{p^p} H_\kappa(aR)^p + \sum_{j \le p} \binom{p}{j} \frac{(n-1)^{p-j} H_\kappa(aR)^{p-j}}{p^{p-j}} \frac{2^j \pi^j}{R^j} \\
&+ \sum_{j \in N_2} \binom{p}{j} \frac{(n-1)^{p-j} H_\kappa(bR)^{p-j}}{p^{p-j}} \frac{2^j \pi^j}{R^j}.
\end{aligned}$$

For $\kappa = 0$,

$$\begin{aligned}
g_0(R) &:= \tfrac{(n-1)^p}{p^p} H_0(aR)^p + \sum_{j \le p} \binom{p}{j} \frac{(n-1)^{p-j} H_0(aR)^{p-j}}{p^{p-j}} \frac{2^j \pi^j}{R^j} |\cos(2\pi(a-1/2))|^j \\
&+ \sum_{j \in N_2} \binom{p}{j} \frac{(n-1)^{p-j} H_0(bR)^{p-j}}{p^{p-j}} \frac{2^j \pi^j}{R^j} |\cos(2\pi(b-1/2))|^j
\end{aligned}$$

and

$$h(R) := \int_{aR}^{bR} (\ln Q(t))' dt.$$

Applying Lemma 5.4.2 and using the fact that $\dfrac{\mathrm{Vol}(B_t^\kappa)}{\mathrm{Vol}(S_t^\kappa)}$ is a non-decreasing function, we have

$$\begin{aligned}
\lambda_p^*(M) &\le \frac{Q(bR)}{Q(aR)} \frac{\int_{aR}^{bR} |\phi'(t)|^p (\ln Q(t))' \mathrm{Vol}(B_t^\kappa) dt + \int_{aR}^{bR} |\phi'(t)|^p \mathrm{Vol}(S_t^\kappa) dt}{\int_{aR}^{bR} |\phi(t)|^p \mathrm{Vol}(S_t^\kappa) dt} \\
&\le \frac{Q(bR)}{Q(aR)} \left(\frac{2\pi}{\lambda(p) R} \int_{aR}^{bR} |\phi'(t)|^p \big(\ln Q(t)\big)' \mathrm{Vol}(B_t^\kappa) dt + \theta_\kappa(R) \right) \\
&\le \frac{Q(bR)}{Q(aR)} \left(\frac{2\pi}{\lambda(p) R} \frac{\mathrm{Vol}\big(B_{bR}^\kappa\big)}{\mathrm{Vol}\big(S_{bR}^\kappa\big)} g_\kappa(R) h(R) + \theta_\kappa(R) \right). \qquad (5.4.7)
\end{aligned}$$

Note that

$$\begin{aligned}
\lim_{R\to\infty}\frac{Q(bR)}{Q(aR)} &= 1,\\
\lim_{R\to\infty}\frac{\mathrm{Vol}\big(B^{\kappa}_{bR}\big)}{\mathrm{Vol}\big(S^{\kappa}_{bR}\big)}\frac{2\pi}{\lambda(p)R} &= \begin{cases} 0 \ \text{ if } \ \kappa<0,\\ \frac{2b\pi}{n\lambda(p)} \ \text{ if } \ \kappa=0,\end{cases}\\
\lim_{R\to\infty} g_\kappa(R) &= \frac{(n-1)^p\sqrt{-\kappa}^p}{p^p},\\
\lim_{R\to\infty} h(R) &= 0,\\
\lim_{R\to\infty}\theta_\kappa(R) &= \frac{(n-1)^p\sqrt{-\kappa}^p}{p^p}.
\end{aligned}$$

Passing to the limit as $R\to\infty$ in (5.4.7), we obtain the upper bound which gives the conclusion.

□

Since the function $Q(t)$ is non-decreasing, we see that $\frac{Q(bR)}{Q(aR)}\leq\frac{Q(R)}{Q(\frac{R}{2})}$ for $[a,b]\subset[1/2,1]$. If we replace the hypothesis on the volume growth in Theorem 5.4.1, we are able to get a more general result as follows:

Theorem 5.4.2 *Let M^n be an n-dimensional complete properly immersed minimal submanifold in a Cartan-Hadamard manifold N of sectional curvature K_N bounded from above by $K_N\leq\kappa<0$. Suppose that the immersion has an extrinsic doubling property, namely*

$$\frac{Q(R)}{Q(\frac{R}{2})}<C.$$

Then,

$$\frac{(n-1)^p\sqrt{-\kappa}^p}{p^p}\leq\lambda^*_p(M)\leq\frac{C(n-1)^p\sqrt{-\kappa}^p}{p^p}.$$

As a consequence of Theorem 5.4.1, we get the following interesting intrinsic result in the direction of the generalized McKean's theorem obtained by Lima, Montenegro and Santos in (LIMA *et al.*, 2010).

Theorem 5.4.3 *Let M^n be a complete simply connected manifold with sectional curvature bounded from above $K_M\leq\kappa<0$. Furthermore, suppose that there exists a point $q\in M$ such that*

$$\sup_{R>0}\frac{\mathrm{Vol}(B^q_R)}{\mathrm{Vol}(B^\kappa_R)}<+\infty, \tag{5.4.8}$$

where B_R^q is the geodesic ball in M centered at q of radius R, and B_R^κ is the geodesic ball in $\mathbb{H}^n(\kappa)$ of the same radius R. Then

$$\lambda_p^*(M) = \frac{(n-1)^p \sqrt{-\kappa}^p}{p^p}. \tag{5.4.9}$$

In the following three theorems, the assumptions on the norm of the second fundamental form guarantee that the minimal submanifold has finite volume growth, i.e., $\sup_{R>0} Q(R) < +\infty$. Hence the following three theorems are immediate consequences of Theorem 5.4.1, (see also Theorem 2.7.2, Theorem 2.7.3 and Theorem 2.7.4).

Theorem 5.4.4 *Let M be an immersed minimal surface in the hyperbolic space $H^m(\kappa)$ of constant sectional curvature $\kappa < 0$. Suppose that M^2 has finite total extrinsic curvature, i.e., $\int_M |\alpha|^2 dM < \infty$. Then the p-fundamental tone satisfies*

$$\lambda_p^*(M) = \frac{\sqrt{-\kappa}^p}{p^p}.$$

Remark 5.4.2 *We note tha the finite total scalar curvature condition implies that the immersion is proper (Cf. (ANDERSON, 1984)).*

Theorem 5.4.5 *Let M^n be a properly immersed minimal submanifold in the hyperbolic space $H^m(\kappa)$ of constant sectional curvature κ with second fundamental form α. Suppose that $n > 2$ and the submanifold is of faster than exponential decay of its extrinsic curvature. Namely, there exists a point $q \in M$ such that*

$$|\alpha|(x) \leq \frac{\delta(r_q(x))}{e^{2\sqrt{-\kappa}r_q(x)}},$$

where $\delta(r)$ is a function such that $\delta(r) \to 0$ when $r \to \infty$ and r_q is the extrinsic distance function. Then

$$\lambda_p^*(M) = \frac{(n-1)^p \sqrt{-\kappa}^p}{p^p}.$$

Theorem 5.4.6 *Let M^n be a minimal submanifold immersed in $\mathbb{R}^m$ with finite total scalar curvature, i.e., $\int_M |\alpha|^n dM < +\infty$. Then*

$$\lambda_p^*(M) = 0.$$

5.5 Bernstein-Heinz-Chern-Flanders type inequalities

In this section, we study transversely oriented codimension one C^2-foliations of open subsets Ω of Riemannian manifolds M and obtain lower bounds estimates for the infimum of the mean curvature of the leaves in terms of the p-fundamental $\lambda_p^*(\Omega)$ tone of Ω, which are called Bernstein-Heinz-Chern-Flanders type inequalities. The motivation of this study is summarized as follows. Let $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < r\}$ and $z(x,y)$ be a C^2 function defined on B . Denote by $H(x,y)$ and $K(x,y)$ the mean and Gauss curvatures of the graph of $z = z(x,y)$. In 1955 Heinz proved that if $|H| \geq \beta > 0$, then $r \leq \frac{1}{\beta}$ and if $K \geq \beta > 0$, then $r \leq \frac{1}{\sqrt{\beta}}$, which improves the result of Bernstein and Efimov .

Later, Chern and Flanders obtained independently a natural generalization for graphs of C^2 functions $z(x_1,\ldots,x_n)$ defined on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. In fact they proved that if $\inf|H_{(x_1,\ldots,x_n)}| \geq b > 0$ then $b \leq Vol_{n-1}(\partial\Omega)/Vol_n(\Omega)$, and and if the scalar curvature $R(x_1,\ldots,x_n) \geq b > 0$ then $\sqrt{b} \leq Vol_{n-1}(\partial\Omega)/Vol_n(\Omega)$. These inequalities are called Chern-Heinz inequalities for graphs.

In (SALAVESSA, 1989) Isabel Salavessa generalized the Chern–Heinz inequality for graphs $G(f)$ of smooth function $f : M \to \mathbb{R}$ for a Riemannian manifold, proving that $\inf|H_{G(f)}| \leq Vol_{n-1}(\partial\Omega)/Vol_n(\Omega)$ for every oriented domain $\Omega \subset M$, where $H_{G(f)}$ is the mean curvature of the graph of f.

Barbosa and others in (BARBOSA *et al.*, 2008) realized that a graph of a function C^2 function can be regarded as a leave of a transversaly oriented C^2-foliaton of the product $M \times \mathbb{R}$ obtained by vertical translation of the graph. In this context, the Chern-Heinz inequality says that the infimum of the mean curvatures of these leaves are bounded above by the Cheeger constant $h(M) = \inf_\Omega Vol_{n-1}(\partial\Omega)/Vol_n(\Omega)$, where the infimum is taken over all relatively compact subsets Ω of M with smooth boundary. Hence, they were able to estimate the infimum of the mean curvature H^F of the leaves of a C^2 foliation of an open set Ω of a Riemannian manifold M in terms of the fundamental tone of the open set Ω. In this section we extend their results for the p-fundamental tone, that is, we prove that the infimum of the mean curvatures of the leaves is bounded above by the p-fundamental tone of M. Therefore, we have the following result.

Theorem 5.5.1 *Let $\mathscr{F}$ be a transversely oriented codimension one C^2-foliation of a connected open set Ω of $(n+1)$-dimensional Riemannian manifold M. Then*

$$p\sqrt[p]{\lambda_p^*(\Omega)} \geq \inf_{\mathscr{F}} \inf_{x\in F} |H^F(x)|,$$

where H^F denotes the mean curvature function of the leaf F.

Proof: Since Ω admits a transversely oriented C^2 foliation of codimension one, we can choose a continuous unit vector field η on M that is normal to the leaves of F. Denote by H^F the value of the mean curvature of the leaf F at x computed with respect to η. Define by $b = \inf_{\mathscr{F}} \inf_{x\in F} |H^F(x)|$. Since if $b = 0$ then it is trivial, we may assume that $b > 0$. This assumption implies that H^F does not change sign. Thus it is possible to choose the unite normal vector field η such that $H^F(x) > 0$ for every $x \in \Omega$. Then, we have the well known relation between the divergence of η and the mean curvature H^F

$$\mathrm{div}\,\eta = H^F.$$

Since $\mathrm{div}\,\eta = H^F \geq b$ and $\|\eta\| = 1$, we may apply (5.2.4) to get

$$p\sqrt[p]{\lambda_p^*} \geq c(\Omega) \geq \inf_{\Omega} \frac{\mathrm{div}\,\eta}{\|\eta\|} \geq b,$$

which gives the desired result. □

In particular, if $p = 2$, then Theorem 5.5.1 is exactly the same as the result obtained by Barbosa, Bessa, and Montenegro (BARBOSA *et al.*, 2008). Moreover, Theorem 5.5.1 gives the following interesting consequences for a Riemannian manifold M satisfying $\lambda_p^*(M) = 0$.

Corollary 5.5.1 *Let $\mathscr{F}$ be a transversely oriented codimension one C^2-foliation of a Riemannian manifold M for which $\lambda_p^*(M) = 0$ for some $p \in (1,\infty)$. If the leaves of $\mathscr{F}$ have the same constant mean curvature, then they are minimal submanifolds of M.*

Corollary 5.5.2 *Let $\mathscr{F}$ be a transversely oriented codimension one C^2-foliation of a Riemannian manifold M with the Ricci curvature $\mathrm{Ric}_M \geq (n-1)\kappa$. Then,*

1. $p\sqrt[p]{\lambda_p^*(\mathbb{M}^n(\kappa))} \geq \inf_{\mathscr{F}} \inf_{x\in F} |H^F(x)|$, *where $\mathbb{M}^n(\kappa)$ is the simply connected space form of constant sectional curvature κ and dimension n.*
2. *If $\kappa \leq 0$ and $H^F \geq b > 0$, then $\kappa = -a^2$ for some constant $a > 0$ satisfying $(n-1)a \geq b$.*

Proof: Takeuchi (TAKEUCHI, 1998) proved that if $\mathrm{Ric}_M \geq \kappa(n-1)$, then

$$\lambda_p^*(B_M(r)) \leq \lambda_p^*(B_{\mathbb{M}^n(\kappa)}(r)),$$

where $B_{\mathbb{M}^n(\kappa)}(r)$ is a geodesic ball of radius r in the simply connected n-dimensional space form $\mathbb{M}^n(\kappa)$ and $B_M(r)$ is a geodesic ball of radius r in M. Noting that the p-fundamental tone can be also given by $\lambda_p^*(M) = \lim_{r\to\infty} \lambda_p^*(B_M(r))$, we obtain

$$\lambda_p^*(M) \leq \lambda_p^*(\mathbb{M}^n(\kappa)). \tag{5.5.1}$$

Then, by (5.5.1) and Theorem 5.5.1 we have

$$\inf_{\mathscr{F}} \inf_{x \in F} |H^F(x)| \leq p \sqrt[p]{\lambda_p^*(M)} \leq p \sqrt[p]{\lambda_p^*(\mathbb{M}^n(\kappa))},$$

which proves 1.

On the other hand, we note that if $H^F \geq b > 0$, we can use (1) to get $\sqrt[p]{\lambda_p^*(\mathbb{M}^n(\kappa))} \geq \frac{b}{p}$. Since, $\kappa \leq 0$ we have $\lambda_p^*(\mathbb{M}^n(\kappa)) \neq 0$ if and only if $\kappa = -a^2$ where a is a positive number, in this case it follows that $\lambda_p^*(\mathbb{M}^n(\kappa))) = \frac{(n-1)^p a^p}{p^p}$. Then, the inequality $\sqrt[p]{\lambda_p^*(\mathbb{M}^n(\kappa))} \geq \frac{b}{p}$ is possible if and only if $\kappa = -a^2$ and $(n-1)a \geq b$, which proves 2. □

Now we give a version of Theorem 5.5.1 for the scalar curvature when the ambient manifold has nonpositive sectional curvature.

Theorem 5.5.2 *Let M be a $(n+1)$-dimensional Riemannian manifold with nonpositive sectional curvature and let $\mathscr{F}$ be a transversely oriented codimension one C^2 foliation of a connected open set $\Omega \subset M$. Suppose that the scalar curvature R of each leaf is nonnegative. Then*

$$\sqrt{\inf R} \leq p \sqrt[p]{\lambda_p^*(\Omega)}.$$

In particular, if $\lambda_p^*(M) = 0$, $\Omega = M$ and all the leaves have the same constant nonnegative scalar curvature $R \geq 0$, then $R = 0$.

Proof: We may assume that $\inf R = c > 0$. Let $q \in F$ be a point of the leave $F \in \mathscr{F}$ and $\{e_1, \ldots, e_n\}$ be an orthonormal basis for the tangent space T_qF. By Gauss equation (2.7.1) we have

$$\widetilde{K}(e_i, e_j) = \overline{K}(e_i, e_j) + \langle \alpha(e_i, e_i), \alpha(e_j, e_j) \rangle - |\alpha(e_i, e_j)|^2, \tag{5.5.2}$$

where $\overline{K}$ denotes the sectional curvature of the space $\Omega \times \mathbb{R}$ and $\widetilde{K}$ denotes the Gaussian curvature of F. Taking summation over i, j in (5.5.2) we get

$$R(q) = \sum_{i,j} \overline{K}(e_i, e_j) + H^2 - \|\alpha\|^2.$$

Note that $\overline{K} \leq 0$, since the sectional curvatures K of M are nonpositive. This implies

$$R(q) \leq H^2.$$

Since $R \geq c > 0$, we have $H \geq \sqrt{c} > 0$. Applying Theorem 5.5.1, we get

$$p \sqrt[p]{\lambda_p^*(\Omega)} \geq \inf_{\mathscr{F}} \inf_{x \in F} |H^F(x)| \geq \sqrt{c} = \sqrt{\inf R}.$$

□

The r-th order mean curvature H_r of an n-dimensional oriented hypersurface $M \subset N$ is defined by the elementary symmetric polynomial of degree r in the principal curvatures $\lambda_1, \lambda_2, \cdots, \lambda_n$ on M as follows:

$$\binom{n}{r} H_r = \sum_{1 \leq i_1 < \ldots < i_r \leq n} \lambda_{i_1} \ldots \lambda_{i_r}.$$

Note that our mean curvature H satisfies $H = nH_1$. Since $H_1^2 \geq H_2$, we immediately get a consequence of Theorem 5.5.1 for the second order mean curvature H_2 as follows:

Corollary 5.5.3 *Let $\mathscr{F}$ be a transversely oriented codimension one C^2-foliation of a connected open set Ω of $(n+1)$-dimensional Riemannian manifold M. Suppose that the leaves have the second order mean curvature $H_2 \geq 0$. Then,*

$$p \sqrt[p]{\lambda_p^*(\Omega)} \geq n \inf_{\mathscr{F}} \inf_{x \in F} (H_2^F)^{\frac{1}{2}},$$

where H_2^F stands for the second order mean curvature function of the leaf F. In particular, if $\lambda_p^*(M)$, $\Omega = M$ and all the leaves have the same constant second order mean curvature $H_2 \geq 0$, then $H_2 = 0$.

For higher-order mean curvature H_r, we have the following result.

Corollary 5.5.4 *Let $\mathscr{F}$ be a transversely oriented codimension one C^2-foliation of a connected open set Ω of $(n+1)$-dimensional Riemannian manifold M. Suppose that $\lambda_p^*(M) = 0$, and all the leaves have the same constant r-th order mean curvature $H_r \geq 0$. If $r \geq 3$ suppose additionally that there is a point $q \in F$ for some leave $F \in \mathscr{F}$ such that all principal curvatures $k_i(q) \geq 0$. Then, $H_r = 0$.*

Proof: If $H_r = 0$ there is nothing to prove. Now let us prove that H_r can not be strictly positive. In fact, suppose by contradiction that $H_r > 0$. Since $k_i(q) \geq 0$, we have that the Gårding inequalities hold (for details about the validity of Gårding inequalities under the hypothesis $H_r > 0$ and $k_i(q) \geq 0$ we recommend (FONTENELE; SILVA, 2005) proof of Theorem 1.3), it means that $H_1 \geq H_r^{\frac{1}{r}} > 0$. Using that $\lambda_p^*(M) = 0$ and applying Theorem 5.5.1 we get

$$0 = \frac{p}{n} \sqrt[p]{\lambda_p^*(M)} \geq \inf_M H_1 = \inf_M (nH_1) \geq nH_r^{\frac{1}{r}} > 0,$$

which is a contradiction. □

5.6 Haymann-Makai-Osserman type inequality

Let Ω be a connected open set of M , we define the inradius $\rho(\Omega)$ of Ω by $\rho(\Omega) = \sup\{r > 0 : B_M(r) \subset \Omega\}$, where $B_M(r)$ is the geodesic ball of radius r of M. Let $\Omega \subset \mathbb{R}^2$ be a connected open set, Makai in (MAKAI, 1965) proved that the fundamental tone $\lambda^*(\Omega)$ is bounded from above by $\frac{1}{4\rho^2}$, where ρ is the inradius of Ω. Makai's estimate for the fundamental tone was improved latter by Haymann (HAYMAN, 1978) and Osserman (OSSERMAN, 1977). In (BARBOSA *et al.*, 2008) Barbosa, Bessa, and Montenegro extended Haymann-Makai-Osserman inequality for embedded tubular neighbourhoods of simple smooth curves in $\mathbb{R}^n$. In this section, we prove a Haymann-Makai-Osserman type inequality for the p-fundamenal tone.

Theorem 5.6.1 *Let $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ be a simple smooth curve and $T_\gamma(\rho(t))$ be an embedded tubular neighborhood of γ with variable radius $\rho(t)$ and a smooth boundary $\partial T_\gamma(\rho(t))$. Let $\rho_0 = \sup_t \rho(t) > 0$ be its inradius. Then*

$$\lambda_p^*(T_\gamma(\rho(t))) \geq \frac{(n-1)^p}{p^p \rho_0^p}. \tag{5.6.1}$$

Proof: Barbosa, Bessa, and Montenegro in (BARBOSA *et al.*, 2008) showed that $T_\gamma(\rho(t))$ admits a smooth codimension one transversally oriented foliation such that the mean curvature of the leaves is constant $\frac{n-1}{\rho_0}$ (for details about this foliation see (BARBOSA *et al.*, 2008) proof of Theorem 4.1). Hence, by Theorem 5.5.1 we have

$$\lambda_p^*(T_\gamma(\rho(t))) \geq \frac{(n-1)^p}{p^p \rho_0^p},$$

which gives the desired result. □

6 CONCLUSION

In this work we studied compact almost Ricci soliton with null Cotton tensor and proved that under some conditions on the symmetric functions associated to the Schouten tensor the almost Ricci soliton is isometric to a standard sphere. Furthermore, it would be interesting to remove the hypothesis on the symmetric functions of the Schouten tensor and prove that an almost Ricci soliton with null cotton tensor is isometric to a sphere. This is a interesting subject for future studies. We highlight that the same problem for locally conformally flat compact almost Ricci soliton is still open. Then, the weak condition on the cotton tensor considered in this work is an interesting result.

On the other hand we provided a partial solutition for the CPE conjecture which improve, some known results on this subjet. However, the conjecture is still an open problem and an interesting subject to future studies.

Finally on the last part we obtained some lower bounds for the p-fundametal tone of geodesic ball with bounded sectional curvature and of submanifolds with bounded mean curvature. Moreover, we provided upper bounds for the p -fundamental tone of minimal submanifolds of a Hadamard-Cartan manifold. Finally, we study transversely oriented co-dimension one C^2-foliations of open subsets of Riemannian manifolds M and obtain lower bounds estimates for the infimum of the mean curvature of the leaves in terms of the p-fundamental tone of . As an application we get a Haymann-Makai-Osserman inequality for embedded tubular neighbourhoods of simple smooth curves in $\mathbb{R}^n$.

The present thesis is divided in three parts because each part is a paper. We studied three different subject which give us three different papers. The three papers present on this thesis are already accepted to publish in international journals.

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